NON-LOOSE LEGENDRIAN HOPF LINKS IN LENS SPACES

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ABSTRACT. We give a complete classification of non-loose Legendrian Hopf links in $L(p,q)$ generalizing a result of the author with Geiges and Onaran [CGO25]. The classification is for non-loose Hopf links for both zero and non-zero Giroux torsion in their complement. We also give an explicit algorithm for the contact surgery diagrams for all these Legendrian representatives with no Giroux torsion in their complement.

1. INTRODUCTION

In recent years, there has been extensive studies on Legendrian and transverse knots in overtwisted contact manifolds with tight complements, known as non-loose knots (also known as exceptional). Most of these results are about partial and complete classification of certain knot types [EF09, GO20, Mat22, EMM22]. The author along with Etnyre, Min, and Mukherjee also proved the existence of these knots in general contact 3 manifolds and studied their behavior under cabling [CEMM25]. These knots are extremely important in understanding certain questions in contact topology, for example they can be used to construct and classify tight contact structures on manifolds obtained by surgeries. Although there has been some progress in understanding non-loose knots, results concerning non-loose links remain scarce due to the lack of techniques and tools. The only link type that has been studied is the Hopf link. Geiges–Onaran gave the first classification of non-loose Hopf links in S^3 in all contact structures [GO19] and later in [CGO25], the author together with Geiges and Onaran proved a classification result for Hopf links in $L(p,1)$ with any contact structure. Note that, this is also the first Legendrian classification of a link type in a manifold other than S^3. Here by Hopf links we mean the core of the two Heegaard tori of $L(p,1)$. In [CGO25], we used the classification of tight contact structures on $T^2 \times I$ [Gir00, Hon00] to give an upper bound on the number of non-loose representatives and then explicitly found the contact surgery diagrams for those representatives. It turns out that the same technique becomes lot complicated when $q > 1$. In this paper, we used convex surface theory to extend the classification result in every $L(p,q)$. This method also allowed us to fully understand how the link components are related via stabilizations thus completing the classification. Though we did not use the contact surgery diagrams in our proof, we gave an algorithm on how to extract the contact surgery diagrams of the Legendrian representatives of the Hopf links. This algorithm shows a beautiful connection of the geometry of the Farey graph and contact surgeries.

While in [CGO25] we only proved a classification for Hopf links having zero Giroux torsion in their complement (these are also known as strongly exceptional), in this paper we expand the classification to include Hopf links with Giroux torsion as well. All the classification mentioned in this paper are up to coarse equivalence.

By Hopf link we mean the core of the two Heegaaard tori of $L(p,q)$. Figure 1 shows the smooth Hopf link in $L(p,q)$. In this paper by Hopf link we also refer to the positive Hopf link (in the sense of [CGO25]) unless otherwise stated. A classification of negative Hopf links will follow similarly after changing the orientation of one of the components. First we recall, the smooth classification of rational unknots in lens spaces. Let K_i denote the cores of the two Heegaard tori of $L(p,q)$.

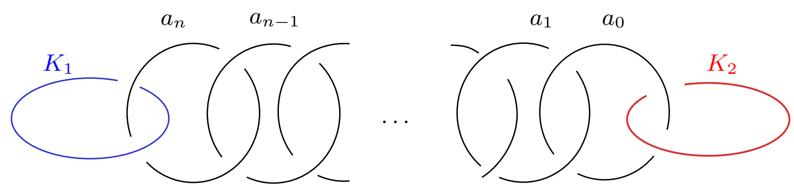

FIGURE 1. Hopf link in $L(p,q)$ where $-p/q = [a_0, a_1 \cdots, a_n]$

$$\text{rational unknots in } L(p,q) = \begin{Bmatrix} K_1 & p = 2 \\ K_1, -K_1 & p \neq 2,\ q \equiv \pm 1 \pmod p \\ K_1, -K_1, K_2, -K_2 & \text{otherwise} \end{Bmatrix}$$

We begin by mentioning some basic notions of classification of Legendrian knots in a contact manifold. Given a knot type K in an overtwisted manifold (M, ξ), we denote the coarse equivalence class of its Legendrian representatives by $\mathcal{L}(K)$. Consider the map $\Phi\colon \mathcal{L}(K) \to \mathbb{Z}^2$ that sends a null-homologous Legendrian knot L to $(\mathrm{rot}(L), \mathrm{tb}(L))$ where tb and rot denote the Thurston–Bennequin invariant and rotation number of the Legendrian knot. The image of ϕ along with the number of elements that has been sent to a single point is known as the *mountain range* of K. If K is rationally null-homologous then we replace tb and rot by $\mathrm{tb}_{\mathbb{Q}}$ and $\mathrm{rot}_{\mathbb{Q}}$, the rational Thurston–Bennequin invariant and the rational rotation number and consider a similar map $\Phi\colon \mathcal{L}(K) \to \mathbb{Q}^2$ that sends L to $(\mathrm{rot}_{\mathbb{Q}}(L), \mathrm{tb}_{\mathbb{Q}}(L))$.

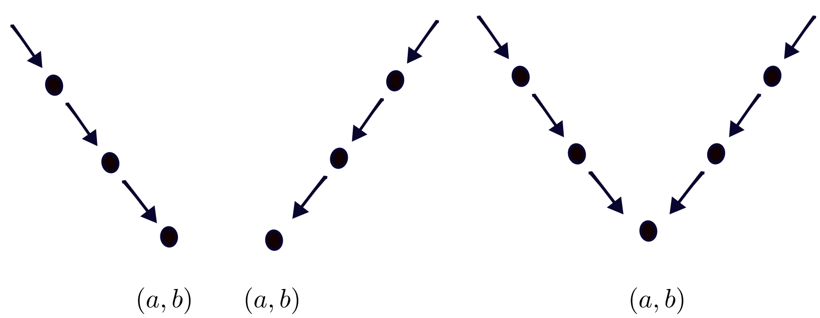

FIGURE 2. (a) A backward slash and a forward slash based at (a,b). (b) On the right we see a V based at (a,b).

We say that a mountain range for a knot contains a *non-loose* V based at (a,b) if the image of Φ contains non-loose Legendrian knots $L^0, L^i_\pm$ for $i \in \mathbb{N}$ such that

$$\mathrm{tb}(L^i_\pm) = b + i \text{ and } \mathrm{rot}(L^i_\pm) = a \pm i,$$
$$\mathrm{tb}(L^0) = b \text{ and } \mathrm{rot}(L^0) = a$$

and satisfy

$$S_\pm(L^{i+1}_\pm) = L^i_\pm, \text{ and } S_\pm(L^1_\pm) = L_0,$$
$$S_\mp(L^i) \text{ and } S_\pm(L^0) \text{ are loose.}$$

Here $S_\pm(L)$ denotes the positive (respectively negative) stabilization of L.

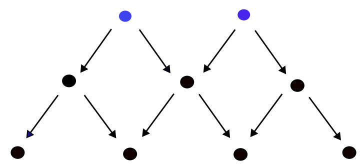

FIGURE 3. Part of a loose cone with two peaks.

One the other hand we call a V a *loose* V if the image of Φ contains loose Legendrian knots $L^0, L^i_\pm$ for $i \in \mathbb{N}$ which when included in a link gives a non-loose link, satisfies exact same relations are before, except that

$$S_\mp(L^i) \text{ and } S_\pm(L^0) \text{ will no longer be a part of the non-loose link.}$$

See the right side of Figure 2. We say the mountain range of L contains a *non-loose back slash based at* (a, b) if the image of Φ contains non-loose Legendrian knots L^i for $i \in \mathbb{Z}_{\geqslant 0}$ such that

$$\mathrm{tb}(L^i) = b + i \text{ and } \mathrm{rot}(L^i) = a - i$$

and satisfy

$$S_+(L^{i+1}) = L^i,$$

and

$$S_-(L^i) \text{ and } S_+(L^0) \text{ are loose.}$$

We say the it's a *loose back slash* if all the vertices are loose but could be included as a component of a non-loose link, relations are same as before and

$$S_-(L^i) \text{ and } S_+(L^0) \text{ are not part of the non-loose link.}$$

Similarly we say a mountain range for K contains a *non-loose forward slash based at* (a, b) if the image of Φ contains non-loose Legendrian knots L^i for $i \in \mathbb{Z}_{\geqslant 0}$ such that

$$\mathrm{tb}(L^i) = b + i \text{ and } \mathrm{rot}(L^i) = a + i$$

and satisfy

$$S_-(L^{i+1}) = L^i,$$

and

$$S_+(L^i) \text{ and } S_-(L^0) \text{ are loose.}$$

Check the middle picture of Figure 2. We can define a *loose forward slash* exactly as we defined the loose back slash.

Next we will define a *loose cone* of a Legendrian knot. A *cone* C of a Legendrian knot L is defined as follows:

$$C(L) = \{S^i_+ S^j_-(L) \text{ where } i, j \geqslant 0\}$$

that is the set which contains all possible stabilizations of the Legendrian L.

We call C a loose cone if every vertex in the cone is a loose knot but can be included in a non-loose link. We call a cone *double peaked* if there are two Legendrian knots L_+ and L_- at the peak such that $\mathrm{tb}(L+) = \mathrm{tb}(L_-), \mathrm{rot}(L_+) = \mathrm{rot}(L_-) + 2$ and $S_+(L_-) = S_-(L_+)$. See Figure 3.

We start by stating our result for $L(p, 1)$. Note that, in $L(p, 1)$ there are only two rational unknots K_1 and $-K_1$. Thus we will have two rational positive Hopf links $K_1 \sqcup -K_1$ and $-K_1 \sqcup K_1$. We will state our result for $K_1 \sqcup -K_1$. The classification for the other one can be obtained by interchanging the components.

Theorem 1.1. *Suppose $L_1 \sqcup L_2$ denote the positive Hopf-link in $L(p, 1)$. The non-loose realizations of $L_1 \sqcup L_2$ (up to switching the components) are as follows:*

(1) *(loose–non-loose pairing): In this case the Thurston– Bennequin invariants for the pairs are given by $\mathrm{tb}_{\mathbb{Q}}(L_1) = -(k_1 + 1) + \frac{1}{p}$ and $\mathrm{tb}_{\mathbb{Q}}(L_2) = k_2 + \frac{1}{p}$. Fixing L_1, the non-loose realizations of L_2 forms one non-loose V based at $(0, \frac{1}{p})$ and $p - 1$ non-loose Vs based at $(-1 + \frac{2m}{p}, 1 + \frac{1}{p})$ for $m = 1, 2, \cdots, p-1$. The based vertex of the first V (corresponding to $k_2 = 0$) pairs with each loose L_1 that forms a loose cone with two peaks at $(-1, -1+\frac{1}{p})$ and $(1, -1+\frac{1}{p})$ to give $|k_1 - 2|$ non-loose realizations. On the other hand, each of the vertex of the non-loose Vs with $k_2 \geqslant 1$ pairs with a loose cone of L_1 based at $(-1 + \frac{2m}{p}, -1 + \frac{1}{p})$ to give non-loose realizations. The first non-loose V lives in Euler class 0 and the other $p - 1$ non-loose Vs live in Euler class $-p + 2m$ for $m = 1, 2, \cdots, p-1$.*

(2) *(non-loose and non-loose pairing) There exists a unique non-loose Hopf link with both components being non-loose. The classical invariants of the components are given by $(0, \frac{1}{p})$ and $(0, \frac{1}{p})$. The contact structure has Euler class 0. Fixing L_2, the positive stabilization of L_1 is the peak of the loose cone of L_1 from (1).*

(3) *(loose and loose pairing) The Thurston–Bennequin invariants in this case are given by $\mathrm{tb}_{\mathbb{Q}}(L_i) = k_i + \frac{1}{p}$. The rotation numbers of the individual components are given in Table 2. The all possible non-loose realizations are: Fixing the loose unknot L_2 with*

   (a) *$k_2 = 0$, the (loose) mountain range for L_1 is given by: 1 loose back slash and 1 loose forward slash based at $(-\frac{p+2}{p}, 1 + \frac{1}{p})$ and $(\frac{p+2}{p}, 1 + \frac{1}{p})$ respectively and 1 loose V based at $(0, 2 + \frac{1}{p})$. The forward and back slashes live in Euler class 2 and the V lives in Euler class 0.*

   (b) *$k_2 = 1$, the (loose) mountain range for L_1 is given by: 1 loose back and 1 loose forward slash based at $(-\frac{2}{p}, \frac{1}{p})$ and $(\frac{2}{p}, \frac{1}{p})$ and $p + 1$ loose Vs based at $(-\frac{p+2-2m}{p}, 1 + \frac{1}{p})$ for $m = 1, 2, \cdots, p+1$. The forward and backward slash live in Euler class ± 2 and the Vs live in Euler class $-(p + 2 - 2m)$ for $m = 1, 2, \cdots p + 1$.*

   (c) *$k_2 = 2$, the (loose) mountain range for L_1 is given by: 1 loose back slash and 1 loose forward slash based at $(-\frac{2}{p}, \frac{1}{p})$ and $(\frac{2}{p}, \frac{1}{p})$ respectively and 1 loose V based at $(0, \frac{1}{p})$. Moreover, there are $2p$ loose Vs based at $(-\frac{p+1-2m}{p} \mp \frac{1}{p}, 1 + \frac{1}{p})$ for $m = 1, 2, \cdots, p$. The forward and back slashes live in Euler class ± 2 and the first V lives in Euler class 0. The later $2p$ Vs live in Euler class $-(p + 1 - 2m) \mp 1$.*

   (d) *$k_2 > 2$, the (loose) mountain range for L_1 is given by: 2 loose back slashes based at $(-\frac{2}{p}, \frac{1}{p}), (0, \frac{1}{p})$, 2 loose forward slashes based at $(\frac{2}{p}, \frac{1}{p}), (0, \frac{1}{p})$ and $2p$ loose Vs based at $(-\frac{p+1-2m\mp 1}{p}, 1 + \frac{1}{p})$ where $m = 1, 2, \cdots p$. The 2 back slashes live in Euler class -2 and 0, the two forward slashes live in Euler class 2 and 0 and the $2p$ Vs live in Euler class $-(p+1-2m) \mp 1$ for $m = 1, 2, \cdots p$.*

   Fixing loose L_1, we will see an exactly same mountain range for L_2 as well.

Remark 1.2. In the above statement, some of the candidates will have same Euler classes. But those have different d_3 invariants as shown in [CGO25]. So they live in distinct overtwisted contact structures. Also, pairwise they have distinct rotation numbers.

Remark 1.3. The above theorem recovers the result of [CGO25]. Moreover, it also tells us how the individual components of the non-loose representatives are related via stabilizations. Note that, all of the candidates from above have zero Giroux torsion in their complement.

Next we give a complete classification of non-loose Hopf links in $L(2n + 1, 2)$. Here we need to consider both unknots K_1 and K_2, and $K_1 \sqcup -K_2$ will give us a positive Hopf link. For a negative Hopf link, one needs to switch the orientation of K_2 and thus the rotation numbers L_2 (L_i denote the Legendrian representatives of K_i). In the mountain range for L_2 one needs to switch the forward and the back slashes as well. In the following theorem L_2 is oriented opposite of L_1.

Theorem 1.4. *The non-loose realizations of* $L_1 \sqcup L_2$ *in* $L(2n+1,2)$ *with* $\mathrm{tb}_{\mathbb{Q}}(L_1) = k_1 + \frac{n+1}{2n+1}$ *and* $\mathrm{tb}_{\mathbb{Q}}(L_2) = k_2 + \frac{2}{2n+1}$ *are as follows:*

(1) *(loose/non-loose pairing) For* $k_1 < 0, k_2 \geqslant 0$*: Fixing loose* L_1*, the non-loose realizations of* L_2 *form one forward slash based at* $(\frac{1}{2n+1}, \frac{2}{2n+1})$*, one back slash based at* $(-\frac{1}{2n+1}, \frac{2}{2n+1})$ *and* n V*s based at* $(\frac{2(-n-1+2m)}{2n+1}, 1+\frac{2}{2n+1})$ *for* $m = 1, \cdots, n$*. The forward and back slashes live in Euler class* $\pm(n+1)$ *and the* V*s live in Euler class* $(-n-1+2m)$ *for* $m = 1, 2, \cdots n$*. Each of these non-loose candidates pairs with a loose* L_1 *cone based at* $(r, -1+\frac{n+1}{2n+1})$ *to give non-loose Hopf links. More detail about this pairing and* r *will be given in Section 3.*

(2) *(non-loose/loose pairing) For* $k_1 \geqslant 0, k_2 < 0$*: Fixing loose* L_2*, the non-loose realizations of* L_1 *form* 1 *forward slash based at* $(\frac{n}{2n+1}, \frac{n+1}{2n+1})$*, one back slash based at* $(-\frac{n}{2n+1}, \frac{n+1}{2n+1})$*,* $n-1$ V*s based at* $(\frac{(-n+2m_1)}{2n+1}, \frac{n+1}{2n+1})$ *where* $m_1 = 1, \cdots n-1$ *and* n V*s based at* $(\frac{(-n+1+2m_2)}{2n+1}, 1+\frac{n+1}{2n+1})$ *for* $m_2 = 0, 1, \cdots n-1$*. The forward and back slashes live in Euler class* $\pm n$*, the first* $n-1$ V*s live in Euler class* $-n+2m_1$ *for* $m_1 = 1, 2, \cdots n-1$ *and the later* n V*s live in Euler class* $-n+1+2m_2$ *for* $m_2 = 0, 1, \cdots n-1$*. Each of these non-loose candidates pairs with a loose* L_2 *cone based at* $(r, -1+\frac{2}{2n+1})$ *to give non-loose Hopf links. More detail on this pairing and* r *will be given in Section 3.*

(3) *(loose/loose pairing)* $k_1 \geqslant 0, k_2 \geqslant 0$*, both the components are loose.*
   - *Fixing* L_2*, the loose mountain range for* L_1 *follows:*
     (a) *For* $k_2 = 0$*, we have* 1 *loose forward and* 1 *loose back slash based at* $(\pm\frac{n+2}{2n+1}, \frac{n+1}{2n+1})$ *respectively and* 2 *loose* V *'s based at* $(\mp\frac{n+1}{2n+1}, 1+\frac{n+1}{2n+1})$*. The forward and back slashes live in Euler class* $\pm(n+2)$ *and the* V*s live in Euler class* $\mp(n+1)$*.*
     (b) *For* $k_2 = 1$*, the loose mountain range of* L_1 *is given by the same forward and back slash from before,* $n+1$ *loose* V *s based at* $(\frac{-n-2+2m_1}{2n+1}, \frac{n+1}{2n+1})$ *and* $n+2$ *loose* V*s based at* $(\frac{-n-1+2m_2}{2n+1}, 1+\frac{n+1}{2n+1})$ *where* $m_1 = 0, \cdots n$ *and* $m_2 = 0, 1, \cdots n+1$*. The* $n+1$ V*s live in Euler class* $(-n-2+2m_1)$ *and the* $n+2$ V*s from above live in Euler class* $(-n-1+2m_2)$*.*
     (c) *Finally for* $k_2 > 1$*, we have* 2 *forward and* 2 *back slashes based at* $(\frac{n+1}{2n+1} \mp \frac{1}{2n+1}, \frac{n+1}{2n+1})$ *and* $(-\frac{n+1}{2n+1} \mp \frac{1}{2n+1}, \frac{n+1}{2n+1})$ *respectively,* $2n$ V*'s based at* $(\frac{-n-1+2m_1}{2n+1} \mp \frac{1}{2n+1}, \frac{n+1}{2n+1})$ *and* $2(n+1)$ V*s based at* $(\frac{-n+2m_2}{2n+1} \mp \frac{1}{2n+1}, 1+\frac{n+1}{2n+1})$ *where* $m_1 = 1, 2, \cdots n$ *and* $m_2 = 0, 1, \cdots n$*. The two forward slashes live in Euler classes* $n, n+2$*, the two back slashes live in Euler class* $-n, -n-2$*, the* $2n$ V*s live in Euler class* $-n-1+2m_1 \mp 1$ *and the* $2(n+1)$ V*s live in Euler class* $-n+2m_2 \mp 1$*.*

     For fixed L_2 *with* $k_2 \geqslant 0$*, the loose mountain range of* L_1 *is given in Figure 7.*
   - *Fixing* L_1*, the loose mountain range for* L_2 *follows:*
     (a) *For* $k_1 = 0$*, there is* 1 *forward and* 1 *back slash based at* $(\pm\frac{3}{2n+1}, \frac{2}{2n+1})$ *respectively and* $n+1$ V*'s based at* $(\frac{2(-n-2+2m)}{2n+1}, 1+\frac{2}{2n+1})$ *where* $m = 1, 2, \cdots n+1$*. The forward and backward slashes live in Euler classes* $\pm(n+2)$ *and the* V*s live in* $(-n-2+2m)$ *for* $m = 1, 2, \cdots n+1$*.*
     (b) *For* $k_1 = 1$*, there is* 1 *forward and* 1 *backward slash from the previous case,* 2 V*'s based at* $(\pm\frac{1}{2n+1}, \frac{2}{2n+1})$ *and* $3n$ V*s based at* $(\frac{-2+2m_1}{2n+1} + \frac{2(-n-1+2m_2)}{2n+1})$ *where* $m_1 = 0, 1, 2$ *and* $m_2 = 1, \cdots n$*. The two* V*s live in Euler class* $\pm(1+n)$ *and the* $3n$ V*s live in Euler class* $(n+1)(-3+2m_1) + 2m_2$*.*
     (c) *Finally for* $k_1 > 1$*, there are two forward slashes based at* $(\frac{3}{2n+1}, \frac{2}{2n+1})$ *and at* $(\frac{1}{2n+1}, \frac{2}{2n+1})$*,* 2 *back slashes based at* $(-\frac{3}{2n+1}, \frac{2}{2n+1})$ *and at* $(-\frac{1}{2n+1}, \frac{2}{2n+1})$ *and* $4n$ V*s based at* $(\mp\frac{1}{2n+1} \mp \frac{1}{2n+1} + \frac{2(-n-1+2m)}{2n+1}, 1+\frac{2}{2n+1})$ *for* $m = 1, 2, \cdots n$*. The Euler class of*

the forward slashes are $n+1$, *for the back slashes are* $-n-1$, *for the* $4n$ V*s are* $\pm n \mp (n+1) + (-n-1+2m)$.

For fixed L_1 *with* $k_1 \geqslant 0$ *the loose mountain range of* K_2 *is given in Figure 6. The rotation numbers of the pairs are given in Table 3.*

Remark 1.5. Note that, in case (3) when L_1 is fixed at $k_1 > 2$ among the $4n$ loose Vs, $2n$ Vs have same Euler class, but they can be distinguished by their d_3 invariant. The d_3 invariants of these overtwisted contact structures can be easily computed from the explicit contact surgery diagrams given in Section 5. But as they can be all distinguished by their pairwise distinct rotation numbers we refrained ourselves from the algebraic calculations.

