Twisted homological stability for handlebody mapping class groups

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Abstract

We prove twisted homological stability for handlebody mapping class groups. Using the categorical framework developed by Randal-Williams and Wahl, we establish that the homology of the handlebody groups stabilises with respect to both genus and the number of marked boundary discs, for all coefficient systems of finite degree. Our first main theorem refines and extends the twisted stability result for handlebodies outlined by Randal-Williams and Wahl, allowing any number of marked discs and boundary points. We then introduce the notion of coefficient bisystem to treat stability under variation of boundary markings. As an application, we deduce homological stability for moduli spaces of 3-dimensional handlebodies equipped with tangential structures.

1. Introduction

Homological stability is a prevalent phenomenon in topology, asserting that the homology of a sequence of groups (or spaces) $G_0 \to G_1 \to G_2 \to \cdots$ becomes independent of the index in a range increasing with homological degree. Beyond the classical case with constant coefficients, its twisted form considers functorial coefficient systems, allowing one to capture interactions between geometric and representation-theoretic information. In particular, twisted homological stability is thus a fundamental tool for computing group homology via the stable value, and for constructing universal characteristic classes of bundles of families of manifolds. In this paper, we study twisted homological stability for *handlebody groups*. Namely, for $g \geq 0$, let V_g denote the 3-dimensional handlebody of genus g, i.e. the iterated boundary connected sum $V_g := \natural_g S^1 \times \mathbb{D}^2$ for some integer $g \geq 0$, where $\mathbb{D}^2$ denotes the 2-dimensional unit disc. Denoting by $\mathcal{D}$ the image of an embedded copy of $\mathbb{D}^2$ in the boundary of V_g, we consider the *handlebody mapping class group* (a.k.a. *handlebody group*)

$$\mathcal{H}_{g,1} := \pi_0\mathrm{Diff}(V_g; \mathcal{D}), \tag{1.1}$$

that is the group of isotopy classes of diffeomorphisms of V_g that pointwise fix the disc $\mathcal{D}$.

By taking the boundary connected sum $V_g \natural V_1$ along some marked disc in ∂V_g as in Figure 1.1, we can extend any diffeomorphism of V_g that fixes $\mathcal{D}$ to one of V_{g+1}, by defining it to be the identity on V_1. By picking any embedded 2-disc in the boundary of $V_{g+1} \setminus V_g$, this defines a *stabilisation homomorphism*

$$\sigma_{g,1} \colon \mathcal{H}_{g,1} \to \mathcal{H}_{g+1,1}.$$

Hatcher and Wahl [HW10, Cor. 1.9] proved that the induced sequence of homomorphisms satisfies integral *homological stability*, i.e. that in degrees sufficiently small compared to g, $\sigma_{g,1}$ induces an isomorphism in integral homology. A proof generalising this result to hold with various twisted coefficient was outlined by Randal-Williams and Wahl [RW17, §5.7], recovering in particular a previous stability result for the first homology with a specific twisted coefficient system found by

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Ishida and Sato in [IS17, Th. 1.2] (see Example 3.2). In this paper, we generalise these results further, in several directions. The first main result fills in the details of the proof of twisted homological stability for handlebody mapping class groups of [RW17, Th. 5.31], and generalises it to hold with any number of marked discs and points in the boundary of V_g; see Theorem A.

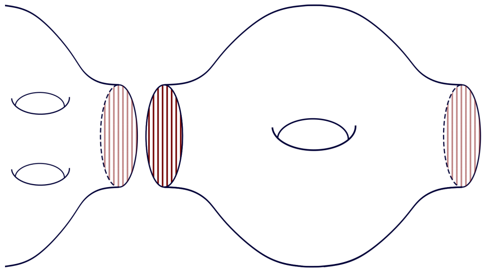

Figure 1.1 The handlebody V_{g+1} is obtained from V_g by gluing on a genus 1 handlebody along the marked disc and the stabilisation map $\sigma_{g,1}$ is defined by extending diffeomorphisms to the new part by the identity.

Additional marked discs and points. A more general flavour of the above handlebody mapping class group (1.1) is given by replacing the marked disc $\mathcal{D}$ in the boundary of V_g with any number $r \geq 1$ of disjointly embedded discs and $s \geq 0$ distinct points in the boundary. We denote by $\mathcal{H}^s_{g,r}$ the group of the isotopy classes of diffeomorphisms of V_g that pointwise fix all marked discs and points. If s is zero, we drop it from the notation. Also, we write

$$\sigma^s_{g,r} \colon \mathcal{H}^s_{g,r} \to \mathcal{H}^s_{g+1,r}$$

for the analogously defined stabilisation homomorphism; see also §2.4.3 for the precise definition of this map.

Twisted coefficients. For integers $r \geq 1$ and $s \geq 0$, let $\{(F^s_{g,r}, c^s_{g,r})\}_{g\geq 0}$ be a sequence of pairs where $F^s_{g,r}$ is a $\mathbb{Z}[\mathcal{H}^s_{g,r}]$-module and

$$c^s_{g,r} : F^s_{g,r} \to F^s_{g+1,r}$$

is a $\mathbb{Z}[\mathcal{H}^s_{g,r}]$-linear map, where the target is considered a $\mathbb{Z}[\mathcal{H}^s_{g,r}]$-module via the stabilisation map $\sigma^s_{g,r}$. We call such a sequence a *coefficient system* for the sequence $\{(\mathcal{H}^s_{g,r}, \sigma^s_{g,r})\}_{g\geq 0}$.

Stability with respect to handles. A coefficient system $\{(F^s_{g,r}, c^s_{g,r})\}_{g\geq 0}$ for the sequence $\{(\mathcal{H}^s_{g,r}, \sigma^s_{g,r})\}_{g\geq 0}$ induces a map in homology

$$(\sigma^s_{g,r}, c^s_{g,r})_* \colon H_*(\mathcal{H}^s_{g,r}; F^s_{g,r}) \to H_*(\mathcal{H}^s_{g+1,r}; F^s_{g+1,r}) \tag{1.2}$$

for each $g \geq 0$. Using the general framework for homological stability developed by [RW17], we prove that if the coefficient system is of *finite degree*, which means that it satisfies a certain polynomiality condition (see §2.3 for details), the sequence of the maps (1.2) satisfies homological stability:

Theorem A (Theorem 3.1) *If $\{(F^s_{g,r}, c^s_{g,r})\}_{g\geq 0}$ for integers $r \geq 1$ and $s \geq 0$, is a coefficient system of finite degree d, with respect to the sequence $\{(\mathcal{H}^s_{g,r}, \sigma^s_{g,r})\}_{g\geq 0}$, the map*

$$(\sigma^s_{g,r}, c^s_{g,r})_* \colon H_*(\mathcal{H}^s_{g,r}; F^s_{g,r}) \to H_*(\mathcal{H}^s_{g+1,r}; F^s_{g+1,r})$$

is an isomorphism in degrees $* \leq \frac{g-1}{2} - d - 1$.

Remark 1.1 As detailed in the more technical Theorem 3.1 below, the stability range can be improved if the coefficient system is *split* (see §2.3 for the definition), an additional assumption which roughly boils down to the assumption that $c^s_{g,r} : F^s_{g,r} \to F^s_{g+1,r}$ is split injective for each $g \geq 0$; see §2.3.

Some examples of finite degree coefficients $\{(F^s_{g,r}, c^s_{g,r})\}_{g\geq 0}$ can be obtained from the homology of V_g and of its boundary, the surface ∂V_g. More specifically, if we let $\mathcal{D}_r$ denote the union of the marked discs and $\mathcal{P}_s$ the union of the marked points in the boundary of V_g, the relative homology groups $H_1(V_g, \mathcal{D}_r \sqcup \mathcal{P}_s; \mathbb{Z})$ and $H_1(\partial V_g \setminus (\mathring{\mathcal{D}}_r \sqcup \mathcal{P}_s); \mathbb{Z})$ are the components of coefficient systems of degree 1; see Example 3.2 for more details. This implies that taking the r-th tensor powers of these coefficient systems give us coefficient systems of degree r (see [LS25, Cor. 1.5]).

Stability with respect to marked discs. Hatcher and Wahl [HW10] also proved that when g is large compared to the homological degree, the integral homology of $\mathcal{H}_{g,r}$ is independent of r, the number of marked discs in the boundary of V_g. More specifically, there is a group homomorphism

$$\mu^s_{g,r} \colon \mathcal{H}^s_{g,r} \to \mathcal{H}^s_{g,r+1},$$

defined by gluing a sphere with three marked discs to the first marked disc of V_g (see Figure 1.2) and extending diffeomorphisms by the identity, and the result of Hatcher and Wahl says (in the case $s = 0$) that for g large enough to the homological degree, the induced map in homology is an isomorphism.

Our second main theorem generalises this result to homology with twisted coefficients. Our assumption on the twisted coefficients is somewhat technical, so let us leave out the details in the introduction. However, it essentially says that the coefficients need to be of finite degree in two different ways: we call this being a "coefficient bisystem"; see Definition 2.28.

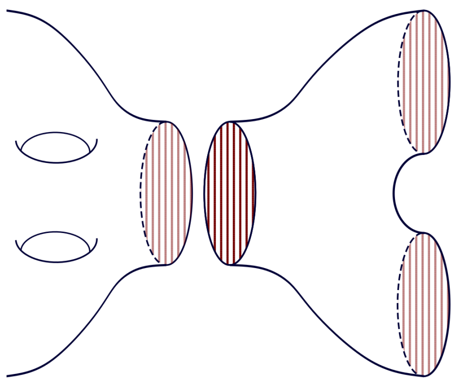

Figure 1.2 By gluing a "solid pair of pants" to the first marked disc of our handlebody, we obtain a handlebody with an additional marked disc.

Theorem B (Corollary 3.18) *For integers $r \geq 1$ and $s \geq 0$, let $m^s_{g,r} \colon F^s_{g,r} \to F^s_{g,r+1}$ be a $\mathbb{Z}[\mathcal{H}^s_{g,r}]$-linear map, where the target is consider a $\mathbb{Z}[\mathcal{H}^s_{g,r}]$-module via $\mu^s_{g,r}$. If $m^s_{g,r}$ is induced by a coefficient bisystem of degree d (see Definition 2.28 below), the map*

$$(\mu^s_{g,r}, m^s_{g,r})_* \colon H_*(\mathcal{H}^s_{g,r}; F^s_{g,r}) \to H_*(\mathcal{H}^s_{g,r+1}; F^s_{g,r+1}),$$

is an isomorphism in degree $* \leq \frac{g-1}{2} - d - 1$.

Remark 1.2 As in Theorem A, the stability range can be improved if the coefficient bisystem is further assumed to be split.

Some examples of finite degree coefficient bisystems can again be obtained from the $\mathcal{H}^s_{g,r}$-representations $H_1(V_g, \mathcal{D}_r \sqcup \mathcal{P}_s; \mathbb{Z})$ and $H_1(\partial V_g \setminus \mathring{\mathcal{D}}_r, \mathcal{P}_s; \mathbb{Z})$, considered above; see Example 3.19 for details.

Moduli spaces of handlebodies with tangential structures. Using our stability results for handlebody mapping class groups with twisted coefficients, we also generalise homological stability results for handlebody mapping class groups in a slightly different direction. Letting $R^s_{g,r} \subset \partial V_g$ denote the set of $r \geq 1$ marked discs and $s \geq 0$ points in the boundary of V_g, we know from the proof of Lemma 2.1 that the diffeomorphism group $\mathrm{Diff}(V_g; R^s_{g,r})$ has contractible path components, so in other words $\mathrm{Diff}(V_g; R^s_{g,r})$ is homotopy equivalent to the discrete group $\mathcal{H}^s_{g,r}$. In particular, this implies that $\mathrm{BDiff}(V_g; R^s_{g,r}) \simeq \mathrm{B}\mathcal{H}^s_{g,r}$, so the group cohomology of $\mathcal{H}^s_{g,r}$ agrees with the singular cohomology of $\mathrm{BDiff}(V_g; R^s_{g,r})$ and homological stability for $\mathcal{H}^s_{g,r}$ may thus equivalently be stated as homological stability of these spaces. The space $\mathrm{BDiff}(V_g; R^s_{g,r})$ can be thought of as a *moduli space* of handlebodies, which can be generalised to moduli spaces of handlebodies which carry various *tangential structures*. For handlebodies of higher dimensions, homological stability of such moduli spaces are studied by Perlmutter [Per18] and their stable homology by Botvinnik and Perlmutter [BP17]. Similar results were previously obtained by Galatius and Randal-Williams [GR18; GR14] for the even dimensional manifolds $W_g := \#S^n \times S^n$, i.e. the boundaries of $2n + 1$-dimensional handlebodies.

A tangential structure is, in this setting, a fibration $\theta : \mathrm{B} \to \mathrm{BO}(3)$, where $\mathrm{O}(3)$ is the group of 3×3 orthogonal matrices, and a θ-structure on the handlebody V_g is a map $\ell : V_g \to \mathrm{B}$ such that the composition $\theta \circ \ell$ is a classifying map for the tangent bundle of V_g. We can thus define the space of θ-structures on V_g, which we denote by $\mathrm{Bun}^\theta(V_g)$ as a subspace of the mapping space $\mathrm{Map}(V_g; \mathrm{B})$. We define the *moduli spaces of genus g handlebodies with a θ-structure* as the space of homotopy orbits

$$\mathrm{BDiff}^\theta(V_g) := \mathrm{Bun}^\theta(V_g)_{h\mathrm{Diff}(V_g)},$$

where the diffeomorphism group acts by precomposition. In §4.1 below, we define these notions in more detail, and introduce appropriate boundary conditions on the set $R^s_{g,r}$ of marked discs and points in ∂V_g, resulting in a moduli space we denote by $\mathrm{BDiff}^\theta(V_g; \ell^s_{g,r})$. In general, this space is not path-connected and, for technical reasons, we therefore restrict our attention to a subspace $\mathrm{BDiff}^\theta_0(V_g; \ell^s_{g,r})$, which is the path component of a certain constant θ-structure.

The boundary condition $\ell^s_{g,r}$ allows us to define a stabilisation map

$$\breve{\sigma}^{\theta,s}_{g,r} : \mathrm{BDiff}^\theta_0(V_g; \ell^s_{g,r}) \to \mathrm{BDiff}^\theta_0(V_{g+1}; \ell^s_{g+1,r}), \tag{1.3}$$

increasing the genus, given by extending by a constant θ-structure, and similarly a stabilisation map

$$\breve{\mu}^{\theta,s}_{g,r} : \mathrm{BDiff}^\theta_0(V_g; \ell^s_{g,r}) \to \mathrm{BDiff}^\theta_0(V_g; \ell^s_{g,r+1}). \tag{1.4}$$

increasing the number of marked discs. With these, we can state our last main theorem:

Theorem C (Theorem 4.5) *For $\theta : \mathrm{B} \to \mathrm{BO}(3)$ a pointed tangential structure, the maps*

$$(\breve{\sigma}^{\theta,s}_{g,r})_* : H_*(\mathrm{BDiff}^\theta_0(V_g; \ell^s_{g,r}); \mathbb{Q}) \to H_*(\mathrm{BDiff}^\theta_0(V_{g+1}; \ell^s_{g+1,r}); \mathbb{Q}) \tag{1.5}$$

and

$$(\breve{\mu}^{\theta,s}_{g,r})_* : H_*(\mathrm{BDiff}^\theta_0(V_g; \ell^s_{g,r}); \mathbb{Q}) \to H_*(\mathrm{BDiff}^\theta_0(V_g; \ell^s_{g,r+1}); \mathbb{Q}), \tag{1.6}$$

induced by (1.3) and (1.4), respectively, are isomorphisms in degrees $* \leq \frac{g-3}{2}$.

Remark 1.3 As Theorem C is an application of Theorems A and B, which both hold over the integers, it may seem surprising that this results is stated with rational coefficients. This is due to a technical issue about finite degree coefficient systems (see Remark 4.9 for more details).

Conventions and notation. We denote by **Ab** the category of abelian groups. We write $\otimes$ for $\otimes_R$ when the ground ring R is clear from the context. Non-specified tensor products are taken over $\mathbb{Z}$. For each $d \geq 1$, the d-fold tensor power of abelian groups functor is denoted by $T^d \colon \mathbf{Ab} \to \mathbf{Ab}$. We denote by I the unit interval $[0, 1]$.

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Contents

2. Background

In this section, we start by recollecting in §2.1 some basics about handlebody groups and their relation to mapping class groups of surfaces. After this, we construct in §2.2 a number of braided monoidal groupoids of handlebodies and modules over these, whose automorphism groups are precisely the handlebody groups, and we recall the Quillen bracket construction. As we study homological stability with twisted coefficients, we review in §2.3 the designed notion of finite degree coefficient system, of which we also introduce interesting examples. We finally recall in §2.4 the general framework of [RW17; Kra19] to prove twisted homological stability, and we verify that the categories we have introduced satisfy some basic properties that make them fit into it.

2.1. Recollections on handlebody groups

We recall that a 3-dimensional compact handlebody is a smooth 3-manifold obtained by attaching a finite number of handles to the 3-ball. Equivalently, it is diffeomorphic to the boundary connected sum $\flat_g(\mathbb{S}^1 \times \mathbb{D}^2)$ for some integer $g \geq 0$, which is denoted by V_g and its boundary by $\Sigma_g := \partial V_g$. For integers $r, s \geq 0$, let $\mathcal{D}_r$ be the image of the disjoint union $\sqcup_r \mathbb{D}^2$ of $r \geq 0$ embedded copies of the 2-disc $\mathbb{D}^2$ in the boundary of V_g and $\mathcal{P}_s$ be the image of an embedded copy of the set of $s \geq 0$ distinct points $\{p_1, \ldots, p_s\}$ in $\Sigma_g \setminus \mathcal{D}_r$. We also denote by $\Sigma^s_{g,r}$ the surface $\Sigma_g \setminus (\mathring{\mathcal{D}}_r \sqcup \mathcal{P}_s)$, where $\mathring{\mathcal{D}}_r$ is the disjoint union of the interiors of the marked discs of $\mathcal{D}_r$.

We consider the topological group $\mathrm{Diff}(V_g; \mathcal{D}_r \sqcup \mathcal{P}_s)$ of orientation-preserving diffeomorphisms of V_g, which pointwise fix the sets $\mathcal{D}_r$ and $\mathcal{P}_s$. Its group of path components is denoted by $\mathcal{H}^s_{g,r}$ and called the *handlebody (mapping class) group*. We also deal with the topological group $\mathrm{Diff}(\Sigma_g; \mathcal{D}_r \sqcup \mathcal{P}_s)$ of orientation-preserving diffeomorphisms of the boundary of V_g fixing both $\mathcal{D}_r$ and $\mathcal{P}_s$ pointwise. We write $\Gamma^s_{g,r}$ for its group of path components, which is called the *mapping class group* of $\sigma^s_{g,r}$. If $r = 0$ (respectively $s = 0$), we may omit r and $\mathcal{D}_r$ (respectively s and $\mathcal{P}_s$) from the notation.

A key property of handlebody groups is that they canonically embed into the mapping class groups of their boundaries. This result is probably known to the experts, but we give a short proof here for the convenience of the reader; see Lemma 2.1. Since diffeomorphisms of smooth manifolds are boundary-preserving (see [Lee13, Th. 2.18] for instance), the restriction map to the boundary $V_g \to \Sigma_g$ induces a fibration

$$\mathrm{Diff}(V_g; \mathcal{D}_r \sqcup \mathcal{P}_s) \to \mathrm{Diff}(\Sigma_g; \mathcal{D}_r \sqcup \mathcal{P}_s), \tag{2.1}$$

with fibre $\mathrm{Diff}(V_g; \partial V_g)$.

Lemma 2.1 *For each $g, r, s \geq 0$, the map (2.1) induces an injective homomorphism*

$$\mathtt{i}^s_{g,r} \colon \mathcal{H}^s_{g,r} \hookrightarrow \Gamma^s_{g,r}. \tag{2.2}$$

Furthermore, if $g \geq 2$, then the homotopy groups $\pi_i(\mathrm{Diff}(V_g; \mathcal{D}_r \sqcup \mathcal{P}_s))$ are trivial for all $i \geq 1$.

Proof. First, the morphism $\mathtt{i}^s_{g,r}$ is defined as the right-most map of the homotopy long exact sequence $\cdots \to \pi_0 \mathrm{Diff}(V_g; \partial V_g) \to \mathcal{H}^s_{g,r} \to \Gamma^s_{g,r}$ of the fibration (2.1). We note that the handlebody V_g is a compact and irreducible 3-manifold, homeomorphic to $\mathbb{D}^2_g \times I$ where $\mathbb{D}^2_g := \mathbb{D}^2 \setminus \sqcup_{1 \leq i \leq g} \mathring{\mathbb{D}}^2_i$ (where $\mathring{\mathbb{D}}^2_i$ are in the interior of $\mathbb{D}^2$). Then we know from [Hat99, Proof of Th. 1, Step 4] that $\pi_0 \mathrm{Diff}(V_g; \partial V_g) = 0$, and so we deduce from the above homotopy long exact sequence that $\mathtt{i}^s_{g,r}$ is injective.

Finally, for $g \geq 2$, the path components of $\mathrm{Diff}(\Sigma_g; \mathcal{D}_r \sqcup \mathcal{P}_s)$ are contractible by [EE67; EE69; ES70], as well as those of $\mathrm{Diff}(V_g; \partial V_g)$ by [Iva76, Th. 2], [Hat76] or [Hat99, Th. 2]. The vanishing of the higher homotopy groups follows from the homotopy long exact sequence of the fibration (2.1). □

Remark 2.2 Since the path component of the identity in $\mathrm{Diff}(V_g; \mathcal{D}_r \sqcup \mathcal{P}_s)$ is trivial by Lemma 2.1, the cohomology of the classifying space $\mathrm{BDiff}(V_g; \mathcal{D}_r \sqcup \mathcal{P}_s)$ agrees with the group cohomology of the handlebody group $\mathcal{H}^s_{g,r}$.

2.2. Categorical framework

In this section, we review the necessary specific categories to apply the framework of [RW17] and [Kra19] to prove our twisted homological stability results for the handlebody groups.

Preliminaries on categorical tools. We refer to [Mac98, §VII] for a complete introduction to the notions of monoidal categories and modules over them. We generically denote a monoidal category by $(\mathcal{C}, \odot, 0)$, where $\mathcal{C}$ is a category, $\odot$ is the monoidal product, and 0 is the monoidal unit. If it is braided, then its braiding is generically denoted by $b^{\mathcal{C}}_{A,B} \colon A \odot B \xrightarrow{\sim} B \odot A$ for all objects A and B of $\mathcal{C}$. A right-module $(\mathcal{M}, \triangleright)$ (resp. left-module $(\mathcal{M}, \triangleleft)$) over a monoidal category $(\mathcal{C}, \odot, 0)$ is a category $\mathcal{M}$ with a functor $\triangleright \colon \mathcal{M} \times \mathcal{C} \to \mathcal{M}$ (resp. $\triangleleft \colon \mathcal{C} \times \mathcal{M} \to \mathcal{M}$) that is unital and associative. For example, a monoidal category $(\mathcal{C}, \odot, 0)$ is equipped with a left or right-module structure over itself, induced by its monoidal product. Each right-module structure $\triangleright$ in this paper is defined from some underlying monoidal structure $\odot$ (see §2.2.1), so we abuse notation by using the same symbol $\odot$ for $\triangleright$ or $\triangleleft$.

2.2.1. Groupoids associated to compact handlebodies

We start by defining some braided monoidal groupoids, where the objects are handlebodies and the automorphism groups are the groups $\mathcal{H}^s_{g,r}$.

Definition 2.3 We define a category $\mathfrak{H}$ as follows.

- The objects are 5-tuples (V, r, s, i, j). Here V is a 3-dimensional compact handlebody (i.e. it is diffeomorphic to $V_g = \natural_g(\mathbb{S}^1 \times \mathbb{D}^2)$ for some integer $g \geq 0$) or a disjoint union of a finite number of 3-balls; $r \geq 1$ and $s \geq 0$ are integers, $i \colon \mathcal{D}_r \hookrightarrow \partial V$ is a smooth embedding and $j \colon \mathcal{P}_s \hookrightarrow \partial V \setminus \mathrm{Im}(i)$ is an injection. In particular, we remember the data of the embedding j and not just its image, so that we have well-defined notions of *left*-half and *right*-half of the image of each disc. The set $\mathcal{D}_r := \sqcup_r \mathbb{D}^2$ has a canonical ordering induced by that of the set $\{1, \ldots, r\}$, and we denote by $\mathcal{D}^k_V \in \mathcal{D}_r$ the image of the k-th embedded disc via the embedding i.
- A morphism $(V_1, r_1, s_1, i_1, j_1) \to (V_2, r_2, s_2, i_2, j_2)$ is given by the isotopy class of a diffeomorphism $\phi \colon V_1 \to V_2$ such that $\phi \circ i_1 = i_2$ and $\phi \circ j_1 = j_2$. Note that in particular, this implies that $r_1 = r_2$ and $s_1 = s_2$.

For $r \geq 1$, let us denote by $\mathfrak{H}^{\geq r}$ the full subcategory with objects of the form (V, k, s, i, j) for $k \geq r$. In particular, we have $\mathfrak{H}^{\geq 1} = \mathfrak{H}$.

We fix the following conventions for the compact handlebodies of $\mathfrak{H}$.

Notation 2.4 For integers $r \geq 1$ and $s \geq 0$, we denote by $V^s_{g,r}$ the genus $g \geq 0$ compact handlebody $V_g = \natural_g(\mathbb{S}^1 \times \mathbb{D}^2)$, along with r marked discs $\mathcal{D}_r$ and s marked points $\mathcal{P}_s$ in the boundary, as well as implicitly associated smooth embedding $i\colon \mathcal{D}_r \hookrightarrow \partial V$ and injection $j\colon \mathcal{P}_s \hookrightarrow \partial V \setminus \mathrm{Im}(i)$, to make it an object of $\mathfrak{H}$; see Definition 2.3. When $s = 0$, we omit it from the notation. In particular, we have $\mathrm{Aut}_{\mathfrak{H}}(V^s_{g,r}) = \mathcal{H}^s_{g,r}$.

Monoidal structures. We continue by defining a monoidal structure on the category $\mathfrak{H}^{\geq r}$ for each $r \geq 1$, each corresponding to a different family of stabilisation maps between handlebody groups. We only consider the cases $r = 1, 2$ later in §2 and §3, but it is more practical to introduce all $r \geq 1$ at once.

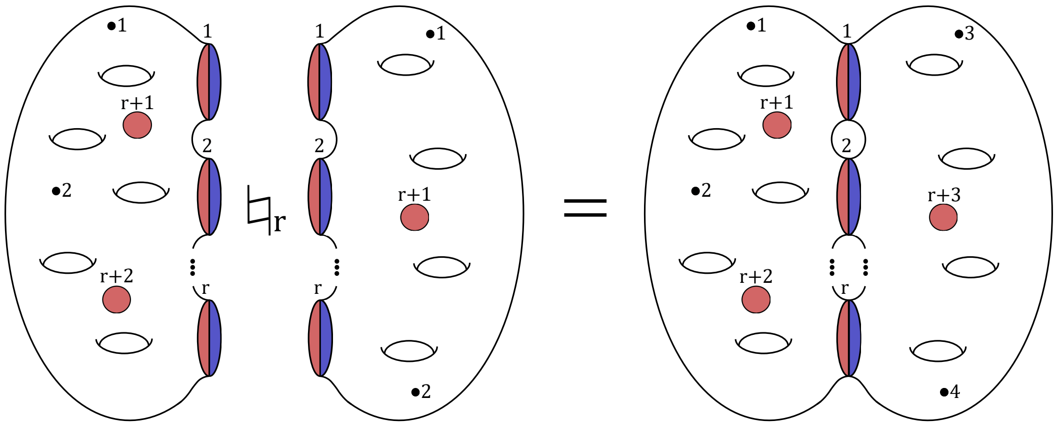

Figure 2.1 The monoidal product $V_1 \natural_r V_2$ of two handlebodies in $\mathcal{G}^{\geq r}_{\mathcal{G}}$.

Definition 2.5 For $r \geq 1$ and for $(V_1, r_1, s_1, i_1, j_1)$ and for objects $(V_2, r_2, s_2, i_2, j_2)$ of $\mathfrak{H}^{\geq r}$, we define the object $V_1 \natural_r V_2$, which will serve as a monoidal product, as follows. We denote by $\frac{1}{2}\mathbb{D}^2$ a half-disc. Let $f_1\colon \sqcup_r \frac{1}{2}\mathbb{D}^2 \hookrightarrow \mathrm{Im}(i_1)$ denote the canonical inclusion into the right-half of the first r marked discs $\sqcup_{1 \leq i \leq r} \mathcal{D}^i_{V_1}$ of V_1. Similarly, we denote by $f_2\colon \sqcup_r \frac{1}{2}\mathbb{D}^2 \hookrightarrow \mathrm{Im}(i_2)$ the canonical inclusion into the left-half of the r first marked discs $\sqcup_{1 \leq i \leq r} \mathcal{D}^i_{V_2}$ of V_2. The monoidal product $V_1 \natural_r V_2$ is then given by the 5-tuple

$$(V_1 \cup_{f_1 \sim f_2} V_2, r_1 + r_2 - r, s_1 + s_2, i_1 \natural_r i_2, j_1 \sqcup j_2).$$

Namely:

- The adjunction space $V_1 \cup_{f_1 \sim f_2} V_2$ is the quotient space $V_1 \sqcup V_2/(f_1 \sim f_2)$, i.e. the pushout of the diagram $(V_1 \overset{f_1}{\hookleftarrow} \sqcup_r \frac{1}{2}\mathbb{D}^2 \overset{f_2}{\hookrightarrow} V_2)$. In other words, it is the boundary connected sum along the right-half of each of the first r marked discs (following the order given by i_1) in V_1 with the left-half of each corresponding marked disc in V_2.
- Identifying $\mathbb{D}^2$ with the unit disc in $\mathbb{R}^2$, we define $i_1 \natural_r i_2\colon \sqcup_{r_1+r_2-r} \mathbb{D}^2 \to V_1 \cup_{f_1 \sim f_2} V_2$ by

$$i_1 \natural_r i_2(x, y) = \begin{cases} i_1(x, y) & \text{if } x \leq 0, \\ i_2(x, y) & \text{if } x \geq 0, \end{cases}$$

  on the first r discs and as before on the remaining discs (ordered according to the order of the monoidal product).

