ON THE CONTINUITY IN TIME AND THE EXPONENTIAL DECAY OF SOLUTIONS TO A GENERALIZED NAVIER–STOKES–FOURIER SYSTEM IN 2D

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Abstract. We consider the flow of a generalized Newtonian incompressible heat-conducting fluid in a bounded domain, subject to homogeneous Dirichlet boundary conditions for velocity and Dirichlet boundary conditions for temperature. For a fluid whose constitutive equation for the Cauchy stress follows a power law with exponent p, we prove that, for $p \geq 2$ and for initial data with finite energy, there always exists a global-in-time weak solution that additionally satisfies the entropy equality. The main novelty of this work is that we rigorously establish the continuity of temperature with respect to time, a property not previously proven in this setting. This time continuity allows us to construct a Lyapunov functional designed specifically for the problem, which in turn implies that the steady solution is nonlinearly stable and attracts all suitable weak solutions, even exponentially.

1. Introduction

In this paper, we consider the following system of partial differential equations

$$\partial_t \boldsymbol{u} + \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) - \operatorname{div} \boldsymbol{\mathcal{S}} + \nabla \pi = \boldsymbol{f}, \tag{1.1}$$

$$\operatorname{div} \boldsymbol{u} = 0, \tag{1.2}$$

$$\partial_t \vartheta + \operatorname{div}(\vartheta \boldsymbol{u}) + \operatorname{div} \boldsymbol{q} = \boldsymbol{\mathcal{S}} : D\boldsymbol{u}, \tag{1.3}$$

$$\partial_t \eta + \operatorname{div}(\eta \boldsymbol{u}) + \operatorname{div}\left(\frac{\boldsymbol{q}}{\vartheta}\right) = \frac{\boldsymbol{\mathcal{S}} : D\boldsymbol{u}}{\vartheta} - \frac{\boldsymbol{q} \cdot \nabla \vartheta}{\vartheta^2}, \tag{1.4}$$

which is satisfied in $Q := (0,T) \times \Omega$, where $T > 0$ and $\Omega \subset \mathbb{R}^2$ is a Lipschitz domain. Here $\boldsymbol{u} : Q \to \mathbb{R}^2$ denotes the velocity field, $D\boldsymbol{u} := (\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)/2$ is the symmetric part of the velocity gradient, $\pi : Q \to \mathbb{R}$ is the pressure, $\vartheta : Q \to \mathbb{R}$ the temperature, and $\eta = \log \vartheta$ the entropy. Furthermore, $\boldsymbol{f} : Q \to \mathbb{R}^2$ is given density of the external body forces, $\boldsymbol{\mathcal{S}} : Q \to \mathbb{R}^{2\times 2}$ denotes the viscous part of the Cauchy stress tensor, and $\boldsymbol{q} : Q \to \mathbb{R}^2$ is the heat flux. We assume that the fluid is homogeneous and set the density equal to one for simplicity. The system (1.1)–(1.4) is completed by the initial and boundary conditions of the form

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{0}, & \vartheta &= \vartheta_b \quad \text{on } \partial\Omega \times (0,T) \\ \boldsymbol{u}(0) &= \boldsymbol{u}_0, & \vartheta(0) &= \vartheta_0 \quad \text{in } \Omega. \end{aligned} \tag{1.5}$$

The above system describes the planar flow of an incompressible, homogeneous, heat-conducting fluid. Equation (1.2) represents the incompressibility constraint, the balance of linear momentum is given by (1.1), and the balance of internal energy by (1.3), where we assume that the heat capacity is identically equal to one. At first glance, the system might appear to be overdetermined; however, this is not the case, since (1.3) and (1.4) are formally equivalent, provided that the solution is sufficiently smooth.

We impose homogeneous Dirichlet boundary conditions for the velocity and spatially inhomogeneous Dirichlet boundary conditions for the temperature. Consequently, no mass passes through the boundary, although thermal interaction with the exterior is allowed.

1.1. Constitutive relations.

We focus on the case where all material parameters may depend on the temperature, and moreover, we assume that the fluid is non-Newtonian. In particular, the viscous part of the Cauchy stress tensor $\boldsymbol{\mathcal{S}}$ depends nonlinearly on the symmetric part of the velocity gradient. In this paper, the dependence is characterized by the power-law index $p \geq 2$. This means that we assume that the heat flux is represented by the Fourier law. That is, $\boldsymbol{q} = \boldsymbol{q}^*(\vartheta)$ and it holds

$$\boldsymbol{q}^*(\vartheta) = -\kappa(\vartheta)\nabla\vartheta, \tag{1.6}$$

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where the heat conductivity $\kappa : \mathbb{R} \to (0,\infty)$ is a continuous function, satisfying for some constants $0 < \underline{\kappa}$, $\overline{\kappa} < \infty$,

$$\underline{\kappa} \le \kappa \le \overline{\kappa}. \tag{1.7}$$

Concerning the viscous part of the Cauchy stress, we assume that $\boldsymbol{\mathcal{S}} = \boldsymbol{\mathcal{S}}^*(\vartheta, D\boldsymbol{u})$, where $\boldsymbol{\mathcal{S}}^* : (0,\infty) \times \mathbb{R}^{2\times 2}_{sym} \to \mathbb{R}^{2\times 2}_{sym}$ is a continuous mapping, and that there exist $0 < \underline{\nu} \le \overline{\nu} < \infty$ and $p \ge 2$, such that for all $\vartheta \in \mathbb{R}_+$, $\boldsymbol{\mathcal{D}}_1, \boldsymbol{\mathcal{D}}_2 \in \mathbb{R}^{2\times 2}_{sym}$ the following set of conditions holds

$$\begin{aligned} (\boldsymbol{\mathcal{S}}^*(\vartheta, \boldsymbol{\mathcal{D}}_1) - \boldsymbol{\mathcal{S}}^*(\vartheta, \boldsymbol{\mathcal{D}}_2)) : (\boldsymbol{\mathcal{D}}_1 - \boldsymbol{\mathcal{D}}_2) &\ge 0, \\ \boldsymbol{\mathcal{S}}^*(\vartheta, \boldsymbol{\mathcal{D}}_1) : \boldsymbol{\mathcal{D}}_1 &\ge \underline{\nu}|\boldsymbol{\mathcal{D}}_1|^p - \overline{\nu}, \\ |\boldsymbol{\mathcal{S}}^*(\vartheta, \boldsymbol{\mathcal{D}}_1)| \le \overline{\nu}(1 + |\boldsymbol{\mathcal{D}}_1|)^{p-1} \quad &\text{and} \quad \boldsymbol{\mathcal{S}}^*(\vartheta, \boldsymbol{0}) = \boldsymbol{0}. \end{aligned} \tag{1.8}$$

The above assumption is sufficient in the case when we deal with the existence and continuity of the solution. In the case where we also want to show its exponential stability, we additionally require that

$$\boldsymbol{\mathcal{S}}^*(\vartheta, \boldsymbol{\mathcal{D}}) : \boldsymbol{\mathcal{D}} \ge \underline{\mu}(1 + |\boldsymbol{\mathcal{D}}|)^{p-2}|\boldsymbol{\mathcal{D}}|^2. \tag{1.9}$$

1.2. **Notation.** In what follows, we adopt the standard notation for the Lebesgue, Sobolev, and Bochner spaces, equipping them with their usual norms. The symbol C_0^∞ is reserved for smooth, compactly supported functions. The function spaces associated with the incompressible setting are denoted by

$$W^{1,p}_{0,\mathrm{div}} := \{\boldsymbol{v} \in W^{1,p}_0(\Omega; \mathbb{R}^2); \operatorname{div} \boldsymbol{v} = 0\},$$

and $L^2_{0,\mathrm{div}}$ denotes its closure in the L^2 topology. For a general Banach space, the duality pairing is denoted by $\langle \cdot\,, \cdot \rangle_X$, while in the case $X = W^{1,p}_{0,\mathrm{div}}$ the subscript is omitted for simplicity. For notational convenience, we also use the abbreviation $V := W^{1,p}_{0,\mathrm{div}}$.

We recall here a few classical inequalities related to the function space setting used in this paper. First, we recall the Korn inequality

$$C\|\boldsymbol{u}\|_{1,q} \le \|D\boldsymbol{u}\|_q + \|\mathrm{Tr}\ \boldsymbol{u}\|_{L^2(\partial\Omega)}, \tag{1.10}$$

which holds true for any Lipschitz domain $\Omega \subset \mathbb{R}^2$ and $q \in (1,\infty)$, see [8, as Lemma 1.11]. Note that for the space V, the trace part of the above inequality vanishes identically.

Second is the classical Lebesgue–Sobolev interpolation inequality

$$\|z\|_r \le C\|z\|_{1,s}^{\frac{2s(q-r)}{r(2q-sq-2s)}} \|z\|_q^{\frac{q(2r-sr-2s)}{r(2q-sq-2s)}}, \tag{1.11}$$

which holds true for $r \ge q \ge 1$ as follows. If $s < 2$, then the inequality is valid for all $r \le \frac{2s}{2-s}$. If $s > 2$, then it holds for all $r \le \infty$. If $s = 2$, then it holds for all $r < \infty$.

1.3. **Assumptions on data.** Here we state the assumptions imposed on the data. We begin with ϑ_b, for which we require

$$\vartheta_b \in W^{\frac{1}{2},2}(\partial\Omega) \cap L^\infty(\partial\Omega).$$

Note that, by classical elliptic theory and (1.6)–(1.7), this assumption guarantees the existence of a function $\hat{\vartheta} : \Omega \to \mathbb{R}$ satisfying

$$\operatorname{div} \boldsymbol{q}^*(\hat{\vartheta}) = 0 \quad \text{in } \Omega, \qquad \hat{\vartheta} = \vartheta_b \quad \text{on } \partial\Omega, \qquad \hat{\vartheta} \in L^\infty(\Omega) \cap W^{1,2}(\Omega). \tag{1.12}$$

Hence, in what follows we impose assumptions only on $\hat{\vartheta}$ and no longer consider ϑ_b.

For the momentum equation, we assume that the right-hand side $\boldsymbol{f}$ and the initial condition $\boldsymbol{u}_0$ satisfy the standard requirements:

$$\boldsymbol{f} \in L^{p'}(0, T; V^*), \quad \boldsymbol{u}_0 \in L^2_{0,\mathrm{div}}(\Omega). \tag{1.13}$$

Regarding the temperature equation, we assume that the initial and boundary temperatures are bounded and strictly positive, i.e.,

$$\vartheta_0 \in L^1(\Omega), \quad \hat{\vartheta} \in W^{1,2}(\Omega) \cap L^\infty(\Omega), \tag{1.14}$$

$$\mu := \min\left\{ \operatorname*{ess\,inf}_{x\in\Omega} \hat{\vartheta}(x), \operatorname*{ess\,inf}_{x\in\Omega} \vartheta_0(x) \right\} > 0. \tag{1.15}$$

1.4. **Main results.** The main results of the paper are summarized in the following theorem.

Theorem 1.1 (Existence of a solution fulfilling entropy equality)**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz boundary and $T > 0$. Assume that $\boldsymbol{\mathcal{S}}^*$ and κ satisfy* (1.7)–(1.8)*. Additionally, assume that $\boldsymbol{f}$, $\boldsymbol{u}_0$, ϑ_0, and $\hat{\vartheta}$ fulfill* (1.13)–(1.15)*.*

Then there exists a quadruplet $(\boldsymbol{u}, \boldsymbol{\mathcal{S}}, \vartheta, \eta)$ fulfilling

$$\boldsymbol{u} \in C([0,T]; L^2_{0,\mathrm{div}}) \cap L^p(0,T;V), \tag{1.16}$$

$$\partial_t \boldsymbol{u} \in L^{p'}(0,T;V^*),\ \boldsymbol{\mathcal{S}} \in L^{p'}(Q, \mathbb{R}^{2\times 2}_{sym}), \tag{1.17}$$

$$\vartheta \in C([0,T]; L^1(\Omega)),\ (\vartheta)^\alpha \in L^2(0,T;W^{1,2}(\Omega)) \qquad \alpha \in (0,1/2)\,, \tag{1.18}$$

$$\vartheta \in L^r(Q) \qquad r \in [1,2)\,, \tag{1.19}$$

$$\vartheta - \hat{\vartheta} \in L^s(0,T;W^{1,s}_0(\Omega)) \qquad s \in [1,4/3)\,, \tag{1.20}$$

$$\eta \in L^2(0,T;W^{1,2}(\Omega)) \cap L^q(Q) \qquad q \in [1,\infty), \tag{1.21}$$

and satisfying (1.1)–(1.5) *in the following sense:*
Momentum equation: *The Cauchy stress is of the form $\boldsymbol{\mathcal{S}} = \boldsymbol{\mathcal{S}}^*(\vartheta, D\boldsymbol{u})$ a.e. in Q, it holds $\boldsymbol{u}(0) = \boldsymbol{u}_0$ in L^2, and for all $\boldsymbol{w} \in L^p(0,T;V)$*

$$\begin{aligned}\int_0^T \langle \partial_t \boldsymbol{u}\,, \boldsymbol{w}\rangle\, \mathrm{d}t + \int_0^T \int_\Omega \boldsymbol{\mathcal{S}} : D\boldsymbol{w}\, \mathrm{d}x\, \mathrm{d}t \\ = \int_0^T \int_\Omega (\boldsymbol{u} \otimes \boldsymbol{u}) : D\boldsymbol{w}\, \mathrm{d}x\, \mathrm{d}t + \int_0^T \langle \boldsymbol{f}\,, \boldsymbol{w}\rangle\, \mathrm{d}t;\end{aligned} \tag{1.22}$$

