

Anomalous Hall effect in rhombohedral graphene

Vera Mikheeva,^{1,2} Daniele Guerci,³ Daniel Kaplan,^{4,*} and Elio J. König^{5,6}

¹*Nanocenter CENN, Ljubljana, Slovenia*

²*University of Ljubljana, Slovenia*

³*Department of Physics, Massachusetts Institute of Technology, Cambridge, USA*

⁴*Department of Physics and Astronomy, Center for Materials Theory, Rutgers University, Piscataway, USA*

⁵*Department of Physics, University of Wisconsin-Madison, Madison, Wisconsin, USA*

⁶*Max-Planck Institute for Solid State Research, Stuttgart, Germany*

Motivated by recent experiments on rhombohedral stacked multilayer graphene and the observation of the anomalous Hall effect in a spontaneous spin-valley polarized quarter metal state, we calculate the anomalous Hall conductivity for this system in the presence of two types of impurities: weak and dense as well as sparse and strong. Our calculation of σ_{xy} is based on the Kubo-Streda diagrammatic approach. In a model with Gaussian disorder applicable to weak dense impurities, this involves all non-crossing diagrams (intrinsic, side-jump and Gaussian skew-scattering contributions) and additionally diagrams with two intersecting impurities, X and Ψ , representing diffractive skew-scattering processes. A "Mercedes star" diagram (non-Gaussian skew scattering) is furthermore included to treat in the case of strong, sparse impurities. We supplement our asymptotically exact analytical solutions for an isotropic model without warping effects by semi-numerical calculations accounting perturbatively for warping, which plays a crucial role in the low-energy band structure.

Graphene [1–3] based structures are an active playground for interacting and correlated phases in condensed matter physics. Experimental and theoretical interest in these materials has particularly sparked since the recent discovery of a rich spectrum of correlated electron phenomena, including superconductivity, correlated insulators, spontaneous spin and valley polarization, anomalous Hall crystals, and fractional quantum anomalous Hall states [4–22]. Since most of the correlated phases are found in the rhombohedral (ABC) stacked configuration, here we focus on stacking n -layer graphene, a metastable but experimentally realizable [23, 24] configuration for multilayer graphene (for $n > 2$, ABA stacking is energetically advantageous). Unlike moiré systems, they exhibit a plethora of phases without breaking the atomic crystal's translation symmetry and still display an increased electronic density of states (DOS) near charge neutrality [25]. This remarkable property is due to the structure of low-energy states permitted by the rhombohedral arrangement of layers [26]: the low energy states are localized on opposite sublattices of the extremal layers (A_1, B_N , where N is the number of layers), with the bulk states gapped out by hybridization.

Thus, the low energy bands flatten with N and are further modified by the application of a nonzero perpendicular displacement field separating the states of the (A_1, B_N) pseudospinor, resulting in a tunable gap at the K/K' points. The sizable valley-dependent Berry curvature inherited from graphene monolayers furthermore allows that both correlated and topological states can emerge from this material, including the spin-polarized half-metal and spin- and valley-polarized quarter-metal phases which effectively result from a Stoner-like instability. These phases spontaneously break time reversal

symmetry and, amongst others, were observed by measurements of the anomalous Hall effect (AHE) [4, 6, 14].

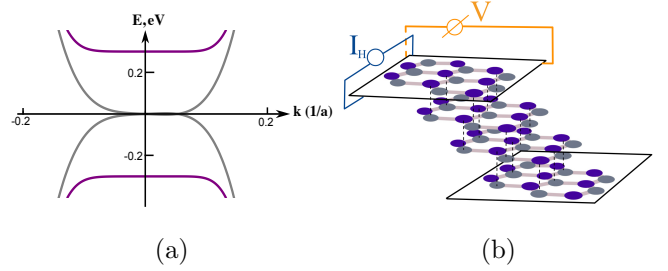


FIG. 1: (a) Schematic band structure of rhombohedral multilayer graphene without (grey) and with displacement field $m = 0.3$ eV (purple). Here $a = 0.246$ nm is the graphene lattice constant; (b) Schematic system and experimental setup for measuring the anomalous Hall effect in tetralayer rhombohedral graphene.

The AHE, i.e. transverse voltage in response to a current, without applying an external magnetic field, [27] was discovered in 1881, by E. Hall in a study of ferromagnets. For this effect to exist, the system must break time reversal symmetry and display a non-trivial chiral texture in momentum space transforming the information of spin or valley polarization to orbital motion. Rhombohedral graphene satisfies these demands in the quarter-metal phase. Multiple physical effects conspire in a non-zero AHE signal. The intrinsic contribution [28, 29] is determined solely by the electronic structure of the material, specifically the Berry curvature of occupied states. Additionally, extrinsic effects are of relevance. Amongst those, impurity scattering is most prominent at lowest temperatures and its contribution is parametrically the same or can even become the leading source of anoma-

* d.kaplan1@rutgers.edu

lous Hall response in realistic systems, where disorder is always present to some extent.

To describe those phenomena, there are two common techniques based on Boltzmann kinetic equation [27, 30] and Kubo-Streda diagrammatic formalism [31, 32]. In this paper, we will focus on the latter approach. However, we note that both techniques are equivalent and both commonly used.

Theoretically, the impurity induced contributions to the total anomalous Hall conductivity σ_{xy} can be decomposed into two distinct contributions associated with different physical mechanisms: side-jump [33–35] and skew-scattering [36]. The side-jump mechanism arises due to a transverse displacement of the electron wave packet during impurity scattering i.e., semiclassical trajectories of incoming and outgoing electrons do intersect away from the scattering center. The skew-scattering mechanism, resulting from an asymmetry (broken detailed balance) in the impurity scattering cross section, is conveniently divided into three distinct components: Gaussian skew scattering (non-crossing diagrams) [37], diffractive (crossed diagrams) [38–40], and non-Gaussian (higher moments of the random potential) [27, 41, 42]. Some authors reserve the word “skew-scattering” to the non-Gaussian effect (“skewness” of the disorder distribution function) but we employ the more general definition here. The Gaussian and diffractive contributions to skew scattering stem from complexes of nearby impurities and are therefore parametrically comparable. Gaussian skew scattering stems from coherent interband transitions. In contrast, diffractive skew scattering is a quantum interference effect.

In this paper, we consider the anomalous Hall effect in rhombohedral multilayer graphene in the presence of disorder using the diagrammatic Kubo-Streda approach. As a disorder, we consider two different cases: Gaussian (weak dense impurities) and non-Gaussian disorder (strong sparse impurities) which is modelled phenomenologically via a finite third moment of the disorder potential. For weak dense impurities we calculate the response to leading order in disorder strength, i.e. we take into account the ladder diagrams and diagrams with two intersecting impurity lines. We present the general analytical result for intrinsic and non-crossing contributions and the general answer including diffractive skew-scattering. Besides the isotropic model, we consider trigonal warping [27, 43–45] (an effect of the crystal lattice), which plays a crucial role in the low-energy band structure. We demonstrate using numerical calculations how it affects the anomalous Hall conductivity. In the following, we consider the general case of rhombohedral multilayer graphene with an arbitrary number of layers n . However, we focus on the cases $n = 2, 3, 4, 5$ for isotropic and $n = 3, 4$ for warped model, which are the most relevant from the experimental point of view.

This paper is structured as follows: in the next section (Sec. I), we describe the effective low-energy Hamiltonian of ABC-stacked multilayer graphene and discuss disorder

types with respect to experimental reality. Then, in Sec. II, we consider an isotropic model and, using diagrams, we present the full result for σ_{xy} . In Sec. III, we provide the same calculations in the presence of interlayer asymmetry that opens a gap in the spectrum. Section III E dedicates the comparison between isotropic and warped models with numerical calculations.

I. MODEL

The low-energy electronic properties of rhombohedral graphene are described by an effective two-component Hamiltonian that describes wavefunctions predominantly localized on atomic sites A of the bottom layer and B of the top layer [44, 46, 47] (Appendix A):

$$H_n^{(w)} = \begin{pmatrix} m & v\mathcal{P}^n + w\mathcal{P}^{n-3} \\ v\bar{\mathcal{P}}^n + w\bar{\mathcal{P}}^{n-3} & -m \end{pmatrix} = \vec{d}(\mathbf{p}) \cdot \vec{\sigma}, \quad (1)$$

where $\mathcal{P} = \xi p_x - ip_y$ and $\xi = \pm 1$ is the valley index. We focus on a single valley and will keep $\xi = 1$; v is a constant with dimensions [energy $\times$ length n]; $\vec{\sigma}$ is the vector of Pauli matrices describing sublattice space; m is the displacement field; the term proportional to w describes trigonal warping. The n -th power $\mathcal{P}^n$ in the momentum dependence of the effective low-energy Hamiltonian originates from the virtual interlayer hopping processes connecting the sublattices in n -layer rhombohedral stack. Trigonal warping arises from subleading interlayer hoppings, which couple equivalent sublattices across adjacent layers and break continuous rotational symmetry. We note that the effective Hamiltonian Eq. (1) describes multilayer graphene in rhombohedral graphene for $n \geq 3$ (the low energy limit of Bernal bilayer graphene is also captured by taking $n = 2$ and $w = 0$).

Impurities are modeled by a random potential $V(\mathbf{x})$ with only two non-vanishing moments

$$\langle V(\mathbf{x}_1)V(\mathbf{x}_2) \rangle = 2\pi\alpha v^2 p_0^{2n-2} \delta(\mathbf{x}_1 - \mathbf{x}_2), \quad (2a)$$

$$\langle V(\mathbf{x}_1)V(\mathbf{x}_2)V(\mathbf{x}_3) \rangle = \beta v^3 p_0^{3n-4} \delta(\mathbf{x}_1 - \mathbf{x}_2) \delta(\mathbf{x}_2 - \mathbf{x}_3), \quad (2b)$$

where α and β are a dimensionless parameter characterizing disorder strength [38, 41]. We introduced the Fermi momentum in the absence of warping

$$p_0 = [(\epsilon_F^2 - m^2)/v^2]^{\frac{1}{2n}}, \quad (3)$$

where ϵ_F is the Fermi energy.

We briefly comment on the experimental relevance of our model of impurity scattering in rhombohedral graphene. First, our model explicitly assumes that the dominant scattering mechanism comes from the short-range disorder. The applicability of delta-function impurity potentials in rhombohedral graphene arises from the fact that the characteristic distance between impurities and the gate ($d \sim nm$) is significantly smaller than the Fermi wavelength ($\lambda_F \sim 100nm$). This condition ensures

that the impurity potential is screened on a length scale much shorter than that of the electronic wave functions, thus supporting the use of a point-like model. Second, we assume that disorder is relevant for transport in our model. To verify this, one can compare the mean free path with the sample size. The mean free path is estimated to be in the range $l \sim 0.1\text{--}1\ \mu\text{m}$ [6, 18], while in experiments the typical sample size is $L \sim 2.5\ \mu\text{m}$ [4, 18]. Thus, samples can be expected slightly on the diffusive side of the ballistic-to-diffusive crossover.

We employ the diagrammatic approach to calculate the anomalous Hall conductivity of rhombohedral multilayer graphene. In the limit $\beta = 0$ we take into account all the diagrams to the leading order in $(p_0 l)^0 \sim \alpha^0$. Note that this involves a non-perturbative resummation of infinitely many diagrams. In addition, we add (perturbatively) the contribution of β .

Technically, the Hall conductivity σ_{xy} using the Kubo-Streda formula [48] can be written as $\sigma_{xy} = \sigma_{xy}^I + \sigma_{xy}^{\text{II}}$ [38] where

$$\sigma_{xy}^I = \frac{e^2}{h} \langle \text{Tr} [\hat{j}_x G^R \hat{j}_y G^A] \rangle, \quad (4a)$$

$$\sigma_{xy}^{\text{II}} = ec \frac{\partial q}{\partial B}. \quad (4b)$$

Here, $G^{R/A}$ are the retarded and advanced Green's functions; $j_{x/y}(p)$ are current operators which can be determined as $\hat{j}_{x/y}(p) = \frac{\partial H_n}{\partial p_{x/y}}$; q is the total electron concentration; B is the external magnetic field; c is the speed of light in vacuum. In this paper we set $\hbar = 1$ unless explicitly highlighted otherwise.

There are two different regimes for Hall conductivity: when the Fermi energy ϵ_F lies in the gap, $|\epsilon_F| < |m|$ and otherwise. In the former case, the term σ_{xy}^{II} in Eq. (4b) provides the dominant topologically quantized contribution and can be interpreted in terms of the Chern number. Outside the gap, i.e., for $|\epsilon_F| > |m|$, this topological contribution becomes negligible [29, 38], and the entire Hall conductivity stems from σ_{xy}^I (diagrammatically a bubble). Most of our calculations will be in the latter regime. The assumption $p_0 l \gg 1$ ceases to be correct near the band edge. Sufficiently below the band edge (inside the gap) corrections to the topological quantized response are negligible. We note that we entirely neglect Anderson localization physics, e.g. weak-localization corrections [49, 50]. This is justified for the present case of sufficiently small samples or when dephasing is strong enough.

We first focus on $\beta = 0$. The leading-order contribution in terms of $(p_0 l)$ involves summing ladder diagrams while taking into account crossed impurity lines within the conductivity bubble represented in Fig. 2 as well as self-energy corrections necessary for a conserving approximation (self-consistent Born approximation supplemented with one cross).

Thin lines in the diagrams correspond to clean Green's function, bold lines describe the disorder averaged,

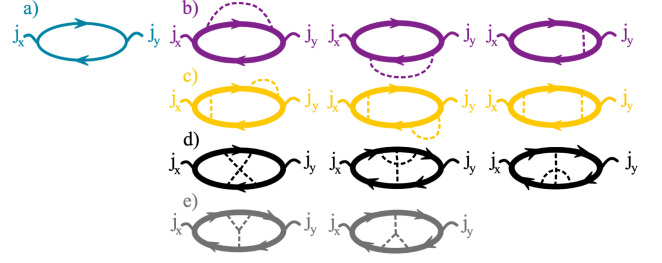


FIG. 2: Exemplary diagrams for the anomalous Hall conductivity: a) intrinsic contribution, b) side jump, c) Gaussian skew-scattering, d) diffractive skew-scattering, e) third moment (skewness).

i.e. dressed, Green's function and dashed lines are disorder correlators, Eq. (2).

II. ISOTROPIC MODEL

In this section we present calculations for the isotropic model, i.e. setting $w = 0$ in Eq. (1).

A. Intrinsic contribution

The intrinsic contribution (bare bubble without impurity lines, Fig. 2, a)) yields:

$$\sigma_{xy}^{\text{int}} = -\frac{e^2}{h} \frac{m}{2\epsilon_F} n; \quad |\epsilon_F| > |m|. \quad (5)$$

The half-integer quantization if the Fermi energy is inside the gap arises from fermion number fractionalization [51]–[50]. (cf. Appendix B 3). As a technical remark to obtain a non-zero Hall conductivity, we note that it is essential that one of the Green's functions in the diagram lies away from the Fermi energy (i.e., is off-shell).

B. Non-Crossing approximation

Next, we study how the presence of impurities influences the anomalous Hall conductivity. We can use the fact that the self-consistent Born approximation supplemented by diagrams with one crossed line yields a self-energy that is essentially the same as in the standard Born approximation, allowing us to obtain the average retarded Green's function:

$$\mathbf{G}_{\mathbf{p}}^R = \frac{\epsilon_F \left(1 + \frac{\pi i \alpha}{2n}\right) + m \sigma_z \left(1 - \frac{\pi i \alpha}{2n}\right) + d_x(\mathbf{p}) \sigma_x + d_y(\mathbf{p}) \sigma_y}{\epsilon_F^2 - m^2 - d_x(\mathbf{p})^2 - d_y(\mathbf{p})^2 + \frac{i \pi \alpha}{n} (\epsilon_F^2 + m^2)}. \quad (6)$$

The advanced Green's function is the Hermitian conjugate of the above expression (Appendix B 2).