Now, for morphisms $\phi_1 \in \mathrm{Hom}_{\mathfrak{H}^{\geq r}}(V_1, V_1')$ and $\phi_2 \in \mathrm{Hom}_{\mathfrak{H}^{\geq r}}(V_2, V_2')$, we define the morphism $\phi_1 \natural_r \phi_2$ by extending (up to isotopy) ϕ_1 and ϕ_2 by the identity on a collar neighbourhood of each disc in $\mathrm{Im}(i_1)$ and $\mathrm{Im}(i_2)$ respectively where the gluing is done.

Proposition 2.6 *The operation $-\natural_r -\colon \mathfrak{H}^{\geq r} \times \mathfrak{H}^{\geq r} \to \mathfrak{H}^{\geq r}$ makes $\mathfrak{H}^{\geq r}$ into a monoidal groupoid. The monoidal unit is the disjoint union of r 3-balls I_r, each with one embedded disc and no marked points, which may be assumed to be* strict *for simplicity.*

Proof. As $-\natural_r-$ is defined as a pushout on objects and by extending by the identity in the glued region for morphisms, the assignments of $-\natural_r- \colon \mathfrak{H}\times\mathfrak{H} \to \mathfrak{H}$ on objects and morphisms are defined in a unique way. The identity and composition axioms for morphisms are then straightforwardly checked from Definition 2.5, and the operation $-\natural_r- \colon \mathfrak{H}^{\geq r} \times \mathfrak{H}^{\geq r} \to \mathfrak{H}^{\geq r}$ is thus a well-defined bifunctor. The associator is defined by the natural isomorphism $V_1\natural_r(V_2\natural_r V_3) \cong (V_1\natural_r V_2)\natural_r V_3$ provided by the universal properties of $\cup_{f_1\sim f_2}$ and $\cup_{f_2\sim f_3}$ as pushouts, for which one easily checks the pentagon diagram axiom. The left and right unitors are given by the natural isomorphisms $V\natural_r I_r \cong V$ and $I_r\natural_r V \cong V$ induced by an evident deformation retraction of the 3-balls in I_r onto their boundary marked discs, for which the triangle diagram axiom is obviously satisfied. Finally, we may turn I_r into a *strict* monoidal unit by slightly modifying the definition of the bifunctor $\natural_r$ through identification $V\natural_r I_r = V = I_r\natural_r V$, since the isomorphisms $V\natural_r I_r \cong V$ and $I_r\natural_r V \cong V$ then induce an obvious natural equivalence between $-\natural_r I_r$ and $I_r\natural_r-$. □

Although $\mathfrak{H}^{\geq 1} = \mathfrak{H}$ by Definition 2.3, we implicitly mean that this groupoid is considered along with its monoidal structure $(\natural_1, I_1)$ when using the notation $\mathfrak{H}^{\geq 1}$. Furthermore, for all integers $g \geq 0$, $r \geq 1$ and $s \geq 0$, it straightforwardly follows from the construction of the monoidal structures in Definition 2.5 that there are evident natural isomorphisms in the groupoid $\mathfrak{H}$

$$V^s_{g,r} \cong V^s_{0,r}\natural_1 V^{\natural_1 g}_{1,1} \cong V^{\natural_1 g}_{1,1}\natural_1 V^s_{0,r} \quad \text{and} \quad V^s_{g,r} \cong V^s_{0,r}\natural_2 V^{\natural_2 g}_{0,2} \cong V^{\natural_2 g}_{0,2}\natural_2 V^s_{0,r}, \tag{2.3}$$

with $r \geq 2$ for the isomorphisms involving $\natural_2$.

Let us now check some first elementary properties of these monoidal structures. We recall that a monoidal category $(\mathcal{C}, \odot, 0)$ is said to have *no zero-divisors* if, for all objects A and B of $\mathcal{C}$, $A \odot B \cong 0$ if and only if $A \cong B \cong 0$.

Proposition 2.7 *The monoidal groupoid $(\mathfrak{H}^{\geq r}, \natural_r, I_r)$ has no zero-divisors and the monoidal unit I_r has no non-trivial automorphisms, i.e.* $\mathrm{Aut}_{\mathfrak{H}^{\geq r}}(I_r) = \{\mathrm{id}_{I_r}\}$.

Proof. First, we note that the condition $A\natural_r B \cong I_r$ for objects in $\mathfrak{H}^{\geq r}$ implies that each component of A and B is a 3-ball, each with one marked disc and no marked points. Since A and B must have r components, we deduce that $A \cong I_r \cong B$, thus proving the first statement. Now, since we require that the automorphisms of I_r fix the marked disc in each component individually, we have group isomorphisms

$$\mathrm{Aut}_{\mathfrak{H}^{\geq r}}(I_r) \cong \mathrm{Aut}_{\mathfrak{H}}(I_1)^{\times r} \cong \mathcal{H}^{\times r}_{0,1}.$$

Recall that $\Gamma_{0,1} \cong 0$ by Alexander's trick (see [FM12, Lem. 2.1] for instance), so $\mathcal{H}_{0,1}$ is also trivial by Lemma 2.1, which then implies the second statement. □

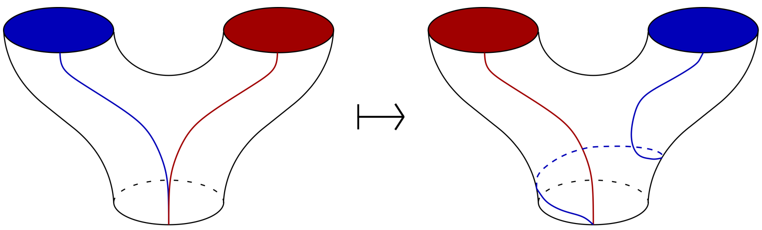

Figure 2.2 Illustration of the braiding in $(\mathfrak{H}^{\geq r}, \natural_r, I_r)$ in a neighborhood of one of the glued disc of $V_1\natural_r V_2$.

Braidings. Finally, we define a braiding on each $\mathfrak{H}^{\geq r}$.

Let us pick a closed neighborhood $\mathcal{N}$ in the handlebody $V_1\natural_r V_2$ of the subsurface induced by the gluing discs, i.e. defined by the union of the glued half-discs $(f_1 \sim f_2)$ along with the new r marked discs defined by $i_1\natural_r i_2$. Note that $\mathcal{N}$ is a disjoint union of pair of pants neighbourhoods of the glued discs as shown in Figure 2.2: namely, it is diffeomorphic to r disjoint 3-balls, each with one marked disc defined by $i_1 \natural_r i_2$, as well as two marked discs in each component of $\mathcal{N}$ defined by the intersection

$$(V_1 \natural_r V_2 \setminus \mathring{\mathcal{N}}) \cap \mathcal{N}.$$

There is a diffeomorphism $b\colon \mathcal{N} \to \mathcal{N}$, unique up to isotopy, which fixes $\mathrm{Im}(i_1 \natural_r i_2)$ pointwise and switches the remaining two discs of each component, as described in Figure 2.2. Extending b by the identity on $V_1 \natural_r V_2 \setminus \mathcal{N}$, this defines an isomorphism:

$$\beta^r_{V_1,V_2}\colon V_1 \natural_r V_2 \to V_2 \natural_r V_1.$$

Note that this construction is clearly independent of the choice of the closed neighborhood $\mathcal{N}$, so β_{V_1,V_2} is a well-defined morphism in $\mathfrak{H}$.

Proposition 2.8 *The natural isomorphism β^r makes $(\mathfrak{H}^{\geq r}, \natural_r, I_r)$ into a braided monoidal category.*

Proof. Note that a diffeomorphism of V_i for $i \in \{1,2\}$ is isotopic to the identity in $V_i \cap \mathcal{N} \subset V_1 \natural_r V_2$. Hence it follows from a disjoint support argument and extension by the identity for these diffeomorphisms that each isomorphism $\beta^r_{V_1,V_2}$ is natural with respect to morphisms of $\mathfrak{H}^{\geq r}$ in both variables V_1 and V_2. Now, note that the braiding is supported on a neighborhood of the glued discs for any monoidal product $V_1 \natural_r V_2$, which can be chosen as a disjoint union of 3-balls on which the braiding looks as in Figure 2.2. Hence it suffices to show that the braid relation (i.e. the hexagon identities) hold in $\mathcal{H}_{0,1,(3)}$, by which we mean the mapping class group of diffeomorphisms of the 3-ball which fix one marked disc pointwise and fix three other marked discs *setwise*, i.e. up to permutation. The analogous mapping class group of diffeomorphisms of the 2-sphere fixing one marked disc pointwise and three other marked discs *setwise* is isomorphic to the braid group on three strands $\mathbf{B}_3$; see [FM12, §9.1.4] for instance. Since $\mathrm{Diff}(V_0; \partial V_0)$ is contractible by the Smale conjecture (see [Hat83, Appendix]), the fibration (2.1) defined by the restriction map to the boundary $V_0 \to \Sigma_0$ induces an injection $\mathcal{H}_{0,1,(3)} \hookrightarrow \mathbf{B}_3$. The claim thus follows from the fact that the braid relation holds in $\mathbf{B}_3$; see for example [RW17, §5.6.1]. $\square$

The monoidal products $\natural_r$ are used below to define stabilisation maps $\mathcal{H}^s_{g,r} \to \mathcal{H}^s_{g+1,r}$ for $r = 1, 2$; see (2.16) and (2.17). Going forward, we only consider these cases, and we thus introduce the following notation:

Notation 2.9 We write $\natural := \natural_1$ and $\# := \natural_2$.

2.2.2. The Quillen bracket construction

We now recall a categorical construction necessary for the twisted homological stability framework in §2.4 that we follow in §3. This notion was originally introduced in [Gra76, p.3], and later developed in [RW17, §1.1] and [Kra19, §4.1], to which we refer the reader for further details.

Let $(\mathcal{G}, \odot, 0)$ be a monoidal groupoid. We consider a left $(\mathcal{G}, \odot, 0)$-module $(\mathcal{M}, \odot)$ and a right $(\mathcal{G}, \odot, 0)$-module $(\mathcal{M}', \odot)$. (For instance, we may take $\mathcal{M} = \mathcal{M}' = \mathcal{G}$ with the left or right-module structure induced by the monoidal product $\odot$.) The *Quillen bracket constructions* $\langle \mathcal{M}, \mathcal{G}]$ and $[\mathcal{M}', \mathcal{G}\rangle$ on $(\mathcal{M}, \odot)$ and $(\mathcal{M}', \odot)$ over the groupoid $(\mathcal{G}, \odot, 0)$ are the categories whose objects are the same as those of $\mathcal{M}$ and whose morphisms are given by

$$\mathrm{Hom}_{\langle \mathcal{G}, \mathcal{M}]}(X, Y) = \operatorname*{colim}_{\mathcal{G}} (\mathrm{Hom}_{\mathcal{M}}(- \odot X, Y)) \text{ and } \mathrm{Hom}_{[\mathcal{M}', \mathcal{G}\rangle}(X, Y) = \operatorname*{colim}_{\mathcal{G}} (\mathrm{Hom}_{\mathcal{M}}(X \odot -, Y)).$$

Hence, a morphism from X to Y in $\langle \mathcal{M}, \mathcal{G}]$ (resp. $[\mathcal{M}', \mathcal{G}\rangle$) is an equivalence class of pairs (A, ϕ) (resp. (ϕ', A')), denoted by $[A, \phi]$ (resp. $[\phi', A']$), where $A \in \mathrm{ob}(\mathcal{G})$ (resp. $A' \in \mathrm{ob}(\mathcal{G})$) and $\phi \in \mathrm{Hom}_{\mathcal{G}}(A \odot X, Y)$ (resp. $\phi' \in \mathrm{Hom}_{\mathcal{G}}(X \odot A', Y)$). Namely, two pairs (A_1, ϕ_1) and (A_2, ϕ_2) (resp. (ϕ'_1, A'_1) and (ϕ'_2, A'_2)) are equivalent if there exists an isomorphism $\chi \in \mathrm{Hom}_{\mathcal{G}}(A_1, A_2)$ (resp. $\chi' \in \mathrm{Hom}_{\mathcal{G}}(A'_1, A'_2)$) such that $\phi_1 = \phi_2 \circ (\chi \odot \mathrm{id}_X)$ (resp. $\phi'_1 = \phi'_2 \circ (\mathrm{id}_X \odot \chi')$). For two morphisms $[A, \phi]\colon X \to Y$ and $[B, \psi]\colon Y \to Z$ in $\langle \mathcal{G}, \mathcal{M}']\rangle$ (resp. $[\phi', A']\colon X \to Y$ and $[\psi', B']\colon Y \to Z$ in $[\mathcal{M}', \mathcal{G}\rangle$), the composition is defined by $[B, \psi] \circ [A, \phi] = [B \odot A, \psi \circ (\mathrm{id}_B \odot \phi)]$ (resp. $[\psi', B'] \circ [\phi', A'] = [\psi' \circ (\phi' \odot \mathrm{id}_{B'}), B' \odot A']$).

There are canonical functors $\mathcal{M} \to \langle \mathcal{G}, \mathcal{M}]$ and $\mathcal{M}' \to [\mathcal{M}', \mathcal{G}\rangle$ defined as the identity on objects, while sending $f \in \mathrm{Hom}_{\mathcal{M}}(X, Y)$ and $f' \in \mathrm{Hom}_{\mathcal{M}'}(X', Y')$ to $[0, f]$ and $[f', 0]$ respectively. We collect here some elementary properties about the Quillen bracket constructions that we will later use in §2.3–§2.4.

Proposition 2.10 *We assume that the modules $\mathcal{M}$ and $\mathcal{M}'$ are groupoids. If the monoidal groupoid $(\mathcal{G}, \odot, 0)$ is braided (with braiding $\beta^{\mathcal{G}} \colon \mathcal{G} \times \mathcal{G} \to \mathcal{G} \times \mathcal{G}$), has no zero-divisors and is such that $\mathrm{Aut}_{\mathcal{G}}(0) = \{\mathrm{id}_0\}$. Then:*

(i) The groupoid $\mathcal{M}$ is the maximal subgroupoid of $\langle \mathcal{G}, \mathcal{M}]$ and $[\mathcal{M}, \mathcal{G}\rangle$.

(ii) For $\mathcal{M} = \mathcal{G}$, the monoidal unit 0 is an initial object in the category $\langle \mathcal{G}, \mathcal{G}]$ and the monoidal structure $(\mathcal{G}, \odot, 0)$ extends to $\langle \mathcal{G}, \mathcal{G}]$ by the same assignments for objects and defined as follows on morphisms

$$[A, \phi] \odot [B, \psi] := [A \odot B, (\phi \odot \psi) \circ (\mathrm{id}_A \odot (b^{\mathcal{G}}_{X_1, B})^{-1} \odot \mathrm{id}_{X_2})] \in \mathrm{Hom}_{\langle \mathcal{G}, \mathcal{G}]}(X_1 \odot X_2, Y_1 \odot Y_2);$$

(iii) The right-module structure $(\mathcal{M}', \odot)$ induces a right $(\mathcal{G}, \odot, 0)$-module structure on $[\mathcal{M}', \mathcal{G}\rangle$, with the same assignments for objects and defined as follows on morphisms

$$[\phi', A'] \odot \psi' := [(\varphi' \odot \psi') \circ (\mathrm{id}_{X_1} \odot (b^{\mathcal{G}}_{A', X_2})^{-1}), A'] \in \mathrm{Hom}_{[\mathcal{M}', \mathcal{G}\rangle}(X_1 \odot X_2, Y_1 \odot X_2).$$

Proof. For the category $\langle \mathcal{G}, \mathcal{M}]$, the proof of (i) repeats verbatim that of [RW17, Prop. 1.7], while that of (ii) is same as that of those of [RW17, Prop. 1.8(i)-(ii)]. Also, the proof of [RW17, Prop. 1.8(ii)] adapts mutatis mutandis to show the analogous property (iii) for the category $[\mathcal{M}', \mathcal{G}\rangle$; see [Kra19, Rem. 4.12]. □

Notation 2.11 In the situation of Proposition 2.10, we abuse notation and write ϕ for $[0, \phi] \in \mathrm{Hom}_{\langle \mathcal{G}, \mathcal{M}]}(X, Y)$ or $[\phi', 0] \in \mathrm{Hom}_{[\mathcal{M}', \mathcal{G}\rangle}(X, Y)$ for all $X, Y \in \mathrm{ob}(\mathcal{G})$, $\phi \in \mathrm{Hom}_{\mathcal{M}}(X, Y)$ and $\phi' \in \mathrm{Hom}_{\mathcal{M}'}(X, Y)$.

Remark 2.12 Monoidal and module structure properties analogous to (ii) and (iii) of Proposition 2.10 also hold for the categories $[\mathcal{G}, \mathcal{G}\rangle$ and $\langle \mathcal{G}, \mathcal{M}]$ respectively. We only stated the results we need for our work, in particular used to introduce the notion of finite degree coefficient systems in our context (see §2.3) or to interpret some objects we introduce (see Remarks 2.36 and 2.40).

Finally, we deduce the following result from Propositions 2.7, 2.8 and 2.10.

Corollary 2.13 *For any $r \geq 1$, the groupoid $\mathfrak{H}^{\geq r}$ is the maximal subgroupoid of the categories $\langle \mathfrak{H}^{\geq r}, \mathfrak{H}^{\geq r}]$ and $[\mathfrak{H}^{\geq r}, \mathfrak{H}^{\geq r}\rangle$. Also, the category $\langle \mathfrak{H}^{\geq r}, \mathfrak{H}^{\geq r}]$ (resp. $[\mathfrak{H}^{\geq r}, \mathfrak{H}^{\geq r}\rangle$) is equipped with a monoidal (resp. right $\mathfrak{H}^{\geq r}$-module) structure $(\natural_r, I_r)$.*

2.2.3. Handlebodies of infinite rank

Because of a technical issue in §3.2, we need to consider handlebodies which are not compact at some point. We therefore make the following definition:

Definition 2.14 For each 3-dimensional compact handlebody V of $\mathfrak{H}$, let $V_{\infty,1} \natural V$ be the directed colimit in the category $\langle \mathfrak{H}^{\geq 1}, \mathfrak{H}^{\geq 1}]$ of the sequential diagram

$$V \xrightarrow{\iota_1(V)} V_{1,1} \natural V \xrightarrow{\iota_2(V)} V_{1,1}^{\natural 2} \natural V \longrightarrow \cdots \xrightarrow{\iota_i(V)} V_{1,1}^{\natural i} \natural V \xrightarrow{\iota_{i+1}(V)} \cdots . \tag{2.4}$$

where $\iota_i(V)$ denotes the morphism $[V_{1,1}, \mathrm{id}_{V_{1,1}^{\natural i} \natural V}]$ of $\mathrm{Hom}_{\langle \mathfrak{H}^{\geq 1}, \mathfrak{H}^{\geq 1}]}(V_{1,1}^{\natural i-1} \natural V, V_{1,1}^{\natural i} \natural V)$. It is called the *infinite object of V*. We construct the groupoid $\overline{\mathfrak{H}}$ as follows:

- The objects of $\overline{\mathfrak{H}}$ are those of $\mathfrak{H}$ along with any infinite object $V_{\infty,1} \natural V$ for V a compact handlebody with marked discs and marked points of $\mathfrak{H}$.

- The morphisms in $\overline{\mathfrak{H}}$ for the objects of $\mathfrak{H}$ are those Definition 2.3. Those involving infinite objects are of the form $\phi\colon V_{\infty,1}\natural V \to V_{\infty,1}\natural V'$, i.e. ϕ is defined by the universal property of the colimit $V_{\infty,1}\natural V$ if there exist integers $j, j' \geq 0$ and a set of morphisms $\{\phi_i \in \mathrm{Hom}_{\mathfrak{H}}(V_{1,1}^{\natural i+j}\natural V, V_{1,1}^{\natural i+j'}\natural V')\}_{i\geq j}$, such that the following diagram is commutative:

$$\begin{tikzcd}
V_{1,1}^{\natural j}\natural V \arrow[r, "\iota_{1+j}(V)"] \arrow[d, "\phi_j"] & V_{1,1}^{\natural 1+j}\natural V \arrow[r, "\iota_{2+j}(V)"] \arrow[d, "\phi_{1+j}"] & \cdots \arrow[r, "\iota_{i+j}(V)"] & V_{1,1}^{\natural i+j}\natural V \arrow[r, "\iota_{i+1+j}(V)"] \arrow[d, "\phi_{i+j}"] & \cdots \\
V^{\natural j'}\natural V' \arrow[r, "\iota_{1+j'}(V')"] & V_{1,1}^{\natural 1+j'}\natural V' \arrow[r, "\iota_{2+j'}(V')"] & \cdots \arrow[r, "\iota_{i+j'}(V')"] & V_{1,1}^{\natural i+j'}\natural V' \arrow[r, "\iota_{i+1+j'}(V')"] & \cdots .
\end{tikzcd} \tag{2.5}$$

We also denote by $\overline{\mathfrak{H}}^{\geq 2}$ the full subcategory of $\overline{\mathfrak{H}}$ with objects those of $\mathfrak{H}^{\geq 2}$ along with the infinite objects $V_{\infty,1}\natural V$ with $V \in \mathrm{ob}(\mathfrak{H}^{\geq 2})$.

Remark 2.15 It is a classical fact (see [Bor94, Prop. 2.13.3] for instance) that a sequential diagram (2.4) in the category $\langle \mathfrak{H}^{\geq 1}, \mathfrak{H}^{\geq 1}]$ corresponds to a quotient of a infinite disjoint union of the form $\coprod_{i\geq 0} V_{1,1}^{\natural i}\natural(V \# V_{0,2}^{\# n})/\sim$, while their morphisms are infinite disjoint unions of diffeomorphisms up to isotopy satisfying the condition that the diagram (2.5) is commutative. Such infinite disjoint unions do exist (for instance in the homotopy category of topological spaces), and so the infinite objects of Definition 2.14 and their morphisms are well-defined.

Notation 2.16 Following Notation 2.4, we write $V_{\infty,r}^{s}$ for the infinite object $V_{\infty,1}\natural V_{0,r}^{s}$ in $\overline{\mathfrak{H}}$ for some fixed integers $r \geq 1$ and $s \geq 0$.

Convention 2.17 Assuming that manifolds are second-countable (thus implying that their cardinality is no larger than $|\mathbb{R}|$), the categories of Definitions 2.3 and 2.14, as well as those built from them with the Quillen bracket construction of §2.2.2, are essentially small following the standard general arguments of [Gal04, Rem. 1.2.7]. Namely, the objects of these categories are the disjoint unions of a finite number of 3-balls, 3-dimensional compact handlebodies and their sequential diagrams of the form (2.4), plus some additional data (such as marked discs, marked points, etc.). Then, fixing a sufficiently large set Ω (i.e. its cardinality is at least $|\mathbb{R}|$), we consider the full subcategory $\mathcal{C}$ whose objects are only those where the underlying set of a 3-manifold is a subset of Ω. This category $\mathcal{C}$ is small and its inclusion into the whole (large) category is essentially surjective on objects, and so it is an equivalence. We shall thus consider this type of corresponding small category whenever we need our categories to be small, for instance when considering functor categories in §2.3. Moreover, assuming the axiom of choice, all the categories associated to handlebodies we consider have a skeleton; see [Ric20, Prop. 2.6.4] for instance. Hence, up to equivalence, we may always work with these categories as if they were skeletal. In particular, similarly to [RW17; Kra19], we use the notation $\mathrm{Aut}_{\mathcal{C}}(X)$ of the automorphisms of an object X to also denote its class of isomorphisms in $\mathcal{C}$ (which coincide when $\mathcal{C}$ is skeletal). We will not dwell further on these considerations, as these minor category theory details and mild identifications do not impact our methods and results, nor add value to our work.

Module structure. We now define a right $(\mathfrak{H}^{\geq 2}, \#, I_2)$-module structure over $\overline{\mathfrak{H}}^{\geq 2}$ as follow. We define $-\#-\colon \overline{\mathfrak{H}}^{\geq 2} \times \mathfrak{H}^{\geq 2} \to \overline{\mathfrak{H}}^{\geq 2}$ by the assignments of Definition 2.5 for the objects of $\mathfrak{H}^{\geq 2}$, and by assigning $(V_{\infty,1}\natural V)\# W$ to be the directed colimit in the category $\langle \mathfrak{H}^{\geq 1}, \mathfrak{H}^{\geq 1}]$ of the diagram

$$V\# W \xrightarrow{\iota_1(V\# W)} V_{1,1}\natural(V\# W) \xrightarrow{\iota_2(V\# W)} \cdots \xrightarrow{\iota_i(V\# W)} V_{1,1}^{\natural i}\natural(V\# W) \xrightarrow{\iota_{i+1}(V\# W)} \cdots \tag{2.6}$$

for any infinite object $V_{\infty,1}\natural V$ and W an object of $\mathfrak{H}^{\geq 2}$. We then set $-\#-\colon \overline{\mathfrak{H}}^{\geq 2} \times \mathfrak{H}^{\geq 2} \to \overline{\mathfrak{H}}^{\geq 2}$ on morphisms in the evident way, i.e. as in Definition 2.5 for morphisms $\mathfrak{H}^{\geq 2}$ and thanks to the universal property of a colimit for a diagram the form (2.5) applied to (2.6) for morphisms involving infinite objects in the first variable.

Lemma 2.18 *The groupoid $\overline{\mathfrak{H}}^{\geq 2}$ has a well-defined right $(\mathfrak{H}^{\geq 2}, \#, I_2)$-module structure with the above assignments.*

Proof. By restriction to the full subcategory $\mathfrak{H}^{\geq 2} \subset \overline{\mathfrak{H}}^{\geq 2}$, it follows from Proposition 2.6 that the operation $-\#-$ gives $\mathfrak{H}^{\geq 2}$ a well-defined right $(\mathfrak{H}^{\geq 2}, \#, I_2)$-module structure. Hence, it suffices to check that this structure holds for the infinite objects $V_{\infty,1}\natural V$ of $\overline{\mathfrak{H}}^{\geq 2}$. Recall that connected objects V and W of $\mathfrak{H}^{\geq 2}$ are of the form $V^{s}_{h,r}$ and $V^{s'}_{h',r'}$ for fixed $h, h', s, s' \geq 0$ and $r, r' \geq 2$, so there is a natural isomorphism $\psi_{i,W} \colon (V_{1,1}^{\natural i}\natural V)\#W \cong V_{1,1}^{\natural i}\natural(V\#W)$ obtained by composing isomorphisms of the form (2.3). Then, by applying these isomorphisms $\psi_{i,W}$ for all $i \geq 0$ to the diagram (2.6), the functoriality of $-\#-$ with respect to diagrams of the form (2.5) for the first summand and the morphisms of $\mathfrak{H}^{\geq 2}$ in the second summand, as well as the associativity and unitality axioms for a module structure, are straightforward consequences of Proposition 2.6 by some obvious diagram chasings along with the universal properties of the colimits involved in the definitions. $\square$

Finally, we will make use of the following property in $\overline{\mathfrak{H}}^{\geq 2}$ at several points.

Lemma 2.19 *For any $V \in \mathrm{ob}(\mathfrak{H}^{\geq 2})$, there exists an isomorphism in $\overline{\mathfrak{H}}^{\geq 2}$*

$$(V_{\infty,1}\natural V)\#V_{0,2} \cong V_{\infty,1}\natural V. \tag{2.7}$$

Proof. By Definition 2.14, the object $V_{\infty,1}\natural V$ is the diagram (2.4) with V an object of $\mathfrak{H}^{\geq 2}$ of the form $V^{s}_{h,r}$ for some fixed $r \geq 2$, $h, s \geq 0$. Note that, for any such $V \in \mathrm{ob}(\mathfrak{H}^{\geq 2})$, there exists an isomorphism $\phi_V \colon V\#V_{0,2} \cong V_{1,1}\natural V$ in $\mathfrak{H}^{\geq 2}$ obtained by composing isomorphisms of the form (2.3). Hence, we have a sequence of isomorphisms

$$\begin{tikzcd}
V\#V_{0,2} \arrow[r] \arrow[d, "\cong"', "\phi_V"] & V_{1,1}\natural(V\#V_{0,2}) \arrow[r] \arrow[d, "\cong"', "\mathrm{id}_{V_{1,1}}\natural\phi_V"] & \cdots \\
V_{1,1}\natural V \arrow[r] & V_{1,1}^{\natural 2}\natural V \arrow[r] & \cdots
\end{tikzcd}$$

which induces the isomorphism (2.7) by the universal properties of the involved colimits. $\square$

2.3. Finite degree coefficient systems

We review here the notion of *finite degree coefficient system*, designed for twisted coefficients in order to deal with twisted homological stability; see §2.4.1. This is a classical concept appearing in the literature on homological stability, in particular in [RW17, §4.1–§4.4] and [Kra19, §4.1] to which we refer the reader for further details. We consider a braided monoidal groupoid $(\mathcal{G}, \odot, 0)$ with no zero-divisors and such that $\mathrm{Aut}_{\mathcal{G}}(0) = \{\mathrm{id}_0\}$, and a right $(\mathcal{G}, \odot, 0)$-module $(\mathcal{M}, \odot)$ such that $\mathcal{M}$ is a groupoid. In order to introduce the notion of coefficient system of finite degree, we need to work over the following groupoids:

Notation 2.20 For objects $A \in \mathrm{ob}(\mathcal{M})$ and $X \in \mathrm{ob}(\mathcal{G})$, we write $A_n := A \odot X^{\odot n}$ for each integer $n \geq 0$. We also denote by $\mathcal{G}_X$ and $\mathcal{M}_{A,X}$ the full subgroupoids of $\mathcal{G}$ and $\mathcal{M}$ on the objects $\{X^{\odot n}\}_{n\in\mathbb{N}}$ and $\{A_n\}_{n\in\mathbb{N}}$ respectively.

