Well-Posedness and Approximation of Weak Solutions to Time Dependent Maxwell's Equations with L^2-Data *†

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Abstract

We study Maxwell's equations in conducting media with perfectly conducting boundary conditions on Lipschitz domains, allowing rough material coefficients and L^2-data. Our first contribution is a direct proof of well-posedness of the first-order weak formulation, including solution existence and uniqueness, an energy identity, and continuous dependence on the data. The argument uses interior-in-time mollification to show uniqueness while avoiding reflection techniques. Existence is via the well-known Galerkin method (cf. Duvaut and Lions [6, Eqns. (4.31)–(4.32), p. 346; Thm. 4.1]). For completeness, and to make the paper self-contained, a complete proof has been provided.

Our second contribution is a structure-preserving semi-discrete finite element method based on the Nédélec/Raviart–Thomas de Rham complex. The scheme preserves a discrete Gauss law for all times and satisfies a continuous-in-time energy identity with stability for nonnegative conductivity. With a divergence-free initialization of the magnetic field (via potential reconstruction or constrained L^2 projection), we prove convergence of the semi-discrete solutions to the unique weak solution as the mesh is refined. The analysis mostly relies on projector consistency, weak-* compactness in time-bounded L^2 spaces, and identification of time derivatives in dual spaces.

1 Introduction

Problem setting. Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and $T > 0$. Material parameters satisfy

$$\varepsilon \in L^\infty(\Omega; \mathbb{R}^{3\times 3}), \quad \mu \in L^\infty(\Omega; \mathbb{R}^{3\times 3}), \quad \sigma \in L^\infty(\Omega; \mathbb{R}^{3\times 3}),$$

all symmetric, with ε and μ uniformly elliptic and σ nonnegative. Given $f \in L^2(0,T; L^2(\Omega)^3)$ and $E_0, B_0 \in L^2(\Omega)^3$ with $\operatorname{div} B_0 = 0$ in $L^2(\Omega)$, we consider

$$\begin{cases} \varepsilon\, \partial_t E + \sigma E - \operatorname{curl}(\mu^{-1} B) = f, \\ \partial_t B + \operatorname{curl} E = 0, \end{cases} \qquad E \times n = 0 \;\text{ on } \partial\Omega \times (0,T), \qquad E(0) = E_0, \quad B(0) = B_0.$$

Our weak solution notion (Definition 1) requires

$$E \in H^1\big(0,T; H_0(\operatorname{curl};\Omega)^*\big) \cap C([0,T]; L^2(\Omega)^3), \qquad B \in H^1\big(0,T; H(\operatorname{curl};\Omega)^*\big) \cap C([0,T]; L^2(\Omega)^3),$$

satisfying the usual variational identities against $\psi \in H_0(\operatorname{curl};\Omega)$ and $\phi \in H(\operatorname{curl};\Omega)$ for a.e. $t \in (0,T)$.

*Dedication: This work is dedicated to the memory of our teacher and mentor Prof. Dr. Ronald H.W. Hoppe.

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Scope and goals. The primary goal of this paper is to prove convergence of a conforming finite element *semi-discretization in space* (first–order $H(\mathrm{curl})/H(\mathrm{div})$ formulation) to the continuous weak solution for data $f \in L^2(0,T;L^2(\Omega)^3)$ and $E_0, B_0 \in L^2(\Omega)^3$. To the best of our knowledge, this result is new. A byproduct of the arguments needed for the convergence analysis is a direct, self-contained proof of well-posedness for the continuous problem with the above stated data regularity.

Positioning within the literature. There are comparatively few references that treat well-posedness for Maxwell's equations at this level of generality. The classical monograph of Duvaut and Lions [6, Eqns. (4.31)–(4.32), p. 346; Thm. 4.1] establishes existence and uniqueness of $L^\infty(0,T;L^2(\Omega)^3)$ solutions. Their concise presentation makes it nontrivial to infer continuity in time, continuous dependence on data, and additional time-regularity. Notice that such results can also be inferred from [15, Thm. 2.4] and [24, Lemma 3.2] via the semigroup/mild-solution framework. The latter result uses the abstract theory of C_0-semigroups (cf. Ball [2]). We also refer to [16, Sec. 8.2] for a discussion on well-posedness using the spectral theorem. See also [8, Sec. 7.8] where the authors consider weak solutions satisfying the free charge density law and Gauss law in the weak sense. Their well-posedness result uses density based arguments.

The present paper gives a complete self-contained proof of well-posedness for the weak formulation of Maxwell's equations. Our approach may be viewed as an extension and clarification of the arguments in [6]: (i) we provide complete details in L^2-data setting; (ii) uniqueness is obtained without resorting to time-reflection, using instead an interior-in-time mollification argument; and (iii) we establish the stated time-regularity and continuous dependence without auxiliary smoothing assumptions on the data. We collect these results here both for completeness and because several steps in the continuous analysis feed directly into the convergence proof of our numerical scheme. This synthesis honors Ronald H.W. Hoppe, who made influential contributions to computational electromagnetics [13].

On the numerical side, early finite element discretizations for Maxwell's equations were developed in the semi-discrete (space-only) setting in [21] and in fully discrete form in [4], typically for second–order (in space/time) formulations. See also [19] for analysis and convergence estimates of a scheme closely related to ours (Nédélec and Raviart-Thomas (RT) discretization for E and B); [18] for a fully time-discrete leapfrog analysis; and [17] for a piecewise-constant (in space) approximation of the electric field.

Recent work has addressed nonlinear and nonsmooth models. For Maxwell variational inequalities (MVIs) of the second kind in type-II superconductivity, [23] uses Nédélec elements for E, piecewise constants for B, and implicit Euler in time. For MVIs of the first kind in electric shielding, [12] employs piecewise constants for E and Nédélec for H (with $B = \mu H$). Both assume sources in $W^{1,\infty}(0,T;L^2)$. The quasi-MVI study [11] treats both implicit Euler and leapfrog: Nédélec for E and piecewise constants for H in the former; piecewise constants for E and Nédélec for H in the latter. Remarkably for leapfrog scheme, the authors can handle sources which are of bounded variation type.

To our knowledge, none of these works consider the specific conforming pairing analyzed here—Nédélec/RT in the first-order $H(\mathrm{curl})/H(\mathrm{div})$ framework—together with *minimal data regularity* $f \in L^2(0,T;L^2)$, $E_0, B_0 \in L^2$, and prove convergence of the *semi-discrete* solution to the weak continuous solution.

Contributions.

- A direct, proof of existence, uniqueness, and continuous dependence for the weak Maxwell system in the minimal-regularity class

$$E \in H^1(0,T; H_0(\mathrm{curl};\Omega)^*) \cap C([0,T]; L^2), \qquad B \in H^1(0,T; H(\mathrm{curl};\Omega)^*) \cap C([0,T]; L^2).$$

- A structure-preserving *semi-discrete* FE formulation based on the Nédélec/RT de Rham complex that (i) preserves a discrete Gauss law *for all times* and (ii) satisfies a continuous-in-time discrete energy identity and stability for $\sigma \geq 0$.
- Convergence of the semi-discrete solutions (E_h, B_h) to the continuous unique weak solution as $h \to 0$ under minimal data assumptions ($f \in L^2(0,T; L^2(\Omega)^3)$, $E_0, B_0 \in L^2(\Omega)^3$), using mainly spatial projector consistency and weak-* compactness in $L^\infty(0,T; L^2)$.

Organization. Section 2 states the functional setting and weak formulation. Section 3 proves uniqueness via time mollification and test-side smoothing and existence using the Galerkin method. Section 4 introduces the Nédélec/RT semi-discrete scheme, proves stability and discrete Gauss law preservation, and establishes convergence. Auxiliary results are collected in Appendix A.

2 Notation and Preliminaries

Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and $0 < T < +\infty$. Consider the time-dependent Maxwell system with anisotropic tensor coefficients:

$$\varepsilon(x)\partial_t E - \mathrm{curl}\,(\mu^{-1}(x)B) + \sigma(x)E = f(x,t), \qquad \text{in } \Omega \times (0,T) \tag{1a}$$

$$\partial_t B + \mathrm{curl}\, E = 0, \qquad \text{in } \Omega \times (0,T) \tag{1b}$$

subject to the initial and boundary conditions:

$$\begin{aligned} E(0,x) &= E_0(x), \quad B(0,x) = B_0(x), && \text{in } \Omega \\ E \times \nu &= 0 && \text{on } \partial\Omega \times (0,T) \end{aligned}$$

Notice that (1a) and (1b) are the Ampére-Maxwell and Faraday laws, respectively.

Assumption 1. *Throughout the article, the following conditions are assumed on the data:*

- $\varepsilon(x), \sigma(x), \mu(x) \in L^\infty(\Omega; \mathbb{R}^{3\times 3})$ *are symmetric tensor fields;*
- *There exist constants* ε_0, μ_0 *such that*

$$\xi^\top \varepsilon(x)\xi \geq \varepsilon_0|\xi|^2, \quad \xi^\top \mu(x)\xi \geq \mu_0|\xi|^2, \quad \textit{for all } \xi \in \mathbb{R}^3, \ \textit{a.e. } x \in \Omega;$$

- σ *is symmetric positive semi-definite,* $f \in L^2(0,T; L^2(\Omega)^3)$*, and* $E_0, B_0 \in L^2(\Omega)^3$ *with* $\mathrm{div}\, B_0 = 0$ *in* $L^2(\Omega)$*.*

Throughout, we use $\|\cdot\|_{L^2(\Omega)}$ to denote the L^2-norm and $(\cdot,\cdot)$ to denote the L^2-scalar product. We denote the space for vector valued functions by $L^2(\Omega)^3$, but will also interchangeably use L^2 (to minimize the notation) when it is clear from the context. For a given Banach space X, we denote its topological dual by X^*, moreover, $\langle\cdot,\cdot\rangle_X$ denotes the duality pairing between X^* and X. We define the following Sobolev spaces

$$
\begin{aligned}
H(\mathrm{div};\Omega) &:= \left\{v \in L^2(\Omega)^3 : \mathrm{div}\ v \in L^2(\Omega)\right\},\\
H_0(\mathrm{div};\Omega) &:= \{v \in H(\mathrm{div};\Omega) : \gamma_\nu(v) := \gamma v\cdot\nu = 0 \text{ on } \partial\Omega\},\\
H(\mathrm{div}^0;\Omega) &:= \{v \in H(\mathrm{div};\Omega) : \mathrm{div}\ v = 0 \text{ a.e. in } \Omega\},\\
H(\mathrm{curl};\Omega) &:= \left\{v \in L^2(\Omega)^3 : \mathrm{curl}\, v \in L^2(\Omega)^3\right\},\\
H_0(\mathrm{curl};\Omega) &:= \{v \in H(\mathrm{curl};\Omega) : \gamma_\tau(v) := \gamma v\times\nu = 0 \text{ on } \partial\Omega\}.
\end{aligned}
$$

Here γ_ν is the normal trace [9] and γ_τ the tangential trace [3, 20].

Definition 1 (Weak solution to Maxwell's Equations)**.** *The electric and magnetic fields* (E,B) *solve* (1) *weakly if and only if*

$$
\begin{aligned}
E &\in H^1(0,T;H_0(\mathrm{curl};\Omega)^*)\cap L^\infty(0,T;L^2(\Omega)^3),\\
B &\in H^1(0,T;H(\mathrm{curl};\Omega)^*)\cap L^\infty(0,T;L^2(\Omega)^3),
\end{aligned}
$$

and satisfy, for almost every $t\in(0,T)$*, the variational formulation:*

$$
\langle\varepsilon\partial_t E,\psi\rangle_{H_0(\mathrm{curl};\Omega)} + (\sigma E,\psi) - (\mu^{-1}B,\mathrm{curl}\,\psi) = (f,\psi), \qquad \forall\psi\in H_0(\mathrm{curl};\Omega), \tag{2a}
$$

$$
\langle\partial_t B,\phi\rangle_{H(\mathrm{curl};\Omega)} + (E,\mathrm{curl}\,\phi) = 0, \qquad \forall\phi\in H(\mathrm{curl};\Omega), \tag{2b}
$$

with initial data $E(0)=E_0\in L^2(\Omega)^3$*,* $B(0)=B_0\in L^2(\Omega)^3$*. Moreover, if* $\mathrm{div}\,B_0 = 0$ *in* $L^2(\Omega)$*, then additionally,* $B\in L^\infty(0,T;H(\mathrm{div}^0;\Omega))$*.*

For the justification of pointwise evaluation of the initial condition in time under the regularity given in Definition 1, we refer to [14, Proposition 2.19]. Notice that in Corollary 1 we establish that (E,B) solving (2) also fulfills $E\in C([0,T];L^2(\Omega)^3)$ and $B\in C([0,T];L^2(\Omega)^3)$.