Internal energy balance: *Temperature satisfies the minimum principle $\vartheta \geq \mu$ a.e. in Q, it holds $\vartheta(0) = \vartheta_0$ in $L^1(\Omega)$, and for all $\varphi \in C^\infty_0((-\infty,T) \times \Omega)$*

$$\begin{aligned}-\int_0^T \int_\Omega \vartheta \partial_t \varphi\, \mathrm{d}x\, dt - \int_0^T \int_\Omega \vartheta \boldsymbol{u} \cdot \nabla\varphi\, \mathrm{d}x\, dt + \int_0^T \int_\Omega \kappa(\vartheta) \nabla\vartheta \cdot \nabla\varphi\, \mathrm{d}x\, dt \\ = \int_0^T \int_\Omega \boldsymbol{\mathcal{S}} : D\boldsymbol{u}\varphi\, dx\, dt + \int_\Omega \vartheta_0 \varphi(0)\, \mathrm{d}x;\end{aligned} \tag{1.23}$$

Entropy equality: *Entropy is given as $\eta = \ln \vartheta$ a.e. in Q, $\eta_0 := \ln \vartheta_0$ and for all $\varphi \in C^\infty_0((-\infty,T) \times \Omega)$*

$$\begin{aligned}-\int_0^T \int_\Omega \eta \partial_t \varphi\, \mathrm{d}x\, \mathrm{d}t - \int_0^T \int_\Omega \eta \boldsymbol{u} \cdot \nabla\varphi\, \mathrm{d}x\, \mathrm{d}t + \int_0^T \int_\Omega \kappa(\vartheta) \nabla\eta \cdot \nabla\varphi\, \mathrm{d}x\, \mathrm{d}t \\ = \int_0^T \int_\Omega \frac{1}{\vartheta} \boldsymbol{\mathcal{S}} : D\boldsymbol{u}\varphi\, \mathrm{d}x\, \mathrm{d}t + \int_0^T \int_\Omega \kappa(\vartheta) \frac{|\nabla\vartheta|^2}{\vartheta^2} \varphi\, \mathrm{d}x\, \mathrm{d}t + \int_\Omega \eta_0 \varphi(0)\, \mathrm{d}x.\end{aligned} \tag{1.24}$$

Foundational works by Consiglieri [10, 11] studied both stationary and time-dependent models of shear-thickening fluids with temperature-dependent viscosity and thermal conductivity. While these studies provided rigorous existence results, they did not address the entropy equality, the long-time behaviour, or the continuity of the temperature with respect to time. More complex existence theory for heat-conducting non-Newtonian fluids was later established in [6, 9], however, without any ambition for establishing further properties such as time continuity or the validity of energy and/or entropy equality.

From a thermodynamic perspective, Dafermos [12] introduced the use of entropy as a Lyapunov functional, establishing a link between the Second Law of Thermodynamics and nonlinear stability of fluid systems, see also [7]. More recently, Abbatiello et al. [1] proved the existence of weak solutions to the three-dimensional Navier–Stokes–Fourier system with non-Newtonian power-law behavior (for $p \geq \frac{11}{5}$) that satisfy the entropy equality. This approach notably avoids the classical DiPerna–Lions renormalization framework [13], by constructing solutions directly fulfilling the entropy equality, even when $p \leq \frac{5}{2}$. Building upon this, the follow-up work [2] established nonlinear stability of such entropy-equality solutions by constructing a Lyapunov functional and demonstrating exponential decay to a steady state, even under inhomogeneous boundary temperatures; see also [14] for similar but conditional results.

Considering only the theory for nonlinear heat equations with irregular data (purely integrable or even measure-valued), we overcome the limitations of traditional renormalization approaches. For instance, [17] addressed entropy solutions that are continuous in time, but relied on entropy-like inequalities with multiple cut-off functions, without clearly establishing a renormalized framework or coupling with other equations. Classical existence results for the heat equation with measure data were established by [4], while [16] introduced renormalized solutions requiring multiple renormalizations in different function spaces. The analysis in [5] extended the existence theory to cases where even the convective term cannot be handled by standard methods, and [3] addressed more complex systems with dissipative heating, albeit without proving continuity with respect to time.

The present article contributes to this line of research by focusing on a two-dimensional non-Newtonian heat-conducting fluid and establishing the existence of entropy-equality solutions. Our main novelty lies in the regularity with respect to time: we show that the temperature is continuous in time, which allows us to avoid classical renormalization procedures altogether. Remarkably, in most works on heat-conducting fluids, such issues are usually omitted, and even the attainment of the initial condition in the strong topology is typically not guaranteed. In this paper, all the above-mentioned drawbacks are resolved, and the validity of the entropy equation appears to be the key step allowing us to avoid the classical renormalized solution notions while still proving the desired strong results.

2. Proof of the existence theorem

In the first four sections, following the methods of [6], we prove the existence of a weak solution satisfying (1.22) and (1.23). The proof requires modification due to the different character of the boundary conditions. We then show that the initial condition for the temperature is attained on a set of full measure.

Next, we prove that the solution satisfies (1.24), adopting the methods from [1]. Finally, we show that the temperature ϑ is continuous in L^1 with respect to time, and hence the initial condition is attained in the sense of Theorem 1.1.

2.1. Galerkin approximations.

We now define the Galerkin approximation. First, we denote

$$W^{2,2}_{0,\operatorname{div}}(\Omega) := \overline{\{\boldsymbol{u} \in (C_0^\infty(\Omega))^2; \operatorname{div} \boldsymbol{u} = 0\}}^{\|\cdot\|_{2,2}}.$$

Then, following [15, Appendix, Theorem 4.11], we obtain the existence of a basis $\{\boldsymbol{w}_j\}_{j=1}^\infty$ that is orthogonal in $W^{2,2}_{0,\operatorname{div}}(\Omega)$ and orthonormal in $L^2(\Omega)$. Note that $\|\boldsymbol{w}_j\|_{1,q}$ is bounded for all $j \in \mathbb{N}$, $q \in [1,\infty)$. Similarly, we can find a sequence $\{w_j\}_{j=1}^\infty$ forming an orthonormal basis of $L^2(\Omega)$, which is orthogonal in $W_0^{1,2}(\Omega)$.

Then, for fixed $N, M \in \mathbb{N}$, we find

$$\begin{aligned}
\boldsymbol{u}^{N,M}(x,t) &= \sum_{i=1}^N c_i^{N,M}(t)\boldsymbol{w}_i(x),\\
\vartheta^{N,M}(x,t) &= \left(\sum_{i=1}^M d_i^{N,M}(t) w_i(x)\right) + \hat{\vartheta}(x),
\end{aligned}$$

such that $(\boldsymbol{c}^{N,M}, \boldsymbol{d}^{N,M}) := (c_1^{N,M}, \dots, c_N^{N,M}, d_1^{N,M}, \dots, d_M^{N,M}) : (0,T^*) \to \mathbb{R}^{N+M}$, for $T^* \in (0,T)$, is the maximal solution to the following system of ordinary differential equations

$$\begin{aligned}
&\int_\Omega \partial_t \boldsymbol{u}^{N,M} \boldsymbol{w}_j \,\mathrm{d}x + \int_\Omega \boldsymbol{S}^*\left(\vartheta^{N,M}, D\boldsymbol{u}^{N,M}\right) : D\boldsymbol{w}_j \,\mathrm{d}x =\\
&\qquad\qquad = \int_\Omega (\boldsymbol{u}^{N,M} \otimes \boldsymbol{u}^{N,M}) : \nabla(\boldsymbol{w}_j)\,\mathrm{d}x + \langle \boldsymbol{f}\ , \boldsymbol{w}_j\rangle\,,\\
&\int_\Omega \partial_t \vartheta^{N,M} w_k \,\mathrm{d}x + \int_\Omega \kappa(\vartheta^{N,M})\nabla\vartheta^{N,M} \cdot (\nabla w_k)\,\mathrm{d}x =\\
&= \int_\Omega \boldsymbol{S}^*\left(\vartheta^{N,M}, D\boldsymbol{u}^{N,M}\right) : D\boldsymbol{u}^{N,M} w_k \,\mathrm{d}x + \int_\Omega \vartheta^{N,M}\boldsymbol{u}^{N,M} \cdot (\nabla w_k)\,\mathrm{d}x,
\end{aligned} \tag{2.1}$$

where $j \in \{1,\dots,N\}$ and $k \in \{1,\dots,M\}$.

We equip the above system of ordinary differential equations (2.1) with the following initial conditions:

$$\begin{aligned}
\boldsymbol{u}^{N,M}(x,0) &= \boldsymbol{u}_0^N(x) &&\text{on } \Omega,\\
\vartheta^{N,M}(x,0) &= \vartheta_0^{N,M}(x) &&\text{on } \Omega,
\end{aligned} \tag{2.2}$$

where $\boldsymbol{u}_0^N(x) = \sum_{i=1}^N c_{i,0}^N \boldsymbol{w}_i(x)$ is the projection of $\boldsymbol{u}_0$ onto the linear hull of $\{\boldsymbol{w}_j\}_{j=0}^N$ in $L^2(\Omega)$. The approximative initial conditions for the temperature $\vartheta_0^{N,M}$ is defined following manner. We define $\vartheta_0^N(x) := \min\{\vartheta_0(x), N\}$ and then define $\vartheta_0^{N,M}$ as a projection of ϑ_0^N onto the linear hull of $\{w_j\}_{j=0}^M$ in $L^2(\Omega)$, hence

$\vartheta_0^{N,M} = \sum_{j=1}^{M} d_{0,j}^{N,M} w_j$. Note, that

$$\begin{aligned} \vartheta_0^{N,M} &\xrightarrow{M\to\infty} \vartheta_0^N \quad \text{in } L^2(\Omega),\\ \boldsymbol{u}_0^N &\xrightarrow{N\to\infty} \boldsymbol{u}_0 \quad \text{in } L^2(\Omega;\mathbb{R}^2),\\ \vartheta_0^N &\xrightarrow{N\to\infty} \vartheta_0 \quad \text{in } L^1(\Omega), \end{aligned} \tag{2.3}$$

where the first two convergence results hold due to the completeness of an orthonormal basis in Hilbert spaces.

By Carathéodory's existence theory, it is not difficult to obtain a maximal solution to the system (2.1)–(2.2) on a time interval $(0,T^*)$. Showing that $\boldsymbol{u}^{N,M}$ and $\vartheta^{N,M}$ are bounded with respect to the time variable then implies that we can set $T^*=T$.

2.2. **Limit passage $M\to\infty$.** To pass to the limit as $M\to\infty$ and to establish existence on the time interval $(0,T)$, we need to derive estimates for $\boldsymbol{u}^{N,M}$ and $\vartheta^{N,M}$ that are uniform with respect to M. For the velocity part, we follow the classical procedure. We multiply the j-th equation in $(2.1)_1$ by $c_j^{N,M}$, sum the results over $j=1,\dots,N$, and use the fact that $\operatorname{div}\boldsymbol{u}^{N,M}=0$ to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}^{N,M}(t)\|_2^2 + \int_\Omega \boldsymbol{\mathcal{S}}(\vartheta^{N,M}, D\boldsymbol{u}^{N,M}) : D\boldsymbol{u}^{N,M}\,\mathrm{d}x = \left\langle \boldsymbol{f}\,, \boldsymbol{u}^{N,M}\right\rangle. \tag{2.4}$$

Then, using the assumption (1.8), the assumptions on the data (1.13), the Korn inequality (1.10), the properties of the approximation of the initial data (2.3), the interpolation (1.11), and the identity $(2.1)_1$ for estimating the time derivative, we deduce the uniform bounds

$$\|\boldsymbol{u}^{N,M}\|_{L^\infty(L^2)} + \|\boldsymbol{u}^{N,M}\|_{L^p(V)} + \|\boldsymbol{u}^{N,M}\|_{L^{2p}(Q)} + \|\partial_t \boldsymbol{u}^{N,M}\|_{L^{p'}((W^{2,2}_{0,\mathrm{div}})^*)} \le C. \tag{2.5}$$

These estimates are crucial for passing to the limit and establishing existence on the interval $(0,T)$.

