Quantum-rigid random quantum graphs

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Abstract

A quantum graph $\mathcal{G}$ housed by a matrix algebra M_n can be encoded as an operator system $\mathcal{S} = \mathcal{S}_{\mathcal{G}} \le M_n$. There are two sensible notions of quantum automorphism group for any such: $\mathrm{Qut}(\mathcal{G})$, capturing the quantum symmetries of the adjacency matrix $A : M_n \to M_n$ attached to $\mathcal{G}$, and $\mathrm{Qut}(\mathcal{S} \le M_n)$, the quantum group acting universally on M_n so as to preserve its C^* structure, standard trace, and subspace $\mathcal{S} \le M_n$. The two quantum groups coincide classically, but diverge in general. We nevertheless show that both are generically trivial in the sense that they are so for $\mathcal{S} \le M_n$ ranging over a non-empty Zariski-open set under all reasonable dimensional constraints on $\dim \mathcal{S}$ and n.

This extends analogous prior results by the first and third authors to the effect that classical symmetry groups of still-quantum graphs are generically trivial, and offers a fully quantum counterpart to the familiar probabilistic almost-rigidity of finite graphs. An auxiliary result sheds some light on the relationship between the two notions of quantum automorphism group, identifying the universal preserver of the quantum adjacency matrix of $\mathcal{G}$ with the quantum automorphism group not of $\mathcal{S} \le M_n$, but rather of the complex conjugate $\overline{\mathcal{S}} \le M_n$.

Key words: Choi matrix; Grassmannian; Zariski topology; comodule; compact quantum group; operator system; quantum automorphism group; quantum graph

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Introduction

Quantum graphs have received much recent attention in the literature; the later is by now substantial enough to make it difficult to survey with any pretense at exhaustion in a short recollection such as this, but we mention [6, 9, 13, 20, 24] as representative sources (with substantive references of their own) along with a number of papers [4, 8, 16, 26, 28, 33, 38] concerned with quantum graphs' relevance to emergent areas of mathematical physics such as quantum computation and information theory.

By way of a quick recollection:

Definition 0.1 Let B be a finite dimensional C^*-algebra equipped with a faithful tracial functional τ such that $mm^* = \mathrm{id}$, where $m : B \otimes B \to B$ is the multiplication map and $m^* : B \to B \otimes B$ is its adjoint with respect to the inner product induced by τ, i.e. $\langle x \mid y \rangle := \tau(x^* y)$. We call a completely positive map $A : B \to B$ a *quantum adjacency matrix* if it satisfies $m(A \otimes A)m^* = A$. The triple $\mathcal{G} := (B, \tau, A)$ will be called a *quantum graph.* ♦

Remark 0.2 Such a functional τ is unique for $B \simeq \bigoplus_{a=1}^k M_{n_a}$, namely $\tau(x) = \bigoplus_{a=1}^k n_a \operatorname{Tr}_{n_a}(x_a)$ for the usual trace Tr_n on M_n. In the non-tracial case such a positive functional is not unique. ♦

An equivalent definition [41, Definition 2.6] is in terms of B'-bimodules inside

$$\mathcal{L}(\mathsf{H}) := \{\text{bounded operators on } \mathsf{H}\},$$

having represented B faithfully on the Hilbert space H. In the special case $B = M_n$ and $\mathsf{H} = \mathbb{C}^n$ (which we are concerned with virtually exclusively) these are just subspaces of M_n. *Operator systems*, i.e. [30, pre Proposition 2.1] unital self-adjoint subspaces correspond to reflexive, undirected quantum graphs. This condition phrased in terms of the quantum adjacency matrix means that $m(A \otimes \mathrm{id})m^* = \mathrm{id}$ and A is self-adjoint. An operator system associated with the quantum graph $\mathcal{G}$ will be denoted by $\mathcal{S}$.

Definition 0.3 Let $\mathcal{G}$ be a quantum graph. Then $D := A\mathbb{1}$ is called the *degree matrix.* If $\mathcal{S}$ is the corresponding operator system and $(X_i)_i$ is its orthonormal basis with respect to the *normalized* trace $\mathrm{tr} = \frac{\mathrm{Tr}}{n}$, then the quantum adjacency matrix is given by $Ax = \sum_i X_i x X_i^*$, so $D = \sum_i X_i X_i^*$.

Many quantum graphs in this paper will arise as linear spans $\langle \mathcal{X} \rangle$ of operator tuples $\mathcal{X}$. ♦

Our primary goal is to establish the generic quantum rigidity of a quantum graph housed by M_n, "fully quantizing"

- the analogous [11, Theorem 3.19], to the effect that "most" operator systems $\mathcal{S} \le M_n$, under only the sensible dimensional constraints, have trivial (*classical*) automorphism group;

- as well as the well-known fully-classical counterpart ([23, Theorem 3.1] or [2, Theorem 9.13] for $p = 1/2$ modeled on the original [17, Theorem 2] etc.): selecting each edge of an n-vertex graph with probability $0 < p < 1$, the probability that the graph have trivial symmetry group approaches 1 as $n \to \infty$.

Quantum rigidity refers to a quantum graph's having trivial *quantum automorphism group*, which notion requires some unpacking of its own. In first instance, all such will be *compact quantum groups*; the background on the latter needed here is not extensive, and we refer the reader to [29, Chapter 1] or [35, Chapters 3 and 5] say, with other sources occasionally making an appearance where needed.

Compact quantum groups that will feature fairly prominently include

- the *quantum automorphism groups* $\mathrm{Qut}(B, \tau)$ of finite-dimensional C^*-algebras equipped with (typically tracial) states τ (e.g. [40, Theorem 6.1(2)] or [19, Definition 1.10]);

- the *free unitary group* $\mathrm{U}^+(n)$, whose underlying *CQG algebra* [15, Definition 2.2] $H := \mathcal{O}\left(\mathrm{U}^+(n)\right)$ is freely generated (as a $*$-algebra) by the entries of a unitary $u := (u_{ij})_{i,j} \in M_n(H)$ with $\overline{u} := \left(u_{ij}^*\right)_{i,j}$ again unitary.

The C^*-completion of $\mathcal{O}\left(\mathrm{U}^+(n)\right)$ is the Hopf $*$-algebra denoted by $A_u(n)$ in [39, Theorem 1.2] and $A_u(I_n)$ in [36, Remark 1.5(2)]. Note that the $\mathrm{U}^+(n)$-action on $\mathbb{C}^n$ affords a conjugation action on $(M_n, \tau := \text{normalized trace})$; we take this for granted repeatedly in the sequel.

We will begin with the more common definition of the quantum automorphism group of a quantum graph, introduced in [27] and then extended to the non-tracial case in [7].

Definition 0.4 Let $\mathcal{G} := (B, \tau, A)$ be a quantum graph. Let $\mathrm{Qut}(B, \tau)$ be the quantum automorphism group of (B, τ) and let $\alpha : B \to B \otimes \mathcal{O}(\mathrm{Qut}(B, \tau))$ be the associated action. We define $\mathcal{O}(\mathrm{Qut}(\mathcal{G}))$ to be quotient of $\mathcal{O}(\mathrm{Qut}(B, \tau))$ by the relation $\alpha \circ A = (A \otimes \mathrm{id}) \circ \alpha$ and call the resulting compact quantum group the quantum automorphism group of the quantum graph $\mathcal{G}$. ♦

Given the description of quantum graphs in terms of operator systems, the following seems natural.

Definition 0.5 Let $\mathcal{S} \leq B$ be an operator system. We call the quantum subgroup of $\mathrm{Qut}(B, \tau)$ preserving $\mathcal{S}$ the quantum automorphism group the quantum automorphism group of $\mathcal{S} \leq B$, denoted by $\mathrm{Qut}(\mathcal{S} \leq B)$. ♦

If $B = M_n$ then $\mathcal{S}$ can be viewed as a quantum graph on M_n, hence we have *two* candidates for the quantum automorphism group. The connection between different notions of the quantum automorphism groups of quantum graphs has been studied by Daws (see [14, Theorem 9.18]) and the relationship is not straightforward (see Example 1.5 below for relatively simple cases of disagreement). Nevertheless, we will show that for our purposes these objects are similar enough: one is generically trivial if and only if the other one is (Corollary 1.9).