Next we give the general classification result for the positive Hopf link $L_1 \sqcup L_2$ in $L(p,q)$ for $q > 1$. L_1 and L_2 are oriented opposite as before. We will denote the negative continued fraction

$$a_0 - \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots - \cfrac{1}{a_n}}}}$$

for $a_i \leqslant -2$ by $[a_0, a_1, \cdots, a_n]$.

Theorem 1.6. *Suppose* (p,q) *is a pair of relatively prime integers with* $p > q > 1$. *Let* $-p/q = [a_0, a_1, \cdots a_n]$ *and* $n \geqslant 1$ *as* $q \neq 1$. *In* $L(p,q)$, *the non-loose Legendrian representatives of* $L_1 \sqcup L_2$ *are as follows:*

(1) *(loose-nonloose pairing) For fixed* L_1, *the non-loose realizations of* L_2 *form*

$$\begin{Bmatrix} 1, & n=1 \\ |a_2+1||a_3+1|\cdots|a_n+1|, & n \geqslant 2 \end{Bmatrix}$$

many non-loose forward slashes and the same number of non-loose back slashes based at $(r, \frac{q}{p})$, *and*

$$\begin{Bmatrix} |a_1+2|, & n=1 \\ |a_1+2||a_2+1|\cdots|a_n+1|, & n \geqslant 2 \end{Bmatrix}$$

many non-loose V*'s based at* $(r, \frac{q}{p})$. *Additionally, there are*

$$|a_0+1||a_1+1|\cdots|a_n+1|$$

V*'s based at* $(r, 1+\frac{q}{p})$. *Each of these pairs with a loose* L_1 *that belongs to a loose cone* C *based at* $(r, -1-\frac{p''}{p})$ *to give us the loose and non-loose pairings where* $1 \leqslant p'' \leqslant p$ *and* $p''q \equiv p-1 \pmod p$.

(2) *(nonloose-loose pairing) For fixed* L_2, *the non-loose realizations of* L_1 *form*

$$\begin{Bmatrix} 1, & n=1 \\ |a_0+1||a_1+1|\cdots|a_{n-2}+1|, & n \geqslant 2 \end{Bmatrix}$$

many non-loose forward slashes and the same number of non-loose back slashes based at $(r, \frac{p'}{p})$, *and*

$$\begin{Bmatrix} |a_0+2|, & n=1 \\ |a_0+1||a_1+1|\cdots|a_{n-1}+2|, & n \geqslant 2 \end{Bmatrix}$$

many non-loose V*'s based at* $(r, \frac{p'}{p})$. *Additionally, there are*

$$|a_0+1||a_1+1|\cdots|a_n+1|$$

V's based at $(r, 1+\frac{p'}{p})$. Each of these pairs with each loose L_1 that belongs to a loose cone C based at $(r, -1+\frac{q}{p})$ to give us the non-loose and loose pairings where $1 \leqslant p' \leqslant p$ and $pq' \equiv 1 \pmod p$.

(3) *(loose-loose pairings) The non-loose Hopf links with both components loose and having $\mathrm{tb}_{\mathbb{Q}}(L_1) = k_1 + \frac{p'}{p}$ and $\mathrm{tb}_{\mathbb{Q}}(L_2) = k_2 + \frac{q}{p}$ form the following mountain range:*

When we fix the loose L_1

(i) *For $k_1 = 0$, the loose Legendrian representatives of L_2 forms*

$$\left\{ \begin{array}{ll} 1, & n = 1, 2 \\ |a_2+1|\cdots|a_{n-2}+1||a_{n-1}|, & n \geqslant 3 \end{array} \right\}$$

many loose forward slashes and the same number of loose back slashes based at $(r, \frac{q}{p})$, and

$$\left\{ \begin{array}{ll} 0, & n = 1 \\ |a_1+1|, & n = 2 \\ |a_1+2||a_2+1|\cdots|a_{n-2}+1||a_{n-1}|, & n \geqslant 3 \end{array} \right\}$$

many loose V's based at $(r, \frac{q}{p})$.

Additionally, there are

$$\left\{ \begin{array}{ll} 2|a_0-1|, & n = 1 \\ |a_0+1||a_1+1|\cdots|a_{n-1}|, & n \geqslant 2 \end{array} \right\}$$

V's based at $(r, 1+\frac{q}{p})$.

(ii) *For $k_1 = 1$, the loose Legendrian representatives of L_2 forms*

$$\left\{ \begin{array}{ll} 1, & n = 1 \\ |a_2+1|\cdots|a_{n-1}+1||a_n-1|, & n \geqslant 2 \end{array} \right\}$$

many loose forward slashes and the same number of loose back slashes based at $(r, \frac{q}{p})$, and

$$\left\{ \begin{array}{ll} |a_1|, & n = 1 \\ |a_1+2||a_2+1|\cdots|a_{n-1}+1||a_n-1|, & n \geqslant 2 \end{array} \right\}$$

many loose V's based at $(r, \frac{q}{p})$. Additionally, there are

$$|a_0+1||a_1+1|\cdots|a_n-1|$$

V's based at $(r, 1+\frac{q}{p})$.

(iii) *For $k_1 \geqslant 2$, the loose Legendrian representatives of L_2 forms*

$$\left\{ \begin{array}{ll} 2, & n = 1 \\ 2|a_2+1|\cdots|a_{n-1}+1||a_n|, & n \geqslant 2 \end{array} \right\}$$

many loose forward slashes and the same number of loose back slashes based at $(r, \frac{q}{p})$, and

$$\left\{ \begin{array}{ll} 2|a_1+1|, & n = 1 \\ 2|a_1+2||a_2+1|\cdots|a_{n-1}+1||a_n|, & n \geqslant 2 \end{array} \right\}$$

many loose V's based at $(r, \frac{q}{p})$. Additionally, there are $2|a_0+1||a_1+1|\cdots|a_n|$ V's based at $(r, 1+\frac{q}{p})$.

An algorithm for r and the Euler classes will be given in Section 2. If we fix L_2, the mountain range is given in Figure 9 for $n \geqslant 2$ and in Figure 4 for $n = 1$.

Note that, all the above results are strongly exceptional as they have no Giroux torsion in their complement. The following theorem gives a complete classification of non-loose Hopf links with non-zero (convex) Giroux torsion.

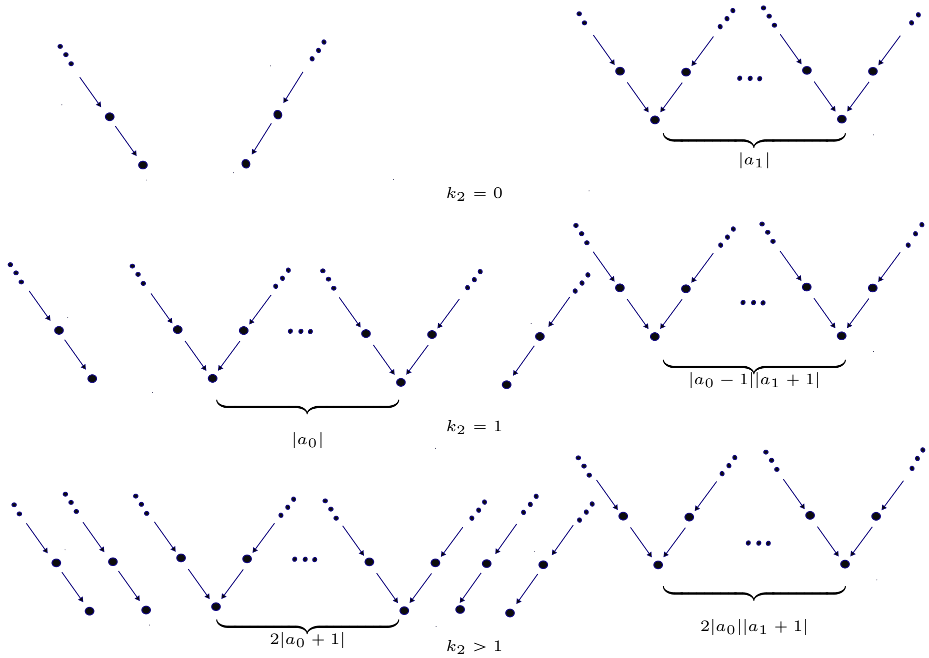

FIGURE 4. The loose mountain range of L_1 when we fix L_2 for $-p/q = [a_0, a_1]$.

Theorem 1.7. *Suppose $L_1 \sqcup L_2$ be the positive Hopf link in $L(p,q)$ where $p > q \geqslant 1$. The classification of exceptional $L_1 \sqcup L_2$ with π Giroux torsion in its complement is as follows:*

(1) For $q = 1$, for each choice of integers $(k_1, k_2) \neq 0$ and a natural number m there is exactly a pair of non-loose Legendrian (positive) Hopf links $L_1 \sqcup L_2$ distinguished by the rotation numbers, with $\mathrm{tb}_{\mathbb{Q}}(L_i) = k_i + \frac{1}{p}$ and with $m\pi$ Giroux torsion in the complement. For $k_1 = k_2 = 0$, there is exactly one non-loose Hopf link with $m\pi$ Giroux torsion in the complement. For $k_1 < 0, k_2 < 0$, the Hopf link is in tight $L(p,1)$. For $k_1 < 0$ and $k_2 \geqslant 0$ L_1 is loose and L_2 is non-loose. For $k_1 \geqslant 0$ and $k_2 < 0$, L_1 is non-loose and L_2 is loose. Finally for $k_1 \geqslant 0, k_2 \geqslant 0$ and $m > 0$ both components are loose.

(2) For $q \neq 1$, for each choice of integers and a natural number m there is exactly a pair of non-loose Legendrian Hopf links $L_1 \sqcup L_2$ distinguished by their rotation numbers and with $m\pi$ Giroux torsion in the complement. The $\mathrm{tb}_{\mathbb{Q}}$ of the components are as follows:

(a) For $k_1 < 0, k_2 < 0$, the Hopf links are in tight $L(p,q)$ with $\mathrm{tb}_{\mathbb{Q}}(L_1) = k_1 - p''/p$ and $\mathrm{tb}_{\mathbb{Q}}(L_2) = k_2 + q/p$.

(b) For $k_1 \geqslant 0, k_2 < 0$, L_1 is non-loose and L_2 is loose with $\mathrm{tb}_{\mathbb{Q}}(L_1) = k_1 + p'/p$ and $\mathrm{tb}_{\mathbb{Q}}(L_2) = k_2 + q/p$.

(c) For $k_1 < 0, k_2 > 0$, L_1 is loose and L_2 is non-loose with $\mathrm{tb}_{\mathbb{Q}}(L_1) = k_1 - p''/p$ and $\mathrm{tb}_{\mathbb{Q}}(L_2) = k_2 + q/p$. and

(d) For $k_1 \geqslant 0, k_2 \geqslant 0$, both the components are loose with $\mathrm{tb}_{\mathbb{Q}}(L_1) = k_1 + p'/p$ and $\mathrm{tb}_{\mathbb{Q}}(L_2) = k_2 + q/p$.

where p', p'' are same as in Theorem 1.6.

Remark 1.8. The above theorem recovers Theorem 1.2 (e) from [GO19] for $p = 1$. Furthermore, the above theorem also gives the classification of non-loose Hopf links with nonzero Giroux torsion in $L(p,1)$, filling the gap left in [CGO25].

1.1. **Organization.** In Section 2 we give some preliminaries and definitions. In Section 3, we prove Theorem 1.1, and 1.4. In Section 4 we give the proof of our main theorems Theorem 1.6 and 1.7. We finish by including an algorithm for the contact surgery diagrams of the non-loose realizations in Section 5.

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2. BACKGROUND

We assume the reader is familiar with basic contact geometry, Legendrian knots and convex surface theory, as can be found in [EH01, Gei06, Hon00]. In this section, we will briefly recall some of the important results for the convenience of the reader and to establish the notations that we use throughtout the paper. In Section 2.1 we recall the Farey graph and discuss its relation with curves on tori, next we discuss the classification of tight contact structures on solid tori, $T^2 \times [0,1]$ and lens spaces. In Section 2.3, we review basic facts about Legendrian knots, such as standard neighborhoods and how these are related by stabilizations. In Section 2.5 and Section 2.6 we discuss rationally null homologous knots and computation of their classical invariants.

2.1. **The Farey graph.** The Farey graph is essential in keeping track of the embedded essential curves on a torus. Recall that once we choose a basic for $H_1(T^2)$, the embedded essential curves on T^2 are in one to one correspondence with $\mathbb{Q} \cup \infty$.

The Farey graph is constructed in the following way. See Figure 5. Consider the unit disk in the xy-plane. Label the point $(0,1)$ as $0 = \frac{0}{1}$ and $(0,-1)$ as $\infty = \frac{1}{0}$. Connect these two points by a straight line. Now if a point on the boundary of the disk has a positive x-coordinate and if it lies between two points labeled $\frac{a}{b}$ and $\frac{c}{d}$ then we label it as $\frac{a+c}{b+d}$. We call this the "Farey sum" of $\frac{a}{b}$ and $\frac{c}{d}$ and denote as $\frac{a}{b} \bigoplus \frac{c}{d}$. Now we connect this point with both $\frac{a}{b}$ and $\frac{c}{d}$ by hyperbolic geodesics (note that, we consider a hyperbolic metric on the interior of the disk). We keep iterating this process until all positive rational numbers are labeled on the boundary of the disk. We do the same thing for all the negative national number by considering ∞ as $\frac{-1}{0}$. We use $\frac{a}{b} \bigoplus k\frac{c}{d}$ to denoted the k^{th} iterated Farey sum i.e. we add $\frac{c}{d}$ to $\frac{a}{b}$, k times. Note that, two embedded curves on the torus with slopes r and s will form a basis if and only if there is exactly an edge between them in the Farey graph. We also introduce the dot product of two rational numbers here $\frac{a}{b} \bullet \frac{c}{d} = ad - bc$ as the minimum number of times the curves can intersect.

We have the following well-known lemma, See [ELT12]

Lemma 2.1. *Suppose $q/p < -1$. Given $q/p = [a_0, \cdots, a_n]$, let $(q/p)^c = [a_1, \cdots, a_n + 1]$ and $(q/p)^a = [a_0, \cdots, a_{n-1}]$. There will be an edge in the Farey graph between each pair of numbers q/p, $(q/p)^c$, and $(q/p)^a$. Moreover, $(q/p)^c$ will be fathest clockwise point from q/p that is larger than q/p with an edge to q/p, while $(q/p)^a$ will be the farthest anti-clockwise point from q/p that is less than q/p with an edge to q/p.*

In the above lemma if $a_n + 1 = -1$, then we consider $[a_0, \cdots, a_n + 1]$ to be $[a_0, \cdots a_{n-1} + 1]$. Also, if q/p is a negative integer then $(q/p)^a = \infty$.

A path in the Farey graph is a sequence of elements $p_1, p_2, \cdots, p_k$ in $\mathbb{Q} \cup \infty$ moving clockwise such that each p_i is connected to p_{i+1} by an edge in the Farey graph, for $i < k$. Let P be the minimal path in the Farey graph that starts at p_1 and goes clockwise to p_k. We say a path in the Farey graph is a *decorated path* if all of the edges are decorated by a $+$ or $-$. We call a path in the Farey graph *a continued fraction block* if there is a change of basis such that the path goes from 0 clockwise to n for some positive n. We say two choices of signs on the continued fraction block are related by *shuffling* if the number of $+$ signs in the continued fraction blocks are the same.

Next we introduce a notation that we frequently use. Given two numbers r, s in $\mathbb{Q} \cup \infty$, we denote by $[r, s]$ all the numbers that are clockwise to r and anti-clockwise to s in the Farey graph.

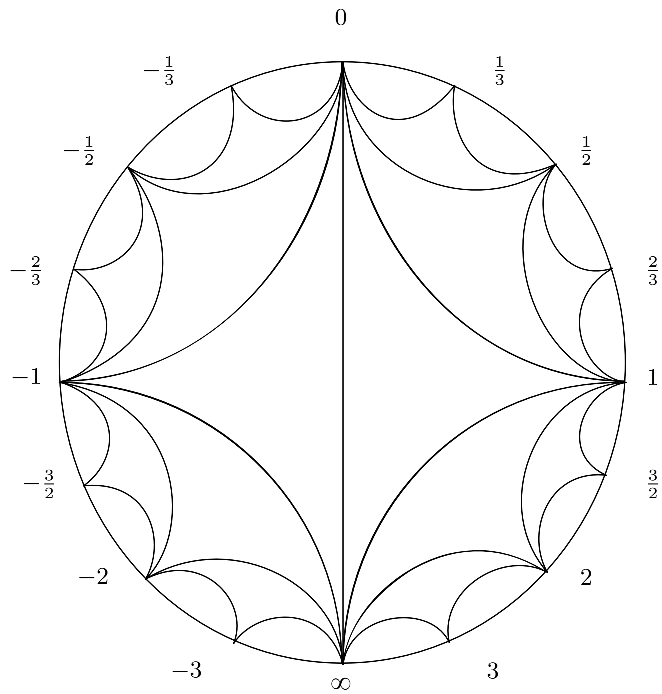

FIGURE 5. The Farey graph.

2.2. **Tight contact structures on solid tori, $T^2 \times [0,1]$ and $L(p,q)$.** Here we briefly recall the classiifcation of tight contact structures on $T^2 \times [0,1]$, $S^1 \times D^2$ and lens spaces due to Giroux [Gir00] and Honda [Hon00]. We discuss the classiifcation results along the lines of Honda.

2.2.1. *Contact structures on $T^2 \times [0,1]$.* Consider a contact structure ξ on $T^2 \times [0,1]$ that has convex boundary with dividing curves of slope s_i on $T^2 \times \{i\}$ for $i = 0, 1$. We also assume that there are exactly two dividing curves on each of the boundary component. We say ξ is *minimally twisting* if any convex torus in $T^2 \times [0,1]$ parallel to the boundary has dividing slope in $[s_0, s_1]$. We denote the minimally twisting contact structures, up to isotopy, on $T^2 \times [0,1]$ with the above boundary conditions as $\text{Tight}^{min}(T^2 \times [0,1]; s_0, s_1)$. Giroux [Gir00] and Honda [Hon00] classified tight contact structures in $\text{Tight}^{min}(T^2 \times [0,1]; s_0, s_1)$ establishing the following result.

Theorem 2.2. *Each decorated minimal path in the Farey graph from s_0 clockwise to s_1 describes an element of Tight$^{min}(T^2 \times [0,1]; s_0, s_1)$. Two such decorated paths will describe the same contact structure if and only if the decorations differ by shuffling in the continued fraction blocks.*

Notice that, if s_0 and s_1 has exactly one edge between them then there are exactly two tight contact structures in $\text{Tight}^{min}(T^2 \times [0,1]; s_0, s_1)$. These are called *basic slices* $B_\pm$ and the correspondence of the theorem can be understood via stacking basic slices according to the decoration in the path that describes the contact structure. The two different contact structures on a basic slice can be distinguished by their relative Euler classes, we call them *positive* and *negative* basic slices.

The relative Euler class of the contact structure in $\text{Tight}^{min}(T^2 \times [0,1]; s_0, s_1)$ can be computed as follows: let $s_0 = r_0, s_1, \cdots, r_k = s_1$ be the vertices of the minimal path from s_0 to s_1 amd ϵ_i be the sign of the basic slice with boundary slopes r_{i-1} and r_i. Then the relative Euler class of the contact structure associated with this path is Poincaré dual to the curve

$$\sum_{i=1}^{k} \epsilon_i (r_i \ominus r_{i-1})$$

where $\frac{a}{b} \ominus \frac{c}{d} = \frac{a-c}{b-d}$.

Next we discuss shortening of a non-minimal path. Suppose P is a non-minimal path in the Farey graph. So there will be a vertex v in P such that there is an edge between its neighboring vertices v' and v''. We can shorten this path by removing v and the two edges and replacing it by the edge between v' and v'' in the path. We call the new path P'. If P were a decorated path, then we call the shortening to get P' *inconsistent* if the edges removed had different signs and *consistent* if the signs are the same. When the shortening is consistent, then we can decorate the new edge in P' by the sign of the removed edges, thus P' is the new decorated path.

For any decorated path in the Farey graph, even non-minimal, one can construct a contact structure on $T^2 \times [0,1]$ by stacking basic slices. The following result due to Honda tells us when these path will lead to tight contact structures.

Theorem 2.3. [Hon00] *Let ξ be a contact structure on $T^2 \times [0,1]$ described by a non-minimal decorated path P in the Farey graph from s_0 to s_1. Then ξ is tight if and only if one may consistently shorten the path to a shortest path from s_0 to s_1.*

Next we discuss convex Giroux torsion. Consider $\xi = \ker(\sin 2\pi z dx + \cos 2\pi z dy)$ on $T^2 \times \mathbb{R}$ where (x, y) is the co-ordinate on T^2 and z is the co-ordinate on $\mathbb{R}$. Consider the region $T^2 \times [0, k]$ for $k \in \frac{1}{2}\mathbb{N}$ and notice that the contact planes twist k times as z goes from 0 to k. We can perturb $T^2 \times \{0\}$ and $T^2 \times \{k\}$ so that they become convex with two dividing curves of slope 0. Let ξ^k denotes the resulting contact structure on $T^2 \times [0,1]$ after we identify $T^2 \times [0,1]$ with $T^2 \times [0,k]$. For $k \in \frac{1}{2}\mathbb{N}$, we call $(T^2 \times [0,1], \xi^k)$ a *convex k Giroux torsion layer* and if it embeds into a contact manifold (M, ξ), we say (M, ξ) has *convex k Giroux torsion*. We say (M, ξ) has exactly k Giroux torsion if one can embed $(T^2 \times [0,1], \xi^k)$ into (M, ξ) but cannot embed $(T^2 \times [0,1], \xi^{k+\frac{1}{2}})$ in (M, ξ). On the other hand (M, ξ) has no convex Giroux torsion or zero Giroux torsion if $(T^2 \times [0,1], \xi^k)$ does not embed in (M, ξ) for any $k \in \frac{1}{2}\mathbb{N}$.

Remark 2.4. There is a difference between a convex Giroux torsion layer and a Giroux torsion layer. When we talk about Giroux torsion in general we do not require the boundary tori to be convex, we just need them to be pre-Lagrangian. But as one can always find a convex tori with same boundary slope inside the layer, every Giroux torsion layer contains a convex Giroux torsion layer of same slope (converse need not be true). In this paper when we refer to Giroux torsion we will be talking about convex Giroux torsion layer.

2.2.2. *Contact structures on solid tori.* Now we discuss tight contact structures on solid tori. While we will usually use "standard" coordinates on a solid torus so that the meridional slope is $-\infty = \infty$, but it will be convenient sometimes to use different coordinates. The notation is as follows: Consider

$T^2 \times [0,1]$ and choose a basis for $H_1(T^2)$ so that we may denote curves on T^2 by rational numbers $\cup\infty$. Given $r \in \mathbb{Q} \cup \infty$ we can foliate $T^2 \times \{0\}$ by curves of slope r. Let S_r be the result of collapsing each leaf in the foliation of $T^2 \times \{0\}$ to a point. One can check that S_r is a solid torus with meridional slope r. We say S_r is a *solid torus with lower meridian* r. One the other hand one could foliate $T^2 \times \{1\}$ similarly by curves of slope r and collapse the curves to obtain S^r. This is called a *solid torus with upper meridian* r. Note that the standard solid torus is S_∞ and unless otherwise stated this is the solid torus we are talking about.

