Complexity of Bilevel Linear Programming with a Single Upper-Level Variable

Nagisa Sugishita[1] and Margarida Carvalho[2]

[1] Département de sciences de la décision, HEC Montréal
[2] Département d'informatique et de recherche opérationnelle, Université de Montréal

Abstract. Bilevel linear programming (LP) is one of the simplest classes of bilevel optimization problems, yet it is known to be NP-hard in general. Specifically, determining whether the optimal objective value of a bilevel LP is at least as good as a given threshold, a standard decision version of the problem, is NP-complete. However, this decision problem becomes tractable when either the number of lower-level variables or the number of lower-level constraints is fixed, which prompts the question: What if restrictions are placed on the upper-level problem? In this paper, we address this gap by showing that the decision version of bilevel LP remains NP-complete even when there is only a single upper-level variable, no upper-level constraints (apart from the constraint enforcing optimality of the lower-level decision) and all variables are bounded between 0 and 1. This result implies that fixing the number of variables or constraints in the upper-level problem alone does not lead to tractability in general. On the positive side, we show that there is a polynomial-time algorithm that finds a local optimal solution of such a rational bilevel LP instance. We also demonstrate that many combinatorial optimization problems, such as the knapsack problem and the traveling salesman problem, can be written as such a bilevel LP instance.

Keywords: Bilevel Programming · Computational Complexity · Lexicographic Linear Programming

1 Introduction

Bilevel programming is a class of hierarchical optimization problems in which a *leader* makes a decision first, and a *follower* subsequently optimizes their own objective subject to the leader's choice. In this paper, we focus on the computational complexity of bilevel linear programming (LP), a setting where both the upper- and lower-level problems are LPs. We consider the optimistic variant: when multiple optimal solutions exist for the follower, the one most favorable to the leader is chosen. Even this simplest form of bilevel programming poses serious challenges in theory and practice. A standard way to analyze the complexity of optimization problems is to consider their decision versions, specifically, determining whether the optimal objective value is equal to or less than a given threshold in a minimization problem. Jeroslow [9] was the first to show that the decision version of bilevel LP is NP-hard. An alternative proof was given by Ben-Ayed and Blair [2]. Later, strong NP-hardness was shown by Hansen *et al.* [6]. More recently, Buchheim [3] proved its inclusion in NP, thereby establishing its NP-completeness. Other decision problems have also been studied in the context of bilevel LP: Rodrigues *et al.* [15] showed that checking whether the problem is unbounded is NP-complete, and Vicente *et al.* [19] showed that verifying local optimality of a given point is NP-hard. Recently, Prokopye and Ralphs [14] studied the complexity of the search of a local optimal solution.

Given these negative complexity results, a natural line of research is to identify conditions under which bilevel LP becomes tractable. Such conditions can guide the design of efficient algorithms for practical instances. For example, in the context of mixed-integer LP, a classical result by Lenstra Jr [10] establishes that problems with a fixed number of integer variables are solvable in polynomial time. Similar results have been derived for bilevel LP. Deng [5] demonstrated that bilevel LP is solvable in polynomial time when the number of lower-level variables is fixed. Likewise, the arguments in [1] and [3] imply polynomial-time solvability when the number of lower-level constraints is fixed. Recently, it was shown that the toll pricing problem can be solved in polynomial time when the number of upper-level variables is fixed, enabling efficient algorithmic strategies [4].

In this paper, we examine the decision version of bilevel LP under a particularly restricted setting: where there is only a single upper-level variable. We show that even under this restricted structure, the decision problem remains NP-complete. In particular, we prove a stronger result: bilevel LP is NP-complete even when there is only one upper-level variable and no upper-level constraints (other than the constraint enforcing the optimality of the lower-level variable with respect to the lower-level problem), and all variables are constrained to lie in the unit cube (i.e., when the lower-level constraints explicitly include bounds of 0 and 1 on each variable). Moreover, we show that there does not exist a polynomial-time approximation scheme (PTAS) for computing the optimal objective value of such bilevel LP instances, unless NP = P. Additionally, we prove that given a candidate solution, the problem of verifying whether it is an optimal solution is also coNP-complete. On the positive side, we show that a locally optimal solution can be computed in polynomial time. Furthermore, we show that many combinatorial problems, such as the knapsack problem and the traveling salesman problem, can be reformulated as bilevel LPs of this form, while preserving their optimal solutions.

Paper Structure. This paper is organized as follows. Section 2 formally introduces the class of bilevel LP instances, its decision version and its hardness, the main result of this paper. Section 3 presents auxiliary results that will be used to prove the main result in Section 4. Section 5 discusses how combinatorial optimization problems can be transformed to bilevel LPs with a single upper-level variable, using the technique developed in the preceding sections. Section 6 is concerned with the complexity of searching for a globally and locally optimal solution.

Notation. We denote the binary encoding size of a rational number p and a rational vector v by $\langle p \rangle$ and $\langle v \rangle$, respectively. For any sequence $a_1, \ldots, a_n$, we write $\langle a_1, \ldots, a_n \rangle$ to denote the sum $\langle a_1 \rangle + \cdots + \langle a_n \rangle$. We follow the encoding conventions described in [16]. In particular, the size of a rational number $x = p/q$, where p and q are relatively prime integers, is given by $1 + \lceil \log_2(|p| + 1) \rceil + \lceil \log_2(|q| + 1) \rceil$. We say that a vector, matrix, or optimization instance is *rational* if and only if all its data are rational numbers. Given an optimization instance P, we use $v(P)$, $\mathcal{F}(P)$, and $\mathcal{S}(P)$ to denote the optimal objective value, the set of feasible solutions, and the set of optimal solutions, respectively. We use $\mathbf{0}$ (respectively, $\mathbf{1}$) to denote the all-zeros (respectively, all-ones) vector, with the dimension clear from the context.

2 Main Result

In this paper, we study the hardness of rational bilevel LPs with a single upper-level variable. In fact, we consider a more restrictive class where the upper-level problem has no constraints and all the variables are bounded to be within the unit cube. Let $\text{BLP}_{\text{SINGLE}}$ denote the set of such bilevel LP instances. An instance in $\text{BLP}_{\text{SINGLE}}$ can be written as:

$$\begin{aligned} \min_{x_1, x_2} \ & c_{21}^\top x_1 + c_{22} x_2 \\ \text{s.t. } & x_1 \in \operatorname*{argmin}_{x_1'} \{ c_{11}^\top x_1' : A_{11} x_1' + A_{12} x_2 \geq b_1, \mathbf{0} \leq x_1' \leq \mathbf{1}, 0 \leq x_2 \leq 1 \}, \end{aligned} \tag{P_{SINGLE}}$$

where $c_{11}, c_{21} \in \mathbb{Q}^n$, $c_{22} \in \mathbb{Q}$, $A_{11} \in \mathbb{Q}^{m \times n}$, and $A_{12}, b_1 \in \mathbb{Q}^m$. A subtle remark is in order. Note that the lower-level constraints impose the bounds on the upper-level variable x_2. If the leader chooses x_2 outside these bounds, the lower-level problem becomes infeasible, consequently resulting into a bilevel infeasible solution to (P_{SINGLE}). In other words, although the upper-level problem has no variable bounds on x_2, effective bounds are implicitly enforced by the lower-level constraints.

Consider the following decision problem:

Given a rational instance (P_{SINGLE}) and a rational number k, is the optimal objective value less than or equal to k? ($\text{VAL}_{\text{SINGLE}}$)

We will prove the following result.

Theorem 1. *The decision problem* $\text{VAL}_{\text{SINGLE}}$ *is NP-complete.*

In Section 4, we prove Theorem 1. To this end, we provide a reduction from 3-SAT to $\text{VAL}_{\text{SINGLE}}$. That is, given a Boolean formula F in 3-conjunctive normal form (3CNF), we construct a $\text{BLP}_{\text{SINGLE}}$ instance such that the optimal objective value is -1 if F is satisfiable and 0 otherwise.

To simplify the exposition, we divide the reduction into several steps. First, we construct a bilevel LP instance with multiple upper-level variables and upper-level constraints (instance $\text{P}_{\text{3SAT}}(F)$). We then reduce the problem to a bilevel

LP with a single upper-level variable (instance $\mathrm{P}_{\mathrm{weight}}(F, \eta)$). Lastly, we eliminate the upper-level constraints via a penalty method (instance $\mathrm{P}(F)$). The final instance is a bilevel LP with a single upper-level variable and no upper-level constraints. We will show that the optimal objective value is preserved throughout these transformations.

To develop our argument, we need to analyse general bilevel LPs, i.e., instances with multiple variables and constraints in the upper-level problem. Thus, in Section 3, we provide some auxiliary results to study general bilevel LPs.

3 Auxiliary Results

This section presents auxiliary results that will be used in Section 4 to prove Theorem 1. Section 3.1 provides an estimate on the size of an optimal solution to a bilevel LP. To the best of our knowledge, such bounds have not been explicitly discussed in the literature. Section 3.2 presents a method for reformulating upper-level constraints as lower-level constraints. A similar idea appears in [7], though it is unclear whether their transformation can be performed efficiently. Here, we provide an alternative, constructive argument.

3.1 Estimate of Optimal Solution Size

We begin by presenting a bound on the size of an optimal solution to an LP, which will be used to estimate the size of solutions for bilevel LPs.

Lemma 1. *Let $A, B \in \mathbb{Q}^{m \times n}$, $a, b \in \mathbb{Q}^m$, and $c \in \mathbb{Q}^n$. Let σ denote the maximum size of the entries in A, B, a, and b. Consider the LP*

$$\min_x \left\{ c^\top x : Ax \geq a, Bx > b \right\}.$$

If an optimal solution exists, then there exists a rational optimal solution $\bar{x}$ of size bounded by a polynomial in n and σ. That is, there exists a polynomial f_{sol} such that $\langle \bar{x} \rangle \leq f_{\mathrm{sol}}(n, \sigma)$.

Note that the optimization instance in Lemma 1 includes strict inequalities. Nonetheless, we refer to such instances as LPs throughout this paper. Strict inequalities are used later in the proof of Lemma 3. For the proof, see [16]. We are now ready to state an analogous result for bilevel LPs.

Lemma 2. *Let $A_{11} \in \mathbb{Q}^{m_1 \times n_1}, A_{12} \in \mathbb{Q}^{m_1 \times n_2}, A_{21} \in \mathbb{Q}^{m_2 \times n_1}, A_{22} \in \mathbb{Q}^{m_2 \times n_2}$, $b_1 \in \mathbb{Q}^{m_1}, b_2 \in \mathbb{Q}^{m_2}$, $c_{11}, c_{21} \in \mathbb{Q}^{n_1}$ and $c_{22} \in \mathbb{Q}^{n_2}$. Let $n = n_1 + n_2$ and σ denote the maximum size of the entries in $A_{11}, A_{12}, A_{21}, A_{22}$, b_1, and b_2. Consider the bilevel LP*

$$\begin{aligned} \min_{x_1, x_2} \ & c_{21}^\top x_1 + c_{22}^\top x_2 \\ \text{s.t.} \ & A_{21} x_1 + A_{22} x_2 \geq b_2, \\ & x_1 \in \operatorname*{argmin}_{x_1'} \{ c_{11}^\top x_1' : A_{11} x_1' + A_{12} x_2 \geq b_1 \}. \end{aligned} \tag{1}$$

If an optimal solution exists, then there exists a rational optimal solution $(\bar{x}_1, \bar{x}_2)$ *such that* $\langle \bar{x}_1, \bar{x}_2 \rangle \leq f_{\text{sol}}(n, \sigma)$.

The proof is provided in Appendix A.