Next, we focus on the estimates for the temperature. We multiply the $(M+k)$-th equation in (2.1) by $d_k^{N,M}$ and sum over $k=1,\dots,M$. This yields (using the fact that $\operatorname{div}\boldsymbol{u}^{N,M}=0$) the identity

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\vartheta^{N,M}(t)-\hat\vartheta\|_2^2 + \int_\Omega \kappa(\vartheta^{N,M})\nabla\vartheta^{N,M}\cdot(\vartheta^{N,M}-\hat\vartheta)\,\mathrm{d}x = \\ &\quad = \int_\Omega \boldsymbol{\mathcal{S}}^*\left(\vartheta^{N,M}, D\boldsymbol{u}^{N,M}\right) : D\boldsymbol{u}^{N,M}(\vartheta^{N,M}-\hat\vartheta)\,\mathrm{d}x + \int_\Omega \hat\vartheta\,\boldsymbol{u}^{N,M}\cdot\nabla\vartheta^{N,M}\,\mathrm{d}x. \end{aligned}$$

Then, using (1.7) and (1.8), the Young inequality, and the Poincaré inequality, and integrating over the time interval $(0,t)$ for any $t\in(0,T)$, we have

$$\begin{aligned} &\|(\vartheta^{N,M}-\hat\vartheta)(t)\|_2^2 + \int_0^t \|\vartheta^{N,M}-\hat\vartheta\|_{1,2}^2\,\mathrm{d}\tau \\ &\quad\le C\|(\vartheta^{N,M}-\hat\vartheta)(0)\|_2^2 + C\int_0^t \left(1+\|D\boldsymbol{u}^{N,M}\|_{2p}^{2p} + \|\hat\vartheta\|_\infty^2\|\boldsymbol{u}^{N,M}\|_2^2 + \|\hat\vartheta\|_2^2\right)\mathrm{d}\tau \\ &\quad\le C(N)\left(1+\int_0^t \|D\boldsymbol{u}^{N,M}\|_{2p}^{2p}\,\mathrm{d}\tau\right), \end{aligned} \tag{2.6}$$

where the last inequality follows form the Gronwall inequality. By the choice of the basis, we have

$$\int_0^t \|D\boldsymbol{u}^{N,M}\|_{2p}^{2p}\,\mathrm{d}\tau \le \sup_{j\le N}\|D\boldsymbol{w}_j\|_{2p}^{2p}\int_0^t\left(\sum_{j=1}^N c_j^{N,M}\right)^{2p}\mathrm{d}\tau \overset{(2.5)}{\le} C(N). \tag{2.7}$$

Consequently, it follows from (2.6) that

$$\sup_{t\in(0,T)}\|(\vartheta^{N,M}-\hat\vartheta)(t)\|_2^2 + \int_0^T \|\vartheta^{N,M}-\hat\vartheta\|_{1,2}^2\,\mathrm{d}t \le C(N). \tag{2.8}$$

Similarly,

$$\|\partial_t\vartheta^{N,M}\|_{(L^2(W_0^{1,2})^*)} \le \int_0^T C(\|D\boldsymbol{u}^{N,M}\|_{2p}^{2p} + \|\vartheta^{N,M}\|_4^4 + \|\boldsymbol{u}^{N,M}\|_4^4 + \|\nabla\vartheta^{N,M}\|_2^2)\,\mathrm{d}t \le C(N). \tag{2.9}$$

Due to the estimates (2.5)–(2.9), which are independent of M, we can, for any fixed $N \in \mathbb{N}$, find a subsequence $\{\boldsymbol{u}^{N,M}, \vartheta^{N,M}\}_{M=1}^{\infty}$ that we do not relabel, such that as $M \to \infty$, we obtain the following convergence results

$$\partial_t \boldsymbol{u}^{N,M} \rightharpoonup \partial_t \boldsymbol{u}^N \qquad \text{in } L^{p'}(0,T;(W^{2,2}_{0,\mathrm{div}})^*), \tag{2.10}$$

$$\boldsymbol{u}^{N,M} \rightharpoonup^* \boldsymbol{u}^N \qquad \text{in } L^{\infty}(0,T;L^2_{0,\mathrm{div}}), \tag{2.11}$$

$$\boldsymbol{u}^{N,M} \rightharpoonup \boldsymbol{u}^N \qquad \text{in } L^{p}(0,T;V), \tag{2.12}$$

$$\partial_t \vartheta^{N,M} \rightharpoonup \partial_t \vartheta^N \qquad \text{in } L^{2}(0,T;(W^{1,2}_{0})^*), \tag{2.13}$$

$$\vartheta^{N,M} \rightharpoonup^* \vartheta^N \qquad \text{in } L^{\infty}(0,T;L^2), \tag{2.14}$$

$$\vartheta^{N,M} \rightharpoonup \vartheta^N \qquad \text{in } L^{2}(0,T;W^{1,2}). \tag{2.15}$$

With the use of the Aubin–Lions lemma (see [19, Corollary 9] and also [18]), we can additionally obtain strong convergence results

$$\vartheta^{N,M} \to \vartheta^N \qquad \text{in } L^r(Q) \text{ for } r \in [1,4). \tag{2.16}$$

$$\boldsymbol{u}^{N,M} \to \boldsymbol{u}^N \qquad \text{in } L^r(Q) \text{ for } r \in [1,2p). \tag{2.17}$$

We can now pass to the limit $M \to \infty$ in (2.1) and obtain

$$\begin{aligned} \left\langle \partial_t \boldsymbol{u}^N, \boldsymbol{w}_j \right\rangle_{W^{2,2}_{0,\mathrm{div}}} &+ \int_\Omega \boldsymbol{S}^*\left(\vartheta^N, D\boldsymbol{u}^N\right) : D\boldsymbol{w}_j \, dx = \\ &= \int_\Omega (\boldsymbol{u}^N \otimes \boldsymbol{u}^N) : \nabla \boldsymbol{w}_j \, dx + \langle \boldsymbol{f}, \boldsymbol{w}_j \rangle, \quad \text{for all } j \in \{1,2,\dots,N\} \end{aligned} \tag{2.18}$$

$$\begin{aligned} \left\langle \partial_t \vartheta^N, \psi \right\rangle_{W^{1,2}_0} &+ \int_\Omega \kappa(\vartheta^N)\nabla\vartheta^N \cdot \nabla\psi \, \mathrm{d}x = \\ &= \int_\Omega \boldsymbol{S}^*\left(\vartheta^N, D\boldsymbol{u}^N\right) : D\boldsymbol{u}^N \psi \, \mathrm{d}x + \int_\Omega \vartheta^N \boldsymbol{u}^N \cdot \nabla\psi \, \mathrm{d}x, \quad \forall \psi \in W^{1,2}_0(\Omega). \end{aligned} \tag{2.19}$$

Furthermore, using the classical parabolic embedding, the fact that $\hat{\vartheta}$ is time independent, and the convergence results (2.13) and (2.15), we have

$$\vartheta^N \in C([0,T];L^2). \tag{2.20}$$

2.3. **Minimum principle.** We show that $\vartheta^N \geq \mu$ for almost all $(t,x) \in (0,T)\times\Omega$, where μ is defined in (1.15). To this end, we set

$$\psi(\tau,x) := \min\{0, \vartheta^N(\tau,x) - \mu\} := (\vartheta^N(\tau,x)-\mu)_- \leq 0$$

in (2.19) and integrate the resulting expression until $t \in (0,T)$. Note that ψ has zero trace due to the choice of μ. This gives us (since $\operatorname{div} \boldsymbol{u}^N = 0$)

$$\int_0^t \int_\Omega \partial_t \left(\frac{\psi^2}{2}\right) + \kappa(\vartheta^N)|\nabla\psi|^2 \, \mathrm{d}x\,\mathrm{d}\tau = \int_0^t \int_\Omega \boldsymbol{S}^*\left(\vartheta^N, D\boldsymbol{u}^N\right) : D\boldsymbol{u}^N \psi \, \mathrm{d}x\,\mathrm{d}\tau \leq 0, \tag{2.21}$$

where we used the nonnegativity of[1] $\boldsymbol{S}^*(\vartheta^N, D\boldsymbol{u}^N) : D\boldsymbol{u}^N$ and the nonpositivity of ψ. This implies

$$0 \geq \int_0^T \partial_t \int_\Omega \psi^2 \, \mathrm{d}x\,\mathrm{d}\tau = \|(\vartheta^N(t)-\mu)_-\|_2^2 - \|(\vartheta^N_0-\mu)_-\|_2^2 = \|(\vartheta^N(t)-\mu)_-\|_2^2,$$

which yields

$$\vartheta^N \geq \mu \qquad \text{for a. a. } (t,x) \in (0,T)\times\Omega. \tag{2.22}$$

To summarize, we found functions

$$\boldsymbol{u}^N : Q \to \mathbb{R}^2 \qquad \vartheta^N : Q \to \mathbb{R}$$

such that

$$\begin{aligned} &\boldsymbol{u}^N \in C([0,T];L^2_{0,\mathrm{div}}) \cap L^p(0,T;V), \quad \partial_t \boldsymbol{u}^N \in L^{p'}(0,T;(W^{2,2}_{0,\mathrm{div}})^*), \\ &\vartheta^N \in C([0,T];L^2) \cap L^2(0,T;W^{1,2}), \quad \partial_t \vartheta^N \in L^2(0,T;(W^{1,2}_0)^*), \\ &\vartheta^N \geq \mu \text{ a.a. on } Q. \end{aligned} \tag{2.23}$$

[1]It follows from (1.8) by using the following inequality

$$0 \leq (\boldsymbol{S}^*(\vartheta^N, D\boldsymbol{u}^N) - \boldsymbol{S}^*(\vartheta^N, \boldsymbol{0})) : (D\boldsymbol{u}^N - \boldsymbol{0}) = \boldsymbol{S}^*(\vartheta^N, D\boldsymbol{u}^N) : D\boldsymbol{u}^N.$$

Furthermore, $\boldsymbol{u}^N$ and ϑ^N fulfill the system (2.18)–(2.19), and satisfy the initial conditions

$$\boldsymbol{u}^N(0) = \boldsymbol{u}_0^N \quad \text{and} \quad \vartheta^N(0) = \vartheta_0^N.$$

We omit the discussion of the attainment of the initial conditions here, but we focus on it in the next section, where a more delicate procedure must be employed.

2.4. **Uniform estimates and limit passages in N.** We provide estimates on $\boldsymbol{u}^N$ and ϑ^N that are uniform with respect to N and obtain weakly convergent subsequences. Since the estimates for $\boldsymbol{u}^{N,M}$ were uniform with respect to N, we can use weak lower semicontinuity and obtain from (2.5) that $\boldsymbol{u}^N$ also satisfies

$$\sup_{t\in(0,T)} \|\boldsymbol{u}^N(t)\|_2 + \|\boldsymbol{u}^N\|_{L^p(V)} + \|\boldsymbol{u}^N\|_{L^{2p}(Q)} + \|\partial_t \boldsymbol{u}^N\|_{L^{p'}((W^{2,2}_{0,\mathrm{div}})^*)} \le C. \tag{2.24}$$

In addition, similarly as (2.4), we obatin the following identity

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}^N(t)\|_2^2 + \int_\Omega \boldsymbol{\mathcal{S}}(\vartheta^N, \boldsymbol{D}\boldsymbol{u}^N) : \boldsymbol{D}\boldsymbol{u}^N \,\mathrm{d}x = \left\langle \boldsymbol{f}\, , \boldsymbol{u}^N \right\rangle. \tag{2.25}$$

The same procedure for ϑ^N cannot be used since (2.8) is not uniform with respect to N. Therefore, we proceed differently. We consider arbitrary $\alpha \in (0,1)$, denote $K := \|\hat{\vartheta}\|_\infty$ and set

$$\psi := 1 - \left(\frac{K}{K + \vartheta^N - \hat{\vartheta}}\right)^\alpha$$

as a test function in (2.19). Note that $\psi \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ thanks to the properties of $\hat{\vartheta}$ and the minimum principle (2.22). Integrating the result over $t \in (0,T)$ and using the positivity of $\boldsymbol{\mathcal{S}}^N : \boldsymbol{D}\boldsymbol{u}^N$, as well as the facts that $\psi \le 1$ and $\boldsymbol{\mathcal{S}}^N : \boldsymbol{D}\boldsymbol{u}^N \ge 0$, we have

$$\begin{aligned}
&\int_0^t \left\langle \partial_t \vartheta^N\, , 1 - \left(\frac{K}{K + \vartheta^N - \hat{\vartheta}}\right)^\alpha \right\rangle_{W_0^{1,2}} \mathrm{d}\tau - \int_0^t \int_\Omega \left(\kappa(\vartheta^N)\nabla\vartheta^N - \vartheta^N \boldsymbol{u}^N\right) \cdot \nabla \left(\frac{K}{K + \vartheta^N - \hat{\vartheta}}\right)^\alpha \mathrm{d}\tau \le \\
&\le \int_0^t \int_\Omega \boldsymbol{\mathcal{S}}^N : \boldsymbol{D}\boldsymbol{u}^N \,\mathrm{d}x\,\mathrm{d}\tau \le C.
\end{aligned} \tag{2.26}$$

The last inequality holds by (2.24), and the third part of (1.8). We define $H^\alpha : (0,\infty)^2 \to \mathbb{R}$ as

$$H^\alpha(s,\sigma) = \int_0^s \left(\frac{K}{K + \tau - \sigma}\right)^\alpha \mathrm{d}\tau$$

and we observe that there exist constants $c_1, c_2 > 0$ such that for all $s \ge \mu$ and $0 \le \sigma \le K$, it holds

$$c_1(\alpha) s^{1-\alpha} \le H^\alpha(s,\sigma) \le c_2(\alpha) s^{1-\alpha}. \tag{2.27}$$

Next, we evaluate all terms in (2.26). For the first term, we have

$$\left\langle \partial_t \vartheta^N\, , 1 - \left(\frac{K}{K + \vartheta^N - \hat{\vartheta}}\right)^\alpha \right\rangle_{W_0^{1,2}} = \frac{\mathrm{d}}{\mathrm{d}t}\, \|\vartheta^N - H^\alpha(\vartheta^N, \hat{\vartheta}))\|_1. \tag{2.28}$$