3 Well Posedness of Maxwell's Equations

This section is organized as follows. First in Theorem 1 we establish that (2) has a unique solution. This is tricky because we cannot use E and B as test functions to use the standard energy argument to establish uniqueness. Instead we develop a mollification in time argument. Corollary 1 shows that E and B are in $C([0,T];L^2(\Omega)^3)$. Next, in Theorem 2 we show existence of solution to (1) via a Galerkin type argument. Proposition 1 shows that B is solenoidal, i.e., $\mathrm{div}\,B=0$ in a certain sense.

The following result will be helpful in showing uniqueness of solution:

Lemma 1 (Characterization of $H(\mathrm{curl})$ via distributional curl)**.** *Let* $v\in L^2(\Omega)^3$*. If there exists* $g\in L^2(\Omega)^3$ *such that*

$$
(v,\ \mathrm{curl}\,\phi)_{L^2(\Omega)^3} \;=\; (g,\ \phi)_{L^2(\Omega)^3} \qquad \forall\,\phi\in C_c^\infty(\Omega)^3,
$$

then $v\in H(\mathrm{curl};\Omega)$ *and* $\mathrm{curl}\,v = g$ *in* $L^2(\Omega)^3$*.*

Proof. Define a linear functional Λ on $C_c^\infty(\Omega)^3$ by $\Lambda(\phi) := (v,\mathrm{curl}\,\phi)_{L^2}$. The hypothesis yields $\Lambda(\phi) = (g,\phi)_{L^2}$, hence $|\Lambda(\phi)|\le\|g\|_{L^2}\|\phi\|_{L^2}$ for all $\phi\in C_c^\infty$. Thus Λ extends continuously (by density) to $L^2(\Omega)^3$ and the Riesz representative of Λ is g. By the definition of the distributional curl, this precisely means that $\mathrm{curl}\,v=g$ as an L^2–field; hence $v\in H(\mathrm{curl};\Omega)$ with $\mathrm{curl}\,v=g$ in $L^2(\Omega)^3$. □

Theorem 1 (Solution to (2) is unique)**.** *Let (E_i, B_i), $i = 1, 2$, be weak solutions to* (2) *with the same data f, E_0, B_0 in the sense of Definition 1. Then $E_1 \equiv E_2$ and $B_1 \equiv B_2$ on $(0, T)$.*

Proof. Let $(\tilde{E}, \tilde{B}) := (E_1 - E_2, B_1 - B_2)$. Then $(\tilde{E}, \tilde{B})$ solves the homogeneous system

$$\begin{aligned} \langle \varepsilon \partial_t \tilde{E}, \psi \rangle_{H_0(\mathrm{curl};\Omega)} + (\sigma \tilde{E}, \psi) - (\mu^{-1} \tilde{B}, \operatorname{curl} \psi) = 0 \quad \forall \psi \in H_0(\mathrm{curl}; \Omega), \\ \langle \partial_t \tilde{B}, \phi \rangle_{H(\mathrm{curl};\Omega)} + (\tilde{E}, \operatorname{curl} \phi) = 0 \quad \forall \phi \in H(\mathrm{curl}; \Omega), \end{aligned} \tag{3}$$

with $\tilde{E}(0) = 0$ and $\tilde{B}(0) = 0$ in $L^2(\Omega)^3$.

Step 1 (interior time–mollification). Let $\rho \in C_c^\infty(\mathbb{R})$ be even, nonnegative, with $\int_{\mathbb{R}} \rho = 1$ and $\operatorname{supp} \rho \subset [-1, 1]$. For $\delta \in (0, T/2)$ set $\rho_\delta(s) := \delta^{-1} \rho(s/\delta)$, so $\operatorname{supp} \rho_\delta \subset [-\delta, \delta]$.

For $t \in (\delta, T - \delta)$ define the (interior) mollifications

$$\tilde{E}^\delta(t) := \int_{\mathbb{R}} \rho_\delta(s)\, \tilde{E}(t - s)\, ds = \int_{-\delta}^{\delta} \rho_\delta(s)\, \tilde{E}(t - s)\, ds, \qquad \tilde{B}^\delta(t) := \int_{-\delta}^{\delta} \rho_\delta(s)\, \tilde{B}(t - s)\, ds.$$

This is well-defined because, for any $t \in (\delta, T - \delta)$ and any s with $\rho_\delta(s) \neq 0$ (hence $|s| \leq \delta$),

$$t - s \;\geq\; t - \delta \;>\; 0, \qquad t - s \;\leq\; t + \delta \;<\; T, \tag{4}$$

so $t - s \in (0, T)$ and only values of $(\tilde{E}, \tilde{B})$ inside their domain are sampled. Standard properties of convolution yield

$$\tilde{E}^\delta, \tilde{B}^\delta \in C^\infty\big((\delta, T - \delta); L^2(\Omega)^3\big), \qquad \partial_t \tilde{E}^\delta = \int_{-\delta}^{\delta} \rho_\delta'(s)\, \tilde{E}(t - s)\, ds, \quad \partial_t \tilde{B}^\delta = \int_{-\delta}^{\delta} \rho_\delta'(s)\, \tilde{B}(t - s)\, ds,$$

so $\partial_t \tilde{E}^\delta \in C^\infty\big((\delta, T - \delta); L^2(\Omega)^3\big)$ and likewise for $\partial_t \tilde{B}^\delta$.

Moreover, for every compact interval $J = [a, b] \Subset (0, T)$ and $\delta < \min\{a, T - b\}$,

$$\tilde{E}^\delta \to \tilde{E} \;\text{ in } L^2\big(J; L^2(\Omega)^3\big), \qquad \tilde{B}^\delta \to \tilde{B} \;\text{ in } L^2\big(J; L^2(\Omega)^3\big),$$

and the mollifications are L^∞–stable:

$$\|\tilde{E}^\delta\|_{L^\infty((\delta, T-\delta); L^2)} \leq \|\tilde{E}\|_{L^\infty((0,T); L^2)}, \qquad \|\tilde{B}^\delta\|_{L^\infty((\delta, T-\delta); L^2)} \leq \|\tilde{B}\|_{L^\infty((0,T); L^2)}. \tag{5}$$

Step 2 (regularized Maxwell system on $(\delta, T - \delta)$). Fix $t \in (\delta, T - \delta)$, $\psi \in H_0(\mathrm{curl}; \Omega)$, and $\phi \in H(\mathrm{curl}; \Omega)$. Using (3) at times $t - s$ and Fubini,

$$\begin{aligned} \langle \varepsilon \partial_t \tilde{E}^\delta(t), \psi \rangle_{H_0(\mathrm{curl};\Omega)} &= \int \rho_\delta(s)\, \langle \varepsilon \partial_t \tilde{E}(t - s), \psi \rangle\, ds = (\mu^{-1} \tilde{B}^\delta(t), \operatorname{curl} \psi) - (\sigma \tilde{E}^\delta(t), \psi), \\ \langle \partial_t \tilde{B}^\delta(t), \phi \rangle_{H(\mathrm{curl};\Omega)} &= \int \rho_\delta(s)\, \langle \partial_t \tilde{B}(t - s), \phi \rangle\, ds = -(\tilde{E}^\delta(t),\, \operatorname{curl} \phi). \end{aligned}$$

Thus, for a.e. $t \in (\delta, T - \delta)$,

$$\begin{aligned} \langle \varepsilon \partial_t \tilde{E}^\delta(t), \psi \rangle_{H_0(\mathrm{curl};\Omega)} + (\sigma \tilde{E}^\delta(t), \psi) - (\mu^{-1} \tilde{B}^\delta(t), \operatorname{curl} \psi) = 0, \\ \langle \partial_t \tilde{B}^\delta(t), \phi \rangle_{H(\mathrm{curl};\Omega)} + (\tilde{E}^\delta(t), \operatorname{curl} \phi) = 0. \end{aligned} \tag{6}$$

Because the time mollification yields $\partial_t \tilde{B}^\delta(t) \in L^2(\Omega)^3$ for a.e. t, in $(6)_2$ we have

$$\big\langle \partial_t \tilde{B}^\delta(t),\ \phi \big\rangle_{H(\mathrm{curl};\Omega)} = (\partial_t \tilde{B}^\delta(t),\ \phi)_{L^2}.$$

Hence $(6)_2$ reads

$$\big(\tilde{E}^\delta(t),\ \operatorname{curl}\phi\big)_{L^2} \;=\; -\,(\partial_t\tilde{B}^\delta(t),\ \phi)_{L^2} \qquad \forall\,\phi\in H(\operatorname{curl};\Omega). \tag{7}$$

In particular, (7) holds for $\phi \in C_c^\infty(\Omega)^3 \subset H(\operatorname{curl};\Omega)$. Then Lemma 1 with $v = \tilde{E}^\delta(t)$ and $g = -\partial_t\tilde{B}^\delta(t)$ implies

$$\tilde{E}^\delta(t)\in H(\operatorname{curl};\Omega), \qquad \operatorname{curl}\tilde{E}^\delta(t) = -\partial_t\tilde{B}^\delta(t) \text{ in } L^2(\Omega)^3. \tag{8}$$

Now apply the Green identity for $H(\operatorname{curl})$ to $\tilde{E}^\delta(t)\in H(\operatorname{curl})$: for any $\phi\in H(\operatorname{curl};\Omega)$,

$$\big(\tilde{E}^\delta(t),\ \operatorname{curl}\phi\big)_{L^2} = \big(\operatorname{curl}\tilde{E}^\delta(t),\ \phi\big)_{L^2} \;-\; \big\langle n\times\tilde{E}^\delta(t),\ \phi\big\rangle_{\partial\Omega}.$$

Using (7) and (8), we find

$$-\big(\partial_t\tilde{B}^\delta(t),\ \phi\big)_{L^2} = \big(\operatorname{curl}\tilde{E}^\delta(t),\ \phi\big)_{L^2} \;-\; \big\langle n\times\tilde{E}^\delta(t),\ \phi\big\rangle_{\partial\Omega} = -\big(\partial_t\tilde{B}^\delta(t),\ \phi\big)_{L^2} \;-\; \big\langle n\times\tilde{E}^\delta(t),\ \phi\big\rangle_{\partial\Omega}.$$

Canceling the equal L^2 terms yields

$$\big\langle n\times\tilde{E}^\delta(t),\ \phi\big\rangle_{\partial\Omega} = 0 \qquad \forall\,\phi\in H(\operatorname{curl};\Omega).$$

Thus the tangential trace vanishes, $n\times\tilde{E}^\delta(t) = 0$ in the trace sense, i.e., $\tilde{E}^\delta(t)\in H_0(\operatorname{curl};\Omega)$ on $t\in(\delta, T-\delta)$.