We refer to $(d+1)$-dimensional operator systems as *operator d-systems* (for the unit will not play much of a role in assessing the size of the space of symmetries). Here and throughout the paper a quantum-graph/operator-d-system property holds *generically* if it does over a non-empty open set in the real Zariski topology on the Grassmannian $\mathbb{G}_{sa}(d, M_n)$ of d-dimensional self-adjoint complex subspaces of M_n.

In order to state Theorem 0.7 we need *spatial* versions of the two quantum groups in Definitions 0.4 and 0.5: quantum groups operating on the quantum graph in the appropriate structure-preserving manner, with an action induced by one on $\mathbb{C}^n$.

Definition 0.6 Let $\mathrm{G} \xrightarrow{\varphi} \mathrm{U}^+(n)$ be a quantum-group morphism inducing

$$\mathbb{C}^n \xrightarrow[\text{G-action}]{} \mathbb{C}^n \otimes \mathcal{O}(\mathrm{G}),$$

hence also a conjugation action

$$\left(M_n \cong \mathbb{C}^n \otimes (\mathbb{C}^n)^* \cong \mathrm{End}(\mathbb{C}^n)\right) \xrightarrow{\mathrm{Ad}_\varphi} M_n \otimes \mathcal{O}(\mathrm{G})$$

with associated morphism

$$\begin{tikzcd} & \mathrm{U}^+(n) \arrow[dr] & \\ \mathrm{G} \arrow[ur, "\varphi"] \arrow[rr, "\varphi_{\mathrm{Ad}}"'] & & \mathrm{Qut}(M_n, \tau) \end{tikzcd}$$

For quantum graphs $\mathcal{G}$ on M_n or operator systems $\mathcal{S} \leq M_n$ we use the collective notation $\mathrm{Qut}(\bullet)$ for the objects introduced in Definitions 0.4 and 0.5.

The *φ- (or $\mathbb{G}$-)relative quantum automorphism group* of a quantum graph $\mathcal{G}$ on M_n or an operator system $\mathcal{S}$ is the object dual to the *pushout* [3, Definition 2.5.1 and post Proposition 2.5.3]

$$\begin{tikzcd} & \mathcal{O}\left(\mathrm{Qut}(\bullet)\right) \arrow[dr] & \\ \mathcal{O}\left(\mathrm{Qut}(M_n, \tau)\right) \arrow[ur] \arrow[dr, "\varphi_{\mathrm{Ad}}"'] & & \mathcal{O}\left(\mathrm{Qut}_\varphi(\bullet)\right) \\ & \mathcal{O}(\mathrm{G}) \arrow[ur] & \end{tikzcd}$$

in the category of CQG algebras. ♦

Theorem 0.7 *Let $1 \leqslant d \leqslant n^2-2$ and $\mathrm{G} \xrightarrow{\varphi} \mathrm{U}^+(n)$ a quantum-group morphism. For an operator d-system $\mathcal{S} \le M_n$ with quantum graph $\mathcal{G}$ we have*

$$\mathrm{Qut}_{\varphi}(\mathcal{G}) = \mathrm{Qut}_{\varphi}\left(\overline{\mathcal{S}} \le M_n\right),$$

$\overline{\mathcal{S}}$ denoting the complex conjugate of $\mathcal{S}$.

That in hand, we can state the announced generic-rigidity result.

Theorem 0.8 *For positive integers $n \ge 3$ and $2 \le d \le n^2-3$ the quantum automorphism group $\mathrm{Qut}(\mathcal{S} \le M_n)$ is trivial for generic operator d-system $\mathcal{S} \le M_n$.*

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1 Two versions of the quantum automorphism group

We begin with an analogue of [11, Proposition 1.13] saying that all symmetries preserve the degree matrix.

Proposition 1.1 *Let $\mathcal{G} := (M_n, \tau, A)$ be a quantum graph and let $\mathcal{S} \le M_n$ be the corresponding operator system. Let $D := A\mathbb{1}$ be the degree matrix of $\mathcal{G}$. Then $\alpha(D) = D \otimes \mathbb{1}$ if α denotes either the action of $\mathrm{Qut}(\mathcal{G})$ or $\mathrm{Qut}(\mathcal{S} \le M_n)$ on M_n.*

Proof For $\mathrm{Qut}(\mathcal{G})$ this is clear because $\alpha \circ A = (A \otimes \mathrm{id}) \circ \alpha$, $D = A\mathbb{1}$ and α is unital.

For $\mathrm{Qut}(\mathcal{S} \le M_n)$, recall that $D = \sum_j e_j e_j^*$, where $(e_j)_j$ is an orthonormal basis of $\mathcal{S}$. Observe that the Hilbert space $(M_n, \langle - \mid - \rangle)$ is a *unitary* [15, (1.8)] H-comodule for the CQG algebra $H := \mathcal{O}(\mathrm{Qut}(M_n, \tau))$, so the same goes for H'-subcomodules $\mathcal{V} \le M_n$ for CQG quotient algebras $H \twoheadrightarrow H'$, in particular for $\mathcal{S} \le M_n$. We have

$$\mathcal{V} \ni e_j \longmapsto \sum_i e_i \otimes u_{ij} \in \mathcal{V} \otimes H' \quad \Longrightarrow \quad \mathcal{V}^* \ni e_j^* \longmapsto \sum_i e_i^* \otimes u_{ij}^* \in \mathcal{V}^* \otimes H',$$

identifying [29, post Definition 1.3.8] $\mathcal{V}^* \le M_n$, also H'-invariant, with the *dual* comodule in the monoidal-categorical sense of [18, Definition 2.10.1]. It follows that

$$\sum_j e_j \otimes e_j^* \in \mathcal{V} \otimes \mathcal{V}^* \le \mathcal{M}_n^{\otimes 2},$$

identifiable with $1 \in \mathrm{End}(\mathcal{V}) \cong \mathcal{V} \otimes \mathcal{V}^*$, is H'-fixed, and hence so is its image through multiplication (an H'-comodule morphism). ■

Similarly to [11, Theorem 3.24] one can now conclude that generically the quantum automorphism group of a quantum graph is abelian. Some terms and notation will be useful in discussing this.

Definition 1.2 (1) The quantum group $\mathrm{D}^+(n) \le \mathrm{U}^+(n)$ is the *Pontryagin dual* [35, §3.3] $\widehat{F_n}$ of the free group

$$F_n \cong \langle s_i,\ 1 \le i \le n \rangle, \quad s_i \text{ free generators,}$$

coacting on $\mathbb{C}^n$ in the obvious fashion: the standard basis $(e_i)_{i=1}^n \subset \mathbb{C}^n$ is homogeneous for an F_n-grading assigning e_i degree s_i.

(2) $\mathrm{D}^+(n)$ acts on $\mathbb{C}^n$ as well as (M_n, τ), by conjugation. We will say that a quantum group G mapping to either $\mathrm{U}^+(n)$ or $\mathrm{Qut}(M_n, \tau)$ acts

- *diagonally* on $\mathbb{C}^n$ or M_n if $\mathrm{G} \to \mathrm{U}^+(n)$ factors through $\mathrm{D}^+(n)$, and similarly for the action on M_n;
- *essentially diagonally* if the same holds up to a change of orthonormal basis for $\mathbb{C}^n$. ♦

Proposition 1.3 *If $X \in M_n$ is such that X^*X has simple spectrum then the action of $\mathrm{Qut}(\mathbb{C}X \subseteq M_n)$ on M_n is essentially diagonal in the sense of Definition 1.2(2).*

Proof The coaction of $\mathcal{O}(\mathrm{G})$, $\mathrm{G} := \mathrm{Qut}(\mathbb{C}X \subseteq M_n)$ is by assumption of the form

$$X \xmapsto{\ \rho\ } X \otimes \delta, \quad \delta \in \mathcal{O}(\mathrm{G}) \text{ *group-like* [32, Definition 2.1.10].}$$

ρ thus fixes the simple-spectrum self-adjoint operator X^*X; assuming that operator diagonal, the coaction will factor through $M_n \to M_n \otimes \mathcal{O}(\mathrm{D}^+(n))$. ■

Corollary 1.4 *For $3 \le n$ and $1 \le d \le n^2 - 2$ $\mathrm{Qut}(\mathcal{S} \le M_n)$ and $\mathrm{Qut}(\mathcal{G})$ generically act essentially diagonally in the sense of Definition 1.2(2).*

Proof In both cases the degree matrix D is fixed with almost-surely simple spectrum [11, proof of Theorem 3.24], hence the conclusion via Proposition 1.3. ■

This in hand, we can revisit the issue of contrasting the two notions of quantum automorphism group discussed in Definitions 0.4 and 0.5.