We now want to simplify the second term in (2.26). Using the Young inequality, we have

$$\begin{aligned}
-\kappa(\vartheta^N)\nabla\vartheta^N \cdot \nabla \left(\frac{K}{K + \vartheta^N - \hat{\vartheta}}\right)^\alpha &= \frac{\alpha\kappa(\vartheta^N)K^\alpha}{\left(K + \vartheta^N - \hat{\vartheta}\right)^{\alpha+1}}|\nabla\vartheta^N|^2 - \frac{\alpha\kappa(\vartheta^N)K^\alpha}{\left(K + \vartheta^N - \hat{\vartheta}\right)^{\alpha+1}}\nabla\vartheta^N \cdot \nabla\hat{\vartheta} \\
&\ge \frac{\alpha\kappa(\vartheta^N)K^\alpha}{2\left(K + \vartheta^N - \hat{\vartheta}\right)^{\alpha+1}}|\nabla\vartheta^N|^2 - \frac{\alpha\kappa(\vartheta^N)K^\alpha}{2\left(K + \vartheta^N - \hat{\vartheta}\right)^{\alpha+1}}|\nabla\hat{\vartheta}|^2 \\
&\ge \frac{\alpha\kappa(\vartheta^N)K^\alpha}{2\left(K + \vartheta^N - \hat{\vartheta}\right)^{\alpha+1}}|\nabla\vartheta^N|^2 - C(\alpha, K, \mu, \overline{\kappa})|\nabla\hat{\vartheta}|^2,
\end{aligned} \tag{2.29}$$

where we also used (1.7). The last term on the left-hand side of (2.26) is estimated with the help of the fact that $\operatorname{div} \boldsymbol{u}^N = 0$ as

$$\begin{aligned}
-\int_0^t \int_\Omega \vartheta^N \boldsymbol{u}^N \cdot \nabla \left(\frac{K}{K+\vartheta^N-\hat{\vartheta}}\right)^\alpha \mathrm{d}x\,\mathrm{d}\tau &= -\int_0^t \int_\Omega \hat{\vartheta} \boldsymbol{u}^N \cdot \nabla \left(\frac{K}{K+\vartheta^N-\hat{\vartheta}}\right)^\alpha \mathrm{d}x\,\mathrm{d}\tau \\
&= \int_0^t \int_\Omega \nabla\hat{\vartheta} \cdot \boldsymbol{u}^N \left(\frac{K}{K+\vartheta^N-\hat{\vartheta}}\right)^\alpha \mathrm{d}x\,\mathrm{d}\tau \\
&\le C(K,\alpha,\mu) \int_0^t \int_\Omega |\nabla\hat{\vartheta}|^2 + |\boldsymbol{u}^N|^2 \,\mathrm{d}x\,\mathrm{d}\tau \le C(K,\alpha,\mu),
\end{aligned}$$

where the last inequality follows from the assumption on $\hat{\vartheta}$, see (1.14), and the uniform estimate (2.24). Putting all the estimates together we can rewrite (2.26) as

$$\|\vartheta^N(t) - H^\alpha(\vartheta^N(t),\hat{\vartheta}))\|_1 + \int_0^t \int_\Omega \frac{\alpha\kappa(\vartheta^N)K^\alpha|\nabla\vartheta^N|^2}{2\left(K+\vartheta^N-\hat{\vartheta}\right)^{\alpha+1}} \,\mathrm{d}x\,\mathrm{d}\tau \le \|\vartheta_0^N - H^\alpha(\vartheta_0^N,\hat{\vartheta}))\|_1 + C(\alpha,\mu,K,\overline{\kappa}).$$

In what follows, we keep in mind that the constant C may depend on the given data and may blow up as $\alpha \to 0_+$, but it is independent of N. Therefore, we do not write the dependence of C on other constants explicitly. With this in mind, and by using the minimum principle (2.22), the assumption (1.7), the assumptions on the initial data (1.13) and (2.3), as well as (2.27), we deduce from the above estimate that

$$\sup_{t\in(0,T)} \|\vartheta^N - H^\alpha(\vartheta^N,\hat{\vartheta}))\|_1 + \int_Q \frac{|\nabla\vartheta^N|^2}{(\vartheta^N)^{\alpha+1}} \,\mathrm{d}x\,\mathrm{d}t \le C. \tag{2.30}$$

Next, using (2.27) and the Young inequality, we obtain the estimate

$$\int_\Omega |\vartheta^N| \,\mathrm{d}x \le \|\vartheta^N - H^\alpha(\vartheta^N,\hat{\vartheta}))\|_1 + \int_\Omega c_2(\vartheta^N)^{1-\alpha} \,\mathrm{d}x \le \|\vartheta^N - H^\alpha(\vartheta^N,\hat{\vartheta}))\|_1 + C + \frac{1}{2}\int_\Omega |\vartheta^N| \,\mathrm{d}x.$$

Consequently, (2.30) implies

$$\|\vartheta^N\|_{L^\infty(L^1)} \le C. \tag{2.31}$$

We recall that we cannot let $\alpha \to 0_+$ in (2.30), since in that case we would have

$$\int_Q \frac{|\nabla\vartheta^N|^2}{(\vartheta^N)^{\alpha+1}} \,\mathrm{d}x\,\mathrm{d}t \overset{\alpha\to 0^+}{\longrightarrow} \infty,$$

and we would thus lose the estimate (2.30). We can now use (2.30) together with (1.14)–(1.15) to estimate

$$\int_Q \left|\nabla\left((\vartheta^N)^{\frac{1-\alpha}{2}} - \hat{\vartheta}^{\frac{1-\alpha}{2}}\right)\right|^2 \mathrm{d}x\,\mathrm{d}t \le \int_Q \frac{\left|\nabla\vartheta^N\right|^2}{(\vartheta^N)^{\alpha+1}} + \frac{|\nabla\hat{\vartheta}|^2}{(\hat{\vartheta})^{\alpha+1}} \,\mathrm{d}x\,\mathrm{d}t \le C,$$

and thanks to the Poincaré inequality and the classical Sobolev embedding, we conclude that for all finite $s \in [1,\infty)$ it holds

$$(\vartheta^N)^{\frac{1-\alpha}{2}} - (\hat{\vartheta})^{\frac{1-\alpha}{2}} \in L^2(0,T;W_0^{1,2}) \hookrightarrow L^2(0,T;L^s(\Omega)).$$

Furthermore, it follows from the upper bound on $\hat{\vartheta}$ that

$$(\vartheta^N)^{\frac{1-\alpha}{2}} \in L^2(0,T;L^s(\Omega)) \qquad \text{for all } s<\infty. \tag{2.32}$$

Interpolating (2.31) and (2.32), and letting $s \to \infty$ and $\alpha \to 0^+$, we obtain

$$\|\vartheta^N\|_{L^r(Q)} \le C \qquad \text{for } r \in [1,2), \tag{2.33}$$

where C may blow up as $r \to 2_-$.

Let us now estimate the gradient of ϑ^N using Hölder's inequality:

$$\begin{aligned}
\int_Q |\nabla\vartheta^N|^t \,\mathrm{d}x\,\mathrm{d}t &= \int_Q \left(\frac{|\nabla\vartheta^N|}{(\vartheta^N)^{\frac{\alpha+1}{2}}}\right)^t (\vartheta^N)^{\frac{t(\alpha+1)}{2}} \,\mathrm{d}x\,\mathrm{d}t \\
&\le \left(\int_Q \frac{|\nabla\vartheta^N|^2}{(\vartheta^N)^{\alpha+1}} \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{t}{2}} \left(\int_Q (\vartheta^N)^{\frac{t(\alpha+1)}{2-t}} \,\mathrm{d}x\,\mathrm{d}t\right)^{\frac{2-t}{2}},
\end{aligned}$$

where the first factor is bounded by (2.30) and the second factor is finite by (2.33) if $t < \frac{4}{3+\alpha}$. Hence, letting $\alpha \to 0^+$, we obtain

$$\|\nabla \vartheta^N\|_{L^t(Q)} \le C \qquad \text{for } t \in \left[1, \frac{4}{3}\right). \tag{2.34}$$

Finally, we focus on the estimate for the time derivative of ϑ^N. It follows from (2.19) that for almost all time $t \in (0,T)$, we have (here $K := \{\psi \in W_0^{1,5}(\Omega);\ \|\psi\|_{1,5} \le 1\}$) thanks to embedding $W^{1,5} \hookrightarrow L^\infty$, Hölder's inequality, the Young inequality and assumption (1.7) that

$$\begin{aligned}
\|\partial_t \vartheta^N(t)\|_{(W_0^{1,5})^*} &= \sup_{\psi \in K} \langle \partial_t \vartheta^N(t)\,, \psi \rangle \\
&= \sup_{\psi \in K} \left(-\int_\Omega \kappa(\vartheta^N) \nabla \vartheta^N \cdot \nabla \psi \,\mathrm{d}x + \int_\Omega \boldsymbol{\mathcal{S}}^N(t) : D\boldsymbol{u}^N \psi \,\mathrm{d}x + \int_\Omega \vartheta^N \boldsymbol{u}^N \cdot \nabla \psi \,\mathrm{d}x \right) \\
&\le \overline{\kappa} \|\nabla \vartheta^N(t)\|_{\frac{5}{4}} + C \int_\Omega \boldsymbol{\mathcal{S}}^N(t) : D\boldsymbol{u}^N(t) \,\mathrm{d}x + \|\vartheta^N(t)\, \boldsymbol{u}^N(t)\|_{\frac{5}{4}} \\
&\le C \int_\Omega 1 + |\nabla \vartheta^N(t)|^{\frac{5}{4}} + \boldsymbol{\mathcal{S}}^N(t) : D\boldsymbol{u}^N(t) + |\vartheta^N(t)|^{\frac{20}{11}} + |\boldsymbol{u}^N(t)|^4 \,\mathrm{d}x.
\end{aligned}$$

Integrating the above inequality over the time interval $(0,T)$, and using the uniform estimates (2.33), (2.34), and (2.24), we obtain

$$\|\partial_t \vartheta^N\|_{L^1((W_0^{1,5})^*)} \le C. \tag{2.35}$$

The uniform estimates (2.24), (2.30), (2.31), (2.33), and (2.34), together with the Aubin–Lions lemma, provide us with the following convergence results for a subsequence that we do not relabel:

$$\partial_t \boldsymbol{u}^N \rightharpoonup \partial_t \boldsymbol{u} \qquad \text{in } L^{p'}(0,T;(W^{2,2}_{0,\mathrm{div}})^*), \tag{2.36}$$

$$\boldsymbol{u}^N \rightharpoonup^* \boldsymbol{u} \qquad \text{in } L^\infty(0,T;L^2_{0,\mathrm{div}}), \tag{2.37}$$

$$\boldsymbol{u}^N \rightharpoonup \boldsymbol{u} \qquad \text{in } L^p(0,T;V), \tag{2.38}$$

$$\boldsymbol{u}^N \to \boldsymbol{u} \qquad \text{in } L^r(0,T;L^r) \text{ for } r \in [1,2p), \tag{2.39}$$

$$\boldsymbol{\mathcal{S}}^N \rightharpoonup \boldsymbol{\mathcal{S}} \qquad \text{in } L^{p'}(0,T;L^{p'}(\Omega;\mathbb{R}^{2\times 2}_{sym})). \tag{2.40}$$

$$\vartheta^N \rightharpoonup \vartheta \qquad \text{in } L^s(0,T;L^s) \text{ for } s \in [1,2), \tag{2.41}$$

$$\nabla \vartheta^N \rightharpoonup \nabla \vartheta \qquad \text{in } L^t(0,T;L^t) \text{ for } t \in [1,4/3), \tag{2.42}$$

$$(\vartheta^N)^\alpha \rightharpoonup (\vartheta)^\alpha \qquad \text{in } L^2(0,T;W^{1,2}) \text{ for } \alpha \in (0,1/2). \tag{2.43}$$

Furthermore, the generalized version of the Aubin–Lions lemma (see [18]), together with the uniform bound (2.35) and the above convergence results, implies

$$\vartheta^N \to \vartheta \qquad \text{almost everywhere in } Q, \tag{2.44}$$

$$\vartheta^N \to \vartheta \qquad \text{in } L^s(0,T,L^s) \text{ for } s \in [1,2). \tag{2.45}$$

Consequently, from (2.31) and (2.45), we can also deduce that

$$\vartheta \in L^\infty(0,T;L^1). \tag{2.46}$$

We now test equation (2.18) by $\varphi \in \mathcal{D}(0,T)$ and take the limit $N \to \infty$. By convergences (2.36)-(2.40), we have

$$\int_0^T \langle \partial_t \boldsymbol{u}\,, \boldsymbol{\psi} \rangle_{W^{2,2}_{0,\mathrm{div}}} \varphi \,\mathrm{d}t + \int_Q \varphi \boldsymbol{\mathcal{S}} : D\boldsymbol{\psi} \,\mathrm{d}x\,\mathrm{d}t = \int_Q \varphi(\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{\psi} \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \langle \boldsymbol{f}\,, \boldsymbol{\psi} \rangle_V \varphi \,\mathrm{d}t, \tag{2.47}$$

for all $\boldsymbol{\psi} \in W^{2,2}_{0,\mathrm{div}}(\Omega)$. We will now show that $\partial_t \boldsymbol{u} \in L^{p'}(0,T;V^*)$. We know that by the definition of the weak time derivative

$$\int_0^T \langle \partial_t \boldsymbol{u}\,, \boldsymbol{\psi} \rangle_{W^{2,2}_{0,\mathrm{div}}} \varphi \,\mathrm{d}t = -\int_Q \boldsymbol{u} \partial_t(\boldsymbol{\psi}\varphi) \,\mathrm{d}x\,\mathrm{d}t = \mathcal{F}(\boldsymbol{\psi}\varphi), \qquad \forall \varphi \in \mathcal{D}(0,T), \forall \boldsymbol{\psi} \in W^{2,2}_{0,\mathrm{div}}(\Omega) \tag{2.48}$$

where

$$\mathcal{F}(\boldsymbol{\psi}\varphi) := \int_Q \varphi(\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{\psi} - \varphi \boldsymbol{\mathcal{S}} : D\boldsymbol{\psi} \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \langle \boldsymbol{f}\,, \varphi \boldsymbol{\psi} \rangle_V \,\mathrm{d}t.$$

Furthermore, $\mathcal{F}$ is bounded for all $\boldsymbol{w} \in L^p(0,T;V)$, since

$$|\mathcal{F}(\boldsymbol{w})| \leq \|\boldsymbol{u}\|^2_{L^{2p'}(Q)} \|\nabla \boldsymbol{w}\|_{L^p(Q)} + \|\boldsymbol{\mathcal{S}}\|_{L^{p'}(Q)} \|D\boldsymbol{w}\|_{L^p(Q)} + \|\boldsymbol{f}\|_{L^{p'}(V^*)} \|\boldsymbol{w}\|_{L^p(V)}.$$