Similarly, from $(6)_1$ and the fact that $\partial_t\tilde{E}^\delta(t), \tilde{E}^\delta(t)\in L^2(\Omega)^3$, we obtain that

$$\mu^{-1}\tilde{B}^\delta(t)\in H(\operatorname{curl};\Omega) \qquad \text{for a.e. } t\in(\delta, T-\delta).$$

Step 3 (energy identity with a time cutoff and $\delta\downarrow 0$). Fix a $\eta\in C_c^\infty(0,T)$ and set

$$K := \operatorname{supp}\eta = \overline{\{\, s\in(0,T) : \eta(s)\neq 0 \,\}} \Subset (0,T).$$

Since η has compact support in $(0,T)$, K is compact and $K\subset(0,T)$. Let

$$a := \inf K, \qquad b := \sup K,$$

so $0 < a \le b < T$ and $K\subset[a,b]$. Define the distances to the endpoints

$$d_0 := a > 0, \qquad d_T := T - b > 0,$$

and set

$$\delta_\eta := \tfrac{1}{2}\min\{d_0, d_T\} > 0.$$

Then for any $\delta\in(0,\delta_\eta)$ and any $s\in K$ we have

$$s \ge a > d_0/2 > \delta,$$

and, since $b = T - d_T$ and $\delta < d_T/2 < d_T$,

$$s \le b = T - d_T < T - \delta.$$

Hence $s\in(\delta, T-\delta)$, and therefore

$$K\subset(\delta, T-\delta).$$

Since $\tilde{E}^\delta(\cdot) \in H_0(\mathrm{curl};\Omega)$ and $\mu^{-1}\tilde{B}^\delta(\cdot) \in H(\mathrm{curl};\Omega)$ a.e. on $(\delta, T-\delta)$ (Step 2), we may use the time–dependent test functions

$$\psi(t) := \eta(t)\,\tilde{E}^\delta(t) \in H_0(\mathrm{curl};\Omega), \qquad \phi(t) := \eta(t)\,\mu^{-1}\tilde{B}^\delta(t) \in H(\mathrm{curl};\Omega)$$

in (6), integrate in $t \in (0,T)$, and add the two relations. Because $\eta = \eta(t)$ has no spatial dependence, the cross terms

$$-(\mu^{-1}\tilde{B}^\delta, \mathrm{curl}(\eta\tilde{E}^\delta)) + (\mathrm{curl}\,\tilde{E}^\delta, \eta\,\mu^{-1}\tilde{B}^\delta) = -\eta(\mu^{-1}\tilde{B}^\delta, \mathrm{curl}\,\tilde{E}^\delta) + \eta(\mathrm{curl}\,\tilde{E}^\delta, \mu^{-1}\tilde{B}^\delta) = 0$$

cancel pointwise. Using that $\tilde{E}^\delta, \tilde{B}^\delta \in C^\infty(\mathbb{R}; L^2(\Omega)^3)$ in time,

$$\langle \varepsilon\partial_t\tilde{E}^\delta, \tilde{E}^\delta\rangle = \tfrac{1}{2}\tfrac{d}{dt}(\varepsilon\tilde{E}^\delta, \tilde{E}^\delta), \qquad \langle \partial_t\tilde{B}^\delta, \mu^{-1}\tilde{B}^\delta\rangle = \tfrac{1}{2}\tfrac{d}{dt}(\mu^{-1}\tilde{B}^\delta, \tilde{B}^\delta),$$

we obtain

$$-\int_0^T \eta'(t)\,\mathcal{E}(\tilde{E}^\delta, \tilde{B}^\delta)(t)\,dt + \int_0^T \eta(t)\,(\sigma\tilde{E}^\delta, \tilde{E}^\delta)\,dt = 0, \tag{9}$$

where $\mathcal{E}(E,B)(t) := \frac{1}{2}\big(\|\sqrt{\varepsilon}\,E(t)\|_{L^2}^2 + \|\sqrt{\mu^{-1}}\,B(t)\|_{L^2}^2\big)$.

Passage $\delta \downarrow 0$ with η fixed. Because $K = \mathrm{supp}\,\eta \Subset (0,T)$ and $0 < \delta < \delta_\eta$, the mollifications satisfy

$$\tilde{E}^\delta \to \tilde{E}, \qquad \tilde{B}^\delta \to \tilde{B} \quad \text{in } L^2\big(K; L^2(\Omega)^3\big) \text{ as } \delta \downarrow 0,$$

Hence,

$$\|\tilde{E}^\delta(\cdot)\|_{L^2}^2 \;\to\; \|\tilde{E}(\cdot)\|_{L^2}^2, \qquad \|\tilde{B}^\delta(\cdot)\|_{L^2}^2 \;\to\; \|\tilde{B}(\cdot)\|_{L^2}^2 \quad \text{in } L^1(K),$$

by the elementary bound $|a^2 - b^2| \le (|a| + |b|)|a-b|$. Therefore,

$$\int_0^T \eta'(t)\,\mathcal{E}(\tilde{E}^\delta, \tilde{B}^\delta)(t)\,dt \;\longrightarrow\; \int_0^T \eta'(t)\,\mathcal{E}(\tilde{E}, \tilde{B})(t)\,dt,$$

and, since $\sigma \in L^\infty$ and $\tilde{E}^\delta \to \tilde{E}$ in $L^2(K; L^2)$,

$$\int_0^T \eta(t)\,(\sigma\tilde{E}^\delta, \tilde{E}^\delta)\,dt \;\longrightarrow\; \int_0^T \eta(t)\,(\sigma\tilde{E}, \tilde{E})\,dt.$$

Letting $\delta \downarrow 0$ in (9) yields, for our fixed $\eta \in C_c^\infty(0,T)$,

$$-\int_0^T \eta'(t)\,\mathcal{E}(\tilde{E}, \tilde{B})(t)\,dt + \int_0^T \eta(t)\,(\sigma\tilde{E}, \tilde{E})\,dt = 0. \tag{10}$$

Since the right–hand side belongs to $L^1(0,T)$, (10) shows that $\mathcal{E}(\tilde{E}, \tilde{B}) \in W^{1,1}(0,T)$ with

$$\frac{d}{dt}\,\mathcal{E}(\tilde{E}, \tilde{B})(t) = -(\sigma\tilde{E}(t), \tilde{E}(t)) \quad \text{in } \mathcal{D}'(0,T),$$

hence $\mathcal{E}(\tilde{E}, \tilde{B})$ is absolutely continuous on $[0,T]$. Integrate both sides to conclude that $\tilde{E} = 0$ and $\tilde{B} = 0$. The proof is complete. □

The same argument from the above result can be used to show that the solution to (2) is in fact continuous in time. We state an auxiliary result before proving this, see [5, Proposition 2.5.1] for details.

Lemma 2 (Differentiability of Scalar Pairings)**.** *Let V be a reflexive Banach space with dual V^*, and let $u \in H^1(0,T;V^*)$. Then, for every $v \in V$, the scalar function*

$$\alpha(t) := \langle u(t), v\rangle_{V^*,V} \in H^1(0,T),$$

and its derivative satisfies

$$\frac{d}{dt}\alpha(t) = \langle \partial_t u(t), v\rangle_{V^*,V} \quad \textit{for a.e. } t \in (0,T).$$

The next result shows time continuity of solution to (2).

Corollary 1 (Strong L^2–continuity in time and energy identity)**.** *Let (E,B) be the weak solution of* (2) *in the sense of Definition 1. Then $E, B \in C([0,T]; L^2(\Omega)^3)$ and, for every $t \in [0,T]$,*

$$\mathcal{E}(t) + \int_0^t (\sigma E(s), E(s))\, ds = \mathcal{E}(0) + \int_0^t (f(s), E(s))\, ds, \qquad \mathcal{E}(t) := \tfrac{1}{2}\big(\|\sqrt{\varepsilon}\, E(t)\|_{L^2}^2 + \|\sqrt{\mu^{-1}}\, B(t)\|_{L^2}^2\big). \tag{11}$$

In particular, $t \mapsto \|E(t)\|_{L^2}$ and $t \mapsto \|B(t)\|_{L^2}$ are continuous on $[0,T]$.

Proof. Step 1 (weak L^2–continuity). The continuous embedding $H_0(\mathrm{curl};\Omega) \hookrightarrow L^2(\Omega)^3$ induces $L^2(\Omega)^3 \hookrightarrow H_0(\mathrm{curl};\Omega)^*$ via $\langle E(t), w\rangle_{H_0(\mathrm{curl})} = (E(t), w)_{L^2}$. Since $E \in H^1(0,T; H_0(\mathrm{curl})^*)$, Lemma 2 yields $(E(\cdot), w)_{L^2} \in H^1(0,T) \subset C([0,T])$ for each $w \in H_0(\mathrm{curl};\Omega)$. Because $H_0(\mathrm{curl};\Omega) \supset C_c^\infty(\Omega)^3$ is dense in $L^2(\Omega)^3$ and $\sup_t \|E(t)\|_{L^2} < \infty$, we approximate any $\phi \in L^2$ by $w_k \in H_0(\mathrm{curl})$ and pass to the limit uniformly in t to conclude that $t \mapsto (E(t), \phi)_{L^2}$ is continuous. Thus $E \in C_w([0,T]; L^2)$, i..e, weakly continuous. The same argument with $H(\mathrm{curl};\Omega)$ shows $B \in C_w([0,T]; L^2)$.

Step 2 (energy identity in distribution form). Repeating the interior time–mollification/cutoff test used in the uniqueness proof, now retaining the forcing term, gives for every $\eta \in C_c^\infty(0,T)$

$$-\int_0^T \eta'(t)\, \mathcal{E}(t)\, dt + \int_0^T \eta(t)\, (\sigma E, E)\, dt = \int_0^T \eta(t)\, (f, E)\, dt.$$

Since $f \in L^2(0,T;L^2)$ and $E \in L^\infty(0,T;L^2)$, the right-hand side is in $L^1(0,T)$, hence $\mathcal{E} \in W^{1,1}(0,T)$ with $\mathcal{E}'(t) = (f(t), E(t)) - (\sigma E(t), E(t))$ a.e. Integrating from 0 to t yields (11).

Step 3 (strong L^2–continuity). From Step 1, $E, B \in C_w([0,T]; L^2)$. From Step 2, $t \mapsto \|\sqrt{\varepsilon} E(t)\|_{L^2}$ and $\|\sqrt{\mu^{-1}} B(t)\|_{L^2}$ are continuous; by uniform ellipticity and boundedness of ε, μ^{-1} these norms are equivalent to $\|E(t)\|_{L^2}$ and $\|B(t)\|_{L^2}$. In a Hilbert space, weak continuity plus continuity of the norm implies strong continuity. Hence $E, B \in C([0,T]; L^2(\Omega)^3)$. □

Now we are ready to state our existence of solution proof.

Theorem 2 (Well-posedness: Existence, Uniqueness, and Continuous Dependence)**.** *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain and $T > 0$. Under the Assumption 1, there exists a unique weak solution (E,B), according to Definition 1, to the Maxwell system* (1)*. Furthermore, the solution satisfies the stability estimate*

$$\begin{aligned} &\|\partial_t E\|_{L^2(0,T;H_0(\mathrm{curl};\Omega)^*)} + \|\partial_t B\|_{L^2(0,T;H(\mathrm{curl};\Omega)^*)} + \|E\|_{C([0,T];L^2(\Omega)^3)} + \|B\|_{C([0,T];L^2(\Omega)^3)} \\ &\qquad \le C\left(\|f\|_{L^2(0,T;L^2(\Omega)^3)} + \|E_0\|_{L^2(\Omega)^3} + \|B_0\|_{L^2(\Omega)^3}\right), \end{aligned} \tag{12}$$

for some constant C depending only on T, ε, μ.

Proof. Uniqueness of the weak solutions from Definition 1 is due to Theorem 1 and $C([0,T];L^2(\Omega)^3)$ regularity is due to Corollary 1. Next we will establish existence and continuous dependence.

Step 1: Galerkin Approximation. Since $H_0(\mathrm{curl};\Omega)$ and $H(\mathrm{curl};\Omega)$ are separable Hilbert spaces, there exists basis $\{\phi_i\}_{i=1}^{\infty} \subset H(\mathrm{curl};\Omega)$ and $\{\psi_i\}_{i=1}^{\infty} \subset H_0(\mathrm{curl};\Omega)$, which can be made orthonormal in $L^2(\Omega)^3$ (e.g., using Gram–Schmidt). Notice that the resulting vectors (after Gram–Schmidt in L^2) still forms a basis of $H_0(\mathrm{curl};\Omega)$ and $H(\mathrm{curl};\Omega)$.