Example 1.5 By Corollary 1.4 it suffices to assume that $\mathrm{Qut}(\langle \mathcal{X} \rangle \le M_n)$ operates via the conjugation of a Γ-grading, where $F_n \twoheadrightarrow \Gamma$ is a quotient of the free group $F_n = \langle s_i \rangle_{i=1}^n$ assigning the standard basis element $e_i \in \mathbb{C}^n$ degree s_i. The claim is that even when Y_j are respectively γ_j-homogeneous for $\gamma_j \in \Gamma$ (which we can always assume under the circumstances),

$$M_n \ni x \xmapsto{\Phi_{\mathcal{S}} \in \mathrm{End}(M_n)} \sum_{j=1}^{\dim \mathcal{S}} Y_j x Y_j^* \in M_n, \quad \mathcal{S} := \mathbb{C} \oplus \langle \mathcal{X} \rangle$$

need not be a graded map.

Indeed, consider a *single* γ-homogeneous $Y = Y_j$ for non-central $\gamma \in \Gamma$ (so that $\langle \mathcal{X} \rangle = \mathrm{span}\,\{Y, Y^*\}$) in such a fashion as to ensure the existence of some γ'-homogeneous $x \in M_n$ with $[\gamma, \gamma'] \ne 1$ and $YxY^* \ne 0$. This is easily arranged: set $n := |\Gamma|$ for a finite group Γ, identify $\mathbb{C}^n \cong \mathbb{C}\Gamma$ (the group algebra), and take for Y and x the left translation operators by non-commuting $\gamma, \gamma' \in \Gamma$ respectively. We then have

$$YxY^* = \text{translation by } \gamma\gamma'\gamma^{-1} \ne \gamma',$$

meaning that $Y \cdot Y^*$ does not preserve degrees. ♦

Classically the automorphism group of M_n is the *projective* unitary group and one cannot go through the usual unitary group, in particular the automorphism group does not act on $\mathbb{C}^n$. In our setting, generically the quantum automorphism group will sit inside $\mathrm{D}^+(n)$, so it acts on $\mathbb{C}^n$. This nice algebraic property will allow us to compare the two versions of the quantum automorphism group.

Proposition 1.6 *Let V and W be finite-dimensional unitary representations of a compact quantum group. Then the Choi correspondence between completely positive maps $\Phi : V \otimes V^* \to W \otimes W^*$ and positive operators $C_\Phi \in \mathrm{End}(W^* \otimes V)$ is equivariant.*

Proof This is essentially [37, Theorem 4.13]. We provide some details for the reader's convenience. Note that

(1-1)
$$\begin{tikzcd}
W^*\otimes V\otimes V^*\otimes V \arrow[rr, "\mathrm{id}_{W^*}\otimes\Phi\otimes\mathrm{id}_V"] & & W^*\otimes W\otimes W^*\otimes V \arrow[d, "\mathrm{ev}_W\otimes\mathrm{id}_{W^*\otimes V}"] \\
W^*\otimes V \arrow[u, "\mathrm{id}_{W^*\otimes V}\otimes\mathrm{db}_{V^*}"] \arrow[rr, "C_\Phi"'] & & W^*\otimes V
\end{tikzcd}$$

where

- asterisks denote duals in *rigid monoidal C^*-categories* [29, Definition 2.2.1];
- and the morphisms

$$\mathbf{1} \xrightarrow[\text{for `dual basis'}]{\mathrm{db}_Z} Z \otimes Z^* \quad \text{and} \quad Z^* \otimes Z \xrightarrow[\text{for `evaluation'}]{\mathrm{ev}_Z} \mathbf{1}$$

($\mathbf{1}$ being the monoidal unit) are those witnessing the duality (what [29, Definition 2.2.1] would denote by $\overline{R}$ and R^* respectively).

This means if Φ is equivariant, i.e. is a morphism in the representation category, then so is C_Φ. A similar formula expresses Φ in terms of C_Φ. ■

The following result is worked out here for completeness; it is easily extracted from [12, Remark 4 and proof of Theorem 1], modulo inessential linguistic differences. It relates the operator system to the Choi matrix of the quantum adjacency matrix equivariantly.

Lemma 1.7 *Let V and W be finite-dimensional unitary representations of a compact quantum group and*

(1-2)
$$V \otimes V^* \xrightarrow{\Phi=\sum_\ell Y_\ell \cdot Y_\ell^*} W \otimes W^*, \quad Y_\ell \in W \otimes V^*$$

an equivariant completely positive map.

The image of the attached equivariant morphism C_Φ of (1-1) is precisely the span of $\overline{Y_\ell}$, using the identification

(1-3)
$$W^* \otimes V \simeq \overline{W} \otimes V \simeq \overline{W \otimes \overline{V}}.$$

Proof We begin by examining the map C_Φ (frequently referred to as the *Choi matrix* [31, proof of Theorem 3] of Φ, in light of [12, Theorem 2]) in the specific case of $\Phi = Y \cdot Y^*$ for $Y \in \mathrm{Hom}(V, W) \cong W \otimes V^*$. Some pertinent notation:

- es and fs denote basis elements of V, V^* and W, W^* respectively;
- lower (upper) induces indicate basis elements for V, W and their dual counterparts in V, W^* respectively;
- repeated indices in the same expression indicate a suppressed sum (the familiar [34] *Einstein summation* convention).

All of this in place, tracing a basis element $f^r \otimes e_i \in W^* \otimes V$ through (1-1) produces

$$\begin{aligned} f^r \otimes e_i &\longmapsto f^r \otimes e_i \otimes e^j \otimes e_j \longmapsto f^r \otimes \Phi(e_{ij}) \otimes e_j \longmapsto f^r \circ \Phi(e_{ij}) \otimes e_j \\ &\overset{\Phi = Y\cdot Y^*}{=\!=} f^r \circ (y_{pi}\overline{y_{qj}})_{p,q} \otimes e_j = (y_{ri}\overline{y}_{qj})_q \otimes e_j = y_{ri}\overline{y}_{qj} f^q \otimes e_j = y_{ri}\overline{Y} \end{aligned}$$

In the general case of (1-2) the range of C_Φ will thus be precisely the span of $\sum_\ell y_{\ell,ri}\overline{Y}_\ell$ for varying i and r. That

$$\mathrm{span}\left\{\sum_\ell y_{\ell,ri}\overline{Y}_\ell\right\}_{i,r} = \mathrm{span}\left\{\overline{Y}_\ell\right\}_\ell$$

is easily seen: the $\leq$ inclusion is self-evident, and the dimensions coincide (with the *rank* [21, Definition 3.35] of the tensor $(y_{\ell,ri})_{\ell,i,r}$). ■

Proof of Theorem 0.7 Let α be the action of $\mathrm{Qut}_\varphi(\mathcal{G})$ on M_n. From Proposition 1.6 we know that the Choi matrix of the quantum adjacency matrix of $\mathcal{G}$ is equivariant. It follows that its range, identified with $\overline{\mathcal{S}}$ in Lemma 1.7, is invariant and hence $\mathrm{Qut}_\varphi(\mathcal{G}) \leq \mathrm{Qut}_\varphi(\overline{\mathcal{S}} \leq M_n)$. On the other hand, the action β of $\mathrm{Qut}_\varphi(\overline{\mathcal{S}} \leq M_n)$ on M_n leaves $\overline{\mathcal{S}}$ invariant. The Choi matrix must thus be equivariant, and with it the associated quantum adjacency matrix. ■

Naturally, being interested in generic behavior, knowledge on almost all complex conjugates $\overline{\mathcal{S}}$ translates to knowledge on almost all operator systems $\mathcal{S}$.

