ON CASAGRANDE-DRUEL FANO VARIETIES WITH LEFSCHETZ DEFECT 2

PIER ROBERTO PASTORINO

Abstract

The larger the Lefschetz defect δ_X of a smooth complex Fano variety X, the more information we can deduce about the geometry of X. By [Cas12], if $\delta_X \geq 4$, then $X \simeq S \times T$ where $\dim(T) = \dim(X) - 2$. The structure of X when $\delta_X = 3$ was described in [CRS22]. In this paper, we study the case $\delta_X = 2$. In particular, we focus on Fano varieties with $\delta_X = 2$ arising from the construction introduced by C. Casagrande and S. Druel in [CD15], which we refer to as **Construction A**. We show that among the 19 families of Fano 3-folds with $\delta_X = 2$ classified by Mori and Mukai, 15 arise from such construction. Moreover, we construct all Fano 4-folds with $\rho_X \geq 4$ and $\delta_X = 2$ admitting such a structure, obtaining 147 distinct families in total. Combined with [Sec23, Theorem 1.1], this yields a complete classification of all Casagrande-Druel Fano 4-folds with $\delta_X = 2$. To broaden the scope, we also study a generalized version of Construction A, which we call **Construction B**, and we find that 18 out of the 19 families of Fano 3-folds with $\delta_X = 2$ arise from it.

Contents

1 Introduction

Let X be a smooth complex Fano variety. The *Lefschetz defect* of X, denoted by δ_X, is an invariant introduced in [Cas12] that relates the Picard number of X to that of its prime divisors. Given a prime divisor D in X, let us consider the natural push-forward $\iota_* : N_1(D) \to N_1(X)$ given by the inclusion $\iota : D \hookrightarrow X$, and set

$$N_1(D, X) := \iota_*((N_1(D)) \subseteq N_1(X).$$

The Lefschetz defect of X is defined as

$$\delta_X = \max\{\operatorname{codim} N_1(D, X) \mid D \subset X \text{ is a prime divisor in } X\}.$$

It was recently proved [Ver25] that the Lefschetz defect is invariant under smooth deformation. If we denote by ρ_X the Picard number of X, it is clear that $\delta_X \in \{0, ..., \rho_X - 1\}$. Recent developements in the study of the Lefschetz defect have highlighted the fact that large values of δ carry strong information about the geometric structure of the variety. For instance, we have the following key result about Fano varieties with $\delta \geq 4$.

Theorem 1.1 ([Cas12], Theorem 3.3)**.** *Let X be a smooth Fano variety with Lefschetz defect $\delta_X \geq 4$. Then $X \simeq S \times T$, where S is a del Pezzo surface with $\rho_S = \delta_X + 1$.*

The case of a Fano variety X with $\delta_X = 3$ was studied in [CRS22], where it was shown that there exists a smooth Fano variety T with $\dim T = \dim X - 2$ such that X is obtained from T by means of two possible explicit constructions. In both cases, the construction consists of the composition of a $\mathbb{P}^2$-bundle $Z \to T$ with the blow-up of three pairwise disjoint smooth, irreducible, codimension 2 subvarieties in Z.

In this paper, we turn to the case $\delta = 2$. We begin by analyzing the structure of Fano varieties with $\delta = 2$ in arbitrary dimension from a theoretical perspective. We then specialize to the case of three and four-dimensional Fano varieties, where our results yield several new examples of Fano 4-folds with $\delta = 2$.

The main known result about the structure of Fano varieties with $\delta = 2$ is the following.

Theorem 1.2 ([Cas14], Theorem 5.22)**.** *Let X be a smooth Fano variety with $\delta_X = 2$, then one of the following holds.*

(i) There exists a sequence of flips $X \dashrightarrow X'$ and a conic bundle $\phi : X' \to Z$ where X' and Z are smooth, $\rho_X - \rho_Z = 2$, and ϕ factors as $X' \xrightarrow{\sigma} Y \xrightarrow{\pi} Z$, where π is a smooth $\mathbb{P}^1$-fibration and σ is the blow-up of a smooth irreducible codimension 2 subvariety A_Y in Y, such that A_Y is a section over its image under π.

(ii) There is an equidimensional fibration in del Pezzo surfaces $\psi : X \to T$, where T has locally factorial, canonical singularities, $\operatorname{codim} \operatorname{Sing}(T) \geq 3$ *and $\rho_X - \rho_T = 3$.*

Examples suggest that (i) could hold for every smooth Fano variety with $\delta = 2$, and this is actually the case for smooth Fano 3-folds. In this paper we focus on smooth Fano varieties X with $\delta = 2$ satisfying a stronger version of (i), namely the existence of a conic bundle $\phi\colon X \to Z$, with $\rho_X - \rho_Z = 2$, such that ϕ factors through a smooth $\mathbb{P}^1$-fibration over Z, that would be $X = X'$ in (i), without the need for any sequence of flips. Let $\pi\colon Y \to Z$ be the $\mathbb{P}^1$-fibration and $\sigma\colon X \to Y$ be the blow-up of $A_Y \subset Y$ such that $\phi = \sigma \circ \pi$. We study two particular instances of this setup. We assume that A_Y lies in a section S_Y of π; in particular this implies that π is a $\mathbb{P}^1$-bundle[1]. Under these assumptions, we say that X arises from *Construction B* (see Section 3.1 for a more precise definition). If moreover there is another section of π disjoint from S_Y, we say that X arises from *Construction A* (see Section 2.1 for a more precise definition). Although this setup might at first sight appear restrictive, we will see that it describes the structure of most families of Fano 3-folds with $\delta = 2$, and yields several examples of families of Fano 4-folds with $\delta = 2$.

We begin by studying Construction A. Notice that Construction A coincides with the so called Casagrande-Druel construction (see [CD15]), and seems a natural starting point for studying the structure of Fano varieties with $\delta = 2$. Indeed, it is known that if X is a Fano variety with $\dim(X) \geq 3$, $\delta_X = 2$ and $\rho_X = 3$ (the minimal possible value for ρ when $\delta = 2$), then X necessarily arises from the Casagrande-Druel construction ([CD15, Theorem 3.8]). This result was also applied in [Sec23] in order to classify all families of Fano 4-folds with $\rho = 3$ and $\delta = 2$. Variations and special cases of such construction have recently appeared in [CDGF$^+$23] and [Mal24], with a focus on K-stability issues.

Given a smooth variety X arising from Construction A, our first goal is to identify criteria ensuring that X is a Fano variety with $\delta_X = 2$, expressed in terms of conditions on Z. Such criteria are provided by the combination of Proposition 2.3 (see also [CDGF$^+$23, Proposition 2.3]) and Lemma 2.4. These results yield effective tools for studying Fano varieties with $\delta = 2$ in an inductive way from lower to higher dimension.

In contrast, the study of Construction B turns out to be more involved. Let X be a smooth variety arising from Construction B. In the pursuit of conditions for X to be Fano that can be tested on Z, we prove the following theorem:

Theorem 1.3. *(Theorem 3.2) Let Z be a smooth Fano variety, $\pi\colon Y = \mathbb{P}(\mathcal{E}) \to Z$ a $\mathbb{P}^1$-bundle and $S_Y \subset Y$ a section of π associated to a short exact sequence of this form:*

$$0 \to \mathcal{O}_Z \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_Z(D) \to 0 \tag{1}$$

Let A be a smooth irreducible codimension 1 subvariety of Z and $\sigma\colon X \to Y$ the blow-up of $A_Y := \pi^{-1}(A) \cap S_Y$ in Y. Consider the following conditions

(I) $-K_Z + D - A$ is ample on Z.

(II) $\mathcal{E} \otimes \mathcal{O}_Z(-K_Z - D)$ is ample on Z.

(III) $\mathcal{E}|_A \otimes \mathcal{O}_Z(-K_Z - D)|_A$ is ample on A.

Then:

$$\begin{cases} (I) \\ (II) \end{cases} \implies X \text{ is Fano} \implies \begin{cases} (I) \\ (III). \end{cases}$$

[1] A morphism $\pi\colon Y \to Z$ will be called a "$\mathbb{P}^1$-bundle over Z" if Y is the projectivization of a rank 2 vector bundle on Z and π is the natural projection.

When specialized to the setting of Construction A, conditions (I) and (II) in Theorem 1.3 are both necessary and sufficient for X to be Fano, and we recover Proposition 2.3. However, examples show that, in general, conditions *(I)*, *(II)* are not necessary and conditions *(I)*, *(III)* are not sufficient for X to be Fano. The difficulty in bridging the gap between necessary and sufficient conditions in Theorem 1.3 is due to the *asymmetric* nature of Construction B as opposed to the perfect symmetry of Construction A. To make this statement precise, we introduce *Maruyama's elementary transformations* ([Mar82]) and explain their relation to our constructions (see Section 4).

Regarding the determination of the Lefschetz defect in this setting, the same tools developed for Construction A remain effective and applicable (see Lemma 3.5).

All these results allow us to list all families of Fano 3-folds with $\delta = 2$ that arise from Construction A and Construction B. It turns out that among the 19 families of Fano 3-folds with $\delta = 2$ in the Iskovskikh-Mori-Mukai classification (see [Isk77], [Isk78], [MM81] and [MM03]), only 4 cannot be obtained via Construction A (see Section 6), and only one family cannot be obtained via Construction B. Members of this latter family (#4-1 in [ACC$^+$23, Table 6.1]) can be described as the blow up of a $(2,2,2)$-curve in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ that is the smooth complete intersection of two $(1,1,1)$-divisors. Thus they admit a similar conic bundle structure as the one described by Construction B, but in this case the blown up locus $A_Y \subset Y := (\mathbb{P}^1)^3$ is a $(2,2,2)$-curve that does not lie in any section of the $\mathbb{P}^1$-bundle $\pi\colon Y \to Z := \mathbb{P}^1 \times \mathbb{P}^1$.

Turning to the four dimensional case, it is known that if X is a Fano 4-fold with $\delta_X = 2$, then $3 \leq \rho_X \leq 6$ (see [CS24, Theorem 6.1]). In this paper, we list all families of Fano 4-folds with $\delta = 2$ and $\rho \geq 4$ that can be obtained through Construction A, while the case $\rho = 3$ was settled in [Sec23]. As a result of Theorem 9.1, Theorem 10.1 and Theorem 11.1, we obtain the following:

Theorem 1.4. *There are 147 distinct families of smooth Fano 4-folds with $\delta = 2$ and $\rho \geq 4$ arising from Construction A. Among them, 96 have $\rho = 4$ (see Table 24), 50 have $\rho = 5$ (see Table 25) and one of them has $\rho = 6$ (the product $\mathrm{Bl}_2\,\mathbb{P}^2 \times \mathrm{Bl}_2\,\mathbb{P}^2$). Moreover, 95 of those families are not toric nor products of lower dimensional varieties. Among them, 68 have $\rho = 4$ and 27 have $\rho = 5$.*

Among the 95 families of non-toric and non-product varieties, we searched for potential correspondences with some known databases of Fano 4-folds families in the literature. We found no matches in [FTT24] or in [BFMT25]. However, we did identify some families in [CKP15] that share some numerical invariants (for example c_1^4 and $h^0(-K)$) as some families in our list. It is worth noticing that the Hodge numbers of Fano families in [CKP15] are not generally known, which makes a precise comparison with our families challenging and therefore beyond the scope of this paper.

Future projects may include the study of Fano 4-folds with $\delta = 2$ arising from Construction B.

In conclusion, Construction A accounts for the majority of smooth Fano 3-folds with $\delta = 2$ and yields 147 families of smooth Fano 4-folds with $\delta = 2$ and $\rho \geq 4$. The more general Construction B captures all but one of the 19 families of smooth Fano 3-folds with $\delta = 2$, and may ultimately provide a more coherent and uniform description of these varieties than the traditional case-by-case classification by Iskovskikh-Mori-Mukai.

Structure of the paper. In Sections 2 and 3 we present Constructions A and B respectively and study conditions for them to yield a Fano variety. In Section 4 we explore the relation of Constructions A and B with the setting of *Maruyama's elementary transformation*. In Section 5 we study the iteration of two constructions of type A. In Sections 6 and 7 we list all smooth Fano 3-folds with $\delta = 2$ arising from Constructions A and B respectively. In Section 8 we give some preliminary results for the study of Fano 4-folds arising from Construction A. In Sections 9, 10 and 11 we list all smooth Fano 4-folds with $\delta = 2$ arising from Construction A with $\rho = 6$, $\rho = 5$ and $\rho = 4$ respectively. Finally, in Section 12 we show the techniques we used to compute the invariants of the Fano 4-folds we constructed in the previous sections, and in Appendix A we give a complete list of all Fano 4-folds arising from Construction A with $\rho \in \{4, 5\}$.

Notation and conventions. We work over the field of complex numbers. Let X be a smooth projective variety of arbitrary dimension.

$N_1(X)$ (respectively $N^1(X)$) is the real vector space of one-cycles (respectively, Cartier divisors) with real coefficients, modulo numerical equivalence, and $\dim N_1(X) = \dim N^1(X) = \rho_X$ is the Picard number of X. We denote by $\mathrm{Nef}(X)$ the nef cone of X and by $\mathrm{Eff}(X)$ the effective cone of X.

The symbol $\sim$ stands for linear equivalence of divisors, while $\equiv$ stands for numerical equivalence of one-cycles, as well as divisors.

$\mathrm{NE}(X) \subset N_1(X)$ is the convex cone generated by the classes of effective curves. An *extremal ray* R is a one-dimensional face of $\mathrm{NE}(X)$. When X is Fano, the *length* of R is $\ell(R) = \min\{-K_X \cdot C \mid C \text{ is a rational curve in } R\}$.

A *contraction* is a surjective morphism $\varphi : X \to Y$ with connected fibers, where Y is normal and projective. A contraction is *elementary* if $\rho_X - \rho_Y = 1$.

A *conic bundle* is a contraction of fiber type $X \to Y$ where every fiber is one-dimensional and isomorphic to a plane conic.

In $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, denote by $p_i \colon X \to \mathbb{P}^1$ the projection onto the i-th factor, and H_i the class of $p_i^*\mathcal{O}_{\mathbb{P}^1}(1)$ in $N^1(X)$ for $i = 1, 2, 3$. Let f_1, f_2, f_3 be the generators of $N_1(X)$ given by the classes of $\mathbb{P}^1 \times \{*\} \times \{*\}$, $\{*\} \times \mathbb{P}^1 \times \{*\}$, $\{*\} \times \{*\} \times \mathbb{P}^1$ respectively, so that $H_i \cdot f_j = \delta_{ij}$. By "$(\alpha_1, \alpha_2, \alpha_3)$-divisor in X" (with $\alpha_i \in \mathbb{Z}$) we mean a Cartier divisor D in X such that $[D] = \alpha_1 H_1 + \alpha_2 H_2 + \alpha_3 H_3$. We also write $\mathcal{O}(D) = \mathcal{O}(\alpha_1, \alpha_2, \alpha_3)$. A curve C in X is called a "$(\beta_1, \beta_2, \beta_3)$-curve" if $[C] = \beta_1 f_1 + \beta_2 f_2 + \beta_3 f_3$.

