ILL-POSEDNESS OF THE 2D EULER EQUATIONS IN A LOGARITHMICALLY REFINED CRITICAL SOBOLEV SPACE

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Abstract. In [BL15], Bourgain and Li establish strong ill-posedness of the 2D Euler equations for initial velocity in the critical Sobolev space $H^2(\mathbb{R}^2)$. In this work, we extend the results of [BL15] by demonstrating strong ill-posedness in logarithmically regularized spaces which are strictly contained in $H^2(\mathbb{R}^2)$ and which contain $H^s(\mathbb{R}^2)$ for all $s > 2$. These spaces are constructed via application of a fractional logarithmic derivative to the critical Sobolev norm. We show that if the power α of the logarithmic derivative satisfies $\alpha \leq 1/2$, then the 2D Euler equations are strongly ill-posed.

1. Introduction

The incompressible Euler equations on $\mathbb{R}^2$ in velocity form are given by

$$\text{(E)} \qquad \begin{cases} \partial_t u + u \cdot \nabla u = -\nabla p & \text{on } \mathbb{R}^2 \times (0, \infty), \\ \nabla \cdot u = 0 & \text{on } \mathbb{R}^2 \times [0, \infty), \\ u|_{t=0} = u_0 & \text{on } \mathbb{R}^2, \end{cases}$$

where $u : \mathbb{R}^2 \times [0, \infty) \to \mathbb{R}^2$ denotes the velocity and $p : \mathbb{R}^2 \times [0, \infty) \to \mathbb{R}$ denotes the internal scalar pressure. The 2D incompressible Euler equations can be recast in terms of the *vorticity*, defined by

$$\omega := \nabla^\perp \cdot u = \partial_{x_2} u_1 - \partial_{x_1} u_2.$$

One can show using the incompressibility condition that (E) is equivalent to the 2D vorticity equation, given by

$$\text{(V)} \qquad \begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0 & \text{on } \mathbb{R}^2 \times (0, \infty), \\ u = \nabla^\perp (-\Delta)^{-1} \omega & \text{on } \mathbb{R}^2 \times [0, \infty), \\ \omega|_{t=0} = \omega_0 & \text{on } \mathbb{R}^2. \end{cases}$$

Here, the velocity can be obtained from the vorticity via the relation $u = K * \omega$, where

$$K(x) = \frac{1}{2\pi} \frac{x^\perp}{|x|^2}$$

is the Biot-Savart kernel. We refer to convolution with K as the Biot-Savart operator. Throughout this work, we also let $X : \mathbb{R}^2 \times [0, \infty) \to \mathbb{R}^2$ be the particle trajectory map defined by

$$\text{(1.1)} \qquad \frac{d}{dt} X^t(x) = u(X^t(x), t), \quad X^0(x) = x, \qquad (x, t) \in \mathbb{R}^2 \times [0, \infty).$$

The question of well-posedness of (E) in different functional settings has a long history. Wolibner [Wol33] establishes the global well-posedness of (E) in $C^{k,r}(\mathbb{R}^2)$, the space of k-times continuously differentiable functions with the k^{th}-order derivatives r-Hölder continuous, when $k \geq 1$ and $r \in (0,1)$. Kato and Ponce [KP86] prove local well-posedness of (E) in the Sobolev spaces $W^{s,p}(\mathbb{R}^2)$, with $s > 2/p + 1$ and $p \in (1, \infty)$. One can obtain global in time solutions in these Sobolev spaces via the so-called BKM criterion [BKM84], with at most double exponential growth in time of the $W^{s,p}$-norm. In the lower regularity setting, Yudovich [Yud63] establishes existence of a unique weak solution to (V) in $L^\infty_{loc}([0,\infty); L^1 \cap L^\infty(\mathbb{R}^2))$. See also [CNSS17], [BBC16] for results on existence of Lagrangian and weak solutions with vorticity in $L^1(\mathbb{R}^2)$. For well-posedness results in the Besov and Triebel-Lizorkin spaces, $B^s_{p,q}(\mathbb{R}^2)$ and $F^s_{p,q}(\mathbb{R}^2)$, we refer the reader to [BCD11, Cha02, Cha03, Cha04, Vis98, PH23]. Notably, for these spaces, the condition $s > 2/p + 1$ is sufficient for global well-posedness, while, when $s = 2/p + 1$, well-posedness can be established in certain cases when the corresponding space embeds into $L^\infty(\mathbb{R}^2)$.

In contrast to the well-posedness results mentioned above, several works in recent years have demonstrated strong ill-posedness of (E) in various function spaces. Here, strong ill-posedness in the Banach space X refers to the existence of initial data $u_0 \in X$ for which the unique (Yudovich) solution u to (E) does not belong to $L^\infty([0,t]; X)$ for *any* $t > 0$. The seminal works of Bourgain and Li [BL15, BL21] establish the strong ill-posedness of (E) for $d = 2$ and $d = 3$ in $C^k(\mathbb{R}^d)$ with k a positive integer and in $B^s_{p,q}(\mathbb{R}^d)$ with $s = d/p + 1$, $p \in (1, \infty)$, and $q > 1$. The authors of [EJ17, JK21] give simplified proofs of the results of [BL15, BL21], with [EJ17] providing a proof on the two-dimensional torus, while [JK21] also establishes ill-posedness in the critical Lorentz spaces. A similar construction to that used in [BL15] is applied in [Kwo21], showing strong ill-posedness of the logarithmically modified Euler equations, where the vorticity is transported by a flow logarithmically smoother than the velocity given by the Biot-Savart law.

In this work, we demonstrate ill-posedness of (E) in logarithmically refined critical Sobolev spaces. Before stating our main result, we briefly discuss a few relevant function spaces (see Section 2 for further discussion of these spaces). We first define a logarithmic derivative using the Fourier transform: we fix $\alpha \in \mathbb{R}$ and set

$$\log^\alpha(D+1)f := \int_{\mathbb{R}^2} \log^\alpha(|\xi|+1)\widehat{f}(\xi)e^{ix\cdot\xi}\, d\xi, \tag{1.2}$$

for a locally integrable function $f : \mathbb{R}^2 \to \mathbb{R}$, where $\widehat{f} : \mathbb{R}^2 \to \mathbb{R}$ denotes the Fourier transform of f. We can now define the logarithmic Sobolev seminorm

$$|f|^2_{H^{s,\alpha}} := \int_{\mathbb{R}^2} |\xi|^{2s} \log^{2\alpha}(|\xi|+1)|\widehat{f}(\xi)|^2 d\xi, \tag{1.3}$$

where $s \geq 0$. Then the logarithmic Sobolev space $H^{s,\alpha}(\mathbb{R}^2)$ is given by

$$H^{s,\alpha}(\mathbb{R}^2) := \{f \in L^1_{loc}(\mathbb{R}^2) : \|f\|^2_{H^{s,\alpha}} := \|f\|^2_{L^2} + |f|^2_{H^{s,\alpha}} < \infty\}.$$

Of course, when $\alpha = 0$, $H^{s,\alpha}(\mathbb{R}^2)$ reduces to the classical Sobolev space $H^s(\mathbb{R}^2)$. Moreover, for any $\alpha > 0$ and $\epsilon > 0$, we have the (strict) embeddings

$$H^{s+\varepsilon,0}(\mathbb{R}^2) \hookrightarrow H^{s,\alpha}(\mathbb{R}^2) \hookrightarrow H^{s,0}(\mathbb{R}^2).$$

Thus, when $\alpha > 0$, the spaces $H^{2,\alpha}(\mathbb{R}^2)$ lie strictly between the supercritical scale, at which (E) is well-posed, and the critical scale, at which (E) is strongly ill-posed. Our main result states that if the extra regularity imposed by the α-logarithmic derivative is to a low enough degree, then (E) is strongly ill-posed in $H^{2,\alpha}(\mathbb{R}^2)$.

Theorem 1.1. *Let $\alpha \in (0, \frac{1}{2}]$ and $\varepsilon > 0$. Then there exists $u_0 \in H^{2,\alpha}(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2)$ such that $\|u_0\|_{H^{2,\alpha}} < \varepsilon$ and the unique smooth solution u to (E) satisfies*

$$\limsup_{t \to 0^+} \|u(t)\|_{H^{2,\alpha}} = \infty.$$

We note that in the recent work [HR25], the second and third authors prove that (E) is globally well-posed in $H^{2,\alpha}(\mathbb{R}^2)$ for $\alpha > \frac{1}{2}$; thus, the degree of logarithmic regularity in Theorem 1.1 is sharp.

We briefly outline our strategy for proving Theorem 1.1, which closely follows the method of large Lagrangian deformation used in the works [BL15, BL21, Kwo21]. First, in Section 3, we construct initial vorticity which is small in the $H^{1,\alpha}$ norm, but which produces a large Lagrangian deformation in short time. We then perturb this initial data by a highly oscillatory function which is also small in $H^{1,\alpha}$-norm. We show that this perturbation, combined with the stretching from the particle trajectory map, is capable of drastically inflating the norm at a later time. In Section 4, we iterate this construction to build a countable collection of solutions satisfying norm inflation, and we patch these local solutions together in such a way that they have little pairwise interaction. The full velocity solution is then shown to blow up in $H^{2,\alpha}$-norm instantaneously.

Our approach most significantly differs from the approaches of [BL15, BL21, Kwo21] in that we must establish existence of initial vorticity which is small in the $H^{1,\alpha}$-seminorm and which leads to norm inflation. To show that $|\omega_0|_{H^{1,\alpha}}$ can be made small, we utilize a real-variable Gagliardo-type equivalent seminorm (see (2.4)), as well as a logarithmic interpolation inequality (see Lemma 2.1). The proof of $H^{1,\alpha}$-norm inflation requires delicate frequency analysis using the Fourier definition of the $H^{1,\alpha}$-seminorm, as given in (1.3).

Finally, we remark that after suitable modification, many of our arguments apply to a more generally regularized critical Sobolev space; that is, where the logarithmic derivative is replaced with a generic function which grows slower than a power at infinity. These more general spaces are considered by the second and third authors in [HR25], where the authors specify a threshold for the growth rate of the Fourier multiplier, above which global well-posedness is guaranteed. In future work, we plan to address the ill-posedness question on the other side of this threshold.

2. Preliminaries and Notation

2.1. **Notation.** In what follows, we let log denote the base-two logarithm, and we utilize the notation $\log^\alpha(t)$ to denote $(\log(t))^\alpha$ for $\alpha \in \mathbb{R}$, $t > 0$. We denote the open disc of radius $r > 0$ around a point $x \in \mathbb{R}^2$ by $B_r(x)$. For a matrix A, $|A|_\infty$ denotes the largest absolute value of the entries of a matrix A. Finally, we let $|\cdot|$ denote the standard Euclidean norm in $\mathbb{R}^2$; that is, for $x = (x_1, x_2) \in \mathbb{R}^2$, $|x| = \sqrt{x_1^2 + x_2^2}$.

Throughout this work, we use $C > 0$ to denote a constant which may change from one line to the next. We may indicate that a constant depends on some quantity if the dependence on that quantity is important to the argument; for example, if C depends on α, we may utilize the notation C_α. Often, to simplify the presentation, we apply the notation $A \lesssim B$ to indicate that $A \leq CB$ for some constant $C > 0$.

2.2. **Sobolev Spaces.** Let k be a non-negative integer. The *Sobolev space* $W^{k,p}(\mathbb{R}^d)$ is the space of distributions f in $L^p(\mathbb{R}^d)$ whose distributional derivatives of all orders up to and including k are also in $L^p(\mathbb{R}^d)$. We equip $W^{k,p}(\mathbb{R}^d)$ with the norm

$$\|f\|_{W^{k,p}} = \|f\|_{L^p} + \sum_{|\sigma|=1}^{k} \|D^\sigma f\|_{L^p}.$$

When $p = 2$, we denote the kth order Sobolev space $W^{k,2}(\mathbb{R}^d)$ by $H^k(\mathbb{R}^d)$.

