

Consumption-Investment Problem in Rank-Based Models

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Abstract

We study a consumption-investment problem in a multi-asset market where the returns follow a generic rank-based model. Our main result derives an HJB equation with Neumann boundary conditions for the value function and proves a corresponding verification theorem. The control problem is nonstandard due to the discontinuous nature of the coefficients in rank-based models, requiring a bespoke approach of independent mathematical interest. The special case of first-order models, prescribing constant drift and diffusion coefficients for the ranked returns, admits explicit solutions when the investor is either (a) unconstrained, (b) abides by open market constraints or (c) is fully invested in the market. The explicit optimal strategies in all cases are related to the celebrated solution to Merton's problem, despite the intractability of constraint (b) in that setting.

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1 Introduction

We study a consumption-investment problem in a large dimensional equity market with a risk-free asset and d equities $X = (X_1, \dots, X_d)$ available for investment. The problem is characterized by the value function

$$v(t, x, w) = \sup_{(\pi, c) \in \mathcal{A}_T^o(t, x, w)} \mathbb{E}_{t, x} \left[\int_t^T e^{-\beta s} U_1(c(s)) ds + U_2(V_{\pi, c}^w(T)) \right], \quad (1.1)$$

which seeks to maximize an investor's utility from consumption and terminal wealth (all of the ingredients in (1.1) are precisely defined in Section 2). Problems of this type have a rich history going back to Merton [7] who solved the problem explicitly with power utility and when the asset process X is a geometric Brownian motion (GBM).

Here, we revisit this problem when the stock returns follow a *rank-based* model. These are reduced-form models where each asset's returns at a given time depend on the rank in the market the

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company occupies at that time (see (2.1) below). Rank-based models have been shown to reproduce certain features of large equity markets that standard models are unable to capture, such as the empirical stability of capital distribution in the market over long periods of time [2, 3]. Moreover, the rank-based drift parameters can be efficiently estimated through a, so-called, collision estimator (see [2, Chapter 5.4]) providing a path to drift parameter calibration, which is a notoriously difficult problem for standard name-based models. On the other hand, rank-based models inherently lead to discontinuous coefficients for the returns process leading to difficulties applying the standard stochastic optimal control machinery to characterize the value function (1.1).

Our main result overcomes this difficulty by deriving the Hamilton–Jacobi–Bellman (HJB) equation that the value function satisfies, (3.3), together with the appropriate Neumann boundary conditions, (3.5), and proving a corresponding verification result, Theorem 3.2. When X follows the *first-order model* of [2, Chapter 5.5], prescribing constant drift and diffusion coefficients for the ranked returns, it turns out that an explicit solution exists. The optimal rule involves (i) the celebrated Merton fraction

$$\tilde{\pi}^* = \tilde{a}^{-1}(\tilde{\mu} - r\mathbf{1}_d),$$

where $\tilde{a}$ is the covariance matrix of the ranked returns, $\tilde{\mu}$ is the drift of the ranked returns and r is the risk-free rate and (ii) the same optimal feedback form consumption rule c^* as in Merton’s problem. The key difference is that the fraction $\tilde{\pi}^*$ specifies the optimal proportion of wealth the investor should hold in the assets according to the rank they occupy, rather than the company name.

Additionally, in Section 4.2, we study *open market* constraints, which have recently gained attention in the literature [4, 5]. These constraints only allow investment in certain ranks and, as such, can serve as a proxy for investors who restrict their investment universe to companies of a certain size (e.g. large cap equities, mid-cap equities, etc.). Remarkably, in the case of first-order models we obtain entirely explicit optimal solutions, whereas the corresponding open market constrained problem in the standard GBM setting is intractable. Section 4.3 derives further explicit optimal strategies when the investor is additionally required to be fully invested in the market.

On the mathematical side, we characterize the value function for this problem by relating it to an HJB equation with Neumann boundary conditions. Equations of this type on general domains were recently studied in [8] and shown to characterize the value function with a *reflected* diffusion, corresponding here to the ranked capitalizations $X_{(\cdot)}$, as one of the state variables. In contrast, the named capitalizations X are state variables in (1.1). As such, the class of admissible controls in [8] are progressively measurable with respect to the filtration generated by $X_{(\cdot)}$, while in our problem the filtration is larger incorporating information about X . Our main result, Theorem 3.2, establishes that the value functions for these two problems coincide despite the fact that the optimal allocations π^* for the problem (1.1) are not adapted to the filtration generated by $X_{(\cdot)}$.

The paper is organized as follows. Section 2 introduces the model and the consumption–investment problem. In Section 3 we study the associated HJB equation. It is heuristically derived in Section 3.1 and the verification theorem is established in Section 3.2. Finally in Section 4 we consider first-order models with power utility preferences, explicitly solving the unconstrained problem in Section 4.1, the open market constrained problem in 4.2 and the fully invested constrained problem in 4.3.

Notation For $d \geq 2$ and a vector $x \in (0, \infty)^d$, we write $x_{(\cdot)}$ for its decreasing order statistics, which is the permutation of x satisfying $x_{(1)} \geq x_{(2)} \geq \dots \geq x_{(d)}$. The set of all ordered vectors is

defined as

$$(0, \infty)_{\geq}^d = \{y \in (0, \infty)^d : y_1 \geq y_2 \geq \dots \geq y_d\}.$$

The subset of $(0, \infty)_{\geq}^d$ consisting of all points where no coordinates are equal will be denoted by $(0, \infty)_{>}^d$ and its complement by $(0, \infty)_{\leq}^d = (0, \infty)_{\geq}^d \setminus (0, \infty)_{>}^d$.

For every $i \in \{1, \dots, d\}$, we define the rank identifying function $\mathcal{R}_i : (0, \infty)^d \rightarrow \{1, \dots, d\}$ via $\mathcal{R}_i(x) = k$ if $x_i = x_{(k)}$. To ensure this is well-defined we break ties using lexicographical ordering. That is, $\mathcal{R}_i(x) = \min\{k \in \{1, \dots, d\} : x_i = x_{(k)}\}$ though the precise tie-breaking rule does not affect the results in this paper.