Lemma 2.21 *The subgroupoid $\mathcal{G}_X$ inherits the braided monoidal structure $(\odot, 0)$ of $\mathcal{G}$, and the groupoid $\mathcal{M}_{A,X}$ is a compatible $(\mathcal{G}_X, \odot)$-submodule of $\mathcal{M}$. Furthermore, the category $[\mathcal{M}_{A,X}, \mathcal{G}_X\rangle$ is isomorphic to the full subcategory of $[\mathcal{M}, \mathcal{G}\rangle$ on the objects $\{A_n\}_{n\in\mathbb{N}}$ denoted by $[\mathcal{M}, \mathcal{G}\rangle_{A,X}$.*

Proof. The first statements are straightforwardly verified from the fact that $\mathcal{G}_X$ and $\mathcal{M}_{A,X}$ are full subgroupoids of $\mathcal{G}$ and $\mathcal{M}$ respectively, generated by the monoidal structure $\odot$ with the object X iterated over 0 and A respectively. Then, the categories $[\mathcal{M}_{A,X}, \mathcal{G}_X\rangle$ and $[\mathcal{M}, \mathcal{G}\rangle_{A,X}$ have the same objects, while $\mathrm{colim}_{\mathcal{G}_X}\left(\mathrm{Hom}_{\mathcal{M}_X}(A_n \odot -, A_{n+m})\right)$ and $\mathrm{colim}_{\mathcal{G}_X}\left(\mathrm{Hom}_{\mathcal{M}_{A,X}}(A_n \odot -, A_{n+m})\right)$ are obviously isomorphic for all integers $n, m \geq 0$ because $\mathcal{G}_X$ and $\mathcal{M}_{A,X}$ are full subgroupoids of $\mathcal{G}$ and $\mathcal{M}$ respectively. This provides a canonical isomorphism between these categories by the definition of the Quillen bracket construction in §2.2.2. $\square$

We fix two objects $A \in \mathrm{ob}(\mathcal{M})$ and $X \in \mathrm{ob}(\mathcal{G})$ for the remainder of §2.3. A *coefficient system* over the category $[\mathcal{M}_{A,X}, \mathcal{G}_X\rangle$ defined in §2.2.2 is a functor $[\mathcal{M}_{A,X}, \mathcal{G}_X\rangle \to \mathbf{Ab}$. For a fixed coefficient system F over $[\mathcal{M}_{A,X}, \mathcal{G}_X\rangle$, it follows from (iii) of Proposition 2.10 that we have a well-defined functor $\tau_X(F) := F(- \odot \mathrm{id}_X) \colon [\mathcal{M}_{A,X}, \mathcal{G}_X\rangle \to \mathbf{Ab}$ (called the *translation functor* or *upper suspension* of F) defined by assigning $A_n \mapsto F(A_n \odot X)$ on objects and $[\phi, X^{\odot m}] \mapsto [\phi, X^{\odot m}] \odot \mathrm{id}_X$ on morphisms. There is also a well-defined natural transformation $i_X(F) \colon F \to \tau_X(F)$ in $\mathbf{Fct}([\mathcal{M}_{A,X}, \mathcal{G}_X\rangle, \mathbf{Ab})$ induced by the canonical morphisms $\{[\mathrm{id}_{A_{n+1}}, X] \colon A_n \to A_{n+1}\}_{n \in \mathbb{N}}$ of $[\mathcal{M}_{A,X}, \mathcal{G}_X\rangle$ (see §2.2.2), i.e. $i_X(F)_n := F([\mathrm{id}_{A_{n+1}}, X])$ for each integer $n \geq 0$. Since the functor category $\mathbf{Fct}([\mathcal{M}_{A,X}, \mathcal{G}_X\rangle, \mathbf{Ab})$ is abelian (see [Gab62, §VIII.3]), the cokernel $\mathrm{Coker}(i_X(F))$ and the kernel $\mathrm{Ker}(i_X(F))$ of the natural transformation $i_X(F)$ are well-defined coefficient systems over $[\mathcal{M}_{A,X}, \mathcal{G}_X\rangle$, called the *difference* functor and the *evanescence* functor of F respectively, and denoted by $\delta_X(F)$ and $\kappa_X(F)$ respectively.

Definition 2.22 We now recursively define the notion of *finite degree* for a coefficient system $F \colon [\mathcal{M}_{A,X}, \mathcal{G}_X\rangle \to \mathbf{Ab}$ as follows.

- A coefficient system F is of finite degree -1 if $F = 0$.
- For an integer $d \geq 0$, a coefficient system F is of finite degree less than or equal to d if the natural transformation $i_X(F) \colon F \to F(- \odot X)$ is injective in $\mathbf{Fct}([\mathcal{M}_{A,X}, \mathcal{G}_X\rangle, \mathbf{Ab})$ (i.e. $\kappa_X(F) = 0$) and the difference functor $\delta_X(F)$ is a finite coefficient system of degree less than or equal to $d - 1$.

Furthermore, a coefficient system F of finite degree r is *split* if the natural transformation $i_X(F)$ is *split* injective in the category of coefficient systems, and the difference functor $\delta_X(F)$ is a split coefficient system of degree $r - 1$.

Example 2.23 The constant functor at $\mathbb{Z}$ defines a split coefficient system $\mathbb{Z} \colon [\mathcal{M}_{A,X}, \mathcal{G}_X\rangle \to \mathbf{Ab}$ of finite degree 0.

Remark 2.24 The definition of finite degree coefficient systems presented here follows that of [Kra19, Def. 4.6]. That of [RW17, Def. 4.10] uses a notion of *lower suspension* instead of the upper suspension $\tau_X(F) := F(- \odot \mathrm{id}_X)$. These two suspensions may be related via the braiding of $\mathcal{G}$ (see [RW17, (4.2)]) and these two approaches are equivalent; we refer the reader to [Kra19, Rem. 4.11, §7.3] for more details. On another note, we consider here finite degree coefficient systems *at* 0 in the terminology of [Kra19, Def. 4.6], i.e. the above defining conditions could rather be set on $F(X^n)$ for all $n \geq N$ for a fix bound $N \in \mathbb{N}$. Although we could extend the framework and results of the present paper using this slightly more general notion of coefficient systems, we prefer this restriction for simplicity and because it seems more natural.

2.3.1. First homologies coefficient systems

Some first examples of non-trivial finite degree coefficient systems are given by the first homology groups of the surfaces and the handlebodies that we introduce in this section; see Proposition 2.26.

First of all, we denote by **hMan** the homotopy category of pairs of topological manifolds, where the objects are pairs (M, A), where M is a topological manifolds and $A \subset M$ is a submanifold, and where morphisms are homotopy classes of continuous maps of pairs $(M, A) \to (M', A')$; see [Spa95, §3] for instance. We recall that the first relative homology defines a functor $H_1(-, -; \mathbb{Z}) \colon \mathbf{hMan} \to \mathbf{Ab}$; see [Hat02, §2.3] for instance. We note from Definition 2.3 that $\mathfrak{H}$ is a category where each object (V, r, s, i, j) has an underlying pair of manifolds $(V, \mathcal{D}_r \sqcup \mathcal{P}_s)$, where $\mathcal{D}_r = \mathrm{Im}(i)$ and $\mathcal{P}_s = \mathrm{Im}(j)$ for $r \geq 1$ and $s \geq 0$ are subsets of ∂V, and whose morphisms are homotopy classes of continuous maps of pairs (plus some additional data), so there is a canonical forgetful functor $\mathfrak{F} \colon \mathfrak{H} \to \mathbf{hMan}$. Furthermore, let $\partial\mathfrak{F} \colon \mathfrak{H} \to \mathbf{hMan}$ be the functor defined by assigning $(V, r, s, i, j) \mapsto (\partial V \setminus (\mathring{\mathcal{D}}_r \sqcup \mathcal{P}_s), \emptyset)$ on objects, and by restriction to the boundary on morphisms (recalling that the diffeomorphisms of smooth manifolds are boundary-preserving, and so are their homotopy classes). Therefore, we consider the composite functors

$$H_1(-, -; \mathbb{Z}) \circ \partial\mathfrak{F} \colon \mathfrak{H} \to \mathbf{hMan} \to \mathbf{Ab} \text{ and } H_1(-, -; \mathbb{Z}) \circ \mathfrak{F} \colon \mathfrak{H} \to \mathbf{hMan} \to \mathbf{Ab},$$

that we denote by H_Σ and $H_\mathcal{V}$ respectively. Namely, these functors are defined by assigning to each object $(V,r,s,i,j) \in \mathrm{ob}(\mathcal{H})$ the first homology groups $H_1(\partial V \setminus (\mathring{\mathcal{D}}_r \sqcup \mathcal{P}_s);\mathbb{Z})$ and $H_1(V, \mathcal{D}_r \sqcup \mathcal{P}_s;\mathbb{Z})$ respectively, together with the natural action of the isotopy classes of the diffeomorphism groups.

Notation 2.25 We write Z for either the handlebody with a marked disc $V_{1,1}$ or the 3-ball with two marked discs $V_{0,2}$. For a fixed compact handlebody V of $\mathcal{H}$, taking up Notation 2.20, we consider the monoidal groupoid $(\mathcal{H}_Z^{\geq i}, \natural_i, I_i)$ and right-module $(\mathcal{H}_{V,Z}^{\geq i}, \natural_i)$ over it, with the following assignments:

- if $Z = V_{1,1}$, then $i = 1$;
- if $Z = V_{0,2}$, then $i = 2$ and $r \geq 2$.

Although we obviously have $\mathcal{H}_{V,Z}^{\geq i} = \mathcal{H}_{V,Z}$ as categories, we use the notation $\mathcal{H}_{V,Z}^{\geq i}$ in order to indicate which monoidal structure is used when relevant.

We restrict the functors $H_\Sigma \colon \mathcal{H} \to \mathbf{Ab}$ and $H_\mathcal{V} \colon \mathcal{H} \to \mathbf{Ab}$ to the subgroupoid $\mathcal{H}_{V,Z}$ of $\mathcal{H}$. Also, we note from (i) of Proposition 2.10 that there is a fully faithful canonical functor.

$$\mathfrak{C}_Z \colon \mathcal{H}_{V,Z} \hookrightarrow [\mathcal{H}_{V,Z}^{\geq i}, \mathcal{H}_Z^{\geq i}\rangle, \tag{2.8}$$

defined as the identity on objects and by $\phi \mapsto [\phi, 0]$ for all $\phi \in \mathrm{Aut}_{\mathcal{H}_{V,Z}}(V \natural_i Z^{\natural_i n})$ where $n \geq 0$. The functors H_Σ and $H_\mathcal{V}$ induce coefficients systems as follows:

Proposition 2.26 *For fixed integers $d \geq 1$ $r \geq 1$ and $s \geq 0$, the d-fold tensor powers of the functors H_Σ and $H_\mathcal{V}$ in $\mathbf{Fct}(\mathcal{H}_{V,Z}, \mathbf{Ab})$ lift along $\mathfrak{C}_Z$ to finite degree d coefficient systems $[\mathcal{H}_{V,Z}^{\geq i}, \mathcal{H}_Z^{\geq i}\rangle \to \mathbf{Ab}$ for each $i \in \{1,2\}$, denoted by $T^d H_\Sigma$ and $T^d H_\mathcal{V}$ respectively. Moreover, the coefficient systems $T^d H_\Sigma \colon [\mathcal{H}_{V,V1,1}^{\geq 1}, \mathcal{H}_{V1,1}^{\geq 1}\rangle \to \mathbf{Ab}$ and $T^d H_\mathcal{V} \colon [\mathcal{H}_{V,Z}^{\geq i}, \mathcal{H}_Z^{\geq i}\rangle \to \mathbf{Ab}$ for $i \in 1,2$ are split.*

Proof. Firstly, we note from §2.2.2 that a morphism $[\phi, Z^{\natural_i g'-g}] \colon V \natural_i Z^{\natural_i g} \to V \natural_i Z^{\natural_i g'}$ in the category $[\mathcal{H}_{V,Z}^{\geq i}, \mathcal{H}_Z^{\geq i}\rangle$ is the equivalence class consisting of the diffeomorphisms of $V \natural_i Z^{\natural_i g'}$ which are isotopic to ϕ on the closed submanifold V of $V \natural_i Z^{\natural_i (g'-g)}$. Hence, the category $[\mathcal{H}_{V,Z}^{\geq i}, \mathcal{H}_Z^{\geq i}\rangle$ is isomorphic to the category with the same objects, and whose morphisms are the isotopy classes of proper smooth embeddings $V \hookrightarrow V \natural_i Z^{\natural_i (g'-g)}$ which preserve the sets $\mathcal{D}_r$ and $\mathcal{P}_s$ associated to V; see also [PS24, Prop. 1.76] for further details. We deduce that there is a canonical forgetful functor $\tilde{\mathfrak{F}} \colon [\mathcal{H}_{V,Z}^{\geq i}, \mathcal{H}_Z^{\geq i}\rangle \to \mathbf{hMan}$, such that by definition $\tilde{\mathfrak{F}} \circ \mathfrak{C}_Z = \mathfrak{F}$. Furthermore, by classification of surfaces (see [Ric63; BM79]), the space $\partial V \setminus (\mathring{\mathcal{D}}_r \sqcup \mathcal{P}_s)$ is homeomorphic to the surface $\Sigma_{h,r}^s$ for some $h \geq 0$. We also recall that the monoidal structure $\natural_i$ of Definition 2.5 has a clear analogous counterpart for the surfaces and their mapping class groups (that we denote in the same way), induced by gluing surfaces along half-circles rather than half-discs; see [RW17, §5.6.1] for further details. Then, we note that a proper smooth embedding $\phi \colon V \hookrightarrow V \natural_i Z^{\natural_i (g'-g)}$ preserving $\mathcal{D}_r \sqcup \mathcal{P}_s$ induces by restriction to the boundary a proper smooth embedding $\varphi_\partial \colon \Sigma_{h,r}^s \hookrightarrow \Sigma_{h,r}^s \natural_i (\partial Z)^{\natural_i (g'-g)}$ preserving $\mathcal{P}_s$. Hence, we define a functor $\tilde{\partial\mathfrak{F}} \colon \colon [\mathcal{H}_{V,Z}^{\geq i}, \mathcal{H}_Z^{\geq i}\rangle \to \mathbf{hMan}$ by assigning each $(V \natural_i Z^{\natural_i (g'-g)}, \mathcal{D}_r \sqcup \mathcal{P}_s)$ to $(\Sigma_{h,r}^s \natural_i (\partial Z)^{\natural_i (g'-g)}, \emptyset)$ for object, and sending the homotopy class of each φ to the homotopy class of $[\varphi_\partial]$. Then, it is straightforward to check from these assignments that we thus have constructed a lift of $\partial\mathfrak{F}$ to $[\mathcal{H}_{V,Z}^{\geq i}, \mathcal{H}_Z^{\geq i}\rangle$, in the sense that $\tilde{\partial\mathfrak{F}} \circ \mathfrak{C}_Z = \partial\mathfrak{F}$. Therefore, the functors H_Σ and $H_\mathcal{V}$ of $\mathbf{Fct}(\mathcal{H}_{V,Z}, \mathbf{Ab})$ along $\mathfrak{C}_Z$ are given by the composites $H_1(-,-;\mathbb{Z}) \circ \tilde{\partial\mathfrak{F}}$ and $H_1(-,-;\mathbb{Z}) \circ \tilde{\mathfrak{F}}$ respectively, which we may postcompose by the d-fold tensor power functor $T^d \colon \mathbf{Ab} \to \mathbf{Ab}$.

Now, we denote by $H \colon \mathcal{H}_{V,Z} \to \mathbf{Ab}$ either H_Σ or $H_\mathcal{V}$, and by V_g the object $V \natural_i Z^{\natural_i g}$ for some integer $g \geq 0$. Let us analyse the natural transformation $i_Z(H)$ in each situation. First, denoting by h the genus of the compact handlebody $V_g \natural_i Z$, we consider the homology classes

$$\mathcal{B}_{h+1}^\Sigma = \{[a_1], [b_1], \ldots, [a_{h+1}], [b_{h+1}], [c_2], \ldots, [c_r], [d_1], \ldots, [d_s]\}; \tag{2.9}$$

$$\mathcal{B}_{h+1}^\mathcal{V} = \{[b_1], \ldots, [b_{h+1}], [c_2'], \ldots, [c_r'], [d_1'], \ldots, [d_s']\}. \tag{2.10}$$

Here:

- $\{a_i, b_i \mid 1 \leq i \leq h+g+1\}$ is a standard system of *meridians* and *longitudes* for the surface $\partial(V_g \natural_i Z)$. Namely for each handle, these are the homotopy classes of simple closed curves in $\partial V_{1,1}$, that respectively bound a nonseparating disc in $V_{1,1}$ for the meridians and the genus for the longitudes.
- $\{c_i, d_j \mid 2 \leq i \leq r, 1 \leq j \leq s\}\}$ is a standard system of simple closed curves of $\partial(V_g \natural_i Z)$ bounding the $r-1$ marked discs $\{\mathcal{D}_V^2, \ldots, \mathcal{D}_V^r\}$ (i.e. there is no generator for the first marked disc $\mathcal{D}_V^1$) and the s marked points of $V_g \natural_i Z$ respectively.
- $\{c'_i, d'_j \mid 2 \leq i \leq r, 1 \leq j \leq s\}\}$ is a standard system of simple closed curves of $\partial(V_g \natural_i Z)$, where each c'_i joins the marked discs $\mathcal{D}_V^i$, while each d'_j connects the marked point $p_V^j \in \mathcal{P}_s$ to the marked disc $\mathcal{D}_V^1$.

It is an elementary cellular homology calculation that $H(V_g \natural_i Z)$ is a finitely generated free abelian group, with a standard basis given by $\mathcal{B}_{h+1}$ for $H = H_\Sigma$, and $\mathcal{B}_{h+1} \setminus \{[a_1], \ldots, [a_{h+1}]\}$ for $H = H_{\mathcal{V}}$. Also, a standard basis for $H(V_g \natural_i Z)$ is similarly provided by the sets $\mathcal{B}_h$ and $\mathcal{B}'_h$ for $H = H_\Sigma$ and $H = H_{\mathcal{V}}$ respectively, i.e. the bases (2.9) (resp. (2.10)) minus the classes $[a_{h+1}]$ and $[b_{h+1}]$ (resp. $[b_{h+1}]$). Futhermore, for any $n \geq 0$, for a morphism $[\phi, Z^{\natural_i n}]\colon V_g \to V_g \natural_i Z^{\natural_i n}$ and a representative $\tilde{\phi}$ of this isotopy class, the map $H([\phi, Z^{\natural_i n}] \natural_i \mathrm{id}_Z)$ is given by the induced action on the homology classes of the embedding $\tilde{\phi}\colon V_g \hookrightarrow V_g \natural_i Z^{\natural_i n}$ extended by the identity over the complement $(n+1)$-st copy of Z. In particular, for $n = 1$ and $\phi = \mathrm{id}_{V_g \natural_i Z}$, the morphism $[\mathrm{id}_{V_g \natural_i Z}, Z]$ may be viewed as the proper smooth embedding $V_g \hookrightarrow V_g \natural_i Z$. Also, we note that the support of the action of $H([\phi, Z^{\natural_i n}] \natural_i \mathrm{id}_Z)$ is disjoint from the submanifold Z. By a straightforward analysis on simple closed curves defining these bases, we deduce that $\delta_Z(H)$ is computed as follows.

For $i = 1$: for each $g \geq 0$, there is a canonical isomorphism $\delta_{V_{1,1}}(H)(V_g) \cong H(V_{1,1})$ and the morphism $H([\mathrm{id}_{V_g \natural V_{1,1}}, V_{1,1}])$ is the canonical inclusion $H(V_g) \hookrightarrow H(V_g) \oplus H(V_{1,1})$. Moreover, by disjointness of the support of the action, the morphism $H([\phi, V_{1,1}^{\natural n}] \natural \mathrm{id}_{V_{1,1}})$ decomposes as $H([\phi, V_{1,1}^{\natural n}]) \oplus \mathrm{id}_{H(V_{1,1})}$ for each morphism $[\phi, V_{1,1}^{\natural n}]\colon V_g \to V_g \natural V_{1,1}^{\natural n}$. Therefore, the natural transformation $i_{V_{1,1}}(H)\colon H \to H(-\natural V_{1,1})$ is split injective in $\mathbf{Fct}([\mathfrak{H}_{V,V_{1,1}}^{\geq 1}, \mathfrak{H}_{V_{1,1}}^{\geq 1}\rangle, \mathbf{Ab})$, and the difference functor $\delta_{V_{1,1}}(H)$ is constant with value $H(V_{1,1})$, hence a split coefficient system of finite degree 0 (see Example 2.23).

For $i = 2$: for each $g \geq 0$, the morphism $H_{\mathcal{V}}([\mathrm{id}_{V_g \# V_{0,2}}, V_{0,2}])$ is the canonical injection of classes $\mathcal{B}'_h \hookrightarrow \mathcal{B}'_{h+1}$, while the map $H_\Sigma([\mathrm{id}_{V_g \# V_{0,2}}, V_{0,2}])$ is induced by the canonical injection of classes of $\mathcal{B}_h \setminus \{[c_2]\} \hookrightarrow \mathcal{B}_{h+1} \setminus \{[c_2]\}$ along with $H([\mathrm{id}_{V_g \# V_{0,2}}, V_{0,2}])([c_2]) = [a_{h+1}]$. We deduce that $\delta_{V_{0,2}}(H_{\mathcal{V}})(V_g) \cong \langle [b_{h+1}] \rangle \cong \mathbb{Z}$ and $\delta_{V_{0,2}}(H_\Sigma)(V_g) \cong \langle [c_2], [b_{h+1}] \rangle \cong \mathbb{Z}^2$, while $\kappa_{V_{0,2}}(H_{\mathcal{V}}) = \kappa_{V_{0,2}}(H_\Sigma) = 0$. Furthermore, we consider the standard generating set of the mapping class group of the surface ∂V_g, given by the Lickorish generators plus the $r+s$ Dehn twists along the meridians of the h-th handle, bounding and separating the marked disc and marked points (similarly to [FM12, Fig. 4.10]); see [FM12, §4.4.2–§4.4.4]. The only one of these generators whose support is not disjoint from the homology classes $[c_2]$ and $[b_{h+1}]$ is the Dehn twist $T(c_{1,2})$ along the meridian of the h-th handle which separates the marked disc $\mathcal{D}_V^1$ and $\mathcal{D}_V^2$. By an elementary analysis on the homology classes, we compute that $T(c_{1,2})([c_2]) = [c_2]$ and $T(c_{1,2})([b_{h+1}]) = [b_{h+1}] + [a_h]$, and so the mapping class group of the surface ∂V_g acts trivially on $\delta_{V_{0,2}}(H)(V_g)$. Collecting all these observations and using Lemma 2.1, we conclude that the map $\delta_{V_{0,2}}(H)([\phi, V_{0,2}^{\# n}])$ is the identity map for each morphism $[\phi, V_{0,2}^{\# n}]\colon V_g \to V_g \natural V_{0,2}^{\# n}$, and that $\delta_{V_{0,2}}(H_{\mathcal{V}})(V_g)$ canonically splits when $H = H_{\mathcal{V}}$. Therefore, the natural transformation $i_{V_{0,2}}(H)\colon H \to H(-\# V_{0,2})$ is injective in $\mathbf{Fct}([\mathcal{M}_{A,V_{0,2}}, \mathcal{G}_{V_{0,2}}\rangle, \mathbf{Ab})$, and the difference functor $\delta_{V_{0,2}}(H)$ is constant with value $\delta_{V_{0,2}}(H)(V_g)$, hence a split coefficient system of finite degree 0 (see Example 2.23).

Hence, in all cases, the functor H is a coefficient system of finite degree 1, and it is split when $H = H_{\mathcal{V}}$, or if $H = H_\Sigma$ and $i = 1$. Finally, since the target of the functor H factors through the full subcategory of free abelian groups $\mathbf{ab}$, the result on the d-fold tensor power $T^d H\colon [\mathfrak{H}_{V,Z}^{\geq i}, \mathfrak{H}_Z^{\geq i}\rangle \to \mathbf{ab}$ is a classical property of the tensor product of coefficient system; see [LS25, Cor. 1.5]. $\square$

Remark 2.27 An alternative functor $\mathfrak{H} \to \mathbf{Ab}$, analogous but not identical to H_Σ, is defined by assigning to each object $(V, r, s, i, j) \in \mathrm{ob}(\mathfrak{H})$ the relative homology group $H_1(\partial V, \mathring{\mathcal{D}}_r \sqcup \mathcal{P}_s; \mathbb{Z})$. Repeating mutatis mutandis the proof of Proposition 2.26, this induces a coefficient system

$[\mathcal{H}^{\geq i}_{V,Z}, \mathcal{H}^{\geq i}_{Z}\rangle \to \mathbf{Ab}$ of finite degree d for each $i \in \{1,2\}$. We favour the introduction of H_Σ, as it corresponds to a coefficient system previously studied in the context of twisted homological stability for mapping class groups of surfaces; see [Iva93], [CM09, §1.1, Ex. 1] and [Bol12, Ex. 4.3(i)].

2.3.2. Double coefficient systems and coefficient bisystems

Finally, we introduce here the notions of *double coefficient systems* and *coefficient bisystems*; see Definition 2.28. These are required to prove the twisted homological stability results of §3.2 and to extend a functor over $\mathcal{H}$ to some subcategories of the Quillen bracket construction $[\overline{\mathcal{H}}^{\geq 2}, \mathcal{H}^{\geq 2}\rangle$.

We consider a functor $F\colon \mathcal{H} \to \mathbf{Ab}$ and a 3-dimensional compact handlebody V of $\mathcal{H}$. By composing isomorphisms of the form (2.3), we have obvious isomorphisms $(V_{1,1}^{\natural g} \natural V) \# V_{0,2}^{\# m} \cong V \# V_{0,2}^{\# g+m} \cong V_{1,1}^{\natural g+m} \natural V \cong V_{1,1}^{\natural g} \natural (V \# V_{0,2}^{\# m})$ for all integers $g, m \geq 0$, and we denote by $V_{g|m}$ this isomorphism class of objects for brevity. We also recall from Definition 2.14 that $\iota_g(V_{0|m})$ denotes the morphism $[V_{1,1}, \mathrm{id}_{V_{g|m}}]$ of $\mathrm{Hom}_{\langle \mathcal{H}^{\geq 1}, \mathcal{H}^{\geq 1}]}(V_{g-1|m}, V_{g|m})$ for each $m \geq 0$. We take up the framework set in Notations 2.20 and 2.25. We note that, by composing isomorphisms of the form (2.3), the groupoid $\mathcal{H}^{\geq 1}_{V,V_{1,1}}$ is isomorphic to the full subgroupoid of $\mathcal{H}$ on the objects $\{V_{1,1}^{\natural n} \natural V\}_{n \in \mathbb{N}}$, so it is naturally equipped with a left $(\mathcal{H}^{\geq 1}_{V_{1,1}}, \natural, I_1)$-module structure $(\mathcal{H}^{\geq 1}_{V,V_{1,1}}, \natural)$.

Definition 2.28 For $d \geq 0$ an integer and V a compact handlebody of $\mathcal{H}^{\geq 2}$, a functor $F\colon \mathcal{H} \to \mathbf{Ab}$ is a *double coefficient system at V of degree D* if it satisfies the following properties.

(i) The restriction of F to $\mathcal{H}^{\geq 2}_{V,V_{0,2}}$ lifts to a coefficient system $[\mathcal{H}^{\geq 2}_{V,V_{0,2}}, \mathcal{H}^{\geq 2}_{V_{0,2}}\rangle \to \mathbf{Ab}$ of finite degree d.

(ii) The restriction of F to $\mathcal{H}^{\geq 1}_{V,V_{1,1}}$ lifts to a coefficient system $\langle \mathcal{H}^{\geq 1}_{V_{1,1}}, \mathcal{H}^{\geq 1}_{V,V_{1,1}}] \to \mathbf{Ab}$ of finite degree d', and the maximum of the finite degrees $\max(d, d')$ is equal to D

(iii) For fixed integers $m \geq 1$ and $n \geq 0$, the functor F defines a commutative diagram of abelian groups:

$$\begin{tikzcd}
\cdots \arrow[r, "F(\iota_{g-1}(V_{0|n}))"] & F(V_{g-1|n}) \arrow[r, "F(\iota_g(V_{0|n}))"] \arrow[d, "F([\mathrm{id}_{V_{g-1|n+m}}, V_{0,2}^{\# m}])"] & F(V_{g|n}) \arrow[r, "F(\iota_{g+1}(V_{0|n}))"] \arrow[d, "F([\mathrm{id}_{V_{g|n+m}}, V_{0,2}^{\# m}])"] & \cdots \\
\cdots \arrow[r, "F(\iota_{g+1}(V_{0|n+m}))"'] & F(V_{g-1|n+m}) \arrow[r, "F(\iota_g(V_{0|n+m}))"'] & F(V_{g|n+m}) \arrow[r, "F(\iota_{g+1}(V_{0|n+m}))"'] & \cdots
\end{tikzcd} \tag{2.11}$$

Namely, the vertical and horizontal arrows are well-defined by Conditions (i) and (ii) respectively, and we assume that the assignments of F are such that the diagram (2.11) is commutative.

Moreover, if the restriction of F to $\mathcal{H}^{\geq 1}_{V,V_{1,1}}$ also lifts to a coefficient system $[\mathcal{H}^{\geq 1}_{V,V_{1,1}}, \mathcal{H}^{\geq 1}_{V_{1,1}}\rangle \to \mathbf{Ab}$ of finite degree D', then we say that the functor $F\colon \mathcal{H} \to \mathbf{Ab}$ is a *coefficient bisystem* at V of degree $\max(D, D')$.

Remark 2.29 Although Definition 2.28(iii) is mild and easy to check in our examples (see Example 2.30), it is not automatic and does not simply follow from the functoriality of F. Indeed, an analogous commutative diagram to (2.11) before applying F does not make sense, since the vertical maps are images of morphisms in $[\mathcal{H}^{\geq 2}_{V,V_{0,2}}, \mathcal{H}^{\geq 2}_{V_{0,2}}\rangle$, while the horizontal ones come from $\langle \mathcal{H}^{\geq 1}_{V_{1,1}}, \mathcal{H}^{\geq 1}_{V,V_{1,1}}]$. We believe it would be possible to introduce a generalisation of §2.2.2 encompassing these two categories, which would contain the necessary morphisms for the diagram (2.11) before applying F to be well-defined and commutative, but this is out of the scope of this paper. Furthermore, the additional property for a double coefficient system to be a coefficient bisystem generally follows automatically from Definition 2.28(ii); see Example 2.30 for an illustration.