When $p \in (1, \infty)$, one can equivalently define $W^{k,p}(\mathbb{R}^d)$ to be the set of distributions f in $L^p(\mathbb{R}^d)$ such that $(-\Delta)^{k/2} f \in L^p(\mathbb{R}^d)$, where $(-\Delta)^{k/2}$ is the fractional Laplacian, with norm

$$\|f\|_{W^{k,p}} = \|(Id - \Delta)^{k/2} f\|_{L^p}.$$

This definition allows for non-integral values of k, enabling us to make sense of Sobolev spaces $W^{k,p}(\mathbb{R}^d)$ for any differentiability index $k \in \mathbb{R}$. When $p = 2$ and $k \in \mathbb{R}$, we can apply Plancherel's identity and obtain

$$\|f\|_{H^k} \simeq \left(\int_{\mathbb{R}^d} (1 + |\xi|^2)^k |\widehat{f}(\xi)|^2 \, d\xi \right)^{1/2},$$

where $\simeq$ denotes equivalence of norms and $\widehat{f} : \mathbb{R}^d \to \mathbb{R}$ denotes the Fourier transform of f defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-ix\cdot\xi} \, d\xi.$$

In what follows, we make use of the Sobolev Embedding Theorem, which states that if $1 < p \leq q < \infty$, $p < d$, and $s \geq t \geq 0$, and if s, p, t, q satisfy

$$\frac{1}{p} - \frac{s}{d} = \frac{1}{q} - \frac{t}{d},$$

then there exists a constant $C > 0$ such that $\|f\|_{W^{t,q}} \leq C\|f\|_{W^{s,p}}$ for all $f \in W^{s,p}(\mathbb{R}^d)$.

We also recall the following interpolation inequality: if $p \in [1, \infty]$ is fixed, when $s_1 > s > s_2 \geq 0$ and $\theta \in (0, 1)$ satisfy $s = \theta s_1 + (1 - \theta)s_2$, there exists $C > 0$ such that

$$\|f\|_{W^{s,p}} \leq C\|f\|_{W^{s_1,p}}^{\theta} \|f\|_{W^{s_2,p}}^{1-\theta}$$

for all $f \in W^{s_1,p}(\mathbb{R}^d)$.

2.3. **Logarithmic Sobolev Spaces.** For $\alpha \in \mathbb{R}$, we define the *α-logarithmic derivative* of a distribution f by

$$\log^\alpha(D + 1)f(x) := \int_{\mathbb{R}^2} \log^\alpha(|\xi| + 1)\widehat{f}(\xi)e^{ix\cdot\xi} \, d\xi.$$

For $p \in (1, \infty)$, $s \geq 0$, and $\alpha \geq 0$, we define the *logarithmic Sobolev space* $W^{s,p,\alpha}(\mathbb{R}^d)$ to be the set of distributions $f \in L^p(\mathbb{R}^d)$ such that $(-\Delta)^{s/2} \log^\alpha(D + 1)f$ belongs to $L^p(\mathbb{R}^d)$. We define the seminorm

$$|f|_{W^{s,p,\alpha}} := \|(-\Delta)^{s/2} \log^\alpha(D + 1)f\|_{L^p}.$$

Then the norm on $W^{s,p,\alpha}(\mathbb{R}^d)$ is defined as

$$\|f\|_{W^{s,p,\alpha}} := \|f\|_{W^{s,p}} + |f|_{W^{s,p,\alpha}}.$$

When $p = 2$, we set $H^{s,\alpha}(\mathbb{R}^d) := W^{s,2,\alpha}(\mathbb{R}^d)$. Note that by Plancherel's identity, $\|f\|_{H^{s,\alpha}} \simeq \|f\|_{H^s} + |f|_{H^{s,\alpha}}$, with the seminorm $|f|_{H^{s,\alpha}}$ defined as in (1.3).

For simplicity, in this work we restrict our attention to the L^2-based spaces $H^{s,\alpha}(\mathbb{R}^2)$, though we expect Theorem 1.1 to hold for initial velocity in $W^{s,p,\alpha}(\mathbb{R}^2)$ with $s = 2/p + 1$, $p \in (1, \infty)$, and $\alpha \in (0, 1 - \frac{1}{p}]$.

In the proof of Theorem 1.1, we make use of several important properties of logarithmic Sobolev spaces. In Lemma 2.1 below, we establish the first of these properties, a key logarithmic interpolation inequality which relates logarithmic Sobolev spaces to the classical Sobolev spaces. Below, we let $|f|_{H^s}$ denote the homogeneous Sobolev norm; that is,

$$|f|^2_{H^s} = \int_{\mathbb{R}^d} |\xi|^{2s} |\widehat{f}(\xi)|^2 \, d\xi.$$

Lemma 2.1. *Let $\alpha \in (0, 1)$ and $s > t \geq 0$. Then there exists $C > 0$, depending on α and $s - t$, such that for any $f \in H^s(\mathbb{R}^d)$,*

$$|f|_{H^{t,\alpha}} \leq C |f|_{H^t} \log^\alpha \left(\frac{\|f\|_{H^s}}{|f|_{H^t}} \right). \tag{2.1}$$

Moreover, if f satisfies

$$0 < |f|_{H^t} \leq a, \quad \|f\|_{H^s} \leq b,$$

then

$$|f|_{H^{t,\alpha}} \leq C a \log^\alpha \left(\frac{b}{a} + 1 \right).$$

Proof. Let $d\mu$ denote the probability measure

$$d\mu(\xi) := |f|^{-2}_{H^t} |\xi|^{2t} |\widehat{f}(\xi)|^2 d\xi.$$

Then by Jensen's inequality,

$$\begin{aligned} |f|^2_{H^{t,\alpha}} &= (2s - 2t)^{-2\alpha} |f|^2_{H^t} \int_{\mathbb{R}^d} \log^{2\alpha} \left((1 + |\xi|)^{2(s-t)} \right) \, d\mu(\xi) \\ &\leq (2s - 2t)^{-2\alpha} |f|^2_{H^t} \log^{2\alpha} \left(\int_{\mathbb{R}^d} (1 + |\xi|)^{2(s-t)} \, d\mu(\xi) \right) \\ &\leq (2s - 2t)^{-2\alpha} |f|^2_{H^t} \log^{2\alpha} \left(\frac{\int_{\mathbb{R}^d} (1 + |\xi|)^{2s} |\widehat{f}(\xi)|^2 d\xi}{|f|^2_{H^t}} \right). \end{aligned} \tag{2.2}$$

Applying the series of inequalities

$$(1 + |\xi|)^{2s} = (1 + 2|\xi| + |\xi|^2)^s \leq (1 + 2 + \frac{1}{2}|\xi|^2 + |\xi|^2)^s \leq 3^s (1 + |\xi|^2)^s$$

to the integrand on right hand side of (2.2) and taking the square root of both sides yields the desired interpolation inequality,

$$|f|_{H^{t,\alpha}} \leq C |f|_{H^t} \log^\alpha \left(\frac{\|f\|_{H^s}}{|f|_{H^t}} \right).$$

This completes the proof of (2.1).

As (2.1) implies that $|f|_{H^{t,\alpha}} \le C|f|_{H^t} \log^\alpha \left(\frac{\|f\|_{H^s}}{|f|_{H^t}} + 1 \right)$, to complete the proof of Lemma 2.1, we observe that whenever $\alpha \in (0,1)$, the function $h : (0,\infty) \to [0,\infty)$ defined by

$$h(x) = x \log^\alpha \left(\frac{1}{x} + 1 \right)$$

is nondecreasing. □

In the proof of Theorem 1.1, we also make use of the real-variable definition of the $H^{s,\alpha}$- norm. We refer to Theorem 1.4 of [BN19] for the equivalence

$$\|f\|_{H^{s,\alpha}} \simeq \|f\|_{H^s} + \int_{\mathbb{R}^d} \int_{B_{1/2}(x)} \frac{|(-\Delta)^{s/2} f(x) - (-\Delta)^{s/2} f(y)|^2}{|x-y|^d \log^{1-2\alpha}(|x-y|^{-1})} \, dy \, dx. \tag{2.3}$$

This equivalent representation becomes particularly useful when $s = 1$ and $d = 2$, as one can verify using (2.3) that

$$\|f\|_{H^{1,\alpha}} \simeq \|f\|_{H^1} + \int_{\mathbb{R}^2} \int_{B_{1/2}(x)} \frac{|\nabla f(x) - \nabla f(y)|^2}{|x-y|^2 \log^{1-2\alpha}(|x-y|^{-1})} \, dy \, dx. \tag{2.4}$$

2.4. **Oscillatory Integral Estimate.** As mentioned in Section 1, to show ill-posedness in $H^{2,\alpha}(\mathbb{R}^2)$, we construct an initial vorticity which produces a large Lagrangian deformation in short time, and we perturb this initial vorticity by a highly oscillatory function. To prove estimates on the composition of the perturbation with the particle trajectory map, we make use of Lemma 2.2 below. One can find similar types of bounds on oscillatory integrals in, for example, [SM93]. We provide a detailed proof of Lemma 2.2, as we must carefully track how the bounds on the oscillatory integral depend on its phase.

Lemma 2.2. *Let $R > 0$, and assume that $f : \mathbb{R}^2 \to \mathbb{R}$ is smooth and supported in $B_R(0)$. Also assume $\phi : \mathbb{R}^2 \to \mathbb{R}$ is smooth, and suppose that for every $x \in supp(f)$,*

$$|\nabla \phi(x)|_\infty \ge \lambda > 0. \tag{2.5}$$

Then

$$\left| \int_{\mathbb{R}^2} f(x) e^{-i\phi(x)} \, dx \right| \le C_N \|\nabla \phi\|_{C^N}^N \lambda^{-2N} \|f\|_{W^{N,1}} \left(1 + \frac{R\|\nabla\phi\|_{C^1}}{\lambda} \right)^{N+4},$$

for a constant C_N which remains bounded as long as $\|\nabla\phi\|_{C^N}$ remains bounded.

Proof. Set

$$r = \frac{\lambda}{2\|\nabla\phi\|_{C^1}}.$$

It follows from (2.5) and the Mean Value Theorem that for each $x \in supp(f)$, either $|\partial_{x_1}\phi(y)| \ge \lambda/2$ for all $y \in B_r(x)$ or $|\partial_{x_2}\phi(y)| \ge \lambda/2$ for all $y \in B_r(x)$. Since f has compact support, we can find a set of points $X = \{x_k : k = 1, ..., n\} \subset \mathbb{R}^2$ such that

$$supp(f) \subseteq \cup_{k=1}^n B_{r/2}(x_k).$$

We now apply a standard procedure to construct a partition of unity subordinate to the slightly modified open cover

$$\mathcal{B} = \{B_r(x_k) : x_k \in X\}.$$

For fixed k, $1 \le k \le n$, and for $x \in \mathbb{R}^2$, set

$$h_k(x) = \chi\left(\frac{x - x_k}{r}\right),$$

where χ is a smooth radial bump function which is identically one on $B_{1/2}(0)$ and vanishes on $B_1(0)^c$. Note that, by construction, $h_k(z) = 1$ for all $z \in B_{r/2}(x_k)$. Now, for $x \in \mathbb{R}^2$, set

$$g_k(x) = \begin{cases} \frac{h_k(x)}{\sum_{l=1}^n h_l(x)}, & x \in \cup_{k=1}^n B_r(x_k) \\ 0, & \text{otherwise.} \end{cases}$$

Then $\mathcal{U} = \{g_k : 1 \le k \le n\}$ is clearly a partition of unity subordinate to $\mathcal{B}$. Moreover, for each $x \in supp(f)$, there exists l between 1 and n such that $x \in B_{r/2}(x_l)$, ensuring that $\sum_{l=1}^n h_l(x) \ge 1$ for all $x \in \cup_{k=1}^n B_r(x_k)$.