3.2 Reformulation of Upper-Level Constraints

In this section, we present our approach to remove some of the upper-level constraints by adding a penalty term in the objective. Let

$$\begin{aligned} \min_{x_1,x_2}\ & c_{21}^\top x_1 + c_{22}^\top x_2 \qquad (Q_{\text{blp}}) \\ \text{s.t. } & A_{21}x_1 + A_{22}x_2 \geq b_2, \\ & \bar{A}_{21}x_1 + \bar{A}_{22}x_2 \geq \bar{b}_2, \\ & x_1 \in \operatorname*{argmin}_{x_1'} \{c_{11}^\top x_1' : A_{11}x_1' + A_{12}x_2 \geq b_1\}, \end{aligned}$$

where $A_{11} \in \mathbb{Q}^{m_1 \times n_1}, A_{12} \in \mathbb{Q}^{m_1 \times n_2}, A_{21} \in \mathbb{Q}^{m_2 \times n_1}, A_{22} \in \mathbb{Q}^{m_2 \times n_2}, \bar{A}_{21} \in \mathbb{Q}^{m_2' \times n_1}, \bar{A}_{22} \in \mathbb{Q}^{m_2' \times n_2}$, $b_1 \in \mathbb{Q}^{m_1}, b_2 \in \mathbb{Q}^{m_2}, \bar{b}_2 \in \mathbb{Q}^{m_2'}$, $c_{11}, c_{21} \in \mathbb{Q}^{n_1}$ and $c_{22} \in \mathbb{Q}^{n_2}$. We are interested in “replacing” constraints $\bar{A}_{21}x_1 + \bar{A}_{22}x_2 \geq \bar{b}_2$ in (Q_{blp}) with a penalty term in the objective with a help of a single variable $e \in \mathbb{R}$ and a real number M:

$$\begin{aligned} \min_{x_1,x_2,e}\ & c_{21}^\top x_1 + c_{22}^\top x_2 + Me \qquad (Q'_{\text{blp}}(M)) \\ \text{s.t. } & A_{21}x_1 + A_{22}x_2 \geq b_2, \\ & (x_1, e) \in \operatorname*{argmin}_{x_1', e'} \left\{ c_{11}^\top x_1' : \begin{array}{c} A_{11}x_1' + A_{12}x_2 \geq b_1, \\ \bar{A}_{21}x_1' + \bar{A}_{22}x_2 + e'\mathbf{1} \geq \bar{b}_2, \\ 0 \leq e' \leq \bar{e} \end{array} \right\}, \end{aligned}$$

where $\bar{e}$ is 1 or ∞.

Lemma 3. *Suppose* (Q_{blp}) *and* ($Q'_{\text{blp}}(M)$) *have an optimal solution for any* $M > 0$. *Let* $n = n_1 + n_2$, σ *denote the maximum size of entries in* A_{11}, A_{12}, A_{21}, A_{22}, $\bar{A}_{21}$, $\bar{A}_{22}$, b_1, b_2, $\bar{b}_2$, *and* $c_\infty = \max\{\|c_{21}\|_\infty, \|c_{22}\|_\infty, 1\}$. *For any* $M \geq 3nc_\infty 4^{f_{\text{sol}}(n+1,\sigma)}$, *the following relations hold:*
(i) $v(Q_{\text{blp}}) = v(Q'_{\text{blp}}(M))$;
(ii) $\mathcal{S}(Q'_{\text{blp}}(M)) = \{(x_1, x_2, 0) : (x_1, x_2) \in \mathcal{S}(Q_{\text{blp}})\}$.

The proof is given in Appendix B.

4 Proof of Theorem 1

We begin by constructing a bilevel LP instance with multiple upper-level variables. Let F be a Boolean formula in 3CNF with n variables $u_1, \ldots, u_n$ and p clauses. Following [11,13], F can be represented by a system of linear inequalities over binary variables: $A_F x \geq \mathbf{1} - a_F,\ x \in \{0,1\}^n$, where $A_F \in \mathbb{Z}^{p \times n}$, $a_F \in \mathbb{Z}^p$, and satisfying truth assignments correspond exactly to solutions of this system ($u_i =$ true corresponds to $x_i = 1$ for each i). The inequality can be constructed as follows. First, A_F and a_F are set to 0. Then, we scan the clauses one by one. A positive literal u_i in the j-th clause increments the (j, i) element of A_F by 1, while a negated literal $\neg u_i$ decrements the (j, i) element of A_F and increments the j-th element of a_F. For example, $F = (u_1 \vee \neg u_2) \wedge (\neg u_1 \vee \neg u_2 \vee u_3)$ corresponds to

$$A_F = \begin{pmatrix} 1 & -1 & 0 \\ -1 & -1 & 1 \end{pmatrix}, \quad a_F = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

yielding inequalities

$$x_1 + (1 - x_2) \geq 1, \quad (1 - x_1) + (1 - x_2) + x_3 \geq 1, \quad x \in \{0, 1\}^3.$$

In the following, we say that a binary vector satisfies F if and only if the corresponding truth assignment satisfies F.

Now, consider the following bilevel LP instance:

$$\begin{aligned}
\min_{x,y,w} \ & 2 \sum_{i=1}^{n} (y - 2w_i) - \frac{2}{n} \sum_{i=1}^{n} w_i && (\mathrm{P}_{3\mathrm{SAT}}(F)) \\
\text{s.t. } & A_F x \geq \left(\frac{3}{2} - \frac{1}{2} y \right) \mathbf{1} - a_F, && (2) \\
& \frac{1}{2} - \frac{1}{2} y \leq x_i \leq \frac{1}{2} + \frac{1}{2} y, \quad \forall i = 1, \ldots, n, && (3) \\
& 0 \leq x_i \leq 1, \ 0 \leq y \leq 1, \quad \forall i = 1, \ldots, n, \\
& w \in \operatorname*{argmin}_{w'} \left\{ \sum_{i=1}^{n} w'_i : -w'_i \leq x_i - 1/2 \leq w'_i, 0 \leq w'_i \leq 1, \ i = 1, \ldots, n \right\}.
\end{aligned}$$

This instance is obtained from the one studied by Vicente *et al.* [19] (see also [15]). Instance ($\mathrm{P}_{3\mathrm{SAT}}(F)$) differs from their instance in that we removed some redundant variables from the lower-level problem for simplification and added variable bounds $0 \leq x_i \leq 1$, $0 \leq y \leq 1$ and $0 \leq w_i \leq 1$ for each $i = 1, \ldots, n$. Without the variable bounds, the instance is unbounded if and only if F is satisfiable. In our case (i.e., with the variable bounds), the optimal objective value of ($\mathrm{P}_{3\mathrm{SAT}}(F)$) is -1 if F is satisfiable and 0 otherwise, as stated below.

Lemma 4.

(i) Suppose F is unsatisfiable. Then, $x_i = 1/2$ for $i = 1, \ldots, n$, and $y = 0$ is optimal for ($\mathrm{P}_{3\mathrm{SAT}}(F)$) and $v(\mathrm{P}_{3\mathrm{SAT}}(F)) = 0$.

(ii) Suppose F is satisfiable and let μ be a binary vector satisfying F. Then, $x = \mu$ and $y = 1$ is optimal for ($\mathrm{P}_{3\mathrm{SAT}}(F)$) and $v(\mathrm{P}_{3\mathrm{SAT}}(F)) = -1$.

See Appendix C for the proof. To simplify the notation in the following argument, we write $z = (x, y) \in \mathbb{R}^{n+1}$ and the constraints (2) and (3) as $B_F z \geq b_F$. Furthermore, we introduce $f \in \mathbb{R}^{n+1}$ such that $f_i = |z_i - 1/2|$ for $i = 1, \ldots, n+1$.

With this notation, ($\mathrm{P}_{\mathrm{3SAT}}(F)$) can be written as

$$\begin{aligned}
&\min_{z,f}\ 2\sum_{i=1}^{n}(z_{n+1}-2f_i)-\frac{2}{n}\sum_{i=1}^{n}f_i && (\mathrm{P}_{\mathrm{mult}}(F))\\
&\text{ s.t. } B_F z \geq b_F,\ 0\leq z_i\leq 1,\ \forall i=1,\dots,n+1,\\
&\qquad f\in \operatorname*{argmin}_{f'}\left\{\sum_{i=1}^{n+1}f_i' : -f_i'\leq z_i-1/2\leq f_i', 0\leq f_i'\leq 1,\ i=1,\dots,n+1\right\}.
\end{aligned}$$

The formulation ($\mathrm{P}_{\mathrm{mult}}(F)$) differs from ($\mathrm{P}_{\mathrm{3SAT}}(F)$) by the inclusion of an additional variable, f_{n+1}. Nonetheless, both problems attain the same optimal objective value. The additional variable is introduced purely for notational convenience in the analysis that follows.

Now, we use a lexicographic LP to replace the upper-level variables z in ($\mathrm{P}_{\mathrm{mult}}(F)$) with a single upper-level variable θ. Consider

$$\begin{aligned}
&\min_{\theta,z,f,s,t,u}\ 2\sum_{i=1}^{n}(z_{n+1}-2f_i)-\frac{2}{n}\sum_{i=1}^{n}f_i && (\mathrm{P}_{\mathrm{lex}}(F))\\
&\quad\text{ s.t. } B_F z\geq b_F, 0\leq\theta\leq 1,\\
&\qquad\quad (z,f,s,t,u)\in\mathcal{S}(\mathrm{P}_{\mathrm{lex}}^{\mathrm{low}}(n+1,\theta)),
\end{aligned}$$

where

$$\begin{aligned}
&\operatorname*{lexmin}_{z,f,s,t,u}\left\{s_{n+1}+t_{n+1},\ \dots,\ s_1+t_1,\ \sum_{i=1}^{n+1}f_i\right\} && && (\mathrm{P}_{\mathrm{lex}}^{\mathrm{low}}(n+1,\theta))\\
&\quad\text{ s.t. } u_{n+1}=\theta,\\
&\qquad s_i\geq (3/2)u_i-1/2, && t_i\geq(3/2)u_i-1, && \forall i=1,\dots,n+1,\\
&\qquad z_i=2(s_i-t_i), && -f_i\leq z_i-1/2\leq f_i, && \forall i=1,\dots,n+1,\\
&\qquad u_{i-1}=3u_i-2z_i, && && \forall i=2,\dots,n+1,\\
&\qquad 0\leq z_i,f_i,s_i,t_i,u_i\leq 1, && && \forall i=1,\dots,n+1,
\end{aligned}$$

with $\theta\in\mathbb{R}$, $z,f,s,t,u\in\mathbb{R}^{n+1}$. In ($\mathrm{P}_{\mathrm{lex}}(F)$), z denotes a lower-level variable. We intentionally use the same symbol z, for reasons that will be clarified later. For each $\theta\in[0,1]$, ($\mathrm{P}_{\mathrm{lex}}^{\mathrm{low}}(n+1,\theta)$) has a unique solution, and can be solved analytically, as stated below.

Lemma 5.
(i) For any $\theta\in[0,1]$, it holds that

$$\mathcal{S}(\mathrm{P}_{\mathrm{lex}}^{\mathrm{low}}(n+1,\theta)) =$$
$$\left\{\begin{pmatrix} z\\ f\\ s\\ t\\ u\end{pmatrix} : \begin{array}{lll} u_{n+1}=\theta, & & \\ s_i=[(3/2)u_i-1/2]^+, & t_i=[(3/2)u_i-1]^+, & \forall i=1,\dots,n+1,\\ z_i=f^z(u_i), & f_i=|z_i-1/2|, & \forall i=1,\dots,n+1,\\ u_{i-1}=f^u(u_i), & & \forall i=2,\dots,n+1, \end{array}\right\},$$

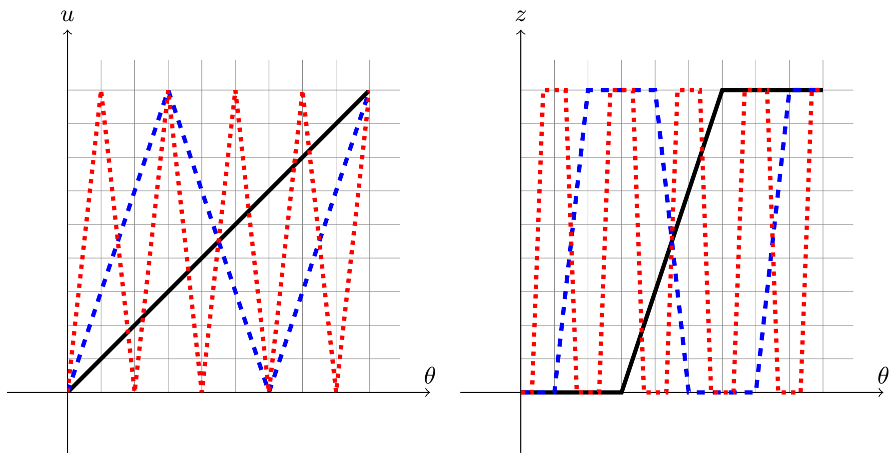

Fig. 1. Plot of u and z in $\mathcal{S}(\mathrm{P}^{\mathrm{low}}_{\mathrm{lex}}(3, \theta))$. The solid black lines represent u_3 and z_3, the dashed blue lines represent u_2 and z_2, and the dotted red lines represent u_1 and z_1.

where $[a]^+ = \max\{a, 0\}$*, and*

$$f^z(\theta) = \begin{cases} 0, & 0 \le \theta < 1/3, \\ 3\theta - 1, & 1/3 \le \theta < 2/3, \\ 1, & 2/3 \le \theta \le 1, \end{cases} \qquad f^u(\theta) = \begin{cases} 3\theta, & 0 \le \theta < 1/3, \\ -3\theta + 1, & 1/3 \le \theta < 2/3, \\ 3\theta - 2, & 2/3 \le \theta \le 1. \end{cases}$$

(ii) Let $\theta = 1/6$ *and* $(z, f, s, t, u) \in \mathcal{S}(\mathrm{P}^{\mathrm{low}}_{\mathrm{lex}}(n + 1, \theta))$*. Then,* $z_{n+1} = 0$ *and* $z_i = 1/2$ *for* $i = 1, \ldots, n$*.*

(iii) For any binary vector $\mu \in \{0, 1\}^n$*, let* $\theta = (2/3)(1 + \sum_{i=1}^{n} \mu_i/3^{n+1-i})$ *and* $(z, f, s, t, u) \in \mathcal{S}(\mathrm{P}^{\mathrm{low}}_{\mathrm{lex}}(n + 1, \theta))$*. Then,* $z_{n+1} = 1$ *and* $z_i = \mu_i$ *for* $i = 1, \ldots, n$*.*

The proof is given in Appendix D. For each θ, the lower-level problem $(\mathrm{P}^{\mathrm{low}}_{\mathrm{lex}}(n + 1, \theta))$ has a unique optimal solution. Plots illustrating the relationship between u, z and θ are provided in Figure 1. As θ varies from 0 to 1, each component of the vector z takes values between 0 and 1. Collectively, the vector z traces all the vertices of the $(n + 1)$-dimensional unit cube, as well as the point $z = (x, y)$, where $x_i = 1/2$ for all $i = 1, \ldots, n$ and $y = 0$. Notably, as θ increases from 0 to 1, z visits all candidate solutions that may be optimal for $(\mathrm{P}_{\mathrm{mult}}(F))$, as established in Lemma 4. Therefore, despite $(\mathrm{P}_{\mathrm{lex}}(F))$ involving only a single upper-level variable, it is capable of "searching" for the solution to $(\mathrm{P}_{\mathrm{mult}}(F))$ by varying θ. Formally, we establish the following result.