2 The consumption-investment problem

2.1 Financial market

We consider a financial market consisting of a risk-free asset $dX_0(t) = rX_0(t)dt$ with risk-free rate $r \in \mathbb{R}$ and d risky assets with market capitalizations $X = (X_1, \dots, X_d)$. We take a rank-based model for the risky assets,

$$\frac{dX_i(s)}{X_i(s)} = \mu_i(s, X(s))ds + \sum_{j=1}^d \sigma_{ij}(s, X(s))dW_j(s), \quad X_i(t) = x_i; \quad i = 1, \dots, d, \quad s \geq t, \quad (2.1)$$

for any initial time $t \geq 0$ and initial value $x \in (0, \infty)^d$, where

$$\mu_i(t, x) = \sum_{k=1}^d \tilde{\mu}_k(t, x_{(\cdot)})1_{\{\mathcal{R}_i(x)=k\}}, \quad \sigma_{ij}(t, x) = \sum_{k, \ell=1}^d \tilde{\sigma}_{k\ell}(t, x_{(\cdot)})1_{\{\mathcal{R}_i(x)=k, \mathcal{R}_j(x)=\ell\}}$$

for some functions $\tilde{\mu}_k, \tilde{\sigma}_{k\ell} : [0, \infty) \times (0, \infty)^d \rightarrow \mathbb{R}$ and where W is a standard Brownian motion. In this model an asset's dynamics only depend on which rank that asset occupies at any time. We impose the following assumption on the inputs

Assumption 2.1. $\tilde{\mu}$ and $\tilde{\sigma}$ are bounded and $\tilde{\sigma}$ is uniformly elliptic. They additionally satisfy the following uniform Lipschitz condition,

$$\|y_1 \circ \tilde{\mu}(t, y_1) - y_2 \circ \tilde{\mu}(t, y_2)\| + \|\text{diag}(y_1)\tilde{\sigma}(t, y_1) - \text{diag}(y_2)\tilde{\sigma}(t, y_2)\| \leq L\|y_1 - y_2\|, \quad (2.2)$$

for some $L > 0$, all $y_1, y_2 \in (0, \infty)_{\geq}^d$ and $t \geq 0$. Here $\circ$ denotes elementwise product and $\text{diag}(y)$ is the $d \times d$ matrix with y on the diagonal and zeros elsewhere.

Despite the high regularity of $\tilde{\mu}$ and $\tilde{\sigma}$, the coefficients μ and σ will typically be discontinuous and fail to be weakly differentiable. As such, we cannot expect strong solutions or uniqueness in law to (2.1), but we will nevertheless be able to work with weak solutions. In contrast, the coefficients of the ranked capitalizations are quite regular. Indeed, the result [1, Theorem 2.5]¹ ensures that the ranked process $Y = X_{(\cdot)}$ satisfies

$$dY_k(s) = Y_k(s) \left(\tilde{\mu}_k(s, Y(s))ds + \sum_{\ell=1}^d \tilde{\sigma}_{k\ell}(s, Y(s))dB_{\ell}(s) \right) + d\Phi_k(s), \quad Y_k(t) = y_k, \quad (2.3)$$

¹Note that the semimartingale decomposition in their Theorem 2.5 is applicable to the process $X_{(\cdot)}$ here due to the remark preceding the statement of the theorem.

for $k = 1, \dots, d$, $s \geq t$, where $y_k = x_{(k)}$, $B_\ell(s) = \int_t^s \sum_{j=1}^d 1_{\{\mathcal{R}_j(X(u))=\ell\}} dW_j(u)$ and Φ is the reflection term given by

$$\Phi_k(s) = \frac{1}{2} \sum_{\ell=k+1}^d \int_t^s \frac{1}{N_k(u)} dL_{Y_k - Y_\ell}(u) - \frac{1}{2} \sum_{\ell=1}^{k-1} \int_t^s \frac{1}{N_k(u)} dL_{Y_\ell - Y_k}(u); \quad k = 1, \dots, d, \quad (2.4)$$

with $N_k(u) = |\{j \in \{1, \dots, d\} : X_j(u) = X_{(k)}(u)\}|$ and $L_{Y_k - Y_\ell}$ being the semimartingale local time at zero of $Y_k - Y_\ell$. In particular, B is a standard Brownian motion so (2.3) is a reflected stochastic differential equation (RSDE) with normal reflection on the domain $(0, \infty)_{\geq}^d$.

Proposition 2.2. *Let Assumption 2.1 hold. Then*

- (i) *there exists a weak solution to (2.1) for every initial $t \geq 0$, $x \in (0, \infty)^d$,*
- (ii) *there exists a pathwise unique strong solution (Y, Φ) to (2.3) for every initial $t \geq 0$, $y \in (0, \infty)_{\geq}^d$. The solution is a strong Markov process and satisfies the moment bound*

$$\tilde{\mathbb{E}}_{t,y} \left[\sup_{t \leq s \leq T} \|Y(s)\|^p \right] \leq C_{T,p} (1 + \|y\|^p) \quad (2.5)$$

for any $T \geq t$, $p \geq 1$ and some constant $C_{T,p} > 0$. Here $\tilde{\mathbb{E}}_{t,y}[\cdot]$ denote expectation with respect to the law of Y when the initial value is $Y(t) = y$.

Proof. For item (i) it is equivalent to establish weak existence on $\mathbb{R}^d$ for $Z = (\log X_1, \dots, \log X_d)$, which satisfies the SDE

$$dZ(s) = ((\mu \circ \exp)(s, Z(s)) + \frac{1}{2}(\text{diag}(a \circ \exp)(s, Z(s)))ds + (\sigma \circ \exp)(s, Z(s))dW(s),$$

where $a = \sigma\sigma^\top$ and $\exp(z) = (\exp(z_1), \dots, \exp(z_d))$. By Assumption 2.1, the drift and volatility coefficients of Z are measurable and bounded and the diffusion matrix is uniformly elliptic. Hence, [6, Theorem 2.6.1] guarantees the existence of a weak solution.