We distribute the positive integers $k = 1, \dots, n$ among two sets K_1 and K_2, where

$$K_j = \{k : |\partial_{x_j}\phi(y)| \ge \lambda/2 \text{ for all } y \in B_{r/2}(x_k)\},$$

with the understanding that $K_1 \cap K_2$ may be nonempty. Now, since for $1 \le k \le n$ and $1 \le j \le 2$, $fg_k e^{-i\phi(x)} = \frac{fg_k}{-i\partial_{x_j}\phi(x)}\partial_{x_j} e^{-i\phi(x)}$, using the identity $f = \sum_{k=1}^n fg_k$ and integration by parts, we may write

$$\begin{aligned} \left|\int_{\mathbb{R}^2} f(x)e^{-i\phi(x)}\,dx\right| &\le \sum_{j=1}^2 \sum_{k\in K_j} \left|\int_{B_r(x_k)} \frac{f(x)g_k(x)}{\partial_{x_j}\phi(x)}\partial_{x_j} e^{-i\phi(x)}\,dx\right| \\ &= \sum_{j=1}^2 \sum_{k\in K_j} \left|\int_{B_r(x_k)} \frac{\partial_{x_j}\left(f(x)g_k(x)\right)\partial_{x_j}\phi(x) - f(x)g_k(x)\partial_{x_j}^2\phi(x)}{\left(\partial_{x_j}\phi(x)\right)^2} e^{-i\phi(x)}\,dx\right|, \end{aligned} \tag{2.6}$$

where the inequality results from the possibility that $K_1 \cap K_2 \ne 0$. Since $|\partial_{x_j}\phi(x)| \ge \lambda/2$ on $B_r(x_k)$ when $k \in K_j$, it follows that

$$\begin{aligned} \left|\int_{\mathbb{R}^2} f(x)e^{-i\phi(x)}\,dx\right| &\le \frac{Cn}{\lambda^2} \sup_{1\le k\le n} \left(\|f\|_{W^{1,1}}\|g_k\|_{C^1}\|\nabla\phi\|_\infty + \|f\|_{L^1}\|g_k\|_{L^\infty}\|\nabla\phi\|_{C^1}\right) \\ &\le \frac{Cn}{\lambda^2}\|\nabla\phi\|_{C^1}\|f\|_{W^{1,1}} \sup_{1\le k\le n} \|g_k\|_{C^1}. \end{aligned} \tag{2.7}$$

To complete the proof of Lemma 2.2 for $N = 1$, we must bound n and $\|g_k\|_{C^1}$. Since $\sum_{l=1}^n h_l(x) \ge 1$ for all $x \in \cup_{k=1}^n B_r(x_k)$, an application of the quotient rule gives

$$\|\nabla g_k\|_\infty \le \frac{Cn}{r} \le \frac{Cn\|\nabla\phi\|_{C^1}}{\lambda} \tag{2.8}$$

for each k. Moreover, since $supp(f) \subseteq B_R(0)$, we can choose n to satisfy

$$n \le C\left(\frac{R}{r} + 1\right)^2 \le C\left(\frac{R\|\nabla\phi\|_{C^1}}{\lambda} + 1\right)^2.$$

Substituting these estimates into (2.7) gives

$$\left|\int_{\mathbb{R}^2} f(x)e^{-i\phi(x)}\,dx\right| \le \frac{C\|\nabla\phi\|_{C^1}}{\lambda^2}\|f\|_{W^{1,1}}\left(1 + \frac{R\|\nabla\phi\|_{C^1}}{\lambda}\right)^5.$$

This verifies the lemma for $N = 1$. Now, returning to (2.6), we once again integrate the right hand side by parts and obtain, in a manner similar to that above, the inequality

$$\left| \int_{\mathbb{R}^2} f(x) e^{-i\phi(x)}\, dx \right| \leq \frac{Cn}{\lambda^4} \|\nabla\phi\|_{C^2}^2 \|f\|_{W^{2,1}} \sup_{1\leq k\leq n} \|g_k\|_{C^2}. \tag{2.9}$$

Again by the quotient rule, for multi-indices $|\beta| = 2$,

$$\|\partial^\beta g_k\|_\infty \leq \frac{Cn^2}{r^2}. \tag{2.10}$$

We substitute the above bound for n into (2.10) and apply the resulting inequality and (2.8) to (2.9), which gives

$$\left| \int_{\mathbb{R}^2} f(x) e^{-i\phi(x)}\, dx \right| \leq \frac{C\|\nabla\phi\|_{C^2}^2}{\lambda^4} \|f\|_{W^{2,1}} \left(1 + \frac{R\|\nabla\phi\|_{C^1}}{\lambda}\right)^6.$$

Iterating this process N times yields the result. □

3. Local Norm Inflation

In this section, we show that there exists smooth initial data ω_0 and a smooth function β, both small in $H^{1,\alpha}$-norm, such that the perturbed initial data $\omega_0 + \beta$ exhibits local norm inflation; specifically, $\omega_0 + \beta$ generates a solution to (V) whose $H^{1,\alpha}$-norm can be made arbitrarily large on the time interval $[0, 1]$.

To prove local norm inflation, we construct initial data that yields a solution with large Lagrangian deformation matrix ∇X^t. We then perturb the initial data by a highly oscillatory function localized around the point at which the large deformation occurs. We will show that, if the frequency of oscillation is large enough, then the solution is well-approximated in $H^{1,\alpha}$-norm by the propagation of the perturbation β by the particle trajectory map of the original solution. The Lagrangian stretching of the oscillation then yields the norm inflation.

We complete this process in two parts, by first showing in Theorem 3.1 that under suitable conditions such a perturbation exists, and then showing in Theorem 3.6 that the conditions of Theorem 3.1 can be met.

Theorem 3.1. *Let $\omega_0 \in C_c^\infty(\mathbb{R}^2)$ satisfy the following properties:*
(i) there exists $R_0 > 0$ such that supp(ω_0)$\subset B_{R_0}(0)$, and
(ii) there exists $t_0 \in [0, 1]$ such that

$$\max_{0\leq t\leq 1} \|\nabla X^t\|_{L^\infty(B_{R_0}(0))} = \|\nabla X^{t_0}\|_{L^\infty(B_{R_0}(0))} = L > 2^{13},$$

where $X^t = (X_1^t, X_2^t)$ denotes the particle trajectory map corresponding to the solution, ω, of (V) with initial data ω_0, i.e.

$$\frac{dX^t}{dt} = (\nabla^\perp(-\Delta)^{-1}\omega)(X^t(\alpha), t), \quad X^0(\alpha) = \alpha, \quad (\alpha, t) \in \mathbb{R}^2 \times [0, \infty).$$

Then there exists a function $\beta \in C_c^\infty(\mathbb{R}^2)$ such that the solution $\widetilde{\omega}$ to (V) with initial data $\widetilde{\omega_0} = \omega_0 + \beta$ satisfies

$$\|\widetilde{\omega}(t_0)\|_{H^{1,\alpha}} > L^{\frac{1}{3}}. \tag{3.1}$$

Moreover, β satisfies

$$\begin{aligned} &supp(\beta) \subset B_{R_0}(0), \\ &\|\beta\|_{L^p} \le \|\omega_0\|_{L^p} \quad \text{for all } p \in [1,\infty], \ \text{and} \\ &|\beta|_{H^{1,\alpha}} \le c_\alpha L^{-\frac{1}{2}}, \end{aligned} \tag{3.2}$$

for a constant $c_\alpha > 0$ which is independent of both ω_0 and L.

Remark 3.2. The existence of a solution ω of (V) satisfying the assumptions of Theorem 3.1 is established in Theorem 3.6 below.

Proof. Let ω denote the solution to (V) with initial data ω_0. First note that if ω is such that $\|\omega(t_0)\|_{H^{1,\alpha}} > L^{\frac{1}{3}}$, then we can take $\beta \equiv 0$ and the theorem holds. Thus, in what follows we assume that

$$\|\omega(t_0)\|_{H^{1,\alpha}} \le L^{\frac{1}{3}}. \tag{3.3}$$

Construction of β satisfying (3.2). Observe that by assumption (ii) and the smoothness of X^t, there exists $x_0 = (x_0^1, x_0^2)$ with $x_0^1 x_0^2 \neq 0$ and $r_0 > 0$ such that $B_{r_0}(x_0) \subset B_{R_0}(0)$ and such that one component of ∇X^{t_0} is at least $L/2$ in magnitude on $B_{r_0}(x_0)$. Without loss of generality, we assume

$$\frac{L}{2} \le \left| \frac{\partial X_2^{t_0}}{\partial x_2}(x) \right| \le L \tag{3.4}$$

for all $x \in B_{r_0}(x_0)$.

We now construct our highly oscillatory perturbation β. First, let $\chi \in C_c^\infty(\mathbb{R}^2)$ be a standard radial bump function with $\operatorname{supp}(\chi) \subset B_1(0)$, $0 \le \chi(x) \le 1$ for all $x \in \mathbb{R}^2$, and $\chi(x) = 1$ for all $x \in B_{1/2}(0)$. Choose $\delta > 0$ so that $\delta \ll \min\{|x_0^1|, |x_0^2|, r_0\}$, and define

$$\varphi(x_1, x_2) = \frac{1}{\delta} \sum_{a_1, a_2 \in \{-1,1\}} a_2 \chi\left(\frac{x_1 - a_1 x_0^1}{\delta}, \frac{x_2 - a_2 x_0^2}{\delta} \right).$$

Note that φ is an odd function of x_2 and an even function of x_1. Moreover, by our choice of δ, the support of φ is contained in four disjoint balls, each of which is contained in $B_{R_0}(0)$. One can then check that $\|\varphi\|_{L^2} = 4\|\chi\|_{L^2}$.

Now, for fixed $k \in \mathbb{N}$, define

$$\beta(x_1, x_2) = \frac{1}{k \log^\alpha(k+1)\sqrt{L}} \sin(kx_1)\varphi(x_1, x_2).$$

It follows immediately that $\operatorname{supp}(\beta)$ is contained in $B_{R_0}(0)$; thus, $(3.2)_1$ is satisfied.