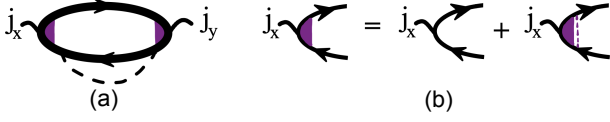


FIG. 3: Example diagram of the side-jump contribution with a vertex correction (a), which is obtained by summing ladder diagrams (b).

We use this in the evaluation of side-jump and skew-scattering terms. First, we focus on non-crossing contribution. It is worth noting that diagrams with the off-shell Green function, which links the off-diagonal current vertex to an impurity, correspond to the side-jump contribution, while in skew-scattering diagrams, the off-shell Green's function connects two impurities.

As the first order correction, Fig. 3, to the vertex

$$\delta j_{x/y} \propto \alpha \int \frac{d^2 p}{(2\pi)^2} \mathbf{G}_{\mathbf{p}}^A j_{x/y} \mathbf{G}_{\mathbf{p}}^R = 0; \quad n \neq 1, \quad (7)$$

vanishes due to the momentum structure in $j_{x/y} = \frac{\partial H}{\partial p_{x/y}}$, bare and dressed vertices coincide. We emphasize that the vertex correction is non-zero for $n = 1$ [27, 37, 38]. The contribution of Gaussian skew-scattering is thus zero and σ_{xy} for the model Eq. (1) in the non-crossing approximation is equal to (see Fig. 2, a-c)):

$$\sigma_{xy}^{\text{nc}} = -\frac{e^2}{h} \frac{m\epsilon_F}{\epsilon_F^2 + m^2} \cdot n; \quad n > 1, \quad |\epsilon_F| > |m|. \quad (8)$$

Details about the evaluation of the diagrams can be found in Appendix B 5. Specifically, the side-jump contributes

$$\sigma_{xy}^{\text{sj}} = -\frac{e^2}{h} \frac{m(\epsilon_F^2 - m^2)}{2\epsilon_F(\epsilon_F^2 + m^2)} \cdot n; \quad n > 1. \quad (9)$$

C. Diffractive Skew scattering

We next consider diffractive skew scattering which originates from pairs of impurities separated by distances on the order of, or smaller than, the Fermi wavelength. In this regime, the scatterers are no longer amenable to a quasiclassical treatment as independent centers. As a result, a rigorous theoretical description of such correlated impurity scattering necessitates inclusion of both X and Ψ diagrammatic contributions. (Fig. 2, d)). However, the inclusion of three or more intersecting impurity lines in the conductivity diagrams yields an additional smallness, (maximally crossed diagrams contribute to a weak localization correction characterized by $\sigma_{xy} \sim \alpha$).

Since the two impurities composing X or Ψ are close by on the order of λ_F , we may evaluate internal lines with bare Green's functions, Eq. (6) at $\alpha = 0$. In real space

	$n = 2$	$n = 3$	$n = 4$	$n = 5$
$\sigma_{xy}^{\text{X}}/f(\epsilon_F, m)$	0.719	0.563	0.542	0.530
$\sigma_{xy}^{\Psi}/f(\epsilon_F, m)$	0.466	0.509	0.494	0.506

TABLE I: Prefactors of diffractive skew scattering contribution $\sigma_{xy}^{\text{X}/\Psi}$ for different n . Here, $f(\epsilon_F, m) = \frac{e^2}{h} \frac{\epsilon_F m (\epsilon_F^2 - m^2)}{(\epsilon_F^2 + m^2)^2}$.

this leads to

$$G_0^{R/A}(\epsilon_F, \mathbf{r}) = [\epsilon_F + \vec{d}(-i\nabla)] \frac{\pi \rho_n(\epsilon_F)}{2\epsilon_F} \left[\tilde{Y}_0(p_0 r) \mp i J_0(p_0 r) \right]. \quad (10)$$

where

$$\tilde{Y}_l(r) = Y_l(r) - \frac{2}{\pi} \sum_{j=1}^{n-1} e^{-i \frac{\pi j}{n} (2+l)} i^l K_l \left(i e^{-i \frac{\pi j}{n}} r \right), \quad (11)$$

J_l and Y_l are the l -th Bessel functions of the first and second kind, respectively; K_l is the modified Bessel function of the second kind (cf. Appendix B 1). We also used the density of states for isotropic model $\rho_n(\epsilon_F)$:

$$\rho_n(\epsilon_F) = \frac{\epsilon_F}{2\pi n v^{2/n} (\epsilon_F^2 - m^2)^{1-1/n}}. \quad (12)$$

Using the real space representation, we obtain

$$\sigma_{xy}^{\text{X}} = \frac{e^2}{h} \frac{\epsilon_F m (\epsilon_F^2 - m^2)}{(\epsilon_F^2 + m^2)^2} \pi n \times \quad (13a)$$

$$\int_0^\infty dx J_1 J_n [J_n \tilde{Y}_0 - J_0 \tilde{Y}_n],$$

$$\sigma_{xy}^{\Psi} = \frac{e^2}{h} \frac{\epsilon_F m (\epsilon_F^2 - m^2)}{(\epsilon_F^2 + m^2)^2} \pi n \times \quad (13b)$$

$$\int_0^\infty dx J_1 J_n [J_n \tilde{Y}_0 + J_0 \tilde{Y}_n] + J_0 J_n \tilde{Y}_0 (J_{n-1} - J_{n+1}),$$

where for the sake of convenience, we use the following compact notations $J_i = J_i(x)$, $\tilde{Y}_i = \tilde{Y}_i(x)$, $x = p_0 R$, R is a distance between two impurities. For detailed calculations use Appendix B 6. While Eqs. (13a)-(13b) determine contributions from diffractive skew-scattering contribution for any $n \geq 1$, the analytical evaluation of these integrals can be obtained for $n = 1$ [38], $n = 2$ (see Appendix B 7), as well as in the limit $n \rightarrow \infty$:

$$\frac{\sigma_{xy}^{\text{X}} + \sigma_{xy}^{\Psi}}{f(\epsilon_F, m)} \simeq \frac{\sqrt{3}}{2\sqrt{2}} + \frac{4}{\pi^2} \begin{cases} 1 & n \text{ odd}, \\ \frac{n^4}{(n^2-1)^2}, & n \text{ even} \end{cases}. \quad (14)$$

Numerical evaluations of the Bessel integrals are, of course, always possible and for the most relevant experimental situations $n \in [2; 5]$ of $\sigma_{xy}^{\text{X}/\Psi}$ are given in Table I.

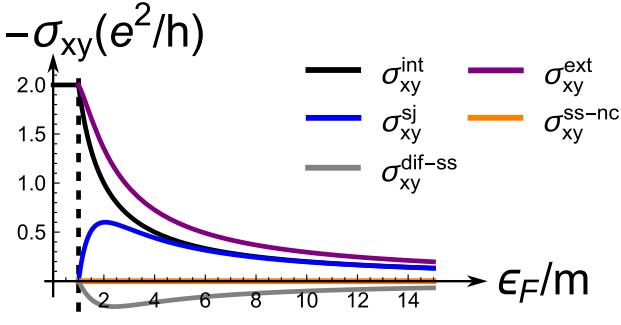


FIG. 4: Intrinsic Eq. (5), side-jump Eq. (9), skew-scattering components Eqs. (13a)-(13b) and the extrinsic part of the anomalous Hall conductivity for $n = 4$. We plot separately the skew scattering parts in the non-crossing approximation (purple) and diffractive (gray). The vertical dashed line marks $\epsilon_F/m = 1$.

D. Summary for weak dense impurities

The full result for the anomalous Hall conductivity in the zeroth order in α from one valley and one spin is given by Eqs. (8), (13a)-(13b):

$$\sigma_{xy} = \sigma_{xy}^{\text{nc}} + \sigma_{xy}^{\text{X}} + \sigma_{xy}^{\Psi}. \quad (15)$$

When we compare these results with the well-known limiting case of $n = 1$, which corresponds to 2D massive Dirac fermions [27, 37, 38], we recover the results of previous works if we artificially removed the vertex correction (which is non-vanishing for $n = 1$), see Eq. (7).

The comparison of different components for anomalous Hall conductivity, Fig. 4, for tetralayer graphene, illustrates that for Fermi energy sufficiently far inside the band the impurity scattering is comparable to the intrinsic contribution, yet upon lowering the Fermi energy, the intrinsic mechanism dominates the anomalous Hall conductivity and ultimately is the sole contribution to the quantized plateau. This is consistent with the topological origin of the intrinsic contribution, which is less sensitive to disorder. These results underscore that in ABC-stacking graphene, the anomalous Hall effect is largely determined by the interplay between intrinsic, side-jump, and diffractive skew-scattering mechanisms, while conventional skew-scattering may be suppressed.

E. Strong sparse impurities

In the rest of the part about an isotropic model, we consider the higher-moment contribution of the disorder distribution Eq. (2b) to the first order in the parameter β (third moment) given by 'Mercedes star' diagram (Fig. 2.

e)). We obtain a vanishing contribution for $n \geq 2$

$$\sigma_{xy}^{\text{MS}} = \sigma_{xy}^{\text{Y}} + \sigma_{xy}^{\text{X}} = \frac{e^2}{h} (\beta v^3 p_0^{3n-4}) \times \int_{\{\mathbf{p}_i\}} \left(\text{Tr}[\mathbf{G}_{\mathbf{p}_1}^A j_x(\mathbf{p}_1) \mathbf{G}_{\mathbf{p}_1}^R \mathbf{G}_{\mathbf{p}_3}^R \mathbf{G}_{\mathbf{p}_2}^R j_y(\mathbf{p}_2) \mathbf{G}_{\mathbf{p}_2}^A] + \text{Tr}[\mathbf{G}_{\mathbf{p}_1}^A j_x(\mathbf{p}_1) \mathbf{G}_{\mathbf{p}_1}^R \mathbf{G}_{\mathbf{p}_2}^R j_y(\mathbf{p}_2) \mathbf{G}_{\mathbf{p}_2}^A \mathbf{G}_{\mathbf{p}_3}^A] \right) = 0, \quad (16)$$

because of angle integration. It should be noted that the higher moment gives us the same result (see Appendix B 8). At the same time note that a non-zero result does exist for $n = 1$ [37].

III. WARPED MODEL

This section addresses the effect of trigonal warping. In experimentally relevant regimes, the displacement field energy is comparable to the characteristic energy associated with the dispersion anisotropy, i.e. warping. This implies that the isotropic calculation may be at the limit of its applicability, in particular at low doping, thus necessitating the inclusion of warping in the analysis.

In the presence of warping in Eq. (1), i.e $w \neq 0$, the system has C_3 symmetry instead of full-rotation symmetry as in isotropic model. It should be emphasized that this effective model is applicable for $n \geq 3$ and microscopically derives from various types of subleading interlayer hopping that were neglected in the isotropic model.

As suggested by its name, the parameter w introduces a warped Fermi surface. The approximate ϕ dependent Fermi momentum $p_F(\phi) \equiv p_\phi$ to second order in the power of warping parameter $\tilde{w} = w/vp_0^3$ is

$$p_\phi = p_0 \left(1 - \frac{\tilde{w} \cos(3\phi)}{n} + \frac{\tilde{w}^2 ((2n-5) \cos(6\phi) - 5)}{4n^2} \right). \quad (17)$$

We emphasize that at smallest doping $\tilde{w}$ is always large and our perturbative scheme breaks down. At the same time, it is worthwhile to note that Eq. (17), provides a quantitatively accurate description of the Fermi surface (within few percents) for all $\tilde{w} < 1$ (Appendix C 1).

For the following calculations we define a ϕ dependent DOS

$$\rho_{n,\phi}(\epsilon_F) = \epsilon_F \frac{p_\phi}{2n\pi f_\phi(p_\phi)}, \quad (18)$$

where

$$f_\phi(p_\phi) = \frac{dF_\phi(p)}{dp}; \quad F_\phi(p) = \frac{|\vec{d}(p, \phi)|^2 - \epsilon_F^2}{2n}. \quad (19)$$

Our focus is on how warping impacts the anomalous Hall conductivity, and we follow the same computational steps as in the isotropic model. From now on, we will focus on the warping effect for trilayer and tetralayer graphene, i.e. $n = 3, 4$. However, similar reasoning can be used for other n .

A. Intrinsic contribution

We derive the intrinsic contribution for the warped model (cf. Appendix C2a, C3a):

$$\sigma_{xy}^{\text{w, int}}|^{n=3} = -\frac{e^2}{h} \frac{m}{\epsilon_F} \frac{3}{2} \mathcal{J}_3^{(4)}, \quad (20a)$$

$$\sigma_{xy}^{\text{w, int}}|^{n=4} = -\frac{e^2}{h} \frac{m}{\epsilon_F} \cdot \left(2\mathcal{J}_4^{(6)} + \tilde{w}\mathcal{I}_4^{(3)} + \frac{\tilde{w}^2}{8}\mathcal{I}_4^{(0)} \right). \quad (20b)$$

Here we introduced

$$\mathcal{I}_n^{(l)} = \left\langle \bar{p}_\phi^l e^{il\phi} \frac{\rho_{n,\phi}(\epsilon_F)}{\rho_n(\epsilon_F)} \right\rangle_\phi; \quad \mathcal{J}_n^{(l)} = \left\langle \bar{p}_\phi^l \frac{\rho_{n,\phi}}{\rho_n} \right\rangle_\phi; \quad \bar{p} = p/p_0, \quad (21)$$

and $\langle \dots \rangle_\phi = \int_0^{2\pi} \dots d\phi$ denotes angular average. These and the following integrals in the warped case can be evaluated numerically and will be discussed in the next section. The isotropic limit ($w = 0$) can always be taken analytically and we can readily verify the agreement with our results for the isotropic results, in this case Eq. (5).

B. Self-energy

We rely on the fact that the self-energy obtained in the self-consistent Born approximation is identical to that in the Born approximation

$$\Sigma^R \simeq \frac{\epsilon_F}{n\rho_n(\epsilon_F)} \alpha \int \frac{dp^2}{(2\pi)^2} \frac{\epsilon_F + \vec{d}(\mathbf{p}) \cdot \boldsymbol{\sigma}}{\epsilon_F^2 - \vec{d}(\mathbf{p})^2}. \quad (22)$$

For $n = 4$ and for n not a multiple of 3, only the $\sigma_{0,z}$ components survive the angular integral due to C_3 symmetry. For $n = 3$ and for n multiple of 3 we have all components. So, in general (see Appendix C1).

$$\text{Im}\Sigma^R = -(\Gamma + \Gamma_x\sigma_x + \Gamma_y\sigma_y + \Gamma_z\sigma_z), \quad (23a)$$

where for $n = 3, 4$

$$\Gamma = -\frac{\pi\alpha}{2n} \epsilon_F \mathcal{I}_n^{(0)}, \quad (23b)$$

$$\Gamma_x = -\delta_{n,3} \frac{\pi\alpha}{2n} v p_0^3 \cdot (\mathcal{I}_n^{(3)} + \tilde{w}\mathcal{J}_n^{(0)}), \quad (23c)$$

$$\Gamma_y = 0, \quad (23d)$$

$$\Gamma_z = -\frac{\pi\alpha}{2n} m \mathcal{I}_n^{(0)}. \quad (23e)$$

C. Non-crossing contribution: vertex corrections

To compute the non-crossing and diffractive skew-scattering contributions, we need to include vertex corrections. Contrary to the isotropic model they generally

do not vanish. We define $j_\pm = [j_x \pm i j_y]/2$, and use bold-face symbols $\mathbf{j}_\pm$ for dressed vertices. We obtain (cf. Appendix C2b, C3b):

$$\mathbf{j}_\pm|^{n=3} = j_\pm|^{n=3}, \quad (24)$$

$$\mathbf{j}_\pm|^{n=4} = j_\pm|^{n=4} + \left[4vp_0^3 \frac{\mathcal{I}_4^{(3)}}{\mathcal{I}_4^{(0)}} + w \right] \frac{A}{1-A} \pm i \left[4vp_0^3 \frac{\mathcal{I}_4^{(3)}}{\mathcal{I}_4^{(0)}} + w \right] \frac{B}{(1-A)^2}, \quad (25)$$

where

$$A = \frac{\epsilon_F^2 - m^2}{2(\epsilon_F^2 + m^2)}, \quad B = -\frac{\epsilon_F m}{\epsilon_F^2 + m^2} \frac{\pi\alpha}{n} \mathcal{I}_n^{(0)}. \quad (26)$$

So, the vertex correction in warped model for $n = 3$ is zero but not for $n = 4$.