Example 2.30 For each $d \geq 1$, the functors $T^d H_\Sigma$ and $T^d H_\mathcal{V}$ of $\mathbf{Fct}(\mathcal{H}, \mathbf{Ab})$ introduced in §2.3.1 are both coefficient bisystems of degree d at any compact handlebody. Indeed, Proposition 2.26 with $Z = V_{0,2}$ and $Z = V_{1,1}$ respectively implies that Definition 2.28(i) and the further property to have a coefficient bisystem are verified. Also, a mutatis mutandis routine adaptation of the first part of the proof of Proposition 2.26 with $Z = V_{1,1}$ shows that Definition 2.28(ii) is also satisfied. Moreover, for all integers $g \geq 1$ and $n \geq 0$, denoting by H either H_Σ or $H_\mathcal{V}$, it directly follows from their assignments that the composites $T^d H(\iota_g(V_{0|n+m})) \circ T^d H([\mathrm{id}_{V_{g-1|n+m}}, V_{0,2}^{\# m}])$ and $T^d H([\mathrm{id}_{V_{g|n+m}}, V_{0,2}^{\# m}]) \circ T^d H(\iota_g(V_{0|n}))$ are both the maps induced in homology by the proper smooth embedding $V_{g-1|n} \hookrightarrow V_{1,1} \natural V_{g-1|n} \# V_{0,2}^{\# m}$, and so they are equal. Hence the abelian group diagram (2.11) is commutative, and thus Definition 2.28(iii) is satisfied.

We consider a double coefficient system $F\colon \mathfrak{H} \to \mathbf{Ab}$ at a compact handlebody V of $\mathfrak{H}^{\geq 2}$ of degree d. We extend the restriction of F along $\mathfrak{H}^{\geq 2} \hookrightarrow \mathfrak{H}$ to the groupoid $\overline{\mathfrak{H}}^{\geq 2}$ introduced in Definition 2.14, denoting it by $\overline{F}$, as follows. Recall that the category of abelian groups $\mathbf{Ab}$ is cocomplete (see [Mac98, §VIII.3.] for instance) and that the functor F defines a coefficient system $\langle \mathfrak{H}^{\geq 1}_{V_{1,1}}, \mathfrak{H}^{\geq 1}_{V,V_{1,1}}] \to \mathbf{Ab}$ by Definition 2.28(ii).

- For each integer $n \geq 0$, we assign $\overline{F}((V_{\infty,1}\natural V)\# V_{0,2}^{\# n})$ to be the colimit of abelian groups defined by the diagram
$$\cdots \xrightarrow{F(\iota_g(V_{0|n}))} F(V_{g|n}) \xrightarrow{F(\iota_{g+1}(V_{0|n}))} F(V_{g+1|n}) \xrightarrow{F(\iota_{g+2}(V_{0|n}))} \cdots. \tag{2.12}$$
In other words, we take the colimit of the sequence defined by applying F to (2.4).
- For an isomorphism ϕ of $(V_{\infty,1}\natural V)\# V_{0,2}^{\# n}$, by Definition 2.14, there exists an integer $j \geq 0$ and a set of morphisms $\{\phi_i \in \mathrm{Hom}_{\mathfrak{H}}(V_{1,1}^{\natural i+j}\natural(V\# V_{0,2}^{\# n}), V_{1,1}^{\natural i+j}\natural(V\# V_{0,2}^{\# n}))\}_{i\geq 0}$ such that the associated diagram of the form (2.5) is commutative. We assign $\overline{F}(\phi)$ to be the colimit induced by the sequence of abelian group morphisms $\{F(\phi_i)\}_{i\geq 0}$, i.e. the image by applying F of a diagram of the form (2.5).
- By Definition 2.28(iii), we assign $\overline{F}([\mathrm{id}_{(V_{\infty,1}\natural V)\# V_{0,2}^{\# n+m}}, V_{0,2}^{\# m}])$ for each $m \geq 1$ and $n \geq 0$ to be the colimit of the vertical abelian group maps defined by diagram (2.11). Since any morphism of $[\overline{\mathfrak{H}}^{\geq 2}_{V_{\infty,1}\natural V, V_{0,2}}, \mathfrak{H}^{\geq 2}_{V_{0,2}}\rangle$ is of the form $[\phi, V_{0,2}^{\# m}] = \phi \circ [\mathrm{id}_{(V_{\infty,1}\natural V)\# V_{0,2}^{\# m}}, V_{0,2}^{\# n+m}]$ with ϕ an automorphism of $(V_{\infty,1}\natural V)\# V_{0,2}^{\# n+m}$ and $m, n \geq 0$, we assign $\overline{F}([\phi, V_{0,2}^{\# m}]) = \overline{F}(\phi) \circ \overline{F}([\mathrm{id}_{(V_{\infty,1}\natural V)\# V_{0,2}^{\# n+m}}, V_{0,2}^{\# m}])$ if $m \geq 1$ while $\overline{F}(\phi)$ is defined at the previous point otherwise.

The composition axiom for the above assignments on morphisms is obviously satisfied by construction, so we have a well-defined functor $\overline{F}\colon [\overline{\mathfrak{H}}^{\geq 2}_{V_{\infty,1}\natural V, V_{0,2}}, \mathfrak{H}^{\geq 2}_{V_{0,2}}\rangle \to \mathbf{Ab}$.

Proposition 2.31 *The coefficient system $\overline{F}\colon [\overline{\mathfrak{H}}^{\geq 2}_{V_{\infty,1}\natural V, V_{0,2}}, \mathfrak{H}^{\geq 2}_{V_{0,2}}\rangle \to \mathbf{Ab}$ is of finite degree d.*

Proof. By the above construction, the map $i_{V_{0,2}}(\overline{F})_{(V_{\infty,1}\natural V)\# V_{0,2}^{\# n}}$ for each integer $n \geq 0$ is the colimit of the abelian group morphisms $\{i_{V_{0,2}}(F)_{V_{g|n}} = F([\mathrm{id}_{V_{g|n+1}}, V_{0,2}])\}_{g\geq 0}$ defining the commutative diagram (2.11) with $m = 1$. We recall that filtered colimits commute with kernels and cokernels; see [Mac98, §IX.2.] for instance. Then, by considering the colimit of the cokernels and kernels of the vertical maps in the commutative diagram (2.11) with $m = 1$, it follows from Definition 2.28(i) that $\delta_{V_{0,2}}\overline{F}$ is a coefficient system of finite degree at most $d-1$ and that $\kappa_{V_{0,2}}\overline{F}$ vanishes. □

2.4. Framework for twisted homological stability

This section presents the meta-method we follow to prove our homological stability results for handlebody groups. For all of §2.4, we consider a braided monoidal groupoid $(\mathcal{G}, \odot, 0)$ with no zero-divisors and such that $\mathrm{Aut}_\mathcal{G}(0) = \{\mathrm{id}_0\}$, and a right-module $(\mathcal{M}, \odot)$ over it, where $\mathcal{M}$ is a groupoid. We also fix objects $A \in \mathrm{ob}(\mathcal{M})$ and $X \in \mathrm{ob}(\mathcal{G})$, and a coefficient system $F\colon [\mathcal{M}_{A,X}, \mathcal{G}_X\rangle \to \mathbf{Ab}$. Recall from Notation 2.20 that $\mathcal{G}_X$ and $\mathcal{M}_{A,X}$ respectively denote the full subgroupoids of $\mathcal{G}$ and $\mathcal{M}$ on the objects $\{X^{\odot n}\}_{n\in\mathbb{N}}$ and $\{A_n\}_{n\in\mathbb{N}}$, where $A_n = A \odot X^{\odot n}$ for each $n \geq 0$.

Notation 2.32 For each integer $n \geq 0$, we write G_n for the group $\mathrm{Aut}_\mathcal{M}(A_n)$, F_n for the abelian group $F(A_n)$ and c_n for the canonical morphism $[\mathrm{id}_{A_{n+1}}, X] \in \mathrm{Hom}_{[\mathcal{M}_{A,X},\mathcal{G}_X\rangle}(A_n, A_{n+1})$ (see §2.2.2).

2.4.1. Twisted homological stability meta-theorem

We recall here the general framework from [RW17] and [Kra19, §7] to prove homological stability for sequences of groups coming from a monoidal groupoid as in §2.2; see Theorem 2.37.

For each integer $n \geq 0$, the right $(\mathcal{G}, \odot, 0)$-module structure of $\mathcal{M}$ induces a canonical group morphism $\mathfrak{s}_n := (-)_n \odot \mathrm{id}_X : G_n \to G_{n+1}$ called a *stabilisation map*, while the functoriality of F ensures that the morphism $F(c_n)$ is G_n-equivariant along the map $\mathfrak{s}_n$. Therefore, it follows from the functoriality of group homology (see [Bro94, Chap. III, §8] for instance) that the maps $\mathfrak{s}_n$ and $F(c_n)$ induce a map

$$(\mathfrak{s}_n; F(c_n))_i : H_i(G_n; F_n) \to H_i(G_{n+1}; F_{n+1}). \tag{2.13}$$

for each $i \geq 0$ and $n \geq 0$. Theorem 2.37 roughly states that if F is of finite degree, the map (2.13) is an isomorphism in a range determined by the connectivity of a semi-simplicial set associated to this data.

Local homogeneity. The framework to prove homological stability for the family of groups $\{G_n\}_{n\in\mathbb{N}}$ requires the mild assumption that the groupoid $\mathcal{M}$ has the following property.

Definition 2.33 We say that $(\mathcal{M}, \odot)$ is *locally homogeneous* at (A, X) if the two following conditions hold:

(i) *Injectivity*: for each $n \geq 1$ and for each $0 \leq p < n$, the map $(-)_{n-p-1} \odot \mathrm{id}_{X^{\odot p+1}} : G_{n-p-1} \to G_n$ is injective.
(ii) *Cancellation*: for each $0 \leq p < n$, if $Y \in \mathrm{ob}(\mathcal{M})$ is such that $Y \odot X^{\odot p+1} \cong A_n$, then $Y \cong A_{n-p-1}$.

Remark 2.34 The notion of local homogeneity was originally introduced in [RW17, Def. 1.2], with conditions equivalent to those of injectivity (i) and cancellation (ii) of Definition 2.33; see [RW17, Th. 1.10(a)–(b)].

Semi-simplicial set of destabilisations. The main hypothesis to prove homological stability for the map (2.13) is the high-connectivity of a semi-simplicial set associated to the sets of morphisms of $\mathcal{M}$ that we now introduce.

Definition 2.35 For any $n \geq 0$, we define the semi-simplicial set of *destabilisations* $W_n(A, X)_\bullet$ to be the semi-simplicial set whose p-simplices are given by

$$W_n(A, X)_p := \operatorname*{colim}_{\mathcal{M}} \left(\mathrm{Hom}_{\mathcal{M}}(- \odot X^{\odot p+1}, A_n) \right),$$

for any $0 \leq p < n$. By an analysis similar to that of the Hom-sets in §2.2.2, we note that any element of $W_n(A, X)_p$ is thus an equivalence class of pairs (B, ϕ), denoted by $[B, \phi]$, where $B \in \mathrm{ob}(\mathcal{G})$ and $\phi \in \mathrm{Aut}_{\mathcal{G}}(B \odot X^{\odot p+1}, A_n)$. The face maps for $0 \leq i \leq p$ are then defined as

$$\begin{aligned} d_i : W_n(A, X)_p &\to W_n(A, X)_{p-1} \\ [B, \phi] &\mapsto [B, \phi] \circ [X, (b^{\mathcal{G}}_{X^{\odot i}, X})^{-1} \odot \mathrm{id}_{X^{\odot p-i}}]. \end{aligned}$$

Remark 2.36 When we consider $\mathcal{G} = \mathcal{M}$ equipped with the canonical right-module structure over itself, there is a canonical isomorphism $W_n(A, X)_p \cong \mathrm{Hom}_{\langle \mathcal{G}, \mathcal{G} \rangle}(X^{\odot p+1}, A_n)$ for all integers $n \geq 1$ and $0 \leq p < n$; see §2.2.2. Also, thanks to the formula of (ii) in Proposition 2.10, we note that the morphism $[X, (b^{\mathcal{G}}_{X^{\odot i}, X})^{-1} \odot \mathrm{id}_{X^{\odot p-i}}]$ defining the face map d_i is equal to the morphism $\mathrm{id}_{X^{\odot i}} \odot [X, \mathrm{id}_X] \odot \mathrm{id}_{X^{\odot p-i}}$ of $\langle \mathcal{G}, \mathcal{G} \rangle$, thus matching [RW17, Def. 2.1].

Twisted homological stability. We now introduce the twisted homological stability "meta-theorem" for a general family of groups, that will be applied to prove our results for handlebody groups in §3.

Theorem 2.37 *Assume that $\mathcal{M}$ is locally homogeneous at (A,X) and that there exist integers $a \geq 2$ and $k \geq 2$ such that $W_n(A,X)_\bullet$ is at least $\frac{n-a}{k}$-connected for each $n \geq 1$. If $F\colon [\mathcal{M}_{A,X}, \mathcal{G}_X\rangle \to \mathbf{Ab}$ is a coefficient system of degree d, then*

$$(\mathfrak{s}_n; F(c_n))_i \colon H_i(G_n; F_n) \to H_i(G_{n+1}; F_{n+1})$$

is an isomorphism for each $i \leq \frac{n+2-a}{k} - d - 1$. If F is in addition split, it is an isomorphism for each $i \leq \frac{n+2-a-d}{k} - 1$.

Proof. The case $a = 2$ is given by [Kra19, Thm. C] as follows. By their definitions in Notation 2.20, the groupoids $\mathcal{G}_X$ and $\mathcal{M}_{A,X}$ have canonical functors $\mathfrak{g}_{\mathcal{G}}\colon \mathcal{G}_X \to \mathbb{N}$ and $\mathfrak{g}_{\mathcal{M}}\colon \mathcal{M}_{A,X} \to \mathbb{N}$ respectively, compatible with each other in the sense that $\mathfrak{g}_{\mathcal{M}}(A_n) = \mathfrak{g}_{\mathcal{M}}(A_{n-1}) + g_{\mathcal{G}}(X)$ for all $n \geq 1$. As explained at the beginning of [Kra19, §7.2], taking the geometric realisations of $\mathcal{M}_{A,X}$ and $\mathcal{G}_X$ gives us a graded E_1-module $\mathrm{B}\mathcal{M}_{A,X}$ over the graded E_2-algebra $\mathrm{B}\mathcal{G}_X$, which is a suitable input for [Kra19, Thm. C]. The ranges of stability in this theorem are determined by the connectivity of the *canonical resolution* $R_\bullet(\mathrm{B}\mathcal{M}_{A,X}) \to \mathrm{B}\mathcal{M}_{A,X}$ defined in [Kra19, §2]: they hold precisely when, for each $n \geq 1$, the restriction $|R_\bullet(\mathrm{B}\mathcal{M}_{A,X})_n| \to (\mathrm{B}\mathcal{M}_{A,X})_n$ to the preimage of the degree n part of $\mathrm{B}\mathcal{M}_{A,X}$ is $\frac{n-2+k}{k}$-connected, i.e. the homotopy fibre of $R_\bullet(\mathrm{B}\mathcal{M}_{A,X}) \to \mathrm{B}\mathcal{M}_{A,X}$ is $\frac{n-2}{k}$-connected; see [Kra19, Rem. 2.17]. By [Kra19, Lem. 7.6], the injectivity condition (i) of Definition 2.33 of $\mathcal{M}$ at (A,X) implies that this homotopy fibre is homotopy equivalent to $|W_n(A,X)_\bullet|$, giving us the stated result.

If $a > 2$, we set $A' := A \odot X^{\odot(a-2)}$ and note that $W_n(A',X)_\bullet = W_{n+2-a}(A,X)_\bullet$ by definition, so the result follows from applying the first case to $W_n(A',X)_\bullet$. □

Remark 2.38 The Cancellation condition (ii) of Definition 2.33 is not necessary to apply [Kra19, Thm. C] and thus prove Theorem 2.37. However, this assumption is needed along with local standardness (see §2.4.2) in order to prove Proposition 2.42 by applying [RW17, Prop. 2.6]; see also [Kra19, §7.3]. Hence we choose to keep that assumption in the standard framework for homological stability in Theorem 2.37.

2.4.2. Categorical standardness and simplicial complex of destabilisations

The main difficulty in applying Theorem 2.37 is to check the connectivity condition on the semi-simplicial set $W_n(A,X)_\bullet$ of Definition 2.35. For this purpose, we present here properties and tools which will be of key use, walking in the footsteps of [RW17, §2.1].

Local standardness property. We recall from [RW17, Def. 2.5] the notion of local standardness for the groupoid $\mathcal{M}$, which provides a description of the simplices of the semi-simplicial set $W_n(A,X)_\bullet$; see Proposition 2.42. We fix objects $A \in \mathrm{ob}(\mathcal{M})$ and $X \in \mathrm{ob}(\mathcal{G})$ and take up Notation 2.20. Similarly to the hom-sets of §2.2.2, we note that an element of $\mathrm{colim}_{\mathcal{M}}(\mathrm{Hom}_{\mathcal{M}}(- \odot X, A_n))$ is an equivalence class $[A_{n-1}, \phi]$ of the pair (A_{n-1}, ϕ) with $\phi \in G_n$, where the pairs (A_{n-1}, ϕ) and (A_{n-1}, ϕ') are equivalent if there exists an isomorphism $\chi \in G_{n-1}$ such that $\phi' = \phi \circ (\chi \odot \mathrm{id}_X)$. Hence there is a natural bijection for each $n \geq 1$

$$\underset{\mathcal{M}}{\mathrm{colim}}\,(\mathrm{Hom}_{\mathcal{M}}(- \odot X, A_n)) \cong G_n/G_{n-1}, \tag{2.14}$$

where G_n/G_{n-1} denotes the quotient set associated to the action of G_{n-1} by precomposition on the left on G_n via the map $(-)_{n-1} \odot \mathrm{id}_X \colon G_{n-1} \to G_n$.

For each integer $n \geq 2$, we consider the morphism

$$\theta^n_{A,X}\colon \underset{\mathcal{M}}{\mathrm{colim}}\,(\mathrm{Hom}_{\mathcal{M}}(- \odot X_1, A_{n-2} \odot X_1)) \to \underset{\mathcal{M}}{\mathrm{colim}}\,(\mathrm{Hom}_{\mathcal{M}}(- \odot X_1, A_{n-2} \odot X_1 \odot X_2))$$

defined by $[A_{n-2}, \phi] \mapsto [A_{n-2} \odot X_2, (\phi \odot \mathrm{id}_{X_2}) \circ (\mathrm{id}_{A_{n-2}} \odot (b^{\mathcal{G}}_{X_1,X_2})^{-1})]$, where we write $A_n \cong A_{n-2} \odot X_1 \odot X_2 \cong A_{n-2} \odot X_2 \odot X_1$ with $X_1 = X_2 = X$ as a notational device to distinguish the two copies of X in the monoidal product.

Definition 2.39 We consider objects $A \in \mathrm{ob}(\mathcal{M})$ and $X \in \mathrm{ob}(\mathcal{G})$ such that $\mathcal{M}$ is locally homogeneous at (A, X). We say that $\mathcal{M}$ is *locally standard* at (A, X) if the two following conditions hold.

(i) The maps $[A \odot X, \mathrm{id}_A \odot (b^{\mathcal{G}}_{X,X})^{-1}]$ and $[A \odot X, \mathrm{id}_{A_2}]$ are distinct in $\mathrm{colim}_{\mathcal{M}}\left(\mathrm{Hom}_{\mathcal{M}}(- \odot X, A_2)\right)$.
(ii) For all $n \geq 2$, the map $\theta^n_{A,X}$ is injective.

Remark 2.40 Following Remark 2.36, we consider the case where $\mathcal{G} = \mathcal{M}$ equipped with the canonical right-module structure over itself. By the formula of (ii) in Proposition 2.10, we deduce that $[A \odot X, \mathrm{id}_A \odot (b^{\mathcal{G}}_{X,X})^{-1}] = [A, \mathrm{id}_A] \odot \mathrm{id}_X \odot [X, \mathrm{id}_X]$ and that $\theta^n_{A,X}([A_{n-2}, \phi]) = [A_{n-2}, \phi] \odot [X_2, \mathrm{id}_{X_2}]$ as morphisms of $\langle \mathcal{G}, \mathcal{G} \rangle$. In particular, Conditions (i) and (ii) exactly match the conditions **LS1** and **LS2** of [RW17, Def. 2.5] respectively.

The following Lemma 2.41 provides an equivalent property to Condition (ii) of Definition 2.39, which will be more convenient to check in practice; see Proposition 2.46. Beforehand, we note that since the braiding $b^{\mathcal{G}}_{-,-}$ is a natural isomorphism in $\mathcal{G}$, it induces an automorphism for each $n \geq 2$

$$\mathfrak{C}_n(X_1, X_2) \colon \mathrm{Aut}_{\mathcal{M}}(A_{n-1} \odot X_2 \odot X_1) \xrightarrow{\cong} \mathrm{Aut}_{\mathcal{M}}(A_{n-1} \odot X_1 \odot X_2),$$

defined by $\phi \mapsto (\mathrm{id}_{A_{n-2}} \odot b^{\mathcal{G}}_{X_1,X_2}) \circ \phi \circ (\mathrm{id}_{A_{n-2}} \odot (b^{\mathcal{G}}_{X_1,X_2})^{-1})$.

Lemma 2.41 *Condition* (ii) *of Definition 2.39 is satisfied if and only if the following diagram is a pullback square for each $n \geq 2$*

$$\begin{tikzcd}
\mathrm{Aut}_{\mathcal{M}}(A_{n-2}) \arrow[r, "(-)_{n-2} \odot \mathrm{id}_{X_2}"] \arrow[d, "(-)_{n-2} \odot \mathrm{id}_{X_1}"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Aut}_{\mathcal{M}}(A_{n-2} \odot X_2) \arrow[d, "(-)_{n-1} \odot \mathrm{id}_{X_1}"] \\
\mathrm{Aut}_{\mathcal{M}}(A_{n-2} \odot X_1) \arrow[r, "\mathfrak{C}_n(X_1, X_2) \circ ((-)_{n-1} \odot \mathrm{id}_{X_2})"] & \mathrm{Aut}_{\mathcal{M}}(A_{n-2} \odot X_2 \odot X_1),
\end{tikzcd} \tag{2.15}$$

Proof. We fix $n \geq 2$. First of all, we note that $\mathrm{id}_{X_1 \odot X_2} \circ (b^{\mathcal{G}}_{X_1,X_2})^{-1} = (b^{\mathcal{G}}_{X_1,X_2})^{-1} \circ \mathrm{id}_{X_2 \odot X_1}$ by definition of the braiding $b^{\mathcal{G}}_{-,-}$. This implies the commutativity of the diagram (2.15) by definition of the conjugate $\mathfrak{C}_n(X_1, X_2)$. Furthermore, using the natural bijection (2.14), it follows from the universal property of a quotient set that the morphism $\theta^n_{A,X}$ is the induced map between the quotients of the vertical maps of the diagram (2.15).

Now, let us prove the equivalence between Condition (ii) and the fact that the commutative diagram (2.15) is a pullback square. We recall that the category of groups is complete (see [Mac98, §V.1.] for instance), so the pullback P of the maps $\mathfrak{C}_n(X_1, X_2) \circ ((-)_{n-1} \odot \mathrm{id}_{X_2})$ and $(-)_{n-1} \odot \mathrm{id}_{X_1} \colon \mathrm{Aut}_{\mathcal{M}}(A_{n-2} \odot X_2) \to \mathrm{Aut}_{\mathcal{M}}(A_{n-2} \odot X_2 \odot X_1)$ targeting the bottom-right corner of the diagram (2.15) is well-defined. By diagram chasing, it follows from the universal properties of the kernel $\mathrm{Ker}(\theta^n_{A,X})$ and of the pullback P that there exist unique maps $\Upsilon \colon \mathrm{Aut}_{\mathcal{M}}(A_{n-2}) \to P$ and $p \colon P \to \mathrm{Ker}(\theta^n_{A,X})$ respecting the kernel, pullback and quotient set structures. In particular, a clear diagram chasing factoring through the quotient $\mathrm{Aut}_{\mathcal{M}}(A_{n-2} \odot X_2)/\mathrm{Aut}_{\mathcal{M}}(A_{n-2})$ shows that the composite $p \circ \Upsilon$ is the zero group morphism $0_{\mathbf{Grp}}$.

Moreover, picking a lift $\xi \colon \mathrm{Ker}(\theta^n_{A,X}) \to \mathrm{Aut}_{\mathcal{M}}(A_{n-2} \odot X_1)$ of the canonical inclusion $\mathrm{Ker}(\theta^n_{A,X}) \subset \mathrm{colim}_{\mathcal{M}}\left(\mathrm{Hom}_{\mathcal{M}}(- \odot X_1, A_{n-2} \odot X_1)\right)$, the composite $\mathfrak{C}_n(X_1, X_2) \circ ((-)_{n-1} \odot \mathrm{id}_{X_2}) \circ \xi$ then lifts to $\mathrm{Aut}_{\mathcal{M}}(A_{n-2} \odot X_2)$ along $(-)_{n-1} \odot \mathrm{id}_{X_1}$, because its image in the quotient $\mathrm{Aut}_{\mathcal{M}}(A_{n-2} \odot X_2 \odot X_1)/\mathrm{Aut}_{\mathcal{M}}(A_{n-2} \odot X_2)$ is zero by the commutativity of the diagram (2.15). Hence, by the universal property of the pullback P, there exists a unique map $\xi' \colon \mathrm{Ker}(\theta^n_{A,X}) \to P$ respecting the pullback structure. Then, it follows from a diagram chasing and the universal property of the kernel $\mathrm{Ker}(\theta^n_{A,X})$ that $p \circ \xi' = \mathrm{id}_{\mathrm{Ker}(\theta^n_{A,X})}$, and so the morphism p is surjective.

By the universal property of the pullback P, if the commutative diagram (2.15) is a pullback square, then the map Υ is an isomorphism. This along with the fact that $p \circ \Upsilon = 0_{\mathbf{Grp}}$ implies that $p = 0_{\mathbf{Grp}}$. It then follows from the surjectivity of p that $\mathrm{Ker}(\theta^n_{A,X}) = 0$, i.e. Condition (ii) is satisfied. Conversely, if $\theta^n_{A,X}$ is injective, then the composite $P \xrightarrow{v} \mathrm{Aut}_{\mathcal{M}}(A_{n-2} \odot X_1) \twoheadrightarrow$

$\mathrm{Aut}_{\mathcal{M}}(A_{n-2} \odot X_1)/\mathrm{Aut}_{\mathcal{M}}(A_{n-2})$ (where υ is the defining map of the pullback P, while the second one is the canonical surjection induced by (2.14)) is equal to $0_{\mathbf{Grp}}$. So there exists a lift $\upsilon' \colon P \to \mathrm{Aut}_{\mathcal{M}}(A_{n-2})$ of υ, such that $\upsilon = ((-)_{n-2} \odot \mathrm{id}_{X_1}) \circ \upsilon'$, and we know by definition that $\upsilon \circ \Upsilon = (-)_{n-2} \odot \mathrm{id}_{X_1}$. By diagram chasings, we deduce from the universal property of the pullback P that $\Upsilon \circ \upsilon' = \mathrm{id}_P$, while it follows from the injectivity of $(-)_{n-2} \odot \mathrm{id}_{X_1}$ (by the injectivity condition (i) of Definition 2.33, since $(\mathcal{M}, \odot)$ is locally homogeneous at (A, X)) that $\upsilon' \circ \Upsilon = \mathrm{Aut}_{\mathcal{M}}(A_{n-2})$. Hence Υ is an isomorphism respecting the pullback structure, and so the commutative diagram (2.15) is a pullback square. □

Furthermore, the following result highlights the significance of the notion of local standardness:

Proposition 2.42 *We consider objects $A \in \mathrm{ob}(\mathcal{M})$ and $X \in \mathrm{ob}(\mathcal{G})$ such that $\mathcal{M}$ is locally homogeneous at (A, X). Then the right $(\mathcal{G}, \odot, 0)$-module $(\mathcal{M}, \odot)$ is locally standard at (A, X) if and only if all simplices of $W_n(A, X)_\bullet$ for all n are determined by their (ordered sets of) vertices and their vertices are all distinct.*

Proof. This result corresponds to [RW17, Prop. 2.6], in particular using [RW17, Lem. 2.7] which gives a equivalent characterisation of Condition (ii) of Definition 2.39. The proofs of these results repeat verbatim for the present framework equivalent to that of [RW17, §2.1]. □

Simplicial complex of destabilisations. In the cases we consider in §3, we see that the connectivity of the semi-simplicial set $W_n(A, X)_\bullet$ of Definition 2.35 is determined by that of its underlying simplicial complex; see §3.1.1 and §3.2.1. We introduce the following notation:

Notation 2.43 For each integer $n \geq 0$, we write $S_n(A, X)_\bullet$ for the underlying simplicial complex of $W_n(A, X)_\bullet$. In other words, this is the simplicial complex whose set of vertices is given by

$$S_n(A, X)_0 := \underset{\mathcal{M}}{\mathrm{colim}} \left(\mathrm{Hom}_{\mathcal{M}}(- \odot X, A_n) \right),$$

and where a $(p+1)$-tuple $([B_0, \phi_0], \ldots, [B_p, \phi_p])$ of vertices spans a p-simplex if and only if there exists $[B, \phi] \in \mathrm{colim}_{\mathcal{M}} \left(\mathrm{Hom}_{\mathcal{M}}(- \odot X^{\odot p+1}, A_n) \right)$ such that

$$[B_i, \phi_i] = [B, \phi] \circ [X^{\odot p}, \mathrm{id}_{X^{\odot i}} \odot (b^{\mathcal{G}}_{X, X^{\odot p-i}})^{-1}]$$

for each $0 \leq i \leq p$. For each $p \geq 0$, there is a surjection $\pi_p \colon W_n(A, X)_p \to S_n(A, X)_p$ and we write

$$\pi \colon |W_n(A, X)_\bullet| \to |S_n(A, X)_\bullet|$$

for the induced map on geometric realisations.