2.2.3. *Contact structures on* $L(p,q)$. The lens space $L(p,q)$ can be defined as $-p/q$ surgery on the unknot in S^3. Equivalently, we can think of $L(p,q)$ as $T^2 \times [0,1]$ with the curves of slope $-p/q$ collaped on $T^2 \times \{0\}$ and curves of slope 0 collaped on $T^2 \times \{1\}$. We can further describe $L(p,q)$ as a result of gluing $S_{-p/q}$, a solid torus with lower meridian $-p/q$ to another solid torus S^0, a solid torus of upper meridian 0. Giroux [Gir00] and Honda [Hon00] classified the tight contact structure on $L(p,q)$ as follows:

Theorem 2.5. *Let P be a minimal path in the Farey graph $-p/q$ clockwise to 0. The tight contact structures on $L(p,q)$ are in one-to-one correspondence with assignments of signs to all but the first and last edge in P up tp shuffling in continued fraction blocks.*

2.3. **Knots in contact maniolds.** A *standard neighborhood* of a Legendrian knot L in (M,ξ) is a solid torus $N(L)$ on which ξ is tight and $\partial N(L)$ is convex with two dividing curves of slope $\mathrm{tb}(L)$. One may arrange via a small isotopy that the characteristic foliation consists of two lines of singularities called *Legendrian divides* and curves of slope $s \neq \mathrm{tb}(L)$. These are called *ruling curves.* Convesely, given a solid torus in a contact manifold (M,ξ) on which ξ is tight and having convex boundary and two dividing curves of slope n, then there exist a Legendrian knot L with $\mathrm{tb} = n$ and S being its standard neighborhood.

Given a Legendrian knot L, one can stabilize it in two ways, $S_\pm(L)$. Note that the standard neighborhood $N(S_\pm(L))$ of $S_\pm(L)$ is in $N(L)$ and $N(L)\backslash N(S_\pm(L))$is a basic slice where the sign of the basic slice depends on the sign of the stabilization. This basic slice has boundary slopes $\mathrm{tb}(L)$ and $\mathrm{tb}(L)-1$.

2.4. **Rationally null-homologous knots.** For deatils on rationally null-homologous knots an interested reader is referred to [BE11]. Recall we say a knot K is *rationally null-homologous* in M if it is trivial in $H_1(M,\mathbb{Q})$. In other words there exists a minimal integer r such that rK is trivial in $H_1(M,\mathbb{Z})$. We call r the order of K. One can build a *rational Seifert surface* Σ for K as explained in [BE11]. Note that, in general Σ might not have connected boundary but it would not concern us in this paper. We assume K is oriented and this will induce an orientation on Σ.

If K' is another oriented knot in M that is disjoint from K then we define the *rational linking number* to be

$$\mathrm{lk}_{\mathbb{Q}}(K,K') = \frac{1}{r}\Sigma' \cdot K$$

where $\Sigma' \cdot K$ denotes the algebraic intersection of Σ' and K. There is some ambiguity in this definition but that will not be an issue here.

Now let K be a Legendrian knot in (M,ξ). As mentioned in section 2.3 K has a standard neighborhood with convex boundary and 2 dividing curves determined by the contact framing. Let K' be one of the Legendrian divides on $\partial N(K)$. We define the *rational Thurston-Bennequien invariant* of K to be

$$\mathrm{tb}_{\mathbb{Q}}(K) = \mathrm{lk}_{\mathbb{Q}}(K,K')$$

Next we define the rational rotation number following [BE11]. We consider the immersion $i\colon \Sigma \to M$ that is an embedding on the interior of Σ and an r to 1 mapping of $\partial\Sigma$. We can now consider $i^*\xi$ as an oriented $\mathbb{R}^2$ bundle over Σ. Since Σ is a surface with boundary we know that $i^*\xi$ can be trivialized as $i^*\xi = \mathbb{R}^2 \times \Sigma$. Let v be a non-zero vector field tangent to $\partial\Sigma$ inducing the orientation of K. Using the trivialization of $i^*\xi$ we can consider v as a map from $\partial\Sigma$ to $\mathbb{R}^2$. Now we can define the *rational rotation number* as follow:

$$\mathrm{rot}_{\mathbb{Q}}(K) = \frac{1}{r}\mathrm{winding}(v, \mathbb{R}^2)$$

Note that winding$(v, \mathbb{R}^2)$ is equivalent to the obstruction to extending v to a non-zero vector field over $i^*\xi$ and thus can be interpreted as the relative Euler number.

2.5. **Hopf links in lens spaces.** Here we will give the basics that we need in the next section. We will consider $L(p,q)$ as the union of two solid tori V_1 and V_2. We can think of V_1 as a solid torus with lower meridian $-p/q$ and V_2 a solid torus with upper meridian 0. If we fix two dividing curves of slope s on $\partial V_1 = \partial V_2$, then a contact structure on $L(p,q)$ is determined by taking a tight contact structure on V_1 in Tight$(S_{-p/q}, s)$ and another tight contact structure in V_2 in Tight$(S^0; s)$ and gluing them together. There is no guarentee that this gluing will give us a tight contact structure (in fact most of the time it will be overtwisted).

We will think of the core of V_i as the rational unknots K_i. If we are looking for Legendrian realizations of K_i then we can take any slope s with an edge to $-p/q$ (respectively 0) in the Farey graph. We call the Legendrian representatives L_i. Now a tight contact structure on V_i with convex boundary having two dividing curves of slope s is a standard neighborhood of L_1. If we are looking for non-loose representative of $L_1 \sqcup L_2$, then the link complement must be tight. By link complement we mean the complement of the standard neighborhood of L_1 and L_2 which is diffeomorphic to $T^2 \times I$. So non-loose representaives of $L_1 \sqcup L_2$ are in one-to -one corrpondence with Tight$^{min}(T^2 \times I; s_1, s_2)$ where s_i are the dividing slopes of the standard neighborhood of V_i. Note that, s_1 can be either in $(-p/q, (-p/q)^c]$ or $[(-p/q)^a, -p/q)$ (recall our notation from section 2.1). On the other hand, s_2 can be in either $(0, 0^c]$ or in $[(0)^a, 0)$.

We refer to all the slopes s as *large slope* for L_1 if we start from the meridional slope of V_1 and traverse clockwise to slope s, we pass the Seifert slope for L_1. Otherwise, we call them *small slopes*. Thus for L_1 the slopes $s \in [(-p/q)^a, -p/q)$ are large slopes and $s \in (p/q, (-p/q)^c]$ are small slopes. On the other hand, for L_2 the slopes $s \in (0, 0^c]$ are large slopes and $s \in [0^a, 0)$ are small slopes (as L_2 is the core of V_2 and we are following the orientation of T^2 as the boundary of V_1, the convention is opposite in this case. In this case we need to start from the meridional slope and have to traverse counter clockwise to s).

Now when gluing V_1 with slope s_1, a tight $T^2 \times [0,1]$ with slopes s_1 and s_2 and V_2 with slope s_2 we will have four possibilities. If $s_1 \in (-p/q, (-p/q)^c]$ and $s_2 \in [0^a, 0)$ the contact structure on $L(p,q)$ is clearly tight. In all three other cases that is when we consider a combination of small and large slopes or large slopes only, the contact structure on $L(p,q)$ will be overtwisted as in all the cases either the torus V_1 will contain a boundary parallel convex torus of slope 0 or V_2 will contain a boundary parallel torus of slope $-p/q$ or both. Thus after gluing V_1, V_2 and the thickened torus the resulting contact structure will contain an overtwisted disk (a Legendrian divide of slope 0 will bound a disk in V_2 and a Legendrian divide of slope $-p/q$ will bound an overtwisted disk in V_1). As we are considering non-loose Hopf links we will only consider those three cases.

2.6. **Computation of classical invariants in $L(p,q)$.** In this section we explain how to compute the classical invariants of the Hopf link components and the Euler class of the contact structure in $L(p,q)$.

Lemma 2.6. *Suppose $r = -\frac{p}{q}$ is the meridional slope of V_1 and L_1 being the core of V_1. Then $\mathrm{tb}_{\mathbb{Q}}(L_1) = -\frac{1}{p}|s \cdot 0|$, if s is a small slope and $\mathrm{tb}(L_1) = \frac{1}{p}|s \cdot 0|$ if s is large slope. Here $|s \cdot 0|$ denotes the number of intersection between s and 0. The same is true for L_2.*

Proof. Notice that the meridional disk for L_2 will provide a Seifert surface for L_1 which has order p in $L(p, q)$. So according to the definition the rational Thurston–Bennequin invariant is the rational linking number of L_1 with a contact push-off of L_1. In other words we count the number of intersection of a Legendrian divide on ∂V_1 with the Seifert slope and divide it by the order. The sign of $\mathrm{tb}_{\mathbb{Q}}$ is determined by whether the slope in small or large. Note that, Thurston–Bennequin number measures the difference between the contact framing and the Seifert framing. As for small slope the contact framing is less that the Seifert framing, $\mathrm{tb}_{\mathbb{Q}}$ is negative. On the other hand, as by definition the large slope is always greater than the Seifert framing and $\mathrm{tb}_{\mathbb{Q}}$ must be positive. □

2.6.1. *Computation of rotation number.* Next we explain how to compute the rotation number of the components. If e is the Euler class of the contact structure on $T^2 \times I$ and D_i be the meridional disk of L_i, then the rational rotation number of the components are given by $\mathrm{rot}_{\mathbb{Q}}(L_1) = \frac{1}{p}e(D_2)$ and $\mathrm{rot}_{\mathbb{Q}}(L_2) = \frac{1}{p}e(D_1)$. For a similar explanation check [CEMM25]. To calculate, the rotation number for the components $L_1 \sqcup L_2$ having standard neighborhood with dividing slopes s_{k_1} and s_{k_2} we follow :

(1) We find the (decorated) shortest path $\{s_{k_1} = p_0, p_1, \ldots, p_n = s_{k_2}\}$ between s_{k_1} clockwise to s_{k_2} on the Farey graph. If s_{k_1}, s_{k_2} both belong to the negative region we continue. (In our case, $s_{k_1} < 0$.) If $s_{k_2} > 0$, we apply a diffeomorphism of T^2 such that the new slope < 0. In fact this can be done by applying the change of basis matrix

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

Thus we have a path in the negative region only.

(2) As mentioned before the relative Euler class of the contact structure associated with this path is Poincaré dual to the curve

$$\sum_{i=0}^{n} \epsilon_i (p_i \ominus p_{i-1})$$

where $\epsilon = \pm 1$ depending on the decoration.

(3) Now we evaluate this on the Seifert disk. In other words, for L_1 we evaluate on slope 0 and for L_2 we use slope $-\frac{p}{q}$. If we used the change of basis matrix from before we need to make the appropriate changes to the Seifert slope as well.

(4) Now $\mathrm{rot}_{\mathbb{Q}}(L_1) = \frac{1}{p}e(D_2)$ and $\mathrm{rot}_{\mathbb{Q}}(L_2) = \frac{1}{p}e(D_1)$.

Remark 2.7. One might notice a little discrepancy of the Euler class when evaluating on D_1 or D_2 in $L(p, q)$ for $q > 1$. The reason behind this is unlike $L(p, 1)$ case there does not exist any isotopy taking D_1 to D_2. Thus when we give the Euler class of $L(p, q)$ where the Hopf link lives, we are evaluating the Poincaré dual on D_2.

Remark 2.8. Note that, one could also use the method from [CK24] to calculate the Euler classes of the contact structures. The algorithm from [DGS04] can be used to calculate the d_3 invariants from the explicit contact surgery diagrams of the candidates given in Section 5.

2.6.2. *Stabilization of Hopf link.* For large slopes s_{k_1} of L_1, s_{k_1} must be in $[(-p/q)^a, -p/q)$ and has an edge to $-p/q$. So $s_{k_1} = (-p/q)^a \bigoplus k_1(-p/q)$. For large slopes of L_2, similarly we will have $s_{k_2} = 1/k_2$. Suppose L_1 be the Legendrian unknot corresponding to the tight contact structure on V_1 with dividing slope s_{k_1} where $k_1 > 0$, L_2 be the Legendrian unknot corresponding to the tight contact structure on V_2 with dividing slope s_{k_2} with $k_2 > 0$ and some tight contact structure on $T^2 \times [0,1]$ in Tight$^{min}(T^2 \times [0,1]; s_{k_1}, s_{k_2})$. Now inside each V_i there are two solid tori $S_i^{\pm}$ (smoothly isotopic to V_i) with convex boundary and 2 dividing curves of slope s_{k_i-1}. Thus $V_i \backslash S_i^{\pm}$ is a Basic slice $B_i^{\pm}$ for $i = 1, 2$. Now $S_i^{\pm}$ is a standard neighborhood of a stabilization of L_i. Note that, as V_1 and V_2 have opposite orientation, if we fix the orientation of V_1 and work with it then while $B_1^{\pm}$corresponds to a positive and negative stabilization of L_1, $B_2^{\pm}$ will correspond to a negative and positive stabilization of L_2. But as we are considering positive Hopf links L_1 and L_1 have opposite orientations. Thus, in our case $B_2^{\pm}$ will actually correspond to a positive and negative stabilization of L_2.

If L_1 and L_2 are both stabilized, then the complement is given by a contact structure on $B_1^{\pm} \cup T^2 \times I \cup B_2^{\pm}$ where we attach B_1 on the front face of T_0 and attach B_2 on the back face T_1. Thus the path in the complement is extended by the two edges describing B_1 and B_2. This new path might not be minimal. If the path can be consistently shortened then we will see the link is still non-loose otherwise loose. Now note that, it is possible that $S_{\pm}(L_1)$ becomes loose as the complement of $S_{\pm}(L_1)$ i.e. $B_{\pm} \cup T^2 \times [0,1] \cup V_1$ is overtwisted (there will be an inconsistent shortening) but the link remains loose. The same is true for L_2 as well. Next we show that any stabilization of L_1 and L_2 with dividing slopes s_{k_1}, s_{k_2} where $k_1 = k_2 = 0$ are loose.

Lemma 2.9. *Suppose $L_i^{s_{k_i}}$ is a non-loose rational unknot that is the core of V_i with dividing slope s_{k_i} where s_{k_i} is a large slope. Then any stabilization of $L_i^{s_0}$ is loose for $i = 1, 2$.*

Proof. We prove it for L_1 and L_2 will follow similarly. Note that, s_0 large corresponds to the slope $(-p/q)^a$. The complement of L_1 is given by V_2. When we stabilize $L_1^{s_0}$ we add a basic slice $B_{\pm}$ of slopes $\{(-p/q)^c, (p/q)^a\}$ to V_1. Notice that, $B_{\pm}$ contains boundary parallel convex tori of any slope between $(-p/q)^c$ clockwise to $(-p/q)^a$. In particular it contains a convex torus of slope 0. Any Legendrian divide on this torus when included in V_2 will contribute to an overtwisted disk. Thus, $B_{\pm} \cup V_2$ will be overtwisted and $S_{\pm}(L_1)$ will be loose. □

3. Classification result for $L(p,1)$ and $L(2n+1,2)$

Proof of Theorem 1.1. Note that, as mentioned before to have an overtwisted lens space we need to consider three cases. (1) small slope $\cup$ large slope (2) large slope $\cup$ small slope and (3) large slope $\cup$ large slope. For $L(p,1)$, we have $(-p)^c = -p+1$ and $(-p)^a = \infty$

3.1. small slope $\cup$ large slope.

In this case the dividing slopes on the standard neighborhood of L_1 and L_2 are given by $s_{k_1} = -\frac{p(k_1+1)-1}{k_1+1}$ and $s_{k_2} = (\frac{1}{0}) \oplus k_2(\frac{0}{1}) = \frac{1}{k_2}$. From now on when we mention a link complement we mean the complement of the standard neighbourhoods of L_1 and L_2. Note that, the link complement with these dividing curves is $T^2 \times [0,1]$ with boundary slopes s_{k_i}. For $k_1 = 0$, a minimum path from s_{k_1} to s_{k_2} consists of a continued fraction block of length k_1+1 from s_{k_1} to ∞ and thus corresponds to exactly k_1+2 tight contact structures. For $k_2 = 1$, a path from s_{k_1} to s_{k_2} consists of a continued fraction block of length k_1 from s_{k_1} to $-p+1$, then a continued fraction block of length p from $-p+1$ to 1. This corresponds to $(k_1+1)(p+1)$ tight contact structures. Finally, for $k_2 > 1$ a minimum path from s_{k_1} to s_{k_2} consists of the following sub-paths, a continued fraction block of length k_1 from s_{k_1} to $-p+1$, followed by a continued fraction block of length $p-1$ from $-p+1$ to 0 and finally a jump from -1 to $\frac{1}{k_2}$. Thus decorations on this path correspond to $2(k_1+1)p$ tight contact structures.

Next we calculate the classical invariants of the components. Note that, by $L_i^{k_i}$ we mean the Legendrian unknot L_i whose standard neighborhood has dividing slope s_{k_i}.

One could easily calculate the rational Thurston–Bennequin invariant of the components to be

$$\operatorname{tb}_{\mathbb{Q}}(L_1^{k_1}) = -(k_1+1) + \frac{1}{p}$$

and

$$\operatorname{tb}_{\mathbb{Q}}(L_1^{k_2}) = k_2 + \frac{1}{p}.$$

To calculate the rotation numbers we follow the technique mentioned in Section 2. For $k_1 \geqslant 0$ and $k_2 = 0$, the rotation numbers of the components are $\operatorname{rot}_{\mathbb{Q}}(L_1^{k_1}) = (-k_1 - 1 + 2n)$ and $\operatorname{rot}_{\mathbb{Q}}(L_2^0) = 0$ where $0 \leqslant n \leqslant k_1 + 1$. The Euler class is 0. (Here n counts the number of negative basic slices in the path from s_{k_1} to ∞.) For $k_2 = 1$, the rotation numbers are $\operatorname{rot}_{\mathbb{Q}}(L_1)^{k_1}) = (-k_1 + 2n) - \frac{(p-2m)}{p}$, $\operatorname{rot}_{\mathbb{Q}}(L_2^1) = -1 + \frac{2m}{p}$, and the Euler class is given by $-p + 2m$ where $0 \leqslant m \leqslant p,\ 0 \leqslant n \leqslant k_1$. In the later case n counts the number of negative basic slice in the path s_{k_1} to $-p+1$ and m counts the negative signs in the following path from $-p+1$ to 1.

It is easy to see that $L_1^{k_1}$ is loose (the complement of L_1^k in this $L(p,1)$ contains a convex torus of slope 0, a Legendrian divide on this torus contributes to an overtwisted disk in V_2) and $L_2^{k_2}$ is non-loose for all $k_1, k_2 \geqslant 0$. Next we will see how the stabilization of $L_2^{k_2}$ are related with one another. We will start with L_2^0. Any stabilization of L_1^0 will be loose as shown in Lemma 2.9. The complement of L_2^0 consists of a union of continued fraction blocks. If all the basic slices in each of the continued fraction block is positive, then a positive stabilization of L_2 will add a half-Giroux torsion in the complement of the link and $L_1^{k_1} \sqcup S_+(L_2^0)$ is still non-loose. Same is true if there are only negative slices and we negatively stabilize L_2^0. In all the other cases, any stabilization of L_2^0 will loosen the link.

For $k_2 = 1$, we have $(k+1)(p+1)$ non-loose representative and the path in the complement of L_2^1 is given by a solid torus with meridional slope $-p$ and dividing slope $s_{k_2} = 1$. We can also think of this as solid torus with boundary slope s_{k_1} and meridional slope $-p$ union a $T^2 \times I$ with boundary slopes $\{s_{k_1}, -p+1\}$ and another $T^2 \times I$ with slopes $\{-p+1, 1\}$. Stabilizing L_2^1 corresponds to adding a basic slice $B_\pm$ with boundary slopes $\{1, \infty\}$. Now adding this basic slice will allow us to shorten the path $\{1, 0, \cdots, -p+1\}$ to $\{-p+1, \infty\}$. If the path from $-p+1$ to 1 consists of only positive (resp. negative) signs then we can consistently shortening the path if the basic slice is positive (resp. negative). If the path has a mixed sign, any stabilization will lead to an inconsistent shortening as we can shuffle the signs in a continued fraction block and arrange it so that the shortening is always inconsistent. If we denote the component L_2^1 with having the complementary path $\{-p+1, \cdots, 1\}$ decorated with all positive signs $(L_2^1)^{all,+}$ then $S_+((L_2^1)^{all,+})$ is non-loose. Similarly, $S_-((L_2^1)^{all,-})$ is non-loose. These two forms the base of the back and forward slashes. If the path has any mix of signs and we denote the number of negative slices in this path by m then $S_\pm((L_2^1)^m)$ for $m = 1, \cdots, p-1$ will be loose and thus $(L_2^1)^m$ will be the bases of $p-1$ Vs. For $k_2 = 2$, the complement of $L_1^{k_1} \sqcup L_2^2$ can be subdivided into a path from $-p+1$ to 0 and then a jump from 0 to $\frac{1}{2}$. Suppose i denotes the number of negative slices in the first part of the path, $i = 0, 1, \cdots p-1$. We denote L_2^2 corresponding to this path as $(L_2^2)^{i,\pm}$ where $\pm$ corresponds to the sign of the jump. Clearly, $S_\pm((L_2^2)^{i,\pm})$ for $i = 0, \cdots p-1$ are all non-loose as this leads to a consistent shortening. We see that $S_+((L_2^2)^{0,+})$ and $S_-((L_2^2)^{p-1,-})$ coincide with $(L_2^1)^{all,+}$ and $(L_2^1)^{all,-}$ and are part of the back and forward slashes. $S_\mp((L_2^2)^{m,\pm})$ are loose. A similar analysis works for $k_2 > 2$ too.

Now putting all these together we have the following mountain range for L_2 fixing L_1, one non-loose V based at $(0, \frac{1}{p})$ and $p-1$ non-loose Vs based at $(-1 + \frac{2m}{p}, 1 + \frac{1}{p})$ for $m = 1, 2, \cdots, p-1$.

k_1	k_2	$T^2 \times I$	number
0	0	I-invariant	1
0	1	$(\infty, 1)$	2
0	2	$(\infty, \frac{1}{2})$	3
0	>2	$(\infty, \frac{1}{k_2})$	4
1	0	$(-p-1, \infty)$	2
1	1	$(-p-1, 1)$	$p+3$
1	>1	$(s_{k_1}, \frac{1}{k_1})$	$2(p+2)$
2	0	(s_{k_1}, ∞)	3
>2	0	(s_{k_1}, ∞)	4
>1	1	$(s_{k_1}, 1)$	$2(p+2)$
>1	>1	$(s_{k_1}, \frac{1}{k_2})$	$4(p+1)$

TABLE 1. Number of non-loose Hopf links in $L(p, 1)$ for large $\cup$ large.