We hence have $\mathcal{F} \in L^{p'}(0,T;V^*)$. Since the set $span\{\varphi\boldsymbol{\psi}; \varphi \in \mathcal{D}(0,T), \boldsymbol{\psi} \in W^{2,2}_{0,\mathrm{div}}(\Omega)\}$ is dense in $L^p(0,T;V)$, there exists a uniquely defined extension of $\partial_t \boldsymbol{u}$ (denoted again $\partial_t \boldsymbol{u}$) such that

$$\partial_t \boldsymbol{u} \in L^{p'}(0,T;V^*). \tag{2.49}$$

This extension is defined by $\langle \partial_t \boldsymbol{u} \,, \boldsymbol{w} \rangle_{L^p(V)} = \mathcal{F}(\boldsymbol{w})$ for all $\boldsymbol{w} \in L^p(0,T;V)$. Hence, we can conclude that for every $\boldsymbol{w} \in L^p(0,T,V)$ it holds

$$\int_0^T \langle \partial_t \boldsymbol{u} \,, \boldsymbol{w} \rangle_V \,\mathrm{d}t + \int_Q \boldsymbol{\mathcal{S}} : D\boldsymbol{w} \,\mathrm{d}x \,\mathrm{d}t = \int_Q (\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{w} \,\mathrm{d}x \,\mathrm{d}t + \int_0^T \langle \boldsymbol{f} \,, \boldsymbol{w} \rangle_V \,\mathrm{d}t. \tag{2.50}$$

Additionally, by (2.38) and (2.49), we have $\boldsymbol{u} \in C([0,T]; L^2_{0,\mathrm{div}})$. Furthermore, it is straightforward to show that $\boldsymbol{u}(0) = \boldsymbol{u}_0$. In addition, if we extend f also for times $t > T$ (e.g., by zero), we see that the above identity remains valid with T replaced by an arbitrary $T^* > T$, and all quantities are well defined on $(0,T^*)$. Consequently, setting $\boldsymbol{w} := \boldsymbol{u}\chi_{[0,\tau]}$ with arbitrary $\tau > 0$ and using $\operatorname{div} \boldsymbol{u} = 0$, we obtain

$$\|\boldsymbol{u}(\tau)\|_2^2 + 2\int_0^\tau \int_\Omega \boldsymbol{\mathcal{S}} : D\boldsymbol{u} \,\mathrm{d}x \,\mathrm{d}t = \|\boldsymbol{u}_0\|_2^2 + 2\int_0^\tau \langle \boldsymbol{f} \,, \boldsymbol{u} \rangle_V \,\mathrm{d}t. \tag{2.51}$$

Next, we identify $\boldsymbol{\mathcal{S}}$ with $\boldsymbol{\mathcal{S}}^*(\vartheta, D\boldsymbol{u})$ by the Minty trick, where we use the zero divergence of $\boldsymbol{u}^N$ and $\boldsymbol{u}$ to avoid the limit passage in the convective term. We recall that $\boldsymbol{\mathcal{S}}^N := \boldsymbol{\mathcal{S}}^*(\vartheta^N, D\boldsymbol{u}^N)$. From the first property in (1.8), we have that for all $\boldsymbol{\mathcal{D}} \in L^p(Q;\mathbb{R}^{2\times2}_{sym})$, it holds (here, following the above note, we extend the time interval also beyond T by extending $\boldsymbol{f}$ and $\boldsymbol{\mathcal{D}}$ by zero and integrating the identity (2.25) over $(0, T+\tau)$)

$$\begin{aligned}
0 &\leq \int_Q (\boldsymbol{\mathcal{S}}^N - \boldsymbol{\mathcal{S}}^*(\vartheta^N, \boldsymbol{\mathcal{D}})) : (D\boldsymbol{u}^N - \boldsymbol{\mathcal{D}}) \,\mathrm{d}x \,\mathrm{d}t \leq \int_0^{T+\tau} \int_\Omega (\boldsymbol{\mathcal{S}}^N - \boldsymbol{\mathcal{S}}^*(\vartheta^N, \boldsymbol{\mathcal{D}})) : (D\boldsymbol{u}^N - \boldsymbol{\mathcal{D}}) \,\mathrm{d}x \,\mathrm{d}t \\
&= \frac{1}{2}\left(\|\boldsymbol{u}_0^N\|_2^2 - \|\boldsymbol{u}^N(T+\tau)\|_2^2\right) \\
&+ \int_0^T \left\langle \boldsymbol{f} \,, \boldsymbol{u}^N \right\rangle_V \,\mathrm{d}t - \int_0^T \int_\Omega \boldsymbol{\mathcal{S}}^N : \boldsymbol{\mathcal{D}} + \boldsymbol{\mathcal{S}}^*(\vartheta^N, \boldsymbol{\mathcal{D}}) : (D\boldsymbol{u}^N - \boldsymbol{\mathcal{D}}) \,\mathrm{d}x \,\mathrm{d}t.
\end{aligned} \tag{2.52}$$

Next, we integrate the above inequality over $\tau \in (0,\varepsilon)$ and divide the result by ε to get

$$\begin{aligned}
0 &\leq \frac{1}{2}\left(\|\boldsymbol{u}_0^N\|_2^2 - \frac{1}{\varepsilon}\int_0^\varepsilon \|\boldsymbol{u}^N(T+\tau)\|_2^2 \,\mathrm{d}\tau\right) \\
&+ \int_0^T \left\langle \boldsymbol{f} \,, \boldsymbol{u}^N \right\rangle_V \,\mathrm{d}t - \int_0^T \int_\Omega \boldsymbol{\mathcal{S}}^N : \boldsymbol{\mathcal{D}} + \boldsymbol{\mathcal{S}}^*(\vartheta^N, \boldsymbol{\mathcal{D}}) : (D\boldsymbol{u}^N - \boldsymbol{\mathcal{D}}) \,\mathrm{d}x \,\mathrm{d}t.
\end{aligned}$$

Using the convergence results (2.38)–(2.40), the convergence (2.44), the assumption (1.8), and the convergence of the initial conditions (2.3), we can let $N \to \infty$ in the above inequality to obtain

$$\begin{aligned}
0 &\leq \frac{1}{2}\left(\|\boldsymbol{u}_0\|_2^2 - \frac{1}{\varepsilon}\int_0^\varepsilon \|\boldsymbol{u}(T+\tau)\|_2^2 \,\mathrm{d}\tau\right) \\
&+ \int_0^T \langle \boldsymbol{f} \,, \boldsymbol{u} \rangle_V \,\mathrm{d}t - \int_0^T \int_\Omega \boldsymbol{\mathcal{S}} : \boldsymbol{\mathcal{D}} + \boldsymbol{\mathcal{S}}^*(\vartheta, \boldsymbol{\mathcal{D}}) : (D\boldsymbol{u} - \boldsymbol{\mathcal{D}}) \,\mathrm{d}x \,\mathrm{d}t \\
&= \frac{1}{2}\left(\|\boldsymbol{u}(T)\|_2^2 - \frac{1}{\varepsilon}\int_0^\varepsilon \|\boldsymbol{u}(T+\tau)\|_2^2 \,\mathrm{d}\tau\right) + (\boldsymbol{\mathcal{S}} - \boldsymbol{\mathcal{S}}^*(\vartheta, \boldsymbol{\mathcal{D}})) : (D\boldsymbol{u} - \boldsymbol{\mathcal{D}}) \,\mathrm{d}x \,\mathrm{d}t,
\end{aligned}$$

where for the second equality, we used the identity (2.51). Letting $\varepsilon \to 0_+$ and using the continuity of $\boldsymbol{u} \in C([0,T^*]; L^2_{0,\mathrm{div}})$ with $T^* > T$, we deduce

$$0 \leq \int_Q (\boldsymbol{\mathcal{S}} - \boldsymbol{\mathcal{S}}^*(\vartheta, \boldsymbol{\mathcal{D}})) : (D\boldsymbol{u} - \boldsymbol{\mathcal{D}}) \,\mathrm{d}x \,\mathrm{d}t. \tag{2.53}$$

We now take $\boldsymbol{\mathcal{D}} := D\boldsymbol{u} \mp \varepsilon \boldsymbol{\mathcal{W}}$ for some $\boldsymbol{\mathcal{W}} \in L^p(Q;\mathbb{R}^{2\times2}_{sym})$ and $\varepsilon \in (0,1)$. Hence, we have

$$0 \leq \pm \int_Q (\boldsymbol{\mathcal{S}} - \boldsymbol{\mathcal{S}}^*(\vartheta, D\boldsymbol{u} \mp \varepsilon\boldsymbol{\mathcal{W}})) : \boldsymbol{\mathcal{W}} \,\mathrm{d}x \,\mathrm{d}t \overset{\varepsilon\to0+}{\to} \pm \int_Q (\boldsymbol{\mathcal{S}} - \boldsymbol{\mathcal{S}}^*(\vartheta, D\boldsymbol{u})) : \boldsymbol{\mathcal{W}} \,\mathrm{d}x \,\mathrm{d}t,$$

since $\boldsymbol{\mathcal{S}}^*$ is continuous. Consequently,

$$0 = \int_Q (\boldsymbol{\mathcal{S}} - \boldsymbol{\mathcal{S}}^*(\vartheta, D\boldsymbol{u})) : \boldsymbol{\mathcal{W}} \,\mathrm{d}x \,\mathrm{d}t$$

for all $\boldsymbol{\mathcal{W}} \in L^p(Q;\mathbb{R}^{2\times 2}_{sym})$. Therefore, $\boldsymbol{\mathcal{S}} = \boldsymbol{\mathcal{S}}^*(\vartheta, D\boldsymbol{u})$ almost everywhere in Q. We thus have $\boldsymbol{u} \in L^p(0,T;V)$ such that $\partial_t \boldsymbol{u} \in L^{p'}(0,T;V^*)$

$$\int_0^T \langle \partial_t \boldsymbol{u} \, , \boldsymbol{w}\rangle_V \, \mathrm{d}t + \int_Q \boldsymbol{\mathcal{S}}^*(\vartheta, D\boldsymbol{u}) : D\boldsymbol{w} \, \mathrm{d}x \, \mathrm{d}t = \int_Q (\boldsymbol{u} \otimes \boldsymbol{u}) : \nabla \boldsymbol{w} \, \mathrm{d}x \, \mathrm{d}t + \int_0^T \langle \boldsymbol{f} \, , \boldsymbol{w}\rangle_V \, \mathrm{d}t, \tag{2.54}$$

for all $\boldsymbol{w} \in L^p(0,T;V)$. We have hence found $\boldsymbol{u}$ fulfilling the *Momentum equation* (1.22).

Next, we even strengthen the convergence result for $\boldsymbol{\mathcal{S}}^N$ and $D\boldsymbol{u}^N$. Namely, we show that

$$\boldsymbol{\mathcal{S}}^N : D\boldsymbol{u}^N \rightharpoonup \boldsymbol{\mathcal{S}} : D\boldsymbol{u} \qquad \text{weakly in } L^1(Q). \tag{2.55}$$

To do so, we set $\boldsymbol{\mathcal{D}} := D\boldsymbol{u}$ in (2.52). Repeating the same procedure as above yields

$$\begin{aligned} &\lim_{N\to\infty} \left| \int_Q (\boldsymbol{\mathcal{S}}^N - \boldsymbol{\mathcal{S}}^*(\vartheta^N, D\boldsymbol{u})) : (D\boldsymbol{u}^N - D\boldsymbol{u}) \right| \mathrm{d}x \, \mathrm{d}t \\ &\quad = \lim_{N\to\infty} \int_Q (\boldsymbol{\mathcal{S}}^N - \boldsymbol{\mathcal{S}}^*(\vartheta^N, D\boldsymbol{u})) : (D\boldsymbol{u}^N - D\boldsymbol{u}) \, \mathrm{d}x \, \mathrm{d}t \le \int_Q (\boldsymbol{\mathcal{S}} - \boldsymbol{\mathcal{S}}^*(\vartheta, D\boldsymbol{u})) : (D\boldsymbol{u} - D\boldsymbol{u}) = 0. \end{aligned}$$

Hence

$$(\boldsymbol{\mathcal{S}}^N - \boldsymbol{\mathcal{S}}^*(\vartheta^N, D\boldsymbol{u})) : (D\boldsymbol{u}^N - D\boldsymbol{u}) \to 0 \qquad \text{in } L^1(Q). \tag{2.56}$$

Next, let $w \in L^\infty(Q)$ be arbitrary. Then, using (2.38)–(2.40), (2.44), the assumption (1.8), and the dominated convergence theorem, we have

$$\boldsymbol{\mathcal{S}}^*(\vartheta^N, D\boldsymbol{u}) \to \boldsymbol{\mathcal{S}} \qquad \text{strongly in } L^{p'}(Q;\mathbb{R}^{2\times 2}), \tag{2.57}$$

$$D\boldsymbol{u}^N w \rightharpoonup D\boldsymbol{u} w \qquad \text{weakly in } L^p(Q;\mathbb{R}^{2\times 2}). \tag{2.58}$$

Therefore

$$\begin{aligned} \lim_{N\to\infty} \int_Q \boldsymbol{\mathcal{S}}^N : D\boldsymbol{u}^N w \, \mathrm{d}x \, \mathrm{d}t &= \lim_{N\to\infty} \int_Q \boldsymbol{\mathcal{S}}^N : (D\boldsymbol{u}^N - D\boldsymbol{u}) w + \boldsymbol{\mathcal{S}}^N : D\boldsymbol{u} w \, \mathrm{d}x \, \mathrm{d}t \\ &\overset{(2.40)}{=} \int_Q \boldsymbol{\mathcal{S}} : D\boldsymbol{u} w \, \mathrm{d}x \, \mathrm{d}t + \lim_{N\to\infty} \int_Q (\boldsymbol{\mathcal{S}}^N - \boldsymbol{\mathcal{S}}^*(\vartheta^N, D\boldsymbol{u})) : (D\boldsymbol{u}^N - D\boldsymbol{u}) w \, \mathrm{d}x \, \mathrm{d}t \\ &\quad + \lim_{N\to\infty} \int_Q \boldsymbol{\mathcal{S}}^*(\vartheta^N, D\boldsymbol{u}) : (D\boldsymbol{u}^N - D\boldsymbol{u}) w \, \mathrm{d}x \, \mathrm{d}t \\ &\overset{(2.56)}{=} \int_Q \boldsymbol{\mathcal{S}} : D\boldsymbol{u} w \, \mathrm{d}x \, \mathrm{d}t - \lim_{N\to\infty} \int_Q (\boldsymbol{\mathcal{S}}^*(\vartheta^N, D\boldsymbol{u}) - \boldsymbol{\mathcal{S}}) : (D\boldsymbol{u}^N - D\boldsymbol{u}) w \, \mathrm{d}x \, \mathrm{d}t \\ &\quad + \lim_{N\to\infty} \int_Q \boldsymbol{\mathcal{S}} : (D\boldsymbol{u}^N - D\boldsymbol{u}) w \, \mathrm{d}x \, \mathrm{d}t \overset{(2.57),(2.58)}{=} \int_Q \boldsymbol{\mathcal{S}} : D\boldsymbol{u} w \, \mathrm{d}x \, \mathrm{d}t \end{aligned}$$

and we see that (2.55) holds true.