Let $W = \mathbb{P}^1 \times \mathbb{P}^1$, and call p_1 and p_2 the projections of W respectively onto the first and second factor. Let $\mathbf{l_i}$ be the numerical class of $p_i^*\mathcal{O}_{\mathbb{P}^1}(1)$ for $i = 1, 2$. Then C is a "(a, b)-curve in W" if $C \equiv a\mathbf{l_1} + b\mathbf{l_2}$. We also write $\mathcal{O}(C) = \mathcal{O}(a, b)$.

Smooth Fano 3-folds have been classified in [Isk77], [Isk78], [MM81], [MM03] into 105 families. Throughout this paper, those families will be labeled as #1-1,#1-2,#1-3,...,#9-1,#10-1, following the classical notation (see, for example, Table 6.1 in [ACC+23]).

In tables 4 to 25 (see Sections 11, 10 and Appendix A), we list all Fano 4-folds X arising from Construction A with $\rho \in \{4, 5\}$. Under the column "Toric?", when X is toric we write the type of the Fano polytope of the family of X described in [Bat99]; under the column "Product", when X is a product of two lower dimensional Fano varieties, we make those varieties explicit. When one of those varieties is a Fano 3-fold, we refer to it by using the classical notation.

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2 Construction A

2.1 Setting

In this section we present Construction A. Let Z be a smooth projective variety with $\dim(Z) = n-1 \geq 2$. Fix a Cartier divisor D and a smooth irreducible hypersurface A on Z. Let $Y := \mathbb{P}_Z(\mathcal{O}_Z \oplus \mathcal{O}_Z(D))$, with the projection $\pi\colon Y \to Z$. Then $\dim(Y) = n$ and $\rho_Y = \rho_Z + 1$. Consider $G_Y \subset Y$ and $\hat{G}_Y \subset Y$ the sections of π corresponding respectively to the quotients $\mathcal{O} \oplus \mathcal{O}(D) \twoheadrightarrow \mathcal{O}$ and $\mathcal{O} \oplus \mathcal{O}(D) \twoheadrightarrow \mathcal{O}(D)$. We have $G_Y \simeq Z \simeq \hat{G}_Y$, and it is not difficult to prove that $G_Y \cap \hat{G}_Y = \emptyset$, $\mathcal{N}_{G_Y/Y} \simeq \mathcal{O}(-D)$ and $\mathcal{N}_{\hat{G}_Y/Y} \simeq \mathcal{O}(D)$.

If X is the blow up of Y along $A_Y := \pi^{-1}(A) \cap \hat{G}_Y$, we say that X arises from **Construction A**. To summarize this construction (which only depends on the choice of Z, and the divisors $A, D \subset Z$), we will adopt the following notation:

$$X = \mathscr{C}_A(Z; A, D).$$

Observe that X is a smooth projective variety with $\dim(X) = n$ and $\rho_X = \rho_Z + 2$. Let $\sigma\colon X \to Y$ denote the blow-up map. Let G and $\hat{G}$ be the strict transforms in X of G_Y and $\hat{G}_Y$ respectively. Then $\mathcal{N}_{G/X} \simeq \mathcal{O}(-D)$ and $\mathcal{N}_{\hat{G}/X} \simeq \mathcal{O}(D-A)$.

Call $D_1 := D$ and $D_2 := A - D$ so that $A \sim D_1 + D_2$. The composition $\phi := \sigma \circ \pi$ is a conic bundle and has another factorization $\phi = \hat{\sigma} \circ \hat{\pi}$ where $\hat{\sigma} : X \to \hat{Y}$ and $\hat{\pi} : \hat{Y} \to Z$ have the following properties:

- $\hat{Y} = \mathbb{P}_Z(\mathcal{O} \oplus \mathcal{O}(D_2))$ and $\hat{\pi}$ is the projection onto Z.
- $\hat{\sigma}(G)$ and $\hat{\sigma}(\hat{G})$ are disjoint sections of $\hat{\pi}$ in $\hat{Y}$.
- $\mathcal{N}_{\hat{\sigma}(G)/\hat{Y}} \simeq \mathcal{O}(D_2)$ and $\mathcal{N}_{\hat{\sigma}(\hat{G})/\hat{Y}} \simeq \mathcal{O}(-D_2)$
- $\hat{\sigma}$ is the blow up of $\hat{Y}$ along $A_{\hat{Y}} := \hat{\pi}^{-1}(A) \cap \hat{\sigma}(G)$.
- Let E and $\hat{E}$ be the exceptional divisors respectively of σ and $\hat{\sigma}$. Then E and $\hat{E}$ coincide respectively with the strict transforms of $\hat{\pi}^{-1}(A)$ and $\pi^{-1}(A)$ in X.

The existence of such an alternative symmetric factorization is proved by M. Maruyama for a more general setting in [Mar82]. The new variety $\hat{Y}$ is called *Maruyama's elementary transformation* of Y and is birational to Y through $\psi := \hat{\sigma} \circ \sigma^{-1}$.

$$\begin{tikzcd}
 & X \arrow[dl, "\sigma"'] \arrow[dr, "\hat{\sigma}"] & \\
Y := \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(D_1)) \arrow[rr, dotted, "\psi"] \arrow[dr, "\pi"'] & & \hat{Y} := \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(D_2)) \arrow[dl, "\hat{\pi}"] \\
 & Z &
\end{tikzcd}$$

Figure 1: Alternative factorization of Construction A.

Using the notation we introduced earlier, we can write $\mathscr{C}_A(Z; A, D_1) \simeq \mathscr{C}_A(Z; A, D_2)$.

Remark 2.1. Let e (resp. $\hat{e}$) be the class of a one-dimensional fiber of σ (resp. of $\hat{\sigma}$) in $\mathrm{NE}(X)$. The classes e and $\hat{e}$ generate the two-dimensional closed subcone $\mathrm{NE}(\phi) \subset \mathrm{NE}(X)$ of the curves contracted by ϕ and the following intersection table holds.

$\cdot$	E	$\hat{E}$	G	$\hat{G}$
e	-1	1	0	1
$\hat{e}$	1	-1	1	0

Table 1: Intersection Table in X

Remark 2.2. From the known description of the anticanonical divisor of a blow-up and of a $\mathbb{P}^1$-bundle, we have $-K_X \sim \sigma^*(-K_Y) - E$ and $-K_Y \sim \pi^*(-K_Z) + G_Y + \hat{G}_Y$. Hence, since $\sigma^* G_Y \sim G$ and $\sigma^*(\hat{G}_Y) \sim \hat{G} + E$, we have

$$-K_X \sim \sigma^*(-K_Y) - E \sim \phi^*(-K_Z) + \sigma^* G_Y + \sigma^* \hat{G}_Y - E \sim \phi^*(-K_Z) + G + \hat{G}.$$

2.2 Conditions for being Fano

Observe that the conic bundle $X \to Z$ given by Construction A has no nonreduced fibers, so that by [Wiś91, Proposition 4.3] if X is Fano, then Z is Fano as well. We now give some conditions for X to be Fano, that can be tested on Z, after assuming that Z is Fano (see also [CDGF+23, Proposition 2.3]).

Proposition 2.3. *Let Z be a smooth Fano variety, A a smooth irreducible hypersurface in Z and D_1, D_2 two Cartier divisors in Z such that $A \sim D_1 + D_2$.*

Consider $X = \mathscr{C}_A(Z; A, D_1) (\simeq \mathscr{C}_A(Z; A, D_2))$. Then X is Fano if and only if $-K_Z - D_i$ is ample for $i = 1, 2$.

Proof. First assume X is Fano. Then $-K_X|_G$ must be ample on G and $-K_X|_{\hat{G}}$ must be ample on $\hat{G}$. Since $\mathcal{N}_{G/X} \simeq \mathcal{O}_Z(-D_1)$ and $\mathcal{N}_{\hat{G}/X} \simeq \mathcal{O}_Z(-D_2)$, by the adjunction formula we get that $-K_Z - D_1$ and $-K_Z - D_2$ are ample on Z.

Conversely, assume that $-K_Z - D_1$ and $-K_Z - D_2$ are ample on Z. We show that $-K_X$ is ample on X.

First we show that $-K_X$ is strictly nef, i.e. it has positive intersection with every irreducible curve on X. Observe that by the adjunction formula, $-K_X|_G$ is ample on G because $-K_Z - D_1$ is ample on Z, and $-K_X|_{\hat{G}}$ is ample on $\hat{G}$ because $-K_Z - D_2$ is ample on Z. Now, let Γ be any irreducible curve in X. Since $-K_X$ is ample on G and $\hat{G}$, if Γ is contained in the support of one of those divisors, we have $-K_X \cdot \Gamma > 0$ by Kleiman's criterion of ampleness. Suppose that Γ is not contained in $G \cup \hat{G}$. Then we have $\phi^*(-K_Z) \cdot \Gamma \geq 0$, $G \cdot \Gamma \geq 0$ and $\hat{G} \cdot \Gamma \geq 0$, because G, $\hat{G}$ are effective and $\phi^*(-K_Z)$ is nef. By Remark 2.2, it follows that

$$-K_X \cdot \Gamma = \phi^*(-K_Z) \cdot \Gamma + G \cdot \Gamma + \hat{G} \cdot \Gamma \geq 0.$$

Now, if Γ is not contracted by ϕ, then $\phi_*\Gamma$ is an irreducible curve in Z and $\phi^*(-K_Z) \cdot \Gamma = -K_Z \cdot \phi_* \Gamma > 0$ because $-K_Z$ is ample. This implies $-K_X \cdot \Gamma > 0$. On the other hand, if Γ is contracted by ϕ, its class in $\mathrm{NE}(X)$ sits in the 2-dimensional subcone $\mathrm{NE}(\phi) = \mathbb{R}_{\geq 0}e + \mathbb{R}_{\geq 0}\hat{e}$, which is K-negative because $-K_X \cdot e = 1 = -K_X \cdot \hat{e}$. This proves that $-K_X$ is strictly nef.

Let us now prove that $-K_X$ is big on X. We know that G_Y is the tautological divisor for $\mathbb{P}_Z(\mathcal{O} \oplus \mathcal{O}(-D_1))$. Hence $W_Y := G_Y + \pi^*(-K_Z)$ is the tautological divisor for $\mathbb{P}_Z(\mathcal{O}(-K_Z) \oplus \mathcal{O}(-K_Z - D_1))$. Since $-K_Z - D_1$ and $-K_Z$ are ample on Z, W_Y must be ample on Y. Therefore there exist $m, c \in \mathbb{N}$ such that

$$W_X := m\sigma^*(W_Y) - cE = m\phi^*(-K_Z) + mG - cE$$

is ample on X. Then

$$-mK_X \sim m\phi^*(-K_Z) + mG + m\hat{G} = W_X + m\hat{G} + cE$$

is the sum of an ample and an effective divisor, which proves that $-K_X$ is big.

In conclusion, since $-K_X$ is strictly nef and big, it is ample on X by the base-point-free theorem. □

We now study the Lefschetz defect of X in relation to the Lefschetz defect of Z.

Lemma 2.4. *Suppose $X = \mathscr{C}_A(Z; A, D)$ is Fano. Then we have:*

(i) $\delta_X \geq 2$.

(ii) $\delta_X \geq \delta_Z$. *Moreover, if* $\operatorname{codim}(N_1(A, Z)) = \delta_Z$, *then* $\delta_X \geq \delta_Z + 1$.

(iii) If $\delta_X \geq 4$, *then* $\delta_X \in \{\delta_Z, \delta_Z + 1\}$.

In particular, if $\delta_Z \leq 2$, *then* $2 \leq \delta_X \leq 3$.

Proof. Let $\iota : G \hookrightarrow X$ be the inclusion of G in X. We notice that, since $\phi \circ \iota : G \to Z$ is an isomorphism, $\dim(N_1(G, X)) = \rho_G = \rho_Z = \rho_X - 2$. Therefore $\delta_X \geq \operatorname{codim}(N_1(G, X)) = 2$, which proves (i).

Since ϕ is surjective, we have $\delta_X \geq \delta_Z$ (see [Cas12, Remark 3.3.18]). Now, consider the exceptional divisor E in X. We know that $\ker(\sigma_* : N_1(X) \to N_1(Y))$ is 1-dimensional (generated by the class of a 1-dimensional fiber of σ) and is contained in $N_1(E, X)$. We also have $\sigma_* N_1(E, X) = N_1(A_Y, Y)$. Moreover, since $\hat{G}_Y$ is a section of π, we have $N_1(\hat{G}_Y) \simeq N_1(\hat{G}_Y, Y)$, which implies that $N_1(A_Y, \hat{G}_Y) \simeq N_1(A_Y, Y)$. Then

$$\dim(N_1(E, X)) = \dim(N_1(A_Y, Y)) + 1 = \dim(N_1(A_Y, \hat{G}_Y)) + 1 = \dim(N_1(A, Z)) + 1.$$

Therefore

$$\operatorname{codim}(N_1(E, X)) = \rho_X - \dim(N_1(A, Z)) - 1 \quad = \rho_Z + 2 - \dim(N_1(A, Z)) - 1 = \operatorname{codim}(N_1(A, Z)) + 1.$$

So if $\operatorname{codim}(N_1(A, Z)) = \delta_Z$, we have $\delta_X \geq \delta_Z + 1$, which completes the proof of (ii).

Suppose now that $\delta_X \geq 4$. Then by Theorem 1.1, $X \simeq S \times X'$ where S is a del Pezzo surface and $\rho_S = \delta_X + 1 \geq 5$. By [Rom19, Lemma 2.10], Z is the product of two Fano varieties $Z_1 \times Z_2$ and ϕ is the product of two contractions $\phi_1 : S \to Z_1$ and $\phi_2 : X' \to Z_2$. Recall that if V_1 and V_2 are two Fano varieties, $\delta_{V_1 \times V_2} = \max\{\delta_{V_1}, \delta_{V_2}\}$ ([Cas23, Lemma 5]). There are three cases for Z_1.

1. $Z_1 = \mathbb{P}^1$. In that case $2 = \rho_X - \rho_Z = (\rho_S - 1) + (\rho_{X'} - \rho_{Z_2}) \geq 4$, a contradiction.