We use the relation $B = \mu H$. For each $N \in \mathbb{N}$, we define the Galerkin approximations:

$$E_N(x,t) := \sum_{i=1}^{N} \alpha_i^N(t)\,\psi_i(x), \qquad H_N(x,t) := \sum_{i=1}^{N} \beta_i^N(t)\,\phi_i(x).$$

and $B_N = \mu H_N$. We require (E_N, H_N) to satisfy the Galerkin system:

$$\begin{aligned} \langle \varepsilon\,\partial_t E_N, \psi_j\rangle_{H_0(\mathrm{curl};\Omega)} + (\sigma E_N, \psi_j) - (H_N, \mathrm{curl}\,\psi_j) &= (f, \psi_j), \\ \langle \mu \partial_t H_N, \phi_j\rangle_{H(\mathrm{curl};\Omega)} + (\mathrm{curl}\,E_N, \phi_j) &= 0, \end{aligned} \tag{13}$$

for all $j = 1,\dots,N$, with initial data projections:

$$\alpha_j^N(0) := (E_0, \psi_j), \qquad \beta_j^N(0) := (H_0, \phi_j),$$

where $H_0 = \mu^{-1}B_0 \in L^2(\Omega)^3$ because $\mu^{-1} \in L^\infty(\Omega;\mathbb{R}^{3\times 3})$. This leads to a system of ODEs for the coefficients with $j = 1,\dots,N$:

$$\sum_{i=1}^{N} \left[(\varepsilon\psi_i,\psi_j)\dot{\alpha}_i^N(t) + (\sigma\psi_i,\psi_j)\alpha_i^N(t) - (\phi_i, \mathrm{curl}\,\psi_j)\beta_i^N(t)\right] = (f(\cdot,t),\psi_j),$$

$$\sum_{i=1}^{N} \left[(\mu\phi_i,\phi_j)\dot{\beta}_i^N(t) + (\mathrm{curl}\,\psi_i,\phi_j)\alpha_i^N(t)\right] = 0.$$

Define the matrices:

$$[M_E]_{ij} := (\varepsilon\psi_i,\psi_j), \qquad [K_E]_{ij} := (\sigma\psi_i,\psi_j),$$
$$[C]_{ij} := (\phi_i, \mathrm{curl}\,\psi_j), \qquad [M_B]_{ij} := (\mu\phi_i,\phi_j),$$

and vectors:

$$\alpha^N(t) := [\alpha_1^N(t),\dots,\alpha_N^N(t)]^\top, \qquad \beta^N(t) := [\beta_1^N(t),\dots,\beta_N^N(t)]^\top,$$
$$F(t) := [(f(\cdot,t),\psi_1),\dots,(f(\cdot,t),\psi_N)]^\top.$$

The Galerkin system can now be written compactly as:

$$\begin{aligned} M_E\dot{\alpha}^N(t) + K_E\alpha^N(t) - C\beta^N(t) &= F(t), \\ M_B\dot{\beta}^N(t) + C^\top\alpha^N(t) &= 0. \end{aligned} \tag{14}$$

Define the combined unknown vector:

$$y(t) := \begin{bmatrix}\alpha^N(t)\\ \beta^N(t)\end{bmatrix}, \qquad y(0) := \begin{bmatrix}\alpha^N(0)\\ \beta^N(0)\end{bmatrix}.$$

Define the block matrix and forcing:

$$A := \begin{bmatrix}-M_E^{-1}K_E & M_E^{-1}C\\ -M_B^{-1}C^\top & 0\end{bmatrix}, \qquad G(t) := \begin{bmatrix}M_E^{-1}F(t)\\ 0\end{bmatrix}.$$

Then the Galerkin ODE system reads:

$$\frac{d}{dt}y(t) = Ay(t) + G(t), \qquad y(0) = y_0.$$

By the Carathéodory existence theorem of ODEs [10, Theorem 5.2] for systems with $A \in \mathbb{R}^{2N\times 2N}$ constant and $G \in L^2(0,T;\mathbb{R}^{2N})$, we obtain:

$$y \in H^1(0,T;\mathbb{R}^{2N}) \quad \Rightarrow \quad \alpha_i^N, \beta_i^N \in H^1(0,T).$$

Thus,

$$(E_N, H_N) \in H^1(0,T;Y_N) \times H^1(0,T;X_N),$$

where $Y_N := \operatorname{span}\{\psi_1, \dots, \psi_N\} \subset H_0(\operatorname{curl};\Omega)$, and $X_N := \operatorname{span}\{\phi_1, \dots, \phi_N\} \subset H(\operatorname{curl};\Omega)$.

Step 2: Energy Estimate. We now derive a uniform a priori energy estimate for the Galerkin approximations (E_N, B_N). Recall the Galerkin system from (13). Multiply the first equation by $\alpha_j^N(t)$ and sum over $j = 1, \dots, N$. Using the expansion $E_N = \sum_{j=1}^N \alpha_j^N(t)\psi_j$, this yields:

$$\langle \varepsilon\partial_t E_N, E_N\rangle_{H_0(\operatorname{curl};\Omega)} + (\sigma E_N, E_N) - (H_N, \operatorname{curl} E_N) = (f, E_N).$$

Similarly, multiply the second equation by $\beta_j^N(t)$, sum over $j = 1, \dots, N$, and use the expansion $H_N = \sum_{j=1}^N \beta_j^N(t)\phi_j$ to obtain:

$$\langle \mu\partial_t H_N, H_N\rangle_{H(\operatorname{curl};\Omega)} + (\operatorname{curl} E_N, H_N) = 0.$$

Adding the two equations gives the energy identity:

$$(\varepsilon\partial_t E_N, E_N) + (\mu\partial_t H_N, H_N) + (\sigma E_N, E_N) = (f, E_N),$$

where we have used the fact that $\mu\partial_t H_N \in L^2(\Omega)^3$ and $\varepsilon\partial_t B_N \in L^2(\Omega)^3$ therefore the duality $\langle\cdot,\cdot\rangle$ coincides with L^2 pairing $(\cdot,\cdot)$. Using the identity $(u', u) = \frac{1}{2}\frac{d}{dt}\|u\|^2_{L^2(\Omega)}$, we get:

$$\frac{1}{2}\frac{d}{dt}\left(\|\sqrt{\varepsilon}E_N\|^2_{L^2} + \|\sqrt{\mu}H_N\|^2_{L^2}\right) + \|\sqrt{\sigma}E_N\|^2_{L^2} = (f, E_N).$$

Apply the Cauchy–Schwarz and Young inequalities:

$$(\varepsilon^{-1}f, \varepsilon E_N) \le \|\sqrt{\varepsilon^{-1}}f\|_{L^2}\|\sqrt{\varepsilon}E_N\|_{L^2} \le \frac{1}{2\varepsilon_0}\|f\|^2_{L^2} + \frac{1}{2}\|\sqrt{\varepsilon}E_N\|^2_{L^2}.$$

We obtain that

$$\frac{1}{2}\frac{d}{dt}\left(\|\sqrt{\varepsilon}E_N\|^2_{L^2} + \|\sqrt{\mu}H_N\|^2_{L^2}\right) \le C\|f\|^2_{L^2} + \left(\|\sqrt{\varepsilon}E_N\|^2_{L^2} + \|\sqrt{\mu}H_N\|^2_{L^2}\right).$$

Define the energy functional:

$$\mathcal{E}(E,H)(t) := \frac{1}{2}\left(\|\sqrt{\varepsilon}E(t)\|^2_{L^2} + \|\sqrt{\mu}H(t)\|^2_{L^2}\right),$$

for a.e. $t \in [0,T)$. We obtain the differential inequality:

$$\frac{d}{dt}\mathcal{E}(E_N, H_N)(t) \le \mathcal{E}(E_N, H_N)(t) + C\|f(t)\|^2_{L^2},$$

Apply Grönwall's estimate in differential form, we obtain that

$$\mathcal{E}(E_N, H_N)(t) \leq C\left(\mathcal{E}(E_N, H_N)(0) + \int_0^t \|f(s)\|_{L^2}^2\, ds\right).$$

Using the initial data projections:

$$\mathcal{E}(E_N, H_N)(0) = \frac{1}{2}\left(\|\sqrt{\varepsilon}E_N(0)\|_{L^2}^2 + \|\sqrt{\mu}H_N(0)\|_{L^2}^2\right) \leq C\left(\|E_0\|_{L^2}^2 + \|B_0\|_{L^2}^2\right),$$

we obtain the uniform energy bound:

$$\|E_N\|_{L^\infty(0,T;L^2)}^2 + \|H_N\|_{L^\infty(0,T;L^2)}^2 \leq C\left(\|E_0\|_{L^2}^2 + \|B_0\|_{L^2}^2 + \|f\|_{L^2(0,T;L^2)}^2\right), \tag{15}$$

where $C > 0$ depends only on L^∞ bounds for ε, μ^{-1}, and the final time T, but not on N. In particular, the energy estimate implies (up to subsequences)

$$\begin{aligned} E_N &\rightharpoonup E \quad \text{weakly-* in } L^\infty(0,T;L^2(\Omega)^3), \\ E_N &\rightharpoonup E \quad \text{weakly in } L^2(0,T;L^2(\Omega)^3), \\ H_N &\rightharpoonup H \quad \text{weakly-* in } L^\infty(0,T;L^2(\Omega)^3). \end{aligned} \tag{16}$$

Recall that since the norm is convex and continuous, it is therefore is weakly lower-semicontinuous. Then using (16) in (15) we obtain the bound (12), except the time derivative part.

Step 3: Convergence of $\partial_t E_N$, $\partial_t H_N$. Since $H_0(\mathrm{curl};\Omega)$ and $H(\mathrm{curl};\Omega)$ are Hilbert spaces, they admit orthogonal projections (e.g., Riesz projection given in Proposition 3):

$$\begin{aligned} &\Pi_N : H(\mathrm{curl};\Omega) \to X_N := \mathrm{span}\{\phi_i\}_{i=1}^N, && \Pi'_N : H_0(\mathrm{curl};\Omega) \to Y_N := \mathrm{span}\{\psi_i\}_{i=1}^N, \\ &\Pi_N\phi \to \phi \text{ in } H(\mathrm{curl};\Omega) \quad \forall \phi \in H(\mathrm{curl};\Omega), && \Pi'_N\psi \to \psi \text{ in } H_0(\mathrm{curl}) \quad \forall \psi \in H_0(\mathrm{curl};\Omega). \end{aligned} \tag{17}$$

Recall that for a Hilbert space X, we have that $C_c^\infty(0,T) \otimes X$ is dense in $L^2(0,T;X)$ (cf. [5, Corollaire 1.3.1]). Therefore for an arbitrary $\phi \in L^2(0,T;H(\mathrm{curl};\Omega))$, we can write $\phi = w(x)\eta(t)$ with $w \in H(\mathrm{curl};\Omega)$ and $\eta \in C_c^\infty(0,T)$ which are arbitrary.