Lemma 6. *It holds that* $v(\mathrm{P}_{\mathrm{mult}}(F)) = v(\mathrm{P}_{\mathrm{lex}}(F))$*.*

Proof. We first show that $v(\mathrm{P}_{\mathrm{mult}}(F)) \ge v(\mathrm{P}_{\mathrm{lex}}(F))$. Suppose F is satisfiable, and let μ be a binary vector satisfying F, $\theta = (2/3)(1 + \sum_{i=1}^{n} \mu_i/3^{n+1-i})$ and $(z, f, s, t, u) \in \mathcal{S}(\mathrm{P}^{\mathrm{low}}_{\mathrm{lex}}(n + 1, \theta))$. By Lemma 5 (iii), we have $z_i = \mu_i$ for $i =$

$1,\dots,n$, $z_{n+1}=1$, $f_i=|z_i-1/2|=1/2$ for $i=1,\dots,n+1$. Therefore, $B_F z \geq b_F$, and (θ,z,f,s,t,u) is feasible for $(\mathrm{P}_{\mathrm{lex}}(F))$ and

$$v(\mathrm{P}_{\mathrm{lex}}(F)) \leq 2\sum_{i=1}^{n}(z_{n+1}-2f_i)-\frac{2}{n}\sum_{i=1}^{n}f_i=-1=v(\mathrm{P}_{\mathrm{mult}}(F)).$$

If F is unsatisfiable, we can show $v(\mathrm{P}_{\mathrm{lex}}(F)) \leq 0 = v(\mathrm{P}_{\mathrm{mult}}(F))$ with $\theta=1/6$.

Next, we show that $v(\mathrm{P}_{\mathrm{mult}}(F)) \leq v(\mathrm{P}_{\mathrm{lex}}(F))$. To this end, we prove that the objective value of any feasible solution to $(\mathrm{P}_{\mathrm{lex}}(F))$ is larger than or equal to $v(\mathrm{P}_{\mathrm{mult}}(F))$. Let (θ,z,f,s,t,u) be any feasible solution to $(\mathrm{P}_{\mathrm{lex}}(F))$. Then, we have $0\leq z_i\leq 1$, $f_i=|z_i-1/2|$ for $i=1,\dots,n+1$, and $B_F z\geq b_F$. Therefore, (z,f) is feasible for $(\mathrm{P}_{\mathrm{mult}}(F))$, and hence

$$2\sum_{i=1}^{n}(z_{n+1}-2f_i)-\frac{2}{n}\sum_{i=1}^{n}f_i \geq v(\mathrm{P}_{\mathrm{mult}}(F)).$$

Since (θ,z,f,s,t,u) was an arbitrary feasible solution to $(\mathrm{P}_{\mathrm{lex}}(F))$, it follows that $v(\mathrm{P}_{\mathrm{mult}}(F)) \leq v(\mathrm{P}_{\mathrm{lex}}(F))$. □

Next, we aim to reformulate the lexicographic LP $(\mathrm{P}^{\mathrm{low}}_{\mathrm{lex}}(n+1,\theta))$ as a single-objective LP. There is a substantial body of literature on transforming lexicographic LPs into LPs. For example, Sherali [17] and Sherali and Soyster [18] explored approaches based on weighted sums of the objectives. However, these techniques are difficult to apply to $(\mathrm{P}^{\mathrm{low}}_{\mathrm{lex}}(n+1,\theta))$: Since the number of objectives increases with n, it is nontrivial to design appropriate weights of polynomial size to combine the objectives. Nevertheless, since $(\mathrm{P}^{\mathrm{low}}_{\mathrm{lex}}(n+1,\theta))$ has a special structure, we can derive an equivalent LP of polynomial size using a tailored argument. Let η be a real number and consider

$$\begin{aligned}\min_{\theta,z,f,s,t,u}\ & 2\sum_{i=1}^{n}(z_{n+1}-2f_i)-\frac{2}{n}\sum_{i=1}^{n}f_i \\ \text{s.t. } & B_F z\geq b_F, 0\leq\theta\leq 1,\\ & (z,f,s,t,u)\in\mathcal{S}(\mathrm{P}^{\mathrm{low}}_{\mathrm{weight}}(n+1,\theta,\eta)),\end{aligned} \qquad (\mathrm{P}_{\mathrm{weight}}(F,\eta))$$

where

$$\begin{aligned}\min_{z,f,s,t,u}\ & \sum_{i=1}^{n+1}16^i(s_i+t_i)+\eta\sum_{i=1}^{n+1}\frac{1}{4^{n-i+1}}f_i \\ \text{s.t. } & (z,f,s,t,u)\in\mathcal{F}(\mathrm{P}^{\mathrm{low}}_{\mathrm{lex}}(n+1,\theta)).\end{aligned} \qquad (\mathrm{P}^{\mathrm{low}}_{\mathrm{weight}}(n+1,\theta,\eta))$$

We show that $v(\mathrm{P}_{\mathrm{lex}}(F))=v(\mathrm{P}_{\mathrm{weight}}(F,\eta))$ given $\eta\in(0,1]$.

Lemma 7.

(i) $\mathcal{S}(\mathrm{P}^{\mathrm{low}}_{\mathrm{lex}}(n+1,\theta))=\mathcal{S}(\mathrm{P}^{\mathrm{low}}_{\mathrm{weight}}(n+1,\theta,\eta))$ *for any* $n\geq 1$, $\theta\in[0,1]$ *and* $\eta\in(0,1]$.

(ii) $v(\mathrm{P}_{\mathrm{lex}}(F)) = v(\mathrm{P}_{\mathrm{weight}}(F, \eta))$ *for any* $\eta \in (0, 1]$.

See Appendix E for the proof. As the last step, we remove $B_F z \geq b_F$ from the upper-level problem using Lemma 3 and obtain

$$\begin{aligned} \min_{\theta, z, f, s, t, u, e} \ & 2 \sum_{i=1}^{n} (z_{n+1} - 2 f_i) - \frac{2}{n} \sum_{i=1}^{n} f_i + Me \\ \text{s.t. } & (z, f, s, t, u, e) \in \mathcal{S}(\mathrm{P}^{\mathrm{low}}(F, \theta)), \end{aligned} \tag{P(F)}$$

where

$$\begin{aligned} \min_{z, f, s, t, u, e} \ & \sum_{i=1}^{n+1} 16^i (s_i + t_i) + \sum_{i=1}^{n+1} \frac{1}{4^{n-i+1}} f_i \\ \text{s.t. } & B_F z + e\mathbf{1} \geq b_F, 0 \leq \theta \leq 1, 0 \leq e \leq 1, \\ & (z, f, s, t, u) \in \mathcal{F}(\mathrm{P}^{\mathrm{low}}_{\mathrm{lex}}(n+1, \theta)), \end{aligned} \tag{$\mathrm{P}^{\mathrm{low}}(F,\theta)$}$$

with $M = 3(5n + 7) 4^{f_{\mathrm{sol}}(5n+3, \langle 6 \rangle)+1}$.

Lemma 8. *It holds that* $v(\mathrm{P}_{\mathrm{weight}}(F, \eta)) = v(\mathrm{P}(F))$ *for* $\eta \leq 1$.

The proof is in Appendix F. Note that the size of (P(F)) is polynomial in n.

Lemma 9.

(i) Suppose F *is satisfiable. Then,* $v(\mathrm{P}(F)) = -1$.
(ii) Suppose F *is unsatisfiable. Then,* $v(\mathrm{P}(F)) = 0$.

Proof. It follows from Lemmata 4-8. □

Theorem 1 follows immediately from the above argument.

Proof (Proof of Theorem 1). The inclusion in NP follows from [3]. By Lemma 9, the above transformation gives a logarithmic-space reduction from 3-SAT to $\textsc{Val}_{\textsc{single}}$. Since 3-SAT is NP-complete [12], $\textsc{Val}_{\textsc{single}}$ is NP-hard. □

A few remarks are in order. First, observe that the optimal objective value of instance (P(F)) is -1 if F is satisfiable and 0 otherwise. Thus, there does not exist a polynomial-time approximation scheme (PTAS) for computing the optimal objective value of such bilevel LP instances, unless NP = P. Second, instance (P(F)) contains constants of exponential magnitude. Consequently, a bilevel LP with a single variable and no upper-level constraints may not be strongly NP-hard. In particular, such instances may admit pseudo-polynomial algorithms. Third, the argument of Basu *et al.* [1] shows that the feasible set of a bilevel LP is the union of at most exponentially many polyhedra. We observe that instance (P(F)), a bilevel LP with a single upper-level variable and no upper-level constraints, requires exponentially many polyhedra to describe its feasible set. Formally, we have the following result, proven in Appendix G.

Proposition 1. *Let* q *be the smallest integer such that there exists a collection of polyhedra* $\{Q_j : j = 1, \ldots, q\}$ *satisfying* $\mathcal{F}(\mathrm{P}(F)) = \cup_{j=1\ldots,q} Q_j$. *Then,* $p \geq 2^n$.