Then, σ_{xy} for warped model in the non-crossing approximation equals to (Appendix C2c, C3c):

$$\sigma_{xy}^{\text{w, nc}}|^{n=3} = -\frac{e^2}{h} \frac{3m\epsilon_F \mathcal{I}_3^{(0)}}{\epsilon_F^2 - m^2} \int_0^{2\pi} \frac{d\phi}{2\pi} \frac{\bar{p}_\phi^5}{f_\phi(\bar{p}_\phi)}. \quad (27)$$

$$\sigma_{xy}^{\text{w, nc}}|^{n=4} = -\frac{e^2}{h} \frac{4m\epsilon_F}{\epsilon_F^2 + m^2} \left\{ \left[\frac{\mathcal{J}_4^{(6)}}{\mathcal{I}_4^{(0)}} - \frac{[\mathcal{I}_4^{(3)}]^2}{[\mathcal{I}_4^{(0)}]^2} \right] + \frac{4(\epsilon_F^2 + m^2)^2}{(\epsilon_F^2 + 3m^2)^2} \left[\frac{\mathcal{I}_4^{(3)}}{\mathcal{I}_4^{(0)}} + \tilde{w}/4 \right]^2 \right\}. \quad (28)$$

In the isotropic limit we recover Eq. (8).

D. Diffractive Skew Scattering

Finally, we can obtain diffractive skew-scattering contribution:

$$\sigma_{xy}^{\text{w, X}}|^{n=3} = -\frac{e^2}{h} \frac{m\epsilon_F}{\epsilon_F^2 - m^2} \cdot N_{X,3}, \quad (29a)$$

$$\sigma_{xy}^{\text{w, X}}|^{n=4} = -\frac{e^2}{h} \frac{m\epsilon_F(\epsilon_F^2 - m^2)}{(\epsilon_F^2 + m^2)^2} \cdot N_{X,4}^{(1)} - \frac{e^2}{h} \frac{m\epsilon_F(\epsilon_F^2 - m^2)^2}{(\epsilon_F^2 + 3m^2)(\epsilon_F^2 + m^2)^2} \cdot N_{X,4}^{(2)} - \frac{e^2}{h} \frac{m\epsilon_F(\epsilon_F^2 - m^2)^3}{(\epsilon_F^2 + 3m^2)^2(\epsilon_F^2 + m^2)^2} \cdot N_{X,4}^{(3)}, \quad (29b)$$

$$\sigma_{xy}^{w,\Psi}|^{n=3} = -\frac{e^2}{h} \frac{m\epsilon_F}{\epsilon_F^2 - m^2} \cdot N_{\Psi,3}, \quad (29c)$$

$$\begin{aligned} \sigma_{xy}^{w,\Psi}|^{n=4} = & -\frac{e^2}{h} \frac{m\epsilon_F(\epsilon_F^2 - m^2)}{(\epsilon_F^2 + m^2)^2} \cdot N_{\Psi,4}|^{(1)} \\ & -\frac{e^2}{h} \frac{m\epsilon_F(\epsilon_F^2 - m^2)^2}{(\epsilon_F^2 + 3m^2)(\epsilon_F^2 + m^2)^2} \cdot N_{\Psi,4}|^{(2)} \\ & -\frac{e^2}{h} \frac{m\epsilon_F(\epsilon_F^2 - m^2)^3}{(\epsilon_F^2 + 3m^2)^2(\epsilon_F^2 + m^2)^2} \cdot N_{\Psi,4}|^{(3)}. \end{aligned} \quad (29d)$$

The dimensionless integrals $N_{X,n}, N_{\Psi,n}$ over three angular variables, are too cumbersome to be represented here, see Eqs. (C16), (C19), (C30)-(C32), (C34)-(C36) of the Appendix C.

E. Comparison isotropic and warped models

In this section, we compare analytical results for the isotropic model with numerical results for the warped model and analyze the effect of trigonal warping on the anomalous Hall conductivity.

Importantly, in the warped model, all contributions depend on $\tilde{w} = \frac{w}{vp_0^3}$, it means that $\tilde{w}$ is a function of (ϵ_F, m) . Thus, we numerically evaluate angular integrals and plot the full anomalous Hall conductivity as a function of ϵ_F/m , at fixed displacement field m , comparing the isotropic and warped models for several values of the warping parameter: Fig. 5. There we used ab-initio estimates of the parameter $w/v \approx 9.97 \times 10^{-3} \text{ nm}^{-3}$ of the dispersion relation of graphene and 'warped' means the model includes the trigonal-warping term of amplitude $w = 20 \text{ meV}$ for $n = 3$ and $w = 34 \text{ meV} \cdot \text{nm}$ for $n = 4$, while for 'isotropic' we set $w = 0$ keeping all other parameters unchanged.

These results suggest that the presence of warping in trilayer graphene reduces the anomalous Hall conductivity, whereas conductivity in tetralayer graphene experiences only a slight increase in the low-energy region. In either case, within the present parameter regime, the impact of warping may be quantitatively relevant, but not qualitatively.

IV. CONCLUSION

In conclusion, we have developed a theoretical framework to describe the anomalous Hall response in graphene samples with rhombohedral stacking. Within our approach we carefully include disorder scattering in a controlled fashion. In particular, we go beyond the conventional non-crossing approximation by incorporating crossed impurity-line diagrams, enabling a complete treatment of impurity scattering effects. We derived an analytical expression for an isotropic model that accounts for all key contributions to the anomalous Hall effect.

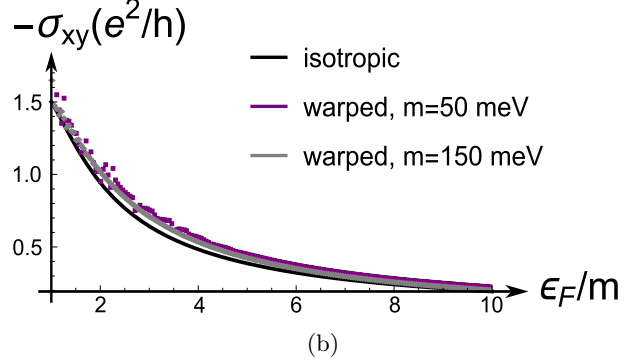
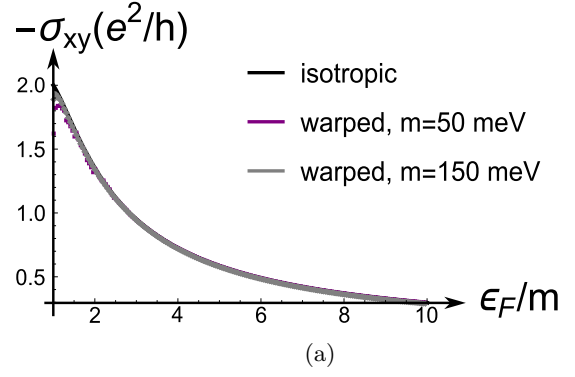


FIG. 5: Full anomalous Hall conductivity (one valley and one spin) for a) $n = 4$ with warping parameter $\tilde{w}_4 = \frac{w}{m} \left(\frac{\epsilon_F^2}{m^2} - 1 \right)^{-1/2}$ and b) $n = 3$ with $\tilde{w}_3 = \frac{w}{v^{1/4} m^{3/4}} \left(\frac{\epsilon_F^2}{m^2} - 1 \right)^{-3/8}$.

In addition, we investigated the role of trigonal warping—via detailed semi-numerical analysis.

While warping introduces only a minor correction deep inside the band, impurity scattering has a pronounced impact on the behavior of σ_{xy} only away from the band gap.

Despite its simplicity, the isotropic model captures the essential physical features and might offer a practical and accurate description of rhombohedral graphene in the limit of sufficiently large displacement fields. The insights gained here may extend to a broader class of multiband materials or nonlinear extensions of the calculations done here [52–55], enhancing our understanding of anomalous transport and guiding future experimental and theoretical studies.

ACKNOWLEDGMENTS

It is a pleasure to thank Pavel Ostrovsky and Alex Levchenko for useful discussions. Support for this research was provided by the Office of the Vice Chancellor for Research and Graduate Education at the University of Wisconsin–Madison with funding from the Wisconsin Alumni Research Foundation. This research was sup-

ported in part by grants NSF PHY-1748958 and PHY-2309135 to the Kavli Institute for Theoretical Physics (KITP). DK, DG and EJK acknowledges hospitality by the KITP, where this project was initiated. VM and EJK acknowledge hospitality by the Max-Planck-Institute for

Solid State Research. DK is supported by the Abrahams Postdoctoral Fellowship of the Center for Materials Theory, Rutgers University and the Zuckerman STEM fellowship.

-
- [1] K. S. Novoselov, A. K. Geim, S. V. Morozov, D. Jiang, M. I. Katsnelson, I. V. Grigorieva, S. V. Dubonos, and A. A. Firsov, Two-dimensional gas of massless dirac fermions in graphene, *nature* **438**, 197 (2005).
 - [2] A. K. Geim and K. S. Novoselov, The rise of graphene, *Nature materials* **6**, 183 (2007).
 - [3] Y. Shi, S. Xu, Y. Yang, S. Slizovskiy, S. V. Morozov, S.-K. Son, S. Ozdemir, C. Mullan, J. Barrier, J. Yin, *et al.*, Electronic phase separation in multilayer rhombohedral graphite, *Nature* **584**, 210 (2020).
 - [4] T. Han, Z. Lu, G. Scuri, J. Sung, J. Wang, T. Han, K. Watanabe, T. Taniguchi, H. Park, and L. Ju, Correlated insulator and chern insulators in pentalayer rhombohedral-stacked graphene, *Nature Nanotechnology* **19**, 181 (2024).
 - [5] T. Han, Z. Lu, Z. Hadjri, L. Shi, Z. Wu, W. Xu, Y. Yao, A. A. Cotten, O. Sharifi Sedeh, H. Weldeyesus, *et al.*, Signatures of chiral superconductivity in rhombohedral graphene, *Nature* **643**, 654 (2025).
 - [6] H. Zhou, T. Xie, A. Ghazaryan, T. Holder, J. R. Ehrets, E. M. Spanton, T. Taniguchi, K. Watanabe, E. Berg, M. Serbyn, *et al.*, Half-and quarter-metals in rhombohedral trilayer graphene, *Nature* **598**, 429 (2021).
 - [7] X. Han, Q. Liu, Y. Wang, R. Niu, Z. Qu, Z. Wang, Z. Li, C. Han, K. Watanabe, T. Taniguchi, *et al.*, Engineering the band topology in a rhombohedral trilayer graphene moiré superlattice, *Nano Letters* **24**, 6286 (2024).
 - [8] F. R. Geisenhof, F. Winterer, A. M. Seiler, J. Lenz, T. Xu, F. Zhang, and R. T. Weitz, Quantum anomalous hall octet driven by orbital magnetism in bilayer graphene, *Nature* **598**, 53 (2021).
 - [9] P. A. Pantaleón, A. Jimeno-Pozo, H. Sainz-Cruz, V. T. Phong, T. Cea, and F. Guinea, Superconductivity and correlated phases in non-twisted bilayer and trilayer graphene, *Nature Reviews Physics* **5**, 304 (2023).
 - [10] Z. Lu, T. Han, Y. Yao, Z. Hadjri, J. Yang, J. Seo, L. Shi, S. Ye, K. Watanabe, T. Taniguchi, *et al.*, Extended quantum anomalous hall states in graphene/hbn moiré superlattices, *Nature* **637**, 1090 (2025).
 - [11] H. Zhou, G. Chang, G. Wang, X. Gui, X. Xu, J.-X. Yin, Z. Guguchia, S. S. Zhang, T.-R. Chang, H. Lin, *et al.*, Enhanced anomalous hall effect in the magnetic topological semimetal $\text{Co}_3\text{Sn}_2\text{S}_2$, *Physical Review B* **101**, 125121 (2020).
 - [12] X. Zhou, R.-W. Zhang, X. Yang, X.-P. Li, W. Feng, Y. Mokrousov, and Y. Yao, Disorder-and topology-enhanced fully spin-polarized currents in nodal chain spin-gapless semimetals, *Physical Review Letters* **129**, 097201 (2022).
 - [13] A. M. Seiler, N. Jacobsen, M. Statz, N. Fernandez, F. Falorsi, K. Watanabe, T. Taniguchi, Z. Dong, L. S. Levitov, and R. T. Weitz, Probing the tunable multi-cone band structure in bernal bilayer graphene, *Nature Communications* **15**, 3133 (2024).
 - [14] Y. Choi, Y. Choi, M. Valentini, C. L. Patterson, L. F. Holleis, O. I. Sheekey, H. Stoyanov, X. Cheng, T. Taniguchi, K. Watanabe, *et al.*, Superconductivity and quantized anomalous hall effect in rhombohedral graphene, *Nature*, 1 (2025).
 - [15] J. Yang, X. Shi, S. Ye, C. Yoon, Z. Lu, V. Kakani, T. Han, J. Seo, L. Shi, K. Watanabe, *et al.*, Impact of spin-orbit coupling on superconductivity in rhombohedral graphene, *Nature Materials*, 1 (2025).
 - [16] X. Mu and J. Zhou, Valley-dependent giant orbital moments and transport features in rhombohedral graphene multilayers, *Physical Review B* **111**, 165102 (2025).
 - [17] Y. Cao, V. Fatemi, S. Fang, K. Watanabe, T. Taniguchi, E. Kaxiras, and P. Jarillo-Herrero, Unconventional superconductivity in magic-angle graphene superlattices, *Nature* **556**, 43 (2018).
 - [18] H. Zhou, T. Xie, T. Taniguchi, K. Watanabe, and A. F. Young, Superconductivity in rhombohedral trilayer graphene, *Nature* **598**, 434 (2021).
 - [19] Z. Lu, T. Han, Y. Yao, A. P. Reddy, J. Yang, J. Seo, K. Watanabe, T. Taniguchi, L. Fu, and L. Ju, Fractional quantum anomalous hall effect in multilayer graphene, *Nature* **626**, 759 (2024).
 - [20] M. Yankowitz, S. Chen, H. Polshyn, Y. Zhang, K. Watanabe, T. Taniguchi, D. Graf, A. F. Young, and C. R. Dean, Tuning superconductivity in twisted bilayer graphene, *Science* **363**, 1059 (2019).
 - [21] X. Liu, Z. Wang, K. Watanabe, T. Taniguchi, O. Vafek, and J. Li, Tuning electron correlation in magic-angle twisted bilayer graphene using coulomb screening, *Science* **371**, 1261 (2021).
 - [22] G. Parra-Martínez, A. Jimeno-Pozo, V. o. T. Phong, H. Sainz-Cruz, D. Kaplan, P. Emanuel, Y. Oreg, P. A. Pantaleón, J. A. Silva-Guillén, and F. Guinea, Band renormalization, quarter metals, and chiral superconductivity in rhombohedral tetralayer graphene, *Phys. Rev. Lett.* **135**, 136503 (2025).
 - [23] Y. Zhang, Y.-Y. Zhou, S. Zhang, H. Cai, L.-H. Tong, W.-Y. Liao, R.-J. Zou, S.-M. Xue, Y. Tian, T. Chen, *et al.*, Layer-dependent evolution of electronic structures and correlations in rhombohedral multilayer graphene, *Nature Nanotechnology* **20**, 222 (2025).
 - [24] M. Kumar, D. Waleffe, A. Okounkova, R. Tejani, V. T. Phong, K. Watanabe, T. Taniguchi, C. Lewandowski, J. Folk, and M. Yankowitz, Superconductivity from dual-surface carriers in rhombohedral graphene, *arXiv preprint arXiv:2507.18598* (2025).
 - [25] A. Ghazaryan, T. Holder, E. Berg, and M. Serbyn, Multilayer graphenes as a platform for interaction-driven physics and topological superconductivity, *Phys. Rev. B* **107**, 104502 (2023).
 - [26] Y. Cao, Rhombohedral graphene goes correlated at four or five layers, *nature nanotechnology* **19**, 139 (2024).