The strong existence and pathwise uniqueness of the RSDE follows from Assumption 2.1 courtesy of [10, Theorem 4.1]. Both the strong Markov property and the moment bound (2.5) are standard results with the former being a consequence of uniqueness in law (see [9, Section 6.2]) and the latter following from Gronwall and Burkholder–Davis–Gundy inequalities (see e.g. the proof of [8, Proposition 2.1]). $\square$

In view of Proposition 2.2, given an initial time $t \geq 0$ and initial value $X(t) = x$ we can work on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_s)_{s \geq t}, \mathbb{P}_{t,x})$ satisfying the usual conditions and supporting a process X satisfying (2.1). Since $X_{(\cdot)}$ solves (2.3) with $y = x_{(\cdot)}$, we have courtesy of the strong existence and pathwise uniqueness for the RSDE that the moment estimate (2.5) holds for $X_{(\cdot)}$. Moreover, $X_{(\cdot)}$ is a strong Markov process and the law of the ranked capitalizations can be represented via the pushforward measure $\tilde{\mathbb{P}}_{t,y} = x_{(\cdot)} \# \mathbb{P}_{t,x}$. In particular, any distinct solutions X to (2.1), even if on different probability spaces, give rise to the same law for the ranked capitalization process $X_{(\cdot)}$.

2.2 The stochastic optimal control problem

When starting with initial wealth $w > 0$, using the trading strategy $\pi(s) = (\pi_1(s), \dots, \pi_d(s))$ and consuming at rate $c(s) \geq 0$ for $s \geq t$ the investors wealth process $V_{\pi,c}^w$ evolves according to

$$\begin{aligned} dV_{\pi,c}^w(s) &= \left(V_{\pi,c}^w(s) \left(\sum_{i=1}^d \pi_i(s) \frac{dX_i(s)}{X_i(s)} + \pi_0(s) \frac{dX_0(s)}{X_0(s)} \right) - c(s) \right) ds \\ &= \left(V_{\pi,c}^w(s) (\pi(s)^\top (\mu(s, X(s)) - r\mathbf{1}_d) + r) - c(s) \right) ds + \pi(s)^\top \sigma(s, X(s)) dW(s), \end{aligned}$$

where $\pi_0(s) = 1 - \sum_{i=1}^d \pi_i(s)$ is the proportion of wealth invested in the risk-free asset and $\mathbf{1}_d$ denotes the d -dimensional vector of all ones. Given a time horizon $T \in (0, \infty)$ we restrict our attention to strictly positive wealth processes

$$\mathcal{A}_T(t, x, w) = \left\{ (\pi(s), c(s))_{s \in [t, T]} : \begin{array}{l} (\pi, c) \text{ are progressively measurable, } c \geq 0 \\ \text{and } V_{\pi,c}^w > 0, \mathbb{P}_{t,x}\text{-a.s. on } [t, T] \end{array} \right\}.$$

Our goal is to solve the investment-consumption problem characterized by the value function (1.1) given in the introduction, where $\beta > 0$ is the patience parameter, $\mathcal{A}_T^\circ(t, x, w) \subset \mathcal{A}_T(t, x, w)$ is the set of admissible controls specified in Section 2.3 and $U_1, U_2 : (0, \infty) \rightarrow \mathbb{R}$ are utility functions, which we assume are concave and increasing. Utility functions on the real line or an infinite time horizon can be handled similarly, but we do not pursue this here.

Remark 2.3. Due to the potential nonuniqueness in law to (2.1) it is not apriori clear that the value function v depends only on (t, x, w) and not the particular solution to (2.1). Part of our main result, Theorem 3.2, establishes that it is independent of the particular solution.

2.3 Control set

Here we define the set of controls $\mathcal{A}_T^\circ(t, x, w)$ over which we optimize. We assume it is of the form

$$\mathcal{A}_T^\circ(t, x, w) = \{(\pi(s), c(s))_{s \in [t, T]} \in \mathcal{A}_T(t, x, w) : (\pi(s), c(s)) \in A(X(s), V_{\pi,c}^w(s)) \quad \forall s \in [t, T]\} \quad (2.6)$$

for some nonempty measurable subsets $A(x, w) \subset \mathbb{R}^d \times [0, \infty)$, which specify constraints on the controls. Next we define the rank-based constraint sets

$$\tilde{A}(x, w) = \left\{ (\tilde{\pi}, c) \in \mathbb{R}^d \times [0, \infty) : \tilde{\pi}_k = \sum_{i=1}^d \pi_i 1_{\{\mathcal{R}_i(x) = k\}} \quad \forall k \text{ and some } (\pi, c) \in A(x, w) \right\}.$$

If at some point in time the investor's portfolio weights are $\pi \in \mathbb{R}^d$ then $\tilde{\pi}_k$ is the proportion of wealth the investor holds in the asset currently occupying rank k . The main condition we will impose is permutation invariance of the rank-based constraint sets.

Assumption 2.4. $\tilde{A}(x, w) = \tilde{A}(x_{(\cdot)}, w)$ for every $(x, w) \in (0, \infty)^{d+1}$.

We now present a few examples of common constraints that satisfy Assumption 2.4.

Example 2.5. (i) (*Unconstrained*). $A(x, w) = \mathbb{R}^d \times [0, \infty)$,
(ii) (*Long-only*). $A(x, w) = [0, 1]^d \times [0, \infty)$,

- (iii) (*Fully invested*). $A(x, w) = \{\pi \in \mathbb{R}^d : \pi_1 + \dots + \pi_d = 1\} \times [0, \infty)$,
- (iv) (*Open market*). $A(x, w) = \{\pi \in \mathbb{R}^d : \pi_i = 0 \text{ if } \mathcal{R}_i(x) \notin [n, N]\} \times [0, \infty) \text{ for some } 1 \leq n \leq N \leq d$.

That items (i)-(iii) satisfy Assumption 2.4 is clear, while for (iv) we note that

$$\tilde{A} = \{(\tilde{\pi}, c) \in \mathbb{R}^d \times [0, \infty) : \tilde{\pi}_k = 0 \text{ if } k \notin [n, N]\}$$

is state independent and, hence, satisfies Assumption 2.4. An example of a constraint that does not satisfy Assumption 2.4 is an asset based long-only constraint $A(x, w) = [0, \infty) \times \mathbb{R}^{d-1} \times [0, \infty)$, which prohibits short selling in asset one, but allows it in the other assets.