Corollary 1.8 *Let $1 \leqslant d \leqslant n^2 - 2$. For generic operator d-systems $\mathcal{S} \leq M_n$ (and associated quantum graphs $\mathcal{G}$) we have*

$$\mathrm{Qut}(\mathcal{G}) = \mathrm{Qut}\left(\overline{\mathcal{S}} \leq M_n\right),$$

$\overline{\mathcal{S}}$ denoting the complex conjugate of $\mathcal{S}$.

Proof It follows from Corollary 1.4 that both actions on M_n are essentially diagonal, so Theorem 0.7 applies to the embedding $\mathrm{D}^+(n) \overset{\varphi}{\hookrightarrow} \mathrm{U}^+(n)$. ■

In particular:

Corollary 1.9 $\mathrm{Qut}(\mathcal{S}\le M_n)$ *is generically trivial if and only if* $\mathrm{Qut}(\mathcal{G})$ *is.* ■

Remark 1.10 It will be instructive to revisit Example 1.5 and illustrate how the complex conjugation (1-3) (related to the opposite action of [14, Theorem 9.18]) affects matters in one particular class of examples.

In the present case we have $V = W = \mathbb{C}\Gamma$, and the index set $I \ni i$ for a basis $(e_i)_I$ is Γ itself. For Y we choose

$$Y = \sum_{i\in\Gamma} e_{\gamma i,i} \quad \text{with} \quad \deg e_{pq} = pq^{-1} \in \Gamma.$$

Now, the complex conjugate $\overline{Y}$ will be $\sum_i f_{\gamma i,i}$ ($e \to f$ change in lettering will help with the bookkeeping), with the caveat that these are now matrix units in $V^*\otimes V \cong \mathrm{End}(V^*)$ instead (cf. (1-3)) and hence

$$\deg f_{pq} = p^{-1}q \Longrightarrow \deg f_{\gamma i,i} = i^{-1}\gamma^{-1}i \Longrightarrow \overline{Y} \in \mathrm{End}\left(V^*\right) \text{ is not (generally) homogeneous.}$$

We thus have

$$(\overline{Y})\bullet(\overline{Y})^* = \left(\sum_j f_{\gamma j,j}\right)\bullet\left(\sum_i f_{i,\gamma i}\right) \quad \text{acting on} \quad \mathrm{End}(V^*),$$

which *is* a graded map: it sends the degree-$j^{-1}i$-element f_{ji} to the element $f_{\gamma j,\gamma i}$, of degree

$$(\gamma j)^{-1}\cdot\gamma i = j^{-1}i.$$

In short:

- $Y \in \mathrm{End}(V)$ is homogeneous, but its associated map $Y\cdot Y^*$ on $\mathrm{End}(V)$ is not;
- on the other hand, the map $\overline{Y}\cdot(\overline{Y})^*$ attached to the *in*homogeneous $\overline{Y}\in\mathrm{End}(V^*)$ *is* homogeneous on $\mathrm{End}(V^*)$. ♦

2 Generic operator systems

In this section we prove that generic quantum graphs do not have quantum symmetries. It follows from Section 1 that it does not matter, which notion of the quantum automorphism group we use for this particular task, so we can choose one; it will be more convenient for us to use $\mathrm{Qut}(\mathcal{S}\le M_n)$.

Definition 2.1 An operator system $\mathcal{S}\le B$ in a finite-dimensional C^*-algebra B equipped with a state τ is (B,τ)-*quantum-rigid* (just *quantum-rigid* when the setting is understood) if the quantum automorphism group $\mathrm{Qut}(\mathcal{S}\le B)$ operates on $\mathcal{S}$ trivially.

B will mostly be M_n, with τ the usual tracial state unless the convention is explicitly overruled. ♦

Definition 2.2 Let $n\in\mathbb{Z}_{\ge 1}$ and $1\le d\le n^2$.

(1) We refer to the claim that for the generic operator d-system $\mathcal{S}\le M_n$ the quantum group $\mathrm{Qut}(\mathcal{S}\le M_n)$ is trivial as ${}_n\mathrm{QRIG}_d$.

(2) We also write ${}_P\mathrm{QRIG}_Q$ for the conjunction of ${}_n\mathrm{QRIG}_d$ with the parameters n and d constrained by various conditions P and Q. The paradigmatic example is $\mathrm{QRIG}_{3\le n\le N,[\alpha(n),\beta(n)]}$, meaning

$$\forall\,(3\le n\le N)\,\forall\,(d\in[\alpha(n),\beta(n)])\quad:\quad {}_n\mathrm{QRIG}_d$$

(for functions $\alpha(n)\le\beta(n)$).

(3) The QRIG statements also specialize to individual tuples of matrices (or subspaces of M_n): $\mathrm{QRIG}(\mathcal{X})=\mathrm{QRIG}(\langle\mathcal{X}\rangle)$ for a collection $\mathcal{X}\subset M_n$ means that the quantum automorphism group $\mathrm{Qut}(\langle\mathcal{X}\rangle\subset M_n)$ is trivial.

(4) It will be convenient, on occasion, to relativize the QRIG conditions: we say that one such holds *in (or relative to)* G for a quantum subgroup $\mathrm{G}\le\mathrm{Qut}(M_n,\tau)$ if the quantum subgroup of G preserving $\langle X\rangle$ is trivial for generic $\mathcal{X}$. ♦

One of the important tools for extending results from [11] will be the following.

Theorem 2.3 *[10, Theorem A] Let V be a finite dimensional representation of a compact quantum group* G *and let $\mathrm{G}_W\le\mathbb{G}$ be the isotropy quantum subgroup of $W\le\mathbb{G}(d,V)$, where $\mathbb{G}(d,V)$ is the complex Grassmannian consisting of d-dimensional subspaces of V. Then the subset $\{W\in\mathbb{G}(d,V):\mathrm{G}_W$ acts trivially on W or $V\}$ is open in the Zariski topology on the* Weil-restricted *[5, §7.6] $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}(\mathbb{G}(d,V))$, hence of full measure if non-empty.*

Before turning to the proof of our main result (Theorem 0.8), we outline its structure. The main inductive step will be presented in Proposition 2.6. Using it and Theorem 2.9 will reduce proving to a number of small cases, which will be covered in §2.1 (see Section 3 for an alternative approach using computer verification).

We will frequently need the following lemma (analogue of [11, Lemma 3.11]).

Lemma 2.4 *Let $\mathcal{S}\le M_n$ be an operator system and let $\mathcal{S}^\perp$ be its* reflexive complement*, we take take its traceless part (i.e. the orthogonal complement of the unit), then the orthogonal complement inside the space of traceless matrices, and finally add the unit. Then $\mathrm{Qut}(\mathcal{S}\le M_n)\simeq\mathrm{Qut}(\mathcal{S}^\perp\le M_n)$. In particular ${}_n\mathrm{QRIG}_d\leftrightarrow{}_n\mathrm{QRIG}_{n^2-d-1}$.*

Proof The action of the quantum automorphism group preserves the unit and the inner product, hence also orthogonal complement. The conclusion easily follows. ■

We will now present the following proposition, in whose proof we show, how the main theorem follows from small cases and it motivates the results we state next.