We wish to show that if $k > 0$ is chosen sufficiently large, then β satisfies the norm bounds in (3.2). To this end, observe that for $p \in [1, \infty]$,

$$\|\beta\|_{L^p} \le \frac{\|\varphi\|_{L^p}}{k \log^\alpha(k+1)\sqrt{L}}, \tag{3.5}$$

so that, for k sufficiently large, $\|\beta\|_{L^p} \le \|\omega_0\|_{L^p}$ for each $p \in [1, \infty]$. To establish the bound on the $H^{1,\alpha}$-seminorm of β, we estimate higher derivatives of β in L^2-norm and apply (2.1). Note that by the product rule and the chain rule,

$$\|\nabla\beta\|_{L^2} \le \frac{1}{\log^\alpha(k+1)\sqrt{L}}\left(\|\varphi\|_{L^2} + \frac{1}{k}\|\nabla\varphi\|_{L^2}\right),$$

$$\|\nabla^2\beta\|_{L^2} \le \frac{1}{\log^\alpha(k+1)\sqrt{L}}\left(k\|\varphi\|_{L^2} + 2\|\nabla\varphi\|_{L^2} + \frac{1}{k}\|\nabla^2\varphi\|_{L^2}\right).$$

So, for large enough k, we have

$$\|\nabla\beta\|_{L^2} \le \frac{2}{\log^\alpha(k+1)\sqrt{L}}\|\varphi\|_{L^2} \tag{3.6}$$

and

$$\|\nabla^2\beta\|_{L^2} \le \frac{2k}{\log^\alpha(k+1)\sqrt{L}}\|\varphi\|_{L^2} \tag{3.7}$$

Applying Lemma 2.1 with (3.6), (3.7), we obtain, for sufficiently large k,

$$\begin{aligned}|\beta|_{H^{1,\alpha}} &\le C_\alpha|\beta|_{H^1}\log^\alpha\left(\frac{\|\beta\|_{H^2}}{|\beta|_{H^1}}+1\right) \le C_\alpha|\beta|_{H^1}\log^\alpha\left(\frac{6k\|\varphi\|_{L^2}}{|\beta|_{H^1}\log^\alpha(k+1)\sqrt{L}}+1\right)\\ &\le \frac{2C_\alpha\|\varphi\|_{L^2}}{\log^\alpha(k+1)\sqrt{L}}\log^\alpha(3k+1) \le \frac{c_\alpha}{\sqrt{L}}.\end{aligned}$$

Thus, (3.2) is satisfied.

Proof of (3.1) Define $\widetilde{\omega}$ by

$$\begin{cases}\partial_t\widetilde{\omega} + \widetilde{u}\cdot\nabla\widetilde{\omega} = 0,\\ \widetilde{u} = \nabla^\perp(-\Delta)^{-1}\widetilde{\omega},\\ \widetilde{\omega}|_{t=0} = \widetilde{\omega}_0 := \omega_0 + \beta,\end{cases}$$

and $\widetilde{X}^t$ by

$$\frac{d}{dt}\widetilde{X}^t(x) = \widetilde{u}(\widetilde{X}^t(x),t), \quad \widetilde{X}^0(x) = x, \quad (x,t)\in\mathbb{R}^2\times[0,\infty).$$

Define the two functions

$$W(\cdot,t) = \omega_0\circ\widetilde{X}^{-t}, \quad p(\cdot,t) = \beta\circ\widetilde{X}^{-t}.$$

Since $\tilde{\omega} = W + p$, it follows from the triangle inequality that

$$\|\widetilde{\omega}\|_{H^{1,\alpha}} \ge \|p\|_{H^{1,\alpha}} - \|W\|_{H^{1,\alpha}}. \tag{3.8}$$

To establish (3.1), we will first show that $\|W\|_{H^{1,\alpha}}$ can be approximated by $\|\omega\|_{H^{1,\alpha}}$. Specifically, we establish (3.9) below. Our assumption (3.3) then gives an upper bound for $\|W(t_0)\|_{H^{1,\alpha}}$ in terms of L. This upper bound, combined with (3.8), reduces the proof of (3.1) to showing that $\|p(t_0)\|_{H^{1,\alpha}}$ is sufficiently large.

Estimate for $\|W-\omega\|_{H^{1,\alpha}}$. We will show that, for sufficiently large k,

$$\max_{0\le t\le 1}\|W-\omega\|_{H^{1,\alpha}} < 1. \tag{3.9}$$

First note that $W-\omega$ satisfies

$$\partial_t(W-\omega)+(\widetilde{u}-u)\cdot\nabla W+u\cdot\nabla(W-\omega)=0.$$

We multiply this equality by $W-\omega$, integrate over $\mathbb{R}^2$, and apply Holder's inequality to write

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|W-\omega\|_{L^2}^2&=-\int_{\mathbb{R}^2}(\widetilde{u}-u)\cdot\nabla W(W-\omega)\,dx-\int_{\mathbb{R}^2}u\cdot\nabla(W-\omega)(W-\omega)\,dx\\&\le\|u-\widetilde{u}\|_{L^4}\|\nabla W\|_{L^4}\|W-\omega\|_{L^2}.\end{aligned}\tag{3.10}$$

Note that we used the divergence free condition on u to conclude that the second integral above vanishes. We now utilize the Hardy-Littlewood-Sobolev inequality and the compact support of ω and $\tilde{\omega}$ to write

$$\|u-\widetilde{u}\|_{L^4}\lesssim\|\omega-\widetilde{\omega}\|_{L^{4/3}}\lesssim\|\omega-\widetilde{\omega}\|_{L^2},\tag{3.11}$$

where the implied constant depends on the measure of the support of ω_0. Substituting this estimate into (3.10) gives

$$\begin{aligned}\frac{d}{dt}\|W-\omega\|_{L^2}^2&\lesssim\|\omega-\widetilde{\omega}\|_{L^2}\|\nabla W\|_{L^4}\|W-\omega\|_{L^2}\\&\lesssim\|\omega-\widetilde{\omega}\|_{L^2}\|W\|_{H^2}\|W-\omega\|_{L^2}\lesssim\|\omega-\widetilde{\omega}\|_{L^2}\|W-\omega\|_{L^2},\end{aligned}\tag{3.12}$$

where we applied the Sobolev Embedding Theorem to obtain the second inequality and Lemma 3.4 to obtain the third inequality.

To complete the bound on $\|W-\omega\|_{L^2}$, we must estimate $\|\omega-\widetilde{\omega}\|_{L^2}$. To this end, note that $\omega-\widetilde{\omega}$ satisfies

$$\partial_t(\omega-\widetilde{\omega})+(u-\widetilde{u})\cdot\nabla\omega+\widetilde{u}\cdot\nabla(\omega-\widetilde{\omega})=0.$$

Multiplying by $\omega-\widetilde{\omega}$, integrating in space, and applying Holder's inequality gives

$$\frac{d}{dt}\|\omega-\widetilde{\omega}\|_{L^2}^2\le2\|u-\widetilde{u}\|_{L^4}\|\nabla\omega\|_{L^4}\|\omega-\widetilde{\omega}\|_{L^2}-\int_{\mathbb{R}^2}\widetilde{u}\cdot\nabla(\omega-\widetilde{\omega})^2\,dx.$$

The integral term above is again zero by the incompressibility of $\widetilde{u}$. By (3.11) and Lemma 3.3,

$$\frac{d}{dt}\|\omega-\widetilde{\omega}\|_{L^2}^2\lesssim\|\nabla\omega\|_{L^4}\|\omega-\widetilde{\omega}\|_{L^2}^2\lesssim\|\omega-\widetilde{\omega}\|_{L^2}^2.\tag{3.13}$$

Grönwall's lemma and (3.5) then yield

$$\max_{0\le t\le1}\|\omega-\widetilde{\omega}\|_{L^2}\lesssim\|\omega_0-\widetilde{\omega_0}\|_{L^2}\lesssim\frac{1}{k\log^\alpha(k+1)}.\tag{3.14}$$

Substituting (3.14) into (3.12), integrating in time, and using that $W(0)=\omega(0)$ gives

$$\max_{0\le t\le1}\|W-\omega\|_{L^2}\lesssim\frac{1}{k\log^\alpha(k+1)}.\tag{3.15}$$

Combining Lemma 3.4, Lemma 3.5, and (3.15) and applying Sobolev interpolation together with the embedding $H^{\frac{3}{2}}(\mathbb{R}^2)\hookrightarrow H^{1,\alpha}(\mathbb{R}^2)$ gives

$$\max_{0\le t\le1}\|W-\omega\|_{H^{1,\alpha}}\lesssim\max_{0\le t\le1}\|W-\omega\|_{H^{\frac{3}{2}}}\lesssim\left(\frac{1}{k\log^\alpha(k+1)}\right)^{\frac{1}{4}}.$$

Thus for sufficiently large k,

$$\max_{0\le t\le 1}\|W-\omega\|_{H^{1,\alpha}}<1. \tag{3.16}$$

Lower bound for $\|p(\cdot,t_0)\|_{H^{1,\alpha}}$. Note that the estimate in (3.16) and the triangle inequality imply that, for sufficiently large k,

$$\|\widetilde{\omega}\|_{H^{1,\alpha}}\ge\|p\|_{H^{1,\alpha}}-\|W\|_{H^{1,\alpha}}\ge\|p\|_{H^{1,\alpha}}-\|\omega\|_{H^{1,\alpha}}-1.$$

We will show that for sufficiently large k,

$$\|p(\cdot,t_0)\|_{H^{1,\alpha}}\ge\frac{1}{2}L^{\frac{1}{2}}. \tag{3.17}$$

This estimate, combined with (3.3), will allow us to conclude that (3.1) holds, as (3.17) implies

$$\|\widetilde{\omega}(t_0)\|_{H^{1,\alpha}}\ge\frac{1}{2}L^{\frac{1}{2}}-L^{\frac{1}{3}}-1>L^{\frac{1}{3}},$$

when $L>2^{13}$.

To establish (3.17), first define

$$q(\cdot,t)=\beta\circ X^{-t}.$$

In what follows, we estimate the size of $q(\cdot,t_0)$ in $H^{1,\alpha}$ and show that, for frequencies near kL with k sufficiently large, q sufficiently approximates p in L^2-norm. This approximation will yield the desired lower bound on $\|p(\cdot,t_0)\|_{H^{1,\alpha}}$.

To estimate $\|q(\cdot,t_0)\|_{H^{1,\alpha}}$, we analyze the frequencies of the Fourier transform of $q(\cdot,t_0)$. Denote

$$\widehat{q}_{t_0}(\xi)=\int_{\mathbb{R}^2}q(x,t_0)e^{-ix\cdot\xi}\,dx.$$

The definition of β combined with a change of variables $x\mapsto X^{t_0}(x)$ gives

$$\widehat{q}_{t_0}(\xi)=\frac{1}{k\log^{\alpha}(k+1)\sqrt{L}}\int_{\mathbb{R}^2}\sin(kx_1)\varphi(x)e^{-iX^{t_0}(x)\cdot\xi}\,dx.$$

Defining $F(\xi):=k\log^{\alpha}(k+1)\sqrt{L}\widehat{q}_{t_0}(\xi)$, we have

$$F(\xi)=\frac{1}{2i}\int_{\mathbb{R}^2}\varphi(x)\left(e^{-iX^{t_0}(x)\cdot\xi+ikx_1}-e^{-iX^{t_0}(x)\cdot\xi-ikx_1}\right)\,dx, \tag{3.18}$$

which is a difference of two oscillatory integrals. In order to estimate the size of F in different ranges of frequencies, we apply the method of nonstationary phase. Consider the phases

$$\phi_\xi^{\pm}(x):=X^{t_0}(x)\cdot\xi\pm kx_1.$$

Since u is divergence free, $\det\nabla X^{t_0}(x)=1$, which implies

$$\nabla\phi_\xi^{\pm}(x)=\nabla X^{t_0}(x)\xi\pm ke_1=\nabla X^{t_0}(x)\left(\xi\pm k\nabla^{\perp}X_2^{t_0}(x)\right),$$

where $e_1=\begin{pmatrix}1\\0\end{pmatrix}$. Hence we may write

$$\xi\pm k\nabla^{\perp}X_2^{t_0}(x)=\left(\nabla X^{t_0}(x)\right)^{-1}\left(\nabla\phi_\xi^{\pm}(x)\right).$$