Consider $(13)_2$, we have that

$$\int_0^T \langle \mu\partial_t H_N, \Pi_N w\rangle_{H(\mathrm{curl};\Omega)}\eta(t)\, dt = -\int_0^T (\mathrm{curl}\, E_N, \Pi_N w)\eta(t)\, dt \quad \forall w \in H(\mathrm{curl};\Omega),\ \forall \eta \in C_c^\infty(0,T).$$

Applying integration-by-parts in-time on the left-hand side and in-space (with vanishing tangential trace for E_N) on the right-hand side, we obtain

$$-\int_0^T (\mu H_N, \Pi_N w)\eta'(t)\, dt = -\int_0^T (E_N, \mathrm{curl}\,\Pi_N w)\eta(t)\, dt \quad \forall w \in H(\mathrm{curl};\Omega),\ \forall \eta \in C_c^\infty(0,T).$$

Now using (16) and (17) we deduce

$$\begin{aligned} -\int_0^T (\mu H, w)\eta'(t)\, dt &= -\int_0^T (E, \mathrm{curl}\, w)\eta(t)\, dt \\ &=: \int_0^T \langle \mathcal{L}(t), w\rangle_{H(\mathrm{curl};\Omega)}\eta(t)\, dt \quad \forall w \in H(\mathrm{curl};\Omega),\ \forall \eta \in C_c^\infty(0,T). \end{aligned}$$

We have that for a.e., $t \in (0,T)$

$$|\langle \mathcal{L}(t), w\rangle_{H(\mathrm{curl};\Omega)}| = |(E, \mathrm{curl}\, w)| \le C\|E(t)\|_{L^2(\Omega)^3}\|w\|_{H(\mathrm{curl};\Omega)}.$$

Since $E \in L^2(0,T;L^2(\Omega)^3)$, therefore $\mathcal{L}(t) : H(\mathrm{curl};\Omega) \to H(\mathrm{curl};\Omega)^*$ is bounded and linear and $\|\mathcal{L}(t)\|_{H(\mathrm{curl};\Omega)^*} \in L^2(0,T)$. Thus we have that

$$\int_0^T \langle \mathcal{L}(t), w\rangle_{H(\mathrm{curl};\Omega)}\eta(t)\,dt = -\int_0^T (\mu H, w)\eta'(t)\,dt \quad \forall w \in H(\mathrm{curl};\Omega)\,, \forall \eta \in C_c^\infty(0,T).$$

Since $H(\mathrm{curl};\Omega)$ is dense in $L^2(\Omega)^3$, using [22, Section 7.2], we can identify $(\mu H, w)$ as $\langle \mu H, w\rangle_{H(\mathrm{curl};\Omega)}$ a.e. in $t \in (0,T)$. Then from the definition of weak derivative, we deduce that

$$\mathcal{L}(t) = \mu\partial_t H \in L^2(0,T;H(\mathrm{curl};\Omega)^*)$$

This proves that for a.e. t

$$\langle \mu\partial_t H, \phi\rangle_{H(\mathrm{curl};\Omega)} + (E, \mathrm{curl}\,\phi) = 0, \quad \forall \phi \in H(\mathrm{curl};\Omega).$$

A similar argument using $C_c^\infty(0,T) \otimes H_0(\mathrm{curl};\Omega)$ shows that

$$\partial_t E \in L^2(0,T;H_0(\mathrm{curl};\Omega)^*),$$

and the first equation of (2) holds.

Moreover, the following a priori bounds hold:

$$\begin{aligned}
\|\mu\partial_t H\|_{L^2(0,T;H(\mathrm{curl};\Omega)^*)} &\le C\left(\|f\|_{L^2(0,T;L^2(\Omega)^3)} + \|E_0\|_{L^2(\Omega)^3} + \|B_0\|_{L^2(\Omega)^3}\right), \\
\|\partial_t E\|_{L^2(0,T;H_0(\mathrm{curl};\Omega)^*)} &\le C\left(\|f\|_{L^2(0,T;L^2(\Omega)^3)} + \|E_0\|_{L^2(\Omega)^3} + \|B_0\|_{L^2(\Omega)^3}\right).
\end{aligned}$$

Step 4: Convergence of Initial Conditions. Since $\{\phi_i\}_{i=1}^\infty$ is an orthonormal basis of $L^2(\Omega)^3$, the projection

$$H_N(0) := \sum_{i=1}^N (H_0, \phi_i)\,\phi_i \to H_0 \quad \text{in } L^2(\Omega)^3 \quad \text{as } N \to \infty.$$

Since $B_N = \mu H_N$, we immediately get that $B_N(0) = \mu H_N(0) \to B_0 = \mu H_0$ in $L^2(\Omega)^3$. Similarly, we can argue for E_N. All the above regularity results directly transfer from H to B. □

In view of Theorem 2, and according to Definition 1, it then remains to show that $\mathrm{div}\, B = 0$ in $L^2(\Omega)$ and a.e. $t \in [0,T)$.

Proposition 1 (Divergence-Free Evolution)**.** *Let (E,B) be weak solution according to Definition 1. Assume further that the initial magnetic field satisfies $\mathrm{div}\, B_0 = 0$ in $L^2(\Omega)$. Then it follows that*

$$\mathrm{div}\, B(\cdot,t) = 0 \qquad \text{in } \mathcal{D}'(\Omega) \text{ for all } t \in [0,T),$$

and thus

$$B \in L^\infty(0,T;H(\mathrm{div}^0;\Omega)).$$

Proof. Let $\psi \in C_c^\infty(\Omega)$. We define the divergence pairing via duality as

$$\langle \operatorname{div} B(\cdot,t), \psi\rangle := -\langle B(\cdot,t), \nabla\psi\rangle.$$

Since $B \in H^1(0,T; H(\text{curl};\Omega)^*)$, the mapping $t \mapsto \langle B(\cdot,t), \nabla\psi\rangle \in H^1(0,T)$ according to Lemma 2. Thus, we may compute its time derivative:

$$\frac{d}{dt}\langle \operatorname{div} B(\cdot,t), \psi\rangle = -\frac{d}{dt}\langle B(\cdot,t), \nabla\psi\rangle = -\langle \partial_t B(\cdot,t), \nabla\psi\rangle.$$

But $\nabla\psi \in H(\text{curl};\Omega)$, and we have the distributional identity $\text{curl}(\nabla\psi) = 0$. Therefore, by the second equation in (2), we obtain

$$\langle \partial_t B(\cdot,t), \nabla\psi\rangle = -(E(\cdot,t), \operatorname{curl}\nabla\psi) = 0.$$

Hence,

$$\frac{d}{dt}\langle \operatorname{div} B(\cdot,t), \psi\rangle = 0 \quad \forall\psi \in C_c^\infty(\Omega).$$

This implies that the map $t \mapsto \langle \operatorname{div} B(\cdot,t), \psi\rangle$ is constant. Since $\operatorname{div} B_0 = 0$ in $L^2(\Omega)$, so $\operatorname{div} B_0 = 0$ in $\mathcal{D}'(\Omega)$, we have

$$\langle \operatorname{div} B(\cdot,t), \psi\rangle = \langle \operatorname{div} B_0, \psi\rangle = 0 \quad \forall\psi \in C_c^\infty(\Omega),\ \text{a.e. } t \in [0,T).$$

Thus $\operatorname{div} B(\cdot,t) = 0$ in the distributional sense, a.e. $t \in [0,T)$. Next, we notice that

$$\langle \operatorname{div} B(\cdot,t), \psi\rangle = -(B(\cdot,t), \nabla\psi) = 0 = (0,\psi), \quad \forall\psi \in C_c^\infty(\Omega),\ \text{a.e. } t \in [0,T)$$

which implies that $\operatorname{div} B(t) = 0$ in $L^\infty(0,T; L^2(\Omega))$ and the proof is complete. □

4 Nédélec/RT spaces and (discrete) de Rham sequence

Having shown the well-posedness of the continuous problem, we now turn our attention to the semi-discrete (in-space) approximation of the weak form (2). We will also establish that the solution to the semi-discrete problem converges to the solution to the continuous problem. Notice that due to the continuous time nature, these results are agnostic to any particular time discretization. From hereon we will assume that the Ω is Lipschitz, simply connected, with connected boundary. Though this is only used in Lemma 7 and all other results are true for Lipschitz domains.

Let $\{\mathcal{T}_h\}_h$ be a shape-regular tetrahedral family of meshes of Ω. Fix an order $k \geq 0$.

- The *Nédélec* space conforming to $H(\text{curl})$ space is given by:

$$\mathcal{N}_h := \{\, v_h \in H(\text{curl};\Omega) : v_h|_K \in \mathcal{N}_k(K)\ \forall K \in \mathcal{T}_h \,\},$$

  and $\mathcal{N}_h^0 := \mathcal{N}_h \cap H_0(\text{curl};\Omega)$.

- The *Raviart–Thomas* space (conforming to $H(\text{div})$):

$$\mathcal{RT}_h := \{\, w_h \in H(\text{div};\Omega) : w_h|_K \in \mathcal{RT}_k(K)\ \forall K \,\}.$$

- Piecewise polynomial scalars $\mathcal{Q}_h := \{q_h \in L^2(\Omega) : q_h|_K \in \mathbb{P}_k(K)\}$.

Discrete de Rham sequence (exactness). There exist commuting diagrams and the exact sequence, on simply connected domains with connected boundary:

$$H_0^1(\Omega) \xrightarrow{\nabla} H_0(\mathrm{curl};\Omega) \xrightarrow{\mathrm{curl}} H(\mathrm{div};\Omega) \xrightarrow{\mathrm{div}} L^2(\Omega) \to 0,$$

and discretely

$$\mathcal{P}_h \xrightarrow{\nabla} \mathcal{N}_h^0 \xrightarrow{\mathrm{curl}} \mathcal{RT}_h \xrightarrow{\mathrm{div}} \mathcal{Q}_h \to 0,$$

where $\mathcal{P}_h$ are conforming Lagrange spaces. In particular,

$$\mathrm{curl}\,\mathcal{N}_h^0 \subset \mathcal{RT}_h, \qquad \mathrm{div}(\mathrm{curl}\,\mathcal{N}_h^0) = 0.$$

Global L^2 projector onto $\mathcal{RT}_h$. Let $P_h : L^2(\Omega)^3 \to \mathcal{RT}_h$ be the L^2-orthogonal projector:

$$(u - P_h u, v_h) = 0 \quad \forall v_h \in \mathcal{RT}_h. \tag{18}$$

P_h is linear, idempotent, $\|P_h\|_{L^2\to L^2} = 1$, and $P_h\Phi \to \Phi$ in L^2 for all $\Phi \in L^2(\Omega)^3$. See Proposition 2 for a proof.

Riesz projection on $H_0(\mathrm{curl};\Omega)$. Define, for each $u \in H_0(\mathrm{curl};\Omega)$, the Riesz projection $R_h u \in \mathcal{N}_h^0$ by

$$a(R_h u, v_h) \;=\; a(u, v_h) \qquad \forall\, v_h \in \mathcal{N}_h^0, \tag{19}$$

where $a(u,v) := (u,v) + (\mathrm{curl}\,u, \mathrm{curl}\,v)$. Such $R_h u \in \mathcal{N}_h^0$ is unique for each u and it fulfills the Galerkin orthogonality $a(u - R_h u, v_h) = 0$ for all $v_h \in \mathcal{N}_h^0$, stability $\|R_h u\|_{H(\mathrm{curl};\Omega)} \le \|u\|_{H(\mathrm{curl};\Omega)}$, and $\|u - R_h u\|_{H_0(\mathrm{curl};\Omega)} \to 0$ as $h \to 0$. See Proposition 3 for a proof.

4.1 Semi-discrete FE scheme

Discrete unknowns/test spaces. We approximate

$$E_h(t) \in \mathcal{N}_h^0, \qquad B_h(t) \in \mathcal{RT}_h,$$

and for all times test Ampère with $\psi_h \in \mathcal{N}_h^0$ and Faraday with $\phi_h \in \mathcal{RT}_h$.