2. $Z_1 = S$, $\phi_1 = id_S$. In that case, $\delta_Z \geq \delta_S = \delta_X$. By (ii), we have $\delta_Z \leq \delta_X$. Hence $\delta_X = \delta_Z$.

3. $\rho_{Z_1} = \rho_S - 1$ and ϕ_1 is divisorial; $\rho_{Z_2} = \rho_{X'} - 1$. In that case Z_1 is still a surface and we have

$$\delta_Z \geq \delta_{Z_1} = \rho_{Z_1} - 1 = \rho_S - 2 = \delta_X - 1.$$

By (ii), we then have $\delta_X \in \{\delta_Z, \delta_Z + 1\}$, which proves (iii). $\square$

Given a smooth Fano variety $X = \mathscr{C}_A(Z; A, D)$, we can also make some comments about curves of anticanonical degree 1 in Z.

Remark 2.5. Let $X = \mathscr{C}_A(Z; A, D)$ be a smooth Fano variety arising from Construction A. Suppose C is an irreducible curve in Z such that $-K_Z \cdot C = 1$. Then by Proposition 2.3 it follows that

$$\begin{cases} 0 < (-K_Z + D - A) \cdot C = 1 + D \cdot C - A \cdot C \\ 0 < (-K_Z - D) \cdot C = 1 - D \cdot C \end{cases}$$

i.e. $A \cdot C \leq D \cdot C \leq 0$. Therefore there are only two possible cases:

1. $A \cdot C = D \cdot C = 0 \implies C \subset A$ or $C \cap A = \emptyset$.

2. $A \cdot C < 0 \implies C \subset A$.

More necessary conditions for Z to be the base of a construction of type A yielding a Fano variety X will be given in Section 8 (see Remark 8.1).

3 Construction B

In this section we present Construction B, which is a generalization of Construction A.

3.1 Setting

Let Z be a smooth projective variety with $\dim(Z) = n - 1 \geq 2$, $\mathcal{E}$ a rank 2 vector bundle on Z, $Y := \mathbb{P}_Z(\mathcal{E})$ and $\pi \colon Y \to Z$ the projection. Let $A \subset Z$ be a smooth irreducible hypersurface and assume that π admits a section, which we denote by $S_Y \subset Y$. Such section S_Y corresponds to a surjection $\beta : \mathcal{E} \twoheadrightarrow \mathcal{L}$ onto a line bundle $\mathcal{L}$ on Z. Call $A_Y := \pi^{-1}(A) \cap S_Y$, let $\sigma \colon X \to Y$ be the blow up of Y along A_Y. The composition $\phi := \pi \circ \sigma$ is a conic bundle. Observe that X is a smooth projective variety, $\dim(X) = \dim(Y) = \dim(Z) + 1$ and $\rho_X = \rho_Y + 1 = \rho_Z + 2$. We will refer to this setting as **Construction B**, and we will use the following notation:

$$X = \mathscr{C}_B(Z; A, S_Y) = \mathscr{C}_B(Z; A, \beta : \mathcal{E} \twoheadrightarrow \mathcal{L}).$$

Observe that if we further assume the existence of a section of π disjoint from S_Y, then we recover the same setting presented in Construction A, which is therefore a special instance of this new more general construction.

More precisely, we know that the existence of a section of π disjoint from S_Y is equivalent to the fact that $\mathcal{E}$ is decomposable into two factors $\mathcal{E} = \mathcal{M} \oplus \mathcal{L}$, and the surjection $\beta : \mathcal{E} \twoheadrightarrow \mathcal{L}$ defining S_Y coincides with the projection p_2 onto the second factor. This yields a construction $\mathscr{C}_B(Z; A, p_2\colon \mathcal{M} \oplus \mathcal{L} \twoheadrightarrow \mathcal{L})$ which is isomorphic to $\mathscr{C}_B(Z; A, p_2\colon \mathcal{O} \oplus (\mathcal{L} \otimes \mathcal{M}^*) \twoheadrightarrow \mathcal{L} \otimes \mathcal{M}^*)$, because $\mathbb{P}_Z(\mathcal{E}) \simeq \mathbb{P}_Z(\mathcal{E} \otimes \mathcal{M}^*)$. By definition, this construction coincides with $\mathscr{C}_A(Z; A, D)$ where $\mathcal{O}(D) \simeq \mathcal{L} \otimes \mathcal{M}^*$.

Remark 3.1. Let $X = \mathscr{C}_B(Z; A, S_Y)$. Consider the exceptional divisor E of $\sigma\colon X \to Y$, and the strict transform $\hat{E}$ of $\pi^{-1}(A)$ in X. Both have a natural $\mathbb{P}^1$-bundle structure over A. In $N_1(X)$, let e (resp. $\hat{e}$) denote the class of a one dimensional fiber in E (resp. in $\hat{E}$). Let S be the strict transform of S_Y in X. The following intersection table holds:

$\cdot$	$-K_X$	E	$\hat{E}$	S
e	1	-1	1	1
$\hat{e}$	1	1	-1	0

The cone $\mathrm{NE}(\phi)$ of the curves contracted by ϕ is a two-dimensional K_X-negative subcone of $\mathrm{NE}(X)$ generated by e and $\hat{e}$, which span two extremal rays. In particular, the curves in $\hat{e}$ are contracted by a divisorial contraction $\hat{\sigma}\colon X \to \hat{Y}$ onto a projective variety $\hat{Y}$, such that $\hat{\sigma}$ is the blow-up of a codimension 2 subvariety in $\hat{Y}$ and $\hat{E}$ is its exceptional divisor.

3.2 Conditions for being Fano

Observe that the conic bundle $X \to Z$ given by Construction B has no nonreduced fibers, so that by [Wiś91, Proposition 4.3] if X is Fano, then Z is Fano as well. We want to study conditions for X to be Fano that can be tested on Z, provided that Z is Fano.

Theorem 3.2. *Let $X = \mathscr{C}_B(Z; A, S_Y)$, where Z is Fano and the section S_Y is associated to a short exact sequence of this form:*

$$0 \to \mathcal{O}_Z \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_Z(D) \to 0 \tag{2}$$

Observe that $\mathcal{O}_Z(D) \simeq \det(\mathcal{E})$ is isomorphic to the normal bundle $\mathcal{N}_{S_Y/Y}$, and $\mathcal{O}_Y(S_Y) \simeq \mathcal{O}_Y(1)$. Consider the following conditions

(I) $-K_Z + D - A$ ample on Z.

(II) $\mathcal{E} \otimes \mathcal{O}_Z(-K_Z - D)$ ample on Z.

(III) $\mathcal{E}|_A \otimes \mathcal{O}_Z(-K_Z - D)|_A$ ample on A.

Then:

$$\begin{cases} \text{(I)} \\ \text{(II)} \end{cases} \implies X \text{ is Fano} \implies \begin{cases} \text{(I)} \\ \text{(III).} \end{cases}$$

Proof. Let E denote the exceptional divisor in X, and S the strict transform of S_Y in X. We know that $\sigma^* S_Y \sim S + E$. Hence

$$\mathcal{N}_{S/X} = \mathcal{O}_S(S) \simeq \mathcal{O}_S(\sigma^* S_Y) \otimes \mathcal{O}_S(-E) \simeq \mathcal{O}_{S_Y}(S_Y) \otimes \mathcal{O}_{S_Y}(-A_Y) \simeq \mathcal{O}_Z(D - A).$$

Therefore, by the adjunction formula, $\mathcal{O}_S(-K_X) = \mathcal{O}_S(-K_S) \otimes \mathcal{N}_{S/X} \simeq \mathcal{O}_Z(-K_Z) \otimes \mathcal{O}_Z(D - A)$. Hence

$$-K_X \text{ ample} \implies -K_X|_S \text{ ample} \iff \text{condition } (I).$$

Let $\hat{E}$ denote the strict transform in X of $\pi^{-1}(A)$. Then $\pi^{-1}(A) \sim \hat{E} + E$, and

$$\begin{aligned}\mathcal{N}_{\hat{E}/X} = \mathcal{O}_{\hat{E}}(\hat{E}) &\simeq \mathcal{O}_{\hat{E}}(\pi^{-1}(A)) \otimes \mathcal{O}_{\hat{E}}(-E)\\ &\simeq \mathcal{O}_{\pi^{-1}(A)}(\pi^{-1}(A)) \otimes \mathcal{O}_{\pi^{-1}(A)}(-A_Y)\\ &\simeq \pi^*\mathcal{O}_A(A) \otimes \mathcal{O}_{\pi^{-1}(A)}(-1)\end{aligned}$$

where we denote by $\mathcal{O}_{\pi^{-1}(A)}(-1)$ the dual of the tautological line bundle of $\pi^{-1}(A) = \mathbb{P}_A(\mathcal{E}|_A)$. Recall that $\hat{E}$ is the exceptional divisor of some blow-up morphism that contracts the fibers of the natural $\mathbb{P}^1$-bundle structure of $\hat{E}$ over A. (see Remark 3.1). Therefore $\mathcal{O}_{\hat{E}}(1) = \mathcal{O}_{\hat{E}}(-\hat{E}) = \pi^*\mathcal{O}_A(-A) \otimes \mathcal{O}_{\pi^{-1}(A)}(1)$, which implies $\hat{E} = \mathbb{P}_A\big(\mathcal{E}|_A \otimes \mathcal{O}_A(-A)\big)$. It follows that

$$\begin{aligned}\mathcal{O}_{\hat{E}}(K_{\hat{E}}) \simeq &\mathcal{O}_{\hat{E}}(2\hat{E}) \otimes \phi|_{\hat{E}}^*\big(\mathcal{O}_A(K_A) \otimes \det(\mathcal{E}|_A \otimes \mathcal{O}_A(-A))\big)\\ \simeq &\mathcal{O}_{\hat{E}}(2\hat{E}) \otimes \phi|_{\hat{E}}^*\big(\mathcal{O}_A(K_Z + A - 2A) \otimes \det(\mathcal{E}|_A)\big)\\ \simeq &\mathcal{O}_{\hat{E}}(2\hat{E}) \otimes \phi|_{\hat{E}}^*\big(\mathcal{O}_A(K_Z + D - A)\big).\end{aligned}$$

Thus

$$\mathcal{O}_{\hat{E}}(-K_X) \simeq \mathcal{O}_{\hat{E}}(-K_{\hat{E}}) \otimes \mathcal{O}_{\hat{E}}(\hat{E}) \simeq \mathcal{O}_{\hat{E}}(-\hat{E}) \otimes \phi_{\hat{E}}^*\big(\mathcal{O}_A(-K_Z - D + A)\big)$$

which is the tautological line bundle for

$$\mathbb{P}_A\big(\mathcal{E}|_A \otimes \mathcal{O}_A(-A) \otimes \mathcal{O}_A(-K_Z - D + A)\big) = \mathbb{P}_A\big(\mathcal{E}|_A \otimes \mathcal{O}_A(-K_Z - D)\big).$$

Thus, $-K_X|_{\hat{E}}$ is ample if and only if condition *(III)* is satisfied. This concludes the proof of the statement: X Fano $\implies$ *(I)+(III)*.

Observe that $E \simeq \mathbb{P}_{A_Y}(\mathcal{N}^*_{A_Y/Y})$, and $\mathcal{N}_{A_Y/Y} \simeq \mathcal{O}_{A_Y}(\hat{S}_Y) \oplus \mathcal{O}_{A_Y}(\pi^{-1}(A)) \simeq \det\mathcal{E}|_A \oplus \mathcal{O}_A(A)$ because A_Y is the smooth complete intersection of the divisors S_Y and $\pi^{-1}(A)$. By similar computation as for $-K_X|_{\hat{E}}$, we can conclude that $-K_X|_E$ is ample if and only if $(-K_Z + D - A)|_A$ is ample on A.

Now, suppose conditions *(I)* and *(II)* are satisfied. Observe that

$$\begin{aligned}\mathcal{O}_X(-K_X) \simeq &\mathcal{O}_X\big(\sigma^*(-K_Y)\big) \otimes \mathcal{O}_X(-E) \simeq \mathcal{O}_X\big(2\sigma^*(S_Y)\big) \otimes \phi^*\big(\mathcal{O}(-K_Z) \otimes \det(\mathcal{E})^*\big) \otimes \mathcal{O}_X(-E)\\ \simeq &\mathcal{O}_X\big(2\sigma^*(S_Y) + \phi^*(-K_Z - D) - E\big).\end{aligned}$$

Now, consider the divisor $W_Y := S_Y + \pi^*(-K_Z - D)$ in Y. It is the tautological divisor for $\mathbb{P}(\mathcal{E} \otimes \mathcal{O}(-K_Z - D))$ and *(II)* implies that W_Y is ample. Then there exist $m, c \in \mathbb{N}$ such that

$$W_X := m\sigma^* W_Y - cE$$

is ample. Observe that $-K_X \sim \sigma^*(W_Y) + \sigma^*(S_Y) - E \sim \sigma^*(W_Y) + S$. Therefore

$$-mK_X \sim m\sigma^*(W_Y) + mS \sim W_X + cE + mS$$

which is the sum of an ample divisor and an effective one. This proves that *(II)* implies that $-K_X$ is big.

Let Γ be an irreducible curve in X.

- If $\Gamma \subset S$, then *(I)* implies that $-K_X \cdot \Gamma > 0$, by Kleiman's criterion of ampleness.
- If $\Gamma \subset E$, then again *(I)* implies that $-K_X \cdot \Gamma > 0$.
- If Γ is not contained in S nor in E, then $\sigma_*\Gamma$ is an irreducible curve in Y and *(II)* implies $-K_X \cdot \Gamma = S \cdot \Gamma + \sigma^*(W_Y) \cdot \Gamma = S \cdot \Gamma + W_Y \cdot \sigma_*\Gamma > 0$, because $S \cdot \Gamma \geq 0$ and W_Y is ample on Y.

This proves that *(I)* and *(II)* imply that $-K_X$ is strictly nef. Since it is also big, we conclude that *(I)* and *(II)* imply that $-K_X$ is ample, by the base-point-free theorem. $\square$

Remark 3.3. Observe that whenever the exact sequence (2) in Theorem 3.2 splits, we have $\mathcal{E} \simeq \mathcal{O} \oplus \mathcal{O}(D)$ and Construction B coincides with Construction A, with $\hat{G}_Y = S_Y$ and D as the Cartier divisor such that $\mathcal{O}(D) = \mathcal{N}_{\hat{G}_Y/Y}$. Observe that if we call $D_1 := D$ and $D_2 := A - D$, condition *(I)* in Theorem 3.2 is the same as asking $-K_Z - D_2$ ample on Z. Moreover, in this setting condition *(II)* in Theorem 3.2 becomes equivalent to requiring that $-K_Z - D_1$ is ample on Z, so that we recover both necessary and sufficient conditions for X to be Fano in Proposition 2.3. Indeed in this case

$$\mathcal{E} \otimes \mathcal{O}(-K_Z - D) \simeq \mathcal{O}(-K_Z - D) \oplus \mathcal{O}(-K_Z)$$

which is ample if and only if both factors are ample line bundles on Z, i.e. $-K_Z - D$ and $-K_Z$ are ample on Z.