Now observe that for every $x \in B_{r_0}(x_0)$, we have

$$\frac{L}{2} \leq |\nabla^\perp X_2^{t_0}(x)| \leq \sqrt{2}L,$$

where the lower bound follows from (3.4) and the upper bound follows from assumption (ii). It follows that if $|\xi| \leq \frac{1}{4}kL$ and $x \in B_{r_0}(x_0)$,

$$|\nabla\phi_\xi^\pm(x)| \geq \frac{1}{2|\left(\nabla X^{t_0}(x)\right)^{-1}|_\infty}|\xi \pm k\nabla^\perp X_2^{t_0}(x)| \geq \frac{1}{2\sqrt{2}L}\frac{kL}{4} \geq \frac{k}{8\sqrt{2}}.$$

Thus, there exists some $c_0 > 0$ such that for any $x \in B_{r_0}(x_0)$ and $\xi \in \mathbb{R}^2$ satisfying $|\xi| \leq \frac{1}{4}kL$, we have

$$|\nabla\phi_\xi^\pm(x)|_\infty \geq c_0 k. \tag{3.19}$$

We refer to Lemma 2.2 to obtain the point-wise estimate on F,

$$|F(\xi)| = \frac{1}{2}\left|\int_{\mathbb{R}^2} \varphi(x)\left(e^{-i\phi_\xi^-(x)} - e^{-i\phi^+(x)}\right)dx\right| \leq C\|\nabla\phi_\xi^\pm\|_{C^2}^2\|\varphi\|_{W^{2,1}}\left(1 + \frac{\|\nabla\phi_\xi^\pm\|_{C^1}}{k}\right)^6 k^{-4}.$$

Here $\|\nabla\phi^\pm\|_{C^n} := \max\{\|\nabla\phi^-\|_{C^n}, \|\nabla\phi^+\|_{C^n}\}$ and $C > 0$ depends on c_0, r_0. Moreover, we have

$$\|\nabla\phi_\xi^\pm\|_{C^2} \leq |\xi|\|\nabla X^{t_0}\|_{C^2} \leq \frac{1}{4}kL\|\nabla X^{t_0}\|_{C^2}.$$

Note that the derivatives of ∇X^{t_0} can be controlled by derivatives of ω, which we can control by the same derivatives of ω_0 by arguments similar to those used in the proofs of Lemmas 3.3 and 3.4. Thus we obtain

$$|F(\xi)| \leq C_1 k^{-2} \quad \text{for all } \xi \in B_{\frac{1}{4}kL}(0),$$

where $C_1 > 0$ is independent of k. It now follows that

$$\|F\mathbb{1}_{|\xi|\leq\frac{1}{4}kL}\|_{L^2} \lesssim k^{-1}. \tag{3.20}$$

Moreover, since X^{-t} is measure-preserving, an application of Plancherel's identity gives

$$\begin{aligned}\|F\|_{L^2}^2 = \left(2\pi k\log^\alpha(k+1)\sqrt{L}\|\beta\|_{L^2}\right)^2 &= 4\pi^2\int_{\mathrm{supp}(\varphi)} |\sin(kx_1)\varphi(x)|^2\,dx \\ &\geq \frac{16\pi^2}{\delta^2}\int_{|x-x_0|<\delta/2}\left(\frac{1}{2} - \frac{1}{2}\cos(2kx_1)\right)dx \\ &= 2\pi^2 - \frac{16\pi^2}{\delta^2}\int_{|x-x_0|<\delta/2}\frac{1}{2}\cos(2kx_1)\,dx \geq 2\pi^2 - C\frac{1}{k}\end{aligned} \tag{3.21}$$

for some constant $C > 0$ which depends on the support of φ. In view of (3.21) and (3.20) we have, for sufficiently large k,

$$\|F\mathbb{1}_{|\xi|\geq\frac{1}{4}kL}\|_{L^2} \geq \|F\|_{L^2} - \|F\mathbb{1}_{|\xi|\leq\frac{1}{4}kL}\|_{L^2} \geq 4. \tag{3.22}$$

We now show that F adequately approximates $k\log^\alpha(k+1)\sqrt{L}p$ in L^2 norm. Define

$$\eta = k\log^\alpha(k+1)\sqrt{L}(p-q).$$

Then η solves

$$\begin{cases} \partial_t \eta + k\log^\alpha(k+1)\sqrt{L}(\widetilde{u}-u)\cdot\nabla q + \widetilde{u}\cdot\nabla\eta = 0, \\ \eta|_{t=0} = 0. \end{cases} \tag{3.23}$$

Multiplying $(3.23)_1$ by η and integrating over $\mathbb{R}^2$ gives

$$\frac{d}{dt}\|\eta\|_{L^2}^2 \leq 2k\log^\alpha(k+1)\sqrt{L}\|\widetilde{u}-u\|_{L^4}\|\nabla q\|_{L^4}\|\eta\|_{L^2}, \tag{3.24}$$

where we used that $\operatorname{div}\widetilde{u} = 0$ to conclude that $\int_{\mathbb{R}^2}\eta\widetilde{u}\cdot\nabla\eta\,dx = 0$. We estimate the terms on the right hand side of (3.24). Recall that by (3.11) and (3.14),

$$\|\widetilde{u}-u\|_{L^4} \lesssim \|\widetilde{\omega}-\omega\|_{L^2} \lesssim \|\widetilde{\omega_0}-\omega_0\|_{L^2} \lesssim \frac{1}{k\log^\alpha(k+1)}. \tag{3.25}$$

Moreover, since q solves

$$\partial_t q + u\cdot\nabla q = 0, \quad q|_{t=0} = \beta,$$

we apply an argument identical to that used to derive (3.31) below to obtain

$$\frac{d}{dt}\|\nabla q\|_{L^4} \lesssim \|\nabla u\|_{L^\infty}\|\nabla q\|_{L^4}.$$

Grönwall's inequality provides, for some constant $c > 0$ independent of k,

$$\|\nabla q\|_{L^4} \lesssim \|\nabla\beta\|_{L^4}\exp\left\{c\int_0^{t_0}\|\nabla u(s)\|_\infty\,ds\right\}. \tag{3.26}$$

By the embedding $W^{1,4}(\mathbb{R}^2) \hookrightarrow C_B^0(\mathbb{R}^2)$, the boundedness of Calderon-Zygmund operators on $L^4(\mathbb{R}^2)$, and the conservation of L^p norms of the vorticity, we have the series of inequalities

$$\|\nabla u\|_\infty \lesssim \|\nabla u\|_{W^{1,4}} \lesssim \|\omega\|_{W^{1,4}} = \|\omega_0\|_{L^4} + \|\nabla\omega\|_{L^4}.$$

Substituting this estimate into (3.26) and applying Lemma 3.3 below, we obtain

$$\|\nabla q\|_{L^4} \lesssim \|\nabla\beta\|_{L^4}. \tag{3.27}$$

Now by arguing as we did to obtain (3.6), we have

$$\|\nabla\beta\|_{L^4} \lesssim \log^{-\alpha}(k+1)L^{-1/2},$$

which, when substituted into (3.27), gives

$$\|\nabla q\|_{L^4} \lesssim \log^{-\alpha}(k+1)L^{-1/2}. \tag{3.28}$$

Substituting (3.25) and (3.28) into (3.24), integrating in time, and using that $\eta|_{t=0} = 0$, we conclude that

$$\|\eta(t_0)\|_{L^2} \lesssim \log^{-\alpha}(k+1). \tag{3.29}$$

Thus, by (3.22), (3.29), and Plancherel's identity, we may take k sufficiently large to obtain

$$\begin{aligned} k\log^\alpha(k+1)L\|\widehat{p}(t_0)\mathbb{1}_{|\xi|\geq\frac{1}{4}kL}\|_{L^2} &\geq k\log^\alpha(k+1)L\|\widehat{q}(t_0)\mathbb{1}_{|\xi|\geq\frac{1}{4}kL}\|_{L^2} - 2\pi\sqrt{L}\|\eta(t_0)\|_{L^2} \\ &= \sqrt{L}\left(\|F\mathbb{1}_{|\xi|\geq\frac{1}{4}kL}\|_{L^2} - 2\pi\|\eta(t_0)\|_{L^2}\right) \geq \sqrt{L}\left(\|F\mathbb{1}_{|\xi|\geq\frac{1}{4}kL}\|_{L^2} - 2\right) \geq 2\sqrt{L}. \end{aligned} \tag{3.30}$$

Finally, we are able to estimate the $H^{1,\alpha}$ norm of p as follows:

$$\begin{aligned}\|p(t_0)\|_{H^{1,\alpha}}^2 &= \int_{\mathbb{R}^2} |\xi|^2 \log^{2\alpha}(|\xi|+1)|\widehat{p}(\xi,t_0)|^2\, d\xi \\ &\geq \frac{1}{16}k^2L^2\log^{2\alpha}\left(\frac{1}{4}kL+1\right)\int_{|\xi|\geq\frac{1}{4}kL}|\widehat{p}(\xi,t_0)|^2\, d\xi \\ &\geq \frac{1}{16}\left(k\log^{\alpha}(k+1)L\|\widehat{p}(t_0)\mathbb{1}_{|\xi|\geq\frac{1}{4}kL}\|_{L^2}\right)^2 \geq \frac{1}{4}L,\end{aligned}$$

where we used that $\frac{1}{4}kL > k$ to obtain the second inequality and (3.30) to obtain the third inequality. Thus, (3.17) holds.

This completes the proof of Theorem 3.1. □

In the proof of Theorem 3.1 above, we made use of the following three lemmas.

Lemma 3.3. *Assume ω is a smooth solution to (V) with initial data $\omega_0 \in C_c^\infty(\mathbb{R}^2)$. Then there exists a constant $C_1 > 0$ such that*

$$\max_{0\leq t\leq 1}\|\nabla\omega(t)\|_{L^4} \leq C_1.$$

Here C_1 depends only on (and increases with respect to) $\|\omega_0\|_\infty$, $\|\nabla\omega_0\|_{L^4}$, and the measure of the support of ω_0,

Proof. The proof is standard, so we only outline the argument. Applying ∂_j for $j = 1, 2$ to the vorticity equation, multiplying the resulting equation by $|\partial_j\omega|^2\partial_j\omega$, and integrating over $\mathbb{R}^2$ gives

$$\frac{1}{4}\frac{d}{dt}\|\nabla\widetilde{\omega}\|_{L^4}^4 + \int_{\mathbb{R}^2}|\partial_j\omega|^2\partial_j\omega(u\cdot\nabla\partial_j\omega)\, dx + \int_{\mathbb{R}^2}|\partial_j\omega|^2\partial_j\omega(\partial_j u\cdot\nabla\omega)\, dx = 0.$$

Since u is divergence free, the second term on the left-hand side of the above equality is 0, while the third term can be bounded using Holder's inequality. This gives

$$\frac{d}{dt}\|\partial_j\omega\|_{L^4}^4 \leq C\|\nabla u\|_\infty\|\nabla\omega\|_{L^4}^4. \tag{3.31}$$

We now substitute the Beale-Kato-Majda logarithmic-Sobolev inequality [BKM84], given by

$$\|\nabla u\|_\infty \leq C(1+\|\omega_0\|_2+\|\omega_0\|_\infty\log(2+\|\nabla\omega\|_{L^4})),$$

into (3.31). Applying a standard argument, one can conclude that for each $t > 0$,

$$\|\nabla\omega(t)\|_{L^4} \leq Ce^{e^{Ct\|\omega_0\|_\infty}}(1+\|\nabla\omega_0\|_{L^4})^{e^{Ct\|\omega_0\|_\infty}}.$$

Taking the maximum over $t \in [0, 1]$ yields the result. □

Lemma 3.4. *Assume $\tilde{\omega}$ is a smooth solution to (V) with corresponding velocity $\tilde{u}$ and initial data $\tilde{\omega}_0 = \omega_0 + \beta$, where $\omega_0, \beta \in C_c^\infty(\mathbb{R}^2)$. Further assume W is a smooth solution to the system*

$$\begin{cases}\partial_t W + \tilde{u}\cdot\nabla W = 0 & \text{on } \mathbb{R}^2\times(0,\infty),\\ \tilde{u} = \nabla^\perp(-\Delta)^{-1}\tilde{\omega} & \text{on } \mathbb{R}^2\times[0,\infty),\\ W|_{t=0} = \omega_0 & \text{on } \mathbb{R}^2.\end{cases}$$

Then there exists a constant $C_0 > 0$ such that

$$\max_{0\le t\le 1} \|W(t)\|_{H^2} \le C_0.$$

Here C_0 depends only on (and increases with respect to) $\|\omega_0\|_\infty$, $\|\beta\|_\infty$, $\|\omega_0\|_{W^{1,4}}$, $\|\beta\|_{W^{1,4}}$, and the measure of the support of ω_0 and β.