2.4.3. Homogeneity and standardness properties for handlebody groups

In this section, we verify that the key properties of Definitions 2.33 and 2.39 are satisfied by the categories associated to the handlebody groups we have defined in §2.2, which are necessary to prove our homological stability results in §3.

Stabilisation maps. We introduce here the suitable stabilisation maps of the form $\mathfrak{s}_n := (-)_n \odot \mathrm{id}_X \colon G_n \to G_{n+1}$ for homological stability, in the context of handlebody groups. Beforehand, we recall that the monoidal structures $\natural$ and $\#$ of Definition 2.5 have clear analogous counterparts for the mapping class groups of surfaces, induced by gluing surfaces along half-intervals rather than half-discs. We refer the reader to [RW17, §5.6.1] for further details and abuse the notation by using the same symbol for these operations. Now, for integers $g, s \geq 0$ and $r \geq 1$, these monoidal structures induce group homomorphisms

$$\sigma^s_{g,r} := (-)_g \natural \mathrm{id}_{V_{1,1}} \colon \mathcal{H}^s_{g,r} \to \mathcal{H}^s_{g+1,r} \quad \text{and} \quad \varsigma^s_{g,r} := (-)_g \natural \mathrm{id}_{\Sigma_{1,1}} \colon \Gamma^s_{g,r} \to \Gamma^s_{g+1,r}, \tag{2.16}$$

and if $r \geq 2$

$$\rho^s_{g,r} := (-)_g \# \mathrm{id}_{V_{0,2}} \colon \mathcal{H}^s_{g,r} \to \mathcal{H}^s_{g+1,r} \quad \text{and} \quad \varrho^s_{g,r} := (-)_g \# \mathrm{id}_{\Sigma_{0,2}} \colon \Gamma^s_{g,r} \to \Gamma^s_{g+1,r}. \tag{2.17}$$

We also recall from Lemma 2.1 that we have a canonical group injection $\mathtt{i}^s_{g,r} \colon \mathcal{H}^s_{g,r} \hookrightarrow \Gamma^s_{g,r}$ relating handlebody groups and mapping class groups of surfaces.

Proposition 2.44 *For integers $g, s \geq 0$ and $r \geq 1$, the maps (2.16) and (2.17) are injective and define the following pullback commutative squares of injective group homomorphisms*

$$\begin{tikzcd}
\mathcal{H}^s_{g,r} \arrow[r, hook, "\sigma^s_{g,r}"] \arrow[d, hook, "\mathtt{i}^s_{g,r}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathcal{H}^s_{g+1,r} \arrow[d, hook, "\mathtt{i}^s_{g+1,r}"] \\
\Gamma^s_{g,r} \arrow[r, hook, "\varsigma^s_{g,r}"] & \Gamma^s_{g+1,r},
\end{tikzcd} \tag{2.18}$$

and if $r \geq 2$

$$\begin{tikzcd}
\mathcal{H}^s_{g,r} \arrow[r, hook, "\rho^s_{g,r}"] \arrow[d, hook, "\mathtt{i}^s_{g,r}"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathcal{H}^s_{g+1,r} \arrow[d, hook, "\mathtt{i}^s_{g+1,r}"] \\
\Gamma^s_{g,r} \arrow[r, hook, "\varrho^s_{g,r}"] & \Gamma^s_{g+1,r}.
\end{tikzcd} \tag{2.19}$$

Proof. First, that the morphisms $\varsigma^s_{g,r}$ and $\varrho^s_{g,r}$ for mapping class groups of surfaces are injective is a classical property of these maps. Proofs (or some mutatis mutandis adaptations) of these properties may be found in [RW17, Prop. 5.18(i)] or [PW24, Lem. 4.2] for $\varsigma^s_{g,r}$, and in [HVW24, Prop. 3.4] for $\varrho^s_{g,r}$. Also, it is a straightforward consequence of the definitions of the stabilisation morphisms (2.16) and (2.17) as extensions by the identity, along with those of the canonical injections $\mathtt{i}^s_{g,r}$ and $\mathtt{i}^s_{g+1,r}$, that the diagrams (2.18) and (2.19) are commutative. In particular, we know that the vertical arrows and the bottom horizontal arrows of both (2.18) and (2.19) are injective, so the maps $\sigma^s_{g,r}$ and $\rho^s_{g,r}$ are also injective (since they are the first factor of a composition which is a monomorphism).

It now remains to prove that the commutative diagrams (2.18) and (2.19) are pullback squares. Let us focus on the case (2.18), as the proof for (2.19) repeats verbatim what follows by replacing $\sigma^s_{g,r}$ and $\varsigma^s_{g,r}$ with $\rho^s_{g,r}$ and $\varrho^s_{g,r}$ respectively. Since monomorphisms are stable under pullback (see [Mit65, Prop. I.7.1] for instance), it follows from the general structure of pullbacks in the category of sets (see [Awo10, §5.3, Ex. 5.9] for example) that the pullback $\Gamma^s_{g,r} \times_{\Gamma^s_{g+1,r}} \mathcal{H}^s_{g+1,r}$ of $\varsigma^s_{g,r}$ and $\mathtt{i}^s_{g+1,r}$ is isomorphic to the intersection subgroup $\Gamma^s_{g,r} \cap \mathcal{H}^s_{g+1,r}$. Note that, for an element $\phi \in \mathcal{H}^s_{g+1,r}$, if the restriction to the boundary $\mathtt{i}^s_{g+1,r}(\phi)$ lifts to $\Gamma^s_{g,r}$ along $\varsigma^s_{g,r} = (-)_g \natural \mathrm{id}_{\Sigma_{1,1}}$, then ϕ is necessarily of the form $\phi' \natural \mathrm{id}_{V_{1,1}} = \sigma^s_{g,r}(\phi')$ where $\phi' \in \mathcal{H}^s_{g,r}$. This defines an injection $\Gamma^s_{g,r} \cap \mathcal{H}^s_{g+1,r} \hookrightarrow \mathcal{H}^s_{g,r}$, which has a canonical inverse induced by the injections $\mathtt{i}^s_{g,r}$ and $\sigma^s_{g,r}$. Hence there is an isomorphism $\Gamma^s_{g,r} \cap \mathcal{H}^s_{g+1,r} \cong \mathcal{H}^s_{g,r}$, such that the induced diagram connecting the pullback square $\Gamma^s_{g,r} \times_{\Gamma^s_{g+1,r}} \mathcal{H}^s_{g+1,r}$ to (2.18) is commutative by construction. We then deduce from the universal property of a pullback that (2.18) is a pullback square. $\square$

Local homogeneity. We now check that the groupoids associated to the handlebody groups satisfy the local homogeneity property of Definition 2.33.

Proposition 2.45 *For any $s \geq 0$, the following categories are locally homogeneous:*

(a) the right $(\mathfrak{H}^{\geq 1}, \natural, I_1)$-module $(\mathfrak{H}^{\geq 1}, \natural)$ at $(V^s_{0,r}, V_{1,1})$ for any $r \geq 1$;

(b) the right $(\mathfrak{H}^{\geq 2}, \#, I_2)$-module $(\mathfrak{H}^{\geq 2}, \#)$ at $(V^s_{0,r}, V_{0,2})$ for any $r \geq 2$;

(c) the right $(\mathfrak{H}^{\geq 2}, \#, I_2)$-module $(\overline{\mathfrak{H}}^{\geq 2}, \#)$ at $(V^s_{\infty,r}, V_{0,2})$ for any $r \geq 2$.

Proof. We fix $r \geq 1$ and $s \geq 0$. We start by proving the local homogeneity properties (a)–(b). Unwinding the definitions, the injectivity condition (i) for the properties (a) and (b) correspond to the injectivities for each $0 \leq p < n$ of the iterates $\sigma^s_{n-1,r} \circ \cdots \circ \sigma^s_{n-p+1,r}$ (for $r \geq 1$) and $\rho^s_{n-1,r} \circ \cdots \circ \rho^s_{n-p+1,r}$ (for $r \geq 2$) of the stabilisation maps (2.16) and (2.17) respectively. These facts then follow from Proposition 2.44. For the cancellation condition (ii), we consider an object Y of either $\boldsymbol{\mathcal{H}}^{\geq 1}$ or $\boldsymbol{\mathcal{H}}^{\geq 2}$, such that $Y \natural V_{1,1}^{\natural p+1} \cong V^s_{g,r}$ or $Y \# V_{0,2}^{\# p+1} \cong V^s_{g,r}$ respectively. By the definitions of $\natural$ and $\#$ as boundary connected sums (see Definition 2.5), it follows from the classification of surfaces (see [Ric63; BM79]) that the boundary of Y is in both cases a compact connected orientable surface of genus $g-p-1$, which has r marked discs and s marked points. Hence the object Y cannot be a disjoint union of 3-balls, and is thus a compact handlebody; see Definition 2.3. Also, it follows from [Bon83, Th. 2.1] that compact handlebodies are determined up to diffeomorphisms by their boundaries. We deduce that $Y \cong V^s_{g-p-1,r}$ in any case, which finish the proof of (a) and (b).

We now deal with the case (c). Let ϕ and ϕ' be elements of $\mathrm{Aut}_{\overline{\boldsymbol{\mathcal{H}}}^{\geq 2}}(V^s_{\infty,r} \# V_{0,2}^{\#(n-p-1)})$ such that $\phi \# \mathrm{id}_{V_{0,2}^{\#(p+1)}} = \phi' \# \mathrm{id}_{V_{0,2}^{\#(p+1)}}$. We deduce from Definition 2.14 that there exist an integer $j \geq 0$ and sets of morphisms $\{\phi_g\}_{g \geq j}$ and $\{\phi'_g\}_{g \geq j}$ with $\phi_g, \phi'_g \in \mathrm{Aut}_{\boldsymbol{\mathcal{H}}^{\geq 2}}(V^s_{g+n-p-1,r})$ making the corresponding diagrams of the form (2.5) commutative, such that $\phi = \mathrm{colim}_{g \geq j}(\phi_g)$ and $\phi' = \mathrm{colim}_{g \geq j}(\phi'_g)$. Since the maps $\iota_i(V^s_{0,r} \# V_{0,2}^{\# n})$ are monomorphisms in the category $\langle \boldsymbol{\mathcal{H}}^{\geq 1}, \boldsymbol{\mathcal{H}}^{\geq 1}]$ for all $i \geq 0$, it directly follows from the explicit description of the form $\coprod_{g \geq 0}(V^s_{g,r} \# V_{0,2}^{\# n})/\sim$ of the colimit $V^s_{\infty,r} \# V_{0,2}^{\# n}$ (see [Bor94, Prop. 2.13.3]) that the structural map $\alpha_i \colon V^s_{i,r} \# V_{0,2}^{\# n} \to V^s_{\infty,r} \# V_{0,2}^{\# n}$ is a monomorphism for each $i \geq 0$. Restricting the equality $\phi \# \mathrm{id}_{V_{0,2}^{\#(p+1)}} = \phi' \# \mathrm{id}_{V_{0,2}^{\#(p+1)}}$ along each α_g for $g \geq j$ then implies that we have an equality $\phi_g \# \mathrm{id}_{V_{0,2}^{\#(p+1)}} = \phi'_g \# \mathrm{id}_{V_{0,2}^{\#(p+1)}}$ for each $g \geq j$. Since the map $(-)_{g+n-p-1} \# \mathrm{id}_{V_{0,2}^{\#(p+1)}}$ is injective by Proposition 2.45(b), we deduce that $\phi_g = \phi'_g$ for all $g \geq j$, and so $\phi = \phi'$ by the universal property of a colimit: this proves the injectivity condition (i) for the case (c). Furthermore, for an object Y of $\overline{\boldsymbol{\mathcal{H}}}^{\geq 2}$ such that $Y \# V_{0,2}^{\#(p+1)} \cong V^s_{\infty,r} \# V_{0,2}^{\# n}$, it is immediate from the definition of $-\#-$ (see Lemma 2.18) that Y is necessarily an infinite object of $\overline{\boldsymbol{\mathcal{H}}}^{\geq 2}$. By applying the isomorphism (2.7) of Lemma 2.19, we deduce that $Y \cong Y \# V_{0,2}^{\#(p-1)} \cong V^s_{\infty,r} \# V_{0,2}^{\# n} \cong V^s_{\infty,r}$, and so the cancellation condition (ii) holds for (c). □

Standardness. We finally verify that the local standardness property of Definition 2.39 hold for the groupoids associated to the handlebody groups.

Proposition 2.46 *For any $s \geq 0$, the following categories are locally standard:*

(a) the right $(\boldsymbol{\mathcal{H}}^{\geq 1}, \natural, I_1)$-module $(\boldsymbol{\mathcal{H}}^{\geq 1}, \natural)$ at $(V^s_{g,r}, V_{1,1})$ for any $g \geq 0$ and $r \geq 1$;

(b) the right $(\boldsymbol{\mathcal{H}}^{\geq 2}, \#, I_2)$-module $(\boldsymbol{\mathcal{H}}^{\geq 2}, \#)$ at $(V^s_{g,r}, V_{0,2})$ for any $g \geq 0$ and $r \geq 2$;

(c) the right $(\boldsymbol{\mathcal{H}}^{\geq 2}, \#, I_2)$-module $(\overline{\boldsymbol{\mathcal{H}}}^{\geq 2}, \#)$ at $(V^s_{\infty,r}, V_{0,2})$ for any $r \geq 2$.

Proof. We first treat Condition (i). We take the conventions of Notation 2.25. It follows from an elementary computation that $H_1(V^s_{g,r} \natural_i Z_1 \natural_i Z_2, \mathcal{D}_r \sqcup \mathcal{P}_s; \mathbb{Z})$ is isomorphic to the free abelian group $\mathbb{Z}^{2g+r-1+s}$, where each Z_i supports the generator of exactly one copy of $\mathbb{Z}$. For any $\phi \in \mathrm{Aut}_{\boldsymbol{\mathcal{H}}^{\geq i}}(V^s_{g,r} \natural_i Z_1)$, the morphism $\phi \natural_i \mathrm{id}_{Z_2}$ acts trivially on the homology class supported on Z_2, whereas the braiding $(b^{\boldsymbol{\mathcal{H}}^{\geq i}}_{Z_1,Z_2})^{-1}$ acts non-trivially on it. Therefore the equivalence classes $[V^s_{g+1,r}, \mathrm{id}_{V^s_{g,r}} \natural (b^{\boldsymbol{\mathcal{H}}^{\geq i}}_{Z_1,Z_2})^{-1}]$ and $[V^s_{g+1,r}, \mathrm{id}_{V^s_{g+2,r}}]$ are distinct in $\mathrm{colim}_{\boldsymbol{\mathcal{H}}^{\geq i}}(\mathrm{Hom}_{\boldsymbol{\mathcal{H}}^{\geq i}}(-\natural_i X, A_2))$, which proves Condition (i) for the cases (a) and (b). Furthermore, if we assume that the equivalence classes $[V^s_{\infty,r} \# V_{0,2}, \mathrm{id}_{V^s_{\infty,r}} \natural (b^{\boldsymbol{\mathcal{H}}^{\geq 2}}_{V_{0,2},V_{0,2}})^{-1}]$ and $[V^s_{\infty,r} \# V_{0,2}, \mathrm{id}_{V^s_{\infty,r}} \# \mathrm{id}_{V_{0,2}} \# \mathrm{id}_{V_{0,2}}]$ are equal in $\mathrm{colim}_{\overline{\boldsymbol{\mathcal{H}}}^{\geq 2}}(\mathrm{Hom}_{\overline{\boldsymbol{\mathcal{H}}}^{\geq 2}}(-\# V_{0,2}, V^s_{\infty,r} \# V_{0,2}^{\# 2}))$, then there exists $\psi \in \mathrm{Aut}_{\overline{\boldsymbol{\mathcal{H}}}^{\geq 2}}(V^s_{\infty,r} \# V_{0,2})$ such that $\psi \# \mathrm{id}_{V_{0,2}} = \mathrm{id}_{V^s_{\infty,r}} \# (b^{\boldsymbol{\mathcal{H}}^{\geq 2}}_{V_{0,2},V_{0,2}})^{-1}$. Following Definition 2.14, there exist an integer $j \geq 0$ and a set of morphisms $\{\psi_g\}_{g \geq j}$ with $\psi_g \in \mathrm{Aut}_{\boldsymbol{\mathcal{H}}^{\geq 2}}(V^s_{g,r} \# V_{0,2})$ making the corresponding diagrams of the form (2.5) commutative, and such that $\psi = \mathrm{colim}_{g \geq j}(\psi_g)$. The maps $\iota_i(V^s_{0,r} \# V_{0,2}^{\# 2})$ are monomorphisms in the category $\langle \boldsymbol{\mathcal{H}}^{\geq 1}, \boldsymbol{\mathcal{H}}^{\geq 1}]$ for all $i \geq 0$, so it directly follows from the explicit description of the form $\coprod_{g \geq 0}(V^s_{g,r} \# V_{0,2}^{\# 2})/\sim$ of the colimit $V^s_{\infty,r} \# V_{0,2}^{\# 2}$ (see [Bor94, Prop. 2.13.3]) that the structural map $\alpha_i\colon V^s_{i,r}\#V_{0,2}^{\#2}\to V^s_{\infty,r}\#V_{0,2}^{\#2}$ is a monomorphism for each $i\geq 0$. Restricting the equality $\psi\#\mathrm{id}_{V_{0,2}}=\mathrm{id}_{V^s_{\infty,r}}\#(b^{\mathfrak{H}^{\geq 2}}_{V_{0,2},V_{0,2}})^{-1}$ along each α_g for $g\geq j$, this implies that the morphisms $\psi_g\#\mathrm{id}_{V_{0,2}}$ and $\mathrm{id}_{V^s_{g,r}}\#(b^{\mathfrak{H}^{\geq 2}}_{V_{0,2},V_{0,2}})^{-1}$ are equal for each $g\geq j$. This contradicts the case (b), and so Condition (i) holds for (c).

Let us now deal with the Condition (ii). We start with the case (a). First, we note from the definitions of the morphisms and repeating verbatim the first part of the proof of Lemma 2.41 that the following diagram is commutative for each $g\geq 0$ and $n\geq 2$

$$\begin{tikzcd}
\mathcal{H}^s_{g+n-2,r} \arrow[r, hook, "\sigma^s_{g+n-1,r}"] \arrow[d, hook, "\sigma^s_{g+n-1,r}"'] & \mathcal{H}^s_{g+n-1,r} \arrow[d, hook, "\sigma^s_{g+n,r}"] \\
\mathcal{H}^s_{g+n-1,r} \arrow[r, hook, "\mathfrak{C}_n(V_{1,1},V_{1,1})\circ\sigma^s_{g+n,r}"] & \mathcal{H}^s_{g+n,r}.
\end{tikzcd} \tag{2.20}$$

It is a classical fact that the analogous diagram to (2.20) for the mapping class groups of surfaces (i.e. replacing $\mathcal{H}$ with Γ, σ with ς, and V with Σ), that we denote by ((2.20)$_\Gamma$, is a pullback square; see [RW17, Prop. 5.6.1] or [PW24, Prop. 4.11(2)] for instance. We consider the cubical commutative diagram (2.20) $\to$ (2.20)$_\Gamma$ induced by the canonical injections of the form $\mathtt{i}^s_{g,r}$. The four commutative squares between (2.20) and (2.20)$_\Gamma$ are of the form of diagram (2.18), and so they are pullback squares by Proposition 2.44. It then follows from the Pullback Lemma (see [Awo10, Lem. 5.10.2.] for instance) that the commutative diagram (2.20) is a pullback square, and we deduce from Lemma 2.41 that Condition (ii) is satisfied for (a). The proof for the case (b) simply repeats mutatis mutandis the above arguments and reasoning, replacing σ with ρ, ς with ϱ, $\natural$ with $\#$, $V_{1,1}$ with $V_{0,2}$, and (2.18) with (2.19). (In this case, that the analogue of (2.20)$_\Gamma$ is a pullback square follows from a straightforward adaptation of [PW24, Prop. 4.3] using $\#$ instead of $\natural$ for instance.) Finally, for the case (c), let us first describe what is an element φ of $\mathrm{colim}_{\mathfrak{H}^{\geq 2}}(\mathrm{Hom}_{\mathfrak{H}^{\geq 2}}(-\#V_{0,2},V^s_{\infty,r}\#V_{0,2}^{\#n-1}))$ for $n\geq 2$. We deduce from Definition 2.14 that there exists an integer $j\geq 0$ and a set of morphisms $\{\varphi_g=[V^s_{g,r}\#V_{0,2}^{\#n-2},\phi_g]\}_{g\geq j}$ with $\phi_g\in\mathrm{Aut}_{\mathfrak{H}^{\geq 2}}(V^s_{g+n-1,r})$ making the following diagram commutative such that $\psi=\mathrm{colim}_{g\geq j}(\psi_g)$

$$\begin{tikzcd}
V_{0,2} \arrow[r, "\mathrm{id}_{V_{0,2}}"] \arrow[d, "\varphi_j"] & V_{0,2} \arrow[r, "\mathrm{id}_{V_{0,2}}"] \arrow[d, "\varphi_{j+1}"] & \cdots \arrow[r, "\mathrm{id}_{V_{0,2}}"] & V_{0,2} \arrow[r, "\mathrm{id}_{V_{0,2}}"] \arrow[d, "\varphi_g"] & \cdots \\
V^s_{j+n-1,r} \arrow[r, "\iota_{j+1}(V_{0,2}^{\#(n-1)})"] & V^s_{j+n,r} \arrow[r, "\iota_{j+2}(V_{0,2}^{\#(n-1)})"] & \cdots \arrow[r, "\iota_g(V_{0,2}^{\#(n-1)})"] & V^s_{g+n-1,r} \arrow[r, "\iota_{g+1}(V_{0,2}^{\#(n-1)})"] & \cdots .
\end{tikzcd}$$

Furthermore, similarly to the above proof of Condition (i) for (c), we again note that the structural map $V^s_{i,r}\#V_{0,2}^{\#n}\to V^s_{\infty,r}\#V_{0,2}^{\#n}$ is a monomorphism for each $i\geq 0$, essentially because since map $\iota_g(V_{0,2}^{\#n})$ is a monomorphism and thanks to the description [Bor94, Prop. 2.13.3] of $V^s_{\infty,r}\#V_{0,2}^{\#n}$.

Now, if $\theta^n_{V^s_{\infty,r},V_{0,2}}(\varphi)=\theta^n_{V^s_{\infty,r},V_{0,2}}(\varphi')$ for $\varphi,\varphi'\in\mathrm{colim}_{\mathfrak{H}^{\geq 2}}(\mathrm{Hom}_{\mathfrak{H}^{\geq 2}}(-\#V_{0,2},V^s_{\infty,r}\#V_{0,2}^{\#(n-1)}))$, it follows from the above description and the restriction along the structural maps that there exists $J\geq j\geq 0$ such that $\theta^n_{V^s_{g,r},V_{0,2}}(\varphi_g)=\theta^n_{V^s_{g,r},V_{0,2}}(\varphi'_g)$ for each $g\geq J$. Since the map $\theta^n_{V^s_{g,r},V_{0,2}}$ is injective by Proposition 2.46(b), we deduce that $\varphi_g=\varphi'_g$ for all $g\geq J$, and so $\varphi=\varphi'$ by the universal property of a colimit. Hence Condition (ii) holds for (c). ☐

3. Homological stability with twisted coefficients

In this section we prove Theorem A and Theorem B. Given the categorical properties we have verified in §2.4.3, we recall from §2.4.1 that proving Theorems A and B in both cases boils down to showing that an associated semi-simplicial set is sufficiently highly connected. In both cases, we are able to obtain the required connectivity from results of [HW10]. We start with Theorem A, concerning the standard genus stabilisation for surfaces, utilising the monoidal structure $\natural$; see §3.1. We next prove twisted homological stability for the second, less standard, genus stabilisation mentioned in the introduction, which comes from the monoidal structure $\#$; see §3.2. Combining these two homological stability results, Theorem B follows as a corollary; see §3.3.

3.1. Handle stabilisation with respect to $\natural$

We consider the standard stabilisation map given by attaching the genus-one handlebody with a marked disc $V_{1,1}$ via the operation $\natural$. For all §3.1, we fix integers $r \geq 1$ and $s \geq 0$ and write for brevity $\mathcal{H}^{\geq 1}_{r,s}$ (resp. $\mathcal{H}^{\geq 1}_{1}$) for $\mathcal{H}^{\geq 1}_{V^s_{0,r},V_{1,1}}$ (resp. $\mathcal{H}^{\geq 1}_{V_{1,1}}$). In terms of our categorical framework set in §2.2, we work in the category $[\mathcal{H}^{\geq 1}_{r,s}, \mathcal{H}^{\geq 1}_{1}\rangle$ defined by the Quillen bracket construction (see §2.2.2) applied to the right-module $(\mathcal{H}^{\geq 1}_{r,s}, \natural)$ over the braided monoidal category $(\mathcal{H}^{\geq 1}_{1}, \natural, I_1)$ (see §2.2.1). We recall that the objects of $[\mathcal{H}^{\geq 1}_{r,s}, \mathcal{H}^{\geq 1}_{1}\rangle$ are the handlebodies $V^s_{0,r}\natural V^{\natural g}_{1,1} \cong V^s_{g,r}$ (see (2.3)) for all $g \geq 0$. In particular, we have isomorphisms (where the second one holds by Corollary 2.13):

$$\mathcal{H}^s_{g,r} \cong \mathrm{Aut}_{\mathcal{H}^{\geq 1}_{1}}(V^s_{g,r}) \cong \mathrm{Aut}_{[\mathcal{H}^{\geq 1}_{r,s}, \mathcal{H}^{\geq 1}_{1}\rangle}(V^s_{g,r}).$$

The stabilisation map is then defined by the group injection $\sigma^s_{g,r} := (-)_g \natural \mathrm{id}_{V_{1,1}} \colon \mathcal{H}^s_{g,r} \hookrightarrow \mathcal{H}^s_{g+1,r}$; see (2.16). Furthermore, we write $c^s_{g,r}$ for the canonical morphism $[\mathrm{id}_{V^s_{g+1,r}}, V_{1,1}] \colon V^s_{g,r} \to V^s_{g+1,r}$ of $[\mathcal{H}^{\geq 1}_{r,s}, \mathcal{H}^{\geq 1}_{1}\rangle$. The following twisted homological stability result, whose proof is done in §3.1.2, is a more precise version of Theorem A:

Theorem 3.1 *We fix integers $r \geq 1$ and $s \geq 0$. Let $F \colon [\mathcal{H}^{\geq 1}_{r,s}, \mathcal{H}^{\geq 1}_{1}\rangle \to \mathbf{Ab}$ be a coefficient system of degree d at 0 (see Definition 2.22). Then*

$$(\sigma^s_{g,r}; F(c^s_{g,r}))_i \colon H_i(\mathcal{H}^s_{g,r}; F(V^s_{g,r})) \to H_i(\mathcal{H}^s_{g+1,r}; F(V^s_{g+1,r}))$$

is an isomorphism for $i \leq \frac{g-1}{2} - d - 1$. If F is split, it is an isomorphism for $i \leq \frac{g-1-d}{2} - 1$.

Example 3.2 Theorem 3.1 is a generalisation of the homological stability result [RW17, Th. 5.31] to any $s \geq 0$. Namely, assigning $s = 0$, Theorem 3.1 then recovers the following previous works:

- [HW10, Th. 1.8(i)] (for handlebodies) by assigning $F = \mathbb{Z}$ (see Example 2.23);
- [IS17, Th. 1.2] by assigning $r = 1$ and $F = H_\Sigma$ (see Proposition 2.26);
- [RW17, Th. 5.31] by assigning $r = 1$ and for any finite degree coefficient system F.

Moreover, for fixed $r \geq 1$ and $s \geq 0$, the functors $T^d H_\Sigma$ and $T^d H_\mathcal{V}$, which encode the homological representations $\{H_1^{\otimes d}(\partial V \setminus (\mathring{\mathcal{D}}_r \sqcup \mathcal{P}_s); \mathbb{Z})\}_{g\geq 0}$ and $\{H_1^{\otimes d}(V, \mathcal{D}_r \sqcup \mathcal{P}_s; \mathbb{Z})\}_{g\geq 0}$ of the handlebody groups $\{\mathcal{H}^s_{g,r}\}_{g\geq 0}$ respectively, are split coefficient systems at $V^s_{0,r}$ of finite degree $d \geq 0$; see Proposition 2.26. Twisted homological stability with these coefficients thus follows from Theorem 3.1.

3.1.1. Properties of the simplicial complex of destabilisations

In order to prove Theorem 3.1, we first study the simplicial complex of destabilisations introduced in Notation 2.43 involved in this context. Namely, we prove that this simplicial complex is isomorphic to a simplicial complex originally defined by Hatcher and Wahl [HW10] (see Proposition 3.7), which satisfies a high-connectivity property (see Theorem 3.5).

An alternative simplicial complex. First, here follows the simplicial complex following that introduced in [HW10, §8.2]:

Definition 3.3 We consider an object (V, r, s, i, j) of $\mathcal{H}$ where V is a compact handlebody, the set $\mathrm{Im}(i) \sqcup \mathrm{Im}(j) = \mathcal{D}_r \sqcup \mathcal{P}_s$ and we fix points $x_0, x_1 \in \partial\mathcal{D}_r$ (where we allow $x_0 = x_1$). Recalling that I denotes the unit interval $[0,1]$, let $I + \mathbb{D}^2$ be the space defined from the union $I \sqcup \mathbb{D}^2$ by identifying $1/2 \in I$ with some base-point $p_0 \in \partial\mathbb{D}^2$. We consider maps $f \colon I + \mathbb{D}^2 \to V$ such that:

- the restriction of f to $\mathring{I} + \mathbb{D}^2$ is an embedding into $V \setminus \mathcal{D}_r \sqcup \mathcal{P}_s$, while the restrictions of f to I and $\mathbb{D}^2$ are smooth;
- $f(I)$ intersects $f(\partial\mathbb{D}^2)$ transversely and $f(\partial\mathbb{D}^2)$ does not separate ∂V;

- $f(I + \partial\mathbb{D}^2) \subset \partial V$ intersects $\mathcal{D}_r \sqcup \mathcal{P}_s$ only at $f(0) = x_0$ and $f(1) = x_1$.