Having the above convergence results, we can now finally focus on the limiting procedure in the temperature equation (2.19). Let us now consider (2.19) with $\psi \in C_0^\infty((-\infty,T)\times\Omega)$ and integrate it over $(0,T)$ to get

$$\int_0^T \left\langle \partial_t \vartheta^N \, , \psi \right\rangle_{W_0^{1,2}} \mathrm{d}t + \int_Q \kappa(\vartheta^N) \nabla \vartheta^N \cdot \nabla \psi \, \mathrm{d}x \, \mathrm{d}t = \int_Q \boldsymbol{\mathcal{S}}^N : D\boldsymbol{u}^N \psi + \vartheta^N \boldsymbol{u}^N \cdot \nabla \psi \, \mathrm{d}x \, \mathrm{d}t. \tag{2.59}$$

Then, integrating by parts in the first term, we can use (2.36)–(2.45), (2.55) and let $N \to \infty$ to conclude that

$$-\int_Q \vartheta \, \partial_t \psi \, \mathrm{d}x \, \mathrm{d}t + \int_Q \kappa(\vartheta) \nabla \vartheta \cdot \nabla \psi \, \mathrm{d}x \, \mathrm{d}t = \int_Q \boldsymbol{\mathcal{S}} : D\boldsymbol{u} \, \psi + \vartheta \, \boldsymbol{u} \cdot \nabla \psi \, \mathrm{d}x \, \mathrm{d}t + \int_\Omega \vartheta_0 \psi(0) \, \mathrm{d}x \tag{2.60}$$

for all $\psi \in C_0^\infty((-\infty,T)\times\Omega)$.

We have thus found $\vartheta \in L^\infty(0,T;L^1) \cap L^r(Q)$, for $r \in [1,2)$, such that

$$\begin{aligned} &\vartheta - \hat{\vartheta} \in L^s(0,T;W_0^{1,s}) \quad \text{for } s \in [1,4/3), \\ &(\vartheta)^\alpha \in L^2(0,T;W^{1,2}) \quad \text{for } \alpha \in (0,1/2), \end{aligned} \tag{2.61}$$

and ϑ solves the *internal energy balance* (1.23). Furthermore, the minimum principle $\vartheta^N \ge \mu$ for $N \in \mathbb{N}$, together with (2.44), implies $\vartheta \ge \mu$ almost everywhere.

2.5. **Initial condition for temperature.** We want to show that the initial condition for the temperature is attained at least in some strong sense. Notice that (2.60) already indicates a kind of weak attainment (without specifying it rigorously). Here, our goal is different; more specifically, we show the strong attainment in $L^1(\Omega)$, but only for the essential limit as $t \to 0_+$. In the next section, when we show that the temperature is continuous, we prove the attainment of the initial condition for the classical limit $t \to 0_+$. Hence, our goal here is to show the existence of a set $S \subset (0,T)$ such that $(0,T) \setminus S$ is of measure zero and

$$\vartheta(t) \to \vartheta_0 \qquad \text{in } L^1(\Omega) \text{ as } S \ni t \to 0_+. \tag{2.62}$$

We note that by (2.46) we have $\sqrt{\theta(t)} \in L^2(\Omega)$ for all $t \in (0,T)$. From (2.45) we know that ϑ^N converges to ϑ almost everywhere on $(0,T)$. Hence, there exists a set $S \subset (0,T)$, such that $(0,T) \setminus S$ is of measure zero and for all $t \in S$ it holds

$$\vartheta^N(t) \to \vartheta(t) \qquad \text{in } L^s(\Omega),\ s < 2. \tag{2.63}$$

Let us now show that

$$\liminf_{S \in t \to 0+} \int_\Omega \sqrt{\vartheta(t)}\varphi \,\mathrm{d}x \geq \int_\Omega \sqrt{\vartheta_0}\varphi \,\mathrm{d}x \qquad \text{for all } \varphi \in L^2(\Omega),\ \varphi \geq 0. \tag{2.64}$$

We set $\psi := \frac{\varphi}{\sqrt{\vartheta^N}}$ in equation (2.19), where $0 \leq \varphi \in C_0^\infty(\Omega)$, and integrate over the time interval $(0,t)$, with $t \in S$ arbitrary. Note that such ψ is an admissible test function for (2.19), since $\frac{\varphi}{\sqrt{\vartheta^N}}(t) \in W_0^{1,2}(\Omega)$ for almost all $t \in (0,T)$. Additionally, $\frac{\varphi}{\sqrt{\vartheta^N}} \in L^\infty(Q)$ due to the minimum principle. Doing so, we obtain

$$\begin{aligned}\int_0^t \partial_t \int_\Omega 2\sqrt{\vartheta^N}\varphi \,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t \int_\Omega \kappa(\vartheta^N)\nabla\vartheta^N \cdot \nabla \frac{\varphi}{\sqrt{\vartheta^N}} \,\mathrm{d}x\,\mathrm{d}\tau = \\ = \int_0^t \int_\Omega \boldsymbol{S}^N : D\boldsymbol{u}^N \frac{\varphi}{\sqrt{\vartheta^N}} \,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t \int_\Omega \vartheta^N \boldsymbol{u}^N \cdot \nabla \frac{\varphi}{\sqrt{\vartheta^N}} \,\mathrm{d}x\,\mathrm{d}\tau,\end{aligned}$$

which can be rewritten into (using integration by parts and the fact that $\operatorname{div} \boldsymbol{u}^N = 0$)

$$\begin{aligned}&2\int_\Omega \sqrt{\vartheta^N(t)}\varphi - \sqrt{\vartheta_0^N}\varphi \,\mathrm{d}x + \int_0^t \int_\Omega \left(\frac{\kappa(\vartheta^N)}{\sqrt{\vartheta^N}}\nabla\vartheta^N - 2\sqrt{\vartheta^N}\boldsymbol{u}^N\right) \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}\tau = \\ &= \int_0^t \int_\Omega \kappa(\vartheta^N)\varphi \frac{|\nabla\vartheta^N|^2}{2(\vartheta^N)^{3/2}} \,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t \int_\Omega \boldsymbol{S}^N : D\boldsymbol{u}^N \frac{\varphi}{\sqrt{\vartheta^N}} \,\mathrm{d}x\,\mathrm{d}\tau \geq 0,\end{aligned} \tag{2.65}$$

We can now let $N \to \infty$ in the inequality (2.65). Using convergence results (2.63), (2.3), (2.44), (2.42) and (2.39) yields to

$$2\int_\Omega \sqrt{\vartheta(t)}\varphi - \sqrt{\vartheta_0}\varphi \,\mathrm{d}x + \int_0^t \int_\Omega \left(\frac{\kappa(\vartheta)}{\sqrt{\vartheta}}\nabla\vartheta - 2\sqrt{\vartheta}\boldsymbol{u}\right) \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}\tau \geq 0. \tag{2.66}$$

We can pass to the essential limit inferior $S \ni t \to 0_+$ in (2.66). The time integral vanishes, and we deduce

$$\liminf_{S \in t \to 0_+} \int_\Omega \sqrt{\vartheta(t)}\varphi \,\mathrm{d}x \geq \int_\Omega \sqrt{\vartheta_0}\varphi \,\mathrm{d}x \qquad \text{for all } \varphi \in C_0^\infty(\Omega),\ \varphi \geq 0. \tag{2.67}$$

Since $\sqrt{\vartheta} \in L^\infty(0,T;L^2)$ we can extend the result for all nonnegative $\varphi \in L^2(\Omega)$ by the density of smooth functions. Hence (2.64) holds.

Next, our goal is to obtain the following result

$$\lim_{S \ni t \to 0+} \int_\Omega \vartheta(t)\varphi \,\mathrm{d}x = \int_\Omega \vartheta_0\varphi \qquad \text{for } \varphi \in C_0^\infty(\Omega),\ 0 \leq \varphi \leq 1. \tag{2.68}$$

We set $\psi := \varphi$ in (2.19), where $\varphi \in C_0^\infty(\Omega)$, $0 \leq \varphi \leq 1$ and integrate the result over time interval $(0,t)$, where $t \in S$ is arbitrary. We have

$$\int_\Omega \vartheta^N(t)\varphi \,\mathrm{d}x = \int_\Omega \vartheta_0^N\varphi \,\mathrm{d}x + \int_0^t \int_\Omega \boldsymbol{S}^N : D\boldsymbol{u}^N\varphi + \vartheta^N\boldsymbol{u}^N \cdot \nabla\varphi - \kappa(\vartheta^N)\nabla\vartheta^N \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}\tau \tag{2.69}$$

We now want to pass to the limit $N \to \infty$ in (2.69). The right-hand side converges by (2.44), (2.42), (2.3), (2.55), (2.39) and (2.41). The left-hand side converges by (2.63). We thus have

$$\int_\Omega \vartheta(t)\varphi \,\mathrm{d}x = \int_\Omega \vartheta_0\varphi \,\mathrm{d}x + \int_0^t \int_\Omega \boldsymbol{S} : D\boldsymbol{u}\varphi + \vartheta\boldsymbol{u} \cdot \nabla\varphi - \kappa(\vartheta)\nabla\vartheta \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}\tau \tag{2.70}$$

We now let $S \ni t \to 0_+$ in (2.70) and obtain the desired result (2.68).

Finally, we show that (2.67) and (2.68) imply the attainment of the initial condition locally in Ω. In fact, for $\varphi \in C_0^\infty(\Omega)$, $0 \le \varphi \le 1$, we can estimate

$$0 \le \lim_{S \ni t \to 0+} \int_\Omega \left(\sqrt{\vartheta(t)} - \sqrt{\vartheta_0} \right)^2 \varphi \,\mathrm{d}x \le \lim_{S \ni t \to 0+} \int_\Omega \vartheta(t)\varphi + \vartheta_0 \varphi \,\mathrm{d}x - \liminf_{S \ni t \to 0+} 2 \int_\Omega \sqrt{\vartheta(t)}\sqrt{\vartheta_0}\varphi \,\mathrm{d}x$$
$$\overset{(2.68),(2.64)}{\le} 0,$$

where we could use the convergence result (2.64), since $\sqrt{\vartheta_0}\varphi$ belongs to $L^2(\Omega)$. For any compact $K \subset \Omega$, we can find $\varphi \in C_0^\infty(\Omega)$, $0 \le \varphi \le 1$, such that $\varphi = 1$ on K. The inequality above hence implies that for $S \ni t \to 0$ and arbitrary $\beta \in (0,1)$ it holds

$$\vartheta^\beta(t) \to \vartheta_0^\beta \qquad \text{in } L^{\frac{1}{\beta}}(K) \text{ for any compact } K \subset \Omega.$$

The above result can be even extended to the whole domain Ω by using the uniform estimate (2.46), which gives that for any $\beta \in (0,1)$ and any $\gamma \in [1, \frac{1}{\beta})$ we have

$$\vartheta^\beta(t) \to \vartheta_0^\beta \qquad \text{in } L^\gamma(\Omega).$$

This and the uniform bound (2.46) then yield into

$$\vartheta^\beta(t) \rightharpoonup \vartheta_0^\beta \qquad \text{in } L^{\frac{1}{\beta}}(\Omega) \text{ as } S \ni t \to 0+. \tag{2.71}$$

Let us now show that

$$\limsup_{S \ni t \to 0_+} \int_\Omega \vartheta(t) \,\mathrm{d}x \le \int_\Omega \vartheta_0 \,\mathrm{d}x, \tag{2.72}$$

which, combined with (2.71), leads to the desired result

$$\vartheta^\beta(t) \to \vartheta_0^\beta \qquad \text{strongly in } L^{\frac{1}{\beta}}(\Omega) \text{ as } S \ni t \to 0_+, \tag{2.73}$$

valid for all $\beta \in (0,1]$.