Semi-discrete in space scheme. Find $(E_h(t), B_h(t)) \in \mathcal{N}_h^0 \times \mathcal{RT}_h$ such that for a.e. $t \in (0,T)$

$$(\varepsilon\partial_t E_h(t), \psi_h) + (\sigma E_h(t), \psi_h) - (\mu^{-1} B_h(t), \mathrm{curl}\,\psi_h) = (f(t), \psi_h) \qquad \forall \psi_h \in \mathcal{N}_h^0, \tag{20a}$$

$$(\partial_t B_h(t), \phi_h) + (\mathrm{curl}\,E_h(t), \phi_h) = 0 \qquad \forall \phi_h \in \mathcal{RT}_h. \tag{20b}$$

Initial data are chosen as

$$E_h(0) = Q_h^N E_0 \in \mathcal{N}_h^0, \qquad B_h(0) \in \mathcal{RT}_h \quad \text{with} \quad (\mathrm{div}\,B_h(0), q_h) = 0\ \forall q_h \in \mathcal{Q}_h,$$

Here, $Q_h^N : L^2(\Omega)^3 \to \mathcal{N}_h^0$ is the L^2-orthogonal projector with $\|E_h(0) - E_0\|_{L^2(\Omega)^3} \xrightarrow[h\to 0]{} 0$ and $\|B_h(0) - B_0\|_{L^2(\Omega)^3} \xrightarrow[h\to 0]{} 0$. An example of such approximation of $B_h(0)$ is the L^2-orthogonal projection of B_0 onto the discrete divergence-free subspace $Z_h := \{v_h \in \mathcal{RT}_h :\ \mathrm{div}\,v_h = 0 \text{ in } \mathcal{Q}_h\}$. See Proposition 4.

Remark 1 (Well-posedness of (20))**.** *Fixing h, (20) is a linear ODE in finite dimensions with a positive definite (block) mass matrix; thus it has a unique solution with $E_h \in H^1(0,T;\mathcal{N}_h^0)$, $B_h \in H^1(0,T;\mathcal{RT}_h)$.*

Lemma 3 (Discrete Gauss law preservation)**.** *Let* (E_h, B_h) *solve the semi-discrete* (20b)*. Assume the compatible Nédélec/RT pair so that* $\operatorname{curl}\mathcal{N}_h^0 \subset \mathcal{RT}_h$ *and* $\operatorname{div} : \mathcal{RT}_h \to \mathcal{Q}_h$ *is the usual (onto) divergence operator. Then, for every* $q_h \in \mathcal{Q}_h$*,*

$$(\operatorname{div} B_h(t), q_h) = (\operatorname{div} B_h(0), q_h) \qquad \textit{for a.e. } t \in [0, T).$$

In particular, if $(\operatorname{div} B_h(0), q_h) = 0$ *for all* $q_h \in \mathcal{Q}_h$*, then* $(\operatorname{div} B_h(t), q_h) = 0$ *for all* $q_h \in \mathcal{Q}_h$ *and a.e.* t*.*

Proof. Step 1: Define the residual and show it vanishes. For a.e. t, define

$$r_h(t) := \partial_t B_h(t) + \operatorname{curl} E_h(t).$$

By construction, $B_h(t) \in \mathcal{RT}_h$ for all t and the coefficient vector of B_h is absolutely continuous in time. Indeed, by well-posedness of the semi-discrete system we have $B_h \in H^1(0, T; \mathcal{RT}_h)$, for example, write $B_h(t) = \sum_{j=1}^{N_B} b_j(t)\,\rho_j$ with $b_j \in H^1(0, T)$ and $\{\rho_j\}$ a basis of $\mathcal{RT}_h$. Hence $\partial_t B_h(t) \in \mathcal{RT}_h$ for a.e. t. By discrete exactness, $\operatorname{curl} E_h(t) \in \mathcal{RT}_h$. Therefore $r_h(t) \in \mathcal{RT}_h$ for a.e. t.

From (20b) we have

$$(r_h(t), \phi_h) = 0 \qquad \forall\, \phi_h \in \mathcal{RT}_h.$$

Thus $r_h(t) \in \mathcal{RT}_h^{\perp}$ (orthogonal complement in $L^2(\Omega)^3$). Since also $r_h(t) \in \mathcal{RT}_h$, we have

$$r_h(t) \in \mathcal{RT}_h \cap \mathcal{RT}_h^{\perp} = \{0\}.$$

Hence

$$\partial_t B_h(t) + \operatorname{curl} E_h(t) = 0 \quad \text{in } L^2(\Omega)^3, \text{ for a.e. } t.$$

Step 2: Apply $\operatorname{div} : \mathcal{RT}_h \to \mathcal{Q}_h$. Because div maps $\mathcal{RT}_h$ into $\mathcal{Q}_h$ and is linear/continuous on $\mathcal{RT}_h$, we may apply div to the identity above (in the sense of $\mathcal{Q}_h$):

$$\partial_t(\operatorname{div} B_h(t)) + \operatorname{div}(\operatorname{curl} E_h(t)) \;=\; 0 \quad \text{in } \mathcal{Q}_h, \text{ for a.e. } t.$$

But $\operatorname{div}(\operatorname{curl} E_h(t)) \equiv 0$ elementwise (and hence in $\mathcal{Q}_h$). Therefore

$$\partial_t(\operatorname{div} B_h(t)) = 0 \quad \text{in } \mathcal{Q}_h, \text{ for a.e. } t.$$

Step 3: Pair with arbitrary $q_h \in \mathcal{Q}_h$. For any fixed $q_h \in \mathcal{Q}_h$, the scalar function

$$g_{q_h}(t) := (\operatorname{div} B_h(t), q_h)$$

is absolutely continuous in t. Indeed, as in Step 1, above we have $B_h \in H^1(0, T; \mathcal{RT}_h)$. Since $\operatorname{div} : \mathcal{RT}_h \to \mathcal{Q}_h$ is bounded linear and $\mathcal{RT}_h$ is finite dimensional, write $B_h(t) = \sum_{j=1}^{N_B} b_j(t)\,\rho_j$ with $b_j \in H^1(0, T)$ and $\{\rho_j\}$ a basis of $\mathcal{RT}_h$. Then

$$\operatorname{div} B_h(t) = \sum_{j=1}^{N_B} b_j(t)\ \operatorname{div} \rho_j \in \mathcal{Q}_h, \qquad \partial_t(\operatorname{div} B_h)(t) = \sum_{j=1}^{N_B} b_j'(t)\ \operatorname{div} \rho_j = \operatorname{div}\big(\partial_t B_h(t)\big)$$

for a.e. $t \in (0, T)$. Hence $\operatorname{div} B_h \in H^1(0, T; \mathcal{Q}_h)$ and $\partial_t(\operatorname{div} B_h) = \operatorname{div}(\partial_t B_h)$ a.e. on $(0, T)$. It then follows that

$$g_{q_h}(t) = (\operatorname{div} B_h(t), q_h) \in H^1(0, T) \subset \mathrm{AC}([0, T]),$$

with, for a.e. $t \in (0,T)$,

$$g'_{q_h}(t) = (\partial_t(\operatorname{div} B_h(t)), q_h) = (\operatorname{div}(\partial_t B_h(t)), q_h).$$

Using the identity $\partial_t B_h + \operatorname{curl} E_h = 0$ in $\mathcal{RT}_h$ and $\operatorname{div} \circ \operatorname{curl} \equiv 0$, we conclude $g'_{q_h}(t) = 0$ a.e. on $[0,T)$. Hence

$$(\operatorname{div} B_h(t), q_h) = (\operatorname{div} B_h(0), q_h) \qquad \forall\, q_h \in \mathcal{Q}_h, \text{ for a.e. } t \in [0,T).$$

This completes the proof. $\square$

Lemma 4 (Discrete energy identity and bounds)**.** *Define the discrete energy*

$$\mathcal{E}_h(t) := \tfrac{1}{2}\left(\|\sqrt{\varepsilon}E_h(t)\|_{L^2}^2 + \|\sqrt{\mu^{-1}}B_h(t)\|_{L^2}^2\right).$$

Then for a.e. t,

$$\frac{d}{dt}\mathcal{E}_h(t) + \|\sqrt{\sigma}E_h(t)\|_{L^2}^2 = (f(t), E_h(t)).$$

Consequently, for any $T > 0$,

$$\sup_{0\le t\le T} \mathcal{E}_h(t) \;\le\; e^T\mathcal{E}_h(0) + \frac{e^T}{2\varepsilon_{\min}}\int_0^T \|f(s)\|_{L^2}^2\, ds,$$

and hence

$$\|E_h\|_{L^\infty(0,T;L^2(\Omega)^3)} + \|B_h\|_{L^\infty(0,T;L^2(\Omega)^3)} \le C_T\left(\|f\|_{L^2(0,T;L^2(\Omega)^3)} + \|E_0\|_{L^2(\Omega)^3} + \|B_0\|_{L^2(\Omega)}\right),$$

with C_T independent of h.

Proof. Set $\psi_h = E_h$ in (20a) and $\phi_h = P_h(\mu^{-1}B_h)$ (recall that $P_h : L^2(\Omega)^3 \to \mathcal{RT}_h$, see (18)) in (20b), we obtain that

$$(\varepsilon\partial_t E_h, E_h) + (\sigma E_h, E_h) - (\mu^{-1}B_h, \operatorname{curl} E_h) = (f, E_h), \tag{21a}$$
$$(\partial_t B_h, P_h(\mu^{-1}B_h)) + (\operatorname{curl} E_h, P_h(\mu^{-1}B_h)) = 0. \tag{21b}$$

Notice that $\partial_t B_h \in \mathcal{RT}_h$ and $\operatorname{curl} E_h \in \mathcal{RT}_h$, therefore (21b) is equivalent to

$$(\partial_t B_h, \mu^{-1}B_h) + (\operatorname{curl} E_h, \mu^{-1}B_h) = 0.$$

Adding the resulting Faraday and Ampére's laws, we obtain that

$$(\varepsilon\partial_t E_h, E_h) + (\sigma E_h, E_h) + (\partial_t B_h, \mu^{-1}B_h) = (f, E_h).$$

Recognize time derivatives of the quadratic terms to obtain the identity. Apply Young's inequality: $(f, E_h) \le \frac{1}{2\varepsilon_0}\|f\|^2 + \frac{1}{2}(\varepsilon E_h, E_h)$, drop the nonnegative damping term, and apply Grönwall to the inequality $\mathcal{E}'_h(t) \le \mathcal{E}_h(t) + \frac{1}{2\varepsilon_0}\|f\|^2_{L^2(0,T;L^2(\Omega)}$ to obtain the result. $\square$

The above result implies that (up to subsequences)

$$\begin{aligned} E_h &\rightharpoonup E \quad \text{weakly-* in } L^\infty(0,T;L^2(\Omega)^3),\\ B_h &\rightharpoonup B \quad \text{weakly-* in } L^\infty(0,T;L^2(\Omega)^3). \end{aligned} \tag{22}$$

These limits will be sufficient to pass through the limit in (20), except the time-derivative terms. In the next result, we use these limits to show the convergence of $\operatorname{div} B_h$. Subsequently, we will analyze the convergence of $\partial_t E_h$ and $\partial_t B_h$.

Lemma 5 (Strong convergence of $\operatorname{div} B_h$ under exact discrete solenoidality)**.** *Let the assumptions of Lemma 3 hold. If moreover* $(\operatorname{div} B_h(0), q_h) = 0$ *for all* $q_h \in \mathcal{Q}_h$*, then*

$$\operatorname{div} B_h \equiv 0 \quad \text{in } L^\infty(0,T;L^2(\Omega)).$$

In particular, $\operatorname{div} B_h \to 0$ *strongly in* $L^\infty(0,T;L^2(\Omega))$*. Furthermore, if* $B_h \rightharpoonup B$ *weak-* in* $L^\infty(0,T;L^2(\Omega)^3)$*, then*

$$\operatorname{div} B_h \rightharpoonup \operatorname{div} B \quad \text{in } L^\infty(0,T;L^2(\Omega)),$$

and by uniqueness of the weak limit we conclude $\operatorname{div} B = 0$ *in* $L^\infty(0,T;L^2(\Omega))$*.*

Proof. For each t, since $B_h(t) \in \mathcal{RT}_h$ we have $\operatorname{div} B_h(t) \in \mathcal{Q}_h$. By the discrete Gauss law and the divergence-free initialization (cf. Lemma 3),

$$(\operatorname{div} B_h(t), q_h) = 0 \qquad \forall\, q_h \in \mathcal{Q}_h.$$

Choosing $q_h = \operatorname{div} B_h(t)$ (admissible because $\operatorname{div} B_h(t) \in \mathcal{Q}_h$) yields $\|\operatorname{div} B_h(t)\|^2_{L^2(\Omega)} = 0$, i.e. $\operatorname{div} B_h(t) \equiv 0$ in $L^2(\Omega)$ for a.e. t. Hence $\|\operatorname{div} B_h\|_{L^\infty(0,T;L^2(\Omega))} = 0$, so $\operatorname{div} B_h \to 0$ strongly in $L^\infty(0,T;L^2(\Omega))$.