5 Reformulation of Combinatorial Optimization Problem

In this section, we briefly present a reformulation of combinatorial optimization problems into instances of $\mathrm{BLP}_{\textsc{single}}$. This relies on the techniques developed in Section 4 to prove Theorem 1. Consider the following 0-1 integer LP:

$$\min_{z}\{c^\top z : Az \geq a, z \in \{0,1\}^r\}, \tag{C}$$

where $c \in \mathbb{Q}^r$, $A \in \mathbb{Q}^{m\times r}$, and $a \in \mathbb{Q}^m$. We assume that (C) has at least one feasible solution, which holds when it is a 0-1 integer LP formulation of the knapsack or traveling salesman problem over a fully-connected graph. By appropriate scaling if necessary, we may assume $a_j \in [-1,1]$ for all $j = 1,\dots,m$. We first replace the binary variables using the lexicographic LP ($\mathrm{P}^{\mathrm{low}}_{\mathrm{lex}}(r,\theta)$) or its LP equivalent ($\mathrm{P}^{\mathrm{low}}_{\mathrm{weight}}(r,\theta,1)$) (Lemma 7 (i) is stated for $r \geq 2$ but it is straightforward to show that it also holds for $r = 1$):

$$\begin{aligned}
\min_{\theta,z,f,s,t,u}\ & c^\top z && (\mathrm{C}_{\mathrm{mult}})\\
\text{s.t. } & Az \geq a, 0 \leq \theta \leq 1, f_i \geq 1/2,\ \forall i = 1,\dots,n, && (4)\\
& (z,f,s,t,u) \in \mathcal{S}(\mathrm{P}^{\mathrm{low}}_{\mathrm{weight}}(r,\theta,1)),
\end{aligned}$$

where $\theta \in \mathbb{R}$, and $z,f,s,t,u \in \mathbb{R}^r$. For each $i = 1,\dots,r$, the optimality of the lower-level problem implies $f_i = |z_i - 1/2|$, thus the condition $f_i \geq 1/2$ enforces θ to be such that $z_i \in \{0,1\}$. Next, we replace the constraints (4) with a penalty term added to the objective:

$$\min_{\theta,z,f,s,t,u,e}\{c^\top z + Me : (z,f,s,t,u,e) \in \mathcal{S}(\mathrm{C}^{\mathrm{low}}_{\mathrm{single}}(\theta))\}, \tag{$\mathrm{C}_{\mathrm{single}}$}$$

where

$$\begin{aligned}
\min_{z,f,s,t,u,e}\ & \sum_{i=1}^{r} 16^i(s_i+t_i) + \sum_{i=1}^{r}\frac{1}{4^{r-i}}f_i && (\mathrm{C}^{\mathrm{low}}_{\mathrm{single}}(\theta))\\
\text{s.t. } & Az + e\mathbf{1} \geq a, f_i + e \geq 1/2,\ \forall i = 1,\dots,r,\\
& 0 \leq \theta \leq 1, 0 \leq e \leq 1,\\
& (z,f,s,t,u) \in \mathcal{F}(\mathrm{P}^{\mathrm{low}}_{\mathrm{lex}}(r,\theta)).
\end{aligned}$$

and M is a penalty coefficient. It is straightforward to verify that both instances ($\mathrm{C}_{\mathrm{mult}}$) and ($\mathrm{C}_{\mathrm{single}}$) admit optimal solutions for any value of M. For example, ($\mathrm{C}_{\mathrm{single}}$) has a feasible solution given by $\theta = 0$, $z = f = s = t = u = \mathbf{0}$, and $e = 1$. Moreover, ($\mathrm{C}_{\mathrm{single}}$) cannot be unbounded since all the variables are bounded, thus guaranteeing the existence of an optimal solution. Therefore, by invoking Lemma 3, one can compute a penalty coefficient M of polynomial size (in the size of (C)) such that $v(\mathrm{C}) = v(\mathrm{C}_{\mathrm{single}})$. In fact, it is straightforward to see the following: $\mathcal{S}(\mathrm{C}_{\mathrm{single}}) = \{(\theta,z,f,s,t,u,e) \in \mathcal{F}(\mathrm{C}_{\mathrm{single}}) : \theta = (2/3)\sum_{i=1}^{r}\mu_i/3^{r-i},\ \mu \in \mathcal{S}(\mathrm{C})\}$. That is, there is a bijection between $\mathcal{S}(\mathrm{C})$ and $\mathcal{S}(\mathrm{C}_{\mathrm{single}})$.

6 Complexity of Local Search

To further understand the computational complexity of $\mathrm{BLP}_{\mathrm{SINGLE}}$, in the remainder of the paper, we study the complexity of searching for a solution. We begin by discussing the coNP-completeness of verifying whether a given point is a globally optimal solution. We then turn our attention to the problem of finding a locally optimal solution. We show that it is possible to compute a locally optimal solution in polynomial time.

Consider the following decision problem:

Given a rational instance ($\mathrm{P}_{\mathrm{SINGLE}}$) and a rational point (x_1, x_2), is it an optimal solution? $\quad (\mathrm{VERIFY}^{\mathrm{GLOBAL}}_{\mathrm{SINGLE}})$

Using instance ($\mathrm{P}(F)$), we obtain the following result, which is proven in Appendix H.

Proposition 2. *The decision problem $\mathrm{VERIFY}^{\mathrm{GLOBAL}}_{\mathrm{SINGLE}}$ is coNP-complete.*

Proposition 2 shows that merely verifying whether a given point is an optimal solution is coNP-complete. Therefore, searching for a globally optimal solution is likely to be computationally intractable. A natural question is whether replacing global optimality with local optimality leads to a tractable problem.

We say that $(\bar{x}_1, \bar{x}_2) \in \mathbb{R}^{n+1}$ is a *locally optimal solution* to ($\mathrm{P}_{\mathrm{SINGLE}}$) if $(\bar{x}_1, \bar{x}_2) \in \mathcal{F}(\mathrm{P}_{\mathrm{SINGLE}})$ and there exists $\epsilon > 0$ such that

$$c_{21}^\top \bar{x}_1 + c_{22}\bar{x}_2 \leq c_{21}^\top x_1 + c_{22}x_2, \quad \forall\, (x_1, x_2) \in \mathcal{F}(\mathrm{P}_{\mathrm{SINGLE}}) \cap \mathcal{B}_\epsilon(\bar{x}_1, \bar{x}_2),$$

where $\mathcal{B}_\epsilon(\bar{x}_1, \bar{x}_2) = \{(x_1, x_2) : \|(x_1, x_2) - (\bar{x}_1, \bar{x}_2)\|_\infty \leq \epsilon\}$. The following result holds.

Theorem 2. *There exists a polynomial-time algorithm that finds a locally optimal solution to a rational instance of* ($\mathrm{P}_{\mathrm{SINGLE}}$).

See Appendix I for the proof. We show that if there is a local optimal solution, there is a rational one of polynomial size. Thus, we apply a binary search on the upper-level variable x_2 to find a local optimal solution. This result contrasts with a well-known result by Vicente *et al.* [19], which shows that verifying local optimality is NP-hard in the general case. Thus, Theorem 2 suggests a fundamental difference between general bilevel LPs and $\mathrm{BLP}_{\mathrm{SINGLE}}$.

Acknowledgments. This study was funded by the FRQ-IVADO Research Chair in Data Science for Combinatorial Game Theory.

Disclosure of Interests. The authors have no competing interests to declare that are relevant to the content of this article.

References

1. Basu, A., Ryan, C.T., Sankaranarayanan, S.: Mixed-integer bilevel representability. Mathematical programming **185**(1-2), 163–197 (2021). https://doi.org/10.1007/s10107-019-01424-w
2. Ben-Ayed, O., Blair, C.E.: Computational difficulties of bilevel linear programming. Operations Research **38**(3), 556–560 (1990). https://doi.org/10.1287/opre.38.3.556
3. Buchheim, C.: Bilevel linear optimization belongs to NP and admits polynomial-size KKT-based reformulations. Operations Research Letters **51**(6), 618–622 (2023). https://doi.org/https://doi.org/10.1016/j.orl.2023.10.006
4. Bui, Q.M., Carvalho, M., Neto, J.: Asymmetry in the complexity of the multi-commodity network pricing problem. Mathematical Programming **208**(1), 425–461 (2024)
5. Deng, X.: Complexity issues in bilevel linear programming. Multilevel optimization: Algorithms and applications pp. 149–164 (1998). https://doi.org/10.1007/978-1-4613-0307-7_6
6. Hansen, P., Jaumard, B., Savard, G.: New branch-and-bound rules for linear bilevel programming. SIAM Journal on scientific and Statistical Computing **13**(5), 1194–1217 (1992). https://doi.org/10.1137/0913069
7. Henke, D., Lefebvre, H., Schmidt, M., Thürauf, J.: On coupling constraints in linear bilevel optimization. Optimization Letters **19**(3), 689–697 (2025). https://doi.org/10.1007/s11590-024-02156-3
8. Isermann, H.: Linear lexicographic optimization. Operations-Research-Spektrum **4**(4), 223–228 (1982)
9. Jeroslow, R.G.: The polynomial hierarchy and a simple model for competitive analysis. Mathematical Programming **32**(2), 146–164 (1985). https://doi.org/10.1007/BF01586088
10. Lenstra Jr, H.W.: Integer programming with a fixed number of variables. Mathematics of operations research **8**(4), 538–548 (1983)
11. Marcotte, P., Savard, G.: Bilevel programming: A combinatorial perspective. In: Graph theory and combinatorial optimization, pp. 191–217. Springer (2005). https://doi.org/10.1007/0-387-25592-3_7
12. Papadimitriou, C.H.: Computational Complexity. Addison-Wesley (1993)
13. Pardalos, P.M., Schnitger, G.: Checking local optimality in constrained quadratic programming is NP-hard. Operations Research Letters **7**(1), 33–35 (1988)
14. Prokopyev, O.A., Ralphs, T.K.: On the complexity of finding locally optimal solutions in bilevel linear optimization. Tech. Rep. 24T-015, Lehigh University, COR@L Technical Report (October 2024), updated September 2025. Available at: https://optimization-online.org/?p=28138
15. Rodrigues, B., Carvalho, M., Anjos, M.F., Sugishita, N.: Unboundedness in bilevel optimization. Optimization Online (2024), https://optimization-online.org/2024/11/unboundedness-in-bilevel-optimization/
16. Schrijver, A.: Theory of Linear and Integer Programming. Wiley-Interscience series in discrete mathematics and optimization, Wiley, Chichester (1998)
17. Sherali, H.D.: Equivalent weights for lexicographic multi-objective programs: Characterizations and computations. European Journal of Operational Research **11**(4), 367–379 (1982)
18. Sherali, H.D., Soyster, A.L.: Preemptive and nonpreemptive multi-objective programming: Relationship and counterexamples. Journal of Optimization Theory and Applications **39**(2), 173–186 (1983). https://doi.org/10.1007/BF00934527

19. Vicente, L.N., Savard, G., Júdice, J.: Descent approaches for quadratic bilevel programming. Journal of Optimization Theory and Applications **81**(2), 379–399 (1994). https://doi.org/10.1007/BF02191670

A Proof of Lemma 2

Proof. The feasible set of (1) is the union of polyhedra [1]. Indeed,

$$\mathcal{F}(1) = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : \exists \lambda \text{ s.t. } \begin{array}{l} A_{21}x_1 + A_{22}x_2 \geq b_2, \\ A_{11}x_1 + A_{12}x_2 \geq b_1, \\ \lambda^\top (A_{11}x_1 + A_{12}x_2 - b_1) = 0, \\ A_{11}^\top \lambda = c_{11}, \\ \lambda \geq 0 \end{array} \right\}$$
$$= \bigcup_{\omega \in \{0,1\}^{m_1}} \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : \exists \lambda \text{ s.t. } \begin{array}{l} A_{21}x_1 + A_{22}x_2 \geq b_2, \\ A_{11}x_1 + A_{12}x_2 \geq b_1, \\ \omega^\top (A_{11}x_1 + A_{12}x_2 - b_1) = 0, \\ (1-\omega)^\top \lambda = 0, \\ A_{11}^\top \lambda = c_{11}, \\ \lambda \geq 0 \end{array} \right\}.$$

That is, $\mathcal{F}(1) = \cup_{\omega \in \Omega} U_\omega$, where

$$\Omega = \{\omega \in \{0,1\}^{m_1} : \exists \lambda \text{ s.t. } (1-\omega)^\top \lambda = 0, A_{11}^\top \lambda = c_{11}, \lambda \geq 0\}$$

and for each $\omega \in \Omega$,

$$U_\omega = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : \begin{array}{l} A_{21}x_1 + A_{22}x_2 \geq b_2, \\ A_{11}x_1 + A_{12}x_2 \geq b_1, \\ \omega^\top (A_{11}x_1 + A_{12}x_2 - b_1) = 0 \end{array} \right\}.$$

Let

$$\min_{x_1, x_2} \{c_{21}^\top x_1 + c_{22}^\top x_2 : (x_1, x_2) \in U_\omega\} \qquad (\mathrm{P}_{\mathrm{decomp}}(\omega))$$

and observe that $v(1) = \min_{\omega \in \Omega} v(\mathrm{P}_{\mathrm{decomp}}(\omega))$ and $\mathcal{S}(1) = \cup_{\omega \in \Omega'} \mathcal{S}(\mathrm{P}_{\mathrm{decomp}}(\omega))$, where $\Omega' = \operatorname{argmin}_{\omega \in \Omega} v(\mathrm{P}_{\mathrm{decomp}}(\omega))$. Note that each U_ω can be defined by inequalities with coefficients of size at most σ. Thus, by Lemma 1, for each $\omega \in \Omega$, $\mathcal{S}(\mathrm{P}_{\mathrm{decomp}}(\omega))$ contains a rational point of size $f_{\mathrm{sol}}(n, \sigma)$ given it is nonempty. □

B Proof of Lemma 3

Lemma 10. *Under the assumptions of Lemma 3, for any* $(x_1, x_2, e) \in \mathcal{S}(Q'_{\mathrm{blp}}(M))$, *it holds that* $e = 0$.