Proposition 2.5 *We have*

$${}_n\mathrm{QRIG}_d \text{ for } (n,d)\in\{(3,3),\ (3,4),\ (4,7)\}\quad\Longrightarrow\quad {}_{3\le n}\mathrm{QRIG}_{2\le d\le n^2-3}.$$

Proof Theorem 2.9 ensures that ${}_3\mathrm{QRIG}_2$ and ${}_4\mathrm{QRIG}_{\{2\ldots6\}}$ hold, and the hypothesis together with the symmetry

$${}_n\mathrm{QRIG}_d\quad\Longleftrightarrow\quad{}_n\mathrm{QRIG}_{n^2-1-d}$$

completes the range for ${}_{n\in\{3,4\}}\mathrm{QRIG}_{2\le d\le n^2-3}$. For $n\ge5$ we have $(n-1)(n-2)\ge 2n$, so the conclusion follows from Corollary 2.7 and Theorem 2.9. ■

We can now state and prove an analogue of [11, Proposition 3.13], which is the main inductive step.

Proposition 2.6 *Let $n \in \mathbb{Z}_{\ge 3}$ and assume generic operator d-systems in M_n are quantum-rigid for $2 \le d \le n^2-3$. The same holds, then, of generic operator d'-systems in M_{n+1} for $2 \le d' \le n^2-2$.*

Proof This will be a simple matter of retracing the steps proving the aforementioned [11, Proposition 3.13], employing Theorem 2.3 in place of [11, Lemma 3.12].

A generic d-tuple $\mathcal{X}$ of traceless self-adjoint operators in $M_n \subset M_{n+1}$ (upper left-hand corner embedding) will generate the corner (non-unital) subalgebra $M_n \subset M_{n+1}$, so the hypothesis ensures that $\mathrm{Qut}(\langle \mathcal{X} \rangle \subseteq M_{n+1})$ operates on $\langle \mathcal{X} \rangle$ trivially. The same holds for a (real-)Zariski-open set of tuples by [10, Theorem A]; the conclusion follows from the fact that generic tuples in M_{n+1} generate the latter as an algebra,

This handles all cases $2 \le d' \le n^2-3$. For the remaining value $d' = n^2-2$, retrace the previous argument with a tuple $\mathcal{X}'$ obtained from $\mathcal{X} \subset M_n$ by appending a diagonal operator with non-zero $(n+1) \times (n+1)$ entry. Generically such gadgets will generate a block-diagonally-embedded subalgebra $M_n \times \mathbb{C} \subset M_{n+1}$, again invariant under $\mathrm{Qut}(\langle \mathcal{X} \rangle \subset M_{n+1})$.

It follows from [19, Proposition 5.4] that the individual blocks M_n, $n \ge 2$ and $\mathbb{C}$ of $M_n \times \mathbb{C}$ are left invariant by the quantum automorphism group of the tracial state inherited from the embedding $M_n \times \mathbb{C} \subset M_{n+1}$, so the argument does still apply. ■

Corollary 2.7 *For $\mathbb{Z}_{\ge 3} \ni M \le N$ we have*

$$_{M}\mathrm{QRIG}_{[2,M^2-3]} \;\wedge\; _{M \le n \le N}\mathrm{QRIG}_{[2,2n]} \Longrightarrow {}_{M \le n \le N}\mathrm{QRIG}_{[2,n^2-3]}.$$

Proof We use induction, whose step will effect the passage from $n \le N-1$ to $n+1$. In first instance, Proposition 2.6 provides $_{n+1}\mathrm{QRIG}_{[2,n^2-2]}$. On the other hand, because $_{n}\mathrm{QRIG}_{d}$ is equivalent to $_{n}\mathrm{QRIG}_{n^2-1-d}$, the missing values $d \in [n^2-1, (n+1)^2-3]$ can be recovered from $d \in [2, 2n+1] = [2, 2(n+1)-1]$, provided by the hypothesis. ■

Remark 2.8 The proof in fact delivers slightly more than the statement claims: one only needs the bounds $2 \le d \le 2n$ for $n = 3$; for $n \ge 4$ it suffices to cover $2 \le d \le 2n-1$. ♦

Other portions of [11] replicate in the quantum setting. An example is the following counterpart to [11, Corollary 3.17], covering part of the range of interest for d.

Theorem 2.9 $_{3 \le n}\mathrm{QRIG}_{2 \le d \le (n-1)(n-2)}$ *holds.*

Proof The argument precisely parallels the one proving [11, Corollary 3.17], relying on the appropriate analogues to [11, Lemmas 3.14 and 3.15]: Proposition 2.10 and Lemma 2.12 respectively. ■

[11, Lemma 3.14] refers to diagonal unitaries, recast in the present context in Definition 1.2. The quantum version of that result is as follows.

Proposition 2.10 *Suppose that for some $n \ge 3$ and $1 \le d$ there is a self-adjoint d-tuple $\mathcal{Y} \subset M_{n-1}$ of zero-diagonal matrices so that $\mathrm{QRIG}(\mathcal{Y})$ holds relative to $\mathrm{D}^+(n-1)$. We then have $_{n}\mathrm{QRIG}_{d+1}$.*

Proof We unwind the proof of [11, Lemma 3.14], paying additional attention to the relevant quantum features.

First, $\mathcal{Y}$ can be chosen generically among self-adjoint zero-diagonal tuples, by [10, Theorem A]. Extend $\mathcal{Y} \subset M_{n-1} < M_n$ (upper left-hand corner) generically to $\mathcal{X} := \mathcal{Y} \cup X$ for a diagonal, self-adjoint, traceless X. $\mathcal{X}$ can be assumed to generate $M_{n-1} \times \mathbb{C} < M_n$, whose factors are left invariant by $\mathrm{G} := \mathrm{Qut}(\langle X \rangle \subset M_n)$ by another application of [19, Proposition 5.4].

Denote by primes the upper left-hand blocks of matrices:

$$M_n \ni Z \longmapsto Z' \in M_{n-1} < M_n.$$

I next claim that G fixes X': indeed, G leaves invariant the trace and hence also the affine space

$$\left\{Z \in \left\langle \mathcal{X}' \right\rangle \ : \ \tau(Z) = \tau(X')\right\}. \tag{2-1}$$

Simply observe, then, that X' is the shortest element of (2-1) with respect to the *Hilbert-Schmidt norm* [22, (5.6.0.2)] (again preserved by G).

Fixing the generic (traceless, self-adjoint) diagonal operator X, G must operate via the conjugation action of $\mathrm{D}^+(n)$. It thus fixes $\langle \mathcal{Y} \rangle$ pointwise by assumption, and hence also $\langle \mathcal{X} \rangle$. The conclusion follows. ■

[11, Lemma 3.15] is concerned with *real* linear subspaces of $\mathbb{C}^p$ invariant under diagonal unitaries in $\mathrm{U}(p)$. The very notion requires a bit of unpacking in the present quantum setup.

Definition 2.11 Let

$$V \xrightarrow{\ \rho\ } V \otimes_{\mathbb{C}} \mathcal{O}(\mathrm{G})$$

be a representation of a compact quantum group G. A real subspace $W \le V$ is G- *(or* $\mathcal{O}(\mathrm{G})$*-)invariant* if the coaction on V factors as

$$\begin{tikzcd}
 & V \arrow[r, "\rho"] & V \otimes_{\mathbb{C}} \mathcal{O}(\mathbb{G}) \\
W \arrow[ur, hook] \arrow[r, "\rho_W"'] & W \otimes_{\mathbb{R}} \mathcal{O}(\mathrm{G})_{sa} \arrow[ur, hook, "\text{obvious inclusion}"'] &
\end{tikzcd}$$

for a coaction ρ_W by the (real) coalgebra $\mathcal{O}(\mathrm{G})_{sa} \le \mathcal{O}(\mathrm{G})$ of self-adjoint elements. ♦

Lemma 2.12 *Let $2 \le p$ and $1 \le m \le 2p-1$ be two positive integers. For a generic m-dimensional subspace $W \le \mathbb{C}^p$ the only quantum subgroups of $\mathrm{D}^+(p)$ leaving W invariant in the sense of Definition 2.11 are those contained in the classical group*

$$\{\pm 1\} \cong \mathbb{Z}/2 \le \mathrm{D}(p) \le \mathrm{D}^+(p).$$

Proof We retain the notation of Definition 1.2, realizing $\mathrm{D}^+(p)$ as $\widehat{F_p}$ for the free group on p generators s_i.