For the identification of the limit, we let $\varphi \in L^1(0,T;C_c^\infty(\Omega))$ and recall that $B_h \rightharpoonup B$ weak-* in $L^\infty(0,T;L^2(\Omega)^3)$, then we have that

$$\int_0^T (\operatorname{div} B_h, \varphi)dt = -\int_0^T (B_h, \nabla\varphi)dt \to -\int_0^T (B, \nabla\varphi)dt = 0$$

where in the last equality we have use the uniqueness of limit. From the definition of weak-derivative the required result follows. □

Lemma 6 (Convergence of $E_h(0)$ in L^2)**.** *Let* $E_0 \in L^2(\Omega)^3$*. With* $E_h(0) = Q_h^{\mathrm{N}} E_0$*,*

$$\|E_h(0) - E_0\|_{L^2(\Omega)^3} \xrightarrow[h\to 0]{} 0.$$

Proof. Since Q_h^{N} is the L^2-orthogonal projector,

$$\|E_0 - Q_h^{\mathrm{N}} E_0\|_{L^2(\Omega)^3} = \min_{v_h \in \mathcal{N}_h^0} \|E_0 - v_h\|_{L^2(\Omega)^3}.$$

The union $\bigcup_h \mathcal{N}_h^0$ is dense in $L^2(\Omega)^3$, hence the best-approximation error vanishes as $h \to 0$. Thus $\|E_h(0) - E_0\|_{L^2(\Omega)^3} \to 0$. □

Lemma 7 (Convergence of $B_h(0)$ in L^2)**.** *Let* $B_0 \in L^2(\Omega)^3$ *with* $\operatorname{div} B_0 = 0$ *in* $L^2(\Omega)$*. Assume* Ω *is simply connected with connected boundary so that there exists* $A_0 \in H_0(\mathrm{curl};\Omega)$ *with* $\operatorname{curl} A_0 = B_0$*.*
(i) Potential-based initialization. *If* $B_h(0) := \operatorname{curl}(R_h A_0)$*, then*

$$\|B_h(0) - B_0\|_{L^2(\Omega)^3} \;\le\; \|R_h A_0 - A_0\|_{H(\mathrm{curl};\Omega)} \xrightarrow[h\to 0]{} 0.$$

(ii) Constrained L^2-projection. *Let* $Z_h := \{v_h \in \mathcal{RT}_h^0 : \operatorname{div} v_h = 0 \text{ in } \mathcal{Q}_h\}$ *and define* $B_h(0) \in Z_h$ *as the* L^2*-orthogonal projection of* B_0 *onto* Z_h*. Then*

$$\|B_h(0) - B_0\|_{L^2(\Omega)^3} \xrightarrow[h\to 0]{} 0.$$

Proof. (i) By definition of $B_h(0)$ and $\operatorname{curl} A_0 = B_0$,

$$\|B_h(0) - B_0\|_{L^2(\Omega)^3} = \|\operatorname{curl}(R_h A_0 - A_0)\|_{L^2(\Omega)^3} \le \|R_h A_0 - A_0\|_{H(\mathrm{curl};\Omega)} \to 0,$$

using the $H(\mathrm{curl})$-convergence of R_h.

(ii) Since $B_h(0)$ is the L^2-orthogonal projection onto Z_h,

$$\|B_0 - B_h(0)\|_{L^2(\Omega)^3} = \min_{z_h \in Z_h} \|B_0 - z_h\|_{L^2(\Omega)^3} \le \inf_{w_h \in \operatorname{curl} \mathcal{N}_h^0} \|B_0 - w_h\|_{L^2(\Omega)^3}.$$

With $B_0 = \operatorname{curl} A_0$ and $w_h = \operatorname{curl}(R_h A_0) \in \operatorname{curl} \mathcal{N}_h^0 \subset Z_h$, the right-hand side tends to 0 by part (i). The proof is complete. □

Theorem 3 (Convergence)**.** *Let $E_h \in H^1(0,T;\mathcal{N}_h^0)$, $B_h \in H^1(0,T;\mathcal{RT}_h)$ solve* (20)*. Then as $h \to 0$, we have* (22) *where E, B satisfy Definition 1 (in particular* (2)*) with $E(0) = E_0$, $B(0) = B_0$, and* $\operatorname{div} B(t) = 0$ *in* $L^2(\Omega)$ *a.e.* $t \in (0,T)$.

Proof. Let $\Phi \in H(\mathrm{curl};\Omega)$ be arbitrary. Define $\phi_h = P_h\Phi$ where P_h as is as in (18), then

$$\langle \partial_t B_h, \phi_h \rangle = -(\operatorname{curl} E_h, P_h\Phi) = -(\operatorname{curl} E_h, \Phi) = -(E_h, \operatorname{curl}\Phi),$$

where in the second equality we have used that definition of P_h and the fact that $\operatorname{curl} E_h \in \mathcal{RT}_h$ from the de Rham sequence. Moreover, in the last equality we have applied the integration-by-parts and have used the boundary conditions on E_h. Next, test the equation with $\eta \in C_c^\infty(0,T)$, after integration-by-parts in time, we obtain that

$$-\int_0^T (B_h, \phi_h)\eta'(t)\,dt = -\int_0^T (E_h, \mathrm{curl}\Phi)\,\mathrm{dt}, \quad \forall \Phi \in \mathrm{H(curl;\Omega)}$$

Now we can take limit as $h \to 0$ and use that $\phi_h \to \Phi$ in L^2 (cf. Proposition 2) together with (22) to obtain that, for all $\Phi \in H(\mathrm{curl};\Omega)$

$$-\int_0^T (B, \Phi)\eta'(t)\,dt = -\int_0^T (E, \mathrm{curl}\Phi)\,\mathrm{dt} =: \int_0^{\mathrm{T}} \langle \mathrm{L(t)}, \Phi\rangle_{\mathrm{H(curl;\Omega)}}\eta(\mathrm{t})\,\mathrm{dt}.$$

It is straight forward to see that $L(t) : H(\mathrm{curl};\Omega) \to H(\mathrm{curl};\Omega)^*$ is a bounded linear operator and $\|L(t)\|_{H(\mathrm{curl};\Omega)^*} \in L^2(0,T)$. Moreover, since $H(\mathrm{curl};\Omega)$ is dense in L^2, therefore we can treat (B,Φ) as the duality pairing on $H(\mathrm{curl};\Omega)$. Then from the definition of weak derivative, we have that

$$\mathrm{L}(t) = \partial_t B \in L^2(0,T;H(\mathrm{curl};\Omega)^*).$$

Namely, the first equation in (2) holds.

Next, we show that the second equation also holds. From (20a), with $\psi_h = R_h\Phi$, where R_h is the Riesz projection (cf. 19) and $\Phi \in H_0(\mathrm{curl};\Omega)$, $\eta \in C_c^\infty(0,T)$

$$\int_0^T \langle \varepsilon\partial_t E_h, R_h\Phi\rangle\eta(t)dt = \int_0^T \big(-(\sigma E_h, R_h\Phi) + (\mu^{-1}B_h, \operatorname{curl} R_h\Phi) + (f, R_h\Phi)\big)\,\eta(t)\,dt.$$

Applying, integration-by-parts on the left-hand-side, we obtain that

$$-\int_0^T (\varepsilon E_h, R_h\Phi)\eta'(t)dt = \int_0^T \big(-(\sigma E_h, R_h\Phi) + (\mu^{-1}B_h, \operatorname{curl} R_h\Phi) + (f, R_h\Phi)\big)\,\eta(t)\,dt.$$

Taking the limits from (22) and Proposition 3, we arrive at

$$-\int_0^T (\varepsilon E, \Phi)\eta'(t)dt = \int_0^T \left(-(\sigma E, \Phi) + (\mu^{-1}B, \operatorname{curl}\Phi) + (f, \Phi)\right)\eta(t)\,dt$$
$$=: \int_0^T \langle \mathcal{L}(t), \Phi\rangle_{H_0(\mathrm{curl};\Omega)}.$$

It is straight-forward to show that for a.e., $t \in (0,T)$, $\mathcal{L} : H_0(\mathrm{curl};\Omega) \to H_0(\mathrm{curl};\Omega)^*$ is bounded and linear and $\|\mathcal{L}\|_{H_0(\mathrm{curl};\Omega)^*} \in L^2(0,T)$. Thus we have

$$\int_0^T \langle \mathcal{L}(t), \Phi\rangle_{H_0(\mathrm{curl};\Omega)}\eta(t)\,dt = -\int_0^T (E, \Phi)\eta'(t)\,dt \quad \forall \Phi \in H_0(\mathrm{curl};\Omega)\,, \forall \eta \in C_c^\infty(0,T).$$

Since $H_0(\mathrm{curl};\Omega)$ is dense in $L^2(\Omega)$, using [22, Section 7.2] we can identify (E, Φ) as $\langle E, \Phi\rangle_{H_0(\mathrm{curl};\Omega)}$ a.e. in $t \in (0,T)$. Then from the definition of weak derivative, we deduce that

$$\mathcal{L}(t) = \partial_t E \in L^2(0,T;H_0(\mathrm{curl};\Omega)^*)$$

and the second equation in (2) holds. Notice that uniqueness to (2) implies that the entire sequence converges. Finally, the convergence of $\operatorname{div} B_h$ was shown in Lemma 5. □

A Appendix

Proposition 2 (Global L^2-orthogonal projection onto $\mathcal{RT}_h$ and its convergence)**.** *Let $\mathcal{RT}_h$ denote the Raviart–Thomas space of order k on $\mathcal{T}_h$, conforming in $H(\mathrm{div};\Omega)$.*
(a) Construction (global L^2-orthogonal projector). *For $u \in L^2(\Omega)^3$, define $P_h u \in \mathcal{RT}_h$ by*

$$(u - P_h u,\, v_h)_{L^2(\Omega)} = 0 \qquad \forall\, v_h \in \mathcal{RT}_h. \tag{23}$$

Equivalently, if $\{\rho_i\}_{i=1}^M$ is any basis of $\mathcal{RT}_h$, let $M_{ij} := (\rho_j, \rho_i)$ and $b_i := (u, \rho_i)$ and solve $Mc = b$; then $P_h u := \sum_{j=1}^M c_j\, \rho_j$.
Then $P_h : L^2(\Omega)^3 \to \mathcal{RT}_h$ is a well-defined linear projector with

$$P_h^2 = P_h, \qquad \mathrm{Range}(P_h) = \mathcal{RT}_h, \qquad \|P_h u\|_{L^2} \le \|u\|_{L^2} \quad \text{and} \quad \|u - P_h u\|_{L^2} = \min_{v_h \in \mathcal{RT}_h} \|u - v_h\|_{L^2}.$$

(b) Convergence for L^2-data. *For every $\Phi \in L^2(\Omega)^3$,*

$$\|\, \Phi - P_h \Phi \,\|_{L^2(\Omega)} \xrightarrow[h\to 0]{} 0. \tag{24}$$

Proof. (a) Pick any basis $\{\rho_i\}$ of $\mathcal{RT}_h$. The matrix $M = [(\rho_j, \rho_i)]$ is symmetric positive definite (SPD) because $(\cdot,\cdot)$ is an inner product and the basis is linearly independent, hence the linear system $Mc = b$ has a unique solution c. This defines a linear map P_h. From (23), using Pythagoras' theorem gives, for every $v_h \in \mathcal{RT}_h$,

$$\|u - v_h\|_{L^2}^2 = \|u - P_h u\|_{L^2}^2 + \|P_h u - v_h\|_{L^2}^2.$$

Taking the infimum over v_h yields the best-approximation identity. Choosing $v_h = 0$ gives the contractivity $\|P_h u\|_{L^2} \le \|u\|_{L^2}$.