Proof. Since W satisfies the transport equation

$$\partial_t W + \widetilde{u}\cdot\nabla W = 0,$$

letting γ denote a multi-index with $|\gamma| \le 2$ and applying ∂^γ gives

$$\partial_t\partial^\gamma W + \widetilde{u}\cdot\nabla\partial^\gamma W = [\widetilde{u}\cdot\nabla, \partial^\gamma]W.$$

Multiplying by $D^\gamma W$ and integrating over $\mathbb{R}^2$ gives

$$\frac{1}{2}\frac{d}{dt}\|\partial^\gamma W\|_{L^2}^2 = \int_{\mathbb{R}^2} \partial^\gamma W[\widetilde{u}\cdot\nabla, \partial^\gamma]W\,dx.$$

The triangle and Hölder's inequalities imply that for all γ satisfying $|\gamma| \le 2$,

$$\frac{1}{2}\frac{d}{dt}\|\partial^\gamma W\|_{L^2}^2 \le C\|\partial^\gamma W\|_{L^2}\left(\|\widetilde{u}\|_{W^{2,4}}\|\nabla W\|_{L^4} + \|\nabla\widetilde{u}\|_\infty\|W\|_{H^2}\right).$$

Combining this estimate with the conservation $\|W(t)\|_{L^2} = \|\omega_0\|_{L^2}$ for all $t \ge 0$ and summing over γ with $|\gamma| \le 2$, we conclude that

$$\frac{d}{dt}\|W\|_{H^2} \le C(\|\widetilde{u}\|_{W^{2,4}}\|\nabla W\|_{L^4} + \|\nabla\widetilde{u}\|_\infty\|W\|_{H^2}). \tag{3.32}$$

By the boundedness properties of Calderon-Zygmund operators and Lemma 3.3,

$$\|\tilde{u}(t)\|_{W^{2,4}} \le \|\tilde{\omega}(t)\|_{L^4} + \|\nabla\tilde{\omega}(t)\|_{L^4} \le \|\tilde{\omega}_0\|_{L^4} + Ce^{e^{Ct\|\tilde{\omega}_0\|_\infty}}(1 + \|\nabla\tilde{\omega}_0\|_{L^4})^{e^{Ct\|\tilde{\omega}_0\|_\infty}}$$

for each $t \in [0,1]$. Also, an application of Sobolev's inequality, boundedness of Calderon-Zygmund operators, and Lemma 3.3 gives

$$\|\nabla W(t)\|_{L^4} \le C\|W(t)\|_{H^2},$$

$$\|\nabla\tilde{u}(t)\|_\infty \le C\|\nabla\tilde{u}(t)\|_{W^{1,4}} \le C\|\tilde{\omega}(t)\|_{W^{1,4}} \le \|\tilde{\omega}_0\|_{L^4} + Ce^{e^{Ct\|\tilde{\omega}_0\|_\infty}}(1 + \|\nabla\tilde{\omega}_0\|_{L^4})^{e^{Ct\|\tilde{\omega}_0\|_\infty}}.$$

Substituting these bounds into (3.32) and applying Grönwall's inequality provides

$$\max_{0\le t\le 1}\|W\|_{H^2} \le C_0, \tag{3.33}$$

where $C_0 > 0$ is a constant dependent only on $\|\omega_0\|_\infty$, $\|\beta\|_\infty$, $\|\omega_0\|_{W^{1,4}}$, $\|\beta\|_{W^{1,4}}$, and the supports of ω_0 and β. □

Lemma 3.5. *Assume ω is a smooth solution to (V) with initial data $\omega_0 \in C_c^\infty(\mathbb{R}^2)$. Then there exists a constant $C_0 > 0$ such that*

$$\max_{0\le t\le 1}\|\omega(t)\|_{H^2} \le C_0.$$

Here C_0 depends only on (and increases with respect to) $\|\omega_0\|_\infty$, $\|\omega_0\|_{W^{1,4}}$, and the measure of the support of ω_0.

Proof. The proof is virtually identical to that of Lemma 3.4, so we omit the details. □

We spend the remainder of this section constructing initial data that both satisfies the conditions of Theorem 3.1 and is small in $H^{1,\alpha}(\mathbb{R}^2)$. The initial data resulting from our construction will serve as a sort of building block for the initial vorticity used to generate strong ill-posedness in the proof of the Theorem 1.1.

Theorem 3.6. *Given $\epsilon > 0$, $\tau > 0$, and $\Gamma > 0$, there exists $f \in C_c^\infty(\mathbb{R}^2)$ such that*

(i) $f(x_1, x_2)$ is odd in x_1 and x_2 with $f(x_1, x_2) \geq 0$ if $x_1, x_2 \geq 0$,
(ii) $\|f\|_\infty + \|f\|_{H^{1,\alpha}} < \epsilon$,
(iii) the smooth, compactly supported solution ω to (V) generated from initial data $\omega_0 = f$ has corresponding particle trajectory map X^t satisfying

$$\max_{0\leq t\leq\tau} \|\nabla X^t\|_{L^\infty(B_R(0))} > \Gamma,$$

where $R > 0$ satisfies supp$(\omega_0) \subset B_R(0)$.

Proof. Let $\chi \in C_c^\infty(\mathbb{R}^2)$ be a standard radial bump function such that $0 \leq \chi(x) \leq 1$ for all $x \in \mathbb{R}^2$, $\chi(x) = 1$ for all $x \in B_{1/2}(0)$, and $\text{supp}(\chi) \subset B_1(0)$. Define

$$\eta(x_1, x_2) = \sum_{a_1, a_2 \in \{-1,1\}} a_1 a_2 \chi\left(64(x_1 - a_1), 64(x_2 - a_2)\right).$$

For $M \in \mathbb{N}, M \geq 2$ define

$$f_M = \frac{1}{\sqrt{M}\log M} \sum_{k=a_{\alpha,M}}^{b_{\alpha,M}} \frac{1}{k^{2\alpha}} \eta(2^k \cdot),$$

where

$$(a_{\alpha,M}, b_{\alpha,M}) = \begin{cases} \left(\left\lfloor (2^M)^{\frac{1}{1-2\alpha}} \right\rfloor, \left\lceil (2^M + M)^{\frac{1}{1-2\alpha}} \right\rceil\right) & 0 < \alpha < \frac{1}{2}, \\ \left(M, 2^M + M\right) & \alpha = \frac{1}{2}. \end{cases}$$

Note that f_M satisfies condition (i) of Theorem 3.6. To complete the proof, we show that given $\epsilon > 0$, $\tau > 0$, and $\Gamma > 0$, one can choose M sufficiently large so that f_M also satisfies conditions (ii) and (iii).

For the remainder of this section, we will use the notation $A \lesssim_\alpha B$ when $A \lesssim B$ and the implied constant depends on α, but not M.

f_M **satisfies (iii)** We begin by showing f_M satisfies (iii) for sufficiently large M. To do this, we take an approach similar to that used in [BL15] and [Kwo21]; specifically, we bound the Lagrangian deformation in the L^∞-norm locally from below at positive times by an integral functional which approximates the initial velocity gradient at the origin.

Note first that the odd-odd symmetry in f_M gives rise to the same odd-odd symmetry in the solution ω to (V) with initial data f_M. Denoting by $X^t = (X_1^t, X_2^t)$ the particle trajectory map corresponding to this solution, we see that for all $x_1, x_2 \in \mathbb{R}$ and for all $t > 0$,

$$\begin{aligned} X_1^t(0, x_2) = X_2^t(x_1, 0) = 0, \\ X^t(0) = 0, \end{aligned} \tag{3.34}$$

so that the origin and axes are invariant under the flow. It follows that

$$\operatorname{sgn}(x_i) = \operatorname{sgn}(X_i^t(x)), \qquad i = 1, 2, t > 0, x \in \mathbb{R}^2. \tag{3.35}$$

Moreover, if $u = (u_1, u_2)$ denotes the velocity field corresponding to ω, then (3.34) implies that for all $x_1, x_2 \in \mathbb{R}$ and for all $t > 0$,

$$u_2(X_1^t(x_1, 0), 0, t) = 0 = u_1(0, X_2^t(0, x_2), t). \tag{3.36}$$

Therefore, $\nabla u(0, t)$ is diagonal and satisfies

$$\nabla u(0,t) = \begin{bmatrix} \left(-\partial^2_{x_1,x_2}(-\Delta)^{-1}\omega(t)\right)(0) & 0 \\ 0 & \left(\partial^2_{x_1,x_2}(-\Delta)^{-1}\omega(t)\right)(0) \end{bmatrix},$$

and

$$\nabla X^t(0) = \exp\left\{ \int_0^t \begin{bmatrix} \left(-\partial^2_{x_1,x_2}(-\Delta)^{-1}\omega(s)\right)(0) & 0 \\ 0 & \left(\partial^2_{x_1,x_2}(-\Delta)^{-1}\omega(s)\right)(0) \end{bmatrix} ds \right\}.$$

Next, $X^t \circ X^{-t} = Id$ implies $\nabla X^t(X^{-t}(x)) = (\nabla X^{-t})^{-1}(x)$ for all $x \in \mathbb{R}^2$. Thus, using $\det \nabla X^{-t} = 1$, we have

$$\|\nabla X^{-t}(\cdot)\|_{L^\infty(X^t(supp(f_M)))} = \|\nabla X^t(X^{-t}(\cdot))\|_{L^\infty(X^t(supp(f_M)))} = \|\nabla X^t(\cdot)\|_{L^\infty(supp(f_M))}.$$

So, if $x \in supp(f_M) \subset B_R(0)$ with $x_1, x_2 \geq 0$ and $t > 0$, using (3.35) we have

$$\begin{aligned} \frac{1}{\|\nabla X^t\|_{L^\infty(B_R(0))}} X_1^t(x) \leq x_1 \leq X_1^t(x) \|\nabla X^t\|_{L^\infty(B_R(0))}, \\ \frac{1}{\|\nabla X^t\|_{L^\infty(B_R(0))}} X_2^t(x) \leq x_2 \leq X_2^t(x) \|\nabla X^t\|_{L^\infty(B_R(0))}. \end{aligned} \tag{3.37}$$