3.2. **Large slope $\cup$ small slope.** This case corresponds to dividing slopes $s_{k_1} = -\frac{k_1 p+1}{k_1}$ and $s_{k_2} = -\frac{1}{k_2}$. Except for $k_1 = k_2 = 0$, any path from s_{k_1} to s_{k_2} will consist of a path from s_{k_1} to -1 and then followed by a continued fraction block of length $k_2 - 1$ from -1 to $-\frac{1}{k_2}$. We call the first part P_1 and the second part of the path P_2. Decorations on P_2 correspond to k_2 tight contact structures. For $k_1 = 0$ and $k_2 \geqslant 1$, there is a continued fraction block of length k_2 between ∞ and $-\frac{1}{k_2}$. This gives us k_2+1 tight contact structures. On the other hand, for $k_1 = 1$ and $k_2 \geqslant 1$ this path consists of a continued fraction block of length p followed by P_2. This will correspond to $(p+1)k_2$ tight contact structures. For $k_1 \geqslant 1$, there is a single jump from s_{k_1} to $-p$, followed by a continued fraction block of length $p-1$ from $-p$ to -1 and then P_2. Decorations on this path correspond to $2pk_2$ contact structures. These can also be obtained by switching the role of L_1 and L_2 as this case the Hopf link is symmetric.

The rational Thurston-Bennequin invariant of the components are given by

$$\mathrm{tb}_{\mathbb{Q}}(L_1^{s_{k_1}}) = k_1 + \frac{1}{p}$$

and

$$\mathrm{tb}_{\mathbb{Q}}(L_1^{s_{k_2}}) = -k_2 + \frac{1}{p}.$$

The rational rotation number are same as the previous case after switching the components.

Remark 3.1. We need to be careful here as there is a shift in the Thurston Bennequin number. After swiching L_1 and L_2 one needs to replace k_1 by k_2 but k_2 by $k_1 + 1$ to adjust the range.

It is easy to see that in this case L_1 is non-loose and L_2 is loose for every $k_1 \geqslant 0, k_2 \geqslant 1$. Stabilizations of L_1 will be exactly same as stabilizations L_2 as before after switching their roles. A detailed analysis of the stabilizations of L_1 is given in Theorem 1.1 [CEMM25] as well.

3.3. **Case 3 large slope $\cup$ large slope.** The dividing slopes in this case are given by $s_{k_1} = -\frac{k_1 p+1}{k_1}$ and $s_{k_2} = \frac{1}{k_2}$.

For $k_1 = k_2 = 0$, we will see that the complementary $T^2 \times I$ has both boundary slopes ∞. As $T^2 \times I$ is minimally twisting, this must be an I-invariant contact structure and thus gives a unique tight

k_1	k_2	$\mathrm{rot}_{\mathbb{Q}}(L_1)$	$\mathrm{rot}(L_2)$	range of m
0	1	$\mp\frac{2}{p}$	$\mp\frac{p+2}{p}$	
0	2	$\frac{2m-2}{p}$	$\frac{(2m-2)(p+1)}{p}$	$0 \leqslant m \leqslant 2$
0	>1	$\mp\frac{1}{p} \mp \frac{1}{p}$	$\mp\frac{p+1}{p} \mp ((k_2-1)+\frac{1}{p})$	
1	1	$-\frac{p+2-2m}{p}$	$-\frac{p+2-2m}{p}$	$0 \leqslant m \leqslant p+2$
1	>1	$-\frac{p+1-2m}{p} \mp \frac{1}{p}$	$-\frac{p+1-2m}{p} \mp \frac{(1+p(k_2-1))}{p}$	$0 \leqslant m \leqslant p+1$
>1	>1	$\mp((k_1-1)+\frac{1}{p}) - \frac{p-2m}{p} \mp \frac{1}{p}$	$\mp\frac{1}{p} - \frac{p-2m}{p} \mp ((k_2-1)+\frac{1}{p})$	$0 \leqslant m \leqslant p$

TABLE 2. Rotation numbers for the components with $\mathrm{tb}(L_i) = k_i + \frac{1}{p}$

contact structure. This is the unique Hopf link with both components non-loose. The Thurston–Bennequin invariants are given by $\mathrm{tb}_{\mathbb{Q}}(L_i) = \frac{1}{p}$ and rotation number $\mathrm{rot}_{\mathbb{Q}}(L_i) = 0$. The Euler class of this contact structure is zero as well. As before any stabilization of L_1^0 or L_2^0 is loose. $S_\pm(L_1^0) \sqcup L_2^0$ is non-loose and corresponds to the 2 Hopf links (loose/non-loose pairing) from case 3.1. In fact, $S_\pm^k(L_1) \sqcup L_2^0$ is isotopic to the Hopf link $L_1^{k-1} \sqcup L_2^0$ from case 3.1 for $k \geqslant 1$ where $S_\pm^k(L_1)$ denotes the $k-$fold stabilization. Similarly, $L_1^0 \sqcup S_\pm^k(L_2)$ is isotopic to the non-loose Hopf link $L_1^0 \sqcup L_2^k$ from case 3.2 for $k \geqslant 1$. Finally, $S_+(L_1^0) \sqcup S_+(L_2^0)$ is still non-loose but has a half Giroux torsion in the complement. The same is true when both the components are negatively stabilized. The mixed stabilization of the components (i.e. if L_1 is stabilized postively and L_2 is stabilized negatively) will loosen the link.

Now for $k_2 = 0$ and $k_1 = 1$, there is exactly one edge between $-p-1$ and ∞ and thus corresponds to 2 tight contact structure. For $k_2 = 0, k_1 > 1$, the path will consist of one jump from s_{k_1} to $-p$ followed by another jump from $-p$ to ∞. For $k_1 = 2$ this is a continued fraction block of length 2 so we will have 3 tight contact structures in the complement. For $k_1 > 2$, this is not a continued fraction block, thus corresponds to 4 tight contact structures. It is easy to check that both the components are loose.

The rational Thurston–Bennequin invariants of the components are given by

$$\mathrm{tb}_{\mathbb{Q}}(L_i) = k_i + \frac{1}{p}$$

and the rational rotation numbers are given in Table 2.

Next we analyze the loose mountain range for L_1 fixing L_2^0. The two loose L_1^0 are denoted as $(L_1^0)^\pm$ respectively. $S_\pm((L_1^0)^\pm) \sqcup (L_2^0)^\pm$ are non-loose but will have a half Giroux torsion in the complement. But $S_\pm((L_1^0)^\mp) \sqcup L_2^0$ are loose. For $k_1 = 1$, we denote the corresponding L_1^1 as $(L_1^1)^0, (L_1^1)^1$ and $(L_1^1)^2$ where the superscript denotes the number of negative slices in the path from s_{k_1} to ∞ which is a continued fraction block of length 2. $S_+(((L_1^1)^0))$ and $S_-((L_1^1)^2)$ coincide with $(L_1^0)^\pm$ respectively. On the other hand, $S_\pm(((L_1^1)^{+-})) \sqcup L_2^0$ will be loose. Finally, for $k_1 = 2$, we have 2 jumps which is not a continued fraction block. So, we denote the corresponding L_1^2 as $(L_1^2)^{ij}$, where i is the sign of the basic slice from s_{k_1} to $-p$ and j is the sign of basic slice from $-p$ to ∞. $S_+((L_1^2)^{++}) \sqcup L_2^0$ and $S_-((L_1^2)^{--}) \sqcup L_2^0$ are contactomorphic to $(L_1^1)^0 \sqcup L_2^0$ and $(L_1^1)^2 \sqcup L_2^0$ respectively. Moreover, $S_+((L_1^2)^{+-}) \sqcup L_2^0$ and $S_-((L_1^2)^{-+}) \sqcup L_2^0$ are both contactomorphic to $(L_1^1)^{+-} \sqcup L_2^0$. So fixing L_2^0, putting all these together, we have 1 forward slash, 1 back slash based at $(\frac{p+2}{p}, 1+\frac{1}{p})$ and $(-\frac{p+2}{p}, 1+\frac{1}{p})$ respectively and one V based at $(0, 2+\frac{1}{p})$.

A similar analysis of the paths can be done fixing $k_2 = 1$ and $k_2 > 1$. The number of non-loose Hopf links are given in Table 1. Analyzing the stabilizations as before we see that for fixed L_2 with $k_2 = 1$ we have 1 forward slash, 1 backward slash based at $(\frac{2}{p}, \frac{1}{p})$ and $(-\frac{2}{p}, \frac{1}{p})$ respectively and $p+1$

Vs based at $(-\frac{p+2-2m}{p}, 1+\frac{1}{p})$ where $1 \leqslant m \leqslant p+1$ for L_1. Here m denotes the number of negative basic slice in the length $p+2$ continued fraction block from $-p-1$ to 1. For fixed L_2 with $k_2 > 1$, there are 2 forward slashes based at $(\frac{2}{p}, \frac{1}{p}), (0, \frac{1}{p})$ and 2 backward slashes based at $(-\frac{2}{p}, \frac{1}{p}), (0, \frac{1}{p})$ respectively and $2p$ Vs based at $(-\frac{p-2m}{p} \mp \frac{1}{p}, 1+\frac{1}{p})$ where $1 \leqslant m \leqslant p$ for L_1. In this case m denotes the number of negative basic slices in the path from $-p-1$ to 0 and the $\mp$ sign corresponds to the $\pm$ basic slice from 0 to $\frac{1}{k_2}$. The Euler classes can be computed as described in Section 2.

One could do a similar analysis fixing L_1 with $k_1 = 0, 1$ and $k_1 > 1$ to get the exact same loose mountain range for L_2.

$\square$

Proof. Proof of Theorem 1.4 Notice that, for $L(2n+1, 2)$, $(-\frac{2n+1}{2})^a = -n-1$ and $(-\frac{2n+1}{2})^c = -n$. Like before, we will subdivide the cases in three subcases and work with them separately.

3.4. **Case 1: small $\cup$ large slope.** In this case, the dividing slopes of the standard neighborhood L_1 and L_2 are given by $s_{k_1} = -\frac{n(1+2k_1)+k_1}{1+2k_1}$ and $s_{k_2} = \frac{1}{k_2}$. For $k_2 = 0$, any path from s_{k_1} to $s_{k_2} = \infty$ consists of a continued fraction block of length k_1 from s_{k_1} to $-n$ and then a jump from $-n$ to ∞ . This corresponds to $2(k_1+1)$ tight contact structures. For $k_2 = 1$, a path from s_{k_1} to $s_{k_2} = 1$ consists of a continued fraction block of length k_1 from s_{k_1} to $-n$ followed by another continued fraction block of length $n+1$ from $-n$ to 1 and thus corresponds to $(n+2)(k_1+1)$ tight contact structures. For $k_2 > 1$, we have a continued fraction block of length k_1 from s_{k_1} to $-n$, followed by a continued fraction block of length n from $-n$ to 0 and finally an edge from 0 to $\frac{1}{k_2}$. Decorations on this path correspond to $2(n+1)(k_1+1)$ tight contact structures. It is easy to see that L_1 is loose and L_2 is non-loose.

The rational Thurston–Bennequin invariant of the components are given by

$$\operatorname{tb}_{\mathbb{Q}}(L_1^{k_1}) = -(k_1+1) + \frac{n+1}{2n+1}$$

and

$$\operatorname{tb}_{\mathbb{Q}}(L_1^{k_2}) = k_2 + \frac{2}{2n+1}$$

One can easily calculate the rotation number for the $2(k_1+1)$ candidates as $\operatorname{rot}_{\mathbb{Q}}(L_1^{k_1}) = (-k_1 + 2m) \mp \frac{n+1}{2n+1}$ where $m = 0, 1. \cdots k_1$ and $\operatorname{rot}(L_2^0) = \mp\frac{1}{2n+1}$. The rotation numbers for the $(n+2)(k_1+1)$ candidates are $\operatorname{rot}_{\mathbb{Q}}(L_1^{k_1}) = (-k_1+2m_1) - \frac{(-n-1+2m_2)}{2n+1}$ and $\operatorname{rot}_{\mathbb{Q}}(L_2^1) = \frac{2(-n-1+2m_2)}{2n+1}$ where $m_1 = 0, 1, \cdots k_1$ and $m_2 = 0, 1, \cdots n+1$. Finally, $\operatorname{rot}_{\mathbb{Q}}(L_1^{k_1}) = (-k_1+2m_1) + \frac{(-n+2m_2 \mp 1)}{2n+1}$ and $\operatorname{rot}_{\mathbb{Q}}(L_2^{k_2}) = \frac{2(-n+2m_2)}{2n+1} \mp (k_2 - \frac{2n-1}{2n+1})$ where $m_1 = 0, 1, \cdots k_1$ and $m_2 = 0, 1, \cdots n$ for the $2(n+1)(k_1+1)$ non-loose representatives. Here m, m_1, m_2 counts the number of negative signs in the corresponding continued fraction blocks as shown above.

As before $L_1^{k_1}$ is loose and $L_2^{k_2}$ is non-loose for all $k_1, k_2 \geqslant 0$. Any stabilization of L_2^0 is loose as observed in other cases. For L_2^1, there are $(n+2)(k+1)$ non-loose representatives. Now note that, the complement of these candidates consist of a thickened torus with boundary slope s_{k_1} and $s_{k_2} = 1$. We could break this path into a path from s_{k_1} to $-n$ and then from $-n$ to 1. The last part in a continued fraction block of length $(n+2)$, let us call this P. Now stabilizing L_2^1 corresponds to adding a basic slice of slopes $\{1, \infty\}$ to this complement. Note that, we will only get a consistent shortening after adding a positive (resp. negative) basic slice if the path P has only '+' (resp. '-') signs. Thus we will have $2(k_1+1)$ stabilizations that are non-loose. These coincides with L_2^0. Now for L_2^2, the complement contains a thickened torus of slopes s_{k_1} and $\frac{1}{2}$. We can break this path into a path from s_{k_1} to -1, then a jump from -1 to 0 and finally another jump from 0 to $\frac{1}{2}$. Adding a basic slice of slope $\{\frac{1}{2}, 1\}$ to the complement will shorten the path. If the edge between 0 and $\frac{1}{2}$

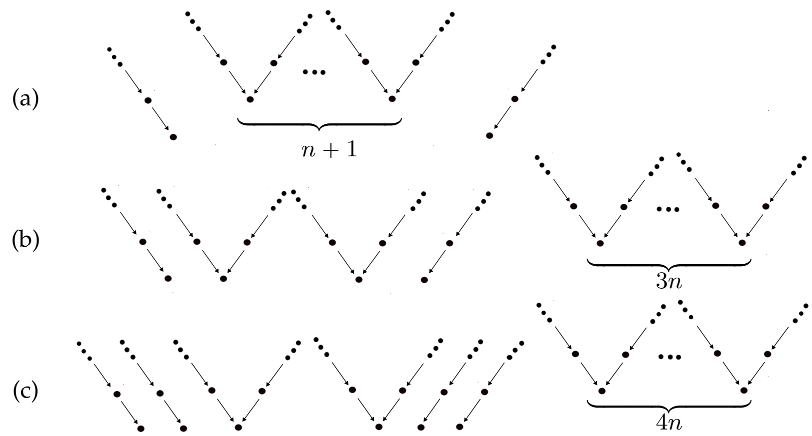

FIGURE 6. Fix K_1 in $L(2n+1,2)$. (a), (b) and (c) show loose mountain range for K_2 when $k_1 = 0, 1, > 1$ respectively.

has a positive sign then a positive stabilization will be non-loose and if it has negative sign then the negative stabilization will be non-loose. Also, notice that after adding the basic slice we will now have a continued fraction block of length $n+1$ and thus there will be $(n+2)(k_1+1)$ non-loose stabilizations that coincide with L_2^1. For $k_2 > 2$, the shortening gives us an edge between 0 to $\frac{1}{k_2-1}$ which is not a part of continued fraction block and thus we will have $2(k_1+1)(n+1)$ non-loose stabilizations that coincide with $L_2^{k_2-1}$. As a pair all of these non-loose Hopf links will be distinguished by their rotation numbers.

Putting all these together we will have: fixing the loose L_1, the non-loose mountain range for L_2 is given by one forward slash based at $(\frac{1}{2n+1}, \frac{2}{2n+1})$, one back slash based at $(-\frac{1}{2n+1}, \frac{2}{2n+1})$ and n Vs based at $(\frac{2(-n-1+2m)}{2n+1}, 1+\frac{2}{2n+1})$ for $m = 1, \cdots, n$. The forward and back slash live in Euler class $\pm(n+1)$ and the Vs live in Euler class $(-n-1+2m)$ for $m = 1, 2, \cdots n$ (we use the 0 sloped disk to evaluate the Euler class). A further careful observation of the pairwise rotation numbers give us the following pairing information:

For non-loose $L_1^{k_1} \sqcup L_2^0$, the corresponding vertices (L_2) from the forward and back slashes pair with a loose cone (L_1) peaked at $(\pm\frac{n+1}{2n+1}, -1+\frac{n+1}{2n+1})$ to give non-loose realizations of Hopf links. For $L_1^{k_1} \sqcup L_2^1$, the corresponding vertices from the forward and back slashes pairs with the same loose cones as before and each based vertex of the Vs pairs with loose cone peaked at $(\frac{2(-n-1+2m)}{2n+1}, -1+\frac{n+1}{2n+1})$ for $m = 1, 2, \cdots, n$. Finally for $L_1^{k_1} \sqcup L_2^{k_2}$, the corresponding vertex from the back and forward slashes pair with the same loose cones, the n vertices from the right wings of the n Vs pair up with loose cones based at $(\frac{-n+2m}{2n+1} + \frac{1}{2n+1}, -1+\frac{n+1}{2n+1})$ for $m = 0, \cdots, n-1$, and the n vertices from the left wings of the n Vs pair up with the loose cones based at $(\frac{-n+2m}{2n+1} - \frac{1}{2n+1}, -1+\frac{n+1}{2n+1})$ for $m = 1, 2, \cdots, n$.

3.5. **Case 2: large slope $\cup$ small slope.** The dividing slopes on the standard neighborhoods of L_1 and L_2 are given by $s_{k_1} = -\frac{n(1+2k_1)+k_1+1}{1+2k_1}$ and $s_{k_2} = -\frac{1}{k_2}$. Note that, $k_1 \geqslant 0$ and $k_2 = 0$ overlaps with the next case.

So we assume $k_2 \geqslant 1$. For $k_1 = 0$, there is a continued fraction block of length n from $-n-1$ to -1 followed by a continued fraction block of length $k_2 - 1$ from -1 to $-\frac{1}{k_2}$, thus corresponds to $(n+1)k_2$ tight contact structures. For $k_1 = 1$, a path from s_{k_1} consists of a continued fraction block of length 2 from $s_{k_1} = -\frac{3n+2}{3}$ to $-n$, followed by a continued fraction block P_1 of length $n-1$

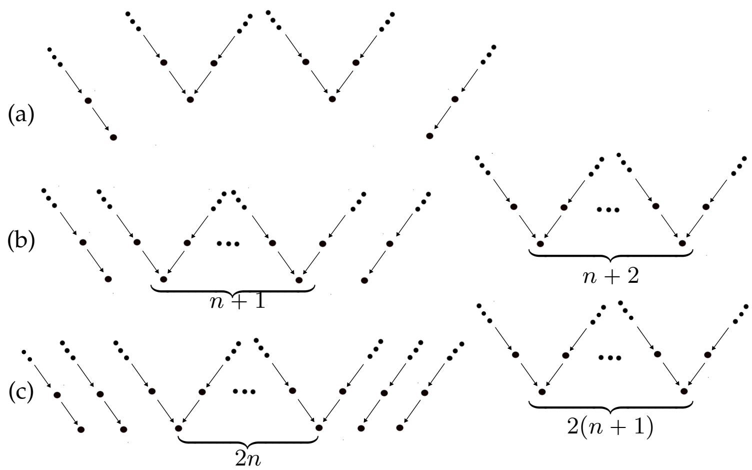

FIGURE 7. Fix K_2. (a), (b) and (c) shows loose mountain range for K_1 when $k_2 = 0, 1, > 1$ respectively.

from $-n$ to -1 and then another continued fraction block P_2 of length $k_2 - 1$ from -1 to $-\frac{1}{k_2}$. This corresponds to $3nk_2$ tight contact structures. Finally, assume $k_1 > 1$. In this case a path from s_{k_1} to s_{k_2} consists of a jump from s_{k_1} to $-\frac{2n+1}{2}$, then another jump from $-\frac{2n+1}{2}$ to $-n$, followed by $P_1 \cup P_2$. This gives us $4nk_2$ tight contact structures. Observe that, in all the cases L_1 is non-loose and L_2 is loose.

Next we calculate the classical invariants. The rational Thurston–Bennequin invariant of the components are given by

$$\operatorname{tb}_{\mathbb{Q}}(L_1^{k_1}) = k_1 + \frac{n+1}{2n+1}$$

and

$$\operatorname{tb}_{\mathbb{Q}}(L_2^{k_2}) = -k_2 + \frac{2}{2n+1}.$$

The rotation numbers are given as $\operatorname{rot}_{\mathbb{Q}}(L_1^0) = \frac{(-n+2m_1)}{2n+1}$ and $\operatorname{rot}_{\mathbb{Q}}(L_2^{k_2}) = \frac{2(-n+2m_1)}{2n+1} + (-k_2 + 1 + 2m_2)$ for $k_2 \geqslant 1, m_1 = 0, 1, \cdots n$ and $m_2 = 0, 1, \cdots, k_2 - 1$. m_1 denotes the number of negative basic slice in the path from $-n-1$ to -1 and m_2 denotes the number of negative basic slices in the path from -1 to $-\frac{1}{k_2}$. For $k_1 = 1, k_2 \geqslant 1$, the rotation numbers are $\operatorname{rot}_{\mathbb{Q}}(K_1^1) = \frac{(n+1)(-2+2m_1)}{2n+1} + \frac{-n+1+2m_2}{2n+1}$ and $\operatorname{rot}_{\mathbb{Q}}(K_2^{k_2}) = \frac{-2+2m_1}{2n+1} + \frac{2(-n+1+2m_2)}{2n+1} + (-k_2 + 1 + 2m_3)$ where $m_1 = 0, 1, 2;\ m_2 = 0, 1, \ldots n-1$ and $m_3 = 0, 1, \cdots k_2 - 1$. m_1 denotes the number of negative basic slice from s_{k_1} to $-n$, m_2 denotes the negative signs in P_1 and m_3 denotes the negative signs in path P_2. Finally for $k_1 > 1$ the rotation numbers are $\operatorname{rot}_{\mathbb{Q}}(L_1^{k_1}) = \mp(k_1 - \frac{n}{2n+1}) \mp \frac{n+1}{2n+1} + \frac{-n+1+2m_2}{2n+1}$ and $\operatorname{rot}_{\mathbb{Q}}(L_2^{k_2}) = \frac{\mp 1 \mp 1 + 2(-n+1+2m_2)}{2n+1} + (-k_1 + 1 + 2m_3)$ where $m_1 = 0, 1, \cdots n-1, m_2 = 0, 1, \cdots k_2 - 1$. m_2 and m_3 counts the number of negative basic slices in the paths P_1 and P_2 as before. The Euler classes can be computed using the techniques mentioned before.