Similarly, if $\mathcal{E} \simeq \mathcal{O} \oplus \mathcal{O}(D)$, condition *(III)* in Theorem 3.2 becomes equivalent to requiring that $(-K_Z - D)|_A$ is ample on A.

From this observation, it follows that conditions *(I)* and *(III)* in Theorem 3.2 are NOT sufficient for X to be Fano. For example, take $Z = \mathbb{P}^1 \times \mathbb{P}^1$, $A \equiv \mathbf{l_1}$, $D \equiv 2\mathbf{l_1}$, where $\mathbf{l_1}$ and $\mathbf{l_2}$ are the classes of the two rulings in $N^1(Z)$, and consider $X = \mathscr{C}_B(Z; A, \mathcal{O} \oplus \mathcal{O}(D) \twoheadrightarrow \mathcal{O}(D)) = \mathscr{C}_A(Z; A, D)$. Conditions (I) and (III) are satisfied, but $-K_Z - D = 2\mathbf{l_2}$ is not ample, thus X is not Fano by Proposition 2.3.

As for conditions *(I)* and *(II)*, we will see in Remark 7.1 that they are NOT necessary for X to be Fano.

Remark 3.4. If a $\mathbb{P}^1$-bundle $Y = \mathbb{P}(\mathcal{E}) \to Z$ has a section S_Y defined by the exact sequence (2), we say that $\mathcal{E}$ is "normalized" with respect to the section S_Y.

Suppose now that a section S_Y of π is given by a general short exact sequence

$$0 \to \mathcal{I} \longrightarrow \mathcal{E} \longrightarrow \mathcal{L} \to 0 \tag{3}$$

where $\mathcal{I}$ and $\mathcal{L}$ are two line bundles on Z. Call $\mathcal{N} := \mathcal{N}_{S_Y/Y} \simeq \mathcal{I}^* \otimes \mathcal{L}$ and $\mathcal{E}' := \mathcal{I}^* \otimes \mathcal{E}$. Now, tensoring (3) by $\mathcal{I}^*$, we get the sequence

$$0 \to \mathcal{O}_Z \longrightarrow \mathcal{E}' \longrightarrow \mathcal{N} \to 0 \tag{4}$$

which corresponds to a section of $\mathbb{P}(\mathcal{E}') \longrightarrow Z$ isomorphic to S_Y through the isomorphism $\mathbb{P}(\mathcal{E}') \simeq \mathbb{P}(\mathcal{E})$. This shows that it is always possible to describe a $\mathbb{P}^1$-bundle as the projectivization of a "normalized" vector bundle with respect to a given section.

As for the Lefschetz defect of X, Lemma 2.4 also holds in the setting of Construction B. Indeed, the proof is still valid after replacing $\hat{G}_Y$ with S_Y and $\hat{G}$ with S.

Lemma 3.5. *Suppose $X = \mathscr{C}_B(Z; A, S_Y)$ is Fano. Then we have:*

(i) $\delta_X \geq 2$.

(ii) $\delta_X \geq \delta_Z$*. Moreover, if* $\operatorname{codim}(N_1(A, Z)) = \delta_Z$*, then* $\delta_X \geq \delta_Z + 1$.

(iii) If $\delta_X \geq 4$*, then* $\delta_X \in \{\delta_Z, \delta_Z + 1\}$.

In particular, If $\delta_Z \leq 2$ *then* $2 \leq \delta_X \leq 3$.

4 Maruyama's elementary transformation

In this section we explore the connections between Construction A, Construction B, and the setting of Maruyama's elementary transformations (see [Mar82]). It turns out that working in Maruyama's framework provides an explicit embedding for any variety arising from Construction B into the fibered product of two projective bundles. This embedding sheds light on the symmetries and asymmetries of the structure of varieties X arising from Construction B, and might be a good starting point for trying to fill the gap between necessary and sufficient conditions for X to be Fano in Theorem 3.2, or at least for better understanding the reason behind such a gap.

4.1 Setting

We present the setting of the Maruyama construction described in [Mar82]. Let $\mathcal{E}$ be a locally free sheaf on a locally noetherian scheme Z and $\mathcal{F}$ be a locally free sheaf on an effective Cartier divisor A of Z. If there is a surjective homomorphism $\psi : \mathcal{E} \to \mathcal{F}$ of $\mathcal{O}_Z$-modules, then $\mathcal{E}' = \ker(\psi)$ is a locally free sheaf on Z. Indeed, consider $p \in Z - A$. Then clearly $\mathcal{F}_p = 0$ and $\ker(\psi)_p \simeq \mathcal{E}_p$, which is a free $\mathcal{O}_{Z,p}$-module. On the other hand, if $p \in A$, then let R denote the local ring $\mathcal{O}_{Z,p}$. We have $\mathcal{O}_{A,p} = R/(f)$ for some nonzero divisor $f \in R$, $\mathcal{E}_p \simeq R^{\oplus k}$ and $\mathcal{F}_p \simeq (R/(f))^{\oplus l}$ for some $k, l \in \mathbb{N}$. Recall that

$$0 \to R \xrightarrow{\cdot f} R \to R/(f) \to 0$$

is a projective resolution of $R/(f)$, so that the projective dimension of $R/(f)$ is $pd_R(R/(f)) \leq 1$, which yields $pd_R(\mathcal{F}_p) \leq 1$. Since $\mathcal{E}_p$ is projective, it follows that $\mathcal{E}'_p$ is also projective, hence free with rank k.

The procedure to obtain $\mathcal{E}'$ from $\mathcal{E}$ is said to be the *(sheaf theoretic) Maruyama's elementary transformation* of $\mathcal{E}$ along $\mathcal{F}$ and we denote it by $\mathcal{E}' = \operatorname{elm}_Z(\mathcal{E}, \mathcal{F})$. The sheaf $\mathcal{E}'$ is said to be the *elementary transform* of $\mathcal{E}$ along $\mathcal{F}$. Such elementary transformation yields the exact commutative diagram in Figure 2,

$$
\begin{tikzcd}
 & 0 & 0 & & \\
0 \arrow[r] & \mathcal{F}' \arrow[r] \arrow[u] & \mathcal{E}|_A \arrow[r] \arrow[u] & \mathcal{F} \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{E}' \arrow[r] \arrow[u, "\psi'"] & \mathcal{E} \arrow[r, "\psi"] \arrow[u] & \mathcal{F} \arrow[r] \arrow[u, equal] & 0 \\
 & \mathcal{E}(-A) \arrow[r, equal] \arrow[u] & \mathcal{E}(-A) \arrow[u] & & \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}
$$

Figure 2: Maruyama's elementary transformation diagram.

where the leftmost vertical exact sequence gives us the *inverse* of the given transformation, i.e. $\mathcal{E}(-A) = \mathrm{elm}_Z(\mathcal{E}', \mathcal{F}')$. Observe that $\mathbb{P}(\mathcal{F})$ and $\mathbb{P}(\mathcal{F}')$ are closed subschemes of $\mathbb{P}(\mathcal{E})$ and $\mathbb{P}(\mathcal{E}')$ respectively, because ψ and ψ' are surjective. We have the following geometric interpretation of the above operation:

Theorem 4.1 ([Mar73], Theorem 1.1)**.** *Let $\sigma\colon X \to \mathbb{P}(\mathcal{E})$ (or $\sigma' : X' \to \mathbb{P}(\mathcal{E}')$) be the blowing-up along $\mathbb{P}(\mathcal{F})$ (or $\mathbb{P}(\mathcal{F}')$ resp.).Then we have an isomorphism $h : X \to X'$ of Z-schemes. For the projection $\pi\colon \mathbb{P}(\mathcal{E}) \to Z$ (or $\pi' : \mathbb{P}(\mathcal{E}') \to Z$), the proper transform of $\sigma^{-1}\big(\pi^{-1}(A)\big)$ (or $(\sigma')^{-1}\big((\pi')^{-1}(A)\big)$ resp.) is equal to the exceptional divisor of σ' (resp. of σ).*

$$
\begin{tikzcd}
X \arrow[rr, "\sim", "h"'] \arrow[d, "\sigma"'] & & X' \arrow[d, "\sigma'"] \\
\mathbb{P}(\mathcal{E}) \arrow[rr, dashed] \arrow[dr, "\pi"'] & & \mathbb{P}(\mathcal{E}') \arrow[dl, "\pi'"] \\
 & Z &
\end{tikzcd}
$$

The birational map $\sigma' \circ h \circ \sigma^{-1} : \mathbb{P}(\mathcal{E}) \to \mathbb{P}(\mathcal{E}')$ is called *(geometric) Maruyama's elementary transformation* of $\mathbb{P}(\mathcal{E})$ along $\mathbb{P}(\mathcal{F})$.

Call $Y := \mathbb{P}(\mathcal{E})$ and $Y' := \mathbb{P}(\mathcal{E}')$. Consider $\mathcal{Q} := Y \underset{Z}{\times} Y'$ and let $p : \mathcal{Q} \to Y$ and $p' : \mathcal{Q} \to Y'$ be the projections onto its factors. Then we have canonical isomorphisms

$$\mathbb{P}_Y(\pi^*\mathcal{E}') \simeq \mathcal{Q} \simeq \mathbb{P}_{Y'}({\pi'}^*\mathcal{E})$$

so that the following diagram commutes:

$$
\begin{tikzcd}
\mathcal{Q} \arrow[r, "p'"] \arrow[d, "p"'] & \mathbb{P}(\mathcal{E}') \arrow[d, "\pi'"] \\
\mathbb{P}(\mathcal{E}) \arrow[r, "\pi"'] & Z
\end{tikzcd}
$$

The idea of the proof of Theorem 4.1 is based on the fact that there are natural embeddings $i : X \hookrightarrow \mathcal{Q}$ and $i' : X' \hookrightarrow \mathcal{Q}$ whose images coincide in $\mathcal{Q}$.

Remark 4.2. Let $q : \mathcal{Q} \to Z$ be the composition $p \circ \pi = p' \circ \pi'$. If $\mathcal{E}$ is a rank 2 vector bundle on Z and $\mathcal{F}$ is a line bundle on A, then X is embedded in $\mathcal{Q}$ as a codimension 1 subvariety, and

$$X \sim p^*\mathcal{O}_Y(1) + (p')^*\mathcal{O}_{Y'}(1) + q^*(A - \det \mathcal{E}) \sim p^*\mathcal{O}_Y(1) + (p')^*\mathcal{O}_{Y'}(1) - q^* \det \mathcal{E}'$$

as a Cartier divisor in $\mathcal{Q}$.

4.2 Relation to Construction A

In the setting of Section 4.1, take Z a smooth Fano variety, D_1 and D_2 two Cartier divisors in Z such that $A \sim D_1 + D_2$ is the class of a smooth irreducible hypersurface. Consider $\mathcal{E} = \mathcal{O}_Z \oplus \mathcal{O}_Z(D_1)$, $\mathcal{F} = \mathcal{O}_A(D_1)$ and $\psi : \mathcal{E} \to \mathcal{F}$ the surjection given by the projection onto the second factor and the restriction to A. The kernel of this morphism of $\mathcal{O}_Z$-modules is $\mathcal{E}' = \mathcal{O}_Z \oplus \mathcal{O}_Z(-D_2)$. Therefore, in this case we have $\mathrm{elm}_Z(\mathcal{O}_Z \oplus \mathcal{O}_Z(D_1), A) = \mathcal{O}_Z \oplus \mathcal{O}_Z(-D_2)$.

Call $Y_i = \mathbb{P}(\mathcal{O}_Z \oplus \mathcal{O}_Z(D_i))$, $i = 1, 2$. Observe that in this setting, the first row of the commutative diagram in Figure 2 is a short exact sequence that defines $\mathbb{P}(\mathcal{F})$ as a section of the $\mathbb{P}^1$-bundle $\pi|_{\mathbb{P}_A(\mathcal{E}|_A)} : \mathbb{P}_A(\mathcal{E}|_A) \to A$. That sequence is nothing but the restriction to A of the short exact sequence representing the section $\hat{G}_{Y_1} = c_1(\mathcal{O}_{Y_1}(1))$ in Y_1:

$$0 \to \mathcal{O}_Z \to \mathcal{O}_Z \oplus \mathcal{O}_Z(D_1) \to \mathcal{O}_Z(D_1) \to 0.$$

This means that $\mathbb{P}(\mathcal{F})$ coincides with the complete intersection of $\pi^{-1}(A) = \mathbb{P}_A(\mathcal{E}|_A)$ and $\hat{G}_{Y_1}$ in Y_1. By an analogous argument on $\mathbb{P}(\mathcal{F}')$ in $Y_2 \simeq \mathbb{P}(\mathcal{E}')$, it is clear that Theorem 4.1 applied to this setting yields $\mathscr{C}_A(Z; A, D_1) \simeq \mathscr{C}_A(Z; A, D_2)$, as we mentioned in Section 2.1.

4.3 Relation to Construction B

We now work in the setting of Construction B (see Section 3.1). Consider the surjection $\psi : \mathcal{E} \twoheadrightarrow \mathcal{L}|_A$ given by the composition of the surjection $\mathcal{E} \twoheadrightarrow \mathcal{L}$ defining the section S_Y, and the restriction to A. Then we have the exact commutative diagram in Figure 3.

$$\begin{tikzcd}
 & 0 & 0 & & \\
0 \arrow[r] & \mathcal{F}' \arrow[r] \arrow[u] & \mathcal{E}|_A \arrow[r] \arrow[u] & \mathcal{L}|_A \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{E}' \arrow[r] \arrow[u, "\psi'"] & \mathcal{E} \arrow[r, "\psi"] \arrow[u] & \mathcal{L}|_A \arrow[r] \arrow[u, equal] & 0 \\
 & \mathcal{E}(-A) \arrow[r, equal] \arrow[u] & \mathcal{E}(-A) \arrow[u] & & \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

Figure 3: Maruyama's elementary transformation in the setting of Construction B.