A quantum subgroup $\mathrm{G} \le \mathrm{D}^+(p)$ will be Pontryagin dual to a quotient $F_p \twoheadrightarrow \Gamma$, with the G-action corresponding to a Γ-grading on $\mathbb{C}^p$. If at least two generators s_i, say s_1 and s_2 have distinct images in Γ, we have a splitting

$$\mathbb{C}^p = V_0 \oplus V_1,$$

where $V_0 := \mathrm{span}\{e_k : e_k \text{ has degree } s_1\}$ and V_1 is its orthogonal complement. If W is G-invariant, it must be compatible with this splitting, that is

$$W = (W \cap V_0) \oplus (W \cap V_1).$$

Writing $d_j := \dim_{\mathbb{C}} V_j$, $j = 0, 1$ (so that $d_0 + d_1 = p$) and $\mathbb{G}_{\mathbb{R}}$ for real Grassmannians, observe that

$$\begin{aligned} m(2p-m) = \dim \mathbb{G}_{\mathbb{R}}(m, \mathbb{C}^p) &> \max_{\substack{V_0, V_1 \\ 0 \le \ell_j \le 2d_j \\ \ell_0 + \ell_1 = m}} \dim\left(\mathbb{G}(\ell_0, V_0) \times \mathbb{G}(\ell_1, V_1)\right) \\ &= \max_{\substack{V_0, V_1 \\ 0 \le \ell_j \le 2d_j \\ \ell_0 + \ell_1 = m}} \left(\sum_{j=0,1} \ell_j (2d_j - \ell_j) \right). \end{aligned}$$

Generic W, then, can only be invariant under subgroups G of the classical group of scalars $\mathbb{S}^1 \le \mathrm{D}^+(p)$ dual to the surjection $F_p \twoheadrightarrow \mathbb{Z}$ identifying all s_i with a generator of $\mathbb{Z}$.

We can now switch to classical language: generic $W \le \mathbb{C}^p$ will fail to be compatible with the complex structure (i.e. invariant under complex multiplication), so cannot be invariant under scalars other than ± 1. ■

The discussion thus far suffices to whittle down the problem to a small number of cases: precisely those addressed classically by [11, Proposition 3.21].

We can now turn to the definitive version of Theorem 2.9, covering the entire expected range.

Proof of Theorem 0.8 We only have to cover the three cases listed in Proposition 2.5:

- the cases $n = 3$, $d \in \{3, 4\}$ are settled by Proposition 2.13 and Lemma 2.14 respectively;
- while Proposition 2.17 handles ${}_{4\le n}\mathrm{QRIG}_{(n-1)^2-2}$ (and hence $(4, 7)$). ■

2.1 Small cases

Proposition 2.13 *${}_{3\le n}QRIG_n$ holds.*

Proof Set $\mathrm{G} := \mathrm{Qut}(\langle \mathcal{X} \rangle \subset M_n)$ for the tuples $\mathcal{X}$ we work with. The crucial case is the base $n = 3$ of the induction facilitated by Proposition 2.6, but the argument can be phrased uniformly for all $n \ge 3$.

In both cases the tuples $\mathcal{X}$ we consider will consist of $n-1$ self-adjoint traceless diagonal matrices $\mathcal{D} = (T_i)_{i=1}^{n-1}$ in M_n (thus spanning the traceless part $\langle \mathcal{D} \rangle$ of the diagonal algebra D_n), supplemented by one other matrix X. We may as well assume the T_i and X mutually orthogonal with respect to the inner product defined by τ.

X will be generic subject to that orthogonality requirement, so in particular with both X and $X^*X = X^2$ having simple spectrum and being diagonal with respect to an orthonormal basis (f_j) in general position with respect to the standard (e_j) (shorthand phrasing such as (f_j)-*diagonal* always has the obvious meaning). We will also have occasion to work with the corresponding matrix units $f_{ij} := |f_i\rangle \langle f_j|$.

Note that the degree matrix attached to X is (f_j)-diagonal. Since on the other hand the tuple (T_i) contributes a scalar to the overall degree matrix D of $\mathcal{X}$, D generates the same commutative C^*-algebra as X. For that reason, D and X are both G-fixed. It follows that G leaves invariant the orthogonal complement $X^{\langle \mathcal{X} \rangle \perp} = \langle \mathcal{D} \rangle$ in $\langle \mathcal{X} \rangle$.

All in all, G operates on $\langle \mathcal{D} \rangle$, (f_j)-diagonally on M_n because it fixes D. Were that (f_j)-diagonal G-action on M_n non-trivial, it would effect a splitting

$$M_n = M_{n,F} \oplus M_{n,F'}, \quad M_{n,\bullet} := \mathrm{span}\left\{ f_{ij} \ : \ (i,j) \in \bullet \right\}, \quad F' := [n]^2 \setminus F$$

with both F and F' non-empty. The generic position of (e_j) and (f_j), though, ensures that no such decomposition can be compatible with $\langle\mathcal{D}\rangle$:

$$F, F' \neq \emptyset \Longrightarrow \langle\mathcal{D}\rangle \neq (\langle\mathcal{D}\rangle \cap M_{n,F}) \oplus \left(\langle\mathcal{D}\rangle \cap M_{n,F'}\right).$$

This contradicts $\langle\mathcal{D}\rangle$'s G-invariance, finishing the proof. ■

Lemma 2.14 ${}_3\mathrm{QRIG}_4$ *holds.*

Proof Again set $\mathrm{G} := \mathrm{Qut}(\langle\mathcal{X}\rangle \subset M_n)$. We borrow much from the classical version of the claim, proven as part of [11, Proposition 3.21]. Consider tuples $\mathcal{X}$ consisting of

- two traceless diagonal matrices T_i, $i = 1, 2$ (which thus span the traceless diagonal subspace of M_3);
- and two matrices of the form

$$Y_i := \begin{pmatrix} 0 & |\alpha_i\rangle \\ \langle\alpha_i| & 0 \end{pmatrix}, \quad i = 1, 2,$$

with $|-\rangle$ and $\langle-|$ indicating column and row vectors respectively.

Hilbert-Schmidt orthonormality for the Y_i means

$$\frac{2}{3}\mathrm{Re}\,\langle\alpha_i \mid \alpha_j\rangle = \delta_{ij}, \quad 1 \le i, j \le 2.$$

We work with the (scaled) contribution

$$D := \sum_i Y_i^* Y_i = \begin{pmatrix} \sum_i |\alpha_i\rangle\langle\alpha_i| & 0 \\ 0 & \sum_i \langle\alpha_i \mid \alpha_i\rangle \end{pmatrix} = \begin{pmatrix} D' & 0 \\ 0 & \lambda_0 \end{pmatrix}$$

of Y_i to the degree matrix $D_{\langle\mathcal{X}\rangle}$, for that of the T_i is in any case a scalar. D's eigenvalues/vectors are

$\lambda_0 = 3$	$\lambda_1 = \frac{3}{2}(1-c)$	$\lambda_2 = \frac{3}{2}(1+c)$
$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	$\begin{pmatrix} \|\alpha_1\rangle + i\,\|\alpha_2\rangle \\ 0 \end{pmatrix}$	$\begin{pmatrix} \|\alpha_1\rangle - i\,\|\alpha_2\rangle \\ 0 \end{pmatrix}$

for $\langle\alpha_1 \mid \alpha_2\rangle = \frac{3}{2}ci \in \mathbb{R}i$, so in particular the generic D is invertible and the eigenvalues $\lambda_j\lambda_k^{-1}$ of Ad_D (conjugation by D on M_3) are distinct. Because

$$\text{(2-2)} \qquad \begin{aligned} \mathbb{C}\left(E_+ := Y_1 + iY_2\right) &= \langle\mathcal{X}\rangle \cap \left(\sum_{\eta \in \left\{\lambda_1\lambda_0^{-1},\ \lambda_2^{-1}\lambda_0\right\}} \ker\left(\eta - \mathrm{Ad}_D\right)\right) \\ \mathbb{C}\left(E_- := Y_1 - iY_2\right) &= \langle\mathcal{X}\rangle \cap \left(\sum_{\eta \in \left\{\lambda_2\lambda_0^{-1},\ \lambda_1^{-1}\lambda_0\right\}} \ker\left(\eta - \mathrm{Ad}_D\right)\right) \end{aligned}$$

the lines $\mathbb{C}E_\pm$ are both G-invariant; so too, then, is their span $\langle E_i\rangle_i = \langle Y_i\rangle_i$ and its orthogonal complement $\langle T_i\rangle_i$ in $\langle\mathcal{X}\rangle$.