Now we can estimate $|\nabla u(0, t)|$ from below by

$$\begin{aligned} |\nabla u(0,t)|_\infty &= \frac{1}{2\pi} \int_{\mathbb{R}^2} \omega(x,t) \partial_{x_1} \frac{-x_2}{|x|^2}\, dx = \frac{1}{\pi} \int_{\mathbb{R}^2} \omega(x,t) \frac{x_1 x_2}{|x|^4}\, dx \\ &= \frac{4}{\pi} \int_{x_1,x_2>0} f_M(x) \frac{X_1^t(x) X_2^t(x)}{|X^t(x)|^4}\, dx \\ &\geq \frac{4}{\pi} \|\nabla X^t\|^{-4}_{L^\infty(B_R(0))} \int_{x_1,x_2>0} f_M(x) \frac{x_1 x_2}{|x|^4}\, dx, \end{aligned} \tag{3.38}$$

where we used (3.37) to obtain

$$\frac{X_1^t(x) X_2^t(x)}{|X^t(x)|^4} = \frac{|X^t(x)|^{-2}}{\frac{X_1^t(x)}{X_2^t(x)} + \frac{X_2^t(x)}{X_1^t(x)}} \geq \frac{|X^t(x)|^{-2}}{\left(\frac{x_1}{x_2} + \frac{x_2}{x_1}\right) \|\nabla X^t\|^2_{L^\infty(B_R(0))}} \geq \|\nabla X^t\|^{-4}_{L^\infty(B_R(0))} \frac{x_1 x_2}{|x|^4},$$

for $x = (x_1, x_2) \in supp(f_M)$ with $x_1, x_2 > 0$ and $t \geq 0$. Denoting the integral

$$I_M := \int_{x_1,x_2>0} f_M(x) \frac{x_1 x_2}{|x|^4}\, dx,$$

we have by (3.38)

$$\|\nabla X^t\|_{L^\infty(B_R(0))} \geq |\nabla X^t(0)| \geq \exp\left\{ \frac{4}{\pi} I_M \int_0^t \|\nabla X^s\|^{-4}_{L^\infty(B_R(0))}\, ds \right\}.$$

Raising both sides to the fourth power and rewriting gives

$$\begin{aligned}&\frac{d}{dt}\exp\left\{\frac{16}{\pi}I_M\int_0^t\|\nabla X^s\|_{L^\infty(B_R(0))}^{-4}\,ds\right\}\\&=\frac{16}{\pi}I_M\|\nabla X^t\|_{L^\infty(B_R(0))}^{-4}\exp\left\{\frac{16}{\pi}I_M\int_0^t\|\nabla X^s\|_{L^\infty(B_R(0))}^{-4}\,ds\right\}\leq\frac{16}{\pi}I_M.\end{aligned}$$

Thus,

$$\exp\left\{\frac{16}{\pi}I_M\int_0^t\|\nabla X^s\|_{L^\infty(B_R(0))}^{-4}\,ds\right\}\leq 1+\frac{16}{\pi}I_Mt,$$

which is equivalent to

$$\int_0^t\|\nabla X^s\|_{L^\infty(B_R(0))}^{-4}\,ds\leq\frac{\pi\log\left(1+\frac{16}{\pi}I_Mt\right)}{16I_M\log e}.$$

This implies the existence of some constant $C>0$ independent of M such that

$$\max_{0\leq s\leq t}\|\nabla X^s\|_{L^\infty(B_R(0))}^4\geq\frac{CI_Mt}{\log(1+CI_Mt)},\tag{3.39}$$

for any $t>0$.

We next estimate I_M. Let

$$I:=\int_{x_1,x_2>0}\eta(x)\frac{x_1x_2}{|x|^4}\,dx>0.$$

Then, by a change of variables,

$$\begin{aligned}I_M&=\int_{x_1,x_2>0}f_M\frac{x_1x_2}{|x|^4}\,dx=\frac{1}{\sqrt{M}\log M}\sum_{k=a_{\alpha,M}}^{b_{\alpha,M}}\frac{1}{k^{2\alpha}}\int_{x_1,x_2>0}\eta(2^kx)\frac{x_1x_2}{|x|^4}\,dx\\&=\frac{1}{\sqrt{M}\log M}\sum_{k=a_{\alpha,M}}^{b_{\alpha,M}}\frac{1}{k^{2\alpha}}\int_{x_1,x_2>0}\eta(x)\frac{x_1x_2}{|x|^4}\,dx=\frac{I}{\sqrt{M}\log M}\sum_{k=a_{\alpha,M}}^{b_{\alpha,M}}\frac{1}{k^{2\alpha}}.\end{aligned}\tag{3.40}$$

Since $1/k^{2\alpha}$ is decreasing in k, we can write

$$\begin{aligned}\sum_{k=a_{\alpha,M}}^{b_{\alpha,M}}\frac{1}{k^{2\alpha}}&\geq\int_{a_{\alpha,M}}^{b_{\alpha,M}}\frac{1}{k^{2\alpha}}\,dk\\&=\begin{cases}\frac{1}{1-2\alpha}\left(\left\lceil(2^M+M)^{\frac{1}{1-2\alpha}}\right\rceil^{1-2\alpha}-\left\lfloor 2^{\frac{M}{1-2\alpha}}\right\rfloor^{1-2\alpha}\right)&0<\alpha<\frac{1}{2},\\\log(2^M+M)-\log(M)&\alpha=\frac{1}{2},\end{cases}\\&\geq C_\alpha M\end{aligned}\tag{3.41}$$

for some constant $C_\alpha>0$ depending on α, but not M. Substituting this lower bound into (3.40) gives

$$I_M=\int_{x_1,x_2>0}f_M\frac{x_1x_2}{|x|^4}\,dx\geq C_\alpha\frac{\sqrt{M}}{\log M}.$$

Thus, substituting this bound for I_M into (3.39) with $t = \frac{1}{\log M}$, we obtain

$$\max_{0\le t\le \frac{1}{\log M}} \|\nabla X^t\|_{L^\infty(B_R(0))} \ge \left(\frac{CC_\alpha\sqrt{M}}{\log^2 M \log\left(1+\frac{CC_\alpha\sqrt{M}}{\log^2 M}\right)} \right)^{1/4}$$

$$= \log M \left(\frac{CC_\alpha\sqrt{M}}{\log^6 M \log\left(1+\frac{CC_\alpha\sqrt{M}}{\log^2 M}\right)} \right)^{1/4} \gtrsim_\alpha \log M.$$

Finally choosing M sufficiently large yields (iii).

f_M **satisfies (ii)** We now show that for sufficiently large M, f_M satisfies (ii). First note that since $a_{\alpha,M} \ge M$ for each α, by a change of variables, $\|f_M\|_{L^1}$ satisfies

$$\|f_M\|_{L^1} \le \frac{1}{\sqrt{M}\log M} \sum_{k=a_{\alpha,M}}^{b_{\alpha,M}} \frac{1}{k^{2\alpha}} 2^{-2k}\|\eta\|_{L^1} \le \frac{2^{-M}\|\eta\|_{L^1}}{\sqrt{M}\log M} \sum_{k=a_{\alpha,M}}^{b_{\alpha,M}} 2^{-(k-M)} \lesssim 2^{-M}. \tag{3.42}$$

Moreover, since $\eta(2^k x)\eta(2^j x) = 0$ whenever $j \ne k$,

$$\|f_M\|_\infty \le \frac{1}{\sqrt{M}\log M}, \tag{3.43}$$

so that $\|f_M\|_\infty$ can be made arbitrarily small for sufficiently large choice of M.

To complete the proof that f_M satisfies (ii), it remains to estimate $\|f_M\|_{H^{1,\alpha}}$. We first use interpolation and the above estimates to write

$$\|f_M\|_{L^2} \le \|f_M\|_\infty^{1/2}\|f_M\|_{L^1}^{1/2} \lesssim 2^{-M/2}. \tag{3.44}$$

We once again use that $\eta(2^k x)\eta(2^j x) = 0$ whenever $j \ne k$, and that

$$\sum_{k=a_{\alpha,M}}^{b_{\alpha,M}} \frac{1}{k^{4\alpha}} < \sum_{k=a_{\alpha,M}}^{b_{\alpha,M}} \frac{1}{k^{2\alpha}} < \int_{a_{\alpha,M}-1}^{b_{\alpha,M}} \frac{1}{k^{2\alpha}}\,dk \lesssim_\alpha M, \tag{3.45}$$

where the integral is computed in (3.41), to write

$$\|\nabla f_M\|_{L^2} = \frac{1}{\sqrt{M}\log M} \left(\sum_{k=a_{\alpha,M}}^{b_{\alpha,M}} \frac{1}{k^{4\alpha}} \|\nabla\eta\|_{L^2}^2 \right)^{\frac12} \lesssim_\alpha \frac{1}{\log M}. \tag{3.46}$$

For the $H^{1,\alpha}$-seminorm of f_M, we have by (2.4),

$$M\log^2 M |f_M|_{H^{1,\alpha}}^2 \lesssim_\alpha \int_{\mathbb{R}^2}\int_{|x-y|\le\frac12} \frac{\left|\sum_{k=a_{\alpha,M}}^{b_{\alpha,M}} \frac{2^k}{k^{2\alpha}} \left(\nabla\eta(2^k x) - \nabla\eta(2^k y)\right)\right|^2}{|x-y|^2 \log^{1-2\alpha}(|x-y|^{-1})}\,dy\,dx$$

$$\lesssim_\alpha \sum_{k=a_{\alpha,M}}^{b_{\alpha,M}} (L_k + C_k), \tag{3.47}$$

where

$$L_k := \int_{\mathbb{R}^2} \int_{|x-y|\le \frac{1}{2}} \frac{\frac{2^{2k}}{k^{4\alpha}} \left|\nabla\eta(2^k x) - \nabla\eta(2^k y)\right|^2}{|x-y|^2 \log^{1-2\alpha}(|x-y|^{-1})}\, dy\, dx$$

and

$$C_k := 2 \sum_{j=a_{\alpha,M}}^{k} \frac{2^{j+k}}{(kj)^{2\alpha}} \int_{\mathbb{R}^2} \int_{|x-y|\le \frac{1}{2}} \frac{\left|\nabla\eta(2^k x) - \nabla\eta(2^k y)\right| \left|\nabla\eta(2^j x) - \nabla\eta(2^j y)\right|}{|x-y|^2 \log^{1-2\alpha}(|x-y|^{-1})}\, dy\, dx. \tag{3.48}$$

To estimate L_k, we utilize the Fourier definition of the $H^{0,\alpha}$-seminorm and apply a change of variables to write

$$\begin{aligned}
L_k &\lesssim_\alpha \frac{2^{2k}}{k^{4\alpha}} |\nabla\eta(2^k \cdot)|^2_{H^{0,\alpha}} \simeq_\alpha \frac{2^{2k}}{k^{4\alpha}} \int_{\mathbb{R}^2} \log^{2\alpha}(|\xi|+1) 2^{-4k} |\widehat{\nabla\eta}(2^{-k}\xi)|^2\, d\xi \\
&\simeq_\alpha \frac{1}{k^{4\alpha}} \int_{\mathbb{R}^2} \log^{2\alpha}(2^k|\xi|+1) |\widehat{\nabla\eta}(\xi)|^2\, d\xi \\
&\lesssim_\alpha \frac{1}{k^{4\alpha}} \left(\int_{\mathbb{R}^2} \left(\log^{2\alpha}(2^k) + \log^{2\alpha}(|\xi|+1)\right) |\widehat{\nabla\eta}(\xi)|^2\, d\xi \right) \\
&\lesssim_\alpha \frac{1}{k^{2\alpha}} \left(\|\nabla\eta\|^2_{L^2} + |\nabla\eta|^2_{H^{0,\alpha}} \right).
\end{aligned}$$

Thus, an application of (3.45) gives

$$\sum_{k=a_{\alpha,M}}^{b_{\alpha,M}} L_k \lesssim_\alpha M \|\nabla\eta\|^2_{H^{0,\alpha}}. \tag{3.49}$$