Let $\mathcal{E}' = \mathrm{elm}_Z(\mathcal{E}, \mathcal{L}|_A)$. Observe that $\mathbb{P}(\mathcal{L}_A)$ coincides with the codimension 2 subvariety A_Y in $Y = \mathbb{P}(\mathcal{E})$. Call $Y' := \mathbb{P}(\mathcal{E}')$. By Theorem 4.1, we have a commutative diagram

$$\begin{tikzcd}
X \arrow[rr, "\sim", "h"'] \arrow[d, "\sigma"'] & & X' \arrow[d, "\sigma'"] \\
Y \arrow[rr, dashed] \arrow[dr, "\pi"'] & & Y' \arrow[dl, "\pi'"] \\
& Z &
\end{tikzcd}$$

where σ is the blowing up of $A_Y = \mathbb{P}(\mathcal{L}_A)$ in Y and σ' is the blowing up of $\mathbb{P}(\mathcal{F}')$ in Y'. Observe that A_Y is the complete intersection of $\pi^{-1}(A)$ and the section S_Y corresponding to the surjection $\mathcal{E} \twoheadrightarrow \mathcal{L}$, so that X actually coincides with $\mathscr{C}_B(Z; A, \mathcal{E} \twoheadrightarrow \mathcal{L})$, while $\mathbb{P}(\mathcal{F}')$ might very well not lie in any section of π'. This explains the asymmetrical nature of the conditions appearing in Theorem 3.2 in contrast to the symmetry of the conditions appearing in Proposition 2.3. Even the natural embedding of X in $\mathcal{Q} = Y \underset{Z}{\times} Y'$ does not help in finding more satisfying conditions for X to be Fano in the setting of Construction B.

We now give an example of a Fano 3-fold X arising from a construction of type B, whose factorization given by Theorem 4.1 does not describe another construction of type B, i.e. the blown-up locus in $Y' = \mathbb{P}(\mathcal{E}')$ does not lie in any section of the $\mathbb{P}^1$-bundle $\pi' \colon Y' \to Z$.

Example 4.3. Let X be the blow up of a curve of degree $(1,1,1)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. As we will see in Section 7, a curve of degree $(1,1,1)$ in $Y := \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is contained in a section S_Y of the projection onto the second and third factors $\pi \colon Y \to Z = \mathbb{P}^1 \times \mathbb{P}^1$, and such section corresponds to the surjection $\mathcal{O}_Z \oplus \mathcal{O}_Z \twoheadrightarrow \mathcal{O}_Z(1,0)$ (unique up to multiplication by scalars). If we let A be the diagonal in $Z = \mathbb{P}^1 \times \mathbb{P}^1$, we have $X = \mathscr{C}_B(Z; A, S_Y)$. In order to find the alternative factorization of the composition $\phi = \sigma \circ \pi$ provided by Maruyama, we need to consider the surjection $\mathcal{O}_Z \oplus \mathcal{O}_Z \twoheadrightarrow \mathcal{O}_Z(1,0)|_A$. Then we have

$$\mathcal{E}' = \mathrm{elm}_Z(\mathcal{O}_Z \oplus \mathcal{O}_Z, \mathcal{O}_Z(1,0)|_A) = \ker(\mathcal{O}_Z \oplus \mathcal{O}_Z \twoheadrightarrow \mathcal{O}_Z(1,0)|_A) = \mathcal{O}_Z(-1,0) \oplus \mathcal{O}_Z(0,-1)$$

and

$$\mathcal{F}' = \ker(\mathcal{O}_A \oplus \mathcal{O}_A \twoheadrightarrow \mathcal{O}_Z(1,0)|_A) = \mathcal{O}_Z(-1,0)|_A \simeq \mathcal{O}_{\mathbb{P}^1}(-1).$$

Call $Y' = \mathbb{P}(\mathcal{E}')$, $\pi' : Y' \to Z$ the projection and σ' the blow up of $\mathbb{P}(\mathcal{F}')$ in Y'. Theorem 4.1 yields $\sigma \circ \pi = \phi = \sigma' \circ \pi'$.

$$\begin{tikzcd}
X \arrow[r, "\sigma'"] \arrow[d, "\sigma"'] \arrow[dr, "\phi"] & Y' \arrow[d, "\pi'"] \\
Y \arrow[r, "\pi"'] & Z
\end{tikzcd}$$

The blown up locus $A_{Y'} = \mathbb{P}(\mathcal{F}')$ in Y' does not lie in any section of π'. Indeed, suppose there is a section $S_{Y'}$ that contains $A_{Y'}$. We know that $A_{Y'}$ is disjoint from the section $\sigma'(S)$, where S is the strict transform of S_Y in X. If $\sigma'(S) \cap S_{Y'} = \emptyset$, then the composition $\sigma' \circ \pi'$ would yield a construction of type A for X, and $X = \mathscr{C}_A(Z; A, D)$ with $\mathcal{O}(D) \simeq \mathcal{N}_{S_{Y'}/Y'}$. However X does not appear in the list of Fano 3-folds arising from Construction A obtained in Section 6. This implies that $\sigma'(S) \cap S_{Y'} \neq \emptyset$, and since A is ample on Z, this yields $A_{Y'} \cap \sigma'(S) \neq \emptyset$, which is a contradiction.

Remark 4.4. Consider the following Fano 3-folds:

- X_1= Blow up of a $(1, 1, 2)$-curve in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.
- X_2= Blow up of a $(1, 1, 3)$-curve in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

If we consider their descriptions as varieties arising from Construction B (see Section 7), it is clear that their factorizations through the construction given by Maruyama do not yield another construction of type B. This can be seen with similar computations as in Example 4.3.

5 Recursion

In this section, we study varieties obtained by performing two successive constructions of type A.

5.1 Setting

Consider a smooth variety $V = \mathscr{C}_A(X; B, L)$. Call $W := \mathbb{P}_X(\mathcal{O}_X \oplus \mathcal{O}_X(L))$, $\rho : W \to X$ the projection, and H_W, $\hat{H}_W$ the two disjoint sections of ρ, $\hat{H}_W$ being the positive one (as described in Section 2.1). Call $B_W := \rho^{-1}(B) \cap \hat{H}_W$, $\tau : V \to W$ the blow up of B_W in W, and $\psi := \rho \circ \tau$.

Suppose that X also arises from Construction A, i.e. $X = \mathscr{C}_A(Z; A, D)$, with Z Fano. Call $Y := \mathbb{P}_Z(\mathcal{O}_Z \oplus \mathcal{O}_Z(D))$, $\pi : Y \to Z$ the projection, and G_Y, $\hat{G}_Y$ the two disjoint sections of π, $\hat{G}_Y$ being the positive one. Call $A_Y := (\pi)^{-1}(A) \cap \hat{G}_Y$, $\sigma : X \to Y$ the blow up of A_Y in Y, and $\phi := \pi \circ \sigma$. We are therefore in the presence of a tower like the one in Figure 4.

$$\begin{array}{ccc}
V & & \\
\downarrow^{\tau} & & \\
W = \mathbb{P}_X(\mathcal{O} \oplus \mathcal{O}(L)) & \supset & B_W := \rho^{-1}(B) \cap \hat{H}_W \\
\downarrow^{\rho} & & \\
X & \supset & B,\, L \\
\downarrow^{\sigma} & & \\
Y = \mathbb{P}_Z(\mathcal{O} \oplus \mathcal{O}(D)) & \supset & A_Y := (\pi)^{-1}(A) \cap \hat{G}_Y \\
\downarrow^{\pi} & & \\
Z & \supset & A, D
\end{array}$$

Figure 4: Iteration of two constructions of type A.

Our goal is to find conditions for V to be Fano that can be tested on Z. This will be particularly useful in Section 10.2. In order to do so, we first establish some preliminary results.

One important consequence of Remark 2.5 is the following lemma, which holds more generally when X arises from Construction B.

Lemma 5.1. *Let $V = \mathscr{C}_A(X; B, L)$ be a smooth Fano variety and assume that $X = \mathscr{C}_B(Z; A, S_Y)$. Then there exist A' and D' in Z such that $B = \phi^*A'$ and $L = \phi^*D'$.*

Proof. For X we use the same notation as in Section 3.1. We know that $\mathrm{Pic}(X) = \mathbb{Z}E \oplus \sigma^*\mathrm{Pic}(Y) = \mathbb{Z}E \oplus \mathbb{Z}\sigma^*S_Y \oplus \phi^*\mathrm{Pic}(Z) = \mathbb{Z}E \oplus \mathbb{Z}S \oplus \phi^*\mathrm{Pic}(Z)$. Recall that $\phi|_E : E \to A$ and $\phi|_{\hat{E}} : \hat{E} \to A$ are both $\mathbb{P}^1$-bundles. Let e denote the class of a 1-dimensional fiber in E and $\hat{e}$ the class of a one-dimensional fiber in $\hat{E}$. We know that $-K_X \cdot e = 1 = -K_X \cdot \hat{e}$ (see Remark 3.1). Therefore, by Remark 2.5 we have $L \cdot e = B \cdot e = 0$ or $B \cdot e < 0$ and analogously $L \cdot \hat{e} = B \cdot \hat{e} = 0$ or $B \cdot \hat{e} < 0$. However if $B \cdot e < 0$, then $F \subset B$ for every fiber F of σ, i.e. $E \subset B$, which implies $E = B$ since they are both irreducible surfaces in X. But then $B \cdot \hat{e} = E \cdot \hat{e} = 1$, which is a contradiction. Therefore $L \cdot e = B \cdot e = 0$ and similarly it can be proved that $L \cdot \hat{e} = B \cdot \hat{e} = 0$ and $B \neq \hat{E}$, so that $B, L \in \mathrm{NE}(\phi)^{\perp}$. Thus there exists an effective Cartier divisor $A' \subset Z$ such that $B = \phi^*A'$ and there exists a Cartier divisor $D' \subset Z$ such that $L = \phi^*D'$. □

Proposition 5.2. *Let Z be a smooth Fano variety. Consider A, A' two distinct smooth irreducible hypersurfaces and D, D' two Cartier divisors in Z. Call $\phi : X = \mathscr{C}_A(Z; A, D) \to Z$ and $\phi' : X' = \mathscr{C}_A(Z; A', D') \to Z$ the conic bundles induced by Construction A. Then*

$$\mathscr{C}_A\big(\mathscr{C}_A(Z, A, D); \phi^*A', \phi^*D'\big) \simeq \mathscr{C}_A\big(\mathscr{C}_A(Z, A', D'); (\phi')^*A, (\phi')^*D\big)$$

i.e. the following diagram commutes:

$$\begin{tikzcd} V \simeq V' \arrow[r, "\psi'"] \arrow[d, "\psi"'] \arrow[dr, "\Phi"] & X' \arrow[d, "\phi'"] \\ X \arrow[r, "\phi"'] & Z \end{tikzcd} \tag{5}$$

where $\psi : V = \mathscr{C}_A(X; \phi^*A', \phi^*D') \to X$ and $\psi' : V' = \mathscr{C}_A(X'; (\phi')^*A, (\phi')^*D) \to X'$ are the conic bundles induced by Construction A.

Proof. In $X = \mathscr{C}_A(Z; A, D)$ consider the exceptional divisors E, $\hat{E}$, and the two disjoint sections G, $\hat{G}$ as in Section 2.1. In $\mathrm{NE}(X)$ consider the classes e and $\hat{e}$ of the irreducible one-dimensional fibers in E and $\hat{E}$ respectively, so that $\mathrm{NE}(\phi) = \mathbb{R}_{\geq 0}e + \mathbb{R}_{\geq 0}\hat{e}$. Analogously, in $V = \mathscr{C}_A(X; \phi^*A', \phi^*D')$ consider the exceptional divisors F, $\hat{F}$, and the two disjoint sections H, $\hat{H}$. Let f and $\hat{f}$ be the classes of the irreducible one-dimensional fibers in the exceptional divisors F and $\hat{F}$ respectively, so that $\mathrm{NE}(\psi) = \mathbb{R}_{\geq 0}f + \mathbb{R}_{\geq 0}\hat{f}$.

Call $\Phi := \phi \circ \psi$. Observe that $\mathrm{NE}(\Phi)$ has dimension 4. We now show that Φ is K_V-negative, so that $\mathrm{NE}(\Phi)$ is a rational polyhedral extremal face of $\mathrm{NE}(V)$.

If $[C] \in \mathrm{NE}(\Phi)$ is the class of an irreducible curve C that is contracted by Φ, then $[C] \in \mathrm{NE}(\Phi)$ and by construction it is already known that $-K_V \cdot C > 0$. On the other hand, if C is an irreducible curve contracted by Φ but not by ψ, it means that ψ_*C is an irreducible component T of a fiber of ϕ and its class in $\mathrm{NE}(X)$ lies in $\mathrm{NE}(\phi) = \mathbb{R}_{\geq 0}e + \mathbb{R}_{\geq 0}\hat{e}$. In order to prove that $-K_V \cdot C > 0$, it suffices to show that $-K_V$ is positive on $\psi^{-1}(T) \subset V$. Observe that there are only two possible cases for T:

1. $T \subset \phi^*A'$ ($\iff$ $\phi(T) \in A'$).
2. $T \cap \phi^*A' = \emptyset$ ($\iff$ $\phi(T) \notin A'$).

Denote $B := \phi^* A'$, $L := \phi^* D'$, consider the $\mathbb{P}^1$-bundle $\rho : W = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(L)) \to X$ and the blow up $\tau : V \to W$ along the subvariety $B_W := \rho^{-1}(B) \cap \hat{H}_W$.

In Case 1, $\rho^{-1}(T) = \mathbb{P}_T(\mathcal{O}_T \oplus \mathcal{O}_T(L)) = \mathbb{P}^1 \times \mathbb{P}^1$ because $L \cdot T = D' \cdot \phi_* T = 0$. Call $T_W := \rho^{-1}(T) \cap \hat{H}_W \subset B_W$. Then $\psi^{-1}(T) = \mathrm{Bl}_{T_W}(\rho^{-1}(T))$. However T_W is a codimension 1 subvariety of $\rho^{-1}(T)$, hence $\psi^{-1}(T) \simeq \rho^{-1}(T) \simeq \mathbb{P}^1 \times \mathbb{P}^1$. Let $\mathbf{l}_1$, $\mathbf{l}_2$ be the embeddings in $\mathrm{NE}(V)$ of the two rulings in $\mathrm{NE}(\psi^{-1}(T))$ such that $\psi_*(\mathbf{l}_1) = [T]$ and $\psi_*(\mathbf{l}_2) = 0$, where $\mathbf{l}_1$ is the class of the transform of T_W in V. Then clearly

- $-K_V \cdot \mathbf{l}_1 = \phi^*(-K_X) \cdot \mathbf{l}_1 + H \cdot \mathbf{l}_1 + \hat{H} \cdot \mathbf{l}_1 = -K_X \cdot T + 0 + L \cdot T = -K_X \cdot T > 0$
- $-K_V \cdot \mathbf{l}_2 > 0$

because $-K_X$ is positive on $\mathrm{NE}(\phi)$ and $-K_V$ is positive on $\mathrm{NE}(\psi)$.