Generic choices of α_i will ensure that D and the T_i generate the block-triangularly-embedded $M_2 \times \mathbb{C} \le M_3$, which is thus also G-invariant. The fixed subalgebra M_2^{G} of its upper left-hand 2×2 block consists of at least the commutant of D' (maximal abelian in M_2). The G-invariance of the span

$$\langle T_i' \rangle \le M_2, \quad T_i' := \text{upper } 2 \times 2 \text{ block of } T_i$$

implies, because the T_i' are in general position with respect to D', that in fact $M_2^{\mathrm{G}} = M_2$ and hence G fixes the T_i. The G-action on M_3 thus amounts is a grading

- fixing the block-diagonal algebra $M_2 \times \mathbb{C} \le M_3$ pointwise;
- and operating on its orthogonal complement $\langle E_\pm \rangle = \langle Y_i \rangle$ by

$$E_\pm \xmapsto{\text{for } E_- = E_+^*} E_\pm \otimes \delta^{\pm 1} \quad \text{for a group-like } \delta \in \mathcal{O}(\mathrm{G}).$$

The generically-non-zero E_+^2 belonging to the fixed-point algebra M_3^{G}, we have $\delta^2 = 1$ and hence the G-action factors through the classical $\mathbb{Z}/2$-action given by $\mathrm{Ad}_{\mathrm{diag}(1,1,-1)}$.

Proposition 2.16 below now shows that the generic action of G on M_n is that induced by a $\mathbb{Z}/2$-grading on $\mathbb{C}^3$, so in particular classical; the conclusion thus follows from [11, Theorem 3.19].■

The following result is a variant of [10, Theorem A], and is presumably of some independent interest. We denote by $\mathbb{G}(d, V)$ the *Grassmannian* [25, §5.1] of dimension-d subspaces of V.

Theorem 2.15 *Let* $\mathrm{G} \circlearrowright V$ *be a finite-dimensional representation of a compact quantum group and denote by* $\mathrm{G}_W \le \mathrm{G}$ *the isotropy quantum group of a subspace* $W \le V$.

(1) For $d, \ell \in \mathbb{Z}_{>0}$ *the subset*

$$\left\{ W \in \mathbb{G}(d,V) \ : \ \mathrm{End}_{\mathrm{G}_W} V \cong \prod_i M_{n_i}, \ \forall i\, (n_i \ge \ell) \right\} \subseteq \mathbb{G}(d,V) \tag{2-3}$$

is Zariski-open, so in particular dense if non-empty.

(2) The same holds of

$$\left\{ W \in \mathbb{G}(d,V) \ : \ \mathrm{End}_{\mathrm{G}_W} V \cong \prod_i M_{n_i}, \ \exists i\, (n_i \ge \ell) \right\} \subseteq \mathbb{G}(d,V) \tag{2-4}$$

Proof Density follows from non-emptiness from the *irreducibility* [25, Theorem 5.4] of the Grassmannian. We focus mainly on the first claim, indicating afterwards the small alterations needed to obtain the second.

(1) Recall [1, Definition 2.2.1] that a *polynomial identity (PI)* for a ring R is a non-commutative polynomial returning 0 no matter how elements of R are substituted for the variables. It follows from [1, Theorem 10.3.2, Lemma 10.3.1 and Proposition 10.2.2] that the simple quotients of a product $A = \prod_i M_{n_i}$ are all of dimension $\ge \ell^2$ precisely when A is generated, as an ideal, by the images of the PIs of $M_{\ell-1}$.

In the spirit of Tannaka-Krein duality, the representation category of G_W is generated by the orthogonal projection $P_W : V \mapsto W$ and the representation category of G, closed under duality, tensor products etc.

The condition that $W \in \mathbb{G}(d,V)$ belong to (2-3) is now expressible as follows:

- for some PIs f_i of $M_{\ell-1}$;

- and elements a_i, b_i and x_{ij} in the algebra generated by the orthogonal projection $V \xrightarrow{P_W} W$ and

$$\bigcup_{N\in\mathbb{Z}_{\ge 0}} \mathrm{End}_{\mathrm{G}}\left(\mathbb{C}\oplus(V\oplus V^*)\oplus\cdots\oplus(V\oplus V^*)^{\otimes N}\right)$$

(generation including algebraic operations along with duality and tensoring);

- the element

$$P_V\left(\sum_i a_i\cdot f_i\left(x_{ij}\right)_j\cdot b_i\right)P_V\in\mathrm{End}(V)$$

is invertible.

Indeed, the last condition says that the ideal generated by the images of the PIs of M_{l-1} contains an invertible element, which is equivalent to saying that the ideal is equal to the whole algebra. As invertibility is an open condition, the conclusion follows.

(2) $\mathrm{End}_{\mathrm{G}_W} V$ is in any case a finite product of matrix algebras, and has at least one simple factor of dimension $\ge \ell^2$ precisely when it does *not* satisfy all PIs of $M_{\ell-1}$. This is again an open condition, so the preceding argument replicates. ■

Even though superseded by generic rigidity, the following principle (employed in the proof of Lemma 2.14) can presumably be of some use in similar circumstances.

Proposition 2.16 *Let $3\le n$ and $1\le d\le n^2-2$. If the action of $\mathrm{Qut}(\mathcal{S}\le M_n)$ factors through a $\mathbb{Z}/2$-grading on $\mathbb{C}^n$ for at least one operator d-system $\mathcal{S}\le M_n$, then it does so generically.*

Proof We already know (Corollary 1.4) that the action is generically essentially diagonal and hence (induced by) a Γ-grading on $\mathbb{C}^n$ for some group Γ. If for some $\mathcal{S}_0$ that group is contained in $\mathbb{Z}_2$, we have

$$\begin{aligned}\mathrm{End}_{\mathrm{Qut}(\mathcal{S}_0\le M_n)} M_n &\cong M_{n_+}\oplus M_{n_-}\\ n_\pm &:= \dim\left(\text{degree-}\pm 1\text{ component of the grading on } M_n\right).\end{aligned}$$

Theorem 2.15 ensures that for generic $\mathcal{S}\le M_n$ the analogous endomorphism algebra $\mathrm{End}_{\mathrm{Qut}(\mathcal{S}\le M_n)} M_n$

- is a product of matrix algebras of dimension $\ge \min\left(n_+^2,\ n_-^2\right)$;

- at least one of which is of dimension $\ge \max\left(n_+^2,\ n_-^2\right)$

Because $n_+ + n_- = n^2$, the only possibilities are

$$\mathrm{End}_{\mathrm{Qut}(\mathcal{S}_0\le M_n)} M_n \cong M_{n_+}\oplus M_{n_-}\quad\text{or}\quad M_{n^2}.$$

There being at most two simple factors, this is a $\mathbb{Z}/2$-grading. ■

Proposition 2.17 ${}_{4\le n}\mathrm{QRIG}_{2\le d\le(n-1)^2-2}$ *holds.*

Proof This is a simple application of Proposition 2.6: we know from Theorem 2.9 that ${}_{3\le m}\mathrm{QRIG}_2$ is valid, so by symmetry so is ${}_{3\le m}\mathrm{QRIG}_{m^2-3}$. ${}_{4\le m+1}\mathrm{QRIG}_{m^2-2}$ follows from Proposition 2.6, i.e. the sought-after conclusion with $m=n-1$. ■

3 Computer verification for small cases

Here we will use and extend the fact that the degree matrix is fixed (see Proposition 1.1). In the same way one shows that $D_2 := A^2\mathbb{1}$ is fixed under the action of the quantum automorphism group.