We define $Y^A(V, \mathcal{D}_r \sqcup \mathcal{P}_s, x_0, x_1)$ to be the simplicial complex whose vertices are isotopy classes of such maps f, where $(f_0, \ldots, f_p)$ forms a p-simplex if:

- the maps have pairwise disjoint images (outside of the endpoints);
- the union $f_0(\partial\mathbb{D}^2) \cup \cdots \cup f_p(\partial\mathbb{D}^2)$ is a *coconnected* system of discs in V, i.e. it does not separate ∂V.

Hatcher and Wahl do not consider marked points in [HW10, §8.2], i.e. they consider the simplicial complex of Definition 3.3 with the assignment $\mathcal{P}_s = \emptyset$. However, their restricted definition recovers our more general one as follows:

Lemma 3.4 *For any object (V, r, s, i, j) of $\mathfrak{H}$, there is an isomorphism of simplicial complexes*

$$\Psi^s_{V,r} \colon Y^A(V, \mathcal{D}_r \sqcup \mathcal{P}_s, x_0, x_1) \xrightarrow{\cong} Y^A(V, \mathcal{D}_{r+s}, x_0, x_1).$$

Proof. For any $f \in Y^A(V, \mathcal{D}_r \sqcup \mathcal{P}_s, x_0, x_1)$, we define $\Psi^s_{V,r}(f)$ as follows. By the conditions of Definition 3.3, the set $\mathrm{Im}(f)$ does not intersect $\mathcal{P}_s$ and, for each marked point $p_i \in \mathcal{P}_s$, we may choose an open neighbourhood $\mathcal{N}(p_i) \subset V$ of p_i such that the map f is a smooth embedding in $\mathcal{N}(p_i)$. Also, since V is a smooth manifold with boundary, we may pick a closed disc $\mathbb{D}^2_{p_i} \subset \partial V \cap \mathcal{N}(p_i)$ centered on p_i for each $1 \leq i \leq s$. Therefore, either $\mathrm{Im}(f) \cap \mathbb{D}^2_{p_i} = \emptyset$, or we may apply an isotopy of $\mathcal{N}(p_i)$ to f defining isotopic map f' so that $\mathrm{Im}(f')$ does not intersect $\mathbb{D}^2_{p_i}$. Denoting by $[f]$ the isotopy class of f, we then assign $\Psi^s_{V,r}(f)$ to be the isotopy class of the map obtained by this procedure. It is a straightforward routine to check from the conditions of Definition 3.3 that $\Psi^s_{V,r}(f)$ belongs to $Y^A(V, \mathcal{D}_{r+s}, x_0, x_1)$, and $\Psi^s_{V,r}([f]) = \Psi^s_{V,r}([g])$ for $g \colon I + \mathbb{D}^2 \to V$ such that $[f] = [g]$, and so $\Psi^s_{V,r}$ is a well-defined morphism of simplicial complexes. Now, the inverse map of $\Psi^s_{V,r}$ is the obvious morphism of simplicial complexes induced by fixing the marked point p_i to be center of the marked disc $\mathcal{D}^i_V$ and forgetting all the other points $\mathcal{D}^i_V \setminus p_i$ for each $1 \leq i \leq s$, and so the map $\Psi^s_{V,r}$ is clearly bijective. $\square$

Then, the following fundamental connectivity property is a direct consequence of [HW10, Thm. 8.5], using the isomorphism of simplicial complexes of Lemma 3.4:

Theorem 3.5 *Let (V, r, s, i, j) be an object of $\mathfrak{H}$, $\mathcal{D}_r \sqcup \mathcal{P}_s$ and $x_0 = x_1 \in \partial\mathcal{D}_r$ as in Definition 3.3. For $h \geq 0$ the number of $\mathbb{S}^1 \times \mathbb{D}^2$-boundary connected summands in V, the simplicial complex $Y^A(V, \mathcal{D}_r \sqcup \mathcal{P}_s, x_0, x_0)$ is $\frac{h-3}{2}$-connected.*

A simplicial complex isomorphism. Another key property of the simplicial complex of Definition 3.3 is that it is isomorphic to the destabilisation complex of Notation 2.43 in this context. From now on, we fix an integer $g \geq 1$ for the rest of §3.1.1, and we set $x_0 = x_1$ as the image of the point $(1, 0) \in \mathbb{D}^2$ under the parametrisation i_1 of the first marked disc $\mathcal{D}^1_{V^s_{g,r}}$ in $\partial V^s_{g,r}$. We define a map of simplicial complexes

$$\Phi^s_{g,r} \colon S_g(V^s_{0,r}, V_{1,1})_\bullet \to Y^A(V_g, \mathcal{D}_r \sqcup \mathcal{P}_s, x_0, x_0), \tag{3.1}$$

as follows. First, let us once and for all choose a map $\Phi_0 \colon I + \mathbb{D}^2 \to V_{1,1}$ satisfying the conditions of Definition 3.3. Namely, we may assume that the parametrisation map $i_1 \colon \mathbb{D}^2 \to \partial V_{1,1}$ defining the marked disc $\mathcal{D}^1_{V^s_{g,r}}$ to be such that if $x_0 = i_1(1, 0) = (t, y) \in \mathbb{S}^1 \times \partial\mathbb{D}^2$, then $\mathbb{S}^1 \times \{y\}$ does not intersect $\mathrm{Im}(i_1)$ elsewhere. Then, we define

$$\Phi_0(x) = \begin{cases} (x, y) & \text{if } x \in I \\ (1/2, x) & \text{if } x \in \mathbb{D}^2. \end{cases}$$

We recall from Notation 2.43 that a vertex of $S_g(V^s_{0,r}, V_{1,1})_\bullet$ is a morphism $[B, \phi]$ in the colimit over $\mathfrak{H}^{\geq 1}_{r,s}$ of $\mathrm{Hom}_{\mathfrak{H}^{\geq 1}_{r,s}}(- \odot V_{1,1}, V^s_{g,r})$, i.e. an equivalence class of pairs (B, ϕ) where $\phi \colon B \natural V_{1,1} \to V^s_{g,r}$ is a morphism in $\mathfrak{H}^{\geq 1}_{r,s}$. We denote by Φ'_0 the isotopy class of the map $I + \mathbb{D}^2 \to B \natural V_{1,1}$ defined by postcomposing the above map $\Phi_0 \colon I + \mathbb{D}^2 \to V_{1,1}$ with the canonical embedding $V_{1,1} \hookrightarrow B \natural V_{1,1}$ induced by the boundary connected sum $\natural$.

Lemma 3.6 *The assignment $[B,\phi]\mapsto\phi\circ\Phi'_0$ for the map (3.1) defines a morphism of simplicial complexes.*

Proof. First, let us verify that the assignment for $\Phi^s_{g,r}([B,\phi])$ does not depend on the choice of representative (B,ϕ). We note that $(B',\phi')\sim(B,\phi)$ in the colimit over $\mathcal{H}^{\geq 1}_{r,s}$ of $\mathrm{Hom}_{\mathcal{H}^{\geq 1}_{r,s}}(-\odot V_{1,1},V^s_{g,r})$ if and only if there is an isomorphism $\psi\colon B\to B'$ in $\mathcal{H}^{\geq 1}_{r,s}$ such that $\phi=\phi'\circ(\psi\natural\mathrm{id}_{V_{1,1}})$. Since ϕ and ϕ' may differ only on the left summand, we deduce that $\phi'_{|V_{1,1}}=\phi_{|V_{1,1}}$ and so $\Phi^s_{g,r}([B,\phi])=\Phi^s_{g,r}([B',\phi'])$ by definition of Φ'_0.

Next, it remains to show that $\Phi^s_{g,r}$ is a well-defined map of simplicial complexes, i.e. that it sends p-simplices to p-simplices. We recall from Notation 2.43 that a $(p+1)$-tuple $([B_0,\phi_0],\ldots,[B_p,\phi_p])$ of vertices of $S_g(V^s_{0,r},V_{1,1})_\bullet$ defines a p-simplex if there is $[B,\phi]\in\mathrm{colim}_{\mathcal{H}^{\geq 1}_{r,s}}\left(\mathrm{Hom}_{\mathcal{H}^{\geq 1}_{r,s}}(-\odot V_{1,1}^{\natural p+1},V^s_{g,r})\right)$ such that for each $0\leq i\leq p$

$$[B_i,\phi_i]=[B,\phi]\circ[V_{1,1}^{\natural p},\mathrm{id}_{V_{i,1}}\natural(b_{V_{1,1},V_{p-i,1}})^{-1}],$$

where $b_{V_{1,1},V_{p-i,1}}$ is the braiding of $(\mathcal{H}^{\geq 1}_1,\natural,I_1)$ and $[V_{1,1}^{\natural p},\mathrm{id}_{V_{i,1}}\natural(b_{V_{1,1},V_{p-i,1}})^{-1}]$ is the canonical map $V_{1,1}\to V_{i,1}\natural V_{1,1}\natural V_{p-i,1}$ induced by equivalence classes of isomorphisms of the target fixing the source summand. Therefore, the image of $([B_0,\phi_0],\ldots,[B_p,\phi_p])$ by $\Phi^s_{g,r}$ induces maps $f_0,\ldots,f_p\colon I+\mathbb{D}^2\to B\natural V_{1,1}^{\natural p+1}$ whose isotopy classes belong to $Y^A(V_g,\mathcal{D}_r\sqcup\mathcal{P}_s,x_0,x_0)$, with the relation $\Phi^s_{g,r}([B_i,\phi_i])=\phi\circ f_i$ and such that the image of f_i is in the i-th $V_{1,1}$-summand (except the endpoints). In particular, the maps $f_0,\ldots,f_p$ have pairwise disjoint images (outside of the endpoints), and their restrictions to $\partial\mathbb{D}^2$ define a coconnected system of discs $f_0(\partial\mathbb{D}^2)\cup\cdots\cup f_p(\partial\mathbb{D}^2)$ in $B\natural V_{1,1}^{\natural p+1}$. Since ϕ is the isotopy class of a diffeomorphism, it preserves nonseparating simple closed curves, and so the union $\phi\circ f_0(\partial\mathbb{D}^2)\cup\cdots\cup\phi\circ f_p(\partial\mathbb{D}^2)$ is a coconnected system of $p+1$ discs in $V^s_{g,r}$. Hence, by Definition 3.3, the system $(\Phi^s_{g,r}([B_0,\phi_0]),\ldots,\Phi^s_{g,r}([B_p,\phi_p]))$ is a p-simplex in $Y^A(V_g,\mathcal{D}_r\sqcup\mathcal{P}_s,x_0,x_0)$. $\square$

Proposition 3.7 *The morphism of simplicial complexes $\Phi^s_{g,r}$ is an isomorphism.*

Proof. We start by showing that the map $\Phi^s_{g,r}$ is surjective. Let $f\colon I+\mathbb{D}^2\to V^s_{g,r}$ be a vertex of $Y^A(V_g,\mathcal{D}_r\sqcup\mathcal{P}_s,x_0,x_0)$. First, we note from Definition 3.3 that any regular neighborhood of $\mathrm{Im}(f)$ is diffeomorphic to $V_{1,1}$. We can pick such a neighborhood $\mathcal{N}(f)$, with a disc $\mathcal{D}_{\mathcal{N}(f)}$ in its boundary which intersects the first marked disc $\mathcal{D}^1_{V^s_{g,r}}$ of $V^s_{g,r}$ exactly in its right-half. We set $B=V^s_{g,r}\setminus\mathcal{N}(f)$, with a first marked disc given by the left-half of $\mathcal{D}^1_{V^s_{g,r}}$, while its right-half given by the left hand side of $\mathcal{D}_{\mathcal{N}(f)}$. We also equip B with $r-1$ more marked discs in the boundary given by the remaining marked discs in $V^s_{g,r}$, and with the marked points $\mathcal{P}_s$. This construction thus induces a diffeomorphism $\phi\colon B\natural\mathcal{N}(f)\to V^s_{g,r}$, preserving marked discs $\mathcal{D}_r$ and points $\mathcal{P}_s$. Hence this defines a vertex $[B,\phi]$ in $S_g(V^s_{0,r},V_{1,1})_\bullet$, such that $\phi([B,\phi])=f$.

Now, let us show that $\Phi^s_{g,r}$ is injective. For $[B,\phi]$ and $[B',\phi']$ in the colimit over $\mathcal{H}^{\geq 1}_{r,s}$ of $\mathrm{Hom}_{\mathcal{H}^{\geq 1}_{r,s}}(-\odot V_{1,1},V^s_{g,r})$, suppose that $\Phi^s_{g,r}([B,\phi])=\phi\circ\Phi'_0=\phi'\circ\Phi'_0=\Phi^s_{g,r}([B',\phi'])$. Since $V_{1,1}$ is a regular neighborhood of the image of Φ'_0, we deduce from the equality $\phi\circ\Phi'_0=\phi'\circ\Phi'_0$ that there exist mapping classes $\varphi\in\mathrm{Aut}(\mathcal{H}^{\geq 1}_{r,s})(B,B')$ and $\psi\in\mathcal{H}_{1,1}$ such that $\phi^{-1}\circ\phi'=\varphi\natural\psi$. More precisely, the map ψ is the isotopy class of a diffeomorphism $\tilde{\psi}$ of $V_{1,1}=\mathbb{S}^1\times\mathbb{D}^2$ pointwise fixing the marked disc in the boundary where x_0 lies. Moreover, since $(\phi^{-1}\circ\phi')\circ\Phi'_0=\Phi'_0$, the representative $\tilde{\psi}$ setwise fixes the embedded disc $\{1/2\}\times\mathbb{D}^2$, as well as a simple closed curve $\gamma\subset\partial V_{1,1}$ along the longitude (i.e. bounding the genus), based at x_0 and intersecting $\{1/2\}\times\mathbb{D}^2$ transversely. We cut $V_{1,1}$ along the embedded disc $\{1/2\}\times\mathbb{D}^2$ and since setwise fixing a disc is equivalent to fixing a point up to isotopy, this implies that ψ is an element of $\mathcal{H}^2_{0,1}$. We recall from Lemma 2.1 that $\mathcal{H}^2_{0,1}\hookrightarrow\Gamma^2_{0,1}$. It is a classical fact (see [FM12, §9.1] for instance) that $\Gamma^2_{0,1}$ is isomorphic to $\mathbb{Z}$, which generator T_m is the Dehn twist along a meridian of $\Sigma_{1,1}$ (i.e. a simple closed curve in $\partial V_{1,1}$ that bounding a nonseparating disc in $V_{1,1}$). Since the mapping class ψ fixes the longitude simple closed curve γ, we deduce that ψ cannot be a non-zero power of T_m. Hence we have $\psi=\mathrm{id}_{V_{1,1}}$, and so $\phi'=(\varphi\natural\mathrm{id}_{V_{1,1}})\circ\phi$. Therefore, by the equivalence relation in the colimit over $\mathcal{H}^{\geq 1}_{r,s}$ of $\mathrm{Hom}_{\mathcal{H}^{\geq 1}_{r,s}}(-\odot V_{1,1},V^s_{g,r})$, we conclude that $[B,\phi]=[B,\phi']$. $\square$

3.1.2. Proof of Theorem 3.1

We now prove Theorem 3.1 by applying Theorem 2.37 with the setting $(\mathcal{G}, \odot, 0) = (\mathfrak{H}_1^{\geq 1}, \natural, I_1)$, $(\mathcal{M}, \odot) = (\mathfrak{H}_{r,s}^{\geq 1}, \natural)$, $A = V_{0,r}^s$ and $X = V_{1,1}$. First of all, we recall that the necessary hypotheses of having no zero-divisors, $\mathrm{Aut}_{\mathfrak{H}_1^{\geq 1}}(I_1) = \{\mathrm{id}_{I_1}\}$ and local homogeneity at $(V_{0,r}^s, V_{1,1})$ are verified in Propositions 2.7 and 2.45(a) respectively. It thus only remains to check the high-connectivity condition of Theorem 2.37 for the semi-simplicial set of destabilisations $W_g(V_{0,r}^s, V_{1,1})_\bullet$ for each $g \geq 1$. This is a consequence of the following result:

Proposition 3.8 *For each $g \geq 1$, there is a homeomorphism*

$$|W_g(V_{0,r}^s, V_{1,1})_\bullet| \xrightarrow{\cong} |Y^A(V_g, \mathcal{D}_r \sqcup \mathcal{P}_s, x_0, x_0)|.$$

Proof. We consider the map $\pi \colon |W_g(V_{0,r}^s, V_{1,1})_\bullet| \to |S_g(V_{0,r}^s)_\bullet|$ of Notation 2.43. For a fixed $p \geq 0$, we recall that a p-simplex in $S_g(V_{0,r}^s, V_{1,1})_p$ has a canonical order on its vertices, induced by the orientation near the first marked disc in the boundary of $V_{0,r}^s \natural V_{1,1}^{\natural g}$. (In terms of the complex $Y^A((V_g, \mathcal{D}_r \sqcup \mathcal{P}_s, x_0, x_0)$, this is the canonical order on the arcs induced by the orientation.) Since the right $(\mathfrak{H}_1^{\geq 1}, \natural, I_1)$-module $(\mathfrak{H}_{r,s}^{\geq 1}, \natural)$ is locally standard at $(V_{g,r}^s, V_{1,1})$ by Proposition 2.46(a), it follows from Proposition 2.42 that, for each $\sigma \in S_g(V_{0,r}^s, V_{1,1})_p$, each simplex in $\pi_p^{-1}(\sigma)$ is determined by its ordered set of vertices. Therefore, the preimage $\pi_p^{-1}(\sigma)$ contains exactly one simplex for each $\sigma \in S_g(V_{0,r}^s, V_{1,1})_p$, so the map π_p is a bijection for each $p \geq 0$ and then π is thus a bijection. Combining this with Proposition 3.7, we deduce the intended homeomorphism as the composite $\pi \circ |\Phi_{g,r}^s|$. $\square$

Combining Proposition 3.8 with Theorem 3.5, we deduce that the semi-simplicial set $W_g(V_{0,r}^s, V_{1,1})_\bullet$ is $\frac{g-3}{2}$-connected for each $g \geq 1$, which thus proves Theorem 3.1 as a consequence of Theorem 2.37.

3.2. Handle stabilisation with respect to

We now prove a twisted homological stability result for the less standard genus stabilisation induced by attaching the genus-zero handlebody with two marked discs $V_{0,2}$ via the operation $\#$; see Theorem 3.9. For all §3.2, we fix integers $r \geq 2$ and $s \geq 0$ and write for brevity $\mathfrak{H}_{r,s}^{\geq 2}$ (resp. $\mathfrak{H}_2^{\geq 2}$) for $\mathfrak{H}_{V_{0,r}^s, V_{0,2}}^{\geq 2}$ (resp. $\mathfrak{H}_{V_{0,2}}^{\geq 2}$). We consider the category $(\langle \mathfrak{H}_{r,s}^{\geq 2}, \mathfrak{H}_2^{\geq 2} \rangle, \#, I_2)$ constructed from the right-module $(\mathfrak{H}_{r,s}^{\geq 2}, \#)$ over the braided monoidal category $(\mathfrak{H}_2^{\geq 2}, \#, I_2)$ (see §2.2.1) via the Quillen bracket construction (see §2.2.2). The objects of $\langle \mathfrak{H}_{r,s}^{\geq 2}, \mathfrak{H}_2^{\geq 2} \rangle$ are the handlebodies $V_{0,r}^s \# V_{0,2}^{\# g} \cong V_{g,r}^s$ (see (2.3)) for all $g \geq 0$, so we again deduce from Corollary 2.13 that we have isomorphisms:

$$\mathcal{H}_{g,r}^s \cong \mathrm{Aut}_{\mathfrak{H}_{r,s}^{\geq 2}}(V_{g,r}^s) \cong \mathrm{Aut}_{\langle \mathfrak{H}_{r,s}^{\geq 2}, \mathfrak{H}_2^{\geq 2} \rangle}(V_{g,r}^s).$$

Here, the stabilisation map is given by the group injection $\rho_{g,r}^s := (-)_g \# \mathrm{id}_{V_{0,2}} \colon \mathcal{H}_{g,r}^s \to \mathcal{H}_{g+1,r}^s$ (see (2.17)). We also write $d_{g,r}^s$ for the canonical morphism $[\mathrm{id}_{V_{g+1,r}^s}, V_{0,2}] \colon V_{g,r}^s \to V_{g+1,r}^s$ of $\langle \mathfrak{H}_{r,s}^{\geq 2}, \mathfrak{H}_2^{\geq 2} \rangle$. We have the following twisted homological stability result with respect to this stabilisation, whose proof is done in §3.2.2:

Theorem 3.9 *We fix integers $r \geq 2$ and $s \geq 0$. Let $F \colon \mathfrak{H} \to \mathbf{Ab}$ be a double coefficient system at $V_{0,r}^s$ of degree d (see Definition 2.28). Then*

$$(\rho_{g,r}^s; F(d_{g,r}^s))_i \colon H_i(\mathcal{H}_{g,r}^s; F(V_{g,r}^s)) \to H_i(\mathcal{H}_{g+1,r}^s; F(V_{g+1,r}^s))$$

is an isomorphism for $i \leq \frac{g-1}{2} - d - 1$. If both coefficient systems are split, it is an isomorphism for $i \leq \frac{g-1-d}{2} - 1$.

Remark 3.10 The assumption for the coefficients to form a double coefficient system is due to leveraging (a mild alternative of) Theorem 3.1 in the proof; see Theorem 3.15. Indeed, a key point to prove Theorem 3.9 is to use a “stabilised version” of the destabilisation complex introduced in §3.2.1, and then to utilise stability with respect to the analogue of $\sigma_{g,r}^s$ adding a handle on the left; see Theorem 3.15.

3.2.1. Stable twisted homological stability

For the proof of Theorem 3.9, we cannot proceed quite as for that of Theorem 3.1, essentially because the connectivity of the corresponding complex of Definition 3.3 is not known. We rather follow here the example of Hatcher and Wahl [HW10, §8.2]: instead of directly considering the stablisation maps $\rho^s_{g,r}$, we first prove a twisted homological stability result for infinite handlebodies (in the sense of Definition 2.4); see Theorem 3.14.

Stable simplicial complex of destabilisations. We begin with studying the simplicial complex of destabilisations $S_g(V_\infty, V_{0,2})_\bullet$ for each $g \geq 1$ (see Notation 2.43) associated to the infinite object $V^s_{\infty,r}$, showing that it is contractible; see Theorem 3.13. To do this, we again use the suitable simplicial complexes of Definition 3.3 for this situation. Namely, for each $g \geq 0$, we consider the simplicial complex $Y^A(V_g, \mathcal{D}_r \sqcup \mathcal{P}_s, x_0, x_1)$ associated to the object $V^s_{g,r} = (V_g, r, s, i, j)$ with $x_0 = i_1(1,0)$ and $x_1 = i_2(1,0)$, where $i_j \colon \mathbb{D}^2 \to \partial V_g$ for $j \in \{1,2\}$ denotes the parametrisation map defining the marked disc $\mathcal{D}^j_{V^s_{g,r}}$. For brevity, we denote by R_g the set $\mathcal{D}_r \sqcup \mathcal{P}_s$ of marked discs and marked points associated to V_g, and we write $\mathfrak{H}^{\geq 1}_{r,s}$ and $\mathfrak{H}^{\geq 1}_{1}$ for $\mathfrak{H}^{\geq 1}_{V^s_{0,r}, V_{1,1}}$ and $\mathfrak{H}^{\geq 1}_{V_{1,1}}$ respectively.

For each $i \geq 1$, we consider the morphism $\iota_i(V^s_{0,r}) = [V_{1,1}, \mathrm{id}_{V^s_{i,r}}] \colon V^s_{i-1,r} \to V^s_{i,r}$ of $\langle \mathfrak{H}^{\geq 1}_1, \mathfrak{H}^{\geq 1}_{r,s}]$. We note from Definition 2.5 that the image of $(1,0) \in \mathbb{D}^2$ in the parametrisation of any marked disc is the same in each V_i, as we are adding $V_{1,1}$ from the left, and so x_0 and x_1 are left invariant by $\iota_i(V^s_{0,r})$. Also, we remark that, although they are isomorphic, R_i is obtained from R_{i-1} by gluing the left hand side of the marked disc $V_{1,1}$ to the right hand half-disc $\mathcal{D}^1_{V^s_{i,r}}$. Therefore, the morphism $\iota_i(V^s_{0,r})$ induces by postcomposition a map of simplicial complexes $\iota^Y_i \colon Y^A(V_{i-1}, R_{i-1}, x_0, x_1) \to Y^A(V_i, R_i, x_0, x_1)$. Hence we obtain the following directed system of simplicial complexes:

$$Y^A(V_0, R_0, x_0, x_1) \xrightarrow{\iota^Y_1} Y^A(V_1, R_1, x_0, x_1) \xrightarrow{\iota^Y_2} \cdots \xrightarrow{\iota^Y_i} Y^A(V_i, R_i, x_0, x_1) \xrightarrow{\iota^Y_{i+1}} \cdots \quad (3.2)$$

Since the category of simplicial complexes is cocomplete, the following definition makes sense similarly to [HW10, §8.2]:

Definition 3.11 We define $Y^A(V^s_{\infty,r}, R_\infty, x_0, x_1)$ as the directed colimit of the diagram (3.2).

In order to study the connectivity of $S_g(V^s_{\infty,r}, V_{0,2})_\bullet$, we start by proving the following property:

Proposition 3.12 *For each integer $g \geq 1$, there is an isomorphism of simplicial complexes $S_g(V^s_{0,r}, V_{0,2})_\bullet \cong Y^A(V^s_{g,r}, R_g, x_0, x_1)$.*

Proof. The definition of the morphism simplicial complexes $S_g(V^s_{0,r}, V_{0,2})_\bullet \to Y^A(V^s_{g,r}, R_g, x_0, x_1)$ is very similar to that of (3.1), while the proof that this is well-defined and an isomorphism is almost identical to those of Lemma 3.6 and Proposition 3.7. Let us only sketch most of the proof and only include the details for the points that differ.