Any stabilization of L_1^0 is loose, but if we consider L_1^0 having all positive signs in the complement then $S_+(L_1^0) \sqcup L_2^{k_2}$ will still be non-loose but will have half-Giroux torsion in the complement. Same is true for $S_-(L_1^0) \sqcup L_2^{k_2}$ when we consider L_1^0 with only negative sign. All other cases, any stabilization will loosen the link. The other stabilizations of $L_1^{k_1}$ are exactly same as in Theorem 1.14 in [CEMM25]. So we omit the details here.

Putting everything together we have the following: Fixing the loose L_2, the non-loose realizations of L_1 form 1 forward slash based at $(\frac{n}{2n+1}, \frac{n+1}{2n+1})$, one back slash based at $(-\frac{n}{2n+1}, \frac{n+1}{2n+1})$, $n-1$

Vs based at $(\frac{(-n+2m_1)}{2n+1}, \frac{n+1}{2n+1})$ where $m_1 = 1, \cdots n-1$ and n Vs based at $(\frac{(-n+1+2m_2)}{2n+1}, 1+\frac{n+1}{2n+1})$ for $m_2 = 0, 1, \cdots n-1$.

For non-loose $L_1^0 \sqcup L_2^{k_2}$ each of the based vertex from the forward slash, the $n-1$ V and the back slash pairs with a loose cone peaked at $(\frac{2(-n+2m_1)}{2n+1}, -1+\frac{2}{2n+1})$ for $m_1 = 0, 1, \cdots n$ respectively. For the non-loose $L_1^1 \sqcup L_2^{k_2}$the corresponding vertex from the forward and back slash pair with loose cones peaked at $(\pm\frac{2n}{2n+1}, -1+\frac{2}{2n+1})$ respectively, the corresponding vertices from the right wings of $n-1$ Vs pair with the loose cones based at $(\frac{2}{2n+1} + \frac{2(-n+1+2m_1)}{2n+1}, -1+\frac{2}{2n+1})$ respectively for $m_1 = 0, \cdots n-2$, the corresponding vertices from the left wings of $n-1$ Vs pair with the loose cones based at $(-\frac{2}{2n+1} + \frac{2(-n+1+2m_1)}{2n+1}, -1+\frac{2}{2n+1})$ for $m_1 = 1, \cdots n-1$, the based vertices of the n Vs pair up with loose cone peaked at $(\frac{2(-n+1+2m_2)}{2n+1}, -1+\frac{2}{2n+1})$ for $m_2 = 0, 1, \cdots, n-1$. Finally, for the non-loose $L_1^{k_1} \sqcup L_2^{k_2}$ for $k_1 > 1$,the corresponding vertices from the forward and back slashes pair up with the loose cones based at $(\pm\frac{2n}{2n+1}, -1+\frac{2}{2n+1})$, the $n-1$ vertices from the right wing of the $n-1$ Vs pair with $n-1$ loose L_2 cones peaked at $(\frac{2}{2n+1} + \frac{2(-n+1+2m_1)}{2n+1}, -1+\frac{2}{2n+1})$ for $m_1 = 0, \cdots, n-2$ and the $n-1$ vertices from the left wing of $n-1$ Vs pair with $n-1$ loose cones peaked at $(-\frac{2}{2n+1} + \frac{2(-n+1+2m_1)}{2n+1}, -1+\frac{2}{2n+1})$ for $m_1 = 1, \cdots, n-1$. The n corresponding vertices from the left wing of the n Vs pair up with a loose cone peaked at $(\frac{2(-n+1+2m_1)}{2n+1}, -1+\frac{2}{2n+1})$ $m_1 = 0, \cdots n-1$, same for the n candidates of the right wings of the n Vs. All of the Hopf links can be distinguished by their pairwise rotation numbers.

3.6. **Case 3: large slope $\cup$ large slope.** The dividing slopes on the standard neighborhoods of L_1 and L_2 are given by $s_{k_1} = -\frac{n(1+2k_1)+k_1+1}{1+2k_1}$ and $s_{k_2} = \frac{1}{k_2}$.

First we assume $k_2 = 0$. For $k_1 = 0$, there is exactly one edge between $-n-1$ to ∞ and this gives 2 tight contact structures and thus 2 non-loose representatives (corresponding to the positive and negative basic slice) whose rotation numbers are $\mathrm{rot}_{\mathbb{Q}}((L_1^0)^{\pm}) = \mp\frac{n+2}{2n+1}$and $\mathrm{rot}_{\mathbb{Q}}((L_2^0)^{\pm}) = \mp\frac{3}{2n+1}$. We denote the two L_1^0 as $L_1^{\pm}$. For $k_1 = 1$, a path from $-\frac{3n+2}{3}$ to ∞ consists of a continued fraction block of length 3 and thus this corresponds to 4 tight contact structures. These 4 L_1^1 we denote as $(L_1^1)^m$ where m denotes the negative basic slices in the above path. For $k_1 > 1$, a path from s_{k_1} to ∞ consists of a jump from s_{k_1} to $-\frac{2n+1}{2}$, then a continued fraction block of length 2 from $-\frac{2n+1}{2}$to ∞. Thus we will have 6 tight contact structures. These 6 candidates are $(L_1^{k_1})^{\pm,m}$ where $\pm$ sign in the sign of the first jump and m counts the number of negative signs in the later path. The rotation numbers of the 4 candidates are $\mathrm{rot}_{\mathbb{Q}}((L_1^1)^m) = \frac{(n+1)(-3+2m)}{2n+1}$ and $\mathrm{rot}_{\mathbb{Q}}((L_2^0)^m) = \frac{-3+2m}{2n+1}$ where $m = 0, 1, 2, 3$. The rotation numbers of the 6 loose/loose pairs are given by $\mathrm{rot}_{\mathbb{Q}}((L_1^{k_1})^{\pm,m}) = \mp(k_1 - \frac{n}{2n+1}) - \frac{(2-2m)(n+1)}{2n+1}$ and $\mathrm{rot}_{\mathbb{Q}}((L_2^0)^{\pm,m}) = \mp\frac{1}{2n+1} - \frac{(2-2m)}{2n+1}$ for $m = 0, 1, 2$. It is easy to check that all of these candidates are componentwise loose. The rational Thurston Bennequin invariants are

$$\mathrm{tb}_{\mathbb{Q}}(L_1^{k_1}) = k_1 + \frac{n+1}{2n+1}$$

and

$$\mathrm{tb}_{\mathbb{Q}}(L_2^0) = \frac{2}{2n+1}.$$

Clearly, for $m = 0$, $S_+((L_1^1)^0) \sqcup (L_2^0)^0$ is non-loose as adding a basic slice in the complement leads to a consistent shortening in the $T^2 \times [0, 1]$ and thus give us a non-loose link. This coincides with the link $(L_1^0)^+ \sqcup (L_2^0)^+$. Same is true when we negatively stabilize $(L_1^1)^3$. These two loose components $(L_1^1)^0$ and $(L_1^1)^3$ are parts of the back and forward slashes. For $m = 1, 2$, any stabilization of $(L_1^1)^m$ will loosen the whole link. Thus these candidates form the base of the two Vs. Now when we stabilize $(L_1^2)^{\pm,m}$ we see that a positive stabilization of $(L_1^2)^{+,m}$ will coincide with $(L_1^2)^{m-1}$ and a negative stabilization of $(L_1^1)^{-,m}$ coincides with $(L_1^1)^{m+1}$.

So putting everything together, we see that fixing the loose L_2 with $k_2 = 0$ the loose L_1 constitutes 1 loose forward slash based at $(\frac{n+2}{2n+1}, \frac{n+1}{2n+1})$, 1 loose back slash based at $(-\frac{n+2}{2n+1}, \frac{n+1}{2n+1})$ and 2 loose Vs based at $(\mp\frac{n+1}{2n+1}, 1+\frac{n+1}{2n+1})$.

If one fixes L_2 with $k_2 = 1$, for $k_1 = 0$ we will see a continued fraction block of length $n+2$ from $-n-1$ to 1, for $k_2 = 1$, we have one continued fraction block of length 2 from $-\frac{3n+2}{3}$ to $-n$ and another continued fraction block of length $n+1$ from $-n$ to 1. Finally for $k_1 > 1$, we have one jump from s_{k_1} to $-\frac{2n+1}{2}$, another jump from $-\frac{2n+1}{2}$ to $-n$ and then the same path as before. So we will have $n+3$, $3(n+2)$ and $4(n+2)$ non-loose Hopf links respectively. The rotation numbers of the components are given in Table 3. We now look at the stabilizations of L_1. Like before we denote the $(n+3)$ L_1^0 as $(L_1^0)^m$ where m denote the number of negative signs in the length $n+2$ path. For $k_1 = 1$, we denote the $3(n+2)$ candidates as $(L_1^1)^{m_1,m_2}$ where m_1 counts the number of negative signs in the path from $-\frac{3n+2}{3}$ to $-n$ and m_2 counts the number of negative signs in the next $n+1$ length path. Note that, $S_+((L_1^1)^{0,0}$ coincides with $(L_1^0)^0$ and $S_-((L_1^1)^{2,n+1}$ coincides with $(L_1^0)^3$. These form the base of the back and the forward slash respectively. The $S_+((L_1^1)^{0,m_2}$ and $S_-((L_1^1)^{2,m_2}$ will form the bases of the $n+1$ Vs for $m_2 = 1, \cdots n+1$. Any stabilization of $(L_1^1)^{1,m_2}$ will loosen the whole link. Thus $(L_1^1)^{1,m_2}$ are the bases of the $n+2$ Vs.

Putting everything together, we have the following loose mountain range for L_1 fixing L_2 with $k_2 = 1$, 1 forward and 1 back slash based at $(\pm\frac{n+2}{2n+1}, \frac{n+1}{2n+1})$ respectively , $n+1$ loose V s based at $(\frac{-n-2+2m_1}{2n+1}, \frac{n+1}{2n+1})$ and $n+2$ loose Vs based at $(\frac{-n-1+2m_2}{2n+1}, 1+\frac{n+1}{2n+1})$ where $m_1 = 0, \cdots n$ and $m_2 = 0, 1, \cdots n+1$.

One can fix L_2 with $k_2 > 1$ and can do a similar analysis to see how the mountain range changes for L_1. Check (c) of Figure 7. The mountain range for L_2 fixing L_1 is given in Figure 6.

For fixed k_1 and k_2, they pair with each other to give non-loose Hopf links with zero Giroux torsion in the complement. Note that, the complement of $S_+{}^k(L_1^0) \sqcup S_+{}^{k'}(L_2^0)$ contains a half Giroux torsion layer if $k+k' = 1$ and it contains a full Giroux torsion layer if $k+k' > 1$. If one component is stabilized positively and the other one negatively that loosens the link.

k_1	k_2	numbers	$\mathrm{rot}_{\mathbb{Q}}(L_1)$	$\mathrm{rot}_{\mathbb{Q}}(L_2)$	range of m
0	0	2	$\mp\frac{n+2}{2n+1}$	$\mp\frac{3}{2n+1}$	
0	1	$(n+3)$	$\frac{-n-2+2m}{2n+1}$	$\frac{2(-n-2+2m)}{2n+1}$	$0 \leqslant m \leqslant n+2$
0	>1	$2(n+2)$	$\frac{-n-1+2m}{2n+1} \mp \frac{1}{2n+1}$	$\mp(k_2 - \frac{2n-1}{2n+1}) + \frac{2(-n-1+2m_1)}{2n+1}$	$0 \leqslant m \leqslant n+1$
1	0	4	$\frac{(2m-3)(1+n)}{2n+1}$	$\frac{2m-3}{2n+1}$	$0 \leqslant m \leqslant 3.$
1	1	$3(n+2)$	$\frac{(-2+2m_1)(n+1)}{2n+1} + \frac{(-n-1+2m_2)}{2n+1}$	$\frac{(-2+2m_1)}{2n+1} + \frac{2(-n-1+2m_2)}{2n+1}$	$0 \leqslant m_1 \leqslant 2$ $0 \leqslant m_2 \leqslant n+1$
1	>1	$6(n+1)$	$\frac{(-2+2m_1)(n+1)}{2n+1} + \frac{(-n+2m_2)}{2n+1} \mp \frac{1}{2n+1}$	$\frac{(-2+2m_1)}{2n+1} + \frac{2(-n+2m_2)}{2n+1} \mp (k_2 - \frac{2n-1}{2n+1})$	$0 \leqslant m_1 \leqslant 2$ $0 \leqslant m_2 \leqslant n$
>1	0	6	$\mp k_1 \pm \frac{n}{2n+1} - \frac{(n+1)(2-2m)}{2n+1}$	$\mp\frac{1}{2n+1} - \frac{2-2m}{2n+1}$	$0 \leqslant m \leqslant 2$
>1	1	$4(n+2)$	$\mp(k_1 - \frac{n}{2n+1}) \mp \frac{n+1}{2n+1} + \frac{-n-1+2m}{2n+1}$	$\mp\frac{1}{2n+1} \mp \frac{1}{2n+1} + \frac{2(-n-1+2m)}{2n+1}$	$0 \leqslant m \leqslant n+1$
>1	>1	$8(n+1)$	$\mp(k_1 - \frac{n}{2n+1}) \mp \frac{n+1}{2n+1} + \frac{-n+2m}{2n+1} \mp \frac{1}{2n+1}$	$\mp\frac{1}{2n+1} \mp \frac{1}{2n+1} + \frac{2(-n+2m)}{2n+1} \mp (k_2 - \frac{2n-1}{2n+1})$	$0 \leqslant m \leqslant n$

TABLE 3. Rotation numbers of non-loose Hopf links in $L(2n+1, 2)$ for $n \geqslant 1$ and for large $\cup$ large slope.

$\square$

4. The general Classification result

In this section we prove the general classification of rational Hopf links. As we are classifying the positive Hopf links, we consider L_1, L_2 with opposite orientations. A classification for the negative Hopf link can be done by switching the orientation of L_2, thus changing the signs of the rotation numbers and switching the back and the forward slashes.

Proof of Theorem 1.6. Consider the lens space $L(p.q)$ where $q \neq 1$ or $p-1$. Let $-\frac{p}{q} = [a_0, a_1, \cdots a_n]$. Note that, $(-\frac{p}{q})^a = [a_0, a_1, \cdots a_{n-1}]$ and we denote this as $-\frac{p'}{q'}$ where $1 \leqslant p' \leqslant p$ and $p'q \equiv 1 \pmod p$. On the other hand, we denote $(-\frac{p}{q})^c = [a_0, a_1, \cdots, a_n + 1]$ as $-\frac{p''}{q''}$ where $1 \leqslant p'' \leqslant p$ and $p''q \equiv p-1 \pmod p$.

Like in the previous section, for non-loose representatives of $L_1 \cup L_2$ we need to consider the following pairs of slopes. small slope $\cup$ large slope, large slope $\cup$ small slope and large slope $\cup$ large slope.

4.1. **Case 1: small slope $\cup$ large slope.** Note that, in this case, the possible dividing slopes for a standard neighborhood of a non-loose representative $L_1 \sqcup L_2$ are $-\frac{p''}{q''} \oplus k_1(-\frac{p}{q}) = -\frac{p''+k_1p}{q''+k_1q}$ for $k_1 \geqslant 0$ and $\frac{1}{k_2}$ for $k_2 \geqslant 0$. To understand the non-loose representatives we need to understand the signed paths describing the tight contact structures on $L(p,q)\backslash(V_1 \cup V_2) = T^2 \times I$ with boundary conditions s_{k_1} and s_{k_2}. Now for $n \geqslant 2$ ($n = 1$ case is slightly different which is left for the reader), we will divide this path into two sub-paths. One from s_{k_1} clockwise to $-\frac{p''}{q''}$ which is a continued fraction block of length k_1 , then a path P_1 from $-\frac{p''}{q''}$ clockwise to $s_{k_2} = \frac{1}{k_2}$. Note that, depending on the value of k_2 the path from $-\frac{p''}{q''}$ to $\frac{1}{k_2}$ will be different. For $k_2 = 0$, decorations of the path P_1 will correspond to $A_1 = |a_1| \cdots |a_n + 1|$ tight contact structures and thus we have $A_1(k_1 + 1)$ non-loose representatives. For $k_2 = 1$, decorations of this path correspond to $A_2 = |a_0 - 1||a_1 + 1| \cdots |a_n + 1|$ tight contact structures. Thus we have $A_2(k_1 + 1)$ non-loose representatives in this case. Finally, when we consider $k_2 > 1$, we can subdivide P_1 into two parts: one from $-\frac{p''}{q''}$ to 0 and then a jump from 0 to $\frac{1}{k_2}$. Thus decorations of this path gives us $A_3 = 2|a_0||a_1 + 1| \cdots |a_n + 1|$ tight contact structures and we will have a total $A_3(k_1 + 1)$ non-loose Hopf links. It is easy to see from the Farey graph that with these dividing slopes, we will have L_1 loose and L_2 non-loose.

Next we will show how these non-loose candidates are related by stabilizations. Note that, L_1 is already loose and we will have a loose mountain peaked at $(r, -1 - \frac{p''}{p})$ where r is the rotation number that can be calculated from the algorithm given in Section 2. We will now consider the stabilizations of L_2. We will denote the component L_2 with its standard neighborhood having slope s_{k_2} as $L_2^{k_2}$.

By Lemma 2.9 any stabilization of L_2^0 is loose. Note that, the link can still be non-loose. Suppose P' be the path that describes the complement of $L_1 \sqcup L_2^0$. This is a union of continued fraction blocks. Stabilizing L_2 means adding a $\pm$ basic slice $B_\pm$ with boundary slopes ∞ and -1 to the back face $T^2 \times \{1\}$. If all the continued fraction blocks have the same sign, say positive then $L_1 \sqcup S_+(L_2^0)$ is still non-loose but will have a half- Giroux torsion in its complement. The same is true if the path contains only negative signs and L_2 is negatively stabilized. But if any of the continued fraction blocks has a mix of sign, adding $B_\pm$ will lead to an inconsistent shortening in $T^2 \times I$ and thus the link will be loose. The link component $(L_2^0)^{m,n}$ with $m = |a_1 + 1|$ and $m = 0$, is the base of $|a_2 + 1| \cdots |a_n + 1|$ forward slashes and $|a_2 + 1| \cdots |a_n + 1|$ back slashes respectively. For $1 \leqslant m, n \leqslant |a_1 + 2|$, $(L_2^0)_{m,n}$ are the base of $|a_1 + 2||a_2 + 1| \cdots |a_n + 1|$ non-loose $V's$.

Now we consider the stabilizations of L_2^1. Note that there are $A_2(k_1 + 1)$ non-loose representatives. The path P from $a_0 + 1$ to 1 has $|a_0|$ basic slices and suppose the decoration of this path has i positive signs and $|a_0|-i$ negative signs. We call the L_2^1 whose complement correspond to these decorations as $(L_2^1)^{i,j}$ where i and j are the number of positive and negative basic slices in P. Clearly, the only two cases where we will see consistent shortening are when we positively stabilize $(L_2^1)^{|a_0|,0}$ or when we negatively stabilize $(L_2^1)^{0,|a_0|}$. Any stabilization of $(L_2^1)^{i,j}$ where $1 \leqslant i,j \leqslant |a_0+1|$ is loose and these $(L_2^1)^{i,j}$ also give us the base of $|a_0+1||a_1+1|\cdots|a_n+1|$ non-loose V's.

Finally for $L_2^{k_2}$ with $k_2>1$, we need to consider two cases. For L_2^2, the path from a_0+1 to $\frac{1}{2}$ can have the following decorations: k positive basic slices and l negative basic slices between a_0+1 to 0 where $0 \leqslant k,l \leqslant |a_0+1|$ and then a $\pm$ basic slice with boundary slopes 0 and $\frac{1}{2}$. Let us call the corresponding L_2^2 as $(L_2^2)^{i,j,\pm}$. In this case, stabilizing corresponds to adding a basic slice $B_\pm$ with boundary slopes $\frac{1}{2}$ and 1. For the same reason as before we will have $S_+((L_2^2)^{k,l,+})$, $S_-((L_2^2)^{k,l,-})$ non-loose and correspond to $(L_2^1)^{k+1,l}$ and $(L_2^1)^{k,l+1}$ where $0 \leqslant k,l \leqslant |a_0+1|$. On the other hand $S_-((L_2^2)^{k,l,+})$, $S_-((L_2^2)^{k,l,+})$ will be loose. For $k_2>2$, the path describing the contact structure of the complement contains a continued fraction block of length $|a_0+1|$ from a_0+1 to 0 with k positive and l negative basic slices as before and a single jump from 0 to $\frac{1}{k_2}$. Note that, stabilizing in this case corresponds to adding a basic slice of slope $\frac{1}{k_2}$ and $\frac{1}{k_2-1}$ in the complement. Like before we denote the link component L_2 corresponding to these paths as $(L_2^{k_2})^{k,l,\pm}$. We will have $S_+((L_2^{k_2})^{k,l,+})$, $S_-((L_2^{k_2})^{k,l,-})$ non-loose and these coincides with $(L_2^{k_2-1})^{k,l,+}$ and $(L_2^{k_2-1})^{k,l,-}$.

Using the method mentioned in Section 2 one could easily compute the rational Thurston-Bennequin invariants as $\mathrm{tb}_{\mathbb{Q}}(L_1^{s_{k_1}})=-k_1-\frac{p''}{p}$ and $\mathrm{tb}_{\mathbb{Q}}(L_1^{s_{k_2}})=k_2+\frac{q}{p}$.

Now putting these together , we have the following mountain range for $L_2^{s_{k_2}}$ fixing L_1 and with zero Giroux torsion in the complement of $L_1 \sqcup L_2$: We have $|a_2+1||a_3+1|\cdots|a_n+1|$ many forward and $|a_2+1||a_3+1|\cdots|a_n+1|$ many backward slashes based at $(r_i,\frac{q}{p})$, $|a_1+2||a_2+1|\cdots|a_n+1|$ non-loose V's based at $(r_j,\frac{q}{p})$. Finally there are $|a_0+1||a_1+1|\cdots|a_n+1|$ many V's based at $(r_k,1+\frac{q}{p})$. The d_3 invariants can be computed from the surgery diagrams given in Section 5 using the techniques of [DGS04]. The rotation number and Euler classes can be computed using the algorithm mentioned in Section 2. Note that, as L_1 is loose, we will have a loose cone peaked at $(r,-1-\frac{p''}{p})$ where value of r can be computed using the technique mentioned before. A similar analysis can be done for $n=1$. For $n=1, k_2=0$, we note that we have a continued fraction block of length $|a_1+1|$ from $-\frac{p'}{q'}$ to ∞ and thus we will have $|a_1|(k_2+1)$ non-loose representatives in this case. For $k_2 \geqslant 1$, the analysis is exactly the same as $n \geqslant 2$. Putting everything together for L_2, we will have, 1 forward and 1 back slash, $|a_1+2|$ $V's$ based at $(r,\frac{q}{p})$ and $|a_0+1||a_1+1|$ V's based at $(r,1+\frac{q}{p})$ when L_1 is fixed.