Lemma 3.1 *Let $\mathcal{G}$ be a quantum graph. If the algebra generated by D and D_2 is equal to M_n then the quantum automorphism group of $\mathcal{G}$ is trivial.*

Proof The whole algebra generated by D and D_2 is fixed by the action of the quantum automorphism group, so if this algebra is equal to M_n then the action has to be trivial. ■

Proposition 3.2 *Let $n \geqslant \mathbb{N}$ and $d \in \{2, \ldots, n^2-3\}$. We think of a d-tuple $\mathcal{X}$ of traceless Hermitian matrices as an operator d-system by adding the unit and taking the linear span, hence a quantum graph. For each $n \leqslant 8$ and $d \in \{2, \ldots, n^2-3\}$ we can find a d-tuple such that the algebra generated by D and D_2 is equal to M_n. More precisely, the matrices $(D^i D_2^j)_{i,j\in\{0,\ldots,n-1\}}$ form a basis of M_n.*

Proof Let $\mathcal{X} = \{X_1, \ldots, X_d\}$. Then we can write the formula for $D = \sum_{i,j=1}^{d} (\Gamma^{-1})_{ij} X_i X_j$, where $\Gamma_{ij} := \tau(X_i X_j)$ is the Gram matrix of the tuple $\mathcal{X}$. Indeed, one can easily verify that the matrices $\widetilde{X_j} := \sum_i (\Gamma^{-\frac{1}{2}})_{ij} X_i$ form an orthonormal basis of $\langle \mathcal{X} \rangle$, so the formula for D follows from the standard formula $D = \sum_j \widetilde{X}_j^2$. We similarly obtain the formula for $D_2 = \sum_{i,j,k,l} (\Gamma^{-1})_{ij} (\Gamma^{-1})_{kl} X_k X_i X_j X_l$. Verification that the matrices $(D^i D_2^j)_{i,j\in\{0,\ldots,n-1\}}$ form a basis of M_n can be done on a computer and this is exactly what we did. For instance, for $n = 7$ and $d = 4$ we have the following example:

$$X_1 = \begin{bmatrix} -0.142491 & 0.0251255 & 0.16238 & 0.0555206 & 0.175618 & -0.0938962 & -0.0929846 \\ 0.0251255 & 0.167523 & -0.0250914 & 0.0559988 & -0.0593764 & 0.0799041 & -0.165092 \\ 0.16238 & -0.0250914 & 0.0679049 & 0.189596 & 0.0681018 & 0.0698056 & 0.28165 \\ 0.0555206 & 0.0559988 & 0.189596 & -0.166191 & -0.191064 & 0.0831966 & 0.0652995 \\ 0.175618 & -0.0593764 & 0.0681018 & -0.191064 & 0.154533 & 0.0791232 & 0.176272 \\ -0.0938962 & 0.0799041 & 0.0698056 & 0.0831966 & 0.0791232 & 0.186155 & 0.256028 \\ -0.0929846 & -0.165092 & 0.28165 & 0.0652995 & 0.176272 & 0.256028 & -0.267433 \end{bmatrix},$$

$$X_2 = \begin{bmatrix} 0.0710277 & -0.149662 & 0.200781 & -0.167936 & 0.0346357 & -0.290504 & -0.109015 \\ -0.149662 & -0.120797 & -0.206312 & 0.0435404 & 0.19233 & 0.0415093 & 0.026054 \\ 0.200781 & -0.206312 & -0.0635957 & -0.0823084 & -0.0903822 & 0.151709 & -0.188229 \\ -0.167936 & 0.0435404 & -0.0823084 & -0.117987 & -0.0552701 & 0.22579 & -0.070853 \\ 0.0346357 & 0.19233 & -0.0903822 & -0.0552701 & 0.31879 & -0.0875183 & -0.107513 \\ -0.290504 & 0.0415093 & 0.151709 & 0.22579 & -0.0875183 & -0.0229677 & -0.0643047 \\ -0.109015 & 0.026054 & -0.188229 & -0.070853 & -0.107513 & -0.0643047 & -0.0644701 \end{bmatrix},$$

$$X_3 = \begin{bmatrix} 0.150074 & 0.101414 & -0.0925115 & 0.23272 & 0.271942 & 0.00275495 & -0.235919 \\ 0.101414 & 0.205297 & 0.0416222 & -0.049573 & 0.0364604 & 0.172456 & 0.105198 \\ -0.0925115 & 0.0416222 & 0.230496 & -0.0894059 & 0.126021 & 0.0379802 & -0.103049 \\ 0.23272 & -0.049573 & -0.0894059 & 0.0720574 & 0.100826 & 0.0606621 & -0.158897 \\ 0.271942 & 0.0364604 & 0.126021 & 0.100826 & -0.0963457 & 0.0647688 & -0.125808 \\ 0.00275495 & 0.172456 & 0.0379802 & 0.0606621 & 0.0647688 & -0.195335 & -0.060983 \\ -0.235919 & 0.105198 & -0.103049 & -0.158897 & -0.125808 & -0.060983 & -0.366243 \end{bmatrix},$$

$$X_4 = \begin{bmatrix} 0.0152974 & 0.0445368 & 0.146435 & 0.0931627 & 0.115207 & 0.0059397 & -0.114823 \\ 0.0445368 & 0.0559448 & -0.0651526 & 0.254707 & 0.0377037 & 0.0989529 & -0.148023 \\ 0.146435 & -0.0651526 & -0.14917 & 0.176034 & 0.122183 & 0.00138746 & 0.072744 \\ 0.0931627 & 0.254707 & 0.176034 & 0.0718361 & 0.116681 & -0.0276206 & -0.0653339 \\ 0.115207 & 0.0377037 & 0.122183 & 0.116681 & -0.144456 & -0.308034 & -0.00932981 \\ 0.0059397 & 0.0989529 & 0.00138746 & -0.0276206 & -0.308034 & -0.301774 & -0.0260149 \\ -0.114823 & -0.148023 & 0.072744 & -0.0653339 & -0.00932981 & -0.0260149 & 0.452321 \end{bmatrix}.$$

■

This provides an alternative to the approach presented in §2.1.

Proposition 3.3 *Let $n \geqslant \mathbb{N}$ and $d \in \{2, \ldots, n^2 - 3\}$. If there exists a d-tuple $\mathcal{X}$ of traceless Hermitian matrices such that the matrices $(D^i D_2^j)_{i,j \in \{0,\ldots,n-1\}}$ form a basis of M_n then the same happens for almost all choices of such d-tuples.*

Proof Because of the formula $D = \sum_{i,j=1}^{d} (\Gamma^{-1})_{ij} X_i X_j$ and an analogous one for D_2, we see they are built from $\mathcal{X}$ in an algebraic fashion. To be precise they are rational function, because Γ^{-1} contains an inverse of the determinant, but we can replace D by $\det(\Gamma)D$ to obtain a truly algebraic expression and it does not affect the condition we want D and D_2 to satisfy. The matrices $(D^i D_2^j)_{i,j \in \{0,\ldots,n-1\}}$ form a basis of M_n if and only if they are linearly independent, which can be stated as non-vanishing of a certain determinant. It means the set of d-tuples $\mathcal{X}$ for which it holds is Zariski open, so it is dense as soon as it is non-empty, and Zariski open and dense subsets are of full measure. ■

This simple proposition shows a possible strategy for proving triviality of the quantum automorphism group of a random quantum graph in a different manner.

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