To estimate the cross terms C_k, we utilize the the support of $\nabla\eta$ to manipulate the integrand. First note that $j<k$ implies $\nabla\eta(2^k x)\nabla\eta(2^j x) = 0$ for all $x \in \mathbb{R}^2$. Also, the support of $\nabla\eta$ ensures that $\nabla\eta(2^k x)\nabla\eta(2^j y) \neq 0$ only if

$$\begin{aligned}
|x| &< 2^{-k}(\sqrt{2} + 2^{-6}), \\
|y| &> 2^{-j}(\sqrt{2} - 2^{-6}).
\end{aligned}$$

Thus by the triangle inequality,

$$\begin{aligned}
|2^{-j} y - 2^{-k} x| &\ge 2^{-2j}(\sqrt{2} - 2^{-6}) - 2^{-2k}(\sqrt{2} + 2^{-6}) \\
&> 2^{-2j+1/4} - 2^{-2k+3/4} = 2^{-2j-1/2}(2^{3/4} - 2^{-2(k-j)+5/4}) \ge 2^{-2j-1/2},
\end{aligned}$$

where we used that $k-j \ge 1$. Finally, by the same reasoning, if $j<k$ and $\nabla\eta(2^j x)\nabla(2^k y) \neq 0$, interchanging the roles of x and y gives $|2^{-j}x - 2^{-k}y| \ge 2^{-2j-1/2}$. Thus, for $j<k$, a change of variables in x and y followed by a change of variables in $|x-y|$ gives

$$\begin{aligned}
&\int_{\mathbb{R}^2} \int_{|x-y|\le \frac{1}{2}} \frac{\left|\nabla\eta(2^k x) - \nabla\eta(2^k y)\right| \left|\nabla\eta(2^j x) - \nabla\eta(2^j y)\right|}{|x-y|^2 \log^{1-2\alpha}(|x-y|^{-1})}\, dy\, dx \\
&\le \int_{\mathbb{R}^2} \int_{|x-y|\le \frac{1}{2}} \frac{\left|\nabla\eta(2^k x)\nabla\eta(2^j y)\right|}{|x-y|^2 \log^{1-2\alpha}(|x-y|^{-1})}\, dy\, dx + \int_{\mathbb{R}^2} \int_{|x-y|\le \frac{1}{2}} \frac{\left|\nabla\eta(2^j x)\nabla\eta(2^k y)\right|}{|x-y|^2 \log^{1-2\alpha}(|x-y|^{-1})}\, dy\, dx
\end{aligned}$$

$$\le 2^{-2k-2j+1}\|\nabla\eta\|_\infty^2 \int_{\mathrm{supp}(\eta)} \left(\int_{2^{-2j-1/2}}^{\frac{1}{2}} \frac{2\pi}{h\log^{1-2\alpha}(h^{-1})} dh \right) dx$$
$$= 2^{-8-2k-2j}\pi^2\|\nabla\eta\|_\infty^2 \left(\left(2j+\frac{1}{2}\right)^{2\alpha} - 1 \right) \lesssim_\alpha 2^{-2k-2j} j^{2\alpha}\|\nabla\eta\|_\infty^2.$$

Substituting the above estimate into (3.48) and summing over k, we conclude that

$$\begin{aligned} \sum_{k=a_{\alpha,M}}^{b_{\alpha,M}} C_k &\lesssim_\alpha \|\nabla\eta\|_\infty^2 \sum_{k=a_{\alpha,M}}^{b_{\alpha,M}} \sum_{j=a_{\alpha,M}}^{k} \frac{1}{2^{k+j}k^{2\alpha}} \\ &\lesssim_\alpha \|\nabla\eta\|_\infty^2 \sum_{k=a_{\alpha,M}}^{b_{\alpha,M}} \sum_{j=a_{\alpha,M}}^{k} \frac{1}{2^{k+j}} \lesssim_\alpha 2^{-2a_{\alpha,M}}\|\nabla\eta\|_\infty^2 \lesssim_\alpha 4^{-M}\|\nabla\eta\|_\infty^2, \end{aligned} \tag{3.50}$$

where we again used that $a_{\alpha,M} \ge M$ for all α to obtain the last inequality. Combining (3.47), (3.49), and (3.50) gives

$$|\nabla f_M|_{H^{0,\alpha}} \lesssim_\alpha \frac{\|\eta\|_{H^{1,\alpha}} + \|\nabla\eta\|_\infty}{\log M}. \tag{3.51}$$

Finally, (3.44), (3.46), and (3.51) together imply that $\|f_M\|_{H^{1,\alpha}}$ can be made small for sufficiently large choice of M. This completes the proof of Theorem 3.6. $\square$

As a corollary of Theorems 3.1 and 3.6, we have the existence of a solution to (V) experiencing arbitrarily large growth in $H^{1,\alpha}$-norm.

Corollary 3.7. *Let $\varepsilon > 0$, $A > 0$, and $\tau > 0$ be arbitrary. Then there exists a function $\omega_0^{\varepsilon,\tau} \in C_c^\infty(\mathbb{R}^2)$ satisfying:*

(i) $\|\omega_0^{\varepsilon,\tau}\|_\infty + \|\omega_0^{\varepsilon,\tau}\|_{H^{1,\alpha}} < \varepsilon$,
(ii) supp$(\omega_0^{\varepsilon,\tau}) \subset B_1(0)$,
(iii) The solution $\omega^{\varepsilon,\tau}$ of (V) with initial data $\omega_0^{\varepsilon,\tau}$ satisfies

$$\max_{0\le t\le\tau} \|\omega^{\varepsilon,\tau}(t)\|_{H^{1,\alpha}} > A.$$

Proof. Let c_α be as in Theorem 3.1. It follows from Theorem 3.6 with $\Gamma = \max\{2^{13}, c_\alpha^2/\varepsilon^2, A^3\}$ that there exists $f = \omega_0$ satisfying conditions (i) and (ii) of Theorem 3.1. Thus, by Theorem 3.1, there exists β satisfying (3.2), and, in particular,

$$|\beta|_{H^{1,\alpha}} \le \frac{c_\alpha}{\sqrt{L}} \le \frac{c_\alpha}{\sqrt{\Gamma}} \le \varepsilon.$$

Set $\omega_0^{\varepsilon,\tau} = f + \beta$. Then $\omega_0^{\varepsilon,\tau}$ satisfies conditions (i) and (ii) of Corollary 3.7; specifically,

$$\|\omega_0^{\varepsilon,\tau}\|_\infty + \|\omega_0^{\varepsilon,\tau}\|_{H^{1,\alpha}} < 4\varepsilon$$

and supp$(\omega_0^{\varepsilon,\tau}) \subset B_1(0)$. Moreover, it follows from Theorem 3.1 and the definition of Γ that $\omega_0^{\varepsilon,\tau}$ satisfies condition (iii). $\square$

4. Interaction of Vorticity Patches and Proof of Theorem 1.1

In this section, we prove Theorem 1.1. Our first step is to use Corollary 3.7 to construct an infinite collection of vorticity patches, each with small $H^{1,\alpha}$-norm. We construct this collection in such a way that the infinite sum of these patches yields a non-compactly supported initial vorticity $\omega_0 \in H^{1,\alpha}(\mathbb{R}^2)$ whose corresponding solution exhibits blow-up in $H^{1,\alpha}$-norm at positive times. In order to control the interaction of these patches, we rely on the following proposition.

Proposition 4.1 ([BL15], Proposition 5.3)**.** *Let $\{\omega_0^j\}_{j=1}^{\infty}$ satisfy $\omega_0^j \in C_c^\infty(\mathbb{R}^2)$ and supp$(\omega_0^j) \subset B_1(0)$ for each $j \in \mathbb{N}$, and let $\{\omega_0^j\}_{j=1}^{\infty}$ be such that*

$$\sum_{j=1}^{\infty} \left(\|\omega_0^j\|_{H^1}^2 + \|\omega_0^j\|_{L^1} \right) + \sup_{j \geq 1} \|\omega_0^j\|_\infty < 1. \tag{4.1}$$

Then there exists a sequence of points $\{x_j\}$ in $\mathbb{R}^2$ such that the following hold:

(1) For $k \neq j$,

$$B_{64}(x_j) \cap B_{64}(x_k) = \emptyset.$$

(2) If we define

$$\omega_0(x) = \sum_{j=1}^{\infty} \omega_0^j(x - x_j),$$

then $\omega_0 \in L^1 \cap C^\infty(\mathbb{R}^2)$.
(3) If ω is the solution to

$$\begin{cases} \partial_t \omega + \nabla^\perp (-\Delta)^{-1} \omega \cdot \nabla \omega = 0 & \text{on } \mathbb{R}^2 \times (0, \infty), \\ \omega|_{t=0} = \omega_0 & \text{on } \mathbb{R}^2, \end{cases}$$

then for all $t \in [0,1]$, $\omega(\cdot, t) \in L^1 \cap C^\infty(\mathbb{R}^2)$ and

$$\text{supp}\,(\omega(\cdot, t)) \subset \bigcup_{j=1}^{\infty} B_4(x_j).$$

(4) If we denote by ω_j the solution to (V) with initial data $\omega_0^j(\cdot - x_j)$, i.e.

$$\begin{cases} \partial_t \omega_j + \nabla^\perp (-\Delta)^{-1} \omega_j \cdot \nabla \omega_j = 0 & \text{on } \mathbb{R}^2 \times (0, \infty), \\ \omega_j(x, 0) = \omega_0^j(x - x_j) & \text{on } \mathbb{R}^2, \end{cases}$$

then

$$\lim_{j \to \infty} \max_{0 \leq t \leq 1} \|\omega_j(\cdot, t) - \omega(\cdot, t)\|_{H^2(B_4(x_j))} = 0.$$

Proof of Theorem 1.1. We first apply Corollary 3.7 to obtain a sequence of functions $\{\omega_0^j\}_{j=1}^{\infty}$ such that for every $j \in \mathbb{N}$, $\omega_0^j \in C_c^\infty(B_1(0))$ and the following hold:

$$\begin{aligned} &\|\omega_0^j\|_{L^\infty} + \|\omega_0^j\|_{L^1} + \|\omega_0^j\|_{H^{1,\alpha}}^2 < 2^{-(j+1)}, \\ &\max_{0 \leq t \leq \frac{1}{j}} \|\omega_j\|_{H^{1,\alpha}} > j, \end{aligned} \tag{4.2}$$

where ω_j solves (V) with initial data ω_0^j. Thus, $\{\omega_0^j\}$ satisfy the hypotheses of Proposition 4.1, which gives us a sequence $\{x_j\}_{j=1}^\infty$ in $\mathbb{R}^2$ such that the solution ω to

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0 & \text{on } \mathbb{R}^2 \times (0,\infty), \\ u = \nabla^\perp (-\Delta)^{-1}\omega & \text{on } \mathbb{R}^2 \times [0,\infty), \\ \omega|_{t=0} = \sum_{j=1}^\infty \omega_0^j(\cdot - x_j) & \text{on } \mathbb{R}^2, \end{cases}$$

satisfies

$$\lim_{j\to\infty} \max_{0\le t\le \frac{1}{j}} \|\omega(\cdot,t) - \omega_j(\cdot,t)\|_{H^2(B_4(x_j))} = 0.$$

Because $\|f\|_{H^{1,\alpha}} \lesssim_\alpha \|f\|_{H^2}$ for all $f \in H^2(\mathbb{R}^2)$, it follows from (4.2) that

$$\limsup_{t\to 0^+} \|\omega(\cdot,t)\|_{H^{1,\alpha}} = \infty.$$

Clearly, this gives $\limsup_{t\to0^+} \|u(\cdot,t)\|_{H^{2,\alpha}} = \infty$, which concludes the proof. $\square$

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