Recalling that I denotes the unit interval $[0,1]$, let us make the identification $V_{0,2} = I \times \mathbb{D}^2$, with $\{0\} \times \mathbb{D}^2$ and $\{1\} \times \mathbb{D}^2$ the marked discs parametrised in the obvious way. Let $y \in \partial\mathbb{D}^2$ be the point such that $i_1(1,0) = (0,y)$ and $i_2(1,0) = (1,y)$, where $i_1, i_2 \colon \mathbb{D}^2 \to \partial V_{0,2}$ are the defining maps of the marked discs $\mathcal{D}^1_{V_{0,2}}$ and $\mathcal{D}^2_{V_{0,2}}$. We then define $\Phi_0 \colon I + \mathbb{D}^2 \to V_{0,2}$ by

$$\Phi_0(x) = \begin{cases} (x, y) & \text{if } x \in I, \\ (1/2, x) & \text{if } x \in \mathbb{D}^2. \end{cases}$$

We recall from Notation 2.43 that a vertex of $S_g(V^s_{0,r}, V_{0,2})_\bullet$ is a morphism $[B, \phi]$ in the colimit over $\mathfrak{H}^{\geq 2}_{r,s}$ of $\mathrm{Hom}_{\mathfrak{H}^{\geq 2}_{r,s}}(- \odot V_{0,2}, V^s_{g,r})$, i.e. an equivalence class of pairs (B, ϕ) where $\phi \colon B\#V_{0,2} \to V^s_{g,r}$ is a morphism in $\mathfrak{H}^{\geq 2}_{r,s}$. We denote by Φ'_0 the isotopy class of the map $I + \mathbb{D}^2 \to B\#V_{0,2}$ defined by postcomposing the map $\Phi_0 \colon I + \mathbb{D}^2 \to V_{0,2}$ with the canonical embedding $V_{0,2} \hookrightarrow B\#V_{0,2}$ induced by the monoidal product $\#$. We may then define a map of simplicial complexes

$\Phi^s_{g,r}\colon S_g(V^s_{0,r}, V_{0,2})_\bullet \to Y^A(V_g, \mathcal{D}_r \sqcup \mathcal{P}_s, x_0, x_1)$ by sending $[B, \phi]$ to the isotopy class $\phi \circ \Phi'_0$. That this gives a well-defined, surjective map of simplicial complexes is proved by repeating mutatis mutandi the proof of Lemma 3.6 and the first half of Proposition 3.7. The proof of injectivity also goes through in a similar way to the second half of the proof of Proposition 3.7: by a routine verbatim adaptation, the proof boils down to showing that given $\psi \in \mathrm{Aut}_{\mathcal{H}_2^{\geq 2}}(V_{0,2}) = \mathcal{H}_{0,2}$ which fixes the isotopy class of the embedding $\Phi_0\colon I + \mathbb{D}^2 \to V_{0,2}$, we have $\psi = \mathrm{id}_{V_{0,2}}$. Recall from Lemma 2.1 that $\mathcal{H}_{0,2} \hookrightarrow \Gamma_{0,2}$ and it is a classical fact (see [FM12, §9.1] for instance) that $\Gamma_{0,2} \cong \mathbb{Z}$, generated by a Dehn twist around the meridian $\{t\} \times \mathbb{D}^2$ for any $0 < t < 1$. However, such a Dehn twist does not fix the isotopy class of the embedding Φ_0, so the claim follows. ☐

A fundamental theorem of [HW10] is that the simplicial complex $Y^A(V^s_{\infty,r}, R_\infty, x_0, x_1)$ is contractible; see [HW10, Thm. 8.6]. By using Proposition 3.12, we are now ready to show the following key result:

Theorem 3.13 *For each $g \geq 1$, the simplicial complex $S_g(V_\infty, V_{0,2})_\bullet$ is contractible.*

Proof. For each $g \geq 1$, we recall that $V^s_{\infty,r} \cong V^s_{\infty,r} \# V_{0,2}^{\# g}$ thanks to the isomorphism (2.7) of Lemma 2.19, and that we have an isomorphism $V^s_{g,r} \cong V^s_{0,r} \# V_{0,2}^{\# g}$ (see (2.3)). Then, we deduce from the definition of the simplicial complex of destabilisations (see Notation 2.43) the following simplicial complex isomorphisms

$$S_0(V^s_{\infty,r}, V_{0,2})_\bullet \cong S_g(V^s_{\infty,r}, V_{0,2})_\bullet \tag{3.3}$$

and

$$S_g(V^s_{0,r}, V_{0,2})_\bullet \cong S_0(V^s_{g,r}, V_{0,2})_\bullet. \tag{3.4}$$

Now, each morphism $\iota_g(V^s_{0,r})$ for $g \geq 1$ induces by postcomposition a map of simplicial complexes $\iota^S_i\colon S_0(V^s_{g-1,r}, V_{0,2})_\bullet \to S_0(V^s_{g,r}, V_{0,2})_\bullet$, and we denote by $\mathrm{colim}_g \left(S_0(V^s_{g,r}, V_{0,2})_\bullet\right)$ the directed colimit of the sequential system defined by these maps $\{\iota^S_g\}_{g\geq 1}$. By the universal property of a colimit, it then follows from Proposition 3.12 and the isomorphism (3.4) that we have an isomorphism of simplicial complexes

$$\operatorname*{colim}_g \left(S_0(V^s_{g,r}, V_{0,2})_\bullet\right) \cong Y^A(V^s_{\infty,r}, R_\infty, x_0, x_1). \tag{3.5}$$

Furthermore, since the postcomposition by the structural maps $V^s_{g,r} \to V^s_{\infty,r}$ for each $g \geq 0$ induces a map of simplicial complex of destabilisations $S_0(V^s_{g,r}, V_{0,2})_\bullet \to S_0(V^s_{\infty,r}, V_{0,2})_\bullet$, we deduce that there is a canonical map of simplicial complexes induced by the universal property of a colimit

$$\Xi_\bullet\colon \operatorname*{colim}_g \left(S_0(V^s_{g,r}, V_{0,2})_\bullet\right) \to S_0(V^s_{\infty,r}, V_{0,2})_\bullet.$$

Let us describe more precisely what happens for the vertices. By Definition 2.14, a morphism $\varphi \in S_0(V^s_{\infty,r}, V_{0,2})_0$ is the colimit map induced by a set of morphisms $\{\varphi_g = [V^s_{g-1,r}, \phi_g]\}_{g\geq j}$ for some integer $j \geq 0$, with $\phi_g \in \mathrm{Aut}_{\mathcal{H}^{\geq 2}}(V^s_{g,r})$ for some integer $j \geq 0$ and making the following diagram commutative

$$\begin{tikzcd}
V_{0,2} \arrow[r, "\mathrm{id}_{V_{0,2}}"] \arrow[d, "\varphi_j"] & V_{0,2} \arrow[r, "\mathrm{id}_{V_{0,2}}"] \arrow[d, "\varphi_{j+1}"] & \cdots \arrow[r, "\mathrm{id}_{V_{0,2}}"] & V_{0,2} \arrow[r, "\mathrm{id}_{V_{0,2}}"] \arrow[d, "\varphi_g"] & \cdots \\
V^s_{j,r} \arrow[r, "\iota_{j+1}(V_{0,2})"] & V^s_{j+1,r} \arrow[r, "\iota_{j+2}(V_{0,2})"] & \cdots \arrow[r, "\iota_g(V_{0,2})"] & V^s_{g,r} \arrow[r, "\iota_{g+1}(V_{0,2})"] & \cdots .
\end{tikzcd}$$

In particular, each $\varphi_g = [V^s_{g-1,r}, \phi_g]$ belongs to $S_0(V^s_{g,r}, V_{0,2})_0$ by definition (see Notation 2.43), and so the morphisms $\{\varphi_g\}_{g\geq j}$ induce an element $[\varphi]$ in $\mathrm{colim}_g \left(S_0(V^s_{g,r}, V_{0,2})_0\right)$ by the universal property of that colimit. Assigning $\varphi \mapsto [\varphi]$ then clearly induces a well-defined map $\Upsilon\colon S_0(V^s_{\infty,r}, V_{0,2})_0 \to \mathrm{colim}_g \left(S_0(V^s_{g,r}, V_{0,2})_0\right)$. Writing the elementary explicit assignment of the map Ξ_0 on an element $\mathrm{colim}_g \left(S_0(V^s_{g,r}, V_{0,2})_0\right)$, it is a clear routine to check that is an Υ is the inverse of Ξ_0. Therefore, the map Ξ is a bijection on vertices, and so it is an isomorphism of simplicial complexes. Hence, composing Ξ with the isomorphisms (3.3) and (3.5), we obtain a simplicial complex isomorphism $S_g(V^s_{\infty,r}, V_{0,2})_\bullet \cong Y^A(V^s_{\infty,r}, R_\infty, x_0, x_1)$. The contractibility of $S_g(V^s_{\infty,r}, V_{0,2})_\bullet$ thus follows from that of $Y^A(V^s_{\infty,r}, R_\infty, x_0, x_1)$ proved in [HW10, Thm. 8.6]. ☐

Twisted homological stability for infinite handlebodies. We are now ready to prove the twisted homological stability result Theorem 3.14 for infinite handlebodies. We follow the method of Theorem 2.37, with the setting $(\mathcal{G}, \odot, 0) = (\mathfrak{H}_2^{\geq 2}, \natural, I_2)$, $(\mathcal{M}, \odot) = (\overline{\mathfrak{H}}_{r,s}^{\geq 2}, \natural)$, $A = V_{\infty,r}^s$ and $X = V_{0,2}$, where we write $\overline{\mathfrak{H}}_{r,s}^{\geq 2}$ for $\overline{\mathfrak{H}}_{V_{\infty,r}^s, V_{0,2}}^{\geq 2}$ for brevity. Beforehand, let us fix some notation. We recall that the category of groups is cocomplete (see [Mac98, §V.1.] for instance). We denote by $\mathcal{H}_{\infty,r}^s$ the group of automorphisms $\mathrm{Aut}_{\overline{\mathfrak{H}}_{r,s}^{\geq 2}}(V_{\infty,r}^s)$, i.e. equivalently the colimits of the groups $\{\mathcal{H}_{g,r}^s\}_{g\geq 0}$ with respect to the maps $\{\iota_g(V_{0,r}^s)\}_{g\geq 0}$; see Definition 2.14. We write $\rho_{\infty,g,r}^s \colon \mathrm{Aut}_{\overline{\mathfrak{H}}_{r,s}^{\geq 2}}(V_{\infty,r}^s \# V_{0,2}^{\# g}) \to \mathrm{Aut}_{\overline{\mathfrak{H}}_{r,s}^{\geq 2}}(V_{\infty,r}^s \# V_{0,2}^{\# g+1})$ for the colimit of the stabilisation maps $\{\rho_{i,r}^s\}_{i\geq 0}$ (see (2.17)) with respect to the maps $\{\iota_j(V_{0,r}^s \# V_{0,2}^{\# g})\}_{j\geq 0}$. Finally, let $d_{\infty,g,r}^s$ denote the canonical morphism $[\mathrm{id}_{V_{\infty,r}^s \# V_{0,2}^{\# g}}, V_{0,2}]$ of $[\overline{\mathfrak{H}}_{r,s}^{\geq 2}, \mathfrak{H}_2^{\geq 2}\rangle$.

Theorem 3.14 *We fix integers $r \geq 2$ and $s \geq 0$. Let $F \colon [\overline{\mathfrak{H}}_{r,s}^{\geq 2}, \mathfrak{H}_2^{\geq 2}\rangle \to \mathbf{Ab}$ be a coefficient system of degree d. For any $i \geq 0$, the map*

$$(\rho_{\infty,0,r}^s; F(d_{\infty,1,r}^s))_i \colon H_i(\mathcal{H}_{\infty,r}^s; F(V_{\infty,r}^s)) \to H_i(\mathrm{Aut}_{\overline{\mathfrak{H}}_{r,s}^{\geq 2}}(V_{\infty,r}^s \# V_{0,2}); F(V_{\infty,r}^s \# V_{0,2})).$$

is an isomorphism.

Proof. First, we note that the same argument as in the proof of Proposition 3.8 works to show that the map $\pi \colon |W_g(V_\infty, V_{0,2})_\bullet| \to |S_g(V_\infty, V_{0,2})_\bullet|$ (see Notation 2.43) is a homeomorphism for each $g \geq 1$. Hence it follows from Theorem 3.13 that the semi-simplicial set $W_g(V_{\infty,r}^s, V_{0,2})_\bullet$ is contractible for each $g \geq 1$, and so it is k-connected for any $k \geq 0$. Moreover, the necessary hypotheses of having no zero-divisors, $\mathrm{Aut}_{\mathfrak{H}_2^{\geq 2}}(I_2) = \{\mathrm{id}_{I_2}\}$ and local homogeneity at $(V_{\infty,r}^s, V_{0,2})$ are verified in Propositions 2.7 and 2.45(c) respectively. Therefore, by applying Theorem 2.37, we deduce that for any $i \geq 0$, there exists $g \gg i+d$ such that the map $(\rho_{\infty,g,r}^s; F(d_{\infty,g,r}^s))_i$ of the form

$$H_i(\mathrm{Aut}_{\overline{\mathfrak{H}}_{r,s}^{\geq 2}}(V_{\infty,r}^s \# V_{0,2}^{\# g}); F(V_{\infty,r}^s \# V_{0,2}^{\# g})) \to H_i(\mathrm{Aut}_{\overline{\mathfrak{H}}_{r,s}^{\geq 2}}(V_{\infty,r}^s \# V_{0,2}^{\#(g+1)}); F(V_{\infty,r}^s \# V_{0,2}^{\# g+1}))$$

is an isomorphism. Then, we deduce from the isomorphism (2.7) of Lemma 2.19 that we have an isomorphism $V_{\infty,r}^s \# V_{0,2}^{\# g} \cong V_{\infty,r}^s$ for each $g \geq 0$, and thus also $V_{\infty,r}^s \# V_{0,2}^{\#(g+1)} \cong V_{\infty,r}^s \# V_{0,2}$. By the functoriality of F, we also have isomorphisms $F(V_{\infty,r}^s \# V_{0,2}^{\# g}) \cong F(V_{\infty,r}^s)$ and similarly $F(V_{\infty,r}^s \# V_{0,2}^{\# g+1}) \cong F(V_{\infty,r}^s \# V_{0,2})$, proving the theorem. $\square$

3.2.2. Proof of Theorem 3.9

We can now prove Theorem 3.9 from Theorem 3.1 and Theorem 3.14.

$V_{1,1}\natural(-)$-stabilisation version of Theorem 3.1. Actually, we need an alternative version of Theorem 3.1, where we stabilise with $V_{1,1}$ on the left; see Theorem 3.15. Namely, we now work in the category $\langle \mathfrak{H}_1^{\geq 1}, \mathfrak{H}_{r,s}^{\geq 1}]$ defined by the Quillen bracket construction (see §2.2.2) applied to $(\mathfrak{H}_{r,s}^{\geq 1}, \natural)$ seen as a *left*-module over the braided monoidal category $(\mathfrak{H}_1^{\geq 1}, \natural, I_1)$ (see §2.2.1). The stabilisation map is defined by the map $\bar{\sigma}_{g,r}^s := \mathrm{id}_{V_{1,1}}\natural(-) \colon \mathcal{H}_{g,r}^s \hookrightarrow \mathcal{H}_{g+1,r}^s$, i.e. analogously to (2.16) by extending by the identity on the left, for which Proposition 2.44 repeats mutatis mutandis. Also, we write $\bar{c}_{g,1}$ for the canonical morphism $[V_{1,1}, \mathrm{id}_{V_{g+1,r}^s}] \colon V_{g,r}^s \to V_{g+1,r}^s$ of $\langle \mathfrak{H}_1^{\geq 1}, \mathfrak{H}_{r,s}^{\geq 1}]$. Moreover, the notion of finite degree coefficient system from Definition 2.22 as well as the examples of §2.3.1 adapt verbatim for functors of the form $\langle \mathfrak{H}_1^{\geq 1}, \mathfrak{H}_{r,s}^{\geq 1}] \to \mathbf{Ab}$.

Theorem 3.15 *We fix integers $r \geq 1$ and $s \geq 0$. Let $F \colon \langle \mathfrak{H}_{r,s}^{\geq 1}, \mathfrak{H}_1^{\geq 1}] \to \mathbf{Ab}$ be a coefficient system of degree d at 0. Then*

$$(\bar{\sigma}_{g,r}^s; F(\bar{c}_{g,1}))_i \colon H_i(\mathcal{H}_{g,r}^s; F(V_{g,r}^s)) \to H_i(\mathcal{H}_{g+1,r}^s; F(V_{g+1,r}^s))$$

is an isomorphism for $i \leq \frac{g-1}{2} - d - 1$. If F is split, it is an isomorphism for $i \leq \frac{g-1-d}{2} - 1$.

Proof. First of all, the work of §2.4.1 repeats verbatim to provide an analogous version of Theorem 2.37 for a functor $F\colon \langle \mathfrak{H}^{\geq 1}_{r,s}, \mathfrak{H}^{\geq 1}_{1}] \to \mathbf{Ab}$ and a suitable corresponding semi-simplicial set $\bar{W}_g(V^s_{0,r}, V_{1,1})_\bullet$. Then, the hypotheses of having no zero-divisors, $\mathrm{Aut}_{\mathfrak{H}^{\geq 1}_1}(I_1) = \{\mathrm{id}_{I_1}\}$ are done in Proposition 2.7, while the alternative version of local homogeneity at $(V^s_{0,r}, V_{1,1})$ in this context is checked by adapting mutatis mutandis Proposition 2.45(a). Also, the work of §3.1.1 carries over mutatis mutandis as well as the proof of Proposition 3.8, which proves that the semi-simplicial set $\bar{W}_g(V^s_{0,r}, V_{1,1})_\bullet$ is $\frac{g-3}{2}$-connected for each $g \geq 1$. The result thus follows from the adapted version of Theorem 2.37. $\square$

Furthermore, we need the following commutativity result to prove Theorem 3.9:

Lemma 3.16 *For each $g \geq 0$, there is a commutative diagram of group injections*

$$\begin{tikzcd} \mathcal{H}^s_{g,r} \arrow[r, hook, "\bar{\sigma}^s_{g,r}"] \arrow[d, hook, "\rho^s_{g,r}"'] & \mathcal{H}^s_{g+1,r} \arrow[d, hook, "\rho^s_{g+1,r}"] \\ \mathcal{H}^s_{g+1,r} \arrow[r, hook, "\bar{\sigma}^s_{g+1,r}"] & \mathcal{H}^s_{g+2,r}. \end{tikzcd} \tag{3.6}$$

Proof. We consider the canonical inclusions $\mathfrak{i}_1(g)\colon V^s_{g,r} \hookrightarrow V_{1,1}\natural V^s_{g,r}$ and $\mathfrak{i}_2(g)\colon V^s_{g,r} \hookrightarrow V^s_{g,r}\# V_{0,2}$. For an element $\phi \in \mathcal{H}^s_{g,r}$, recall that a representative diffeomorphism $\tilde{\phi}$ is isotopic to the identity in a neighbourhood of the gluing discs. By definition, the morphisms $\bar{\sigma}^s_{g,r}$ and $\rho^s_{g,r}$ consists in extending (up to isotopy) ϕ by the identity over the complement of the canonical inclusions $\mathfrak{i}_1(g)$ and $\mathfrak{i}_2(g)$ respectively. Then we note that the extension along $\mathfrak{i}_1(g)$ and then that along $\mathfrak{i}_2(g+1)$ is the same as extending along $\mathfrak{i}_2(g)$ and then along $\mathfrak{i}_1(g+1)$, because in both cases we simply prolongate isotopy classes of diffeomorphisms by the identity in regions which have disjoint support up to isotopy. Therefore, we have an equality $(\rho^s_{g+1,r} \circ \bar{\sigma}^s_{g,r})(\phi) = (\mathrm{id}_{V_{1,1}}\natural\phi)\#\mathrm{id}_{V_{0,2}} = \mathrm{id}_{V_{1,1}}\natural(\phi\#\mathrm{id}_{V_{0,2}}) = (\bar{\sigma}^s_{g+1,r} \circ \rho^s_{g,r})(\phi)$ for each $\phi \in \mathcal{H}^s_{g,r}$, and so the diagram (3.6) is commutative. $\square$

Proving Theorem 3.9. We now use a trick from [HW10, §8.4] to prove Theorem 3.9. We fix an integer $i \geq 0$. By the commutativity of the diagrams (3.6) (for the groups) and (2.11) (for the coefficients) Definition 2.28 and the functoriality of group homology (see [Bro94, Chap. III, §8] for instance), we obtain the following commutative diagram:

$$\begin{tikzcd} H_i(\mathcal{H}^s_{0,r}; F(V^s_{0,r})) \arrow[r, "\Psi^s_{0,r}"] \arrow[d, "(\rho^s_{0,r}{,}d^s_{0,r})_*"] & \cdots \arrow[r, "\Psi^s_{g-1,r}"] & H_i(\mathcal{H}^s_{g,r}; F(V^s_{g,r})) \arrow[r, "\Psi^s_{g,r}"] \arrow[d, "(\rho^s_{g,r}{,}d^s_{g,r})_*"] & H_i(\mathcal{H}^s_{g+1,r}; F(V^s_{g+1,r})) \arrow[r, "\Psi^s_{g+1,r}"] \arrow[d, "(\rho^s_{g+1,r}{,}d^s_{g+1,r})_*"] & \cdots \\ H_i(\mathcal{H}^s_{1,r}; F(V^s_{1,r})) \arrow[r, "\Psi^s_{1,r}"] & \cdots \arrow[r, "\Psi^s_{g,r}"] & H_i(\mathcal{H}^s_{g+1,r}; F(V^s_{g+1,r})) \arrow[r, "\Psi^s_{g+1,r}"] & H_i(\mathcal{H}^s_{g+2,r}; F(V^s_{g+2,r})) \arrow[r, "\Psi^s_{g+2,r}"] & \cdots \end{tikzcd}$$

where $\Psi^s_{g,r} = (\bar{\sigma}^s_{g,r}, \bar{c}_{g,1})_i$. By Theorem 3.15, the map $(\bar{\sigma}^s_{g,r}, \bar{c}_{g,1})_i$ is an isomorphism for $i \leq \frac{g-1}{2} - r - 1$. Also, we recall from §2.3.2 and Proposition 2.31 that the double coefficient system F induces by definition a coefficient system $\overline{F}\colon [\overline{\mathfrak{H}}^{\geq 2}_{V^s_{\infty,r}, V_{0,2}}, \mathfrak{H}^{\geq 2}_{V_{0,2}}\rangle \to \mathbf{Ab}$, whose assignment on $[\mathrm{id}_{V^s_{\infty,r}\# V_{0,2}}, V_{0,2}]$ is defined as the colimit of the vertical abelian group maps defined by diagram (2.11) for $m = n - 1 = 0$. Since twisted group homology commutes with filtered colimits (as a consequence of [Wei94, Th. 2.6.15] for instance), we deduce that the colimit of the vertical arrows in the above diagram is the isomorphism $(\rho^s_{\infty,0,r}; \overline{F}(d^s_{\infty,1,r}))_i$ of Theorem 3.14. Hence $(\rho^s_{g,r}, d^s_{g,r})_*$ is an isomorphism once $(\bar{\sigma}^s_{g,r}, \bar{c}_{g,1})_i$ has reached its range of stability, whence the result of Theorem 3.9.

3.3. Independence of number of marked discs

Combining Theorems 3.1 and 3.9, we may now prove the twisted homological result of Theorem B. In order to state it, let us start by fixing some notation. For all §3.3, we fix integers $r \geq 1$ and $s \geq 0$. Let $\bar{\natural} := \bar{\natural}_1$ and $\bar{\#} := \bar{\natural}_2$ be the operations analogous to $\natural = \natural_1$ and $\# = \natural_2$ respectively where we glue handlebodies along the full marked discs rather than just along half-discs Definition 2.5, i.e. $V_1 \natural_r V_2$ with $r \in \{1,2\}$ is the adjunction space $V_1 \cup_{\bar{f}_1 \sim \bar{f}_2} V_2$ where $\bar{f}_k \colon \sqcup_r \mathbb{D}^2 \hookrightarrow \mathrm{Im}(i_k)$ denotes the canonical inclusion into the first r marked discs of V_k; see Figures 1.2 and 1.1.

Remark 3.17 The operations $\natural$ and $\#$ do not define monoidal structures on $\mathfrak{H}^{\geq 1}$ and $\mathfrak{H}^{\geq 2}$ respectively, which is the reason why we do not work with them in §3.1 and §3.2.

For each $g \geq 0$, repeating the framework of Proposition 2.44, we consider the injective group morphisms $\mu^s_{g,r} := (-)_g \natural \mathrm{id}_{V_{0,3}} \colon \mathcal{H}^s_{g,r} \hookrightarrow \mathcal{H}^s_{g,r+1}$ and $\nu^s_{g,r+1} := (-)_g \# \mathrm{id}_{V_{0,3}} \colon \mathcal{H}^s_{g,r+1} \hookrightarrow \mathcal{H}^s_{g+1,r}$ induced by extending the isotopy classes of diffeomorphisms by the identity on $V_{0,3}$, analogous to the injections $\sigma^s_{g,r}$ and $\rho^s_{g,r}$ (see (2.16) and (2.17)). Furthermore, following Notation 2.32, we write $m^s_{g,r}$ and $n^s_{g,r+1}$ for the equivalence classes of pairs $[\mathrm{id}_{V^s_{g,r+1}}, V_{0,3}]_{\natural} \colon V^s_{g,r} \to V^s_{g,r+1}$ and $[\mathrm{id}_{V^s_{g,r}}, V_{0,3}]_{\#} \colon V^s_{g,r+1} \to V^s_{g+1,r}$ respectively, where the pair $(\phi, V_{0,3})$ is equivalent to $(\mathrm{id}_{V^s_{g,r+1}}, V_{0,3})$ (resp. $(\mathrm{id}_{V^s_{g+1,r}}, V_{0,3})$) if there exists an isomorphism $\chi \in \mathcal{H}_{0,3}$ such that $\phi = \mathrm{id}_{V^s_{g,r+1}} \natural \chi$ (resp. $\phi = \mathrm{id}_{V^s_{g+1,r}} \# \chi$). Since the classes of the diffeomorphisms in the handlebody groups are isotopic to the identity on a collar neighbourhood of each marked disc, it is a routine to check from these definitions that we have equalities of the following group morphisms for each $g \geq 0$: For each $g \geq 0$, we have equalities

$$\sigma^s_{g,r} = \nu^s_{g,r+1} \circ \mu^s_{g,r} \quad \text{and} \quad c^s_{g,r} = n^s_{g,r+1} \circ m^s_{g,r} \tag{3.7}$$

while

$$\rho^s_{g,r} = \mu^s_{g,r} \circ \nu^s_{g,r+1} \quad \text{and} \quad d^s_{g,r} = m^s_{g,r} \circ n^s_{g,r+1}. \tag{3.8}$$

We deduce the following more detailed version of Theorem B:

Corollary 3.18 *For integers $r \geq 1$ and $s \geq 0$, let $F \colon \mathfrak{H} \to \mathbf{Ab}$ be a coefficient bisystem at $V^s_{0,r}$ of degree $d \geq 0$ (see Definition 2.28). Then both of the maps*

$$(\mu^s_{g,r}; F(m^s_{g,r}))_i \colon H_i(\mathcal{H}^s_{g,r}; F(V^s_{g,r})) \to H_i(\mathcal{H}^s_{g,r+1}; F(V^s_{g,r+1})),$$
$$(\nu^s_{g,r+1}; F(n^s_{g,r+1}))_i \colon H_i(\mathcal{H}^s_{g,r+1}; F(V^s_{g,r+1})) \to H_i(\mathcal{H}^s_{g,r}; F(V^s_{g,r})),$$

are isomorphisms for $i \leq \frac{g-1}{2} - d - 1$. If both coefficient systems are split, they are isomorphisms for $i \leq \frac{g-1-d}{2} - 1$.

Proof. By Theorem 3.1, it follows from the decomposition of an isomorphism into two composite factors and the equalities (3.7) that $(\mu^s_{g,r}; F(m^s_{g,r}))_i$ is injective and that $(\nu^s_{g,r+1}; F(n^s_{g,r+1}))_i$ is surjective for the ranges stated in the result. Similarly, we deduce from Theorem 3.9 combined with the equalities (3.8) that $(\mu^s_{g,r}; F(m^s_{g,r}))_i$ is surjective and that $(\nu^s_{g,r+1}; F(n_{g,r+1}))_i$ is injective in the same ranges, whence the result. ☐

Example 3.19 Assigning $s = 0$ and $F = \mathbb{Z}$, Corollary 3.18 corresponds to [HW10, Th. 1.8(ii)] and may be seen as a generalisation of that previous result to any $s \geq 0$ and to twisted coefficients given by coefficient bisystems. For instance, as explained in Example 2.30, the functors $T^d H_\Sigma$ and $T^d H_\mathcal{V}$ define coefficient bisystems at $V^s_{0,r}$ of degree $d \geq 0$, and thus Corollary 3.18 applies to these functors.

Remark 3.20 The theorem [HW10, Th. 1.8(ii)] also shows that the map $(\nu_{g,1}; \mathbb{Z})_i$ (i.e. $s = r = 0$ and $F = \mathbb{Z}$) is an isomorphism for $g \geq 2i + 4$. However, Corollary 3.18 does not hold for $r = 0$ with finite degree coefficient bisystem. Indeed, Ishida and Sato [IS17, Th. 1.1-1.2] provide a counterexample, by showing that $(\nu_{g,1}; H_\Sigma(n^s_{g,r}))_1$ is not an isomorphism for $g \geq 4$.

Remark 3.21 Analogous results to those of Corollary 3.18 above for mapping class groups of surfaces are proven in [Bol12, Th. 4.13], but under the stronger assumption that the involved coefficient systems are *split* (following the terminology of Definition 2.22, see [Bol12, Def. 4.2]). However, the framework of §2 and studies of §3 leading to Corollary 3.18 may be repeated mutatis mutandis for mapping class groups of surfaces, and so this assumption appears to us to be unnecessary.

4. Moduli spaces of handlebodies with tangential structures

In this section, we apply the results of §3 to prove Theorem C (see §4.2) after some preliminary recollections on tangential structures (see §4.1).

4.1. Tangential structures

We start by recalling the definition and some basics on tangential structures. Throughout the section, $\mathrm{O}(d)$ denotes the group of $n \times n$ orthogonal matrices and $\mathrm{BO}(d)$ its classifying space.

Definition 4.1 For $d \geq 1$, a d-tangential structure is a fibration $\theta\colon \mathrm{B} \to \mathrm{BO}(d)$. For M an m-manifold with $m \leq d$, a θ-structure on M is a lift

$$\begin{tikzcd} & \mathrm{B} \arrow[d, "\theta"] \\ M \arrow[ur, dashed, "\ell"] \arrow[r, "\ell_0"'] & \mathrm{BO}(d), \end{tikzcd}$$

where the horizontal arrow ℓ_0 is a classifying map for the bundle $TM \oplus \mathbb{R}^{\oplus(d-m)}$, where TM is the tangent bundle on M, $\mathbb{R}$ denotes the trivial line bundle on M and $\oplus$ denotes the Whitney sum of vector bundles.

Now, let us define a moduli space of manifolds equipped with a θ-structure restrict to θ-structures on M which satisfy some condition with respect to a submanifold of its boundary, to allow us to stabilise. Considering a topological group G, we denote by $\mathbf{Top}_G$ the category of G-spaces and by EG a functorial model of the total space of the universal principal G-bundle $EG \to BG$. Then, for X a G-space, the *homotopy orbit* functor $(-)_{hG} : \mathbf{Top}_G \to \mathbf{Top}$ is defined by $(-)_{hG} := (- \times EG)_G$, i.e. the set of the orbits of $(-) \times EG$ under the action of G. We use this to make the following definition:

Definition 4.2 Let $\theta\colon B \to \mathrm{BO}(d)$ be a tangential structure, M be a d-manifold and $\ell_R : R \to \mathrm{B}$ be a θ-structure on $R \subset \partial M$ a $(d-1)$-submanifold of the boundary. We define $\mathrm{Bun}^\theta(M;\ell_R)$ to be the space of θ-structures on M (topologised as a subspace of $\mathrm{Map}(M,\mathrm{B})$) which restrict to ℓ_R on R. Also, let $\mathrm{BDiff}^\theta(M;\ell_R)$ be the homotopy orbit space $\mathrm{Bun}^\theta(M;\ell_R)_{h\mathrm{Diff}(M,R)}$, where a diffeomorphism $f \in \mathrm{Diff}(M,R)$ acts on $\mathrm{Bun}^\theta(M;\ell_R)$ by precomposition.

Remark 4.3 The notation $\mathrm{Bun}^\theta(M;\ell_R)$ comes from the fact that θ-structures correspond to vector bundle maps from the tangent bundle of M to the pullback along θ of the universal d-vector bundle over $\mathrm{BO}(d)$. Let us also remark that the notation $\mathrm{BDiff}^\theta(M;\ell_R)$ is slightly misleading, as there is generally no group $\mathrm{Diff}^\theta(M;\ell_R)$ such that $\mathrm{BDiff}^\theta(M;\ell_R)$ is its classifying space.

4.2. Proof of Theorem C

Now let us restrict to the case where $d = 3$, B is a *pointed* space and $\theta : \mathrm{B} \to \mathrm{BO}(3)$ is a pointed map (with the basepoint of $\mathrm{BO}(3)$ induced by the identity element of the group), $M = V$ where $(V,r,s,i,j) \in \mathfrak{H}$ is a compact 3-dimensional handlebody and $R = \mathrm{Im}(i) \sqcup \mathrm{Im}(j)$ (see §2.1). Recall that $V \cong V^s_{g,r}$, for some $g \geq 0$, $r \geq 1$ and $s \geq 0$, and so we write $R(V) = R^s_{g,r} = \mathcal{D}_r \sqcup \mathcal{P}_s \subset \partial V_g$. Since the tangent bundle TV_g is trivial as V_g is parallellisable, the constant map to the basepoint of $\mathrm{BO}(3)$ is a classifying map for TV_g. The constant map to the basepoint thus defines a canonical θ-structure $\ell_0(V_g) : V_g \to \mathrm{B}$. Similarly, we denote by $\ell_0(R^s_{g,r})$ also the constant θ-structure on $R^s_{g,r}$.