To prove the range, we first notice that by definition $P_h u \in \mathcal{RT}_h$ for every $u \in L^2$. Hence $Range(P_h) \subset \mathcal{RT}_h$. Next let $v_h \in \mathcal{RT}_h$. Applying orthogonality condition with $u = v_h$, we obtain that

$$(v_h - P_h v_h, w_h) = 0 \quad \forall w_h \in \mathcal{RT}_h.$$

Since $v_h - P_h v_h \in \mathcal{RT}_h$, choose $w_h = v_h - P_h v_h$ to get

$$\|v_h - P_h v_h\|_{L^2}^2 = 0 \quad \Longrightarrow \quad P_h v_h = v_h.$$

Thus every $v_h \in \mathcal{RT}_h$ is in the range of P_h, so $\mathcal{RT}_h \subset Range(P_h)$. Finally, the idempotence follows immediately.

(b) We prove density of $\bigcup_h \mathcal{RT}_h$ in $L^2(\Omega)^3$, which combined with best approximation gives (24). Let $\varepsilon > 0$ and choose $\phi \in C_c^\infty(\Omega)^3$ with $\|\Phi - \phi\|_{L^2} < \varepsilon$ (density of C_c^∞ in L^2). For $\phi \in H^1(\Omega)^3$ there exist standard Raviart–Thomas interpolants (or smoothed quasi-interpolants, [1] and [7, Thms. 16.4, 16.6]) $I_h^{\mathrm{RT}}\phi \in \mathcal{RT}_h$ such that

$$\|\phi - I_h^{\mathrm{RT}}\phi\|_{L^2(\Omega)} \le C\, h\, \|\phi\|_{H^1(\Omega)}. \tag{25}$$

Using the optimality of P_h and stability $\|P_h\| \le 1$,

$$\|\Phi - P_h\Phi\|_{L^2} \le \|\Phi - \phi\|_{L^2} + \|\phi - P_h\phi\|_{L^2} + \|P_h(\phi - \Phi)\|_{L^2} \le 2\varepsilon + \|\phi - I_h^{\mathrm{RT}}\phi\|_{L^2}.$$

By (25), the last term tends to 0 as $h \to 0$. Since $\varepsilon > 0$ is arbitrary, $\|\Phi - P_h\Phi\|_{L^2} \to 0$. This proves (24). □

Proposition 3 (Riesz projection on $H_0(\mathrm{curl})$: stability and convergence)**.** *Define, for each $u \in H_0(\mathrm{curl};\Omega)$, the element $R_h u \in \mathcal{N}_h^0$ by*

$$a(R_h u, v_h) = a(u, v_h) \qquad \forall\, v_h \in \mathcal{N}_h^0, \tag{26}$$

where $a(u,v) := (u,v) + (\mathrm{curl}\, u, \mathrm{curl}\, v)$. Then the following hold.

(i) Well-posedness and linearity. *For each u, there exists a unique $R_h u \in \mathcal{N}_h^0$ solving (26); the map $R_h : H_0(\mathrm{curl};\Omega) \to \mathcal{N}_h^0$ is linear.*

(ii) Projection and Galerkin orthogonality. *R_h is a projection: $R_h|_{\mathcal{N}_h^0} = \mathrm{Id}$. Moreover,*

$$a(u - R_h u,\, v_h) = 0 \qquad \forall\, v_h \in \mathcal{N}_h^0. \tag{27}$$

(iii) Stability (contractivity) in $H(\mathrm{curl})$. *For all $u \in H_0(\mathrm{curl};\Omega)$,*

$$\|R_h u\|_{H(\mathrm{curl})} \le \|u\|_{H(\mathrm{curl})}, \qquad \|u - R_h u\|_{H(\mathrm{curl})} = \min_{v_h \in \mathcal{N}_h^0} \|u - v_h\|_{H(\mathrm{curl})}. \tag{28}$$

(iv) Convergence (general). *For every $u \in H_0(\mathrm{curl})$,*

$$\|u - R_h u\|_{H(\mathrm{curl})} \xrightarrow[h\to 0]{} 0, \quad \textit{hence} \quad \|u - R_h u\|_{L^2} \to 0, \quad \|\mathrm{curl}(u - R_h u)\|_{L^2} \to 0. \tag{29}$$

Proof. (i) Well-posedness. Endow $H_0(\mathrm{curl};\Omega)$ with the inner product $a(\cdot,\cdot)$ and induced norm $\|v\|_{H(\mathrm{curl};\Omega)}^2 = a(v,v)$. On $\mathcal{N}_h^0$, $a(\cdot,\cdot)$ is symmetric and coercive: $a(v_h, v_h) = \|v_h\|_{H(\mathrm{curl};\Omega)}^2 > 0$ for $v_h \neq 0$. Thus, by the Riesz representation theorem in the finite-dimensional subspace $\mathcal{N}_h^0$, for each continuous linear functional $v_h \mapsto a(u, v_h)$ there exists a unique $R_h u \in \mathcal{N}_h^0$ solving (26). Linearity follows from linearity of (26) in u.

(ii) Projection and orthogonality. If $u \in \mathcal{N}_h^0$, then taking $v_h = R_h u - u \in \mathcal{N}_h^0$ in (26) gives $a(R_h u - u, R_h u - u) = 0$, hence $R_h u = u$. Subtracting (26) from itself yields $a(u - R_h u, v_h) = 0$ for all $v_h \in \mathcal{N}_h^0$.

(iii) Stability and best approximation. Since R_h is the orthogonal projector onto $\mathcal{N}_h^0$ in the Hilbert space $\big(H_0(\mathrm{curl};\Omega), a(\cdot,\cdot)\big)$, Pythagoras' theorem gives, for every $v_h \in \mathcal{N}_h^0$,

$$\|u - v_h\|_{H(\mathrm{curl};\Omega)}^2 = \|u - R_h u\|_{H(\mathrm{curl};\Omega)}^2 + \|R_h u - v_h\|_{H(\mathrm{curl};\Omega)}^2.$$

Taking the infimum over v_h yields the best-approximation identity in (28). Choosing $v_h = 0$ shows $\|u\|_{H(\mathrm{curl};\Omega)}^2 = \|u - R_h u\|_{H(\mathrm{curl};\Omega)}^2 + \|R_h u\|_{H(\mathrm{curl};\Omega)}^2 \geq \|R_h u\|_{H(\mathrm{curl};\Omega)}^2$, hence contractivity.

(iv) Convergence. Let $u \in H_0(\mathrm{curl})$ and $\varepsilon > 0$. By density of $\bigcup_h \mathcal{N}_h^0$ in $H_0(\mathrm{curl};\Omega)$ there exists $v_h^\varepsilon \in \mathcal{N}_h^0$ with $\|u - v_h^\varepsilon\|_{H(\mathrm{curl};\Omega)} < \varepsilon$. By best approximation,

$$\|u - R_h u\|_{H(\mathrm{curl};\Omega)} \;=\; \min_{w_h \in \mathcal{N}_h^0} \|u - w_h\|_{H(\mathrm{curl};\Omega)} \;\leq\; \|u - v_h^\varepsilon\|_{H(\mathrm{curl};\Omega)} \;<\; \varepsilon.$$

As ε was arbitrary, $\|u - R_h u\|_{H(\mathrm{curl};\Omega)} \to 0$. The two component convergences in (29) follow since the $H(\mathrm{curl};\Omega)$-norm is $\|u - R_h u\|_{H(\mathrm{curl})}^2 = \|u - R_h u\|_{L^2}^2 + \|\mathrm{curl}(u - R_h u)\|_{L^2}^2$. □

Proposition 4 (Constrained L^2-projection onto discrete divergence-free RT fields)**.** *Let $\mathcal{RT}_h \subset H(\mathrm{div};\Omega)$ be a Raviart–Thomas space (order $k \geq 0$) on a shape-regular mesh $\mathcal{T}_h$, and let $\mathcal{Q}_h := \{q_h \in L^2(\Omega) : q_h|_K \in \mathbb{P}_k(K)\ \forall K \in \mathcal{T}_h\}$. Define the discrete divergence-free subspace*

$$Z_h := \ker(\mathrm{div}\,|_{\mathcal{RT}_h}) = \{v_h \in \mathcal{RT}_h :\ \mathrm{div}\, v_h = 0 \text{ in } \mathcal{Q}_h\}.$$

For any $B_0 \in L^2(\Omega)^3$ there exists a unique $B_h^0 \in Z_h$ such that

$$(B_h^0, z_h) = (B_0, z_h) \qquad \forall\, z_h \in Z_h. \tag{30}$$

Equivalently, $(B_h^0, p_h) \in \mathcal{RT}_h \times \mathcal{Q}_h$ solves the mixed system

$$\begin{aligned} (B_h^0, v_h) + (p_h, \mathrm{div}\, v_h) &= (B_0, v_h) & \forall\, v_h \in \mathcal{RT}_h, \\ (\mathrm{div}\, B_h^0, q_h) &= 0 & \forall\, q_h \in \mathcal{Q}_h. \end{aligned} \tag{31}$$

Moreover, B_h^0 is the unique minimizer of the constrained problem

$$\min\Big\{\tfrac{1}{2}\|v_h - B_0\|_{L^2(\Omega)^3}^2 :\ v_h \in \mathcal{RT}_h,\ \mathrm{div}\, v_h = 0 \text{ in } \mathcal{Q}_h\Big\}.$$

Proof. Step 1: Equivalence "orthogonality ⇔ minimizer". Let Z_h be a finite-dimensional subspace of the Hilbert space $L^2(\Omega)^3$. The (unconstrained) best approximation problem $\min_{z_h \in Z_h} \|z_h - B_0\|^2$ has a unique solution characterized by the L^2-orthogonality $(B_0 - B_h^0, z_h) = 0$ for all $z_h \in Z_h$, i.e. (30). This is the standard projection theorem in Hilbert spaces.

Step 2: Equivalence with the mixed (saddle-point) system. Consider the constrained minimization $\min_{v_h \in \mathcal{RT}_h} \frac{1}{2}\|v_h - B_0\|^2$ subject to $\mathrm{div}\, v_h = 0 \in \mathcal{Q}_h$. Introduce a Lagrange multiplier $p_h \in \mathcal{Q}_h$ and the Lagrangian

$$\mathcal{L}(v_h, p_h) := \tfrac{1}{2}(v_h, v_h) - (B_0, v_h) + (p_h, \mathrm{div}\, v_h).$$

Stationarity w.r.t. v_h and p_h yields (31). Conversely, if (B_h^0, p_h) solves (31), then for any $z_h \in Z_h$ the first line with $v_h = z_h$ gives $(B_h^0 - B_0, z_h) = 0$, i.e. (30), so $B_h^0 \in Z_h$ is the (unique) orthogonal projection.

Step 3: Existence and uniqueness of (31). Let $a(v_h, w_h) := (v_h, w_h)$ on $\mathcal{RT}_h$ and $b(v_h, q_h) := (\operatorname{div} v_h, q_h)$. The pair $(\mathcal{RT}_h, \mathcal{Q}_h)$ satisfies the uniform Babuška–Brezzi conditions: (i) $a(\cdot,\cdot)$ is coercive on $\ker b = Z_h$ because $a(z_h, z_h) = \|z_h\|_{L^2}^2$; (ii) the discrete inf–sup holds:

$$\inf_{0\neq q_h\in\mathcal{Q}_h} \sup_{0\neq v_h\in\mathcal{RT}_h} \frac{(\operatorname{div} v_h, q_h)}{\|v_h\|_{L^2}\,\|q_h\|_{L^2}} \;\geq\; \beta > 0,$$

with β independent of h (standard for RT spaces; e.g. via a Fortin operator that commutes with div). Thus (31) is well-posed, giving a unique pair (B_h^0, p_h) and hence a unique $B_h^0 \in Z_h$. $\square$

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