- [27] N. Nagaosa, J. Sinova, S. Onoda, A. H. MacDonald, and N. P. Ong, Anomalous hall effect, *Reviews of modern physics* **82**, 1539 (2010).
- [28] R. Karplus and J. Luttinger, Hall effect in ferromagnetics, *Physical Review* **95**, 1154 (1954).
- [29] T. Jungwirth, Q. Niu, and A. MacDonald, Anomalous hall effect in ferromagnetic semiconductors, *Physical review letters* **88**, 207208 (2002).
- [30] N. Sinitsyn, Semiclassical theories of the anomalous hall effect, *Journal of Physics: Condensed Matter* **20**, 023201 (2007).
- [31] P. Streda, Theory of quantised hall conductivity in two dimensions, *Journal of Physics C: Solid State Physics* **15**, L717 (1982).
- [32] T. S. Nunner, N. Sinitsyn, M. F. Borunda, V. Dugaev, A. Kovalev, A. Abanov, C. Timm, T. Jungwirth, J.-i. Inoue, A. MacDonald, *et al.*, Anomalous hall effect in a two-dimensional electron gas, *Physical Review B—Condensed Matter and Materials Physics* **76**, 235312 (2007).
- [33] L. Berger, Influence of spin-orbit interaction on the transport processes in ferromagnetic nickel alloys, in the presence of a degeneracy of the 3d band, *Physica* **30**, 1141 (1964).
- [34] V. Belinicher, E. Ivchenko, and B. Sturman, Kinetic theory of the displacement photovoltaic effect in piezoelectrics, *Zh. Eksp. Teor. Fiz* **83**, 649 (1982).
- [35] N. Sinitsyn, Q. Niu, and A. MacDonald, Coordinate shift in the semiclassical boltzmann equation and the anomalous hall effect, *Physical Review B—Condensed Matter and Materials Physics* **73**, 075318 (2006).
- [36] J. Smit, The spontaneous hall effect in ferromagnetics i, *Physica* **21**, 877 (1955).
- [37] N. Sinitsyn, A. MacDonald, T. Jungwirth, V. Dugaev, and J. Sinova, Anomalous hall effect in a two-dimensional dirac band: The link between the kubo-streda formula and the semiclassical boltzmann equation approach, *Physical Review B—Condensed Matter and Materials Physics* **75**, 045315 (2007).
- [38] I. Ado, I. Dmitriev, P. Ostrovsky, and M. Titov, Anomalous hall effect with massive dirac fermions, *Europhysics Letters* **111**, 37004 (2015).
- [39] I. Ado, I. Dmitriev, P. Ostrovsky, and M. Titov, Anomalous hall effect in a 2d rashba ferromagnet, *Physical Review Letters* **117**, 046601 (2016).
- [40] E. König, P. Ostrovsky, M. Dzero, and A. Levchenko, Anomalous hall effect on the surface of topological kondo insulators, *Physical Review B* **94**, 041403 (2016).
- [41] R. M. Lutchyn, P. Nagornykh, and V. M. Yakovenko, Frequency and temperature dependence of the anomalous ac hall conductivity in a chiral px+ipy superconductor with impurities, *Physical Review B—Condensed Matter and Materials Physics* **80**, 104508 (2009).
- [42] J. Goryo, Impurity-induced polar kerr effect in a chiral p-wave superconductor, *Physical Review B—Condensed Matter and Materials Physics* **78**, 060501 (2008).
- [43] T. Ando, T. Nakanishi, and R. Saito, Berry's phase and absence of back scattering in carbon nanotubes, *Journal of the Physical Society of Japan* **67**, 2857 (1998).
- [44] M. Koshino and E. McCann, Trigonal warping and berry's phase $n\pi$ in abc-stacked multilayer graphene, *Physical Review B—Condensed Matter and Materials Physics* **80**, 165409 (2009).
- [45] M. Koshino, Interlayer screening effect in graphene multilayers with aba and abc stacking, *Physical Review B—Condensed Matter and Materials Physics* **81**, 125304 (2010).
- [46] F. Guinea, A. Castro Neto, and N. Peres, Electronic states and landau levels in graphene stacks, *Physical Review B—Condensed Matter and Materials Physics* **73**, 245426 (2006).
- [47] C.-L. Lu, H.-C. Lin, C.-C. Hwang, J. Wang, M.-F. Lin, and C.-P. Chang, Absorption spectra of trilayer rhombohedral graphite, *Applied physics letters* **89** (2006).
- [48] N. Sinitsyn, J. Hill, H. Min, J. Sinova, and A. MacDonald, Charge and spin hall conductivity in metallic graphene, *Physical review letters* **97**, 106804 (2006).
- [49] P. Mitra, R. Misra, A. Hebard, K. Muttalib, and P. Wölfe, Weak-localization correction to the anomalous hall effect in polycrystalline fe films, *Physical review letters* **99**, 046804 (2007).
- [50] E. König, P. Ostrovsky, I. Protopopov, I. Gornyi, I. Burmistrov, and A. Mirlin, Half-integer quantum hall effect of disordered dirac fermions at a topological insulator surface, *Physical Review B* **90**, 165435 (2014).
- [51] R. Jackiw and C. Rebbi, Solitons with fermion number $1/2$, *Physical Review D* **13**, 3398 (1976).
- [52] Z. Du, C. Wang, S. Li, H.-Z. Lu, and X. Xie, Disorder-induced nonlinear hall effect with time-reversal symmetry, *Nature Communications* **10**, 3047 (2019).
- [53] D. Kaplan, T. Holder, and B. Yan, Unifying semiclassics and quantum perturbation theory at nonlinear order, *SciPost Phys.* **14**, 082 (2023).
- [54] D. Kaplan, T. Holder, and B. Yan, General nonlinear hall current in magnetic insulators beyond the quantum anomalous hall effect, *Nature communications* **14**, 3053 (2023).
- [55] D. Kaplan, K. P. Lucht, P. A. Volkov, and J. H. Pixley, *Quantum geometric photocurrents of quasiparticles in superconductors* (2025), arXiv:2502.12265 [cond-mat.supr-con].
- [56] I. Ado, I. Dmitriev, P. Ostrovsky, and M. Titov, On-line supplemental material, anomalous hall effect in 2d rashba ferromagnet, *Physical Review Letters* **117**, 046601 (2016).

Appendix A: The low-energy Hamiltonian

In this appendix we provide details in the derivation of Eq. (1) of the main text, i.e. an effective two-component Hamiltonian that describes hopping between atomic sites A_1 and B_n .

We start from the Hamiltonian in the basis $\Phi = [\phi_1, \phi_2, \dots, \phi_n]^T$ with $\phi_l = [\psi_{A_l}, \psi_{B_l}]$ where l is the layer number, n is the number of layers, ψ_A and ψ_B are the weight on the A, B sublattices:

$$\mathcal{H}_n = \begin{pmatrix} D_1 & V & W & & \\ V^\dagger & D_2 & V & W & \\ W^\dagger & V^\dagger & D_3 & \ddots & \\ & W^\dagger & \ddots & \ddots & \\ & & \ddots & \ddots & V & W \\ & & & V^\dagger & D_{n-1} & V \\ & & & W^\dagger & V^\dagger & D_n \end{pmatrix}, \quad (\text{A1})$$

$$D_i = \begin{pmatrix} U_i & v_F \mathcal{P} \\ v_F \bar{\mathcal{P}} & U_i \end{pmatrix}; \quad V = \begin{pmatrix} -v_4 \mathcal{P} & v_3 \bar{\mathcal{P}} \\ \gamma_1 & -v_4 \mathcal{P} \end{pmatrix}; \quad W = \begin{pmatrix} 0 & \gamma_2/2 \\ 0 & 0 \end{pmatrix},$$

where $v_i = \sqrt{3}a\gamma_i/2\hbar$ are effective velocities, $a = 0.246$ nm is the lattice spacing, $v_F = \sqrt{3}a\gamma_0/2\hbar$; $\mathcal{P} = \xi p_x - ip_y$ and $\xi = \pm 1$ is the valley index (we will keep $\xi = 1$); U_i describe on-site energies of the atoms on layers in the presence of an external displacement field; γ_0 and γ_1 describe nearest-neighbor intralayer and interlayer hopping respectively, γ_3 represents hopping between the low-energy sites of a AB -stacked bilayer, γ_4 couples low- and high-energy sites located on different layers, γ_2 describes the vertical coupling between sites A_1 and B_3 in a different unit cell. We will use the following values for energies: $\gamma_0 = 3.16$ eV, $\gamma_1 = 0.39$ eV, $\gamma_2 = -0.020$ eV, $\gamma_3 = 0.315$ eV, $\gamma_4 = 0.044$ eV [44].

We can derive an effective two-component Hamiltonian for $n = 3, 4$ that describes hopping between atomic sites A_1 and B_n using the procedure from Ref. [44], where case for $n = 3$ was considered:

$$\mathcal{H}_{2 \times 2}(\mathcal{P})^{n=3} = \underbrace{\frac{v_F^3}{\gamma_1^2} \begin{pmatrix} 0 & \mathcal{P}^3 \\ \bar{\mathcal{P}}^3 & 0 \end{pmatrix}}_{H_3} + \underbrace{\left(\frac{\gamma_2}{2} - \frac{2v_F v_3}{\gamma_1} p^2 \right) \sigma_x + \frac{2v_F v_4 p^2}{\gamma_1} \sigma_0}_{H_w} + \underbrace{\frac{U_1 - U_3}{2} \left(1 - \frac{v_F^2 p^2}{\gamma_1^2} \right) \sigma_z}_{H_{a1}} + \quad (\text{A2})$$

$$\underbrace{\left(\frac{U_1 + U_3}{2} + \frac{2U_2 - U_1 - U_3}{2} \frac{v_F^2 p^2}{\gamma_1^2} \right) \sigma_0}_{H_{a2}},$$

$$\mathcal{H}_{2 \times 2}(\mathcal{P})^{n=4} = \underbrace{-\frac{v_F^4}{\gamma_1^3} \begin{pmatrix} 0 & \mathcal{P}^4 \\ \bar{\mathcal{P}}^4 & 0 \end{pmatrix}}_{H_4} + \underbrace{\left(\frac{3v_3 v_F^2}{\gamma_1^2} p^2 - \frac{\gamma_2 v_F}{\gamma_1} \right) \begin{pmatrix} 0 & \mathcal{P} \\ \bar{\mathcal{P}} & 0 \end{pmatrix} + \frac{2v_F v_4 p^2}{\gamma_1} \sigma_0}_{H_w} + \quad (\text{A3})$$

$$\underbrace{\left[\frac{U_1 - U_4}{2} \left(1 - \frac{v_F^2 p^2}{\gamma_1^2} \right) + \frac{U_2 - U_3}{2} \frac{v_F^2 p^2}{\gamma_1^2} \left(1 - \frac{2v_F^2 p^2}{\gamma_1^2} \right) \right] \sigma_z}_{H_{a1}} + \underbrace{\left[\frac{U_1 + U_4}{2} \left(1 - \frac{v_F^2 p^2}{\gamma_1^2} \right) + \frac{U_2 + U_3}{2} \frac{v_F^2 p^2}{\gamma_1^2} \right] \sigma_0}_{H_{a2}},$$

where H_n describes hopping between sites A_1 and B_n , H_w is the term which correspond to warping, H_c introduces electron-hole asymmetry, H_{a1} and H_{a2} stems from the interlayer asymmetries. To simplify the effective Hamiltonian, which can be written as

$$H = d_0(\mathbf{p}) + \vec{d}(\mathbf{p}) \cdot \vec{\sigma}, \quad (\text{A4})$$

we consider the relative magnitude of the various terms in eqs. (A2)-(A3).

In the mass term

$$d_z(\mathbf{p}) = m \left(1 + m_2 p^2 + (m_2 p^2)^2 + \dots \right), \quad (\text{A5})$$

terms involving $m_2 \sim v_F^2/\gamma_1^2 \sim 100a^2$ are negligible so long as $n \lesssim \frac{1}{100a^2} \sim 16 \times 10^{12} \text{cm}^{-2}$. In the intersublattice terms

$$d_{x-iy}(\mathbf{p}) = v\mathcal{P}^n + w(\mathbf{p})\mathcal{P}^{n-3}, \quad (\text{A6})$$

the term $w(\mathbf{p}) = w(1 + w_2 p^2)$ contains $w_2 \sim \frac{v_F v_3}{\gamma_1 \gamma_2}$ is about a factor of 2 larger than the m_2 term. As such, it can be neglected so long as $n \lesssim \frac{1}{100a^2} \sim 8 \times 10^{12} \text{cm}^{-2}$.

Finally, we compare $d_0(\mathbf{p}) = \frac{v_F v_4 \mathbf{p}^2}{\gamma_1}$ (the constant part is absorbed into the chemical potential) to $|\vec{d}(\mathbf{p})| \sim m$ so that

$$\frac{d_0}{|\vec{d}|} = \underbrace{\frac{v_F^2 p^2}{\gamma_1^2}}_{=m_2 p^2} \underbrace{\frac{v_4}{v_F}}_{=1/100=4eV/10m} \underbrace{\frac{\gamma_1}{m}}_{=1/100=4eV/10m}. \quad (\text{A7})$$

We can estimate $m \sim eV$ so that $\frac{d_0}{|\vec{d}|} \sim m_2 p^2 \frac{1}{250}$ so that $d_0(\mathbf{p})$ can be dropped wherever we dropped m_2 . Typical densities in experiments are a fraction of 10^{12} cm^{-2} . This concludes the derivation of Eq. (1)

1. Importance of impurities

In this section, we discuss when impurities are important in experimental samples.

First, the experimental article [18] quotes a mean free path $l \sim 1 \mu\text{m}$ (it was deduced from a Drude expression for R_{xx}) at density $n_e \approx 2 \cdot 10^{12} \text{ cm}^{-2}$ and the sample size $L \sim 2.5 \mu\text{m}$. Alternatively, as quantum oscillations [6] become visible for a magnetic field $\sim .1 \text{ T}$, we estimate a mean-free path $l \sim 100 \text{ nm}$ at density $n_e \approx 0.25 \cdot 10^{12} \text{ cm}^{-2}$.