3 Dynamic programming approach

In this section, we will work with the domains

$$D = [0, T) \times (0, \infty)^d \times (0, \infty), \quad D_{\geq} = [0, T) \times (0, \infty)_{\geq}^d \times (0, \infty).$$

We will similarly write $D_{>}$ and $D_{=}$ as in $D_{\geq}$ with $(0, \infty)_{\geq}^d$ replaced by $(0, \infty)_{>}^d$ and $(0, \infty)_{=}^d$ respectively. When the right endpoint $\{T\}$ is included we will append the corresponding set with a bar, such as $\bar{D}$. For a function $v : D \rightarrow \mathbb{R}$, we will write $\partial_t v$ for the derivative in the first argument, ∇v for the $(d+1)$ -dimensional gradient in the last two arguments, $\nabla_x v$ for the gradient in the second argument and $\partial_w v$ for the derivative in the third argument. Similarly $\nabla^2 v$ will denote the $(d+1) \times (d+1)$ Hessian while $\nabla_x^2 v$, $\partial_w \nabla_x v$ and $\partial_{ww} v$ will denote its respective components. For a function $\tilde{v} : D_{\geq} \rightarrow \mathbb{R}$ we will use an analogous convention and refer by $\nabla_y \tilde{v}$ and $\nabla_y^2 \tilde{v}$ to the derivatives in the second variable.

3.1 Heuristic discussion

The HJB associated to the value function (1.1) is

$$\begin{cases} 0 = (\partial_t + L^X)v(t, x, w) + \sup_{(\pi, c) \in A(x, w)} H_{\pi, c}(t, x, w, \nabla v(t, x, w), \nabla^2 v(t, x, w)), & \text{in } D, \\ v(T, x, w) = U_2(w), & \text{in } (0, \infty)^{d+1}, \end{cases} \quad (3.1)$$

where $L^X v = \sum_{i=1}^d x_i \mu_i \partial_{x_k} v + \frac{1}{2} \sum_{i, j=1}^d x_i x_j a_{ij} \partial_{x_i x_j} v$ and the Hamiltonian is given by

$$\begin{aligned} H_{\pi, c}(t, x, w, \xi, M) &= w \xi_{d+1} (r + \pi^\top (\mu(t, x) - r \mathbf{1}_d)) + w \pi^\top a(t, x) M_{d+1, 1:d} \\ &\quad + \frac{w^2}{2} M_{d+1, d+1} \pi^\top a(t, x) \pi - \xi_{d+1} c + e^{-\beta t} U_1(c) \end{aligned}$$

for $(t, x, w, \xi, M) \in D \times \mathbb{R}^{d+1} \times \mathbb{R}^{(d+1) \times (d+1)}$ and where $a(t, x) = \sigma(t, x) \sigma(t, x)^\top$ and $M_{d+1, 1:d}$ denotes the first d components in the $(d+1)$ st row of M . The HJB (3.1) is a second order nonlinear PDE where the coefficients μ and a are, in general, only measurable and bounded. Due to lack of regularity, the standard viscosity solution theory is not directly applicable.

To make progress we make the ansatz

$$v(t, x, w) = \tilde{v}(t, x_\circ, w) \quad (3.2)$$

for some function $\tilde{v} : D_{\geq} \rightarrow \mathbb{R}$. For any k the map $x \mapsto x_{(k)}$ is Lipschitz and differentiable outside the Lebesgue null-set $\{x : x_i = x_j \text{ for some } i \neq j\}$ with derivative given by $1_{\{\mathcal{R}_i(x)=k\}}$. As such, by making this substitution into (3.1), we formally expect $\tilde{v}$ to satisfy

$$\begin{cases} 0 = (\partial_t + L^Y)\tilde{v}(t, y, w) + \sup_{(\tilde{\pi}, c) \in \tilde{A}(y, w)} \tilde{H}_{\tilde{\pi}, c}(t, y, w, \nabla \tilde{v}(t, y, w), \nabla^2 \tilde{v}(t, y, w)), & \text{in } D_{>}, \\ \tilde{v}(T, y, w) = U_2(w), & \text{in } (0, \infty)_{\geq}^d \times (0, \infty), \end{cases} \quad (3.3)$$

where $L^Y \tilde{v} = \sum_{k=1}^d y_k \tilde{\mu}_k \partial_{y_k} \tilde{v} + \frac{1}{2} \sum_{k, \ell=1}^d y_k y_{\ell} \tilde{a}_{k\ell} \partial_{y_k y_{\ell}} \tilde{v}$ and

$$\begin{aligned} \tilde{H}_{\tilde{\pi}, c}(t, y, w, \xi, M) &= w \xi_{d+1} (r + \tilde{\pi}^{\top} (\tilde{\mu}(t, y) - r \mathbf{1}_d)) + w \tilde{\pi}^{\top} \tilde{a}(t, y) M_{d+1, 1:d} \\ &\quad + \frac{w^2}{2} M_{d+1, d+1} \tilde{\pi}^{\top} \tilde{a}(t, y) \tilde{\pi} - \xi_{d+1} c + e^{-\beta t} U_1(c). \end{aligned} \quad (3.4)$$

The diffusion matrix $\tilde{a}$ does not degenerate on the boundary $D_{=}$ so the equation (3.3) needs to be appended with boundary conditions. As $X_{(\cdot)}$ satisfies an RSDE with normal reflection, we will impose Neumann boundary conditions on $\tilde{v}$,

$$\nabla_y \tilde{v}(t, y, w)^{\top} \mathbf{n}(y) = 0, \quad \text{for } (t, y, w) \in D_{=} \quad (3.5)$$

for any normal vector $\mathbf{n}(y)$ at $y \in D_{=}$. The important role of the boundary condition (3.5) will become apparent in the proof of the verification theorem in the next section.

Remark 3.1. For any $y \in (0, \infty)_{\geq}^d$ with exactly two equal components $y_k = y_{k+1}$, the unique (up to multiplicative constant) normal vector is $\mathbf{n}(y) = e_k - e_{k+1}$. Points with three or more indices coinciding (e.g. $y_{k+1} = y_k = y_{k-1}$) admit additional normal vectors. If the process $Y = X_{(\cdot)}$ of (2.4) is such that $L_{Y_k - Y_{\ell}} = 0$ whenever $k - \ell > 1$ then (3.5) only needs to hold for normal vectors of the type $e_k - e_{k+1}$. A sufficient condition for these local times to vanish is if triple collisions of components of X do not occur.