In Case 2, it is easy to see that $\psi^{-1}(T) \simeq \rho^{-1}(T) \simeq \mathbb{P}^1 \times \mathbb{P}^1$, and through similar computations as in Case 1, we can conclude that $-K_V$ is positive in $\psi^{-1}(T)$.

This concludes the proof that $\mathrm{NE}(\Phi)$ is a (4-dimensional) closed rational polyhedral extremal face of $\mathrm{NE}(V)$.

We have $\psi_* \mathrm{NE}(\Phi) = \mathrm{NE}(\phi)$ through ψ, and faces of $\mathrm{NE}(\phi)$ are in bijection with faces of $\mathrm{NE}(\Phi)$ containing $\mathrm{NE}(\psi)$ (see [Cas08, §2.5]). Consider the extremal ray $\alpha := \mathbb{R}_{\geq 0} e$ of $\mathrm{NE}(\phi)$, and let $\hat{\alpha}$ be the unique face of $\mathrm{NE}(\Phi)$ containing $\mathrm{NE}(\psi)$ such that $\psi_*(\hat{\alpha}) = \alpha$. We have $\dim \hat{\alpha} = \dim \alpha + \dim \mathrm{NE}(\psi) = 3$. Let $\tilde{\alpha}$ be an extremal ray in $\hat{\alpha}$ not contained in $\mathrm{NE}(\psi)$ such that $\psi_*(\tilde{\alpha}) = \alpha$. By definition, there exists an irreducible curve whose class $f' \in \mathrm{NE}(V)$ spans $\tilde{\alpha}$ and $\psi_*(f') = e$ in $\mathrm{NE}(\phi)$. Therefore any irreducible representative T' of f' must lie in $\psi^{-1}(T) \simeq \mathbb{P}^1 \times \mathbb{P}^1$, for some irreducible representative T of e, so that in $\mathrm{NE}(\psi^{-1}(T))$ the class of T' spans the ray generated by the *horizontal* ruling $\mathbf{l_1}$. This is true because of the extremality of $\tilde{\alpha}$. In other words, we can take $T' = \psi^{-1}(T) \cap \hat{H}$ as a curve representing f'. Call $F' := \psi^*(E)$ and observe that $F' \cdot f' = E \cdot e = -1$, $H \cdot f' = 0$ and $\hat{H} \cdot f' = \phi^* D \cdot e = 0$. Clearly F' is covered by curves in f', and actually coincides with the locus of $\tilde{\alpha}$. Therefore the elementary contraction $\tau' : V \to W'$ associated to $\tilde{\alpha}$ (which exists because $\mathrm{NE}(\Phi)$ is K_V-negative) is divisorial and has fibers of dimension at most 1, so that it is in fact the blow up of a smooth irreducible codimension 2 subvariety in W', with exceptional divisor F'.

In the same way, we can prove that there exists a class $\hat{f}'$ in $\mathrm{NE}(V)$ that spans an extremal ray and such that $\psi_*(\hat{f}') = \hat{e}$, $\hat{F}' \cdot \hat{f}' = -1$ (where $\hat{F}' = \psi^*(\hat{E})$), and $H \cdot \hat{f}' = 0 = \hat{H} \cdot \hat{f}'$. Let $\hat{\tau}' : V \to \hat{W}'$ be the blow up induced by the extremal ray spanned by $\hat{f}'$. Observe that f' and $\hat{f}'$ are linearly independent in $\mathrm{NE}(\Phi)$ because e and $\hat{e}$ are linearly independent in $\mathrm{NE}(\phi)$.

Consider the classes f and $\hat{f}$ in F and $\hat{F}$ respectively. Observe that the following intersection table

$\cdot$	f	$\hat{f}$	f'	$\hat{f}'$
$\hat{H}$	1	0	0	0
H	0	1	0	0
$\phi^*\hat{G}$	0	0	1	0
ϕ^*G	0	0	0	1

ensures that f, $\hat{f}$, f', $\hat{f}'$ are linearly independent in $\mathrm{NE}(\Phi)$, which has dimension 4, hence every element in $\mathrm{NE}(\Phi)$ can be expressed in a unique way as a linear combination of those four classes. Also, observe that $F \cdot f' = 0 = F \cdot \hat{f}'$ and $\hat{F} \cdot f' = 0 = \hat{F} \cdot \hat{f}'$ as a consequence of the fact that $B \cdot e = 0 = B \cdot \hat{e}$ in X.

We already know that the 2-dimensional face $\mathbb{R}_{\geq 0}f + \mathbb{R}_{\geq 0}\hat{f}$ in $\mathrm{NE}(\Phi)$ is extremal and coincides with $\mathrm{NE}(\psi)$. We are now going to prove that $\mathrm{NE}(\Phi) = \mathbb{R}_{\geq 0}f + \mathbb{R}_{\geq 0}\hat{f} + \mathbb{R}_{\geq 0}f' + \mathbb{R}_{\geq 0}\hat{f}'$.

Suppose $a \in \mathrm{NE}(\Phi)$. Then there exist coefficients $a_1, a_2, a_3, a_4 \in \mathbb{R}$ such that

$$a = a_1 f + a_2 \hat{f} + a_3 f' + a_4 \hat{f}'$$

and we show that $a_i \geq 0$ for all $i = 1, .., 4$.

Observe that $\psi_*(a) = a_3 e + a_4 \hat{e}$ belongs to $\psi_* \mathrm{NE}(\Phi) = \mathrm{NE}(\phi)$, hence $a_3, a_4 \geq 0$. Suppose $a_1 < 0$. Then $\hat{H} \cdot a = a_1 < 0$ implies that every representative of a lies in $\hat{H}$, hence $0 = H \cdot a = a_2$. But then $\hat{F} \cdot a = a_1 - a_2 = a_1 < 0$, so that every representative of a lies in $\hat{H} \cap \hat{F}$, which is empty by construction. This proves that $a_1 \geq 0$, and similarly it can be proved that $a_2 \geq 0$.

In particular, the 2-dimensional face $\mathbb{R}_{\geq 0}f' + \mathbb{R}_{\geq 0}\hat{f}'$ in $\mathrm{NE}(\Phi)$ is extremal. This yields the existence of a contraction $\psi' : V \to X'$ such that $\mathrm{NE}(\psi') = \mathbb{R}_{\geq 0}f' + \mathbb{R}_{\geq 0}\hat{f}'$, which by rigidity implies the existence of another contraction $\phi' : X' \to Z$ that makes the following diagram commute.

$$\begin{tikzcd}
V \arrow[r, "\psi'"] \arrow[d, "\psi"'] \arrow[dr, "\Phi"] & X' \arrow[d, dashed, "\phi'"] \\
X \arrow[r, "\phi"'] & Z
\end{tikzcd}$$

Observe that $\hat{H}' := \psi^*(\hat{G})$ and $H' := \psi^*(G)$ are two disjoint sections of ψ', so that ψ' realizes V as a result of Construction A, with $V = \mathscr{C}_A(X', (\phi')^*A, (\phi')^*D)$. Moreover, since X' is a copy of H' (and $\hat{H}'$), it can be seen that $\phi' : X' \to Z$ yields $X' = \mathscr{C}_A(Z; A', D')$, where $G' := {\psi'}_*(H' \cap H)$ and $\hat{G}' := {\psi'}_*(H' \cap \hat{H})$ are two disjoint sections. $\square$

5.2 Conditions for being Fano

Consider a smooth variety $V = \mathscr{C}_A(X; B, L)$ and call $L_1 := L$, $L_2 := B - L_1$. Suppose that X also arises from Construction A, so that $X = \mathscr{C}_A(Z; A, D)$, with $D_1 := D$, $D_2 := A - D$. We assume that Z is Fano. By Lemma 5.1 we know that there exist two divisors A' and D' in Z such that $B = \phi^*A'$, $L = \phi^*D'$. Call $D'_1 := D'$ and $D'_2 := A' - D'$. Let $\phi' : X' = \mathscr{C}_A(Z; A', D') \to Z$ and $\psi' : V' = \mathscr{C}_A(Z; (\phi')^*A, (\phi')^*D) \to Z$ be the conic bundles induced by Construction A. By Proposition 5.2, $V \simeq V'$ and the diagram (5) commutes.

We want to find conditions for V to be Fano that can be tested on Z. By Proposition 2.3 and [Wiś91, Proposition 4.3] we have that V is Fano if and only if $-K_X - L_i$ is ample on X for $i = 1, 2$ and X is Fano. Observe that the equality $-K_X \sim \phi^*(-K_Z) + G + \hat{G}$ implies that

$$-K_X - L_i \sim \phi^*(-K_Z - D'_i) + G + \hat{G} \quad \text{for } i = 1, 2.$$

Therefore, if we assume that $-K_Z - D'_i$ is ample for a fixed $i \in \{1, 2\}$, following similar steps as in the proof of Proposition 2.3 we can show that

$$-K_X - L_i \text{ is ample on X} \iff -K_Z - D'_i - D_j \text{ are ample on } Z \text{ for } j = 1, 2.$$

In conclusion, we obtain the following proposition:

Proposition 5.3. *V is Fano if and only if $-K_Z - D'_i - D_j$ are ample for all $i, j \in \{1,2\}$ and $-K_Z - D'_i$ are ample for $i = 1, 2$ (or equivalently $-K_Z - D_j$ are ample for $j = 1, 2$).*

Remark 5.4. As for the Lefschetz defect, by Lemma 2.4, we have $2 \leq \delta_X \leq \delta_V$ so if we want $\delta_V = 2$ we need $\delta_X = 2$.

6 Fano 3-folds arising from Construction A

In this section we study how many families of Fano 3-folds with $\delta = 2$ arise from Construction A.

Let X be a Fano 3-fold with $\delta_X = 2$. By [DN14, Lemma 5.1], $\delta_X \in \{\rho_X - 1, \rho_X - 2\}$, so $\rho_X \in \{3, 4\}$. If X admits a construction of type A arising from a del Pezzo surface Z, we have $\rho_Z = \rho_X - 2$, hence $\rho_Z \in \{1, 2\}$.

- The case $\rho_Z = 1$ (i.e. $Z \simeq \mathbb{P}^2$) has already been studied by C. Casagrande and S. Druel. With such a choice for Z, Construction A yields all families of Fano 3-folds X with $\delta_X = 2$ and $\rho_X = 3$ [CD15, Theorem 3.8]. These are 6 among the 31 families of Fano 3-folds with $\rho_X = 3$ (see [IP99, Table 12.4]). The remaining ones have $\delta_X = 1$.
- If $\rho_Z = 2$, then $Z \simeq \mathbb{F}_1$ or $Z \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

For each of the latter two del Pezzo surfaces, we study all possible choices of D and A that yield $X = \mathscr{C}_A(Z; A, D)$ Fano, by using Proposition 2.3. We then assign to every resulting Fano 3-fold its number in [ACC+23, Table 6.1].

This study enables us to fully answer the question of how many families of Fano 3-folds with $\delta_X = 2$ arise from Construction A.

- $\rho_X = 3 \longrightarrow 6/6$ families arise from Construction A.
- $\rho_X = 4 \longrightarrow 9/13$ families arise from Construction A.

Remark 6.1. Given $X = \mathscr{C}_A(Z; A, D_i)$, using that $-K_X \sim \phi^*(-K_Z) + G + \hat{G}$ we can compute:

$$-K_X^3 = D_1^2 + D_2^2 + 6(-K_Z)^2 - 3(-K_Z) \cdot (D_1 + D_2). \tag{6}$$

6.1 The case $Z \simeq \mathbb{F}_1$

In $N_1(Z)$, let $\mathbf{f}$ be the class of the strict transform of a line passing through the blown-up point $p \in \mathbb{P}^2$ and denote by $\mathbf{e}$ the class of the exceptional divisor. Let $\mathbf{l}$ be the class of the transform of a generic line in $\mathbb{P}^2$, so that $\mathbf{l} = \mathbf{f} + \mathbf{e}$. We have $\mathrm{Eff}(Z) = \mathrm{NE}(Z) = \mathbb{R}_{\geq 0}\mathbf{f} \oplus \mathbb{R}_{\geq 0}\mathbf{e}$, $\mathrm{Nef}(Z) = \mathbb{R}_{\geq 0}\mathbf{f} \oplus \mathbb{R}_{\geq 0}\mathbf{l}$ and $-K_Z = 3\mathbf{f} + 2\mathbf{e}$. This is enough to find all couples (A, D) satisfying conditions in Proposition 2.3.

In the end we get the following 4 known families of Fano 3-folds, where we used Remark 6.1 to compute $-K_X^3$.

A	D	$-K_X^3$	# of the family of $X = \mathscr{C}_A(Z; A, D)$ in [ACC+23, Table 6.1]
$\mathbf{e}$	$-\mathbf{f}/\mathbf{f}+\mathbf{e}$	46	#4-12
	$\mathbf{e}/0$	44	#4-11
$\mathbf{f}+\mathbf{e}$	$\mathbf{f}+\mathbf{e}/0$	40	#4-9
$2\mathbf{f}+2\mathbf{e}$	$\mathbf{f}+\mathbf{e}$	32	#4-4

Table 2: Choices of A, D in $Z = \mathbb{F}_1$.

Remark 6.2. When $-K_X^3$ fails to determine the corresponding family unumbiguously (e.g. when $-K_X^3 = 32$), we are still able to solve the ambiguity through geometric arguments.

6.2 The case $Z \simeq \mathbb{P}^1 \times \mathbb{P}^1$.

In $N_1(Z)$, let $\mathbf{l_1}$ and $\mathbf{l_2}$ be the classes of the two rulings of Z. They generate $N_1(Z)$. Moreover, we have $\mathrm{Eff}(Z) = \mathrm{Nef}(Z) = \mathrm{NE}(Z) = \mathbb{R}_{\geq 0}\mathbf{l_1} \oplus \mathbb{R}_{\geq 0}\mathbf{l_2}$, and $-K_Z = 2\mathbf{l_1} + 2\mathbf{l_2}$. This is enough to find all couples (A, D) satisfying conditions in Proposition 2.3.

We recover the following known 7 families of Fano 3-folds.