Appendix B: Isotropic model

1. Real space clean Green's function

In this appendix we present details in the derivation of Eq. (10). We adopt the convention $\epsilon \equiv \epsilon_F > 0$ throughout the appendices to streamline the derivations. We start from the definition of Green's function:

$$G_0^{R/A}(\epsilon, \mathbf{p}) = [\epsilon \pm i0 - H_0]^{-1}, \quad (\text{B1})$$

$$G_0^{R/A}(\epsilon, \mathbf{r}) = [\epsilon + \vec{d}(-i\nabla)] I_n(\mathbf{r}),$$

where

$$I_n(\mathbf{r}) = \frac{2\pi n \rho_n(\epsilon)}{\epsilon} \underbrace{\int \frac{d\vec{p}_x d\vec{p}_y}{(2\pi)^2} \frac{e^{i\vec{p}_x \mathbf{p}_0 \mathbf{r}}}{1 \pm i0 - (\vec{p}_x^2 + \vec{p}_y^2)^n}}_{\bar{I}_n(p_0 r)},$$

$$\sigma_{xy}^{\text{int}} = \frac{e^2}{h} \int_0^\infty \frac{p dp}{2\pi} \int_0^{2\pi} \frac{d\theta}{2\pi} \frac{v^2 n^2 (4\epsilon m i(i0) p^{2n-2} + i v^2 p^{4n-2} (e^{-2i\theta} - e^{2i\theta}))}{(v^2 p_0^{2n} - v^2 p^{2n})^2 - 4(i0)^2 \epsilon^4} = -\frac{e^2}{h} \frac{m}{2\epsilon} n.$$

Consistently, using the Berry curvature $\Omega_\pm^{(n)}(\mathbf{p}) =$

where $p^2 = \vec{p}_x^2 + \vec{p}_y^2$. We use:

$$\frac{1}{1 + i0 - p^{2n}} = \frac{1}{n} \sum_{j=0}^{n-1} \frac{1}{1 + i0 - \zeta^j p^2}; \quad \zeta = e^{i2\pi/n}, \quad (\text{B2})$$

to define

$$\bar{I}_n(p_0 r) = \frac{1}{4n} \left[Y_0(p_0 r) \mp i J_0(p_0 r) - \frac{2}{\pi} \sum_{j=1}^{n-1} e^{-i \frac{2\pi j}{n}} K_0 \left(i e^{-i \frac{\pi j}{n}} p_0 r \right) \right]. \quad (\text{B3})$$

2. Self-consistent Born approximation

In this appendix we provide details in the derivation of Eq. (6). The self-consistent Born approximation leads to a self-energy:

$$\Sigma = \frac{\epsilon}{n \rho_n(\epsilon)} \alpha. \\ \int \frac{d^2 p}{(2\pi)^2} \frac{\epsilon + m \sigma_z + v \sigma_x p^n \cos(n\theta) + v \sigma_y p^n \sin(n\theta)}{v^2 p_0^{2n} - v^2 p^{2n} \pm i \Gamma(\epsilon + m)} = \frac{-\pi i \alpha(\epsilon + m \sigma_z)}{2n} = -i(\Gamma + \Gamma_z \sigma_z) \quad (\text{B4})$$

Here we only kept the imaginary part of the self-energy, the real part is absorbed into the chemical potential and mass. From this, we can readily get Eq. (6).

3. Intrinsic contribution for isotropic model

In this appendix, we show details for the intrinsic contribution, Eq. (5), which in the Kubo-Streda approach is given by

$$\sigma_{xy}^{\text{int}} = \frac{e^2}{h} \int \frac{dp^2}{(2\pi)^2} \text{Tr} \left(j_x(p) G_p^R j_y(p) G_p^A \right). \quad (\text{B5})$$

After taking the trace we can get

$\pm \frac{1}{2} \frac{n^2 v^2 m p^{2n-2}}{(m^2 + v^2 p^{2n})^{3/2}}$, we obtain

$$\sigma_{xy} = \frac{e^2}{h} \sum_{\pm} \int \frac{d^2 p}{(2\pi)^2} f_{FD}(\pm \xi - \epsilon) \Omega_{\pm}^{(n)}(\mathbf{p}) = -\frac{e^2}{h} \frac{n}{2} \left(\Theta(m - \epsilon) + \Theta(\epsilon - m) \cdot \frac{|m|}{\epsilon} \right) \cdot \text{sgn}(m). \quad (\text{B6})$$

Here $\xi = |\vec{d}(\mathbf{p})|$.

$$\begin{aligned} \delta j_{x/y} &= \alpha \frac{\epsilon}{n\rho_n(\epsilon)} \int \frac{d^2p}{(2\pi)^2} \mathbf{G}_{\mathbf{p}}^A j_{x/y} \mathbf{G}_{\mathbf{p}}^R \propto \\ &\int \frac{d^2p}{(2\pi)^2} (a(|\vec{\mathbf{p}}|^2) + b(|\vec{\mathbf{p}}|^2) \mathcal{P}^n + c(|\vec{\mathbf{p}}|^2) \bar{\mathcal{P}}^n) \mathcal{P}^{n-1} + \text{h.c.} = 0. \end{aligned} \quad (\text{B7})$$

The matrix-valued isotropic functions $a(|\vec{\mathbf{p}}|^2), b(|\vec{\mathbf{p}}|^2), c(|\vec{\mathbf{p}}|^2)$ may be read off from $\mathbf{G}_{\mathbf{p}}^R$ given in Eq. (6).

Due to angle integration, all ladder diagrams will be equal to zero if n is not equal to 1, as a consequence, the contribution from skew-scattering diagrams in non-crossing approximation is equal to zero too.

4. Vertex correction

In this appendix, we calculate a single-impurity vertex correction, which is given by Eq. (7) of the main text. Let us consider the first order correction:

5. σ_{xy} in non-crossing approximation

Now, we show details for the contribution from non-crossing diagrams of Eq. (8), which is given by

$$\begin{aligned} \sigma_{xy}^{\text{nc}} &= \frac{e^2}{h} \int \frac{d^2p}{(2\pi)^2} \text{Tr} (j_x(p) \mathbf{G}^{\mathbf{R}}(p) j_y(p) \mathbf{G}^{\mathbf{A}}(p)) = \\ &\frac{e^2}{h} \int_0^\infty \frac{pd p}{2\pi} \int_0^{2\pi} \frac{d\theta}{2\pi} v^2 n^2 \frac{-4\epsilon m \frac{\pi\alpha}{n} p^{2n-2} + iv^2 p^{4n-2} (e^{-2i\theta} - e^{2i\theta})}{(v^2 p_0^{2n} - v^2 p^{2n})^2 + \frac{\pi^2 \alpha^2 (\epsilon^2 + m^2)^2}{n^2}} = -\frac{e^2}{h} \frac{m\epsilon n}{\epsilon^2 + m^2}. \end{aligned} \quad (\text{B8})$$

The non-crossing contribution includes, in principle, the intrinsic, side-jump and Gaussian skew-scattering components. The latter vanishes, Eq. (7), so we can easily find side-jump components:

$$\sigma_{xy}^{\text{sj}} = \sigma_{xy}^{\text{nc}} - \sigma_{xy}^{\text{int}} = -\frac{e^2}{h} \frac{m(\epsilon^2 - m^2)}{2\epsilon(\epsilon^2 + m^2)} \cdot n; \quad n > 1. \quad (\text{B9})$$

6. Diffractive skew-scattering

In this appendix we provide calculations about diffractive skew-scattering contribution from X and Ψ diagram Eq. (13a), (13b) of the main text.

$$\sigma_{xy}^{\text{X}} = \frac{e^2}{h} \left(\frac{\epsilon}{n\rho_n(\epsilon)} \alpha \right)^2 \int_{\{\mathbf{p}_i\}} \text{Tr} [\mathbf{G}_{\mathbf{p}_1}^A j_x(\mathbf{p}_1) \mathbf{G}_{\mathbf{p}_1}^R \mathbf{G}_{\mathbf{p}_3}^R \mathbf{G}_{\mathbf{p}_2}^R j_y(\mathbf{p}_2) \mathbf{G}_{\mathbf{p}_2}^A \mathbf{G}_{\mathbf{p}_4}^A] (2\pi)^2 \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4). \quad (\text{B10})$$

We rewrite the average Green's function as

$$\begin{aligned} \mathbf{G}_{\mathbf{p}}^R &= N(\mathbf{p}) \mathcal{G}_{\mathbf{p}}^R, \\ N(\mathbf{p}) &= \epsilon + \vec{d}(\vec{\mathbf{p}}) \vec{\sigma}, \quad \mathcal{G}_{\mathbf{p}}^R = \frac{1}{\epsilon^2 - \xi^2 + \frac{i\pi\alpha}{n} (\epsilon^2 + m^2)}. \end{aligned}$$

In this notation, we use that within our approximation we can keep only the lowest order in α in $N(\mathbf{p})$, i.e. zeroth order. Then, using partial fraction decomposition and keeping overall $O(\alpha^0)$

$$\sigma_{xy}^{\text{X}} = \frac{e^2}{h} \left(\frac{\epsilon}{n\rho_n(\epsilon)} \alpha \right)^2 \cdot \frac{n^2}{\pi^2 \alpha^2 (\epsilon^2 + m^2)^2} \cdot \int_{\{\mathbf{p}_i\}} (2\pi)^2 \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4).$$

$$\left(\text{Tr}[N(\mathbf{p}_1)j_x(\mathbf{p}_1)N(\mathbf{p}_1)N(\mathbf{p}_3)N(\mathbf{p}_2)j_y(\mathbf{p}_2)N(\mathbf{p}_2)N(\mathbf{p}_4)] \cdot \mathcal{I}\mathcal{G}_{\mathbf{p}_1}\mathcal{I}\mathcal{G}_{\mathbf{p}_2}(\Re\mathcal{G}_{\mathbf{p}_3} - i\mathcal{I}\mathcal{G}_{\mathbf{p}_3})(\Re\mathcal{G}_{\mathbf{p}_4} + i\mathcal{I}\mathcal{G}_{\mathbf{p}_4}) \right),$$

where $\Re$ and $\mathcal{I}$ denote real and imaginary parts.

After taking the trace and nondimensionalization, we can rewrite it using Green's functions in real space, App. B 1:

$$\sigma_{xy}^X = \Xi_n \int_{\{\mathbf{r}_i\}, \{\mathbf{R}\}} T_X \cdot \mathcal{I}(r_1)\mathcal{I}(r_2)\mathcal{I}(r_3)\Re(r_4) \cdot \delta(\mathbf{R} - \mathbf{r}_1)\delta(\mathbf{R} - \mathbf{r}_2)\delta(\mathbf{R} + \mathbf{r}_3)\delta(\mathbf{R} + \mathbf{r}_4),$$

where

$$\Xi_n = -\frac{e^2}{h} \frac{m\epsilon(\epsilon^2 - m^2)}{8(\epsilon^2 + m^2)^2},$$

$$T_X = \hat{\mathcal{P}}_1^{n-1}\hat{\mathcal{P}}_2^{n-1}(\hat{\mathcal{P}}_2^n\hat{\mathcal{P}}_1^n - 1)[\hat{\mathcal{P}}_1^n(\mathcal{P}_3^n - \mathcal{P}_4^n) + \mathcal{P}_2^n(\hat{\mathcal{P}}_3^n - \hat{\mathcal{P}}_4^n)]$$

We use complex polar space coordinates $\hat{\mathcal{P}} = \hat{p}_x - i\hat{p}_y = -ie^{-i\phi}(\partial_r - \frac{i}{r}\partial_\phi)$, so that

$$\hat{\mathcal{P}}^m = (-i)^m e^{-im\phi} \times \underbrace{\left[\left(\partial_r - \frac{n-1}{r} - \frac{i}{r}\partial_\phi \right) \dots \left(\partial_r - \frac{1}{r} - \frac{i}{r}\partial_\phi \right) \left(\partial_r - \frac{i}{r}\partial_\phi \right) \right]}_{m \text{ differ. operators}}$$

Finally, general answer for any n :

$$\sigma_{xy}^X = \frac{e^2}{h} \frac{\epsilon_F m(\epsilon_F^2 - m^2)}{(\epsilon_F^2 + m^2)^2} \cdot \pi n \times \int_0^\infty dx J_1 J_n [J_n \tilde{Y}_0 - J_0 \tilde{Y}_n].$$

The contribution from the Ψ diagram can be calculated in the same way as for the X diagram, so we can get the following:

$$\sigma_{xy}^\Psi = \frac{e^2}{h} \frac{\epsilon_F m(\epsilon_F^2 - m^2)}{(\epsilon_F^2 + m^2)^2} \cdot \pi n \times \int_0^\infty dx J_1 J_n [J_n \tilde{Y}_0 + J_0 \tilde{Y}_n] + J_0 J_n \tilde{Y}_0 (J_{n-1} - J_{n+1}),$$

This concludes the derivation of Eqs. (13a)-(13b) of the main text.

7. Calculation of integral over Bessel functions

In this section, we present calculations of the Bessel-function integrals appearing in (13a)-(13b) and obtain asymptotic expressions in the limit $n \rightarrow \infty$.

We need the following integrals:

$$I_{n10n} = \int_0^\infty dx J_n J_1 J_0 \tilde{Y}_n, \quad (\text{B11a})$$

$$I_{nn10} = \int_0^\infty dx J_n^2 J_1 \tilde{Y}_0, \quad (\text{B11b})$$

$$I_{nn01} = \int_0^\infty dx J_n^2 J_0 \tilde{Y}_1, \quad (\text{B11c})$$

$$I_{n\pm 00} = \int_0^\infty dx J_n (J_{n-1} - J_{n+1}) J_0 \tilde{Y}_0. \quad (\text{B11d})$$

The integral I_{n10n} drops out of the final expression for the diffractive skew-scattering contribution and $I_{n\pm 00}$ can be expressed in terms of the others as $I_{n\pm 00} = I_{nn10} + I_{nn01}$. Since $\tilde{Y}$ (see Eq. (11)) decomposes into a Y term and a K term, we split each integral accordingly, for example:

$$I_{nn10} = I_{nn10}^Y + I_{nn10}^K, \quad (\text{B12})$$

where

$$I_{nn10}^Y = \int_0^\infty dx J_n^2 J_1 Y_0, \\ I_{nn10}^K = \int_0^\infty dx J_n^2 J_1 \left(\frac{2}{\pi} \sum_{j=1}^{n-1} (-e^{i\frac{2\pi j}{n}}) K_0(i e^{-i\frac{\pi j}{n}} x) \right).$$

The general method we use to handle these integrals is as follows [56]:

- Reduce the number of Bessel functions from 4 to 3 using the following expression:

$$J_n^2 = \int_0^2 \frac{da x^n}{\pi^{1/2} \Gamma(n+1/2)} \left(\frac{a}{2} \right)^n \left[1 - \left(\frac{a}{2} \right)^2 \right]^{n-1/2} J_n(ax).$$

- Integrate over x using one of these four identities

$$(u(a, c) = \frac{a^2 + c^2 - 1}{2ac}):$$

$$\begin{aligned} I_{110}^Y(a, c) &= \int_0^\infty dx x J_1(ax) J_1(x) Y_0(cx) \\ &= \frac{1}{\pi a} \begin{cases} 1 - \frac{u-a/c}{\sqrt{u^2-1}}, & 1 < c - a, \\ 1 + \frac{u-a/c}{\sqrt{u^2-1}}, & 1 < (a - c) \vee 1 > (a + c), \\ 1, & |a - c| < 1 < a + c, \end{cases} \\ I_{101}^Y(a, c) &= \int_0^\infty dx x J_1(ax) J_0(x) Y_1(cx) \\ &= -\frac{1}{\pi ac} \begin{cases} 1 - \frac{u}{\sqrt{u^2-1}}, & 1 < c - a, \\ 1 + \frac{u}{\sqrt{u^2-1}}, & 1 < (a - c) \vee 1 > (a + c), \\ 1, & |a - c| < 1 < a + c, \end{cases} \\ I_{110}^K(a, c) &= \int_0^\infty dx x J_1(ax) J_1(x) K_0(cx) \\ &= \frac{a^2 + c^2 + 1}{2a\sqrt{(a^2 + c^2 + 1)^2 - 4a^2}} - \frac{1}{2a}, \\ I_{101}^K(a, c) &= \int_0^\infty dx x J_1(ax) J_0(x) K_1(cx) \\ &= \frac{a^2 - c^2 - 1}{2ac\sqrt{(a^2 + c^2 + 1)^2 - 4a^2}} + \frac{1}{2ac}. \end{aligned}$$

• Integrate over a .

Employing this approach, we find

$$I_{nn10}^Y = \frac{1}{2\pi n}; \quad I_{nn01}^Y = -\frac{1}{2\pi n}. \quad (\text{B13})$$

Although obtaining I^K in general is analytically prohibitive, the method outlined above allows us to derive the $n = 2$ case:

$$I_{2210}^K = \frac{1}{4\pi} \left(-{}_3F_2 \left(\frac{1}{2}, \frac{5}{4}, \frac{7}{4}; 2, \frac{5}{2}; -4 \right) + 2{}_3F_2 \left(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}; 1, \frac{3}{2}; -4 \right) - 1 \right), \quad (\text{B14})$$

$$I_{2201}^K = \frac{1}{4\pi} \left(1 - 2{}_3F_2 \left(\frac{1}{4}, \frac{1}{2}, \frac{3}{4}; 1, \frac{3}{2}; -4 \right) - {}_3F_2 \left(\frac{1}{2}, \frac{5}{4}, \frac{7}{4}; 2, \frac{5}{2}; -4 \right) + 2{}_3F_2 \left(\frac{1}{2}, \frac{3}{4}, \frac{5}{4}; 2, 2; -4 \right) \right). \quad (\text{B15})$$

Thus, the $n = 2$ diffractive skew-scattering contribution—accurate up to an overall prefactor—reads:

$$\frac{\sigma_{xy}^X + \sigma_{xy}^Y}{f(\epsilon_F, m)} = 3\pi n I_{2210} + \pi n I_{2201} \approx 1.18617.$$

These values agree with the values obtained by straightforward numerical calculation in Table I in the main text. We next focus on $n \geq 3$ and develop a controlled large- n expansion, yielding the $n \rightarrow \infty$ asymptotics.