3.2 Verification theorem

HJB equations of the type (3.3) on fairly general domains with Neumann boundary conditions (3.5) have recently been studied in [8]. Under certain assumptions, the authors are able to establish the existence of a unique viscosity solution to (3.3) and (3.5) and characterize it as the value function of a certain stochastic control problem. In our setting this value function is given by

$$\tilde{v}(t, y, w) = \sup_{(\tilde{\pi}, c) \in \tilde{\mathcal{A}}_T^{\circ}(t, y, w)} \tilde{\mathbb{E}}_{t, y} \left[\int_t^T U_1(c(s)) ds + U_2(\tilde{V}_{\tilde{\pi}, c}^w(T)) \right], \quad (3.6)$$

where Y is as in (2.3), the process $\tilde{V}_{\tilde{\pi}, c}^w$ is given by

$$d\tilde{V}_{\tilde{\pi}, c}^w(s) = \left(\tilde{V}_{\tilde{\pi}, c}^w(s) (\tilde{\pi}(s)^{\top} (\tilde{\mu}(s, Y(s)) - r \mathbf{1}_d) + r) - c(s) \right) ds + \tilde{V}_{\tilde{\pi}, c}^w(s) \tilde{\pi}(s)^{\top} \tilde{\sigma}(s, Y(s)) dB(s),$$

for $s \geq t$ with $\tilde{V}_{\tilde{\pi}, c}^w(t) = w$ and $\tilde{\mathcal{A}}_T^{\circ}(t, y, w)$ denotes the set of admissible control processes. This set is akin to (2.6) with $\tilde{A}(y, w)$ in place of $A(x, w)$ except additionally restricted to consisting only of processes $(\tilde{\pi}(s), c(s))_{s \in [t, T]}$ which are progressively measurable with respect to $\mathcal{F}^B$.

It is far from clear if the value function v satisfies the relationship (3.2) with $\tilde{v}$ solving (3.3). Indeed, the representation (3.6) only allows for controls adapted to $\mathcal{F}^B$, while the original filtration $\mathcal{F} = \mathcal{F}^{X,W}$ is larger. Nevertheless, we will show below that the relationship (3.2) holds where $\tilde{v}$ is a solution to (3.3) and (3.5). For clarity of exposition, and since it is sufficient for our main example in Section 4, we will prove a verification theorem in the case of a classical solution to the HJB.

Theorem 3.2 (Verification theorem). *Let Assumptions 2.1 and 2.4 hold. Suppose there exists a nonnegative function $\tilde{v} \in C^{1,2}(D_{\geq}) \cap C(\overline{D}_{\geq})$ satisfying (3.3), (3.5) and the polynomial growth condition*

$$|\tilde{v}(t, y, w)| \leq C(1 + \|y\|^p + |w|^p), \quad \text{for all } (t, y, w) \in \overline{D}_{\geq}, \quad (3.7)$$

for some $C > 0$ and $p \geq 1$. Suppose, further, that there exists a measurable maximizer

$$(\tilde{\pi}^*(t, y, w), \tilde{c}^*(t, y, w)) \in \arg \max_{(\tilde{\pi}, c) \in \tilde{\mathcal{A}}(y, w)} \tilde{H}_{\tilde{\pi}, c}(t, y, w, \nabla \tilde{v}(t, y, w), \nabla^2 \tilde{v}(t, y, w)) \quad (3.8)$$

for every $(t, y, w) \in D_{\geq}$ satisfying the following uniform Lipschitz and linear growth conditions

$$\begin{aligned} \|w_1 \tilde{\pi}^*(t, y, w_1) - w_2 \tilde{\pi}^*(t, y, w_2)\| + |\tilde{c}^*(t, y, w_1) - \tilde{c}^*(t, y, w_2)| &\leq L|w_1 - w_2|, \\ \|w_1 \tilde{\pi}^*(t, y, w_1)\| + |\tilde{c}^*(t, y, w_1)| &\leq C(1 + \|y\| + |w_1|) \end{aligned} \quad (3.9)$$

for some $C, L > 0$ and all $y \in (0, \infty)_{\geq}^d$, $w_1, w_2 \in (0, \infty)$. Then the value function (1.1) is given by $v(t, x, w) = \tilde{v}(t, x_{\square}, w)$ and an optimal strategy (π^*, c^*) is given in feedback form by

$$\begin{aligned} \pi_i^*(t, X(t), V_{\pi^*, c^*}^w(t)) &= \sum_{k=1}^d \tilde{\pi}_k^*(t, X_{\square}(t), V_{\pi^*, c^*}^w(t)) 1_{\{\mathcal{R}_i(X(t))=k\}}, \quad i = 1, \dots, d, \quad t \geq 0, \\ c^*(t, X(t), V_{\pi^*, c^*}^w(t)) &= \tilde{c}^*(t, X_{\square}(t), V_{\pi^*, c^*}^w(t)), \quad t \geq 0. \end{aligned} \quad (3.10)$$

Remark 3.3. $\tilde{\pi}^*, \tilde{c}^*$ and c^* are $\mathcal{F}^B$ measurable, but the optimal control π^* is not.