A	D	$-K_X^3$	# of the family of $X = \mathscr{C}_A(Z; A, D)$ in [ACC+23, Table 6.1]
$\mathbf{l_1}$	$\mathbf{l_1}+\mathbf{l_2}/-\mathbf{l_2}$	44	#4-11
	$\mathbf{l_1}/0$	42	#4-10
	$\mathbf{l_1}-\mathbf{l_2}/\mathbf{l_2}$	40	#4-9
$\mathbf{l_1}+\mathbf{l_2}$	$0/\mathbf{l_1}+\mathbf{l_2}$	38	#4-8
	$\mathbf{l_1}/\mathbf{l_2}$	36	#4-7
$\mathbf{l_1}+2\mathbf{l_2}$	$\mathbf{l_2}/\mathbf{l_1}+\mathbf{l_2}$	32	#4-5
$2\mathbf{l_1}+2\mathbf{l_2}$	$\mathbf{l_1}+\mathbf{l_2}$	28	#4-2

Table 3: Choices of A, D in $Z = \mathbb{P}^1 \times \mathbb{P}^1$

7 Fano 3-folds arising from Construction B

All remaining families of Fano 3-folds with $\delta = 2$ that do not arise from Construction A can be described as blow-ups of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ along a smooth curve:

(#4-6) X = Blow up of a curve of degree $(1, 1, 1)$.

(#4-3) X = Blow up of a curve of degree $(1, 1, 2)$.

(#4-13) X = Blow up of a curve of degree $(1, 1, 3)$.

(#4-1) X = Blow up of a curve of degree $(2, 2, 2)$ that is the complete intersection of two divisors of type $(1, 1, 1)$.

Observe that the blow up of a curve of degree $(1, 1, 3)$ was missing in the first classification by Mori and Mukai ([MM81]), and was only discovered years later (see [MM03]).

In this section, we study each of those families in order to determine whether they arise from Construction B or not. We will show that families #4-6,#4-3 and #4-13 all arise from Construction B, while family #4-1 cannot be obtained through any construction of type B, and is therefore the *only* family of Fano 3-folds with $\delta = 2$ for which that happens.

Throughout this section, in order to mirror the notation used to describe Construction B in Section 3.1, Y will always denote the variety $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, $\sigma\colon X \to Y$ will denote the blow up of the curve that defines X.

$\boxed{X = \textbf{Blow up of a } (1,1,k) \textbf{ curve in } Y = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1\textbf{, for } k = 1,2,3.}$ Call $Z = \mathbb{P}^1 \times \mathbb{P}^1$ and consider the projection $\pi\colon Y \to Z = \mathbb{P}^1 \times \mathbb{P}^1$ onto the second and third factors. Now consider the following section of π:

$$\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$
$$(y,z) \mapsto (y,y,z)$$

and call the image $S_Y := \{(y,y,z)|y,z \in \mathbb{P}^1\} = \{([x_0 : x_1],[y_0 : y_1],[z_0 : z_1])|x_0y_1 = x_1y_0\}$. Observe that $Y = \mathbb{P}(\mathcal{O}_Z \oplus \mathcal{O}_Z)$ and the short exact sequence of $\mathcal{O}_Z$-sheaves associated to the section S_Y is the following:

$$0 \to \mathcal{O}_Z(-1,0) \longrightarrow \mathcal{O}_Z \oplus \mathcal{O}_Z \longrightarrow \mathcal{O}_Z(1,0) \to 0.$$

Let A be a $(k,1)$-curve in Z (with $k \in \mathbb{N}$), and let A_Y be the complete intersection of $\pi^{-1}A$ and S_Y. A straightforward computation shows that A_Y is a curve of degree $(1,1,k)$ in Y.

This shows that if X is obtained through the blow up of a $(1,1,k)$ curve in Y, then $X = \mathscr{C}_B(Z; A, S_Y)$, and the families #4-6, #4-3 and #4-13 all arise from Construction B. Observe that in all these constructions conditions *(I)* and *(II)* of Theorem 3.2 are satisfied.

Remark 7.1. We now show that the second family (the blow up of a $(1,1,2)$-curve in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$) can be obtained through another construction of type B, for which condition *(II)* of Theorem 3.2 is no longer satisfied. Call $Y = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, $Z = \mathbb{P}^1 \times \mathbb{P}^1$, and consider the projection $\pi_{12} : Y \to Z$ onto the first and second factors. Now consider the following section of π_{12}:

$$\mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$$
$$(x,y) \mapsto (x,y,y^2)$$

and call the image $S_Y := \{(x,y,y^2)|x,y \in \mathbb{P}^1\} = \{([x_0 : x_1],[y_0 : y_1],[z_0 : z_1])|y_0^2z_1 = y_1^2z_0\}$. Observe that $Y = \mathbb{P}(\mathcal{O}_Z \oplus \mathcal{O}_Z)$ and the short exact sequence of $\mathcal{O}_Z$-sheaves associated to the section S_Y is the following:

$$0 \to \mathcal{O}_Z(-2,0) \longrightarrow \mathcal{O}_Z \oplus \mathcal{O}_Z \longrightarrow \mathcal{O}_Z(2,0) \to 0$$

which becomes

$$0 \to \mathcal{O}_Z \longrightarrow \mathcal{O}_Z(2,0) \oplus \mathcal{O}_Z(2,0) \longrightarrow \mathcal{O}_Z(4,0) \to 0$$

after normalization. Let A be a $(1,1)$-curve in Z, and let A_Y be the complete intersection of $\pi^{-1}(A)$ and S_Y. A straightforward computation shows that A_Y is a curve of degree $(1,1,2)$ in Y. We know that the blow up X of such a curve is a Fano 3-fold, because its description appears in the classification by Mori-Mukai. However, we have just provided a construction of type B for X for which condition *(II)* in Theorem 3.2

$$\textit{(II)}\ \mathcal{E} \otimes \mathcal{O}(-K_Z - D) \text{ is ample on } Z$$

is not satisfied. Indeed, in this setting $\mathcal{E} = \mathcal{O}(2,0) \oplus \mathcal{O}(2,0)$ and $\mathcal{O}(D) = \mathcal{O}(4,0)$, so we have $\mathcal{E} \otimes \mathcal{O}(-K_Z - D) = \mathcal{O}(0,2) \oplus \mathcal{O}(0,2)$, which is not an ample vector bundle on Z. This proves that conditions *(I)* and *(II)* in Theorem 3.2 are sufficient but not necessary for X to be Fano.

$\boxed{X = \textbf{Blow up of a } (2,2,2)\textbf{-curve in } \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1\textbf{.}}$ Call $Z = \mathbb{P}^1 \times \mathbb{P}^1$ and consider the projection $\pi \colon Y \to Z$ onto the second and third factors. Call A_Y a curve of degree $(2,2,2)$ that is the complete intersection of two $(1,1,1)$-divisors in Y, call A its image in Z through π, and X the blow up of A_Y in Y. It is not difficult to prove that A has to be a curve of degree $(2,2)$ in Z. Suppose there exists a section S_Y of π containing A_Y, so that $X = \mathscr{C}_B(Z; A, S_Y)$. Up to normalization, from the study of determinants it follows that the short exact sequence associated to S_Y has to be of the following form

$$0 \to \mathcal{O} \to \mathcal{O}(a,b) \oplus \mathcal{O}(a,b) \to \mathcal{O}(2a,2b) \to 0 \tag{7}$$

for some $a, b \in \mathbb{N}$. If $a = b = 0$, A_Y would lie in a section of π that admits a second disjoint section. This would mean that X arises from Construction A, which we excluded in the previous Section. Therefore we can assume that $a \neq 0$, and the exact sequence (7) does not split. Therefore, one necessary condition for S_Y to exist is that $\operatorname{Ext}^1(\mathcal{O}(2a,2b), \mathcal{O})$ is nonzero. Observe that $\operatorname{Ext}^1(\mathcal{O}(2a,2b), \mathcal{O}) \simeq \big(H^0(\mathbb{P}^1, \mathcal{O}(-2+2a)) \otimes H^0(\mathbb{P}^1, \mathcal{O}(-2b))\big) \oplus \big(H^0(\mathbb{P}^1, \mathcal{O}(-2a)) \otimes H^0(\mathbb{P}^1, \mathcal{O}(-2+2b))\big)$, which is nonzero if and only if

$$\begin{cases} a \geq 1 \\ b \leq 0 \end{cases} \quad \text{or} \quad \begin{cases} a \leq 0 \\ b \geq 1. \end{cases} \tag{8}$$

Now consider condition *(I)* in Theorem 3.2:

$$(I) - K_Z + D - A \text{ is ample on } Z.$$

It is a necessary condition for X to be Fano. In this setting, we have $\mathcal{O}(-K_Z) = \mathcal{O}(2,2)$, $\mathcal{O}(D) = \mathcal{O}(a,b)$, $\mathcal{O}(A) = \mathcal{O}(2,2)$. Then

$$\mathcal{O}(-K_Z + D - A) = \mathcal{O}(2 + a - 2, 2 + b - 2) = \mathcal{O}(a,b)$$

which is ample if and only if $a > 0$ and $b > 0$. However, this is in contrast with conditions (8), so we conclude that X, which we know is Fano, cannot arise from such a construction, and the blown up $(2,2,2)$-curve in Y cannot be contained in any section of π.

Observe that this only proves that X does not arise from Construction B with this specific choice of $\mathbb{P}^1$-bundle Y and blown up locus A_Y, but X could arise from a construction of type B with different data. We now prove that this is not the case. More precisely, we will prove that any divisorial contraction of X must be the blow up of a $(2,2,2)$-curve in $Y = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, which we already proved that cannot be part of a construction of type B.

Consider the alternative description of X as a $(1,1,1,1)$-divisor in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. We now briefly study the elementary contractions of X. Observe that, by Lefschetz's Theorem, since X is an ample divisor in $(\mathbb{P}^1)^4$, we have an isomorphism of the spaces of curves $N_1(X) \simeq N_1((\mathbb{P}^1)^4)$ and the cone of effective curves in $\mathrm{NE}(X)$ is embedded in the cone of effective curves $\mathrm{NE}((\mathbb{P}^1)^4)$. Let l_i with $i = 1,2,3,4$ be the canonical generators of $\mathrm{NE}((\mathbb{P}^1)^4)$, each of which spans an extremal ray. Observe that in every class l_i there is a curve contained in X. Indeed, as a subvariety of $(\mathbb{P}^1)^4$, X can be defined by a polynomial equation of this type: $X_0 \cdot L_0(Y,Z,T) + X_1 \cdot L_1(Y,Z,T) = 0$ with respect to the coordinates ($[X_0\colon X_1]$, $[Y_0{:}Y_1]$, $[Z_0{:}Z_1]$, $[T_0{:}T_1]$), where the factors $L_i(Y,Z,T)$ are tri-homogeneous linear polynomials in the variables Y_i, Z_i, T_i. These polynomials L_i define two subvarieties of $(\mathbb{P}^1)^3$ whose intersection is non-empty, and is in fact a curve of tri-degree $(2,2,2)$. Therefore there exists a point $p = (y,z,t) \in (\mathbb{P}^1)^3$ such that $L_1(p) = 0 = L_2(p)$, hence the line $\mathbb{P}^1 \times \{y\} \times \{z\} \times \{t\}$ (which is equivalent to l_1 in $N_1((\mathbb{P}^1)^4)$) is contained in X.

Thus $\mathrm{NE}(X)$ contains all extremal rays of $\mathrm{NE}((\mathbb{P}^1)^4)$, i.e. $\mathrm{NE}(X) = \mathrm{NE}((\mathbb{P}^1)^4)$. It follows that the only elementary contractions of such variety are blow-downs $\sigma\colon X \to Y := \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ induced by the pojections of $(\mathbb{P}^1)^4$ onto any 3 of its factors. The blown-up locus is always a curve of type $(2,2,2)$ in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ that is the smooth complete intersection of two $(1,1,1)$-divisors. If Construction B applies, Z must necessarily be isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, $\pi\colon Y \to Z$ is the projection onto two of the factors of $(\mathbb{P}^1)^3$, say the second and third factors, and $A := \pi(A_Y)$ is a $(2,2)$-curve in Z. However we already proved that this cannot be the case.

8 Preliminaries for the study of Fano 4-folds arising from Construction A

In this section our goal is to provide some useful theoretic results that will allow us to give a complete list of all families of Fano 4-folds arising from Construction A with $\rho \geq 4$ and $\delta = 2$.

Remark 8.1. Suppose Z is a smooth Fano variety with $\dim(Z) = n-1 \geq 1$ such that there exist two Cartier divisors A and D in Z giving rise to a smooth Fano variety X with $\dim(X) = n$ and $\rho_X = \rho_Z + 2$ through Construction A. Since A is effective, some extremal ray R of $\mathrm{NE}(Z)$ must satisfy $A \cdot R > 0$. Let C be an irreducible curve whose class in $\mathrm{NE}(Z)$ generates R and let $c_R : Z \to W$ be the elementary extremal contraction associated to R (we have $\rho_W = \rho_Z - 1$). By Remark 2.5, C cannot have anticanonical degree 1, so that the length of R is $\ell(R) \geq 2$.

We now fix $n = 4$ (i.e. $\dim(Z) = 3$), and $\rho_Z \geq 2$. In that case, by means of the known classification of elementary contractions of Fano 3-folds (see [IP99, Theorem 1.4.3]), we analyze in detail all possible types of contraction with length larger than 1:

(i) If $c_R\colon Z \to W$ is a fiber type contraction, then $1 \leq \dim(W) < \dim(Z) = 3$. If $\dim(W) = 1$, then $\rho_Z = 2$, $W \simeq \mathbb{P}^1$ and c_R is either a $\mathbb{P}^2$-bundle or a quadric bundle. On the other hand, if $\dim(W) = 2$ then W is a smooth del Pezzo surface and c_R is a $\mathbb{P}^1$-bundle over W.

(ii) Suppose $c_R\colon Z \to W$ is a divisorial contraction. Then c_R has to be the blow up of a smooth point in W. Let E be the exceptional divisor of c_R. We have $E \simeq \mathbb{P}^2$, which implies that $\dim(N_1(E,Z)) = 1$ and $\delta_Z = \rho_Z - 1$.

Theorem 8.2. *Let $X = \mathscr{C}_A(Z; A, D)$ be a Fano 4-fold.*

If $\rho_X = 4$, then one of the following holds:

(A.1) There exists a $\mathbb{P}^1$-bundle $Z \to \mathbb{P}^2$.

(A.2) There exists a quadric bundle $Z \to \mathbb{P}^1$.

(A.3) There exists a $\mathbb{P}^2$-bundle $Z \to \mathbb{P}^1$.

(A.4) $Z \simeq Bl_p\mathcal{Q}_3$, where $\mathcal{Q}_3$ is a smooth quadric threefold in $\mathbb{P}^4$ and p is a point in $\mathcal{Q}_3$.

If $\rho_X = 5$, then one of the following holds:

(B.1) There exists a $\mathbb{P}^1$-bundle $Z \to W$ (where W is a del Pezzo surface with $\rho_W = 2$).

(B.2) Z arises from a construction of type A with base $\mathbb{P}^2$.

Finally, if $\delta_X = 2$, then $\rho_X \leq 6$. Moreover $\rho_X = 6$ and $\delta_X = 2$ if and only if $Z \simeq Bl_2(\mathbb{P}^2) \times \mathbb{P}^1$.