We start from the integral I_{nn10}^K . After completing the first two steps of our approach, we obtain:

$$\begin{aligned} I_{nn10}^K &= \int_0^2 \frac{da}{\pi^{1/2} \Gamma(n+1/2)} \left(\frac{a}{2} \right)^n \left[1 - \left(\frac{a}{2} \right)^2 \right]^{n-1/2} \\ &\quad (-1)^{n-1} \left(\partial_a - \frac{n-1}{a} \right) \left(\partial_a - \frac{n-2}{a} \right) \dots \left(\partial_a - \frac{1}{a} \right) \times \\ &\quad \underbrace{\frac{2}{\pi} \sum_{j=1}^{n-1} (-e^{-i\frac{2\pi j}{n}}) I_{110}^K(a, ie^{-i\frac{\pi j}{n}})}_{\Sigma^{K_0}(a)}. \end{aligned} \quad (\text{B16})$$

First, we replace the discrete sum by a continuum integral

$$\begin{aligned} \frac{1}{n} \sum_{j=1}^n e^{-i\frac{2\pi j}{n}} I_{110}^K(a, ie^{-i\frac{\pi j}{n}}) &\rightarrow \int \frac{d\phi}{2\pi} e^{-i\phi} I_{110}^K(a, ie^{-i\phi/2}) \\ &+ \delta_n(a) \simeq -\frac{1}{\pi} \sqrt{1 - (a/2)^2} - \frac{\kappa}{\sqrt{an}} e^{-2an}, \end{aligned} \quad (\text{B17})$$

where κ is a dimensionless prefactor determined by comparison with numerical data ($\kappa \approx \sqrt{\pi}$); below we keep the $\sqrt{\pi}$ factor explicit.

Thus, for large n we find:

$$\Sigma^{K_0}(a) \simeq -\frac{1}{\pi a} + \frac{2n}{\pi^2} \sqrt{1 - (a/2)^2} + \frac{2\sqrt{n}}{\sqrt{a\pi}} e^{-2an}. \quad (\text{B18})$$

Then, after integrating over a , we obtain the $n \rightarrow \infty$ limit:

$$I_{nn10}^K \simeq -\frac{1}{2\pi n} + \frac{2}{\pi^3} \begin{cases} \frac{1}{n}, & n \text{ odd}, \\ \frac{n}{n^2-1}, & n \text{ even} \end{cases} + \frac{\sqrt{3}}{4\sqrt{2\pi}} \frac{1}{n}. \quad (\text{B19})$$

An analogous calculation for I_{nn01}^K yields the $n \rightarrow \infty$ limit:

$$I_{nn01}^K \simeq \frac{1}{2\pi n} - \frac{2}{\pi^3} \begin{cases} \frac{1}{n}, & n \text{ odd}, \\ \frac{n(n^2-3)}{(n^2-1)^2}, & n \text{ even} \end{cases} - \frac{\sqrt{3}}{4\sqrt{2}\pi} \frac{1}{n}. \quad (\text{B20})$$

Combining Eqs. (B11)–(B13) and (B19)–(B20), we obtain the diffractive skew-scattering contribution in the large- n limit, up to an overall prefactor:

$$\frac{\sigma_{xy}^X + \sigma_{xy}^\Psi}{f(\epsilon_F, m)} \simeq \frac{\sqrt{3}}{2\sqrt{2}} + \frac{4}{\pi^2} \begin{cases} 1, & n \text{ odd}, \\ \frac{n^4}{(n^2-1)^2}, & n \text{ even} \end{cases}. \quad (\text{B21})$$

8. Case of sparse and strong impurities

In this appendix we deduce the contribution of the higher-order moment, Eq. (2b), of the disorder potential Eq. (16).

The contribution from "Mercedes star" diagram:

$$\sigma_{xy}^{\text{MC}} = \sigma_{xy}^Y + \sigma_{xy}^L = \frac{e^2}{h} (\beta v^3 p_0^{3n-4}).$$

$$\int_{\{\mathbf{p}_i\}} \left(\text{Tr}[\mathbf{G}_{\mathbf{p}_1}^A j_x(\mathbf{p}_1) \mathbf{G}_{\mathbf{p}_1}^R \mathbf{G}_{\mathbf{p}_3}^R \mathbf{G}_{\mathbf{p}_2}^R j_y(\mathbf{p}_2) \mathbf{G}_{\mathbf{p}_2}^A] + \right.$$

$$\left. \text{Tr}[\mathbf{G}_{\mathbf{p}_1}^A j_x(\mathbf{p}_1) \mathbf{G}_{\mathbf{p}_1}^R \mathbf{G}_{\mathbf{p}_2}^R j_y(\mathbf{p}_2) \mathbf{G}_{\mathbf{p}_2}^A \mathbf{G}_{\mathbf{p}_3}^A] \right).$$

This expression contains $\mathbf{p}_1$ and $\mathbf{p}_2$ integrals analogously to the vertex correction, Eq. (B7), there

$$\sigma_{xy}^{\text{MC}} = 0, \text{ for } n \neq 1.$$

This cancellation originates from the structure of the angular integrals and is not specific to the third moment: higher-order disorder correlators lead to analogous cancellations. Thus, all non-Gaussian disorder moments beyond the second do not generate a finite contribution to the anomalous Hall conductivity in multilayers.

Appendix C: Warped model

1. Necessary designations

In this section, we introduce all the necessary definitions which we use in this paper for the warped model, see Eqs.(17)–(23).

Firstly, let us define the Fermi momentum. Using the notation $\epsilon^2 - m^2 = v^2 p_0^{2n}$ we find

$$\xi^2 = m^2 + v^2 p^{2n} + 2vwp^{2n-3} \cos(3\phi) + w^2 p^{2n-6}. \quad (\text{C1})$$

Solving $\xi = \epsilon$ using $\bar{p} = p/p_0$, $\tilde{w} = w/vp_0^3$ leads to the condition

$$1 = \bar{p}^{2n} + \tilde{w} \bar{p}^{2n-3} \cos(3\phi) + \tilde{w}^2 \bar{p}^{2n-6}. \quad (\text{C2})$$

We cannot solve this equation exactly, but we can find an approximate expression for the ϕ dependent Fermi momentum $p_F(\phi) \equiv p_\phi$ in powers of $\tilde{w}$, Eq. (17). We remark that $\tilde{w}$ is a function of (ϵ, m) :

$$\tilde{w} = \frac{w}{v^{\frac{n-3}{n}} (\epsilon^2 - m^2)^{3/2n}} = \frac{\gamma_1^{\frac{3}{n}-1} \gamma_2}{m^{3/n}} \cdot \frac{1}{(\frac{\epsilon^2}{m^2} - 1)^{3/2n}}. \quad (\text{C3})$$

In the presence of warping we need a ϕ dependent replacement for the DOS

$$\rho_{n,\phi}(\epsilon) = \int_0^\infty \frac{dpp}{2\pi} \delta(\epsilon - |\vec{d}(\mathbf{p}, \phi)|), \quad (\text{C4})$$

which leads to Eqs. (18)–(19) of the main text.

2. Trilayer graphene in warped model

In this section, we provide calculations for trilayer graphene, i.e. $n = 3$ in warped case, for Eqs. (20a), (25), (27), (29) from the main text. We start from the effective Hamiltonian $H_3^{(w)}$:

$$H_3^{(w)} = \vec{d}(\mathbf{p}) \cdot \vec{\sigma} = \begin{pmatrix} m & d_- \\ d_+ & -m \end{pmatrix}, \quad (\text{C5})$$

with $d_- = d_{x-iy}(\mathbf{p}) = v\mathcal{P}^3 + w$; $d_z = m$.

a. Intrinsic contribution

Firstly, we consider intrinsic contribution:

$$\sigma_{xy}^{\text{w,int}}|_{n=3} = \frac{e^2}{h} \int \frac{dp^2}{(2\pi)^2} \text{Tr} \left(j_x(\mathbf{p}) G_{\mathbf{p}}^{R,(w)} j_y(\mathbf{p}) G_{\mathbf{p}}^{A,(w)} \right), \quad (\text{C6})$$

where $G_{\mathbf{p}}^{R,(w)} = [\epsilon - i0 - H_3^{(w)}]^{-1}$. After taking the trace and with using C_3 symmetry we deduce:

$$\sigma_{xy}^{\text{w,int}}|_{n=3} = -\frac{e^2}{h} \frac{m}{\epsilon} \frac{3}{2} \mathcal{J}_3^{(4)}. \quad (\text{C7})$$

b. Vertex correction

Before considering the non-crossing approximation we should consider vertex correction. It is a bit more convenient to use bare vertices

$$J_{\pm} = \frac{1}{2} (\partial_{p_x} H \pm i \partial_{p_y} H) \equiv \partial_{\pm} H, \quad (\text{C8})$$

which are

$$J_+ = \begin{pmatrix} 0 & \underbrace{3v\mathcal{P}^2}_{j_+} \\ 0 & 0 \end{pmatrix} = \partial_+ d_- \sigma_+; \quad J_- = \begin{pmatrix} 0 & 0 \\ \underbrace{3v\bar{\mathcal{P}}^2}_{j_-} & 0 \end{pmatrix} = \partial_- d_+ \sigma_-, \quad (\text{C9})$$

where $\sigma_{\pm} = [\sigma_x \pm i\sigma_y]/2$.

We will need

$$\mathcal{G}_{\mathbf{p}}^R = \frac{1}{\tilde{\epsilon}^2 - \tilde{m}^2 - \tilde{d}_x^2 - \tilde{d}_y^2}, \quad (\text{C10})$$

with

$$\begin{aligned} \tilde{\epsilon} &= \epsilon + i\Gamma; \quad \tilde{m} = m - i\Gamma_z; \\ \tilde{d}_x &= d_x - i\Gamma_x; \quad \tilde{d}_y = d_y - i\Gamma_y. \end{aligned}$$

Partial fraction decomposition leads to

$$\mathcal{G}_{\mathbf{p}}^R \mathcal{G}_{\mathbf{p}}^A = -\frac{\text{Im}\mathcal{G}_{\mathbf{p}}^R}{2(\Gamma\epsilon + \Gamma_1 d_x + \Gamma_2 d_y + \Gamma_3 m)} \simeq \quad (\text{C11})$$

$$\frac{\delta(p - p_{\phi})}{2\alpha f_{\phi}(p_{\phi})[\mathcal{I}_n^{(0)}(\epsilon^2 + m^2) + vp_0^3 d_x(\mathcal{I}_n^{(3)} + \tilde{w}\mathcal{J}_n^{(0)})]}. \quad (\text{C12})$$

We find the following first order correction to the vertex

$$\delta J_{\pm} = \frac{\epsilon}{3\rho_3(\epsilon)}\alpha \int (dp) j_{\pm} \mathcal{G}_{\mathbf{p}}^R \mathcal{G}_{\mathbf{p}}^A N_{\pm}(\mathbf{p}), \quad (\text{C13})$$

with

$$\begin{aligned} N_{\pm}(\mathbf{p}) &= (d_{\pm}^2 + (\Gamma_x \pm i\Gamma_y)^2)\sigma_{\mp} + [(\epsilon \mp i\Gamma_z)^2 - (m \pm i\Gamma)^2]\sigma_{\pm} + \\ &d_{\pm}(m \pm i\Gamma)\sigma_z \pm (i\Gamma_x \mp \Gamma_y)(\epsilon \mp i\Gamma_z)\sigma_z + \\ &d_{\pm}(\epsilon \mp i\Gamma_z)\sigma_0 \pm (i\Gamma_x \mp \Gamma_y)(m \pm i\Gamma)\sigma_0. \end{aligned}$$

Only terms which have the angular dependence $e^{i3l\phi}$ with $l \in \mathbb{Z}$ give the non-zero contribution. With keeping in mind that $d_{\pm} \sim e^{\pm i3\phi}, 1$; $d_{\pm}^2 \sim e^{\pm i6\phi}, e^{\pm i3\phi}, 1$; $j_{\pm} \sim e^{\mp i2\phi}$ we conclude that vertex correction for $n = 3$ is zero.

c. Non-crossing approximation

For the non-crossing approximation we should consider the following integral:

$$\begin{aligned} \sigma_{xy}^{\text{w, nc}}|^{n=3} &= -i\frac{e^2}{h} \int \frac{d^2 p}{(2\pi)^2} \text{Tr}[j_- \mathbf{G}^R j_+ \mathbf{G}^A - j_+ \mathbf{G}^R j_- \mathbf{G}^A] = \\ &\frac{e^2}{h} \int_0^{\infty} \frac{p dp}{2\pi} \int_0^{2\pi} \frac{d\phi}{2\pi} 36i(i0)v^2 p^4 \epsilon m \frac{\pi}{2(0)\epsilon^2} \frac{\delta(p - p_{\phi})}{6f_{\phi}(p_{\phi})|^{n=3}} = \\ &-\frac{e^2}{h} \frac{3\epsilon m \mathcal{I}_3^{(0)}}{\epsilon^2 - m^2} \int_0^{2\pi} \frac{d\phi}{2\pi} \frac{\bar{p}_{\phi}^5}{f_{\phi}(\bar{p}_{\phi})} \times \\ &\frac{1}{\mathcal{I}_3^{(0)} \frac{\epsilon^2 + m^2}{\epsilon^2 - m^2} + (\bar{p}_{\phi}^3 \cos(3\phi) + \tilde{w})(\mathcal{I}_3^{(3)} + \tilde{w}\mathcal{J}_3^{(0)})}. \end{aligned}$$

We can see if $\tilde{w} = 0$ we get $-\frac{e^2}{h} \frac{3\epsilon m \mathcal{I}_3^{(0)}}{\epsilon^2 - m^2} \frac{1}{\mathcal{I}_3^{(0)} \frac{\epsilon^2 + m^2}{\epsilon^2 - m^2}} = -\frac{e^2}{h} \frac{3\epsilon m}{\epsilon^2 + m^2}$, i.e. the answer without warping.