In general, the space $\mathrm{Bun}^\theta(V_g;\ell_0(R^s_{g,r}))$ may be disconnected, so we restrict our attention to a path component. We make the following definition:

Definition 4.4 We denote by $\mathrm{Bun}^\theta_0(V_g;\ell_0(R^s_{g,r}))$ the path component of $\ell_0(V_g)$ in the space $\mathrm{Bun}^\theta(V_g;\ell_0(R^s_{g,r}))$. Since $\mathrm{Bun}^\theta_0(V_g;\ell_0(R^s_{g,r}))$ is clearly fixed by the $\mathrm{Diff}(V_g;R^s_{g,r})$-action, we define

$$\mathrm{BDiff}^\theta_0(V_g;\ell_0(R^s_{g,r})) := \mathrm{Bun}^\theta_0(V_g,\ell_0(R^s_{g,r}))_{h\mathrm{Diff}(V_g,R^s_{g,r})}.$$

Since the $\mathrm{Diff}(V_g; R^s_{g,r})$-action on $\mathrm{Bun}^\theta(V_g; \ell_0(R^s_{g,r}))$ is continuous, the space $\mathrm{BDiff}^\theta_0(V_g; \ell_0(R^s_{g,r}))$ is the same as the path component of the orbit of $\{\ell_0(V_g)\} \times \mathrm{EDiff}(V_g, R^s_{g,r})$ in $\mathrm{BDiff}^\theta(V_g; \ell_0(R^s_{g,r}))$. We define a map

$$\sigma^{\theta,s}_{g,r} : \mathrm{Bun}^\theta_0(V_g; \ell_0(R^s_{g,r})) \to \mathrm{Bun}^\theta_0(V_{g+1}; \ell_0(R^s_{g+1,r})) \tag{4.1}$$

by $\ell \mapsto \ell \natural \ell_0(V_{1,1})$, where the operation $\natural$ is induced from the monoidal structure $\natural_1$ of Definition 2.5 on the source. Similarly, we can define a map

$$\mu^{\theta,s}_{g,r} : \mathrm{Bun}^\theta_0(V_g; \ell_0(R^s_{g,r})) \to \mathrm{Bun}^\theta_0(V_g; \ell_0(R^s_{g,r+1})) \tag{4.2}$$

by $\ell \mapsto \ell \bar{\natural} \ell_0(V_{0,3})$, where the operation $\bar{\natural}$ is induced from that on the source introduced in §3.3. These maps are both $\mathrm{Diff}(V_g; R^s_{g,r})$-equivariant, with the action on the target defined via the usual stabilisation maps $\mathrm{Diff}(V_g; R^s_{g,r}) \to \mathrm{Diff}(V_{g+1}; R^s_{g+1,r})$ and $\mathrm{Diff}(V_g; R^s_{g,r}) \to \mathrm{Diff}(V_g; R^s_{g,r+1})$ respectively. Applying the homotopy orbits functor thus gives us stabilisation maps

$$\breve{\sigma}^{\theta,s}_{g,r} : \mathrm{BDiff}^\theta_0(V_g; \ell_0(R^s_{g,r})) \to \mathrm{BDiff}^\theta_0(V_{g+1}; \ell_0(R^s_{g+1,r})) \tag{4.3}$$

and

$$\breve{\mu}^{\theta,s}_{g,r} : \mathrm{BDiff}^\theta_0(V_g; \ell_0(R^s_{g,r})) \to \mathrm{BDiff}^\theta_0(V_g; \ell_0(R^s_{g,r+1})). \tag{4.4}$$

Let us recall the statement of Theorem C.

Theorem 4.5 *If $\theta : \mathrm{B} \to \mathrm{BO}(3)$ is a pointed tangential structure, the maps $\breve{\sigma}^{\theta,s}_{g,r}$ and $\breve{\mu}^{\theta,s}_{g,r}$ induce isomorphisms in homology with rational coefficients in degrees $* \leq \frac{g-3}{2}$.*

The main ideas of our proof are the same as those of [Ran18, Thm. 4.2(i)], using the key Proposition 4.6 below. Let us start by explaining the main idea of the proof, to put these lemmas into context. First, we note that by [LS25, Lem. 2.1], applying the homotopy orbits functor to the trivial map $\mathrm{Bun}^\theta_0(V_g; \ell_0(R^s_{g,r})) \to *$, we get a fibre bundle

$$\mathrm{BDiff}^\theta_0(V_g; \ell_0(R^s_{g,r})) \to \mathrm{BDiff}(V_g; R^s_{g,r}), \tag{4.5}$$

with fibre $\mathrm{Bun}^\theta_0(V_g; \ell_0(R^s_{g,r}))$. Furthermore, we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Bun}^\theta_0(V_g; \ell_0(R^s_{g,r})) \times E\mathrm{Diff}(V_g; R^s_{g,r}) \arrow[r] \arrow[d] & \mathrm{Bun}^\theta_0(V_{g+1}, \ell_0(R^s_{g+1,r})) \times E\mathrm{Diff}(V_{g+1}, R^s_{g,r}) \arrow[d] \\
E\mathrm{Diff}(V_g; R^s_{g,r}) \arrow[r] & E\mathrm{Diff}(V_{g+1}, R^s_{g+1,r})
\end{tikzcd} \tag{4.6}$$

where the vertical maps are just the projections, the horizontal maps are defined by the stabilisation map $\sigma^{\theta,s}_{g,r}$ (see (4.1)) and the map of total spaces of universal bundles induced by the usual stabilisation map $\mathrm{Diff}(V_g; R^s_{g,r}) \to \mathrm{Diff}(V_{g+1}; R^s_{g,r})$ (given by extending by the identity on the complement V_1). The maps in the diagram are all obviously $\mathrm{Diff}(V_g; R^s_{g,r})$-equivariant, so we get an induced map on the $\mathrm{Diff}(V_g; R^s_{g,r})$-orbits. Passing to the $\mathrm{Diff}(V_{g+1}; R^s_{g+1,r})$-orbits in the targets of the horizontal maps, the induced horizontal maps are the stabilisation maps $\breve{\sigma}^{\theta,s}_{g,r}$ and $\breve{\sigma}^s_{g,r}$, so we get an induced a map of fibrations

$$\begin{tikzcd}
\mathrm{BDiff}^\theta_0(V_g; \ell_0(R^s_{g,r})) \arrow[r, "\breve{\sigma}^{\theta,s}_{g,r}"] \arrow[d] & \mathrm{BDiff}^\theta_0(V_{g+1}; \ell_0(R^s_{g+1,r})) \arrow[d] \\
\mathrm{BDiff}(V_g; R^s_{g,r}) \arrow[r, "\breve{\sigma}^s_{g,r}"] & \mathrm{BDiff}(V_{g+1}, R^s_{g+1,r}),
\end{tikzcd} \tag{4.7}$$

where the restriction to the fibres $\mathrm{Bun}^\theta_0(V_g; \ell_0(R^s_{g,r})) \to \mathrm{Bun}^\theta_0(V_{g+1}; \ell_0(R^s_{g+1,r}))$ is the map $\sigma^{\theta,s}_{g,r}$. Similarly, the stabilisation maps $\mu^{\theta,s}_{g,r}$ and $\mu^s_{g,r}$ induce a map of fibrations

$$\begin{tikzcd}
\mathrm{BDiff}^\theta_0(V_g; \ell_0(R^s_{g,r})) \arrow[r, "\breve{\mu}^{\theta,s}_{g,r}"] \arrow[d] & \mathrm{BDiff}^\theta_0(V_g; \ell_0(R^s_{g,r+1})) \arrow[d] \\
\mathrm{BDiff}(V_g; R^s_{g,r}) \arrow[r, "\breve{\mu}^s_{g,r}"] & \mathrm{BDiff}(V_g; R^s_{g,r})
\end{tikzcd} \tag{4.8}$$

where the restriction to the fibres $\mathrm{Bun}^\theta_0(V_g;\ell_0(R^s_{g,r}))\to\mathrm{Bun}^\theta_0(V_g;\ell_0(R^s_{g,r+1}))$ is the map $\mu^{\theta;s}_{g,r}$. To prove that the maps on the total spaces in (4.7) and (4.8) induce homology isomorphisms in the range in Theorem 4.5, we use the Serre spectral sequences associated to these fibrations and deduce the theorem from twisted homological stability for the base spaces, with coefficients given by the homology of the fibres. For any $q\geq 0$, we therefore show that the homology groups $H_q(\mathrm{Bun}^\theta_0(V;\ell_0(R^s_{g,r}));\mathbb{Q})$ define the components of a functor $F_q\colon\mathfrak{H}\to\mathbf{Ab}$ and show that this is a coefficient bisystem (in the sense of Definition 2.28); see Proposition 4.6. This allows us to apply Theorems 3.1 and 3.18 to deduce the desired twisted homological stability.

A key coefficient bisystem. Let us start by defining the functor $F_q\colon\mathfrak{H}\to\mathbf{Ab}$. Abusing notation going forward, let us for brevity write ℓ_0 for all boundary conditions, whenever they are clear from context. To define the functor F_q, we start by defining a functor $\mathrm{Bun}^\theta_0(-;\ell_0)\colon\mathfrak{H}\to\mathbf{hTop}$, where $\mathbf{hTop}$ denotes the "naive" homotopy category of spaces (i.e. where morphisms are homotopy classes of maps), by $\mathrm{Bun}^\theta_0(-;\ell_0)(V)=\mathrm{Bun}^\theta_0(V;\ell_0)$ and on morphisms as follows. For $\phi:V\to W$ an isomorphism in $\mathfrak{H}$ and $\tilde\phi$ a diffeomorphism representing ϕ, we define the morphism $\mathrm{Bun}^\theta_0(\phi;\ell_0)\colon\mathrm{Bun}^\theta_0(V;\ell_0)\to\mathrm{Bun}^\theta_0(W;\ell_0)$ to be the homotopy class of the map induced by the precomposition with $\tilde\phi^{-1}$. The functor F_q is then defined by the postcomposition of $\mathrm{Bun}^\theta_0(-;\ell_0)$ with the homology functor $H_q(-;\mathbb{Q})\colon\mathbf{hTop}\to\mathbf{Ab}$ (see [Hat02, §2.3] for instance).

Proposition 4.6 *For any $r\geq 2$ and $s\geq 0$, the functor $F_q\colon\mathfrak{H}\to\mathbf{Ab}$ is a coefficient bisystem at $V^s_{0,r}$ of degree q.*

Following Definition 2.28, the key to proving Proposition 4.6 is understanding the kernels and cokernels of the maps induced in homology by the stabilisation map for $V\in\mathfrak{H}^{\geq 1}$

$$\sigma^\theta_V:\mathrm{Bun}^\theta_0(V;\ell_0)\to\mathrm{Bun}^\theta_0(V\natural V_{1,1};\ell_0),\tag{4.9}$$

given by $\ell\mapsto\ell\natural\ell_0(V_{1,1})$, and by the stabilisation map for $W\in\mathfrak{H}^{\geq 2}$

$$\rho^\theta_W:\mathrm{Bun}^\theta_0(W;\ell_0)\to\mathrm{Bun}^\theta_0(W\#V_{0,2};\ell_0),\tag{4.10}$$

given by $\ell\mapsto\ell\#\ell_0(V_{0,2})$, where the operation $\#$ is induced from the monoidal structure $\#=\natural_2$ of Definition 2.5 on the source. It straightforwardly follows from the definition of the spaces of θ-structures (see Definition 4.2) that the monoidal structure $\natural$ (see Definition 2.5) induces a continuous map

$$\mathrm{Bun}^\theta_0(V;\ell_0)\times\mathrm{Bun}^\theta_0(V_{1,1};\ell_0)\xrightarrow{-\natural-}\mathrm{Bun}^\theta_0(V\natural V_{1,1};\ell_0)\tag{4.11}$$

and similarly, the monoidal structure $\#$ induces a continuous map

$$\mathrm{Bun}^\theta_0(W;\ell_0)\times\mathrm{Bun}^\theta_0(V_{0,2};\ell_0)\xrightarrow{-\#-}\mathrm{Bun}^\theta_0(V\#V_{0,2};\ell_0).\tag{4.12}$$

We also generically write $-\times\ell_0$ for the canonical inclusion into the left summand of a direct product (here ℓ_0 denotes the unique map from the initial object $\emptyset$ to any space of θ-structures, whose image is the constant map ℓ_0). All these maps are by definition compatible with the stabilisation maps, in the sense that the diagrams

$$\begin{tikzcd}
\mathrm{Bun}^\theta_0(V;\ell_0) \arrow[r, hook, "-\times\ell_0"] \arrow[dr, "\sigma^\theta_V"'] & \mathrm{Bun}^\theta_0(V;\ell_0)\times\mathrm{Bun}^\theta_0(V_{1,1};\ell_0) \arrow[d, "-\natural-"] \\
& \mathrm{Bun}^\theta_0(V\natural V_{1,1};\ell_0)
\end{tikzcd}\tag{4.13}$$

and

$$\begin{tikzcd}
\mathrm{Bun}^\theta_0(W;\ell_0) \arrow[r, hook, "-\times\ell_0"] \arrow[dr, "\rho^\theta_W"'] & \mathrm{Bun}^\theta_0(W;\ell_0)\times\mathrm{Bun}^\theta_0(V_{0,2};\ell_0) \arrow[d, "-\#-"] \\
& \mathrm{Bun}^\theta_0(W\#V_{0,2};\ell_0)
\end{tikzcd}\tag{4.14}$$

are both commutative. To understand the kernel and cokernel of the stabilisation maps (in homology), we use the following lemma:

Lemma 4.7 *For any $V \in \mathcal{H}^{\geq 1}$, $W \in \mathcal{H}^{\geq 2}$, the maps* (4.11) *and* (4.12) *are homotopy equivalences.*

Proof. We start by treating the first map in detail. Let D be the closed half disc along which V and $V_{1,1}$ are glued in $V\natural V_{1,1}$ and $\mathcal{N}_D \subset V\natural V_{1,1}$ a neighborhood of D, with a homeomorphism $\phi\colon [-1,1] \times D \to \mathcal{N}_D$ such that the restriction to $\{0\} \times D$ is the identity on D. For brevity, let us freely identify $\mathcal{N}_D$ with $[-1,1] \times D$ along ϕ. Identifying D with the right half of the unit disc, we define a self-map $f_t : \mathcal{N}_D \to \mathcal{N}_D$ for any $t \in I = [0,1]$ by

$$f_t(s,(x,y)) = (s,((1+(|s|-1)t)x,y)).$$

Let $H : I \times \mathcal{N}_D \to \mathcal{N}_D$ be the map defined by $H(t,-) = f_t(-)$. By construction, it is a homotopy between $\mathrm{id}_{\mathcal{N}_D}$ and f_1. Furthermore, we have $f_t(\pm 1,(x,y)) = (\pm 1,(x,y))$ for any $t \in I$. Hence, by extending f_t by the identity on the complement of $\mathcal{N}_D$, we define a self-map $\tilde{f}_t\colon V\natural V_{1,1} \to V\natural V_{1,1}$ and we extend H to a homotopy $\tilde{H}\colon I \times (V\natural V_{1,1}) \to V\natural V_{1,1}$ between $\mathrm{id}_{(V\natural V_{1,1})}$ and $\tilde{f}_1$. This induces a map

$$K\colon I \times \mathrm{Bun}_0^\theta(V\natural V_{1,1};\ell_0(R(V\natural V_{1,1}))) \to \mathrm{Bun}_0^\theta(V\natural V_{1,1};\ell_0(R(V\natural V_{1,1}))),$$

defined by $(t,\ell) \mapsto \ell \circ \tilde{H}(t,-)$. Note that the map K is well-defined since $\tilde{f}_t$ is homotopic to the identity map for each $t \in I$, and so the composite $\ell \circ \tilde{f}_t$ is homotopic ℓ for every $\ell \in \mathrm{Bun}_0^\theta(V\natural V_{1,1};\ell_0(R(V\natural V_{1,1})))$. Now, we consider the inclusion

$$\iota\colon \mathrm{Bun}_0^\theta(V\natural V_{1,1};\ell_0(R(V)\cup R(V_{1,1}))) \hookrightarrow \mathrm{Bun}_0^\theta(V\natural V_{1,1};\ell_0(R(V\natural V_{1,1})))$$

induced by the inclusion $R(V\natural V_{1,1}) \hookrightarrow R(V)\cup R(V_{1,1})$. Then K is a homotopy between the identity and $\iota \circ (- \circ \tilde{f}_1)$. If ℓ belongs to the subspace $\mathrm{Bun}_0^\theta(V\natural V_{1,1};\ell_0(R(V)\cup R(V_{1,1})))$, then $\ell \circ f_t$ is also in $\mathrm{Bun}_0^\theta(V\natural V_{1,1};\ell_0(R(V)\cup R(V_{1,1})))$ for each $t \in I$. Hence, by assigning $(t,\ell') \mapsto \ell' \circ \tilde{H}(t,-)$ for $(t,\ell') \in I \times \mathrm{Bun}_0^\theta(V\natural V_{1,1};\ell_0(R(V)\cup R(V_{1,1})))$, we also define a homotopy between the identity map on $\mathrm{Bun}_0^\theta(V\natural V_{1,1};\ell_0(R(V)\cup R(V_{1,1})))$ and $(- \circ \tilde{f}_1)\circ\iota$. Therefore ι and $- \circ \tilde{f}_1$ are homotopy inverses.

On the other hand, the map (4.11) lands in the subspace $\mathrm{Bun}_0^\theta(V\natural V_{1,1};\ell_0(R(V)\cup R(V_{1,1})))$, and the map

$$\mathrm{Bun}_0^\theta(V\natural V_{1,1};\ell_0(R(V)\cup R(V_{1,1}))) \to \mathrm{Bun}_0^\theta(V;\ell_0) \times \mathrm{Bun}_0^\theta(V_{1,1};\ell_0),$$

given by restriction to the two factors induced by the canonical embeddings $V_{1,1} \hookrightarrow V\natural V_{1,1}$ and $V \hookrightarrow V\natural V_{1,1}$, is an inverse, showing that $-\natural-$ is a homeomorphism onto its image. As $-\natural-$ is by definition the composition with the homotopy equivalence ι, we are done.

This proof repeats almost verbatim to prove the claim for the second sequence, by replacing $\mathcal{N}_D$ with a disjoint union of neighborhoods of the two half-discs along the gluing of $W\#V_{0,2}$, and then applying the same arguments as above. □

We can now use Lemma 4.7 to prove Proposition 4.6.

Proof of Proposition 4.6. We want to show that F_q satisfies all of the hypotheses of Definition 2.28. For brevity, let us write $A = V^s_{0,r}$, $X = V_{1,1}$ and start by verifying that the restriction of F_q to $\mathcal{H}^{\geq 1}_{A,X}$ can be extended to a coefficient system $F_q : [\mathcal{H}^{\geq 1}_{A,X}, \mathcal{H}^{\geq 1}_X\rangle \to \mathbf{Ab}$ of degree d at 0. We show this by first extending the functor $\mathrm{Bun}_0^\theta(-;\ell_0)\colon \mathcal{H}^{\geq 1}_{A,X} \to \mathbf{hTop}$ defined just before Proposition 4.6. Given a representative (ϕ, V_h) of a morphism $V \to W$ in $[\mathcal{H}^{\geq 1}_{A,X}, \mathcal{H}^{\geq 1}_X\rangle$ and a diffeomorphism $\tilde{\phi}$ representing ϕ, recall that (the homotopy class of) a map $\mathrm{Bun}_0^\theta(-;\ell_0)(\phi) : \mathrm{Bun}_0^\theta(V\natural V_h;\ell_0) \to \mathrm{Bun}_0^\theta(W;\ell_0)$. Furthermore, we use the stabilisation map

$$\tilde{\sigma}^h_V : \mathrm{Bun}_0^\theta(V;\ell_0) \to \mathrm{Bun}_0^\theta(V\natural V_h;\ell_0),$$

defined by assigning $\ell\natural\ell_0(V_h)\colon V\natural V_h \to \mathrm{B}$ to $\ell\colon V \to \mathrm{B}$, and we asssign

$$\mathrm{Bun}_0^\theta(-;\ell_0)(\phi,V_h) := \mathrm{Bun}_0^\theta(-;\ell_0)(\phi)\circ\tilde{\sigma}^h_V.$$

For two different representatives (ϕ,V_h) and (ϕ',V_h) of the same class of morphisms, recall that the diffeomorphisms ϕ and ϕ' differ only on V_h: since the θ-structure $\ell\natural\ell_0(V_h)$ is defined by the constant map $\ell_0(V_h)$ on V_h, the two representatives (ϕ, V_h) and (ϕ', V_h) induce the same map for the above assignment. Hence, we have a well-defined functor $\mathrm{Bun}_0^\theta(-;\ell_0)\colon [\mathfrak{H}^{\geq 1}_{A,X}, \mathfrak{H}^{\geq 1}_{X}\rangle \to \mathbf{hTop}$. Again, we define F_q as the postcomposition with the functor $H_1(-;\mathbb{Q}) : \mathbf{hTop} \to \mathbf{Ab}$.

Now, we show that for $q \geq 0$, the functor F_q is a finite degree coefficient system $[\mathfrak{H}^{\geq 1}_{A,X}, \mathfrak{H}^{\geq 1}_{X}\rangle \to \mathbf{Ab}$ of degree q. We prove this by induction on q. Since the space of θ-structures $\mathrm{Bun}_0^\theta(V;\ell_0)$ is connected, the functor F_0 is constant with value $\mathbb{Q}$ and so indeed a coefficient system of degree 0; see Example 2.23. Let us therefore assume the claim holds for F_k for all $k < q$. We note that for any $V \in \mathfrak{H}^{\geq 1}_{A,X}$, the stabilisation map $\sigma_V^\theta : \mathrm{Bun}_0^\theta(V;\ell_0) \to \mathrm{Bun}_0^\theta(V \natural X;\ell_0)$ and the functor F_q are defined precisely so that $H_q(\sigma_V^\theta;\mathbb{Q}) = F_q(\iota_X(V))$, where we recall that $\iota_X(V) := [\mathrm{id}, X] : V \to V \natural X$. Furthermore, we have a commutative diagram

$$\begin{tikzcd} \mathrm{Bun}_0^\theta(V;\ell_0) \arrow[r, hook, "-\times \ell_0"] \arrow[dr, "\sigma_V^\theta"'] & \mathrm{Bun}_0^\theta(V;\ell_0) \times \mathrm{Bun}_0^\theta(V_{1,1};\ell_0) \arrow[d, "-\natural-"] \\ & \mathrm{Bun}_0^\theta(V \natural V_{1,1};\ell_0) \end{tikzcd} \tag{4.15}$$

where the vertical map is a homotopy equivalence by Lemma 4.7. It follows from the Künneth theorem for topological spaces (see [Dol95, Cor. VI.12.10, Prop. VII.2.6] for instance), that there is a natural commutative diagram

$$\begin{tikzcd} F_q(V) \arrow[r] \arrow[dr, "F_q(\iota_X(V))"'] & \bigoplus_{i+j=q} F_i(V) \otimes F_j(X) \arrow[d, "\cong"] \\ & F_q(V \natural X), \end{tikzcd}$$

where the vertical arrow is the Künneth isomorphism and the horizontal arrow is the direct summand inclusion. Thus $\kappa_X(F_q) = 0$ and we have a natural isomorphism

$$\delta_X(F_q)(V) \cong \bigoplus_{i=0}^{q-1} F_i(V) \otimes F_{q-i}(X). \tag{4.16}$$

For each $1 \leq j \leq q$, we note that the functor $-\otimes F_j(X)\colon \mathbf{Ab} \to \mathbf{Ab}$ is exact, because the functor F_j takes values in $\mathbb{Q}$-vector spaces and so $F_j(X)$ is a flat abelian group. By the inductive assumption, it is then immediate that the coefficient system $F_i(-) \otimes F_{q-i}(X)$ is of finite degree at most i for each $0 \leq i \leq q-1$. Since finite degree coefficient systems are closed under direct sums (see [LS25, Lem. 1.6]), it follows from (4.16) that the coefficient system $\delta_X(F_q)$ is of finite degree at most $q-1$, and thereby finally that F_q is a finite degree coefficient system of degree q. Moreover, by repeating mutatis mutandis the above proof by stabilising on the left instead of on the right, we also check that the functor F_q defines a coefficient system $\langle \mathfrak{H}^{\geq 1}_{V_{1,1}}, \mathfrak{H}^{\geq 1}_{V,V_{1,1}}] \to \mathbf{Ab}$ of degree q, i.e. it satisfies Condition (ii) of Definition 2.28.

The proof that F_q satisfies Condition (i) of Definition 2.28 also follows from a similar argument as the above one checking Condition (ii), we thus give an overview and focus on the details that differ. First of all, let $W \in \mathfrak{H}^{\geq 2}$ and consider the restriction of F_q to $\mathfrak{H}^{\geq 2}_{W,Y}$. We extend this functor to $[\mathfrak{H}^{\geq 2}_{W,Y}, \mathfrak{H}^{\geq 2}_{Y}\rangle$ by assigning $F_q(\phi, Y^{\#h}) := H_q(F(\phi) \circ \rho_W^h;\mathbb{Z})$, where

$$\tilde{\rho}_W^h : \mathrm{Bun}_0^\theta(W;\ell_0) \to \mathrm{Bun}_0^\theta(W \# Y^{\#h};\ell_0)$$

is defined by assigning $(\ell\colon V \to \mathrm{B}) \mapsto (\ell \# \ell_0(Y^{\#h})\colon V \natural V_h \to \mathrm{B})$. We again show by induction that for $q \geq 0$, $F_q\colon [\mathfrak{H}^{\geq 2}_{W,Y}, \mathfrak{H}^{\geq 2}_{Y}\rangle \to \mathbf{Ab}$ is a coefficient system of finite degree q. The case $q = 0$ again follows from the connectivity of the space of θ-structures $\mathrm{Bun}_0^\theta(W;\ell_0)$. For the inductive step, we now use the commutative diagram

$$\begin{tikzcd} \mathrm{Bun}_0^\theta(W;\ell_0) \arrow[r, hook, "-\times \ell_0"] \arrow[dr, "\rho_W^\theta"'] & \mathrm{Bun}_0^\theta(W;\ell_0) \times \mathrm{Bun}_0^\theta(V_{0,2};\ell_0) \arrow[d, "-\#-"] \\ & \mathrm{Bun}_0^\theta(W \# V_{0,2};\ell_0) \end{tikzcd}$$

and Lemma 4.7. It again follows from the Künneth theorem that the $\kappa_Y(F_q) = 0$ and that $\delta_Y(F_q)$ is isomorphic to a direct sum of coefficient systems of degrees $\leq q-1$, proving that it is indeed of degree $q-1$ by [LS25, Lem. 1.6].

Finally, we show that F_q satisfies Condition (iii) of Definition 2.28. For any $V \in \mathfrak{H}^{\geq 2}$, it follows from the above definitions of the maps $\tilde{\sigma}^1_V$ and $\tilde{\rho}^1_V$ there is a commutative diagram of spaces

$$\begin{tikzcd} \mathrm{Bun}_0^\theta(V;\ell_0) \arrow[r, "\tilde{\rho}^1_V"] \arrow[d, "\tilde{\sigma}^1_V"] & \mathrm{Bun}_0^\theta(V\#Y;\ell_0) \arrow[d, "\tilde{\sigma}^1_V"] \\ \mathrm{Bun}_0^\theta(X\natural V;\ell_0) \arrow[r, "\tilde{\rho}^1_V"] & \mathrm{Bun}_0^\theta(X\natural V\#Y;\ell_0), \end{tikzcd}$$

and applying the functor $H_q(-;\mathbb{Q})$ gives us the desired diagram of abelian groups. □

Remark 4.8 Going through the proof, it is easy to see that the coefficient systems considered as part of the bisystem F_q are actually *split*, but we will not use this fact, as it is does not improve the stable range of Theorem 4.5, and have therefore left it out of the proof.

Remark 4.9 Replacing the rational homology functor $H_q(-;\mathbb{Q})\colon \mathbf{hTop} \to \mathbf{Ab}$ with the functor $H_q(-;\mathbb{Z})$, most of the proof of this proposition goes through in the same way, so let us point out where the assumption of rational coefficients is used. The issue is where we use that the functor $-\otimes F_j(X)$ is exact, which is not generally true over the integers. This can be resolved by adding the assumption that F_i is *split* (see [LS25, Lem. 1.7]), but leads to another issue, namely that over the integers the Künneth isomorphism only describes $F_q(V\natural X)$ as a (non-naturally split) extension of split coefficient systems. Such extensions are not necessarily split themselves, so this breaks the inductive step.

Proving Theorem C. Finally, putting our intermediate results together, we finish the proof of Theorem C:

Proof of Theorem 4.5. Let $E(g,r,s)$ denote the homological Serre spectral sequence associated to the fibre bundle (4.5), which is of the form

$$E(g,r,s)^2_{p,q} = H_p(\mathrm{BDiff}(V_g;R^s_{g,r});F_q(V^s_{g,r})) \Rightarrow H_{p+q}(\mathrm{BDiff}_0^\theta(V_g;\ell_0(R^s_{g,r}));\mathbb{Q}).$$

By the functoriality of the Serre spectral sequence with respect to maps of fibrations (see [MP12, Th. 24.5.1] for instance), the diagrams (4.7) and (4.8) give us induced maps of spectral sequences $E(g,r,s) \to E(g+1,r,s)$ and $E(g,r,s) \to E(g,r+1,s)$ respectively. We recall that the homology of $\mathrm{BDiff}(V_g;R^s_{g,r})$ with coefficients in $F_q(V^s_{g,r})$ considered as a local system, agrees with the group homology of $\mathcal{H}^s_{g,r}$ with the corresponding representation as coefficients; see Remark 2.2. Therefore, by combining Proposition 4.6 together with Theorem 3.1 and Corollary 3.18, we deduce that the maps $E(g,r,s) \to E(g+1,r,s)$ and $E(g,r,s) \to E(g,r+1,s)$ are spectral sequence isomorphisms on the E^2-pages for $p \leq \frac{g-1}{2} - q$ (i.e. $p+q \leq \frac{g-3}{2}$). This means that the induced maps $(\sigma^{\theta,s}_{g,r})_*$ and $(\mu^{\theta,s}_{g,r})_*$ on the respective targets, are isomorphisms in degrees $* \leq \frac{g-3}{2}$, as desired. □

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