Proof. We fix initial values $(t, x, w) \in D$ and let $(\pi, c) \in \mathcal{A}_T^{\circ}(t, x, w)$ be arbitrary. In the course of this proof, to simplify the exposition, we will write $Z(t) = (X_{\square}(t), V_{\pi, c}^w(t))$. Applying Itô's formula to $\tilde{v}(s, Z(s))$, adding and subtracting $e^{-\beta s} U_1(c(s)) ds$ and using the fact that $X_{\square}$ satisfies (2.3) gives

$$\begin{aligned} d\tilde{v}(s, Z(s)) &= \left(\partial_t \tilde{v}(s, Z(s)) + \tilde{H}_{\tilde{\pi}(s), c(s)}(s, Z(s), \nabla \tilde{v}(s, Z(s)), \nabla^2 \tilde{v}(s, Z(s))) - e^{-\beta s} U_1(c(s)) \right) ds \\ &\quad + dM(s) + \nabla_y \tilde{v}(s, Z(s))^{\top} d\Phi(s), \end{aligned} \quad (3.11)$$

where $dM(s) = V_{\pi, c}^w(s) \partial_w \tilde{v}(s, Z(s)) \pi(s)^{\top} \sigma(s, X(s)) dW(s) + \nabla_y \tilde{v}(s, Z(s))^{\top} \tilde{\sigma}(s, X_{\square}(s)) dB(s)$ is a local martingale and $\tilde{\pi}_k(s) = \sum_{i=1}^d \pi_i(s) 1_{\{\mathcal{R}_i(X(s))=k\}}$ for every k . Here we used the fact that $\pi(s)^{\top} \mu(s, Z(s)) = \tilde{\pi}(s)^{\top} \tilde{\mu}(s, Z(s))$ and $\pi(s)^{\top} a(s, X(s)) \pi(s) = \tilde{\pi}(s)^{\top} \tilde{a}(s, Z(s)) \tilde{\pi}(s)$ to rewrite the Hamiltonian in terms of $(\tilde{\pi}, c)$.

By Assumption 2.4 on the constraint set we have that $(\tilde{\pi}(s), c(s)) \in \tilde{A}(X_{\square}(s), V_{\pi, c}^w(s))$ so we deduce from (3.3) that the sum of the first two terms in the dt integral of the right hand side of (3.11) are nonpositive. Additionally, the Neumann boundary condition (3.5) ensures that the reflection terms vanish. As such, we obtain

$$\tilde{v}(u, Z(u)) \leq \tilde{v}(t, x_{\square}, w) - \int_t^u e^{-\beta s} U_1(c(s)) ds + M(u) - M(t) \quad (3.12)$$

for all $t \leq u \leq T$. We now define the localizing sequence of stopping times $\tau_n = \inf\{t \geq 0 : M(t) \geq n\}$ and note that $M(\cdot \wedge \tau_n)$ is a martingale for every n . Evaluating (3.12) at $u = T \wedge \tau_n$ and taking expectation yields

$$\mathbb{E}_{t,x}[\tilde{v}(T \wedge \tau_n, Z(T \wedge \tau_n))] \leq \tilde{v}(t, x_{(\cdot)}, w) - \mathbb{E}_{t,x} \left[\int_t^{T \wedge \tau_n} e^{-\beta s} U_1(c(s)) ds \right].$$

Sending $n \rightarrow \infty$ we obtain

$$\mathbb{E}_{t,x}[U_2(V_{\pi,c}^w(T))] \leq \tilde{v}(t, x_{(\cdot)}, w) - \mathbb{E}_{t,x} \left[\int_t^T e^{-\beta s} U_1(c(s)) ds \right],$$

where we used Fatou's lemma and the terminal condition from (3.3) on the left hand side and monotone convergence on the right hand side. This establishes that

$$v(t, x, w) = \sup_{(\pi,c) \in \mathcal{A}_T^\circ(t,x,w)} \mathbb{E}_{t,x} \left[\int_t^T e^{-\beta s} U_1(c(s)) ds + U_2(V_{\pi,c}^w(T)) \right] \leq \tilde{v}(t, x_{(\cdot)}, w). \quad (3.13)$$

To obtain the reverse inequality we now consider the feedback control $(\pi^*(s), c^*(s))_{s \in [t,T]}$. Note that for any $x \in (0, \infty)^d$ and $(\tilde{\pi}, c) \in \tilde{A}(x_{(\cdot)}, w)$ we have that $(\pi, c) \in A(x, w)$ where $\pi_i = \sum_{k=1}^d \tilde{\pi}_k 1_{\{\mathcal{R}_i(x)=k\}}$ so that (π^*, c^*) is an admissible control. Its wealth process $V^* = V_{\pi^*, c^*}^w$ has dynamics

$$\begin{aligned} dV^*(s) = & \left(V^*(s) \left((r + \tilde{\pi}^*(s, X_0(s), V^*(s)))^\top (\tilde{\mu}(s, X_0(s)) - r \mathbf{1}_d) \right) - \tilde{c}^*(s, X_0(s), V^*(s)) \right) ds \\ & + V^*(s) \tilde{\pi}^*(s, X_0(s), V^*(s))^\top \tilde{\sigma}(s, X_0(s)) dB(s). \end{aligned}$$

Since X is an autonomous uncontrolled state process this SDE can be written in the form

$$dV^*(s) = \alpha(V^*(s), \omega) ds + \nu(V^*(s), \omega) dB(s), \quad (3.14)$$

where the coefficients α and ν satisfy uniform Lipschitz and linear growth conditions courtesy of (3.9). As such, by the standard Itô theory for SDEs there exists a pathwise unique solution V^* to (3.14), which satisfies the bound

$$\mathbb{E}_{t,x} \left[\sup_{t \leq s \leq T} |V^*(s)|^p \right] \leq C(1 + |w|^p) \quad (3.15)$$

for every $T \geq 0$, $p \geq 1$ and some constant $C > 0$. Now, letting $Z^*(t) = (X_0(t), V^*(t))$ we proceed by Itô's formula to see that

$$d\tilde{v}(s, Z^*(s)) = e^{-\beta s} U_1(c^*(s, Z^*(s))) ds + dM^*(s),$$

where $dM^*(s) = (V^*(s) \partial_w \tilde{v}(s, Z^*(s)) \tilde{\pi}^*(s, Z^*(s)) + \nabla_y \tilde{v}(s, Z^*(s))^\top \tilde{\sigma}(s, Z^*(s)) dB(s)$ is a local martingale, we used the optimality of (π^*, c^*) as in (3.8) and the HJB (3.3) to handle the Hamiltonian term, and, as before, we used the Neumann boundary condition (3.5) to ensure that the reflection terms vanish. Defining the stopping times $\tau_n^* = \inf\{s \geq 0 : M^*(s) \geq n\}$ we see that

$$\mathbb{E}_{t,x}[\tilde{v}(T \wedge \tau_n^*, Z^*(T \wedge \tau_n^*))] = \tilde{v}(t, x_{(\cdot)}, w) - \mathbb{E}_{t,x} \left[\int_t^{T \wedge \tau_n^*} e^{-\beta s} U_1(c^*(s, Z^*(s))) ds \right].$$