d. X diagram

As in the isotropic model, the X conductivity bubble correspond to:

$$\begin{aligned} \sigma_{xy}^X &= \frac{e^2}{h} \left(\frac{\epsilon}{n\rho_n(\epsilon)} \alpha \right)^2 \int_{\{\mathbf{p}_i\}} \text{Tr}[\mathbf{G}_{\mathbf{p}_1}^A j_x(\mathbf{p}_1) \mathbf{G}_{\mathbf{p}_1}^R \mathbf{G}_{\mathbf{p}_3}^R \mathbf{G}_{\mathbf{p}_2}^R j_y(\mathbf{p}_2) \mathbf{G}_{\mathbf{p}_2}^A \mathbf{G}_{\mathbf{p}_4}^A] = \\ &-\frac{e^2}{h} \left(\frac{\epsilon}{\rho_n(\epsilon)\pi} \right)^2 \int_{\{\mathbf{p}_i\}} \frac{(2\pi)^2 \delta(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_3 - \mathbf{p}_4) \Im \mathcal{G}_{\mathbf{p}_1} \Im \mathcal{G}_{\mathbf{p}_2} \Im \mathcal{G}_{\mathbf{p}_3} \Re \mathcal{G}_{\mathbf{p}_4}}{[(\epsilon^2 + m^2)\mathcal{I}_3^{(0)} + d_x(\mathbf{p}_1)vp_0^3(\mathcal{I}_3^{(3)} + \tilde{w}\mathcal{J}_3^{(0)})][(\epsilon^2 + m^2)\mathcal{I}_3^{(0)} + d_x(\mathbf{p}_2)vp_0^3(\mathcal{I}_3^{(3)} + \tilde{w}\mathcal{J}_3^{(0)})]} \\ &\left\{ j_+(\mathbf{p}_2)j_-(\mathbf{p}_1) \left(\text{Tr}[N_-(\mathbf{p}_1)(\epsilon + H)_{\mathbf{p}_3} N_+(\mathbf{p}_2)(\epsilon + H)_{\mathbf{p}_4}] - \text{Tr}[N_-(\mathbf{p}_1)(\epsilon + H)_{\mathbf{p}_4} N_+(\mathbf{p}_2)(\epsilon + H)_{\mathbf{p}_3}] \right) - \right. \\ &\left. j_-(\mathbf{p}_2)j_+(\mathbf{p}_1) \left(\text{Tr}[N_-(\mathbf{p}_2)(\epsilon + H)_{\mathbf{p}_4} N_+(\mathbf{p}_1)(\epsilon + H)_{\mathbf{p}_3}] - \text{Tr}[N_-(\mathbf{p}_2)(\epsilon + H)_{\mathbf{p}_3} N_+(\mathbf{p}_1)(\epsilon + H)_{\mathbf{p}_4}] \right) \right\}, \quad (\text{C14}) \end{aligned}$$

where

$$N_{\pm}(\mathbf{p}) = d_{\pm}^2 \sigma_{\mp} + (\epsilon^2 - m^2)\sigma_{\pm} + d_{\pm} m \sigma_z + d_{\pm} \epsilon \sigma_0. \quad (\text{C15})$$

After taking momentum integrals, we obtain:

$$\sigma_{xy}^X|^{n=3} = -\frac{e^2}{h} \frac{\epsilon m}{\epsilon^2 - m^2} N_{X,3},$$

where

$$N_{X,3} = \frac{1}{6} \int \frac{d\phi_1}{2\pi} \frac{d\phi_2}{2\pi} \frac{d\phi_3}{2\pi} \frac{\bar{p}_{\phi_1} \bar{p}_{\phi_2} \bar{p}_{\phi_3}}{f_{\phi_1}(\bar{p}_{\phi_1}) f_{\phi_2}(\bar{p}_{\phi_2}) f_{\phi_3}(\bar{p}_{\phi_3})} \frac{j_+(\bar{p}_{\phi_2}) j_-(\bar{p}_{\phi_1})}{1 - d_-(\bar{p}_{\phi_1}, \bar{p}_{\phi_2}, \bar{p}_{\phi_3}) d_+(\bar{p}_{\phi_1}, \bar{p}_{\phi_2}, \bar{p}_{\phi_3})} \cdot \frac{[d_-(\bar{p}_{\phi_3}) d_+(\bar{p}_{\phi_2}) - d_-(\bar{p}_{\phi_1}, \bar{p}_{\phi_2}, \bar{p}_{\phi_3}) d_+(\bar{p}_{\phi_2}) + d_-(\bar{p}_{\phi_1}) d_+(\bar{p}_{\phi_3}) - d_-(\bar{p}_{\phi_1}) d_+(\bar{p}_{\phi_2}, \bar{p}_{\phi_3})] \cdot (d_-(\bar{p}_{\phi_1}) d_+(\bar{p}_{\phi_2}) - 1)}{[\frac{(\epsilon^2+m^2)}{v^2 p_0^6} \mathcal{I}_3^{(0)} + d_x(\bar{p}_{\phi_1})(\mathcal{I}_3^{(3)} + \tilde{w} \mathcal{J}_3^{(0)})][\frac{(\epsilon^2+m^2)}{v^2 p_0^6} \mathcal{I}_3^{(0)} + d_x(\bar{p}_{\phi_2})(\mathcal{I}_3^{(3)} + \tilde{w} \mathcal{J}_3^{(0)})]}, \quad (C16)$$

with

e. Ψ diagram

$$d_{\pm}(\bar{p}_{\phi_1}, \bar{p}_{\phi_2}, \bar{p}_{\phi_3}) = (\bar{p}_{\phi_1} e^{\pm i\phi_1} + \bar{p}_{\phi_2} e^{\pm i\phi_2} - \bar{p}_{\phi_3} e^{\pm i\phi_3})^3 + \tilde{w}$$

The contribution from Ψ diagram can be written as

$$\sigma_{xy}^{\Psi} = -\frac{e^2}{h} \left(\frac{\epsilon}{\rho_n(\epsilon)\pi} \right)^2 \int_{\{\mathbf{p}_i\}} \frac{(2\pi)^2 \delta(\mathbf{p}_1 - \mathbf{p}_2 - \mathbf{p}_3 + \mathbf{p}_4) \Im \mathcal{G}_{\mathbf{p}_1} \Im \mathcal{G}_{\mathbf{p}_2} (\Im \mathcal{G}_{\mathbf{p}_4} \Re \mathcal{G}_{\mathbf{p}_3} + \Im \mathcal{G}_{\mathbf{p}_3} \Re \mathcal{G}_{\mathbf{p}_4})}{[(\epsilon^2 + m^2) \mathcal{I}_3^{(0)} + d_x(\mathbf{p}_1) v p_0^3 (\mathcal{I}_3^{(3)} + \tilde{w} \mathcal{J}_3^{(0)})][(\epsilon^2 + m^2) \mathcal{I}_3^{(0)} + d_x(\mathbf{p}_2) v p_0^3 (\mathcal{I}_3^{(3)} + \tilde{w} \mathcal{J}_3^{(0)})]} \times \\ \left\{ j_-(\mathbf{p}_1) j_+(\mathbf{p}_2) \cdot \text{Tr}[N_-(\mathbf{p}_1)(\epsilon + H)_{\mathbf{p}_3}(\epsilon + H)_{\mathbf{p}_4} N_+(\mathbf{p}_2) - N_-(\mathbf{p}_1) N_+(\mathbf{p}_2)(\epsilon + H)_{\mathbf{p}_4}(\epsilon + H)_{\mathbf{p}_3}] - \right. \\ \left. j_+(\mathbf{p}_1) j_-(\mathbf{p}_2) \cdot \text{Tr}[N_+(\mathbf{p}_1)(\epsilon + H)_{\mathbf{p}_3}(\epsilon + H)_{\mathbf{p}_4} N_-(\mathbf{p}_2) - N_+(\mathbf{p}_1) N_-(\mathbf{p}_2)(\epsilon + H)_{\mathbf{p}_4}(\epsilon + H)_{\mathbf{p}_3}] \right\}. \quad (C17)$$

After the same steps as for X diagram, we receive:

$$\sigma_{xy}^{\Psi}|^{n=3} = -\frac{e^2}{h} \frac{\epsilon m}{(\epsilon^2 - m^2)} \cdot N_{\Psi,3}, \quad (C18)$$

where

$$N_{\Psi,3} = \frac{1}{3} \int \frac{d\phi_1}{2\pi} \frac{d\phi_2}{2\pi} \frac{d\phi_3}{2\pi} \frac{\bar{p}_{\phi_1} \bar{p}_{\phi_2} \bar{p}_{\phi_3}}{f_{\phi_1}(\bar{p}_{\phi_1}) f_{\phi_2}(\bar{p}_{\phi_2}) f_{\phi_3}(\bar{p}_{\phi_3})} \frac{j_+(\bar{p}_{\phi_2}) j_-(\bar{p}_{\phi_1})}{1 - d_-(\bar{p}_{\phi_1}, \bar{p}_{\phi_2}, \bar{p}_{\phi_3}) d_+(\bar{p}_{\phi_1}, \bar{p}_{\phi_2}, \bar{p}_{\phi_3})} \cdot \frac{1}{[\frac{(\epsilon^2+m^2)}{\epsilon^2-m^2} \mathcal{I}_3^{(0)} + d_x(\bar{p}_{\phi_1})(\mathcal{I}_3^{(3)} + \tilde{w} \mathcal{J}_3^{(0)})][\frac{(\epsilon^2+m^2)}{v^2 p_0^6} \mathcal{I}_3^{(0)} + d_x(\bar{p}_{\phi_2})(\mathcal{I}_3^{(3)} + \tilde{w} \mathcal{J}_3^{(0)})]} \times \\ \left[d_-(\bar{p}_{\phi_1}) d_-(\bar{p}_{\phi_1}, \bar{p}_{\phi_2}, \bar{p}_{\phi_3}) d_+(\bar{p}_{\phi_2}) \cdot (d_+(\bar{p}_{\phi_2}) - d_+(\bar{p}_{\phi_3})) + d_-(\bar{p}_{\phi_1})^2 d_+(\bar{p}_{\phi_2}) \cdot (d_+(\bar{p}_{\phi_3}) - d_+(\bar{p}_{\phi_2})) + \right. \\ \left. d_-(\bar{p}_{\phi_1}) d_+(\bar{p}_{\phi_1}, \bar{p}_{\phi_2}, \bar{p}_{\phi_3}) \cdot (d_-(\bar{p}_{\phi_3}) d_+(\bar{p}_{\phi_2}) - 1) - (d_-(\bar{p}_{\phi_3}) d_+(\bar{p}_{\phi_2}) - 1) \right]. \quad (C19)$$

This concludes the derivation of Eqs. (29a), (29c) of the main text.

Hamiltonian $H_4^{(w)}$:

3. Tetralayer graphene in warped model

In this section we provide calculations for tetralayer graphene, i.e. $n = 4$ in the warped case, i.e. Eqs. (20b), (25), (28), (29) from the main text. We do the similar steps as for the trilayer case. We start from the effective

$$H_4^{(w)} = \vec{d}(\mathbf{p}) \cdot \vec{\sigma} = \begin{pmatrix} m & d_- \\ d_+ & -m \end{pmatrix}, \quad (C20)$$

with $d_- = d_+^\dagger = d_{x-iy}(\mathbf{p}) = v\mathcal{P}^4 + w\mathcal{P}$; $d_z = m$.

a. Intrinsic contribution

Initially, we consider an intrinsic contribution. Doing the same calculations as for trilayer case, see Appendix C2a, we obtain:

$$\begin{aligned} \sigma_{xy}^{\text{w, int}}|^{n=4} &= -\frac{e^2}{h} \frac{m}{\epsilon} \frac{1}{16\pi} \int_0^{2\pi} d\phi \frac{\bar{p}_\phi}{f_\phi(\bar{p}_\phi)} \times \\ &(16\bar{p}_\phi^6 + 4\tilde{w}\bar{p}_\phi^3 e^{3i\phi} + 4\tilde{w}\bar{p}_\phi^3 e^{-3i\phi} + \tilde{w}^2) = \\ &-\frac{e^2}{h} \frac{m}{\epsilon} \cdot \left(2\mathcal{I}_4^{(6)} + \tilde{w}\mathcal{I}_4^{(3)} + \frac{\tilde{w}^2}{8}\mathcal{I}_4^{(0)} \right). \end{aligned} \quad (\text{C21})$$

b. Vertex correction

For non-crossing approximation, we should deal with the vertex correction. For tetralayer graphene we have different definitions for currents:

$$\begin{aligned} J_+ &= \begin{pmatrix} 0 & \underbrace{nv\mathcal{P}^{n-1} + (n-3)w\mathcal{P}^{n-4}}_{j_+} \\ 0 & 0 \end{pmatrix} = \partial_+ d_- \sigma_+, \\ J_- &= \begin{pmatrix} 0 & 0 \\ \underbrace{nv\bar{\mathcal{P}}^{n-1} + (n-3)w\bar{\mathcal{P}}^{n-4}}_{j_-} & 0 \end{pmatrix} = \partial_- d_+ \sigma_-. \end{aligned}$$

We find the following first order correction to the vertex

$$\begin{aligned} \delta J_\pm &= \frac{\epsilon}{n\rho_n(\epsilon)} \alpha \int (dp) j_\pm \mathcal{G}_\mathbf{p}^R \mathcal{G}_\mathbf{p}^A N_\pm(\mathbf{p}), \\ N_\pm(\mathbf{p}) &= d_\pm^2 \sigma_\mp + [(\epsilon \mp i\Gamma_3)^2 - (m \pm i\Gamma)^2] \sigma_\pm + \\ &d_\pm(m \pm i\Gamma) \sigma_z + d_\pm \epsilon \sigma_0. \end{aligned} \quad (\text{C22})$$

For $n = 4$ we may thus only keep

$$\begin{aligned} j_\pm N_\pm &\doteq j_\pm [(\epsilon \mp i\Gamma_3)^2 - (m \pm i\Gamma)^2] \sigma_\pm \simeq \\ j_\pm [\epsilon^2 - m^2 \mp 2i(\epsilon\Gamma_3 + m\Gamma)] &= j_\pm [\epsilon^2 - m^2 \mp 2i\epsilon m \frac{\pi\alpha}{n} \mathcal{I}_n^{(0)}]. \end{aligned} \quad (\text{C23})$$

This leads to

$$\begin{aligned} \delta J_\pm &= \frac{\sigma_\pm [\epsilon^2 - m^2 \mp 2i\epsilon m \frac{\pi\alpha}{n} \mathcal{I}_n^{(0)}]}{2(\epsilon^2 + m^2) \rho_n \mathcal{I}_n^{(0)}} \times \int (dp) j_\pm \delta(\epsilon - \xi) = \\ \sigma_\pm \left[\frac{\epsilon^2 - m^2}{2(\epsilon^2 + m^2)} \mp i \frac{\epsilon m}{\epsilon^2 + m^2} \frac{\pi\alpha}{n} \mathcal{I}_n^{(0)} \right] &\times \left[nv p_0^{n-1} \frac{\mathcal{I}_n^{(3)}}{\mathcal{I}_n^{(0)}} + w \right]. \end{aligned}$$

For $n = 4$ we can use that $\delta J_\pm \propto \sigma_\pm$, so the matrix structure is unchanged. It will be readily clear that this matrix structure is also preserved at higher orders. We can just generally write $\mathbf{J}_\pm = \mathbf{j}_\pm \sigma_\pm$

$$\mathbf{j}_\pm = \sum_{k=0}^{\infty} j_\pm^{(k)},$$

with $j_\pm^{(0)} = j_\pm, j_\pm^{(1)} = \delta j_\pm$. We see that for $k \geq 2$

$$j_\pm^{(k)} = j_\pm^{(k-1)} \frac{\epsilon}{n\rho_n(\epsilon)} \alpha \int (dp) \mathcal{G}_\mathbf{p}^R \mathcal{G}_\mathbf{p}^A N_\pm(\mathbf{p}). \quad (\text{C24})$$

Using the previously derived expression for $N_\pm(\mathbf{p})$ we see that only the terms $\sigma_\pm$ survive angular integrals. This leads to

$$\begin{aligned} j_\pm^{(k)} &= j_\pm^{(k-1)} \frac{\sigma_\pm [\epsilon^2 - m^2 \mp 2i\epsilon m \frac{\pi\alpha}{n} \mathcal{I}_n^{(0)}]}{2(\epsilon^2 + m^2) \rho_n \mathcal{I}_n^{(0)}} \times \int (dp) \delta(\epsilon - \xi) = \\ j_\pm^{(k-1)} \left[\underbrace{\frac{\epsilon^2 - m^2}{2(\epsilon^2 + m^2)}}_{=A} \pm i \underbrace{\frac{-\epsilon m}{\epsilon^2 + m^2} \frac{\pi\alpha}{n} \mathcal{I}_n^{(0)}}_{=B} \right] &= \\ \left[4vp_0^3 \frac{\mathcal{I}_4^{(3)}}{\mathcal{I}_4^{(0)}} + w \right] [A \pm iB]^k. \end{aligned} \quad (\text{C25})$$