4.2. **Case 2 Large slope $\cup$ small slope.** Note that, we start with $k_1 \geqslant 0$ and $k_2 \geqslant 1$ as $k_2=0$ and $k_1 \geqslant 0$ overlaps with the next case. This particular case is very similar to the proof of Theorem 1.15 in [CEMM25]. Here we have $s_{k_1}=-\frac{p'+k_1p}{q'+k_1q}$ and $s_{k_2}=-\frac{1}{k_2}$ for $k_i \geqslant 0$. For $n \geqslant 2$, denote the path from $r=[a_0,a_1,\cdots a_{n-2}+1]$ to -1 in the Farey graph by P_2. Decorations on this path will correspond to $B=|a_0+1||a_1+1|\cdots|a_{n-2}+1|$ tight contact structures on $T^2\times I$ with boundary slopes r and -1. Note that, any path from s_{k_1} clockwise to -1 will contain this path. Now any path from s_{k_1} clockwise to s_{k_2} can be broken into the following sub-paths: a path from s_{k_1} to r, then P_2 and finally a path from -1 to $-\frac{1}{k_2}$. The last portion of the path is a continued fraction block of length k_2-1 and thus decorations on this path will correspond to k_2 tight contact structures. We denote this path by P_3. Now we will look at the path for different values of k_1. In particular for $k_1=0$, we see that the path from s_0 to s_{k_2} consists of a continued fraction block of length $|a_{n-1}+1|$ from $-\frac{p'}{q'}$ to r and then P_2 followed by P_3. This gives $|a_{n-1}|Bk_2$ tight contact structures. For $k_1=1$, we see that that the path from s_1 to $-\frac{1}{k_2}$ consists of a continued fraction block of length $|a_n|$ from s_1 to $s=[a_0,a_1,\cdots a_{n-1}+1]$, then a continued fraction block of length $|a_{n-1}+2|$ from s to r, followed by P_2 and P_3. This gives $|a_n-1||a_{n-1}+1|Bk_2$ tight contact structures. For $k_1>1$, the path from s_{k_1} to s_{k_2} consists of one jump from s_{k_1} to $-\frac{p}{q}$, followed by a continued fraction block of length $|a_n+1|$ from $-\frac{p}{q}$ to $s=[a_0,a_1,\cdots a_{n-1}+1]$, then a continued fraction block of length $|a_{n-1}+2|$ from s to r, then P_2 and finally P_3. This gives $2|a_n||a_{n-1}+1|Bk_2$ tight contact structures. It is easy to check that in this case, L_1 is non-loose and L_2 is loose. Like Case 4.1, now we will check the different stabilizations of L_1 and see how these stabilizations are related with each other. By $L_j^{k_i}$ we denote the Legendrian unknot whose standard neighborhood has dividing slope s_{k_i}. Note that, any stabilization of L_1^0 will be loose by Lemma 2.9 but the link still can be non-loose. The complement of $L_1^0\sqcup L_2$ contains a union of continued fraction blocks from $-\frac{p'}{q'}$ to s_{k_2}. Lets call it P'. While stabilizing L_1^0 we are adding a basic slice with boundary slopes $-\frac{p''}{q''}$ and $-\frac{p'}{q'}$. Thus whenever P' has a mix of sign, we can inconsistently shorten the path and $T^2\times I$ becomes overtwistsed loosening the link. So, when P' only contains positive signs, $S_+(L_1^0)\sqcup L_2$ is still non-loose but contains a half Giroux torsion in its complement. Same is true if P' just contains negative basic slices and we negatively stabilize L_1^0. Thus there will be two non-loose L_1^0 which stays non-loose when included in the link but the link now will have a half-Giroux torsion in the complement. The stabilizations of $L_1^{s_{k_1}}$ with $k_1\geqslant 1$ are exactly the same as shown in the proof of Theroem 1.15 in [CEMM25]. So, we omit the details. An interested reader is suggested to check that proof.

For $n=1$, we note that $-\frac{p'}{q'}=[a_0]$ and $s=[a_0+1]$ are integers. So, any path from s_{k_1} to s_{k_2} will be broken as follows. For $k_1=0$, a path from s_0 to $-\frac{1}{k_2}$ consists of a continued fraction block of length $|a_0+1|$ from s_0 to -1 followed by another continued fraction block of length $|k_2-1|$, denoted by P_3 from -1 to $-\frac{1}{k_2}$. Decorations on this path correspond to $|a_0|k_2$ tight contact structures. For $k_1=1$, the path from s_{k_1} to $-\frac{1}{k_2}$ consists of a continued fraction block of length $|a_1|$ from s_1 to s, followed by a continued fraction block of length $|a_0+2|$ from s to -1 and finally P_3. This corresponds to $|a_1-1||a_0+1|k_2$ contact structures. For $k_1>1$, the path will consist of one jump from s_{k_1} to $-\frac{p}{q}$, followed by a continued fraction block of length $|a_1+1|$ from $-\frac{p}{q}$ to s, followed by a continued fraction block of length $|a_0+2|$ from s to -1 and finally P_3. This corresponds to $2|a_1||a_0+1|k_2$ contact structures.

The rational Thurston–Bennequin invariant of the components are given by $\operatorname{tb}_{\mathbb{Q}}(L_1^{s_{k_1}})=k_1+\frac{p'}{p}$ and $\operatorname{tb}_{\mathbb{Q}}(L_2^{s_{k_2}})=-k_2+\frac{q}{p}$. Putting everything together we see the following non-loose mountain range for L_1: Fixing the loose component L_2, there are $|a_0+1|\cdots|a_{n-2}+1|$ forward slashes and the same number of backward slashes based at $(r_i,\frac{p'}{p})$, $|a_0+1|\cdots|a_{n-2}+1||a_{n-1}+2|$ V's based at $(r_j,\frac{p'}{p})$ and finally another $|a_0+1|\cdots|a_{n-1}+1||a_n+1|$ V's based at $(r_k,1+\frac{p'}{p})$. For $n=1$, we will have the following mountain range : 1 forward and 1 back slash, $|a_0+2|$ V's based at $(r,\frac{p'}{p})$ and additionally $|a_0+1||a_1+1|$ V's based ar $(r,1+\frac{p'}{p})$.

4.3. **Case 3: Large slope $\cup$ Large slope.** Here we will consider both large slopes. The dividing slope of the standard neighborhod of L_1 and L_2 in this case are given by $s_{k_1}=-\frac{p'+k_1p}{q'+k_1q}$ and $s_{k_2}=\frac{1}{k_2}$ for $k_i\geqslant 0$. Note that, the path from s_{k_1} to s_{k_2} will be very similar to our last case except that now the path will go to $\frac{1}{k_2}$. First we consider $k_2=0$. For $n\geqslant 3$, denote the path from $r=[a_0,a_1,a_{n-2}+1]$ to ∞ in the Farey graph by P_2 (Note that, this path will go from r to a_0+1 and then take a jump to infinity) . Decorations on this path correspond to $A_1=|a_1||a_2+1|\cdots|a_{n-2}+1|$ tight contact structures. Any path from s_{k_1} to s_{k_2} will contain this path for $k_2=0$. So after fixing $k_2=0$,

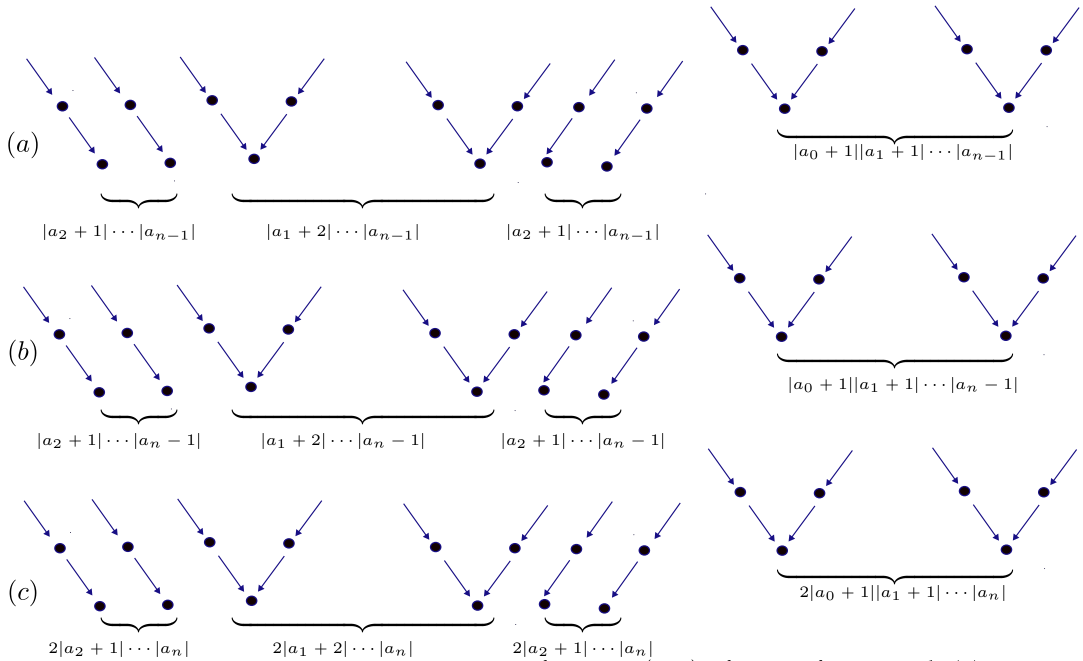

FIGURE 8. Loose mountain range of L_2 in $L(p,q)$ after we fix L_1 with (a) $k_1=0$, (b) $k_1=1$ and (c) $k_1>1$.

we will vary k_1. Now as before, for $k_1=0$, a path from $-\frac{p'}{q'}$ to r consists of a continued fraction block of length $|a_{n-1}+1|$. Thus from $-\frac{p'}{q'}$ to ∞ we have a total $A_1|a_{n-1}|$ tight contact structures. Following same method as Case 4.2 for $k_1=1$ and $k_1>1$, the path gives us $A_1|a_{n-1}+1||a_n-1|$ and $2A_1|a_{n-1}+1||a_n|$ tight contact structures respectively. Now for $k_2=1$ we see that a path from r to 1 will give us $A_2=|a_0-1||a_1+1|\cdots|a_{n-2}+1|$ tight contact structures. Finally for $k_2\geqslant 2$, a path from r to $\frac{1}{k_2}$ can be subdivided into a path from r to 0 and then a jump from 0 to $\frac{1}{k_2}$.The first part gives us $A_3=|a_0-1||a_1+1|\cdots|a_{n-2}+1|$ tight contact structures. So in total this path corresponds to $2A_3$ tight contact structures. Thus for $k_2=1$ and $k_2\geqslant 2$, we can vary k_1 as before and see the numbers of tight contact structures are as shown in Table 4.3. Note that, for $n=1$ and $n=2$, we have few cases where the numbers are different. Those are given in Table 4.3.

It is easy to see that both components in each of the cases are loose. Next, we talk about the stabilizations of the components. Note that when we talk about stabilizations we are basically talking about the stabilizations of the loose component but these stabilizations still keep the link non-loose. As before we will consider the stabilizations for $n\geqslant 3$, the other cases can be analyzed similarly. First, we fix L_1 with $k_1=0$ and analyze the loose mountain range for L_2 while varying k_2. The complement of $L_1^0\sqcup L_2^0$ contains a union of continued fraction block. When all the continued fraction blocks are of the same sign,say positive, then $L_1^0\sqcup S_+(L_2^0)$ is still non-loose but has a half Giroux torsion in the complement. Same is true if all the continued fraction blocks are negative and we negatively stabilize L_2^0. Every other combination will loosen the link. Now note that, the complement of $L_1^0\sqcup L_2^0$ contains a continued fraction block of length $|a_1+1|$ from $|a_0+1|$ to ∞. We call this path P and the corresponding L_2^0, as $(L_2^0)_{i,j}$ where $0\leqslant i,j\leqslant |a_1+1|$. For $i=|a_1+1|, j=0$ and $i=0, j=|a_1+1|$, $(L_2^0)_{i,j}$ are the bases of $|a_2+1|\cdots|a_{n-1}|$ loose forward and the same number of loose back slashes. For $1\leqslant i,j\leqslant |a_1+2|$, $(L_2)^0_{i,j}$ are bases of $|a_1+2||a_2+1|\cdots|a_{n-1}|$ loose V's.

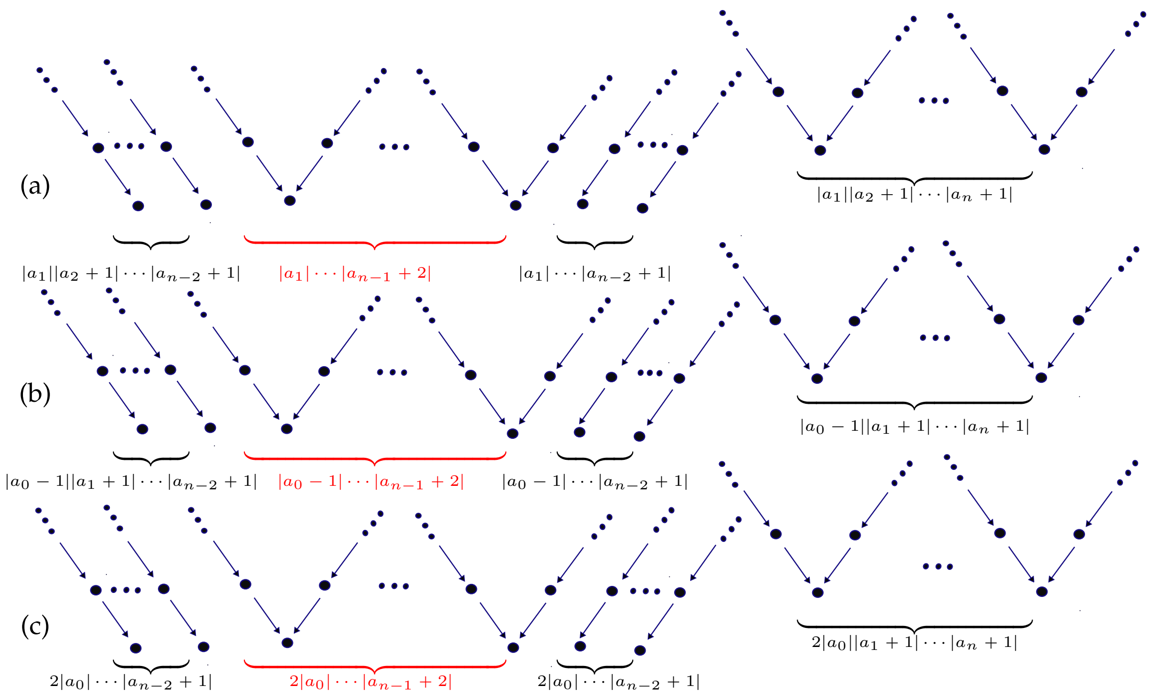

FIGURE 9. (a), (b) and (c) shows the loose mountain range for K_1 when K_2 is fixed with $k_2=0$,$k_2=1$,and $k_2\geqslant 2$. Here $n\geqslant 3$.

Now we consider L_2^1. The complement of L_2^1 contains a path P_1 of length $|a_0|$ from a_0+1 to 1 and another continued fraction block P_2 of length $|a_1+2|$ from $[a_0,a_1+1]$ to a_0+1. Let us suppose now that P_1 contains i positive basic slices, j negative basic slices and P_2 contains k positive basic slices and l negative basic slices. We call the corresponding L_2^1 as $(L_2)^1_{i,j,k,l}$ where $0\leqslant i,j\leqslant |a_0|$, $0\leqslant k,l\leqslant |a_1+2|$. As we can consistently shorten the path in the complement, $L_1\sqcup S_+((L_2^1)_{|a_0|,0,|a_1+2|,0})$ and $L_1\sqcup S_-((L_2^1)_{0,|a_0|,0,|a_1+2|})$ are still non-loose. In fact, one could check $S_+((L_2^1)_{|a_0|,0,|a_1+2|,0})$ coincides with $(L_2^0)_{|a_1+1|,0}$ and $S_-((L_2^1)_{0,|a_0|,0,|a_1+2|})$ coincides with $(L_2^0)_{0,|a_1+1|}$. For $i=|a_0|, j=0$ and $1\leqslant k,l\leqslant |a_1+3|$ and $i=0, j=|a_0|$ and $1\leqslant k,l\leqslant |a_1+3|$, the $S_\pm((L_2^0)_{i,j,k,l}$ corresponds to $(L_2^0)_{i,j}$ where $1\leqslant i,j\leqslant |a_1+2|$. When $1\leqslant i,j\leqslant |a_0+1|$, any stabilization will give us inconsistent shortening in the complement and $L_1^0\sqcup S_\pm((L_2^1)_{i,j,k.l})$ will be loose for any k,l. There are the bases of $|a_0+1||a_1+1|\cdots|a_{n-1}|$ many loose V's.

Now if we fix L_1 with $k_1=1$ or $k_1\geqslant 2$, the mountain range will change. The analysis is exactly the same as before, the only difference is in the number of forward, back slashes and V's as shown in Figure 8.

Now one can fix L_2 with $k_2=0$ and analyze the loose mountain range of L_1. We will do the detailed analysis for this case and the rest will follow similarly. Now if all the continued fraction blocks in the complement of $L_1^0\sqcup L_2^0$ have the same sign, say positive then $S_+(L_1^0)\sqcup L_2^0$ will still be non-loose, but will have a half Giroux torsion in the complement. Same is true if the continued fraction blocks are all negative and we negatively stabilize L_1. Note that, when we stabilize L_1^0 we add a basic slice of boundary slopes $\frac{p'}{q'}$ and $\frac{p''}{q''}$ in the complement. If there is any mix of sign in any of the continued fraction blocks, then any stabilization of L_1 will loosen the link. Note that, in the complement of $L_1^0\sqcup L_2^0$, we have a continued fraction block of length $|a_{n-1}+1|$ from $-\frac{p'}{q'}$ to r. We denote this path by P and suppose there are i positive basic slices and j negative basic

k_1	k_2	$n=1$	$n=2$	$n>2$
0	0	2	$\lvert a_1-1\rvert$	$\lvert a_1\rvert\lvert a_2+1\rvert\cdots\lvert a_{n-1}\rvert$
0	1	$\lvert a_0-2\rvert$		$\lvert a_0-1\rvert\lvert a_1+1\rvert\cdots\lvert a_{n-1}\rvert$
0	>1	$2\lvert a_0-1\rvert$		$2\lvert a_0\rvert\lvert a_1+1\rvert\cdots\lvert a_{n-1}\rvert$
1	0	$\lvert a_1-2\rvert$		$\lvert a_1\rvert\lvert a_2+1\rvert\cdots\lvert a_{n-1}+1\rvert\lvert a_n-1\rvert$
1	1			$\lvert a_0-1\rvert\lvert a_1+1\rvert\cdots\lvert a_n-1\rvert$
1	>1			$2\lvert a_0\rvert\lvert a_1+1\rvert\cdots\lvert a_{n-1}+1\rvert\lvert a_n-1\rvert$
>1	0	$2\lvert a_1-1\rvert$		$2\lvert a_1\rvert\lvert a_2+1\rvert\cdots\lvert a_n\rvert$
>1	1			$2\lvert a_0-1\rvert\lvert a_1+1\rvert\cdots\lvert a_{n-1}+1\rvert\lvert a_n\rvert$
>1	>1			$4\lvert a_0\rvert\lvert a_1+1\rvert\cdots\lvert a_{n-1}+1\rvert\lvert a_n\rvert$

TABLE 4. Number of non-loose Hopf links in $L(p,q)$ with both components loose where $-\frac{p}{q}=[a_0,a_1,\cdots,a_n]$

slices. For $i=|a_{n-1}+1|, j=0$, $(L_1^0)_{i,j}$ are the bases of $|a_1||a_2+1|\cdots|a_{n-2}+1|$ many forward slashes and for $i=0, j=|a_{n-1}+1|$, $(L_1^0)_{i,j}$ are the bases of the same number of backward slashes. For $1\leqslant i,j\geqslant|a_{n-1}+2|$, $(L_1^0)_{i,j}$ they will be the bases of $|a_1|\cdots|a_{n-1}+2|$ many Vs. Now when we consider L_1^1 the complement contains a continued fraction block of length $|a_n|$ from s_1 to s and then another one of length $|a_{n-1}+2|$ from s to r. We call the first path P_1 and then P_2. Suppose P_1 contains i' positive signs and j' negative signs and P_2 contains k positive signs and l negative signs. The corresponding L_1^1 will be denoted as $(L_1^1)_{i',j',k,l}$ For $i'=|a_n|, j'=0$, $S_+((L_1^1)_{i,j,k,l}$ are non-loose for any k,l and for $i'=0, j'=|a_n|$, $S_+((L_1^1)_{i',j',k,l}$ are non-loose for any k,l. These will be part of the forward, back slash and the V's from above. For $i'=|a_n|, j'=0, k=|a_{n-1}+2|$ and $l=0$ $S_+((L_1^1)_{i',j',k,l}$ is same as $(L_1^0)_{|a_{n-1}+1|,0}$. On the other hand, for $i'=0, j'=|a_n|, k=0, l=|a_{n-1}+2|$, $S_-((L_1^1)_{i',j',k,l}$ is same as $(L_1^0)_{0,|a_{n-1}+1|}$. When P_1 has all same (positive or negative) sign and $1\leqslant k,l\leqslant|a_{n-1}+3|$, $S_\pm((L_1^1)_{i',j'k,l})$ will coincide with $(L_1^0)_{i,j}$ where $1\leqslant i,j\leqslant|a_{n-1}+2|$. Finally for $1\leqslant i',j'\leqslant|a_n+1|$, $(L_1^1)_{i',j',k,l}$ are the bases of $|a_1||a_2+1|\cdots|a_n+1|$ loose V's.

For fixed K_2 with $k_2=1, k_2\geqslant 2$, the mountain range is given in Figure 9 (b), (c). The analysis is very similar to the $k_2=0$ case, so we omit the details here.

The rational Thurston–Bennequin invariants of the components are given by $\mathrm{tb}_{\mathbb{Q}}(L_1^{s_{k_1}})=k_1+\frac{p'}{p}$ and $\mathrm{tb}_{\mathbb{Q}}(L_1^{s_{k_2}})=k_2+\frac{q}{p}$. The rotation number and the Euler classes can be computed as before. This completes the proof of Theorem 1.6.