Hence, we focus on (2.72). We set $\psi := 1 - \left(\frac{\hat\vartheta}{\vartheta^N}\right)^{\frac{\alpha+1}{2}}$ in (2.19) and integrate over time interval $(0,t)$, where $t \in S$ is arbitrary. Note, that ψ is an admissible test function since it is zero at the boundary, $\psi(t) \in L^\infty(\Omega)$ for a.a. $t \in S$, and

$$\nabla\psi = -\frac{\alpha+1}{2}\left(\frac{\hat\vartheta}{\vartheta^N}\right)^{\frac{\alpha+1}{2}} \left(\frac{\nabla\hat\vartheta}{\hat\vartheta} - \frac{\nabla\vartheta^N}{\vartheta^N}\right) \in L^2(\Omega) \text{ for a.a. } t \in S,$$

which follows from (2.61) and (1.14) for all $\alpha > 0$. We obtain

$$\int_\Omega \left(\vartheta^N - \frac{2(\vartheta^N)^{\frac{1-\alpha}{2}}(\hat\vartheta)^{\frac{\alpha+1}{2}}}{1-\alpha}\right)(t)\,\mathrm{d}x + \int_0^t \int_\Omega \kappa(\vartheta^N)\nabla\vartheta^N \cdot \nabla\psi \,\mathrm{d}x\,\mathrm{d}\tau =$$
$$= \int_0^t \int_\Omega \boldsymbol{\mathcal{S}}^N : D\boldsymbol{u}^N \psi + \vartheta^N \boldsymbol{u}^N \cdot \nabla\psi \,\mathrm{d}x\,\mathrm{d}\tau + \int_\Omega \left(\vartheta_0^N - \frac{2(\vartheta_0^N)^{\frac{1-\alpha}{2}}(\hat\vartheta)^{\frac{\alpha+1}{2}}}{1-\alpha}\right)\mathrm{d}x.$$

Neglecting the nonnegative term

$$\int_0^t \int_\Omega \frac{\kappa(\vartheta^N)(\alpha+1)(\hat\vartheta)^{\frac{\alpha+1}{2}}}{2(\vartheta^N)^{\frac{\alpha+3}{2}}} |\nabla\vartheta^N|^2 \,\mathrm{d}x\,\mathrm{d}\tau$$

we get (using the fact that ψ is bounded)

$$\begin{aligned} &\int_\Omega \left(\vartheta^N(t) - \frac{2(\vartheta^N(t))^{\frac{1-\alpha}{2}}(\hat\vartheta)^{\frac{\alpha+1}{2}}}{1-\alpha}\right)\mathrm{d}x \le \int_\Omega \left(\vartheta_0^N - \frac{2(\vartheta_0^N)^{\frac{1-\alpha}{2}}(\hat\vartheta)^{\frac{\alpha+1}{2}}}{1-\alpha}\right)\mathrm{d}x + C\int_0^t \int_\Omega \boldsymbol{\mathcal{S}}^N : D\boldsymbol{u}^N \,\mathrm{d}x\,\mathrm{d}\tau \\ &+ \frac{\alpha+1}{1-\alpha}\int_0^t \int_\Omega \kappa(\vartheta^N)(\hat\vartheta)^{\frac{\alpha-1}{2}}\nabla\hat\vartheta \cdot \nabla(\vartheta^N)^{\frac{1-\alpha}{2}} \,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t \int_\Omega \vartheta^N \boldsymbol{u}^N \cdot \nabla\left(\frac{\hat\vartheta}{\vartheta^N}\right)^{\frac{\alpha+1}{2}} \mathrm{d}x\,\mathrm{d}\tau. \end{aligned} \tag{2.74}$$

We want to pass to the limit $N \to \infty$ in (2.74). The term on the left-hand side converges by (2.63). The first term on the right-hand side converges by (2.3). The convergence of the second term on the right-hand side holds due to (2.55). For the third term, it is sufficient to use the weak convergence result (2.43), the assumptions (1.7) and (1.14), and the pointwise convergence (2.44). We use integration by parts and the fact that $\operatorname{div}\boldsymbol{u}^N = 0$ and rewrite the fourth term on the right-hand side of (2.74) as follows

$$\begin{aligned}\int_0^t\int_\Omega \vartheta^N\boldsymbol{u}^N\cdot\nabla\left(\frac{\hat\vartheta}{\vartheta^N}\right)^{\frac{\alpha+1}{2}}\,\mathrm{d}x\,\mathrm{d}\tau &= \frac{\alpha+1}{2}\int_0^t\int_\Omega \hat\vartheta\left(\frac{\hat\vartheta}{\vartheta^N}\right)^{\frac{\alpha-3}{2}}\boldsymbol{u}^N\cdot\nabla\left(\frac{\hat\vartheta}{\vartheta^N}\right)\mathrm{d}x\,\mathrm{d}\tau\\ &= \frac{\alpha+1}{\alpha-1}\int_0^t\int_\Omega \hat\vartheta\boldsymbol{u}^N\cdot\nabla\left(\frac{\hat\vartheta}{\vartheta^N}\right)^{\frac{\alpha-1}{2}}\,\mathrm{d}x\,\mathrm{d}\tau = \frac{\alpha+1}{1-\alpha}\int_0^t\int_\Omega (\vartheta^N)^{\frac{1-\alpha}{2}}\boldsymbol{u}^N\cdot\frac{\nabla\hat\vartheta}{(\hat\vartheta)^{\frac{1-\alpha}{2}}}\,\mathrm{d}x\,\mathrm{d}\tau.\end{aligned}$$

In this integral, we can, however, easily identify the limit due to the assumptions (1.14) and (1.15), the convergence results (2.45) and (2.39), and the facts that $\alpha\in(0,1)$ and $p\ge 2$. We thus obtain

$$\begin{aligned}\int_\Omega\left(\vartheta(t)-\frac{2(\vartheta(t))^{\frac{1-\alpha}{2}}(\hat\vartheta)^{\frac{\alpha+1}{2}}}{1-\alpha}\right)\mathrm{d}x &\le \int_\Omega\left(\vartheta_0-\frac{2(\vartheta_0)^{\frac{1-\alpha}{2}}(\hat\vartheta)^{\frac{\alpha+1}{2}}}{1-\alpha}\right)\mathrm{d}x + C\int_0^t\int_\Omega \boldsymbol{S}:D\boldsymbol{u}\,\mathrm{d}x\,\mathrm{d}\tau\\ &+\frac{\alpha+1}{1-\alpha}\int_0^t\int_\Omega \kappa(\vartheta)(\hat\vartheta)^{\frac{\alpha-1}{2}}\nabla\hat\vartheta\cdot\nabla(\vartheta)^{\frac{1-\alpha}{2}}\,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t\int_\Omega \vartheta\boldsymbol{u}\cdot\nabla\left(\frac{\hat\vartheta}{\vartheta}\right)^{\frac{\alpha+1}{2}}\mathrm{d}x\,\mathrm{d}\tau.\end{aligned}\tag{2.75}$$

Let us take the limit $S\ni t\to 0_+$ in (2.75). Since all the terms on the right-hand side of (2.75) except for the first one converge to zero, we obtain

$$\limsup_{S\ni t\to 0_+}\int_\Omega\left(\vartheta(t)-\frac{2(\vartheta(t))^{\frac{1-\alpha}{2}}(\hat\vartheta)^{\frac{\alpha+1}{2}}}{1-\alpha}\right)\mathrm{d}x \le \int_\Omega\left(\vartheta_0-\frac{2(\vartheta_0)^{\frac{1-\alpha}{2}}(\hat\vartheta)^{\frac{\alpha+1}{2}}}{1-\alpha}\right)\mathrm{d}x.$$

Using (2.73) with $\beta := \frac{1-\alpha}{2}$, it follows from the above inequality that

$$\limsup_{S\ni t\to 0_+}\int_\Omega \vartheta(t)\,\mathrm{d}x \le \int_\Omega \vartheta_0\,\mathrm{d}x,$$

which is precisely the relation (2.72) we aimed to prove. Consequently, we also deduce the strong attainment of the initial condition (2.73).

2.6. **Entropy equality.** Proving that the solution found in the previous sections fulfills the equality (1.24) can follow the approach of [1], i.e., we set $\psi := \frac{\varphi}{\vartheta^N}$, where $\varphi\in C_0^\infty((-\infty,T)\times\Omega)$, in (2.19) and integrate the resulting expression over $(0,T)$ to get

$$\int_Q \kappa(\vartheta^N)\nabla\eta^N\cdot\nabla\varphi-\eta^N\partial_t\varphi-\eta^N\boldsymbol{u}^N\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t = \int_Q \frac{\boldsymbol{S}^N:D\boldsymbol{u}^N\varphi}{\vartheta^N}+\kappa(\vartheta^N)\frac{|\nabla\vartheta^N|^2}{(\vartheta^N)^2}\varphi\,\mathrm{d}x\,\mathrm{d}t+\int_\Omega \eta_0^N\varphi(0)\,\mathrm{d}x,\tag{2.76}$$

where $\eta_0^N := \log\vartheta_0^N$ and $\eta^N := \log\vartheta^N$. Next, we show the convergence as $N\to\infty$. In what follows, we only highlight the key convergence results that are used in the next section and refer the reader to the original paper for a detailed proof. Note that to prove the entropy equality, we need the convergence of ϑ to the initial condition in the sense of (2.62).

By (2.45), for $N\to\infty$, we have

$$\eta^N\to\eta \qquad \text{in } L^q(Q) \text{ for } q\in[1,\infty),\tag{2.77}$$

where $\eta := \log\vartheta$. Moreover, $\nabla\eta^N = \frac{\nabla\vartheta^N}{\vartheta^N}\in L^2(Q)$ by (2.30). We can thus find a converging subsequence

$$\nabla\eta^N \rightharpoonup \nabla\eta \qquad \text{in } L^2(Q),\tag{2.78}$$

and so $\eta\in L^2(0,T;W^{1,2})\cap L^q(Q)$ for $q\in[1,\infty)$. Following [1, Subsection 3.(b)(i)], we have

$$\nabla\vartheta^N\to\nabla\vartheta \qquad \text{in } L^1(Q)\tag{2.79}$$

$$\nabla\vartheta^N\to\nabla\vartheta \qquad \text{a.e. in } Q.\tag{2.80}$$

Moreover, from [1, Subsection 3.(b)(ii)], we know

$$\kappa(\vartheta^N)\frac{|\nabla\vartheta^N|^2}{(\vartheta^N)^2}\to\kappa(\vartheta)\frac{|\nabla\vartheta|^2}{\vartheta^2} \qquad \text{in } L^1(Q).\tag{2.81}$$

The convergence results (2.77)–(2.81) are enough to pass to the limit in (2.76). Consequently, $\eta\in L^2(0,T;W^{1,2})\cap L^q(Q), q\in[1,\infty)$ satisfies *Entropy equation* (1.24).

2.7. **Continuity.** This section is the final part of the proof. It relies on the approach developed in [1], but here we are able to show a stronger result, namely the continuity with respect to time. Our main goal is to show that the sequence ϑ^N is Cauchy in $C([0,T];L^1(\Omega))$, and therefore the limit shares this property. The proof is split into two parts. First, we establish this property for the truncated functions, and second, we prove a kind of uniform equiintegrability.

For any $m>0$ we define

$$\mathcal{T}_m:\mathbb{R}\to[-m,m], \qquad \mathcal{T}_m(z):=\operatorname{sign}(z)\min\{|z|,m\}. \tag{2.82}$$

For $\delta\in(0,m)$, let $\mathcal{T}_{m,\delta}\in C^2(\mathbb{R})$ denote the mollification of $\mathcal{T}_m$, which is given by convolution with a regularization kernel ω_ϵ. For $\mathcal{T}_{m,\delta}$ it holds

$$\begin{aligned}
&\mathcal{T}_{m,\delta}(z)=\mathcal{T}_m(z) && \text{if } |z|\le m-\varepsilon \text{ or } |z|\ge m+\varepsilon,\\
&|\mathcal{T}''_{m,\delta}|\le\frac{C}{\delta} && \text{on } \mathbb{R},\\
&0\le\mathcal{T}'_{m,\delta}\le 1,\quad \mathcal{T}''_{m,\delta}\le 0,\quad \mathcal{T}_{m,\delta}\le\mathcal{T}_m && \text{on } (0,\infty).
\end{aligned}\tag{2.83}$$

Our first goal is to show that $\{\mathcal{T}_{m,\delta}(\vartheta^N)\}_N$ is Cauchy in $C([0,T];L^2)$ for any fixed $m>\|\hat\vartheta\|_\infty$ and δ. This means that for any given $\alpha>0$, there exists N_0 such that for all $N,M>N_0$ it holds

$$\sup_{t\in[0,T]}\|\mathcal{T}_{m,\delta}(\vartheta^N(t))-\mathcal{T}_{m,\delta}(\vartheta^M)(t)\|_2^2\le\alpha. \tag{2.84}$$