Proof. For any $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$, $(x_1, x_2) \in \mathcal{F}(Q_{\mathrm{blp}})$ implies $(x_1, x_2, 0) \in \mathcal{F}(Q'_{\mathrm{blp}}(M))$. Thus, we have

$$v(Q_{\mathrm{blp}}) \geq v(Q'_{\mathrm{blp}}(M)). \tag{5}$$

For the sake of contradiction, assume that there exists $(\bar{x}_1, \bar{x}_2, \bar{e}) \in \mathcal{S}(Q'_{\mathrm{blp}}(M))$ such that $\bar{e} > 0$. Under this assumption, we show that $v(Q_{\mathrm{blp}}) < v(Q'_{\mathrm{blp}}(M))$, which contradicts (5). Let m' be the number of constraints in the lower-level problem of ($Q'_{\text{blp}}(M)$). Following the argument in the proof of Lemma 2, we can write the feasible set of ($Q'_{\text{blp}}(M)$) as the union of polyhedra:

$$\mathcal{F}(Q'_{\text{blp}}(M)) = \bigcup_{\omega \in \Omega'} U'_\omega,$$

where $\Omega' \subset \{0,1\}^{m'}$ and for each $\omega \in \Omega'$, $U'_\omega \subset \mathbb{R}^{n+1}$ is a rational polyhedron defined by inequalities whose coefficients are of size at most σ. Furthermore, there is $\bar{\omega} \in \Omega'$ such that $(\bar{x}_1, \bar{x}_2, \bar{e}) \in U'_{\bar{\omega}}$. Now, consider the LP

$$\min_{x_1, x_2, e} \{c_{21}^\top x_1 + c_{22}^\top x_2 + Me : (x_1, x_2, e) \in U'_{\bar{\omega}}, e > 0\}. \tag{6}$$

Since $(\bar{x}_1, \bar{x}_2, \bar{e})$ is also an optimal solution to (6), we have $\emptyset \neq \mathcal{S}(6) \subset U'_{\bar{\omega}} \subset \mathcal{S}(Q'_{\text{blp}}(M))$. By Lemma 1, there exists a rational solution $(\tilde{x}_1, \tilde{x}_2, \tilde{e}) \in \mathcal{S}(6)$ such that $\langle \tilde{x}_1, \tilde{x}_2, \tilde{e} \rangle \leq f_{\text{sol}}(n+1, \sigma)$. In particular, we have $\tilde{e} > 0$, implying $\tilde{e} \geq 1/2^{f_{\text{sol}}(n+1,\sigma)}$. Let $(\hat{x}_1, \hat{x}_2) \in \mathbb{Q}^n$ be a rational optimal solution to (Q_{blp}) such that $\langle \hat{x}_1, \hat{x}_2 \rangle \leq f_{\text{sol}}(n+1, \sigma)$. Then,

$$\begin{aligned}
&v(Q'_{\text{blp}}(M)) - v(Q_{\text{blp}}) \\
&\quad = (c_{21}^\top \tilde{x}_1 + c_{22}^\top \tilde{x}_2 + M\tilde{e}) - (c_{21}^\top \hat{x}_1 + c_{22}^\top \hat{x}_2) \\
&\quad \geq M\tilde{e} - \|c_{21}\|_\infty \|\tilde{x}_1\|_1 - \|c_{22}\|_\infty \|\tilde{x}_2\|_1 - \|c_{21}\|_\infty \|\hat{x}_1\|_1 - \|c_{22}\|_\infty \|\hat{x}_2\|_1 \\
&\quad \geq M\tilde{e} - c_\infty n_1 \|\tilde{x}_1\|_\infty - c_\infty n_2 \|\tilde{x}_2\|_\infty - c_\infty n_1 \|\hat{x}_1\|_\infty - c_\infty n_2 \|\hat{x}_2\|_\infty \\
&\quad \geq M/2^{f_{\text{sol}}(n+1,\sigma)} - c_\infty n_1 2^{f_{\text{sol}}(n+1,\sigma)} - c_\infty n_2 2^{f_{\text{sol}}(n+1,\sigma)} \\
&\qquad\qquad - c_\infty n_1 2^{f_{\text{sol}}(n+1,\sigma)} - c_\infty n_2 2^{f_{\text{sol}}(n+1,\sigma)} \\
&\quad \geq M/2^{f_{\text{sol}}(n+1,\sigma)} - 2c_\infty n 2^{f_{\text{sol}}(n+1,\sigma)} \\
&\quad \geq M/2^{f_{\text{sol}}(n+1,\sigma)} - (2/3)M/2^{f_{\text{sol}}(n+1,\sigma)} \\
&\quad > 0,
\end{aligned}$$

which contradicts (5). Therefore, for any $(\bar{x}_1, \bar{x}_2, \bar{e}) \in \mathcal{S}(Q'_{\text{blp}}(M))$ we have $\bar{e} = 0$. □

Proof (Lemma 3). We prove relation (ii), from which relation (i) follows directly. We establish the following two statements:

(A-1) If $(x_1, x_2, e) \in \mathcal{S}(Q'_{\text{blp}}(M))$, then $e = 0$ and $(x_1, x_2) \in \mathcal{S}(Q_{\text{blp}})$;
(A-2) If $(x_1, x_2) \in \mathcal{S}(Q_{\text{blp}})$, then $(x_1, x_2, 0) \in \mathcal{S}(Q'_{\text{blp}}(M))$.

The desired relation follows from these two assertions.

We begin with the proof of assertion (A-1). Let $(x_1, x_2, e) \in \mathcal{S}(Q'_{\text{blp}}(M))$. By Lemma 10, $e = 0$. It remains to show that (x_1, x_2) is optimal to (Q_{blp}). We show the feasibility and optimality of (x_1, x_2) in order. Since $(x_1, x_2, 0) \in \mathcal{S}(Q'_{\text{blp}}(M))$, it follows that (x_1, x_2) satisfies all the linear inequalities in (Q_{blp}). Thus, (x_1, x_2) is feasible for (Q_{blp}) if and only if x_1 is optimal to the lower-level problem of (Q_{blp}) given x_2. To derive a contradiction, suppose x_1 is not optimal to the lower-level problem of (Q_{blp}). Then, there exists $\bar{x}_1$ feasible to the lower-level problem of (Q_{blp}) such that $c_{11}^\top \bar{x}_1 < c_{11}^\top x_1$. Let $V \geq 1$ be a real number such that

$$\bar{A}_{21}\bar{x}_1 + \bar{A}_{22}x_2 + V \geq \bar{b}_2.$$

Define

$$\hat{x}_1 = \frac{V-1}{V}x_1 + \frac{1}{V}\bar{x}_1, \hat{e} = 1.$$

Then, $(\hat{x}_1, \hat{e})$ is feasible to the lower-level problem of ($Q'_{\text{blp}}(M)$), and its objective value is strictly smaller than that of $(x_1, 0)$, a contradiction to the feasibility of $(x_1, x_2, 0)$ to ($Q'_{\text{blp}}(M)$). Thus, x_1 is indeed optimal to the lower-level problem of (Q_{blp}), and (x_1, x_2) is a feasible solution to (Q_{blp}). The objective value of (x_1, x_2) in instance (Q_{blp}) is $c_{21}^\top x_1 + c_{22}^\top x_2$. By assumption $(x_1, x_2, 0) \in \mathcal{S}(Q'_{\text{blp}}(M))$, we have $v(Q'_{\text{blp}}(M)) = c_{21}^\top x_1 + c_{22}^\top x_2$. In light of (5), (x_1, x_2) is optimal for (Q_{blp}) and $v(Q_{\text{blp}}) = v(Q'_{\text{blp}}(M))$.

We now prove assertion (A-2). Let $(x_1, x_2) \in \mathcal{S}(Q_{\text{blp}})$. We claim that $(x_1, 0)$ is optimal to the lower-level problem of ($Q'_{\text{blp}}(M)$). Suppose the contrary. Then thre exists a feasible solution $(\hat{x}_1, \hat{e})$ to the lower-level problem of ($Q'_{\text{blp}}(M)$) such that $c_{11}^\top \hat{x}_1 < c_{11}^\top x_1$. Note that $\hat{x}_1$ is feasible to the lower-level problem of (Q_{blp}) with a strictly smaller objective value than x_1, a contradiction to the assumption that $(x_1, x_2) \in \mathcal{F}(Q_{\text{blp}})$. Therefore, $(x_1, 0)$ is optimal to the lower-level problem of ($Q'_{\text{blp}}(M)$), and we have $(x_1, x_2, 0) \in \mathcal{F}(Q'_{\text{blp}}(M))$. Since $v(Q_{\text{blp}}) = v(Q'_{\text{blp}}(M))$, $(x_1, x_2, 0)$ is optimal for ($Q'_{\text{blp}}(M)$). □

C Proof of Lemma 4

Consider the following instance:

$$\min_{x,y,w} \; 2\sum_{i=1}^{n}(y - 2w_i) - \frac{2}{n}\sum_{i=1}^{n} w_i \tag{$\mathrm{P_V}(F)$}$$

$$\text{s.t. } A_F x \geq \left(\frac{3}{2} - \frac{1}{2}y\right)\mathbf{1} - a_F, \tag{7}$$

$$\frac{1}{2} - \frac{1}{2}y \leq x_i \leq \frac{1}{2} + \frac{1}{2}y, \quad \forall i = 1, \ldots, n, \tag{8}$$

$$w \in \operatorname*{argmin}_{w'} \left\{ \sum_{i=1}^{n} w'_i : -w'_i \leq x_i - 1/2 \leq w'_i,\ i = 1, \ldots, n \right\}.$$

Rodrigues *et al.* [15] used this instance to study the unboundedness of the bilevel LP. This instance is based on the one studied by Vicent *et al.* [19][3].

Lemma 11. *The optimal objective value of* ($\mathrm{P_V}(F)$) *is* $-\infty$ *if* F *is satisfiable, and* 0 *otherwise.*

[3] In their lower-level problem, there is another variable z, but it can be eliminated using the relation $z_i = x_0 - w_i$ for all i, which follows from the optimality of the lower-level problem. We also replaced x_0 in their formulation with $y/2$.

See [15] for the proof. If F is unsatisfiable, then the point defined by $x_i = \frac{1}{2}$ for all $i = 1, \dots, n$, $y = 0$, and $w = \mathbf{0}$ attains the optimal objective value of 0. On the other hand, if F is satisfiable, let $\mu \in \{0,1\}^n$ be a binary vector satisfying F. Then for any $t \geq 0$, the point $(x(t), y(t), w(t))$ defined by

$$y(t) = t, \quad x_i(t) = \frac{1}{2} + \left(\mu_i - \frac{1}{2}\right) t, \quad \text{and} \quad w_i(t) = \frac{t}{2}, \quad \text{for } i = 1, \dots, n,$$

is feasible for ($\mathrm{P_V}(F)$), and the objective value improves indefinitely as t grows.

The next instance is also useful to prove Lemma 4:

$$\begin{aligned} \min_{x,y} \ & 2\sum_{i=1}^{n}\left(y - 2\left|x_i - \frac{1}{2}\right|\right) - \frac{2}{n}\sum_{i=1}^{n}\left|x_i - \frac{1}{2}\right| && (\mathrm{P_{nonconvex}}) \\ \text{s.t.} \ & A_F x \geq \left(\frac{3}{2} - \frac{1}{2}y\right)\mathbf{1} - a_F, \\ & \frac{1}{2} - \frac{1}{2}y \leq x_i \leq \frac{1}{2} + \frac{1}{2}y, && \forall i = 1, \dots, n, \quad (9) \\ & 0 \leq x_i \leq 1, && \forall i = 1, \dots, n, \\ & 0 \leq y \leq 1. \end{aligned}$$

For any $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$, (x, y) is feasible for ($\mathrm{P_{nonconvex}}$) if and only if (x, y, w) is feasible for ($\mathrm{P_{3SAT}}(F)$), where $w_i = |x_i - 1/2|$ for $i = 1, \dots, n$. Thus, $v(\mathrm{P_{nonconvex}}) = v(\mathrm{P_{3SAT}}(F))$.

Proof (Lemma 4). Suppose first F is satisfiable. We show that $v(\mathrm{P_{nonconvex}}) = -1$. Constraint (9) implies $y - 2|x_i - 1/2| \geq 0$ and $|x_i - 1/2| \leq (1/2)y \leq 1/2$ for $i = 1, \dots, n$. Thus, for any $(x, y) \in \mathcal{F}(\mathrm{P_{nonconvex}})$,

$$2\sum_{i=1}^{n}\left(y - 2\left|x_i - \frac{1}{2}\right|\right) - \frac{2}{n}\sum_{i=1}^{n}\left|x_i - \frac{1}{2}\right| \geq 2\sum_{i=1}^{n} 0 - \frac{2}{n}\sum_{i=1}^{n}\frac{1}{2} = -1.$$

Thus, $v(\mathrm{P_{nonconvex}}) \geq -1$, but this is attained by $x = \mu$ and $y = 1$, where μ is a binary vector satisfying F.