□

Proof. Proof of Theorem 1.7 Note that, to add Giroux torsion in the link complement we remove a tubular neighborhood of $L_1\sqcup L_2$ and add a Giroux torsion layer with $m\pi$ twisting. The complement of $L_1\sqcup L_2$ is a $T^2\times I$ which is a union of continued fraction blocks. Clearly, if there is any mismatch of sign once we add a layer, that leads to an inconsistent shortening and the complement will become overtwisted. Thus except the one case when $q=1$ and $k_1=0=k_2$, there are exactly two non-loose $L_1\sqcup L_2$, one with the complement having only positive continued fraction blocks and one with only negative union of continued fraction blocks. In these two cases we can add any layer of torsion (consistent with the sign of the continued fraction blocks) and the complement is still tight. Notice that, when $q=1$ and $k_1=k_2=0$, the complement is I-invariant. So there is only one such non-loose Hopf link and we can add any twising in the complement keeping the

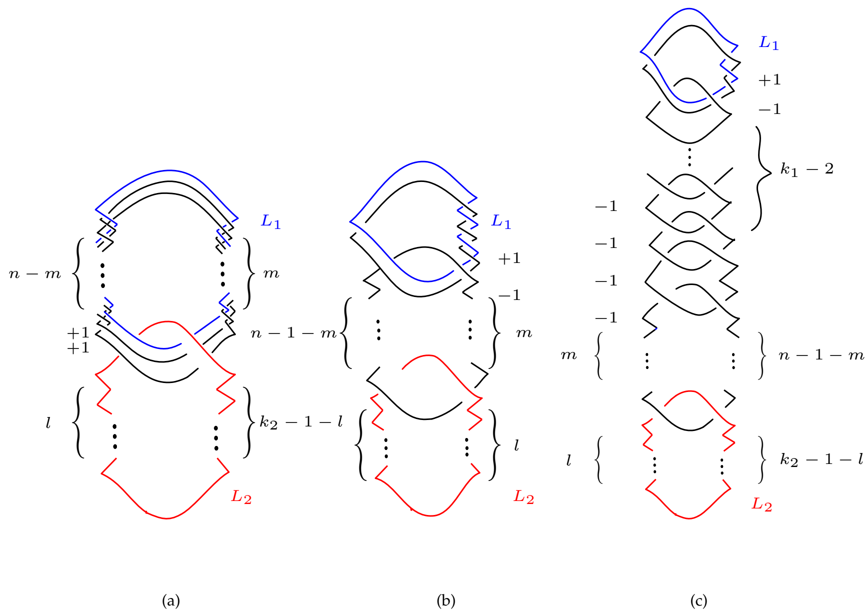

FIGURE 10. Non-loose $L_1 \sqcup L_2$ in $L(2n+1, 2)$ with $\mathrm{tb}_{\mathbb{Q}}(L_1) = k_1 + \frac{n+1}{2n+1}$ and $\mathrm{tb}_{\mathbb{Q}}(L_2) = -k_2 + \frac{2}{2n+1}$. L_1 is non-loose and L_2 is loose.(a) $k_2(n+1)$ candidates with $k_1 = 0$ and $k_2 \geqslant 1$ (b) $3nk_2$ candiadates with $k_1 = 1, k_2 \geqslant 1$ and (c) $4nk_2$ candidates with $k_1 \geqslant 2$ and $k_2 \geqslant 1$.

complement tight. The computations of $\mathrm{tb}_{\mathbb{Q}}$ are same as before. One could check the two $L_1 \sqcup L_2$ can be distinguished by their rotation numbers. This finishes the proof. $\square$

Remark 4.1. The above theorem recovers the result of [GO19] for $p = 1$.

5. CONTACT SURGERY DIAGRAMS FOR NON-LOOSE HOPF LINKS IN $L(p, q)$

5.1. Contact surgery diagrams for non-loose Hopf links in $L(p, 1)$. The explicit diagrams are already shown in [CGO25]. The only explicit surgery diagrams missing in [CGO25] were the 2 realizations with $\mathrm{tb}_{\mathbb{Q}}(L_1) = \frac{1}{p}$ and $\mathrm{tb}_{\mathbb{Q}} = 1 + \frac{1}{p}$. (a) in Figure 18 shows those 2 explicit realizations.

Remark 5.1. We have an explicit algorithm for the Legendrian realizations of the non-loose Hopf links in $L(p, 1)$ as well. The algorithm for $L(p, 1)$ is quite similar to the general case. But as the diagrams are already given in [CGO25], we are not including it here.

5.2. Contact surgery diagrams for non-loose Hopf links in $L(2n + 1, 2)$. Figure 10 shows Legendrian representations of non loose Hopf links in $L(2n + 1, 2)$ where L_1 is non-loose and L_2 is loose. To see this, one could do a sequence of -1 surgeries on L_1 that cancels the $+1$ surgeries and we see a tight manifold in the complement. Figure 11 shows the non-loose Hopf links with L_1 loose and L_2 non-loose. Finally, Figure 12 and 13 show all the non-loose representations with loose-loose component. There does not exist a non-loose Hopf link with non-loose/non-loose components in $L(2n+1, 2)$. One can do a sequence of Kirby moves to see indeed these are Hopf links in $L(2n+1, 2)$.

5.3. General surgery diagrams. Here we give an algorithm on how to find the contact surgery representations of the rational Hopf links. Note that, this is just one way to find the representatives. We could have infinitely many possibilities for the Legendrian Hopf links but all of them are related

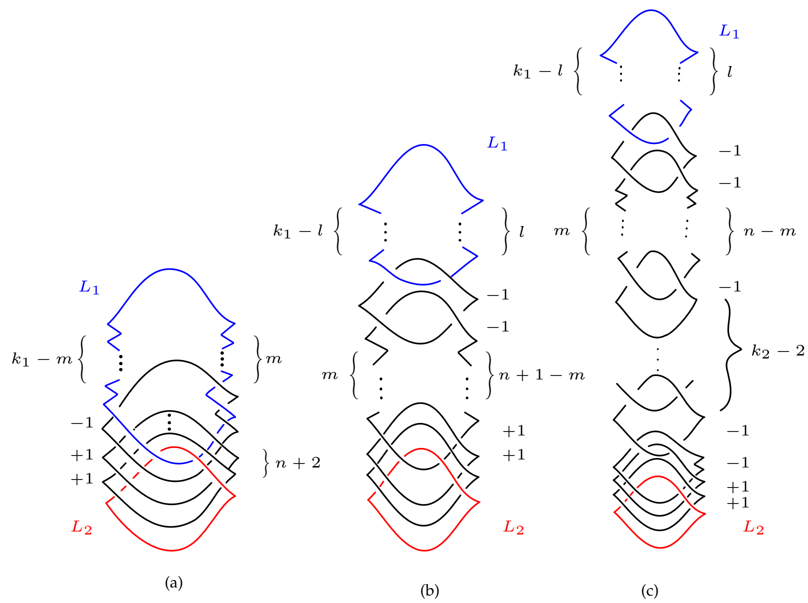

FIGURE 11. The non-loose Hopf link with L_1 loose and L_2 non-loose with $\mathrm{tb}_{\mathbb{Q}} = -(k_1+1) + \frac{n+1}{2n+1}$ and $\mathrm{tb}_{\mathbb{Q}} = k_2 + \frac{2}{2n+1}$. (a) $2(k_1+1)$ candidates with $k_1 = 0, k_2 \geqslant 0$. (b) $(k_1+1)(n+2)$ candidates with $k_1 \geqslant 0, k_2 = 1$ and (c) $2(k_1+1)(n+1)$ candidates with $k_1 \geqslant 0, k_2 \geqslant 2$.

by contact handle slide and Legendrian isotopy. We will give a detailed algorithm for the first and the second case and the rest follow similarly.

Case 1: small $\cup$ large slope. We give a detailed analysis for $\frac{p}{q} = [a_0, a_1, \cdots, a_n]$. When $k_2 = 0$ and $n = 1$ the analysis is slightly different and we leave it to the reader. Note that, when considering small slope for K_1 and large slope for K_2, the path that describes the contact structure of the complement of $L_1 \sqcup L_2$ has endpoints s_{k_1} and s_{k_2} where $s_{k_1} < 0$ but $s_{k_2} > 0$, thus we have a non-uniform twisting in the complement. To solve this issue, we apply $k_2 + 1$ fold Rolfsen twist to the slopes. The $k_2 + 1$ fold Rolfsen twist matrix is given by

$$\begin{pmatrix} 1 & -(k_2+1) \\ 0 & 1 \end{pmatrix}$$

This matrix turns $s_{k_2} = \frac{1}{k_2}$ to -1 and fixes 0. First we do it for $k_1 \geqslant 0, k_2 = 0$ thus we apply 1 Rolfsen twist. Appling 1 Rolfsen twist turns $-\frac{p}{q} = [a_0, a_1, \cdots, a_n]$ to $[-1, a_0 - 1, a_1, \cdots, a_n]$. Now note that, a standard neighborhood of K_1 will have meridional slope $[-1, a_0 - 1, a_2 \cdots a_n]$ and dividing slope s_{k_1}. We start with a max tb Legendrian unknot in S^3. It has standard neighborhood with meridional slope ∞ and dividing slope -1. First we do $|a_0 - 1|$ $+1$ surgeries on it. This will change the meridional slope to $[-1, a_0 - 1]$. After this we take a Legendrian push-off of the surgery curve and do a $|a_1 + 1|$ stabilizations and finally a -1 surgery. This makes the new meridian $[-1, a_0 - 1, a_1]$ and the dividing curve $[-1, a_0 - 1, a_1 + 1]$. We keep doing a sequence of stablizations and -1 surgery as shown in Figure 14 (a). A Legendrian push-off of the last surgery curve will have a standard neighborhood with meridional slope $[-1, a_0 - 1, a_1 \cdots, a_n]$ and dividing slope $[-1, a_0 - 1, a_1, \cdots, a_n + 1]$. Note that, this is the dividing slope corresponding to $k_1 = 0$. Thus we can stabilize this push-off k_1 times to get all the representaives of K_1. For K_2, note that, the meridian

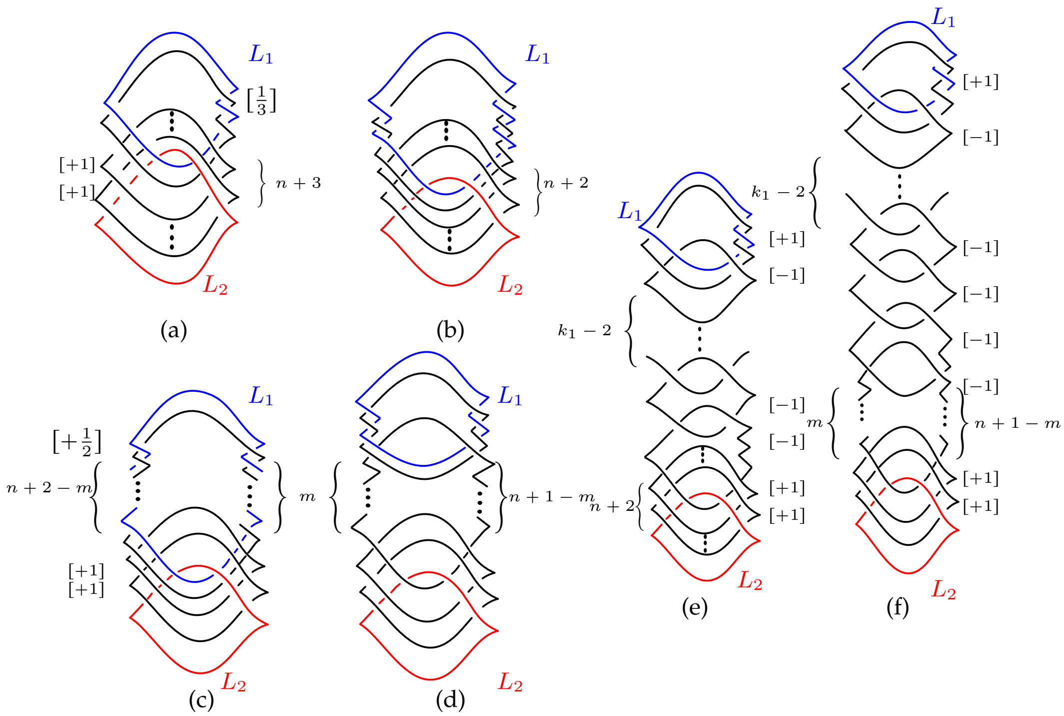

FIGURE 12. The non-loose Hopf link in $L(2n+1,2)$ where both L_1 and L_2 are loose with $\text{tb}_{\mathbb{Q}}(L_1) = k_1 + \frac{n+1}{2n+1}$ and $\text{tb}_{\mathbb{Q}}(L_2) = k_2 + \frac{2}{2n+1}$.(a) 2 candidates with $k_1 = k_2 = 0$ (b) 4 candidates with $k_1 = 1, k_2 = 0$. (c) $n+3$ candidates with $k_1 = 0, k_2 = 1$. (d) $3(n+2)$ candidates with $k_1 = k_2 = 0$. (e) 6 candidates with $k_1 \geqslant 2, k_2 = 0$. (f) $2(n+2)$ candidates with $k_1 \geqslant 2, k_2 = 1$.

of the standard unknot has dividing slope -1 and meridional slope 0 which are actually dividing slope $\frac{1}{k_2}$ and meridinonal slope 0 before the Rolfsen twist. Thus we get our desired Hopf link. It is easy to see that L_2 is non-loose as a sequence of -1 surgeries on L_2 will cancel the $+1$ surgeries and we will get a tight manifold in the complement. We have a total $|a_1||a_2+1|\cdots|a_n+1|(k_1+1)$ candidates in this case.

For $n=1$, as mentioned the analysis will be slightly different. We just replace $|a_1+2|$ by $|a_1+1|$ in (a) of Figure 14.

For $k_2 = 1$. Here we need to apply a 2 fold Rolfsen twist and the meridional slope in this case changes to $[-1,-2,a_0-1,a_1,\cdots,a_n]$. Like the previous case, we start with a standard unknot in S^3 with meridional slope ∞ and dividing slope -1. After 2 $+1$ surgeries the meridional slope becomes $[-1,-2]$. Now stabilizing a Legendrian push-off of this surgery curve $|a_0|$ times the dividing slope changes to $[-1,-2,a_0]$ and doing -1 surgery on this curve makes our new meridional slope $[-1,-2,a_0-1]$. We keep repeating this sequence by stabilizing $|a_i+2|$ times and then a -1 surgery until we reach the meridional slope $[-1,-2,a_0-1,a_1\cdots,a_n]$ and dividing slope $[-1,-2,a_0-1,a_1,\cdots,a_n+1]$. Then following the same procedure as before we get exactly $|a_0-1||a_1+1|\cdots|a_n+1|(k_1+1)$ non-loose candidates. Check (b) of Figure 14.

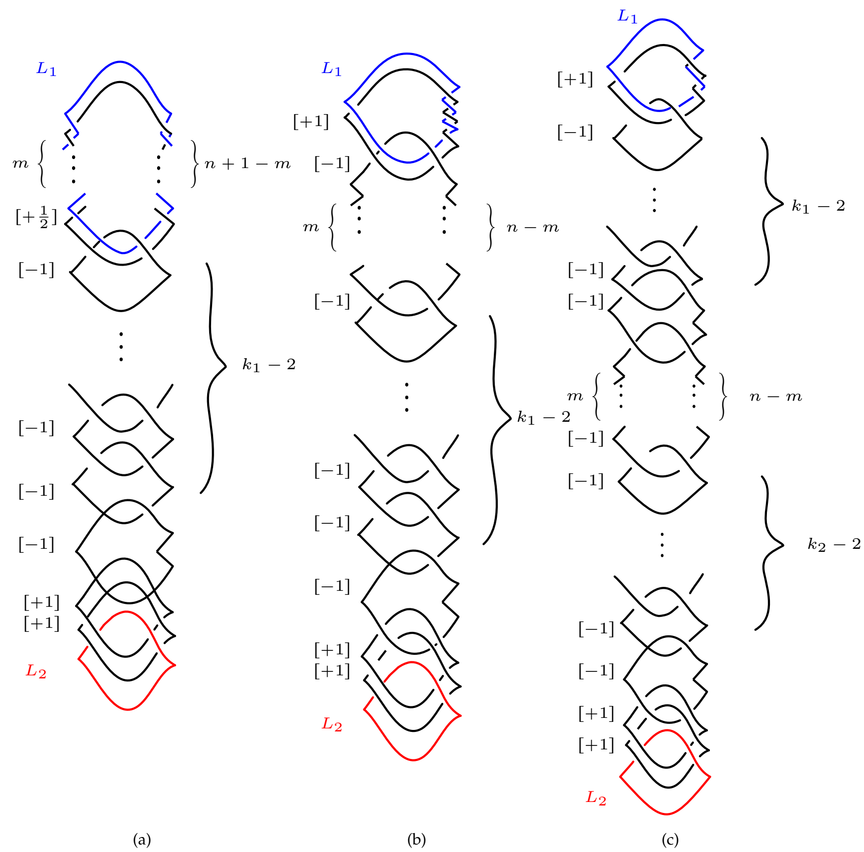

FIGURE 13. The non-loose Hopf link in $L(2n+1,2)$ with both components loose. (a) $2(n+2)$ candidates with $k_1=0, k_2\geqslant 2$ (b) $6(n+1)$ candidates with $k_1=1, k_2\geqslant 2$. (c) $8(n+1)$ candidates with $k_1\geqslant 2$ and $k_2\geqslant 2$.

For $k_2 \geqslant 2$. In this case, we need to apply a k_2+1 fold Rolfsen twist which brings our lower meridian to $[-1,\underbrace{-2,-2,\cdots-2}_{k_2},a_0-1,a_1,\cdots,a_n]$. Doing 2 consecutive $+1$ surgeries on the standard unknot changes our meridional slope to $[-1,-2]$. Next we take a Legendrian push-off of this surgery curve and stabilize it once and then do a -1 surgery. Our meridian now becomes $[-1,-2,-2]$ and dividing slope 0. Now we do a sequence of -1 surgeries of length k_2-2 on the chain of unknots that as shown (Note that, each of these unknots are actually a Legendrian push-off of the once stabilized surgery curve. One could see it via a contact handle slide.) After performing the -1 surgery on the chain our new meridional slope becomes $[-1,\underbrace{-2,-2,\cdots,-2}_{k_2}]$. Once we take a Legendrian push-off of the last surgery curve, a standard neighborhood of this has meridional slope $[-1,\underbrace{-2,-2,\cdots,-2}_{k_2}]$ and dividing slope 0. If now we do a -1 surgery on this curve after stabilizing it $|a_0+1|$ times we see the new meridian and dividing slopes are $[-1,\underbrace{-2,-2,\cdots,-2}_{k_2},a_0-1]$ and $[-1,\underbrace{-2,-2,\cdots,-2}_{k_2},a_0]$ respectively. Then we proceed as before. Note that, in this case we will have $2(k_1+1)|a_0||a_1+1||a_2+1|\cdots|a_n+1|$ non-loose representatives.

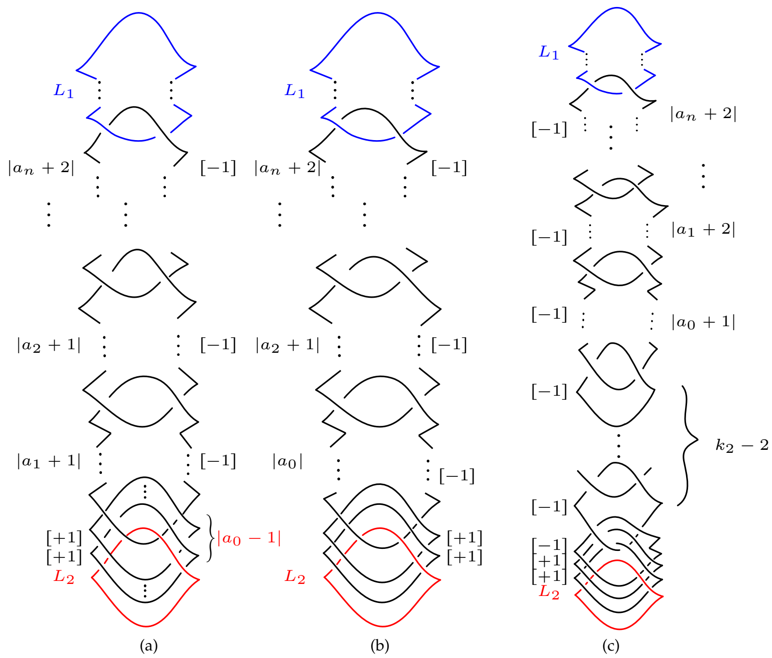

FIGURE 14. Legendrian repressentaives of non-loose Hopf links in $L(p,q)$ where L_1 is loose and L_2 non-loose. (a) $k_2 = 0$ and $n \geqslant 2$ (b)$k_2 = 1$ (c) $k_2 \geqslant 2$. In all the cases $k_1 \geqslant 0$. The value $|a_i+1|$ in the diagram denotes the number of stabilization of that component.

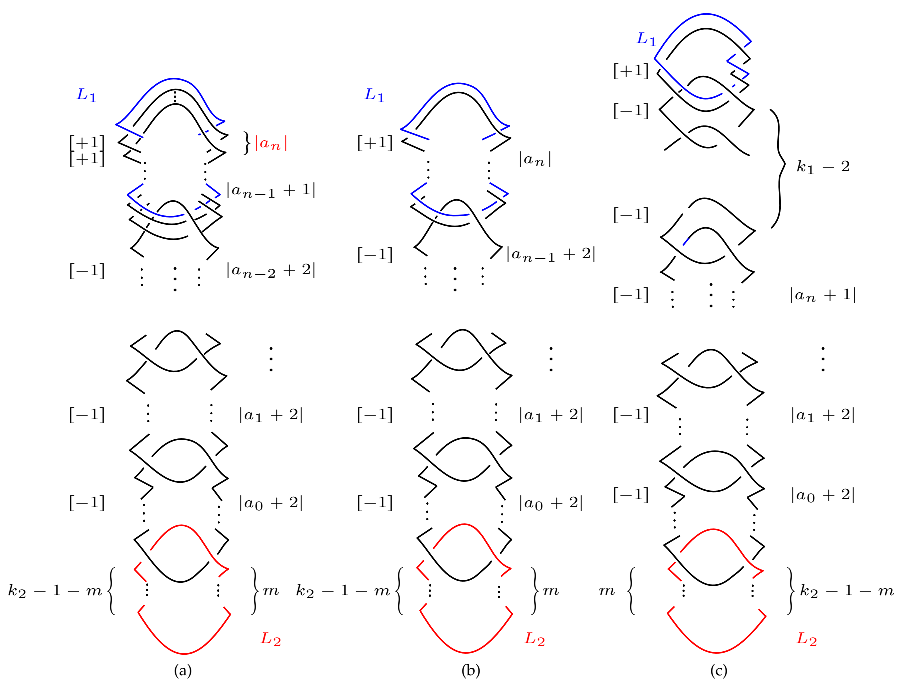

FIGURE 15. Non-loose Hopf links in $L(p,q)$ where L_1 is non-loose and L_2 is loose. (a) $k_1 = 0$ (b) $k_1 = 1$, $n \geqslant 0$ (c) $k_1 \geqslant 2$ and $-p/q = [a_0, a_1, \cdots a_n]$. In all the cases $k_2 \geqslant 1$.