We set $\psi^N:=\mathcal{T}'_{m,\delta}(\vartheta^N)(\mathcal{T}_{m,\delta}(\vartheta^N)-\mathcal{T}_{m,\delta}(\vartheta^M))$, $\psi^M:=\mathcal{T}'_{m,\delta}(\vartheta^M)(\mathcal{T}_{m,\delta}(\vartheta^N)-\mathcal{T}_{m,\delta}(\vartheta^M))$ in (2.19) for N, $M\in\mathbb{N}$, respectively. Subtracting one equation from the other and integrating the result over time interval $(0,t)$ with arbitrary $t\in[0,T]$, we obtain

$$\begin{aligned}
&\frac12\|\mathcal{T}_{m,\delta}(\vartheta^N(t))-\mathcal{T}_{m,\delta}(\vartheta^M(t))\|_2^2=\frac12\|\mathcal{T}_{m,\delta}(\vartheta^N_0)-\mathcal{T}_{m,\delta}(\vartheta^M_0)\|_2^2\\
&\quad+\int_0^t\int_\Omega(\mathcal{T}'_{m,\delta}(\vartheta^N)\boldsymbol{\mathcal{S}}^N:D\boldsymbol{u}^N-\mathcal{T}'_{m,\delta}(\vartheta^M)\boldsymbol{\mathcal{S}}^M:D\boldsymbol{u}^M)(\mathcal{T}_{m,\delta}(\vartheta^N)-\mathcal{T}_{m,\delta}(\vartheta^M))\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad+\int_0^t\int_\Omega(\mathcal{T}'_{m,\delta}(\vartheta^N)\vartheta^N\boldsymbol{u}^N-\mathcal{T}'_{m,\delta}(\vartheta^M)\vartheta^M\boldsymbol{u}^M)\cdot\nabla(\mathcal{T}_{m,\delta}(\vartheta^N)-\mathcal{T}_{m,\delta}(\vartheta^M))\,\mathrm{d}x\,\mathrm{d}\tau\\
&\quad-\int_0^t\int_\Omega(\kappa(\vartheta^N)\nabla\mathcal{T}_{m,\delta}(\vartheta^N)-\kappa(\vartheta^M)\nabla\mathcal{T}_{m,\delta}(\vartheta^M))\cdot\nabla(\mathcal{T}_{m,\delta}(\vartheta^N)-\mathcal{T}_{m,\delta}(\vartheta^M))\,\mathrm{d}x\,\mathrm{d}\tau.
\end{aligned}$$

This implies (using the simple estimate $|\mathcal{T}'_{m,\delta}(\vartheta^N)\vartheta^N|\le m+\delta$ and the assumption (1.7))

$$\begin{aligned}
&\sup_{t\in[0,T]}\|\mathcal{T}_{m,\delta}(\vartheta^N(t))-\mathcal{T}_{m,\delta}(\vartheta^M(t))\|_2^2\le 2m\|\mathcal{T}_{m,\delta}(\vartheta^N_0)-\mathcal{T}_{m,\delta}(\vartheta^M_0)\|_1\\
&\quad+2\int_0^T\int_\Omega\left(\left|\boldsymbol{\mathcal{S}}^N:D\boldsymbol{u}^N\right|+\left|\boldsymbol{\mathcal{S}}^M:D\boldsymbol{u}^M\right|\right)\left|\mathcal{T}_{m,\delta}(\vartheta^N)-\mathcal{T}_{m,\delta}(\vartheta^M)\right|\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+4Cm\int_0^T\int_\Omega\left(|\boldsymbol{u}^N|+|\boldsymbol{u}^M|+|\nabla\mathcal{T}_{m,\delta}(\vartheta^N)|+|\nabla\mathcal{T}_{m,\delta}(\vartheta^M)|\right)\left|\nabla(\mathcal{T}_{m,\delta}(\vartheta^N)-\mathcal{T}_{m,\delta}(\vartheta^M))\right|\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}\tag{2.85}$$

Next, we estimate the right-hand side of (2.85). By (2.3), we have

$$2m\|\mathcal{T}_{m,\delta}(\vartheta^N_0)-\mathcal{T}_{m,\delta}(\vartheta^M_0)\|_1<\frac{\alpha}{3} \tag{2.86}$$

for M, N large enough. To estimate the second term, we can split

$$\left|\mathcal{T}_{m,\delta}(\vartheta^N)-\mathcal{T}_{m,\delta}(\vartheta^M)\right|=\left|\mathcal{T}_{m,\delta}(\vartheta^N)-\mathcal{T}_{m,\delta}(\vartheta)\right|+\left|\mathcal{T}_{m,\delta}(\vartheta)-\mathcal{T}_{m,\delta}(\vartheta^M)\right|.$$

We know that for all $N\in\mathbb{N}$, the function $\mathcal{T}_{m,\delta}(\vartheta)-\mathcal{T}_{m,\delta}(\vartheta^N)$ is bounded in $L^\infty(Q)$ by (2.83), and that it converges almost everywhere to zero by the continuity of $\mathcal{T}_{m,\delta}$ and the convergence (2.44). Furthermore, by (2.55), $\boldsymbol{\mathcal{S}}^N:D\boldsymbol{u}^N$ is uniformly integrable and bounded in $L^1(Q)$ for all $N\in\mathbb{N}$. Using Egorov's theorem, we can conclude that

$$2\int_0^T\int_\Omega\left(\left|\boldsymbol{\mathcal{S}}^N:D\boldsymbol{u}^N\right|+\left|\boldsymbol{\mathcal{S}}^M:D\boldsymbol{u}^M\right|\right)\left|\mathcal{T}_{m,\delta}(\vartheta^N)-\mathcal{T}_{m,\delta}(\vartheta^M)\right|\,\mathrm{d}x\,\mathrm{d}t\le\frac{\alpha}{3} \tag{2.87}$$

for M,N large enough. For the third term, we use that

$$|\boldsymbol{u}^N|+|\boldsymbol{u}^M|+|\nabla\mathcal{T}_{m,\delta}(\vartheta^N)|+|\nabla\mathcal{T}_{m,\delta}(\vartheta^M)|$$

is bounded in $L^2(Q)$, and

$$\left|\nabla(\mathcal{T}_{m,\delta}(\vartheta^N)-\mathcal{T}_{m,\delta}(\vartheta^M))\right|\le\left|\nabla(\mathcal{T}_{m,\delta}(\vartheta^N)-\mathcal{T}_{m,\delta}(\vartheta))\right|+\left|\nabla(\mathcal{T}_{m,\delta}(\vartheta)-\mathcal{T}_{m,\delta}(\vartheta^M))\right|,$$

where each term converges strongly to zero in $L^2(Q)$ by (2.81) and (2.44). For M, N large enough, we can conclude that

$$2\int_0^T \int_\Omega \left(|\vartheta^N \boldsymbol{u}^N| + |\vartheta^M \boldsymbol{u}^M| + \overline{\kappa}|\nabla \mathcal{T}_{m,\delta}(\vartheta^N)| + \overline{\kappa}|\nabla \mathcal{T}_{m,\delta}(\vartheta^M)|\right) \left|\nabla(\mathcal{T}_{m,\delta}(\vartheta^N) - \mathcal{T}_{m,\delta}(\vartheta^M))\right| \,\mathrm{d}x\,\mathrm{d}t < \frac{\alpha}{3}. \tag{2.88}$$

The sequence $\{\mathcal{T}_{m,\delta}(\vartheta^N)\}_N$ is therefore Cauchy in $C([0,T]; L^2)$, since by (2.85)–(2.88) we have

$$\sup_{t\in[0,T]} \|\mathcal{T}_{m,\delta}(\vartheta^N(t)) - \mathcal{T}_{m,\delta}(\vartheta^M(t))\|_2^2 < \alpha, \tag{2.89}$$

for all M, N sufficiently large.

Next, we recall $m > \|\hat{\vartheta}\|_\infty$, consider equation (2.19) with $\psi := \frac{(\vartheta^N - m)_+}{\vartheta^N}$, and integrate over time interval $(0,t)$. Note that ψ has zero trace due to the choice of m. This gives us

$$\begin{aligned} &\int_0^t \int_\Omega \partial_t \vartheta^N \left(1 - \frac{m}{\vartheta^N}\right) \chi_{\{\vartheta^N > m\}} \,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t \int_\Omega \frac{m\kappa(\vartheta^N)}{(\vartheta^N)^2} |\nabla \vartheta^N|^2 \chi_{\{\vartheta^N > m\}} \,\mathrm{d}x\,\mathrm{d}\tau = \\ &= \int_0^t \int_\Omega \boldsymbol{S}^N : D\boldsymbol{u}^N \left(1 - \frac{m}{\vartheta^N}\right) \chi_{\{\vartheta^N > m\}} \,\mathrm{d}x\,\mathrm{d}\tau + \int_0^t \int_\Omega \frac{m}{\vartheta^N} \boldsymbol{u}^N \cdot \nabla \vartheta^N \chi_{\{\vartheta^N > m\}} \,\mathrm{d}x\,\mathrm{d}\tau. \end{aligned} \tag{2.90}$$

We can rewrite

$$\partial_t \vartheta^N \left(1 - \frac{m}{\vartheta^N}\right) \chi_{\{\vartheta^N > m\}} = \partial_t((\vartheta^N - m\log\vartheta^N - m + m\log m)_+).$$

Furthermore, the second term on the left-hand side is positive, so we can omit it from (2.90) and replace the equality sign with an inequality. Using integration by parts we can get rid of the last term on the right-hand side since $\operatorname{div} \vartheta^N = 0$

$$\int_0^t \int_\Omega \frac{m}{\vartheta^N} \boldsymbol{u}^N \cdot \nabla \vartheta^N \chi_{\{\vartheta^N > m\}} \,\mathrm{d}x\,\mathrm{d}\tau = \int_0^t \int_\Omega \boldsymbol{u}^N \cdot \nabla \left(m \log(\vartheta^N) \chi_{\{\vartheta^N > m\}}\right) \,\mathrm{d}x\,\mathrm{d}\tau = 0.$$

Lastly, it holds

$$0 \le \left(1 - \frac{m}{\vartheta^N}\right) \chi_{\{\vartheta^N > m\}} \le \chi_{\{\vartheta^N > m\}},$$

so we can rewrite (2.90) as

$$\int_0^t \partial_t \|(\vartheta^N - m\log\vartheta^N - m + m\log m)_+\|_1 \,\mathrm{d}\tau \le \int_0^t \int_\Omega \boldsymbol{S}^N : D\boldsymbol{u}^N \chi_{\{\vartheta^N > m\}} \,\mathrm{d}x\,\mathrm{d}\tau.$$

Since $t \in (0,T)$ was chosen arbitrarily, we have

$$\begin{aligned} &\sup_{t\in[0,T]} \|(\vartheta^N(t) - m\log\vartheta^N(t) - m + m\log m)_+\|_1 \le \|(\vartheta_0^N - m\log\vartheta_0^N - m + m\log m)_+\|_1 \\ &+ \int_0^T \int_\Omega \boldsymbol{S}^N : D\boldsymbol{u}^N \chi_{\{\vartheta^N > m\}} \,\mathrm{d}x\,\mathrm{d}t \le \|\vartheta_0^N \chi_{\{\vartheta^N > m\}}\|_1 + \int_0^T \int_\Omega \boldsymbol{S}^N : D\boldsymbol{u}^N \chi_{\{\vartheta^N > m\}} \,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{2.91}$$

Since $\boldsymbol{S}^N : D\boldsymbol{u}^N$ converges weakly to $\boldsymbol{S} : D\boldsymbol{u}$ in $L^1(Q)$ by (2.55), it forms a uniformly integrable sequence of functions. Hence, by (2.3) and (2.45), we deduce that $|\{\vartheta^N > m\}| \le C/m$. Consequently, for any $\alpha > 0$, there exists $m_0 > 0$ such that

$$\|\vartheta_0^N \chi_{\{\vartheta_0^N > m\}}\|_1 + \int_0^T \int_\Omega \boldsymbol{S}^N : D\boldsymbol{u}^N \chi_{\{\vartheta^N > m\}} \,\mathrm{d}x\,\mathrm{d}t < \alpha$$

for all $m > m_0$. For any S positive, we can estimate $\frac{(S-m)_+}{2} \le (S - m\log S - m + m\log m)_+$. We can hence conclude that for any $\alpha > 0$ there exists $m_0 > 0$ such that

$$\sup_{t\in[0,T]} \|(\vartheta^N(t) - 2m)_+\|_1 < \alpha \tag{2.92}$$

for all $m > m_0$.

Finally, we show that the sequence $\{\vartheta^N\}$ is a Cauchy sequence in $C([0,T]; L^1)$. Note that each element of the sequence belongs to $C([0,T]; L^1) \subset C([0,T]; L^2)$ by (2.20). For any any $t \in (0,T)$ we can estimate

$$\begin{aligned} \|\vartheta^N(t) - \vartheta^M(t)\|_1 &\le \|\vartheta^N(t) - \mathcal{T}_{m,\delta}(\vartheta^N)(t)\|_1 + \|\vartheta^M(t) - \mathcal{T}_{m,\delta}(\vartheta^M)(t)\|_1 + \|\mathcal{T}_{m,\delta}(\vartheta^N)(t) - \mathcal{T}_{m,\delta}(\vartheta^M)(t)\|_1 \\ &\le \|(\vartheta^N(t) - m)_+\|_1 + \|(\vartheta^M(t) - m)_+\|_1 + \|\mathcal{T}_{m,\delta}(\vartheta^N)(t) - \mathcal{T}_{m,\delta}(\vartheta^M)(t)\|_1 \end{aligned} \tag{2.93}$$

Hence, for a given $\varepsilon > 0$, we can use (2.92) to find m sufficiently large such that for all M, N

$$\sup_{t\in[0,T]} \|(\vartheta^N(t) - m)_+\|_1 < \frac{\varepsilon}{3}, \qquad \sup_{t\in[0,T]} \|(\vartheta^M(t) - m)_+\|_1 < \frac{\varepsilon}{3}. \tag{2.94}$$

Finally, for this choice of m, we set $\delta = 1$ and find N_0 such that for all $M, N > N_0$ we have

$$\|\mathcal{T}_{m,\delta}(\vartheta^N)(t) - \mathcal{T}_{m,\delta}(\vartheta^M)(t)\|_1 < \frac{\varepsilon}{3},$$

by virtue of (2.89). We conclude that for any $\varepsilon > 0$ there exists N_0 such that for all $M, N > N_0$ it holds

$$\sup_{t \in (0,T)} \|\vartheta^N(t) - \vartheta^M(t)\|_1 < \varepsilon. \tag{2.95}$$

Hence, the sequence $\{\vartheta^N\}_{N=0}^{\infty}$ is Cauchy in $C([0,T]; L^1)$, and its limit ϑ indeed belongs to $C([0,T]; L^1)$.

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