Next, suppose F is unsatisfiable. Observe that $\mathcal{F}(\mathrm{P_V}(F)) \supset \mathcal{F}(\mathrm{P_{3SAT}}(F))$ (recall that $\mathcal{F}$ denotes the bilevel feasible set) and therefore ($\mathrm{P_V}(F)$) is a relaxation of ($\mathrm{P_{3SAT}}(F)$). As a result, $v(\mathrm{P_{3SAT}}(F)) \geq v(\mathrm{P_V}(F)) = 0$, where the last equality holds from Lemma 11. Note that $x_i = 1/2$ for $i = 1, \dots, n$, $y = 0$ and $w = \mathbf{0}$ is feasible for ($\mathrm{P_{3SAT}}(F)$) and attains the objective value of 0, hence optimal for ($\mathrm{P_{3SAT}}(F)$).

D Proof of Lemma 5

Proof. (i) For any $\theta \in [0,1]$, the value of the first objective $s_{n+1} + t_{n+1}$ is bounded by $[(3/2)\theta - 1/2]^+ + [(3/2)\theta - 1]^+$ from below. Given $(z, f, s, t, u) \in \mathcal{F}(\mathrm{P_{lex}^{low}}(n+1, \theta))$, this lower bound is attained if and only if $u_{n+1} = \theta$,

$s_{n+1} = [(3/2)u_{n+1} - 1/2]^+$ and $t_{n+1} = [(3/2)u_{n+1} - 1]^+$, and there is at least one point in $\mathcal{F}(\mathrm{P}_{\mathrm{lex}}^{\mathrm{low}}(n+1,\theta))$ satisfying these conditions. Thus, the optimality with respect to the first objective $s_{n+1} + t_{n+1}$ implies $u_{n+1} = \theta$, $s_{n+1} = [(3/2)u_{n+1} - 1/2]^+$, $t_{n+1} = [(3/2)u_{n+1} - 1]^+$, and it follows that $z_{n+1} = f^z(u_{n+1})$ and $u_n = f^u(u_{n+1})$. Therefore,

$$
\begin{aligned}
&\mathcal{S}(\mathrm{P}_{\mathrm{lex}}^{\mathrm{low}}(n+1,\theta)) = \\
&\quad \operatorname*{lexmin}_{z,f,s,t,u} \left\{ s_n + t_n,\ \ldots,\ s_1 + t_1,\ \sum_{i=1}^{n+1} f_i \right\} \\
&\qquad \text{s.t. } u_{n+1} = \theta, \\
&\qquad\quad s_{n+1} = [(3/2)u_{n+1} - 1/2]^+, \quad t_{n+1} = [(3/2)u_{n+1} - 1]^+, \\
&\qquad\quad z_{n+1} = f^z(u_{n+1}), \qquad u_n = f^u(u_{n+1}), \\
&\qquad\quad u_{i-1} = 3u_i - 2z_i, \qquad \forall i = 2, \ldots, n, \\
&\qquad\quad s_i \geq (3/2)u_i - 1/2, \quad t_i \geq (3/2)u_i - 1, \quad \forall i = 1, \ldots, n, \\
&\qquad\quad z_i = 2(s_i - t_i), \qquad \forall i = 1, \ldots, n, \\
&\qquad\quad -f_i \leq z_i - 1/2 \leq f_i, \qquad \forall i = 1, \ldots, n+1, \\
&\qquad\quad 0 \leq z_i, f_i, s_i, t_i, u_i \leq 1, \qquad \forall i = 1, \ldots, n+1,
\end{aligned}
$$

From the optimality with respect to the next objective $s_n + t_n$, we get $s_n = [(3/2)u_n - 1/2]^+$, $t_n = [(3/2)u_n - 1]^+$ $z_n = f^z(u_n)$ and $u_{n-1} = f^u(u_n)$. Repeating this, we obtain $s_i = [(3/2)u_i - 1/2]^+$, $t_i = [(3/2)u_i - 1]^+$ $z_i = f^z(u_i)$ for $i = 1, \ldots, n-1$, and $u_{i-1} = f^u(u_i)$ for $i = 2, \ldots, n-1$. Therefore,

$$
\begin{aligned}
&\mathcal{S}(\mathrm{P}_{\mathrm{lex}}^{\mathrm{low}}(n+1,\theta)) = \\
&\quad \operatorname*{argmin}_{z,f,s,t,u} \sum_{i=1}^{n+1} f_i \\
&\qquad \text{s.t. } u_{n+1} = \theta, \\
&\qquad\quad s_i = [(3/2)u_i - 1/2]^+, \quad t_i = [(3/2)u_i - 1]^+, \quad \forall i = 1, \ldots, n+1, \\
&\qquad\quad z_i = f^z(u_i), \quad -f_i \leq z_i - 1/2 \leq f_i, \quad \forall i = 1, \ldots, n+1, \\
&\qquad\quad u_{i-1} = f^u(u_i), \qquad \forall i = 2, \ldots, n+1, \\
&\qquad\quad 0 \leq z_i, f_i, s_i, t_i, u_i \leq 1, \qquad \forall i = 1, \ldots, n+1,
\end{aligned}
$$

implying the desired results.

(ii) From assertion (i), when $\theta = 1/6$, we have $z_{n+1} = 0$ and $u_n = 1/2$. Thus, for any $i = 1, \ldots, n$, we have $u_i = f^u(u_{i+1}) = f^u(1/2) = 1/2$. We thus obtain $z_i = f^z(u_i) = f^z(1/2) = 1/2$ for any $i = 1, \ldots, n$.

(iii) From assertion (i), since $\theta \geq 2/3$, we have

$$
z_{n+1} = f^z(\theta) = 1, \quad u_n = f^u(\theta) = 3\theta - 2 = \frac{2}{3}\left(\mu_n + \sum_{i=1}^{n-1} \frac{\mu_i}{3^{n-i}}\right).
$$

Since $(2/3)\sum_{i=1}^{n-1}\mu_i/3^{n-i} < 1/3$, we have $u_n \in [0,1/3)$ if $\mu_n = 0$, and $u_n \in [2/3,1]$ if $\mu_n = 1$. Thus,

$$\begin{aligned} u_{n-1} &= \begin{cases} 3\cdot\dfrac{2}{3}\left(0+\displaystyle\sum_{i=1}^{n-1}\dfrac{\mu_i}{3^{n-i}}\right), & \text{if } \mu_n = 0,\\ 3\cdot\dfrac{2}{3}\left(1+\displaystyle\sum_{i=1}^{n-1}\dfrac{\mu_i}{3^{n-i}}\right)-2, & \text{if } \mu_n = 1, \end{cases}\\ &= \frac{2}{3}\left(\mu_{n-1}+\sum_{i=1}^{n-2}\frac{\mu_i}{3^{n-1-i}}\right), \end{aligned}$$

Repeating the argument, we obtain

$$u_{n-2} = \frac{2}{3}\left(\mu_{n-2}+\sum_{i=1}^{n-3}\frac{\mu_i}{3^{n-2-i}}\right),\ \ldots,\ u_1 = \frac{2}{3}\mu_1.$$

Thus, for each $i = 1,\ldots,n+1$, $u_i \in [0,1/3)$ if $\mu_i = 0$, and $u_i \in [2/3,1]$ otherwise, implying $z_i = f^z(u_i) = \mu_i$. □

E Proof of Lemma 7

Proof. Assertion (ii) follows immediately from assertion (i). Thus, we only prove assertion (i).

We show the claim by induction. We first prove the base case: we show that the claim holds for $n = 1$ and any $\eta \in (0,1]$. One can rewrite ($\mathrm{P}^{\mathrm{low}}_{\mathrm{weight}}(n+1,\theta,\eta)$) in standard form with variable upper bounds as

$$\begin{aligned} \min_{z,f,s,t,u,\alpha}\ & 256(s_2+t_2)+16(s_1+t_1)+\frac{\eta}{4}f_1+\eta f_2 \qquad (10)\\ \text{s.t. } & u_2 = \theta,\\ & s_1+\alpha_1 = (3/2)u_1-1/2, \qquad s_2+\alpha_2 = (3/2)u_2-1/2,\\ & t_1+\alpha_3 = (3/2)u_1-1, \qquad t_2+\alpha_4 = (3/2)u_2-1,\\ & z_1-2s_1+2t_1 = 0, \qquad z_2-2s_2+2t_2 = 0,\\ & z_1+f_1-\alpha_5 = 1/2, \qquad z_2+f_2-\alpha_6 = 1/2,\\ & z_1+f_1+\alpha_7 = 1/2, \qquad z_2+f_2+\alpha_8 = 1/2,\\ & u_1 = 3u_2-2z_2,\\ & 0 \le z_1,z_2,f_1,f_2,s_1,s_2,t_1,t_2,u_1,u_2 \le 1,\ \alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6,\alpha_7,\alpha_8 \ge 0. \end{aligned}$$

Suppose $\theta \in [0,1/9]$ and consider the basic solution corresponding to basic variables $(f_1,f_2,s_1,s_2,u_1,u_2,\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_6,\alpha_8)$ and non-basic variables $(z_1,z_2,t_1,t_2,\alpha_5,\alpha_7)$. Suppose all the non-basic variables are fixed to their lower bounds, i.e., 0. The value of the basic variables are

$$\left(\frac{1}{2},\ \frac{1}{2},\ 0,\ 0,\ 3\theta,\ \theta,\ \frac{1}{2}-\frac{9\theta}{2},\ \frac{1}{2}-\frac{3\theta}{2},\ 1-\frac{9\theta}{2},\ 1-\frac{3\theta}{2},\ 1,\ 1\right),$$

and the reduced costs of the non-basic variables are

$$\left(8-\frac{\eta}{4},\ 128-\eta,\ 32,\ 512,\ \frac{\eta}{4},\ \eta\right).$$

When $\theta \in [0, 1/9]$ and $\eta \le 1$, this basic solution is feasible, and all the reduced costs of the non-basic variables are strictly positive. Thus, this is the unique optimal solution to (10). Using Lemma 5 (i), it is straightforward to verify that this point corresponds to the one in $\mathcal{S}(\mathrm{P}^{\mathrm{low}}_{\mathrm{lex}}(n+1,\theta))$. One can show similarly for the cases where $\theta \notin [0, 1/9]$.

Now, suppose the claim holds for some $n-1$ and any $\eta \in (0, 1]$. Pick any $\theta \in [0, 1]$ and $0 < \eta \le 1$ and consider ($\mathrm{P}^{\mathrm{low}}_{\mathrm{lex}}(n+1,\theta)$) and ($\mathrm{P}^{\mathrm{low}}_{\mathrm{weight}}(n+1,\theta,\eta)$). Let $(\hat{z}, \hat{f}, \hat{s}, \hat{t}, \hat{u})$ be the optimal solution to ($\mathrm{P}^{\mathrm{low}}_{\mathrm{lex}}(n+1,\theta)$) and $(\bar{z}, \bar{f}, \bar{s}, \bar{t}, \bar{u})$ be an optimal solution to ($\mathrm{P}^{\mathrm{low}}_{\mathrm{weight}}(n+1,\theta,\eta)$). We show that the two points coincide. Now, consider the instance derived from ($\mathrm{P}^{\mathrm{low}}_{\mathrm{weight}}(n+1,\theta,\eta)$) by fixing the following variables to their optimal values: $z_{n+1} = \bar{z}_{n+1}$, $f_{n+1} = \bar{f}_{n+1}$, $s_{n+1} = \bar{s}_{n+1}$, $t_{n+1} = \bar{t}_{n+1}$, and $u_{n+1} = \bar{u}_{n+1}$. The resulting instance corresponds to ($\mathrm{P}^{\mathrm{low}}_{\mathrm{weight}}(n,\theta',\eta/4)$), where $\theta' = 3\theta - 4(\bar{s}_{n+1} - \bar{t}_{n+1})$. Note that $\theta' \in [0, 1]$ since $(\hat{z}, \hat{f}, \hat{s}, \hat{t}, \hat{u})$ is feasible to ($\mathrm{P}^{\mathrm{low}}_{\mathrm{weight}}(n+1,\theta,\eta)$). The remaining variables must be optimal for this reduced instance. Thus, by the induction hypothesis,

$$\begin{aligned}
&\bar{u}_n = 3\bar{u}_{n+1} - 2\bar{z}_{n+1} = 3\theta - 4(\bar{s}_{n+1} - \bar{t}_{n+1}), && && (11)\\
&\bar{s}_i = [(3/2)\bar{u}_i - 1/2]^+, && \bar{t}_i = [(3/2)\bar{u}_i - 1]^+, && \forall i = 1, \dots, n,\\
&\bar{z}_i = f^z(\bar{u}_i), && \bar{f}_i = |\bar{z}_i - 1/2|, && \forall i = 1, \dots, n,\\
&\bar{u}_{i-1} = f^u(\bar{u}_i), && && \forall i = 2, \dots, n,\\
&0 \le \bar{z}_i, \bar{f}_i, \bar{s}_i, \bar{t}_i, \bar{u}_i \le 1 && && \forall i = 1, \dots, n.
\end{aligned}$$