By the polynomial growth condition (3.7) of $\tilde{v}$ together with the L^p bounds (2.5) and (3.15) we can send $n \rightarrow \infty$ and use dominated convergence on the left-hand side together with the terminal condition in (3.3) and monotone convergence on the right-hand side to obtain

$$\mathbb{E}_{t,x}[U_2(V^*(T))] = \tilde{v}(t, x_{(\cdot)}, w) - \mathbb{E}_{t,x} \left[\int_t^T e^{-\beta s} U_1(c^*(s, Z^*(s))) ds \right].$$

Since $(\pi^*, c^*) \in \mathcal{A}_T^\circ(t, x, w)$ we deduce that $v(t, x, w) \geq \tilde{v}(t, x_{(\cdot)}, w)$, which together with (3.13) shows that $v(t, x, w) = \tilde{v}(t, x_{(\cdot)}, w)$ and that (π^*, c^*) is an optimal control. $\square$

4 First-order models

Here we assume constant coefficients by setting $\tilde{\mu}(t, y) = \tilde{\mu}$ and $\tilde{\sigma}(t, y) = \tilde{\sigma}$ for some $\tilde{\mu} \in \mathbb{R}^d$ and $\tilde{\sigma} \in \mathbb{S}_{++}^d$, the space of symmetric positive definite $d \times d$ matrices. This model for X is known as a *first order model* and was first proposed in [2, Chapter 5.5]. As this model has constant drift and volatility coefficients it is reminiscent of the multivariate version of Merton's problem, though first-order models prescribe constant coefficients for the ranked capitalizations rather than their named counterparts.

One criticism of Merton's problem is the difficulty in estimating the drift parameters for the named assets. This problem is mitigated in first-order models as the rank-based drift parameters can be estimated through the collision local times appearing in the reflection term (2.4) (see [2, Chapter 5.5] for more details). Moreover, as we will see in the following subsection, the optimal investment-consumption problem in first-order models admits an explicit solution which shares similarities to the optimal strategy in Merton's problem. As such, the financial intuition, conclusions and tractability of Merton's problem carry over to the present setting. Moreover, we are able to obtain an explicit solution under open market constraints, which is not a tractable constraint in the standard multivariate version of Merton's problem. In this section we work with the family of power utility functions given for risk-aversion parameter $\gamma \in (0, 1) \cup (1, \infty)$ and for $w > 0$ by

$$U_1(w) = U_2(w) = U(w) = \frac{1}{1-\gamma} w^{1-\gamma}.$$

4.1 Unconstrained problem

Here we let the constraint set $\tilde{A}(x, w) = \mathbb{R}^d \times [0, \infty)$. In this case the HJB (3.3) becomes precisely the HJB for Merton's problem except on the smaller domain $D_{\geq}$, rather than D , and with the addition of the Neumann boundary conditions (3.5). It is well-known that the solution to the HJB for Merton's problem is independent of the asset price variable. Hence, the same solution satisfies (3.3) and (3.5). Indeed, it is straightforward to verify that

$$\tilde{\pi}^* = \frac{1}{\gamma} \tilde{a}^{-1}(\tilde{\mu} - r\mathbf{1}_d), \quad \tilde{c}^*(t, w) = e^{-\frac{\beta}{\gamma}t} f(t; \nu) w, \quad \tilde{v}(t, w) = f(t; \nu)^\gamma \frac{w^{1-\gamma}}{1-\gamma},$$

where

$$f(t; \nu) = e^{\frac{\nu}{\gamma}(T-t)} + \frac{\gamma}{\nu - \beta} e^{-\frac{\nu}{\gamma}t} \left(e^{\frac{\nu-\beta}{\gamma}T} - e^{\frac{\nu-\beta}{\gamma}t} \right) \quad \text{and} \quad \nu = (1-\gamma) \left(r + \frac{1}{2\gamma} (\tilde{\mu} - r\mathbf{1}_d)^\top \tilde{a}^{-1} (\tilde{\mu} - r\mathbf{1}_d) \right) \quad (4.1)$$

satisfy (3.3) and (3.5).² As such, as a consequence of Theorem 3.2, we obtain that the value function is given by $v = \tilde{v}$, since the latter is independent of y , and the optimal controls are of feedback form, pinned down by the functions

$$\pi_i^*(x) = \frac{1}{\gamma} \sum_{k=1}^d \tilde{\pi}_k^* \mathbf{1}_{\{\mathcal{R}_i(x)=k\}}, \quad c^*(t, w) = f(t; \nu)^\gamma \frac{w^{1-\gamma}}{1-\gamma}.$$

Thus the value function and optimal consumption rate in the first-order models are precisely the same as in Merton's problem when the assets have parameters $\tilde{\mu}$ and $\tilde{\sigma}$. The optimal investment strategy π^* also involves the constant Merton fractions $\tilde{\pi}^* = \frac{1}{\gamma} \tilde{a}^{-1}(\tilde{\mu} - r\mathbf{1}_d)$ though, unlike in Merton's problem, they prescribe investment in the ranked assets.

4.2 Open market constraints

Here we study the case when the investor is only allowed to invest in assets whose rank is in between n and N for some integers $1 \leq n \leq N \leq d$. Open markets allow the assets available for investment to change over time. This offers a tractable way to incorporate a changing investment universe into the analysis. Moreover, it can serve as a proxy for investors who restrict themselves to certain subsets of the available investment universe. For example taking $n = 1$ and $N = 500$ restricts investment to the largest 500 assets in the market, which can serve as a proxy for an investor who chooses to only invest in assets that make up the S&P 500.