We thus find in general that

$$\begin{aligned} \mathbf{j}_\pm &\simeq [4vp_0^3 \bar{p}^3 e^{\mp i3\phi} + w] + \left[4vp_0^3 \frac{\mathcal{I}_4^{(3)}}{\mathcal{I}_4^{(0)}} + w \right] \frac{A}{1-A} \pm \\ i \left[4vp_0^3 \frac{\mathcal{I}_4^{(3)}}{\mathcal{I}_4^{(0)}} + w \right] \frac{B}{(1-A)^2} &= j_\pm + \delta \mathbf{j}' \pm i \delta \mathbf{j}''. \end{aligned} \quad (\text{C26})$$

c. Non-crossing approximation

For the non-crossing approximation we should consider the following integral:

$$\sigma_{xy}^{\text{nc}} = -i \frac{e^2}{h} \int \frac{d^2 p}{(2\pi)^2} \text{Tr}[\mathbf{J}_- \mathbf{G}^R J_+ \mathbf{G}^A - \mathbf{J}_+ \mathbf{G}^R J_- \mathbf{G}^A],$$

where we put all the vertex correction only to the left vertex. Then

$$\begin{aligned} \sigma_{xy}^{\text{w, nc}}|^{n=4} &= -i \frac{e^2}{h} \int (dp) \mathcal{G}_\mathbf{p}^R \mathcal{G}_\mathbf{p}^A \times \\ \{\mathbf{j}_- j_+ \text{tr}[N_-(\mathbf{p}) \sigma_+] - \mathbf{j}_+ j_- \text{tr}[N_+(\mathbf{p}) \sigma_-]\} &= -\frac{e^2}{h} \frac{4m\epsilon}{\epsilon^2 + m^2} \times \\ \left\{ \left[\frac{\mathcal{I}_4^{(6)}}{\mathcal{I}_4^{(0)}} - \frac{[\mathcal{I}_4^{(3)}]^2}{[\mathcal{I}_4^{(0)}]^2} \right] + \frac{4(\epsilon^2 + m^2)^2}{(\epsilon^2 + 3m^2)^2} \left[\frac{\mathcal{I}_4^{(3)}}{\mathcal{I}_4^{(0)}} + \tilde{w}/4 \right]^2 \right\}. \end{aligned} \quad (\text{C27})$$

d. X diagram

We should do the similar calculations as for trilayer case. We just notice that for $n = 4$:

$$\mathbf{J}_+(\mathbf{p}_2) \mathbf{J}_-(\mathbf{p}_1) = j_-(\mathbf{p}_1) j_+(\mathbf{p}_2) + \delta \mathbf{j}' (j_-(\mathbf{p}_1) + j_+(\mathbf{p}_2)) + \delta \mathbf{j}'^2.$$

We introduce the following notations for tetralayer graphene:

$$\begin{aligned} \delta \bar{j}' &= \left[4 \frac{\mathcal{I}_4^{(3)}}{\mathcal{I}_4^{(0)}} + \tilde{w} \right] \frac{\epsilon^2 - m^2}{\epsilon^2 + 3m^2}; \\ d_\pm(\bar{p}_{\phi_1}, \bar{p}_{\phi_2}, \bar{p}_{\phi_3}) &= (\bar{p}_{\phi_1} e^{\pm i\phi_1} + \bar{p}_{\phi_2} e^{\pm i\phi_2} - \bar{p}_{\phi_3} e^{\pm i\phi_3})^4 + \\ \tilde{w}(\bar{p}_{\phi_1} e^{\pm i\phi_1} + \bar{p}_{\phi_2} e^{\pm i\phi_2} - \bar{p}_{\phi_3} e^{\pm i\phi_3}). \end{aligned} \quad (\text{C28})$$

And we can get

where

$$\begin{aligned} \sigma_{xy}^X|^{n=4} = & -\frac{e^2}{h} \frac{\epsilon m(\epsilon^2 - m^2)}{(\epsilon^2 + m^2)^2} \cdot N_{X,4}|^{1)} + \\ & -\frac{e^2}{h} \frac{\epsilon m(\epsilon^2 - m^2)^2}{(\epsilon^2 + 3m^2)(\epsilon^2 + m^2)^2} \cdot N_{X,4}|^{2)} + \\ & -\frac{e^2}{h} \frac{\epsilon m(\epsilon^2 - m^2)^3}{(\epsilon^2 + 3m^2)^2(\epsilon^2 + m^2)^2} \cdot N_{X,4}|^{3)}, \end{aligned} \quad (C29)$$

$$\begin{aligned} N_{X,4}|^{1)} = & \frac{1}{8(\mathcal{I}_4^{(0)})^2} \int \frac{d\phi_1}{2\pi} \frac{d\phi_2}{2\pi} \frac{d\phi_3}{2\pi} \frac{\bar{p}_{\phi_1} \bar{p}_{\phi_2} \bar{p}_{\phi_3}}{f_{\phi_1}(\bar{p}_{\phi_1}) f_{\phi_2}(\bar{p}_{\phi_2}) f_{\phi_3}(\bar{p}_{\phi_3})} \frac{j_+(\bar{p}_{\phi_2}) j_-(\bar{p}_{\phi_1})}{1 - d_-(\bar{p}_{\phi_1}, \bar{p}_{\phi_2}, \bar{p}_{\phi_3}) d_+(\bar{p}_{\phi_1}, \bar{p}_{\phi_2}, \bar{p}_{\phi_3})} \\ & (d_-(\bar{p}_{\phi_1}) d_+(\bar{p}_{\phi_2}) - 1) \cdot [d_-(\bar{p}_{\phi_3}) d_+(\bar{p}_{\phi_2}) - d_-(\bar{p}_{\phi_1}, \bar{p}_{\phi_2}, \bar{p}_{\phi_3}) d_+(\bar{p}_{\phi_2}) + d_-(\bar{p}_{\phi_1}) d_+(\bar{p}_{\phi_3}) - d_-(\bar{p}_{\phi_1}) d_+(\bar{p}_{\phi_1}, \bar{p}_{\phi_2}, \bar{p}_{\phi_3})], \end{aligned} \quad (C30)$$

$$\begin{aligned} N_{X,4}|^{2)} = & \frac{1}{8(\mathcal{I}_4^{(0)})^2} \cdot \left[4 \frac{\mathcal{I}_4^{(3)}}{\mathcal{I}_4^{(0)}} + \tilde{w} \right] \int \frac{d\phi_1}{2\pi} \frac{d\phi_2}{2\pi} \frac{d\phi_3}{2\pi} \frac{\bar{p}_{\phi_1} \bar{p}_{\phi_2} \bar{p}_{\phi_3}}{f_{\phi_1}(\bar{p}_{\phi_1}) f_{\phi_2}(\bar{p}_{\phi_2}) f_{\phi_3}(\bar{p}_{\phi_3})} \frac{j_+(\bar{p}_{\phi_2}) + j_-(\bar{p}_{\phi_1})}{1 - d_-(\bar{p}_{\phi_1}, \bar{p}_{\phi_2}, \bar{p}_{\phi_3}) d_+(\bar{p}_{\phi_1}, \bar{p}_{\phi_2}, \bar{p}_{\phi_3})} \\ & (d_-(\bar{p}_{\phi_1}) d_+(\bar{p}_{\phi_2}) - 1) \cdot [d_-(\bar{p}_{\phi_3}) d_+(\bar{p}_{\phi_2}) - d_-(\bar{p}_{\phi_1}, \bar{p}_{\phi_2}, \bar{p}_{\phi_3}) d_+(\bar{p}_{\phi_2}) + d_-(\bar{p}_{\phi_1}) d_+(\bar{p}_{\phi_3}) - d_-(\bar{p}_{\phi_1}) d_+(\bar{p}_{\phi_1}, \bar{p}_{\phi_2}, \bar{p}_{\phi_3})], \end{aligned} \quad (C31)$$

$$\begin{aligned} N_{X,4}|^{3)} = & \frac{1}{8(\mathcal{I}_4^{(0)})^2} \cdot \left[4 \frac{\mathcal{I}_4^{(3)}}{\mathcal{I}_4^{(0)}} + \tilde{w} \right]^2 \int \frac{d\phi_1}{2\pi} \frac{d\phi_2}{2\pi} \frac{d\phi_3}{2\pi} \frac{\bar{p}_{\phi_1} \bar{p}_{\phi_2} \bar{p}_{\phi_3}}{f_{\phi_1}(\bar{p}_{\phi_1}) f_{\phi_2}(\bar{p}_{\phi_2}) f_{\phi_3}(\bar{p}_{\phi_3})} \frac{1}{1 - d_-(\bar{p}_{\phi_1}, \bar{p}_{\phi_2}, \bar{p}_{\phi_3}) d_+(\bar{p}_{\phi_1}, \bar{p}_{\phi_2}, \bar{p}_{\phi_3})} \\ & (d_-(\bar{p}_{\phi_1}) d_+(\bar{p}_{\phi_2}) - 1) \cdot [d_-(\bar{p}_{\phi_3}) d_+(\bar{p}_{\phi_2}) - d_-(\bar{p}_{\phi_1}, \bar{p}_{\phi_2}, \bar{p}_{\phi_3}) d_+(\bar{p}_{\phi_2}) + d_-(\bar{p}_{\phi_1}) d_+(\bar{p}_{\phi_3}) - d_-(\bar{p}_{\phi_1}) d_+(\bar{p}_{\phi_1}, \bar{p}_{\phi_2}, \bar{p}_{\phi_3})]. \end{aligned} \quad (C32)$$

e. Ψ diagram

After the same steps as for X-diagram, we can deduce that:

$$\sigma_{xy}^\Psi|^{n=4} = -\frac{e^2}{h} \frac{\epsilon m(\epsilon^2 - m^2)}{(\epsilon^2 + m^2)^2} \cdot N_{\Psi,4}|^{1)} - \frac{e^2}{h} \frac{\epsilon m(\epsilon^2 - m^2)^2}{(\epsilon^2 + 3m^2)(\epsilon^2 + m^2)^2} \cdot N_{\Psi,4}|^{2)} - \frac{e^2}{h} \frac{\epsilon m(\epsilon^2 - m^2)^3}{(\epsilon^2 + 3m^2)^2(\epsilon^2 + m^2)^2} \cdot N_{\Psi,4}|^{3)}, \quad (C33)$$

$$\begin{aligned} N_{\Psi,4}|^{1)} = & \frac{1}{4(\mathcal{I}_4^{(0)})^2} \int \frac{d\phi_1}{2\pi} \frac{d\phi_2}{2\pi} \frac{d\phi_3}{2\pi} \frac{\bar{p}_{\phi_1} \bar{p}_{\phi_2} \bar{p}_{\phi_3}}{f_{\phi_1}(\bar{p}_{\phi_1}) f_{\phi_2}(\bar{p}_{\phi_2}) f_{\phi_3}(\bar{p}_{\phi_3})} \frac{1}{1 - d_-(\bar{p}_{\phi_1}, \bar{p}_{\phi_2}, \bar{p}_{\phi_3}) d_+(\bar{p}_{\phi_1}, \bar{p}_{\phi_2}, \bar{p}_{\phi_3})} \\ & j_+(\bar{p}_{\phi_2}) j_-(\bar{p}_{\phi_1}) \cdot [d_-(\bar{p}_{\phi_1}) d_-(\bar{p}_{\phi_1}, \bar{p}_{\phi_2}, \bar{p}_{\phi_3}) d_+(\bar{p}_{\phi_2}) \cdot (d_+(\bar{p}_{\phi_2}) - d_+(\bar{p}_{\phi_3})) + d_-(\bar{p}_{\phi_1})^2 d_+(\bar{p}_{\phi_2}) \cdot (d_+(\bar{p}_{\phi_3}) - d_+(\bar{p}_{\phi_2})) + \\ & d_-(\bar{p}_{\phi_1}) d_+(\bar{p}_{\phi_1}, \bar{p}_{\phi_2}, \bar{p}_{\phi_3}) \cdot (d_-(\bar{p}_{\phi_3}) d_+(\bar{p}_{\phi_2}) - 1) - (d_-(\bar{p}_{\phi_3}) d_+(\bar{p}_{\phi_2}) - 1)], \end{aligned} \quad (C34)$$

$$\begin{aligned} N_{\Psi,4}|^{2)} = & \frac{1}{4(\mathcal{I}_4^{(0)})^2} \cdot \left[4 \frac{\mathcal{I}_4^{(3)}}{\mathcal{I}_4^{(0)}} + \tilde{w} \right] \int \frac{d\phi_1}{2\pi} \frac{d\phi_2}{2\pi} \frac{d\phi_3}{2\pi} \frac{\bar{p}_{\phi_1} \bar{p}_{\phi_2} \bar{p}_{\phi_3}}{f_{\phi_1}(\bar{p}_{\phi_1}) f_{\phi_2}(\bar{p}_{\phi_2}) f_{\phi_3}(\bar{p}_{\phi_3})} \frac{1}{1 - d_-(\bar{p}_{\phi_1}, \bar{p}_{\phi_2}, \bar{p}_{\phi_3}) d_+(\bar{p}_{\phi_1}, \bar{p}_{\phi_2}, \bar{p}_{\phi_3})} \\ & (j_+(\bar{p}_{\phi_2}) + j_-(\bar{p}_{\phi_1})) \cdot [d_-(\bar{p}_{\phi_1}) d_-(\bar{p}_{\phi_1}, \bar{p}_{\phi_2}, \bar{p}_{\phi_3}) d_+(\bar{p}_{\phi_2}) \cdot (d_+(\bar{p}_{\phi_2}) - d_+(\bar{p}_{\phi_3})) + d_-(\bar{p}_{\phi_1})^2 d_+(\bar{p}_{\phi_2}) \cdot (d_+(\bar{p}_{\phi_3}) - d_+(\bar{p}_{\phi_2})) + \\ & d_-(\bar{p}_{\phi_1}) d_+(\bar{p}_{\phi_1}, \bar{p}_{\phi_2}, \bar{p}_{\phi_3}) \cdot (d_-(\bar{p}_{\phi_3}) d_+(\bar{p}_{\phi_2}) - 1) - (d_-(\bar{p}_{\phi_3}) d_+(\bar{p}_{\phi_2}) - 1)], \end{aligned} \quad (C35)$$

$$\begin{aligned}
N_{\Psi,4|3)} &= \frac{1}{4\left(\mathcal{I}_4^{(0)}\right)^2} \cdot \left[4\frac{\mathcal{I}_4^{(3)}}{\mathcal{I}_4^{(0)}} + \tilde{w} \right]^2 \int \frac{d\phi_1}{2\pi} \frac{d\phi_2}{2\pi} \frac{d\phi_3}{2\pi} \frac{\bar{p}_{\phi_1}\bar{p}_{\phi_2}\bar{p}_{\phi_3}}{f_{\phi_1}(\bar{p}_{\phi_1})f_{\phi_2}(\bar{p}_{\phi_2})f_{\phi_3}(\bar{p}_{\phi_3})} \frac{1}{1 - d_-(\bar{p}_{\phi_1}, \bar{p}_{\phi_2}, \bar{p}_{\phi_3})d_+(\bar{p}_{\phi_1}, \bar{p}_{\phi_2}, \bar{p}_{\phi_3})} \\
&\left[d_-(\bar{p}_{\phi_1})d_-(\bar{p}_{\phi_1}, \bar{p}_{\phi_2}, \bar{p}_{\phi_3})d_+(\bar{p}_{\phi_2}) \cdot (d_+(\bar{p}_{\phi_2}) - d_+(\bar{p}_{\phi_3})) + d_-(\bar{p}_{\phi_1})^2 d_+(\bar{p}_{\phi_2}) \cdot (d_+(\bar{p}_{\phi_3}) - d_+(\bar{p}_{\phi_2})) + \right. \\
&\left. d_-(\bar{p}_{\phi_1})d_+(\bar{p}_{\phi_1}, \bar{p}_{\phi_2}, \bar{p}_{\phi_3}) \cdot (d_-(\bar{p}_{\phi_3})d_+(\bar{p}_{\phi_2}) - 1) - (d_-(\bar{p}_{\phi_3})d_+(\bar{p}_{\phi_2}) - 1) \right]. \tag{C36}
\end{aligned}$$