Let $\delta^z = \bar{z} - \hat{z}$, $\delta^f = \bar{f} - \hat{f}$, $\delta^s = \bar{s} - \hat{s}$, $\delta^t = \bar{t} - \hat{t}$, and $\delta^u = \bar{u} - \hat{u}$. Furthermore, let $\epsilon^s = \delta^s_{n+1} = \bar{s}_{n+1} - \hat{s}_{n+1}$, $\epsilon^t = \delta^t_{n+1} = \bar{t}_{n+1} - \hat{t}_{n+1}$, and $\epsilon = \epsilon^s + \epsilon^t$. Note that $(\bar{z}, \bar{f}, \bar{s}, \bar{t}, \bar{u}) \in \mathcal{F}(\mathrm{P}^{\mathrm{low}}_{\mathrm{weight}}(n+1,\theta,\eta))$ and Lemma 5 (i) imply $\bar{s}_{n+1} \ge [(3/2)\theta - 1/2]^+ = \hat{s}_{n+1}$ and $\bar{t}_{n+1} \ge [(3/2)\theta - 1]^+ = \hat{t}_{n+1}$. Thus, we have $\epsilon^s = \bar{s}_{n+1} - \hat{s}_{n+1} \ge 0$ and $\epsilon^t = \epsilon^s + \epsilon^t \ge 0$. We have

$$\begin{aligned}
|\delta^z_{n+1}| &= |\bar{z}_{n+1} - \hat{z}_{n+1}|\\
&= |2(\bar{s}_{n+1} - \bar{t}_{n+1}) - 2(\hat{s}_{n+1} - \hat{t}_{n+1})|\\
&= 2|(\bar{s}_{n+1} - \hat{s}_{n+1}) - (\bar{t}_{n+1} - \hat{t}_{n+1})|\\
&\le 2(|\bar{s}_{n+1} - \hat{s}_{n+1}| + |\bar{t}_{n+1} - \hat{t}_{n+1}||)\\
&= 2(\epsilon^s + \epsilon^t)\\
&= 2\epsilon, \qquad (12)
\end{aligned}$$

and

$$\begin{aligned}|\delta_n^u| &= |\bar{u}_n - \hat{u}_n| \\ &= |(3\bar{u}_{n+1} - 2\bar{z}_{n+1}) - (3\hat{u}_{n+1} - 2\hat{z}_{n+1})| \\ &= |(3\theta - 2\bar{z}_{n+1}) - (3\theta - 2\hat{z}_{n+1})| \\ &= 2|\bar{z}_{n+1} - \hat{z}_{n+1}| \\ &= 2|\delta_{n+1}^z| \\ &\leq 4\epsilon. \end{aligned} \tag{13}$$

Observe that f^u and f^z are Lipschitz continuous with the Lipschitz constant 3. Therefore, from (11), for any $i = 1, \dots, n-1$,

$$|\delta_i^u| = |f^u(\bar{u}_{i+1}) - f^u(\hat{u}_{i+1})| \leq 3|\bar{u}_{i+1} - \hat{u}_{i+1}| = 3|\delta_{i+1}^u| \leq 3^{n-i}|\delta_n^u| \leq 3^{n-i}4\epsilon, \tag{14}$$

and for any $i = 1, \dots, n$,

$$|\delta_i^z| = |f^z(\bar{u}_i) - f^z(\hat{u}_i)| \leq 3|\bar{u}_i - \hat{u}_i| \leq 3^{n-i+1}4\epsilon. \tag{15}$$

Similarly, for any $\eta \in \mathbb{R}$, function $\theta \mapsto [(3/2)\theta - \eta]^+$ is Lipschitz continuous with the Lipschitz constant $3/2$. Thus,

$$\begin{aligned} |\delta_i^s| &\leq (3/2)|\delta_i^u|, && \forall i = 1, \dots, n, \\ |\delta_i^t| &\leq (3/2)|\delta_i^u|, && \forall i = 1, \dots, n. \end{aligned} \tag{16}$$

Combining with (13) and (14), we get

$$|\delta_i^s| + |\delta_i^t| \leq 3^{n-i}12\epsilon, \qquad \forall i = 1, \dots, n. \tag{17}$$

Furthermore, from (12) and (15),

$$|\delta_i^f| \leq |\delta_i^z| \leq 3^{n-i+1}4\epsilon, \qquad \forall i = 1, \dots, n+1. \tag{18}$$

Now, we compare $(\bar{z}, \bar{f}, \bar{s}, \bar{t}, \bar{u})$ and $(\hat{z}, \hat{f}, \hat{s}, \hat{t}, \hat{u})$ in terms of the objective value of ($\mathrm{P}_{\mathrm{weight}}^{\mathrm{low}}(n+1, \theta, \eta)$):

$$\begin{aligned} &\left(\sum_{i=1}^{n+1} 16^i(\bar{s}_i + \bar{t}_i) + \eta \sum_{i=1}^{n+1} \frac{1}{4^{n-i+1}} \bar{f}_i\right) - \left(\sum_{i=1}^{n+1} 16^i(\hat{s}_i + \hat{t}_i) + \eta \sum_{i=1}^{n+1} \frac{1}{4^{n-i+1}} \hat{f}_i\right) \\ &\quad = 16^{n+1}((\bar{s}_{n+1} - \hat{s}_{n+1}) + (\bar{t}_{n+1} - \hat{t}_{n+1})) \\ &\qquad + \sum_{i=1}^{n} 16^i((\bar{s}_i - \hat{s}_i) + (\bar{t}_i - \hat{t}_i)) + \eta \sum_{i=1}^{n+1} \frac{1}{4^{n-i+1}} (\bar{f}_i - \hat{f}_i) \\ &\quad = 16^{n+1}\epsilon + \sum_{i=1}^{n} 16^i(\delta_i^s + \delta_i^t) + \eta \sum_{i=1}^{n+1} \frac{1}{4^{n-i+1}} \delta_i^f. \end{aligned}$$

We use (17) and (18) and bound this quantity by ϵ from below:

$$
\begin{aligned}
&16^{n+1}\epsilon + \sum_{i=1}^{n} 16^i(\delta_i^s + \delta_i^t) + \eta \sum_{i=1}^{n+1} \frac{1}{4^{n-i+1}} \delta_i^f \\
&\geq 16^{n+1}\epsilon - \sum_{i=1}^{n} 16^i(|\delta_i^s| + |\delta_i^t|) - \sum_{i=1}^{n+1} \frac{1}{4^{n-i+1}} |\delta_i^f| \\
&\geq \epsilon \left(16^{n+1} - 12 \sum_{i=1}^{n} 16^i 3^{n-i} - 4 \sum_{i=1}^{n+1} \left(\frac{3}{4}\right)^{n-i+1} \right) \\
&= \epsilon \left(16^{n+1} - 12 \frac{16(16^n - 3^n)}{16-3} - 16 \left(1 - \left(\frac{3}{4}\right)^{n+1} \right) \right) \\
&= \epsilon \left(\frac{16^{n+1}}{13} + \frac{192}{13} 3^n - 16 + 16 \left(\frac{3}{4}\right)^{n+1} \right) \\
&\geq \epsilon.
\end{aligned}
$$

Since $(\bar{z}, \bar{f}, \bar{s}, \bar{t}, \bar{u})$ is optimal for ($\mathrm{P}^{\mathrm{low}}_{\mathrm{weight}}(n+1, \theta, \eta)$) and $(\hat{z}, \hat{f}, \hat{s}, \hat{t}, \hat{u})$ is feasible for ($\mathrm{P}^{\mathrm{low}}_{\mathrm{weight}}(n+1, \theta, \eta)$), it follows that $\epsilon = 0$. Thus, $\epsilon^s = \epsilon^t = 0$, and $\bar{s}_{n+1} = \hat{s}_{n+1}$, $\bar{t}_{n+1} = \hat{t}_{n+1}$. Therefore, in light of (11) it holds that $\hat{z} = \bar{z}$, $\hat{f} = \bar{f}$, $\hat{s} = \bar{s}$, $\hat{t} = \bar{t}$, $\hat{u} = \bar{u}$. □

F Proof of Lemma 8

Proof. The entries in the constraint coefficients and constraint RHS in ($\mathrm{P}_{\mathrm{weight}}(F, \eta)$) are $0, \pm 1, \pm 2, \pm 1/2, \pm 3, \pm 1/2, \pm 3/2$, whose sizes are less than or equal to $\langle 6 \rangle$. By applying Lemma 3, we obtain the desired relation. □

G Proof of Proposition 1

Proof. Let $\{Q_j : j = 1, \ldots, q\}$ be a collection of polyhedra such that $\mathcal{F}(\mathrm{P}(F)) = \cup_{j=1,\ldots,q} Q_j$. Let $\mu^{(1)}$ and $\mu^{(2)}$ be two distinct binary vectors in $\{0,1\}^n$. Now, for $k = 1, 2$, let $\theta^{(k)} = (2/3)(1 + \sum_{i=1}^{n} \mu_i^{(k)}/3^{n+1-i})$ and $(z^{(k)}, f^{(k)}, s^{(k)}, t^{(k)}, u^{(k)}) \in \mathcal{S}(\mathrm{P}^{\mathrm{low}}_{\mathrm{lex}}(n+1, \theta^{(k)}))$. By Lemma 5 (iii) and (i), it follows that

$$
z^{(k)} = (\mu^{(k)}, 1) \text{ and } f^{(k)} = (1/2)\mathbf{1}, \;\; k = 1, 2. \tag{19}
$$

In the following, we prove

(B-1) $(\theta^{(k)}, z^{(k)}, f^{(k)}, s^{(k)}, t^{(k)}, u^{(k)}, 1) \in \mathcal{F}(\mathrm{P}(F))$ for $k = 1, 2$;

(B-2) $(\theta^{(1)}, z^{(1)}, f^{(1)}, s^{(1)}, t^{(1)}, u^{(1)}, 1)$ and $(\theta^{(2)}, z^{(2)}, f^{(2)}, s^{(2)}, t^{(2)}, u^{(2)}, 1)$ belong to different polyhedra in $\{Q_j : j = 1, \ldots, q\}$.

The claim follows from the fact that there are 2^n feasible points such that each pair of them belongs to different polyhedra, implying there are at least 2^n distinct polyhedra in $\{Q_j : j = 1, \ldots, q\}$.

First, we show (B-1). Fix k and let $\eta = 1$. By Lemma 7 (i), we have $(z^{(k)}, f^{(k)}, s^{(k)}, t^{(k)}, u^{(k)}) \in \mathcal{S}(\mathrm{P}^{\mathrm{low}}_{\mathrm{weight}}(n+1, \theta, \eta)) = \mathcal{S}(\mathrm{P}^{\mathrm{low}}_{\mathrm{lex}}(n+1, \theta))$. By construction (see (2) and (3)), $B_F z^{(k)} + \mathbf{1} \geq b_F$, implying $(z^{(k)}, f^{(k)}, s^{(k)}, t^{(k)}, u^{(k)}, 1) \in \mathcal{F}(\mathrm{P}^{\mathrm{low}}(F, \theta))$. Now, for contradiction, suppose $(z^{(k)}, f^{(k)}, s^{(k)}, t^{(k)}, u^{(k)}, 1)$ is not optimal. Pick an optimal solution $(\hat{z}^{(k)}, \hat{f}^{(k)}, \hat{s}^{(k)}, \hat{t}^{(k)}, \hat{u}^{(k)}, \hat{e}) \in \mathcal{S}(\mathrm{P}^{\mathrm{low}}(F, \theta))$. Then, it follows that $(\hat{z}^{(k)}, \hat{f}^{(k)}, \hat{s}^{(k)}, \hat{t}^{(k)}, \hat{u}^{(k)}) \in \mathcal{F}(\mathrm{P}^{\mathrm{low}}_{\mathrm{weight}}(n+1, \theta, \eta))$, and has a strictly smaller objective value, contradicting the optimality of $(z^{(k)}, f^{(k)}, s^{(k)}, t^{(k)}, u^{(k)})$ to $\mathcal{S}(\mathrm{P}^{\mathrm{low}}_{\mathrm{weight}}(n+1, \theta, \eta))$. Thus, $(z^{(k)}, f^{(k)}, s^{(k)}, t^{(k)}, u^{(k)}, 1) \in \mathcal{S}(\mathrm{P}^{\mathrm{low}}(F, \theta))$, implying (B-1).