As in part (iv) of Example 2.5 and the discussion following, the open market constraint enforces $\tilde{\pi}_k = 0$ whenever $k \notin [n, N]$. As such, making the ansatz that $\tilde{v}$ is independent of y again, the Hamiltonian in the HJB (3.3) becomes

$$\sup_{\eta \in \mathbb{R}^{N-n+1}} \{w \partial_w \tilde{v}(r + \eta^\top (\tilde{\mu}_{[n:N]} - r\mathbf{1}_{N-n+1})) + \frac{w^2}{2} \partial_{ww} \tilde{v} \eta^\top \tilde{a}_{[n:N]} \eta\} + \sup_{c \geq 0} \{-c \partial_w \tilde{v} + e^{-\beta t} U(c)\}, \quad (4.2)$$

where η corresponds to the non-zero entries of $\tilde{\pi}$ and $\tilde{\mu}_{[n:N]}$, $\tilde{a}_{[n:N]}$ are the truncated vector $(\tilde{\mu}_k)_{n \leq k \leq N}$ and matrix $(\tilde{a}_{k\ell})_{n \leq k, \ell \leq N}$ respectively. We recognize that (4.2) corresponds precisely to the Hamiltonian in Merton's problem when there are $N - n + 1$ risky assets and they possess drift vector $\tilde{\mu}_{[n:N]}$ and diffusion matrix $\tilde{a}_{[n:N]}$. As such, in a similar fashion as to the unconstrained case, we see that the value function is given by

$$v(t, w) = f(t; \nu_{[n:N]})^\gamma \frac{w^{1-\gamma}}{1-\gamma}$$

and the optimal controls are given by the feedback functions

$$\pi_i^*(x) = \sum_{k=n}^N \tilde{\eta}_{k-n+1}^* \mathbf{1}_{\{\mathcal{R}_i(x)=k\}}, \quad \text{where} \quad \tilde{\eta}^* = \frac{1}{\gamma} (\tilde{a}_{[n:N]})^{-1} (\tilde{\mu}_{[n:N]} - r\mathbf{1}_{N-n+1}),$$

$$c^*(t, w) = e^{-\frac{\beta}{\gamma} t} f(t; \nu_{[n:N]}) w,$$

where f is as in (4.1) and

$$\nu_{[n:N]} = (1 - \gamma) \left(r + \frac{1}{2\gamma} (\tilde{\mu}_{[n:N]} - r\mathbf{1}_d)^\top (\tilde{a}_{[n:N]})^{-1} (\tilde{\mu}_{[n:N]} - r\mathbf{1}_d) \right)$$

²If $\nu = \beta$ the correct expression is obtained by taking the limit as $\nu \rightarrow \beta$ in (4.1).

The upshot is that open market constraints, which are intractable in Merton's problem, admit an explicit solution in first-order models. Moreover, the rank-based Merton fraction $\tilde{\eta}^*$ prescribes the optimal holdings. In particular the optimal strategy does not depend on $\tilde{\mu}_k$ or $\tilde{a}_{k\ell}$ for $k, \ell \notin [n, N]$.

4.3 Fully invested constraints

Here we look at the combined open market and fully invested constraints characterized by

$$\tilde{A} = \{\tilde{\pi} \in \mathbb{R}^d : \tilde{\pi}_k = 0 \text{ if } k \notin [n, N] \text{ and } \tilde{\pi}^\top \mathbf{1}_d = 0\} \times [0, \infty).$$

Again writing $\eta \in \mathbb{R}^{N-n+1}$ for the nonzero components of $\tilde{\pi}$ we obtain the same Hamiltonian (4.2) with the additional constraint that $\eta^\top \mathbf{1}_{N-n+1} = 1$. The problem for c is as before, while the problem for η is a quadratic programming problem with linear constraints and can be solved explicitly. Hence, we obtain

$$\eta^* = (a_{[n:N]})^{-1} \left(-\frac{\partial_w \tilde{v}}{w \partial_{ww} \tilde{v}} \tilde{\mu}_{[n:N]} + \frac{(1 + \frac{\partial_w \tilde{v}}{w \partial_{ww} \tilde{v}} \mathbf{1}^\top (\tilde{a}_{[n:N]})^{-1} \tilde{\mu}_{[n:N]})}{\mathbf{1}^\top (\tilde{a}_{[n:N]})^{-1} \mathbf{1}} \mathbf{1} \right), \quad c^* = e^{-\beta t} (\partial_w \tilde{v})^{-\frac{1}{\gamma}},$$

where we write $\mathbf{1}$ in place of $\mathbf{1}_{N-n+1}$ for brevity. Next we make the standard ansatz $\tilde{v}(t, w) = u(t) \frac{w^{1-\gamma}}{1-\gamma}$ for an unknown function u , which leads to the ODE

$$0 = u'(t) + \zeta u(t) + \gamma e^{-\frac{\beta}{\gamma} t} u(t)^{1-\frac{1}{\gamma}}; \quad u(T) = 1,$$

where

$$\zeta = \frac{1-\gamma}{2\gamma} \left(\tilde{\mu}_{[n:N]} - \frac{\gamma - \mathbf{1}^\top (\tilde{a}_{[n:N]})^{-1} \tilde{\mu}_{[n:N]}}{\mathbf{1}^\top (\tilde{a}_{[n:N]})^{-1} \mathbf{1}} \mathbf{1} \right)^\top (\tilde{a}_{[n:N]})^{-1} \left(\tilde{\mu}_{[n:N]} + \frac{\gamma - \mathbf{1}^\top (\tilde{a}_{[n:N]})^{-1} \tilde{\mu}_{[n:N]}}{\mathbf{1}^\top (\tilde{a}_{[n:N]})^{-1} \mathbf{1}} \mathbf{1} \right).$$

This ODE has solution $u(t) = f(t; \zeta)^\gamma$, where f is given by (4.1). As such, the optimal controls are given by the feedback functions

$$\pi_i^*(x) = \sum_{k=n}^N \tilde{\eta}_{k-n+1}^* \mathbf{1}_{\{\mathcal{R}_i(x)=k\}} \quad \text{with} \quad \tilde{\eta}^* = \frac{1}{\gamma} (\tilde{a}_{[n:N]})^{-1} \left(\tilde{\mu}_{[n:N]} + \frac{\gamma - \mathbf{1}^\top (\tilde{a}_{[n:N]})^{-1} \tilde{\mu}_{[n:N]}}{\mathbf{1}^\top (\tilde{a}_{[n:N]})^{-1} \mathbf{1}} \mathbf{1} \right),$$

$$c^*(t, w) = e^{-\frac{\beta}{\gamma} t} f(t; \zeta) w.$$

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