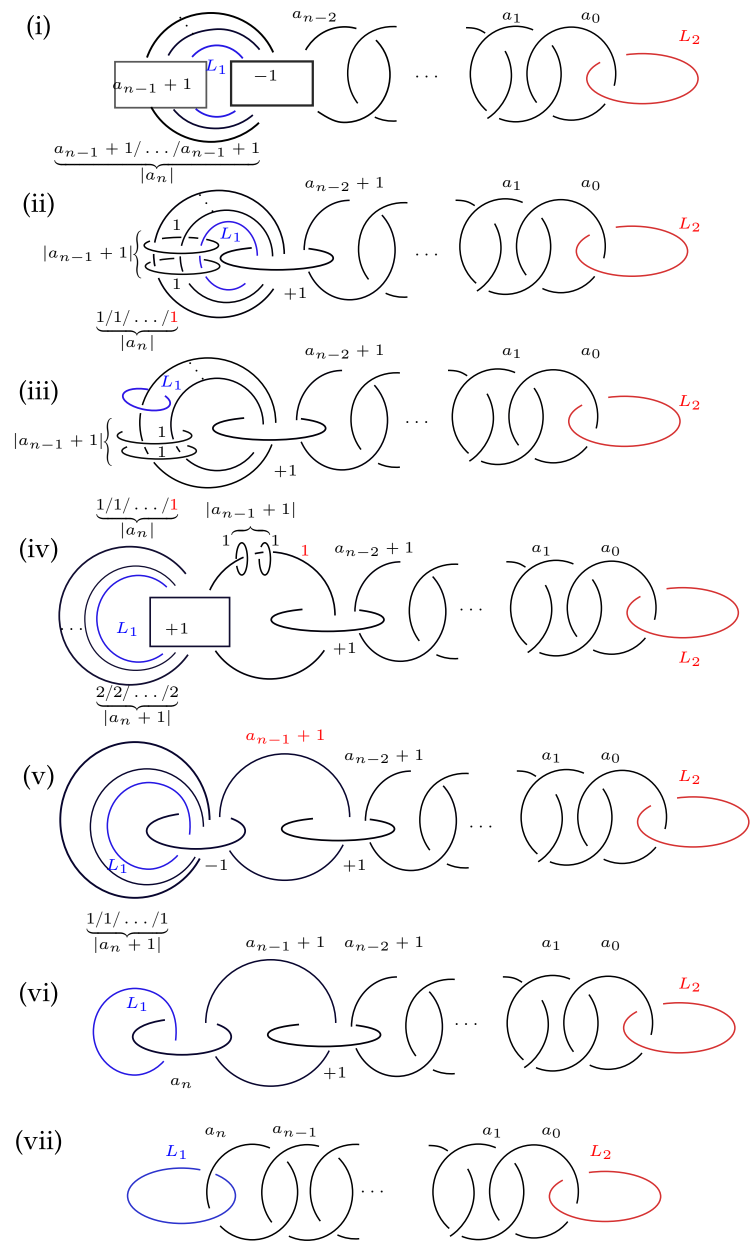

FIGURE 16. A sequence of Kirby moves for candidate Figure 15 (a).

large $\cup$ small slope. In this case we construct the diagrams for $k_1 \geqslant 0$ and $k_2 \geqslant 1$. (Note that, $k_1 \geqslant 0$ and $k_2 = 0$ overleaps with Case 5.3.)

For $k_1 = 0$, $k_2 \geqslant 1$. As usual we start with a standard unknot of slope -1 and stabilize it $|a_0 + 2|$ times. Note that, this is a standard neighborhood of an unknot with dividing slope $a_0 + 1$. A -1 surgery on a Legendrian divide on this neighborhood will change the meridian to a_0. Next we will do a series of stabilization and Legendrian surgery (as shown in Figure 15) to bring us to slope r and meridional slope $[a_0, \cdots a_{n-2}]$. Once we are at dividing slope r and meridional

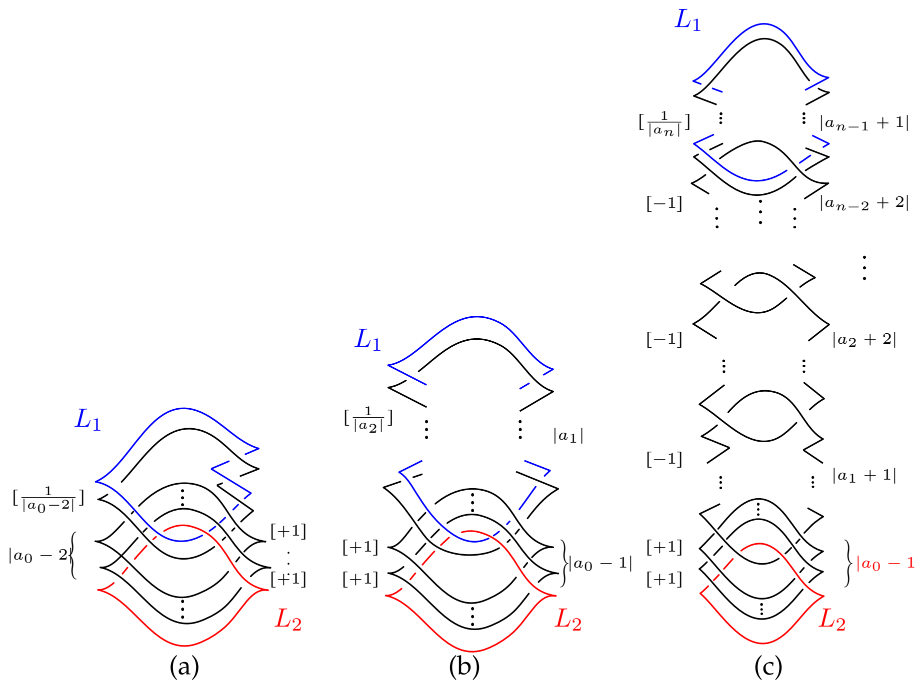

FIGURE 17. These are all non-loose Legendrian representatives in $L(p,q)$ with L_1 and L_2 both loose and $\text{tb}_{\mathbb{Q}}(L_1) = \frac{p'}{p}$ and $\text{tb}_{\mathbb{Q}}(L_2) = \frac{q}{p}$. (a) For $n = 1$ we have 2 representatives. (b) For $n = 2$ there are $|a_1 - 1|$ representatives and (c) For $n \geqslant 3$ we will have $|a_1||a_2+1|\cdots|a_{n-2}+1||a_{n-1}|$ non-loose representatives.

slope $[a_0,\cdots a_{n-2}]$, we do a $|a_{n-1}+1|$ stabilization and finally $|a_n|$ number of $+1$ surgeries to get our desired neighborhood of a Legendrian with dividing slope $-\frac{p''}{q}$ and meridinal slope $-\frac{p}{q}$. A Legendrian divide of this neighborhood is a component of the desired Hopf link. Note that, the meridian of the standard unknot we started with is the other component with slope -1. Stabilizing this unknot (k_2-1) number of times will give us all the non-loose Hopf links with $k_1 = 0$ and $k_2 \geqslant 1$. One could see there are $|a_0+1||a_1+1|\cdots|a_{n-1}|k_2$ such candidates as desired. To see K_1 is non-loose, do a $-\frac{1}{a_n}$ surgery. This cancels all the $+1$ surgeries and thus the complement of K_1 is clearly tight(in fact Stein fillable, thus there cannot exist any Giroux torsion). To see this is indeed a Hopf link, one could do a series of Kirby moves and handle slides as shown in Figure 16.

For $k_1 = 1, k_2 \geqslant 1$. Like the previous case, we will reach the dividing slope r and meridional slope $[a_0,a_1,\cdots,a_{n-2}]$. Next we will stabilize a Legendrian divide of this neighborhood $|a_{n-1}+2|$ number of times and do a -1 surgery on that. This takes our meridional slope to $[a_0,a_1,\cdots a_{n-1}]$. Next we stabilize $|a_n|$ number of times and finally a $+1$ surgery to get to the desired meridional and dividing slope. A Legendrian divide (Legendrian pushoff) of this neighborhood is the component of the non-loose Hopf link with $\text{tb} = 1 + \frac{p''}{q''}$. Clearly this component is non-loose as -1 surgery on K_1 produces a tight manifold. Like the previous case, we can find the other component which is loose. There are $|a_0+1|a_1+1|\cdots a_{n-1}+1||a_n-1|k_2$ candidates as desired.

$k_1 > 1, k_2 \geqslant 1$. We will approach exactly like before to reach dividing slope s and meridional slope $[a_0,a_1,\cdots a_{n-1}]$. Now we do $|a_n+1|$ stabilizations to get dividing slope $-\frac{p}{q}$. Next doing $k_1 - 1$

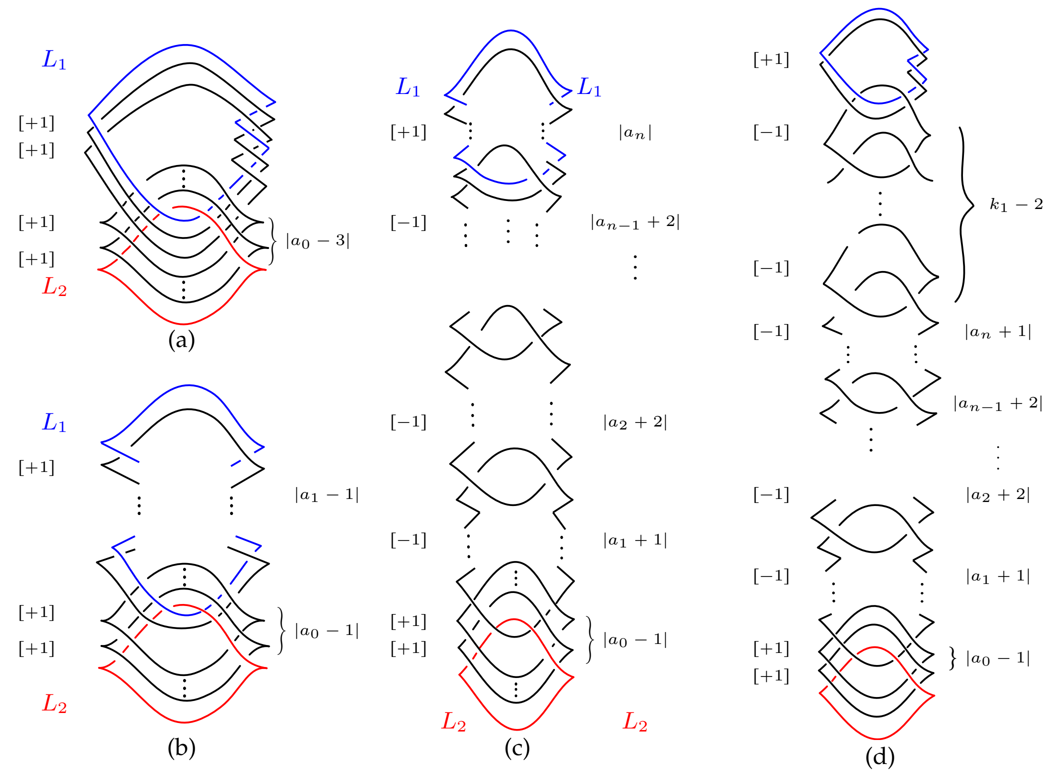

FIGURE 18. All the non-loose Legendrian representatives of the Hopf link in $L(p,q)=[a_0,a_1,\cdots,a_n]$ with $\mathrm{tb}_{\mathbb{Q}}(L_1)=k_1+\frac{p'}{q}$and $\mathrm{tb}_{\mathbb{Q}}(L_2)=\frac{q}{p}$.
(a) $n=0$, $k_1=1$, (b) $n=1$, $k_1=1$ (c) $n\geqslant 2$, $k_1=1$ (d) $n\geqslant 0$,$k_1\geqslant 2$.

Legendrian surgery will bring us to meridional slope $[a_0,a_1,\cdots a_n-k_1+1]$. Finally another stabilization and a $+1$ surgery give us the desired neighborhood. Note that, like before a Legendrian divide on this neighborhood is the non-loose K_1. There are $2k_2|a_0+1||a_1+1|\cdots|a_n|$ non-loose representatives as desired.

large $\cup$ large slope. Note that, like Case 5.3 here too we have an inconsistent twisting and thus we need to apply the k_2+1 fold Rolfsen twist trick. We also see that depending on $n=1$ and $n>1$ there will be cases significantly different as we saw in the proof of Theorem 1.6. The surgery pictures for each of the cases are given in Figure 17, 18, 19 and 20. One could do a similar analysis as the previous cases about the slopes. Thus we omit the details here.

5.4. $k_2=0$. We subdivide this case into various subcases. Note that, we are applying 1 Rolfsen twist to the slopes here and thus $-\frac{p}{q}=[a_0,a_1,\cdots,a_n]$ changes into $[-1,a_0-1,a_1,\cdots,a_n]$.

5.4.1. $k_1=0=k_2$.

(1) $n=0$ In this case, we will have a unique non-loose representative in $L(a_0,1)$ with both components non-loose. The surgery diagram is given in [CGO25]. Just replace p by a_0.
(2) $n=1$. We start with a standard unknot as before with -1 dividing slope and ∞ meridional slope. We do $|a_0-2|$ $+1$ surgeries first. That take us to meridional slope $[-1,a_0-2]$. Now we do 1 stabilization on a Legendrian push-off of the last surgery curve that changes the dividing slope to $[-1,a_0-1]$. And finally we do a sequence of $+1$ surgeries of length $|a_1-1|$

which changes the meridional slope to $[-1, a_0-1, a_1]$ which is our required slope (Note that, we applied a Rofsen twist here). We have 2 such candidates as shown in Figure 17 (a).

(3) $n = 2$. In this case, first we do $|a_0 - 1|$ $+1$ surgeries to the standard unknot that changes the meridional slope to $[-1, a_0 - 1]$, then $|a_1|$ stabilization of it's push-off and finally end with another $|a_2|$, $+1$ surgeries. This gives us $|a_1 - 1|$ candidate as shown in Figure 17 (b).

(4) $n \geqslant 3$. For the general case, we do a sequence of stabilizations and surgeries as shown in (c) of Figure 17. These are all $|a_1||a_2 + 1| \cdots |a_{n-2} + 1||a_{n-1}|$ representatives.

5.4.2. $k_1 = 1, k_2 = 0$.

(1) $n = 0$. This is the $L(a_0, 1)$ case. Note that, this particular contact surgery diagram was missing in [CGO25]. So we add it here. See (a) of Figure 18. We have 2 such representatives.

(2) $n = 1$. We have $|a_1 - 2|$ cases which are coming from the $|a_1 - 1|$ choice of stabilizations. See (b) of Figure 18.

(3) $n \geqslant 2$ This is the general case and we have $|a_1||a_2 + 1| \cdots |a_{n-1} + 1||a_n - 1|$ representatives coming from the choices of stabilizations in (c) of Figure 18.

5.4.3. $k_1 \geqslant 2, k_2 = 0$. (d) of Figure 18 gives all $2|a_1||a_2 + 1| \cdots |a_n|$ representatives.

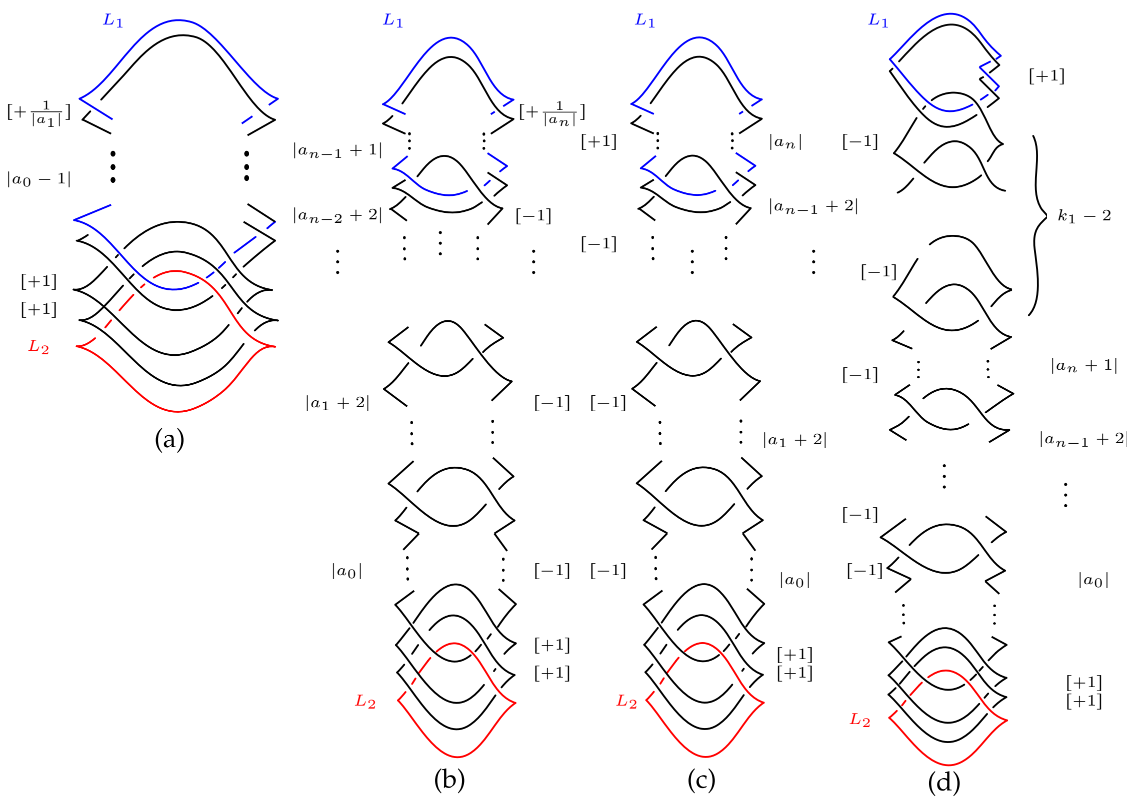

FIGURE 19. The non-loose Hopf links in $L(p, q) = [a_0, a_1, \cdots, a_n]$ where L_0, L_1 are loose with $\mathrm{tb}_{\mathbb{Q}}(L_1) = k_1 + \frac{p'}{p}$ and $\mathrm{tb}_{\mathbb{Q}}(L_2) = 1 + \frac{q}{p}$.
(a) $k_1 = 0,\ n = 1$. (b) $k_1 = 0,\ n \geqslant 2$. (c) $k_1 = 1, n \geqslant 0$. (d) $k_1 \geqslant 2,\ n \geqslant 0$.

5.5. $k_2 = 1$. For this case, we have the following subcases.

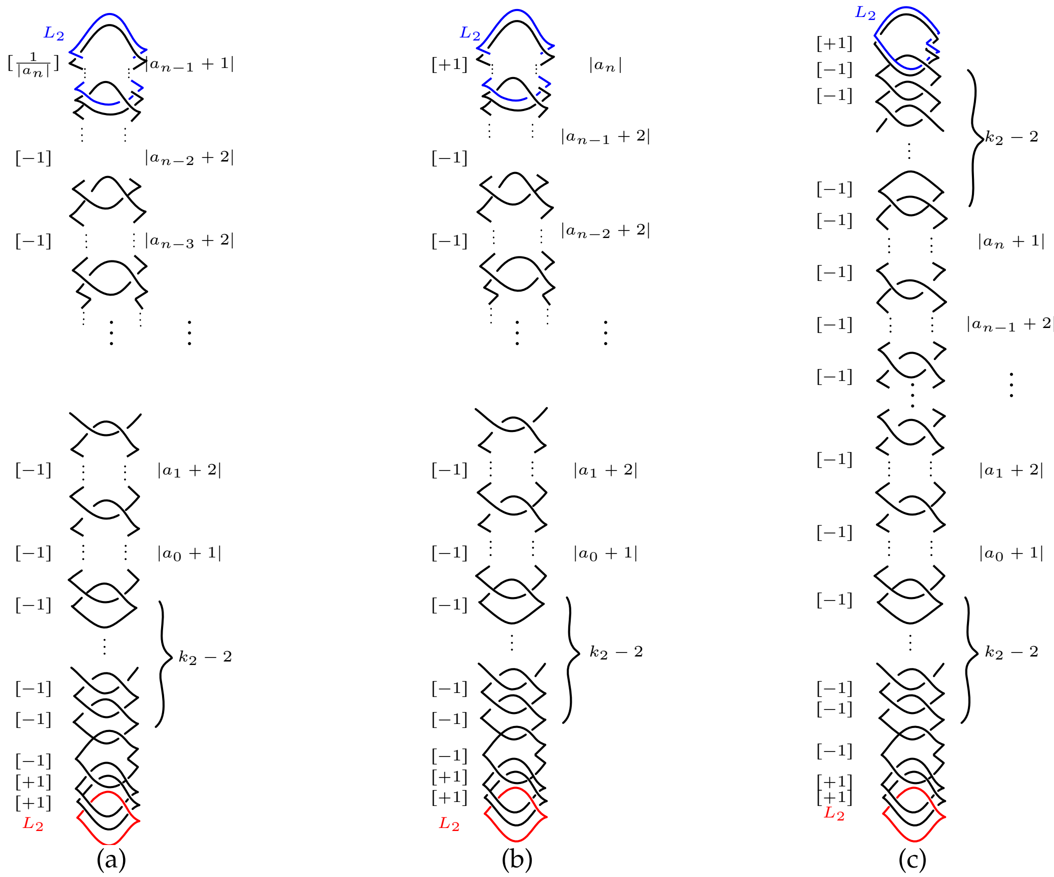

FIGURE 20. Non-loose Hopf links in $L(p,q)$ with both L_1, L_2 loose. $\mathrm{tb}_{\mathbb{Q}}(L_1) = k_1 + \frac{p'}{p}$ and $\mathrm{tb}_{\mathbb{Q}}(L_2) = k_2 + \frac{q}{p}$ with $k_2 \geqslant 2$. (a) $k_1 = 0$, (b) $k_1 = 1$ and (c) $k_1 \geqslant 2$. In all the cases $n \geqslant 0$.

5.5.1. $k_1 = 0, k_2 = 1$.

(1) $n = 1$ This is (a) of Figure 19. There are $|a_0 - 2|$ representatives.
(2) $n \geqslant 2$ For general n we have $|a_0 - 1||a_1 + 1| \cdots |a_{n-1}|$ non-loose representatives as shown in (b) of Figure 19.

5.5.2. $k_1 = k_2 = 1$. Check (c) of Figure 19 for the $|a_0 - 1||a_1 + 1| \cdots |a_n - 1|$ representatives.

5.5.3. $k_1 \geqslant 2, k_2 = 1$. (d) of Figure 19 gives $2|a_0 - 1||a_1 + 1| \cdots |a_{n-1} + 1||a_n|$ representatives.

5.6. $k_2 \geqslant 2$**.** For the final case, we have the three subcases. The Legendrian representaives are given in Figure 20. With these we get all the explicit Legendrian realizations of non-loose Hopf links in $L(p,q)$. To check that all these diagrams are indeed Hopf links one needs to do a sequence of handle slides and Kirby moves. We do not include it here are they are repetitive sequences.

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