Second, we show (B-2). For the sake of contradiction, let us assume that $(\theta^{(1)}, z^{(1)}, f^{(1)}, s^{(1)}, t^{(1)}, u^{(1)}, 1)$ and $(\theta^{(2)}, z^{(2)}, f^{(2)}, s^{(2)}, t^{(2)}, u^{(2)}, 1)$ are in Q, where $Q \in \{Q_j : j = 1, \ldots, q\}$. Let $\bar{\theta} = (\theta^{(1)} + \theta^{(2)})/2$, $\bar{z} = (z^{(1)} + z^{(2)})/2$, $\bar{f} = (f^{(1)} + f^{(2)})/2$, $\bar{s} = (s^{(1)} + s^{(2)})/2$, $\bar{t} = (t^{(1)} + t^{(2)})/2$, and $\bar{u} = (u^{(1)} + u^{(2)})/2$. Since Q is convex, $(\bar{\theta}, \bar{z}, \bar{f}, \bar{s}, \bar{t}, \bar{u}, 1)$ is in $Q \subset \mathcal{F}(\mathrm{P}(F))$. Let i' be such that $\mu^{(1)}_{i'} \neq \mu^{(2)}_{i'}$. From (19), we have $\bar{z}_{i'} = (\mu^{(1)}_{i'} + \mu^{(2)}_{i'})/2 = 1/2$ and $\hat{f}_{i'} = f^{(1)}_{i'} = f^{(2)}_{i'} = 1/2$. However, the optimality of the lower-level problem $(\mathrm{P}^{\mathrm{low}}(F, \theta))$ implies $\bar{f}_{i'} = |\bar{z}_{i'} - 1/2|$, a contradiction. Thus, (B-2) holds. □

H Proof of Proposition 2

Proof. We show the complement $\overline{\textsc{Verify}}^{\textsc{global}}_{\textsc{single}}$ is NP-complete. The inclusion of $\overline{\textsc{Verify}}^{\textsc{global}}_{\textsc{single}}$ in NP is similar to the proof of the inclusion of the decision version of bilevel LP in NP [3]. Thus, we only show $\overline{\textsc{Verify}}^{\textsc{global}}_{\textsc{single}}$ is NP-hard, by reducing 3-SAT to $\overline{\textsc{Verify}}^{\textsc{global}}_{\textsc{single}}$. Let F be a Boolean Formula in 3CNF, and construct $(\mathrm{P}(F))$. By Lemma 5, $\theta = 1/6$, $x_i = 1/2$, $i = 1, \ldots, n$, $y = 0$, $f = \mathbf{0}$, $e = 0$ is feasible for $(\mathrm{P}(F))$. Its objective value is 0, which is not optimal if and only if F is satisfiable. Thus, $\overline{\textsc{Verify}}^{\textsc{global}}_{\textsc{single}}$ is NP-hard.

□

I Proof of Theorem 2

Given instance $(\mathrm{P}_{\textsc{single}})$, one can always transform the lower-level problem into standard form in polynomial time [16]. Thus, we transform instance $(\mathrm{P}_{\textsc{single}})$ to the following:

$$\begin{aligned} \min_{z_1, x_2} \ & \bar{c}_{21}^\top z_1 + c_{22} x_2 \\ \text{s.t.} \ & z_1 \in \mathcal{S}(\mathrm{P}^{\mathrm{lower}}_{\textsc{standard}}(x_2)), \end{aligned} \tag{$\mathrm{P}_{\textsc{standard}}$}$$

where

$$\begin{aligned} \min_{z_1} \ & \bar{c}_{11}^\top z_1 \\ \text{s.t.} \ & \bar{A}_{11} z_1 + \bar{A}_{12} x_2 = \bar{b}_1, \bar{A}'_{12} x_2 \geq \bar{b}'_1, z_1 \geq 0, \end{aligned} \tag{$\mathrm{P}^{\mathrm{lower}}_{\textsc{standard}}(x_2)$}$$

with $\bar{c}_{11}, \bar{c}_{21} \in \mathbb{Q}^{\bar{n}}$, $c_{22} \in \mathbb{Q}$, $\bar{A}_{11} \in \mathbb{Q}^{\bar{m}\times\bar{n}}$, $\bar{A}_{12}, \bar{b}_1 \in \mathbb{Q}^{\bar{m}}$, $\bar{A}'_{12}, \bar{b}'_1 \in \mathbb{Q}^{\bar{m}'}$, and $\bar{A}_{11}$ has full row rank. The decision variables are $z_1 \in \mathbb{R}^{\bar{n}}$ and $x_2 \in \mathbb{R}$. Any locally optimal solution (z_1, x_2) to ($\mathrm{P}_{\text{STANDARD}}$) can be efficiently converted into a locally optimal solution (x_1, x_2) to ($\mathrm{P}_{\text{SINGLE}}$), and vice versa.

Now, consider the bilevel LP instance obtained by fixing x_2 in ($\mathrm{P}_{\text{STANDARD}}$):

$$\psi(x_2) = \min_{z_1} \bar{c}_{21}^\top z_1 + c_{22}x_2 \qquad (\mathrm{P}^{\text{fixed}}_{\text{STANDARD}}(x_2))$$
$$\text{s.t. } z_1 \in \mathcal{S}(\mathrm{P}^{\text{lower}}_{\text{STANDARD}}(x_2)).$$

Given a locally optimal solution x_2 to ψ, we can efficiently recover a locally optimal solution (z_1, x_2) to ($\mathrm{P}_{\text{STANDARD}}$), and vice versa. Moreover, evaluating $\psi(x_2)$ is computationally efficient, since the optimal solution set of ($\mathrm{P}^{\text{fixed}}_{\text{STANDARD}}(x_2)$) coincides with that of the following instance:

$$\operatorname*{lexmin}_{z_1} \{\bar{c}_{11}^\top z_1, \bar{c}_{21}^\top z_1 + c_{22}x_2\} \qquad (\mathrm{P}^{\text{lex}}_{\text{STANDARD}}(x_2))$$
$$\text{s.t. } \bar{A}_{11}z_1 + \bar{A}_{12}x_2 = \bar{b}_1, \bar{A}'_{12}x_2 \geq \bar{b}'_1, z_1 \geq 0.$$

Define

$$l = \inf\{x_2 : \exists z_1 \text{ s.t. } \bar{A}_{11}z_1 + \bar{A}_{12}x_2 = \bar{b}_1, \bar{A}'_{12}x_2 \geq \bar{b}'_1, z_1 \geq 0\} \tag{20}$$

and

$$u = \sup\{x_2 : \exists z_1 \text{ s.t. } \bar{A}_{11}z_1 + \bar{A}_{12}x_2 = \bar{b}_1, \bar{A}'_{12}x_2 \geq \bar{b}'_1, z_1 \geq 0\}. \tag{21}$$

Either both (20) and (21) are infeasible (i.e., $l = \infty$, $u = -\infty$), or both admit optimal solutions with $0 \leq l \leq u \leq 1$. In either case, the function $\psi(x_2)$ is finite if and only if $l \leq x_2 \leq u$. Moreover, by Lemma 1, if l and u are finite, then they are rational numbers of polynomial size.

Lemma 12. *Suppose (20) and (21) are feasible. Then, ψ is continuous and piecewise-linear over $[l, u]$. Furthermore, all breakpoints of ψ (including l and u) are rational numbers of polynomial size.*

Proof. It is known that lexicographic LPs admit basic optimal solutions; that is, for every x_2, if ($\mathrm{P}^{\text{lex}}_{\text{STANDARD}}(x_2)$) is feasible, then there exists a basic optimal solution (see, e.g., [8]). Fix $x_2 \in [\ell, u]$, and pick a basic optimal solution to ($\mathrm{P}^{\text{lex}}_{\text{STANDARD}}(x_2)$) at x_2. The set of values of x_2 for which this basic optimal solution remains feasible defines a closed interval I, whose endpoints are rational numbers of polynomial size. Moreover, the basic solution depends linearly on x_2 over this interval. As long as this solution remains feasible, it also remains optimal, since the reduced cost does not depend on x_2. Thus, over I, the value of ψ is linear in x_2.

Since there are only finitely many basic feasible solutions, $[l, u]$ can be covered by finitely many such intervals, where each corresponds to a different optimal basic solution. Therefore, ψ is piecewise-linear and continuous over $[l, u]$, and all the breakpoints are rational of polynomial size. $\square$

In light of Lemma 12, if a locally optimal solution exists, then there also exists a rational locally optimal solution of polynomial size that corresponds to a breakpoint of ψ. To search for such a solution, we need access to one-sided derivatives of ψ. The following result ensures this is efficient.

Lemma 13. *Given a rational point $x_2 \in \mathbb{Q}$, there is a polynomial-time algorithm that computes the one-sided derivatives of ψ at x_2, whenever they exist.*

To prove this, we use the continued fraction method from ancient Greece.

Lemma 14 (Continued fraction method [16]). *There exists a polynomial algorithm which, for given rational number α and natural number M, tests if there exists a rational number p/q with $1 \le q \le M$ and $|\alpha - p/q| < 1/2M^2$, and if so, finds this (unique) rational number.*

Proof (Lemma 13). We describe the procedure for computing the left derivative at a given point x_2. The right derivative can be computed analogously. First, verify whether $l < x_2 \le u$. If this condition does not hold, return "no." Let s be an integer bounding the size of breakpoints of ψ. Specifically, every breakpoint can be written as p/q, where $-2^s \le p \le 2^s$ and $1 \le q \le 2^s$, and the distance between two successive breakpoints is at least $1/2^{2s}$. We distinguish two cases based on whether the size of x_2 is at most s.

Case 1. $\langle x_2 \rangle \le s$: Let $\epsilon = 1/2^{3s}$ and evaluate ψ at $x_2 - \epsilon$. Since there are no rational numbers of size at most s in the interval $[x_2 - \epsilon, x_2)$, this interval contains no breakpoints of ψ. Furthermore, $x_2 - \epsilon$ is a rational number of polynomial size and satisfies $x_2 - \epsilon > l$. It follows that ψ is linear over $[x_2 - \epsilon, x_2]$, and thus the left derivative at x_2 equals the slope of this segment. Compute this slope and return it.

Case 2. $\langle x_2 \rangle > s$: Let $\epsilon' = 1/2^{2s+1}$ and consider the interval $[x_2 - \epsilon', x_2 + \epsilon']$. This interval contains at most one rational number of size at most s, and hence at most one breakpoint. Apply the continued fraction method (Lemma 14) to determine whether there exists a rational number p/q in $[x_2 - \epsilon', x_2 + \epsilon']$ with $q \le 2^s$. If such a point exists and satisfies $p/q < x_2$, let $v := p/q$. Otherwise, let $v := x_2 - \epsilon'$. Then, the interval $(v, x_2]$ contains no breakpoints, and v has polynomial size. As in Case 1, compute the slope between v and x_2 and return it. □

We are now ready to prove Theorem 2.

Proof (Theorem 2). We describe a procedure for finding a locally optimal solution to ψ. Once such a point x_2 is found, a corresponding z_1 can be computed by solving ($\mathrm{P}^{\mathrm{lex}}_{\mathrm{STANDARD}}(x_2)$), and the pair (z_1, x_2) is a locally optimal solution to ($\mathrm{P}_{\mathrm{STANDARD}}$).

We begin by verifying whether $0 \le l \le u \le 1$ holds. If not, output "no." If $l = u$, then $x_2 = l$ is trivially a locally optimal solution; return this value. In the remainder of the proof, assume $0 \le l < u \le 1$ and proceed to find a locally optimal solution via binary search.

First, check whether either endpoint l or u is a locally optimal solution by evaluating the corresponding one-sided derivatives. If so, return the corresponding endpoint value. Otherwise, we must have that the right derivative of ψ at l is negative, and the left derivative at u is positive. Let $v := (l+u)/2$ be the midpoint of the interval. Evaluate the one-sided derivatives of ψ at v. If v is locally optimal, return it. If the left derivative at v is negative, then by the extreme value theorem, a locally optimal solution (in particular, a locally optimal breakpoint) must lie in the interval $[l, v]$, so let $u := v$. Otherwise, there is a locally optimal solution in $[v, u]$, thus update $l := v$. Repeat this process by evaluating the one-sided derivatives at the new midpoint and updating the interval accordingly.

Suppose this procedure continues until the interval length satisfies $u - l \leq 1/2^{2s+1}$, where s is an upper bound on the size of the breakpoints of ψ. At this point, the interval $[l, u]$ contains at most one rational number of size at most s. Since a locally optimal solution (i.e., a breakpoint of ψ) of size at most s is guaranteed to exist within this interval, it must be the unique such rational point. Apply the continued fraction method (Lemma 14) to identify this point, and return it as the output.

Hence, the algorithm runs in polynomial time. □