Proof. Suppose $\rho_X = 4$ (i.e. $\rho_Z = 2$), and consider the elementary contraction $c_R \colon Z \to W$ introduced in Remark 8.1. Then there are two possibilities:

(i) $c_R : Z \to W$ is a fiber-type contraction.

(ii) $c_R : Z \to W$ is the blow up of a smooth point in W.

Case (i) yields (A.1), (A.2), (A.3). Case (ii) yields (A.4) by the known classification of Fano varieties that can be realized as the blow up of a smooth point ([BCW02]).

Suppose $\rho_X = 5$ (i.e. $\rho_Z = 3$). Again, consider the elementary contraction $c_R \colon Z \to W$ introduced in Remark 8.1, which satisfies (i) or (ii) above. Since $\rho_Z = 3$, Case (i) can only occur with $\dim(W) = 2$, which yields (B.1). Consider case (ii). If $\rho_X = 5$, then $\rho_Z = 3$ and $\delta_Z = \rho_Z - 1 = 2$. We know every family of Fano 3-folds with such invariants: they are 6 families all arising from Construction A over $\mathbb{P}^2$. Among them, only 3 arise from the blow up of a smooth point, namely #3-14, #3-19 and #3-26 in [ACC+23, Table 6.1].

Finally, observe that $\delta_Z \leq \delta_X$ by Lemma 2.4, and $\delta_Z \in \{\rho_Z - 1, \rho_Z - 2\}$ because Z has dimension 3 (see [DN14, Lemma 5.1]). Hence if $\delta_X = 2$, then $\rho_Z - 2 \leq \delta_Z \leq \delta_X = 2$, which yields $\rho_X = \rho_Z + 2 \leq 6$. If $\rho_X = 6$, then $\rho_Z = 4$ and, if case (ii) applies, $\delta_Z = \rho_Z - 1 = 3$. But then $\delta_X \geq \delta_Z = 3$, so we rule out that option. Therefore Z must have a $\mathbb{P}^1$-bundle structure over a del Pezzo surface W with $\rho_W = \rho_Z - 1 = 3$, i.e. $W \simeq \mathrm{Bl}_2\mathbb{P}^2$. By the known classification of Fano projective bundles of rank 2 over a del Pezzo surface (see [SW90]), Z must then be isomorphic to $\mathrm{Bl}_2\mathbb{P}^2 \times \mathbb{P}^1$. This concludes the proof. □

In order to check if the Fano 4-folds we construct are toric, we need the following Lemma.

Lemma 8.3. *Let $X = \mathscr{C}_A(Z; A, D)$. Then X is toric if and ony if Z is toric and $A \subset Z$ is invariant under the torus action.*

Proof. Suppose Z is toric and A is invariant under the action of the torus. We then have that $Y = \mathbb{P}_Z(\mathcal{O} \oplus \mathcal{O}(D))$ is a smooth toric variety. If $\pi \colon Y \to Z$ is the projection onto Z, we have that $\pi^{-1}(A)$ and the sections G_Y, $\hat{G}_Y$ are stable under the action of the torus. Therefore $A_Y := \pi^{-1}(A) \cap \hat{G}_Y$ is also stable under the action of the torus, and the blow up X of A_Y in Y is a toric variety.

Viceversa, if X is toric, then Y is also toric and the blown up locus A_Y is stable under the action of the torus. This then implies that Z is toric and A is stable under the action of the torus on Z. □

9 Fano 4-folds arising from Construction A with $\rho = 6$ and $\delta = 2$

In Theorem 8.2, we proved that the only Fano 4-folds X with $\rho_X = 6$ and $\delta_X = 2$ arising from Construction A are the ones which project onto a smooth Fano base Z that is isomorphic to $\mathrm{Bl}_2\mathbb{P}^2 \times \mathbb{P}^1$. We now determine all choices of A and D in Z that yield a construction of type A. For this purpose, we describe the effective and ample cones of Z. In $N_1(\mathrm{Bl}_2\mathbb{P}^2)$, let $\mathbf{f}$ be the class of the strict transform of the line passing through the blown up points in $\mathbb{P}^2$, and let $\mathbf{e_1}$, $\mathbf{e_2}$ be the classes of the two exceptional divisors.

Call $U_1 := \mathbf{f} \times \mathbb{P}^1$, $U_2 := \mathbf{e_1} \times \mathbb{P}^1$, $U_3 := \mathbf{e_2} \times \mathbb{P}^1$ $U_4 := \mathrm{Bl}_2\mathbb{P}^2 \times \{*\}$. Then $N^1(Z)$ is generated by U_1, U_2, U_3, U_4.

Call $u_1 := \mathbf{f} \times \{*\}$, $u_2 := \mathbf{e_1} \times \{*\}$, $u_3 := \mathbf{e_2} \times \{*\}$ $u_4 := \{*\} \times \mathbb{P}^1$. Then $N_1(Z)$ is generated by u_1, u_2, u_3, u_4.

If $D \in N^1(Z)$, it can be written as $p_1^*(D_1) \otimes p_2^*(D_2)$ where $D_1 \in N^1(\mathbb{F}_1)$ and $D_2 \in N^1(\mathbb{P}^1)$. By the Künneth formula,

$$H^0(Z, \mathcal{O}(D)) = H^0(\mathbb{F}_1, \mathcal{O}(D_1)) \otimes H^0(\mathbb{P}^1, \mathcal{O}(D_2)).$$

Hence, $\mathrm{Eff}(Z) = \{p_1^* D_1 \otimes p_2^* D_2 |\ D_1$ is effective in $\mathrm{Bl}_2\mathbb{P}^2$, and D_2 is effective in $\mathbb{P}^1\} = \sum_{i=1}^4 \mathbb{R}_{\geq 0}[U_i]$. The following intersection table holds:

$\cdot$	u_1	u_2	u_3	u_4
U_1	-1	1	1	0
U_2	1	-1	0	0
U_3	1	0	-1	0
U_4	0	0	0	1

It is then easy to check that $\mathrm{NE}(Z)$ is generated by u_1, u_2, u_3, u_4. By simple computations, we get $\mathrm{Nef}(Z) = \mathbb{R}_{\geq 0}[U_1 + U_2 + U_3] + \mathbb{R}_{\geq 0}[U_1 + U_2] + \mathbb{R}_{\geq 0}[U_1 + U_3] + \mathbb{R}_{\geq 0}[U_4]$.

Recall that

$$-K_Z \sim p_1^*(-K_{\mathrm{Bl}_2\mathbb{P}^2}) + p_2^*(-K_{\mathbb{P}^1}) \sim 3U_1 + 2U_2 + 2U_3 + 2U_4 = (U_1 + U_2 + U_3) + (U_1 + U_2) + (U_1 + U_3) + 2U_4.$$

Let (a_1, a_2, a_3, a_4) be the coordinates of a smooth irreducible surface A in Z with respect to the basis $\{U_i\}$. Let (d_1, d_2, d_3, d_4) be the coordinates of a divisor D, (d_1', d_2', d_3', d_4') be the coordinates of a divisor D'. Let us study all possible choices of A, D, D' such that $D + D' = A$ is a smooth irreducible surface, and $-K_Z - D$ and $-K_Z - D'$ are ample.

Now,

$$\begin{aligned} -K_Z - D \text{ is ample } &\iff (3 - d_1)U_1 + (2 - d_2)U_2 + (2 - d_3)U_3 + (2 - d_4)U_4 \text{ is ample} \\ &\iff d_1 \geq d_2 + d_3,\ d_1 \leq d_2,\ d_1 \leq d_3, \text{ and } d_4 \leq 1. \end{aligned}$$

The same holds for D''s coordinates. Observe that since A is effective, all its coordinates must be non-negative, and since it is the sum of D and D', we must have $0 \leq a_2 + a_3 \leq a_1 \leq \min(a_2, a_3)$ and $0 \leq a_4 \leq 2$.

We may assume $0 \leq a_2 \leq a_3$, so that $a_2 + a_3 \leq a_1 \leq a_2$ implies that $a_3 = 0$, which yields $a_1 = a_2 = 0$ as well. Therefore, the only admissible choices of A are the following: $A = U_4$ and

$A = 2U_4$. However, the latter cannot represent a smooth irreducible hypersurface in Z, because it is the pull back by the projection p_2 of a non-reduced divisor in $\mathbb{P}^1$. Therefore the only possibility for A is $A = U_4$. There is only one choice of D, D' for this hypersurface: $D, D' \in \{0, U_4\}$, which yields $X \simeq \mathrm{Bl}_2\mathbb{P}^2 \times \mathrm{Bl}_2\mathbb{P}^2$. This proves the following theorem:

Theorem 9.1. *The only Fano 4-fold with $\rho_X \geq 6$ and $\delta_X = 2$ arising from Construction A is $X \simeq Bl_2\mathbb{P}^2 \times Bl_2\mathbb{P}^2$.*

10 Fano 4-folds arising from Construction A with $\rho = 5$ and $\delta = 2$

According to Theorem 8.2, given a Fano 4-fold X arising from Construction A with $\rho_X = 5$, there are two possibilities for Z.

(B.1) Z has a $\mathbb{P}^1$-bundle structure $p : Z \to W$ over a del Pezzo surface W.

(B.2) Z arises from a construction of type A with base $\mathbb{P}^2$.

A careful study of those cases leads to a complete classification of Fano 4-folds with $\rho = 5$ that arise from Construction A. All of them have Lefschetz defect $\delta = 2$. Indeed, let $X = \mathscr{C}_A(Z; A, D)$ be a Fano 4-fold with $\rho_X = 5$. The base Z is a Fano 3-fold with $\rho_Z = 3$, therefore $\delta_Z \in \{\rho_Z - 2, \rho_Z - 1\} = \{1, 2\}$. By Lemma 2.4, we know that $2 \leq \delta_X \leq 3$. Now, Fano 4-folds with defect $\delta = 3$ and $\rho = 5$ are known (see [CR22]), and none of them appears in the list of our 50 families.

We ultimately obtain the following theorem.

Theorem 10.1. *There are 50 distinct families of Fano 4-folds X with $\rho_X = 5$ arising from Construction A, and they are all listed in Table 25. All of them have $\delta_X = 2$. Among them, 27 are not toric nor products of lower dimensional varieties.*

We find that 25 of those 27 *new* Fano 4-folds project onto a Fano $\mathbb{P}^1$-bundle Z over a del Pezzo surface (see Section 10.1), while the other 2 project onto a Fano variety Z that is the blowing up of a smooth point on a smooth 3-fold (see Section 10.2).

As a side comment, it is worth noticing that the Fano polytopes of the toric Fano 4-folds arising from Construction A with $\rho = 5$ are all of type Q_n with $n \in \mathbb{N}$ (see [Bat99]).

10.1 Case (B.1)

In this section, we are going to list every Fano 4-fold $X = \mathscr{C}_A(Z; A, D)$ arising from a Fano variety Z that has a $\mathbb{P}^1$-bundle structure $p : Z \to W$ over a del Pezzo surface (case (B.1) of Theorem 8.2). Since $\rho_Z = \rho_X - 2 = 3$, we need $\rho_W = \rho_Z - 1 = 2$. This implies that $W \simeq \mathbb{P}^1 \times \mathbb{P}^1$ or $W \simeq \mathbb{F}_1$. In their paper [SW90], M. Szurek and J. Wiśniewski studied Fano bundles of rank 2 on surfaces. From their analysis, we deduce that there are only seven families of Fano threefolds Z that satisfy our assumptions: #3-17, #3-24, #3-25, #3-27, #3-28, #3-30, #3-31. For each family, it is possible to describe the effective cone, as well as the nef cone. This allows us to find for each Z all choices of A and D that satisfy the conditions in Proposition 2.3 that yield X Fano. We will explain the details of such process in the following paragraphs.

1. $\boxed{Z = \textbf{Divisor of degree } (1,1,1) \textbf{ in } \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2 \ \textbf{(\#3-17).}}$ Since Z is an ample divisor, by the Lefschetz hyperplane theorem we have an isomorphism of the spaces of curves $N_1(\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2) \simeq N_1(Z)$. Similarly to the case of a $(1,1,1,1)$-divisor in $(\mathbb{P}^1)^4$ (see Section 7), it can be proved that the elementary contractions of Z are the restrictions of the projections of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ onto its factors. Call H_1, H_2 and H_3 the canonical generators of the nef cone of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$ and let H_i^Z be their restrictions to Z, which generate $\mathrm{Nef}(Z)$. Observe that $p_{12}|_Z : Z \to \mathbb{P}^1 \times \mathbb{P}^1$ is a $\mathbb{P}^1$-bundle $(\mathcal{O}(Z|_{p_{12}^{-1}(\{*\})}) \simeq \mathcal{O}_{\mathbb{P}^2}(1))$. On the other hand $p_{23}|_Z : Z \to \mathbb{P}^1 \times \mathbb{P}^2$ is a birational divisorial contraction which is the blow up of a curve. Let E_1 be its exceptional divisor. Then $-K_Z = p_{23}^*(-K_{\mathbb{P}^1 \times \mathbb{P}^2}) - E_1$. Hence $E_1 \equiv -H_1^Z + H_2^Z + H_3^Z$ (recall that $-K_Z = -K_{\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2}|_Z - Z|_Z = H_1^Z + H_2^Z + 2H_3^Z$). Similarly $E_2 \equiv H_1^Z - H_2^Z + H_3^Z$ is the exceptional divisor of $p_{13}|_Z$.

Hence $\mathrm{Nef}(Z) = \mathbb{R}_{\geq 0} H_1^Z + \mathbb{R}_{\geq 0} H_2^Z + \mathbb{R}_{\geq 0} H_3^Z$ and $\mathrm{Eff}(Z) = \mathbb{R}_{\geq 0} H_1^Z + \mathbb{R}_{\geq 0} H_2^Z + \mathbb{R}_{\geq 0} E_1 + \mathbb{R}_{\geq 0} E_2$. This is enough to apply Proposition 2.3 and find all possible choices of A, D in Z that yield a Fano 4-fold X (see Table 4).

A	D	#	c_1^4	$c_1^2 c_2$	$h^0(-K_X)$	$h^{1,2}$	$h^{2,2}$	$h^{1,3}$	Toric?	Product?
H_3^Z	$0/H_3^Z$	1	236	152	53	0	10	0	No	No
$2H_3^Z$	H_3^Z	2	184	136	43	0	12	0	No	No

Table 4: Choices of A, D in Z= divisor (1,1,1) in $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$.

2. $\boxed{Z = \textbf{Complete intersection of degree } (1,1,0) \textbf{ and } (0,1,1) \textbf{ in } \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2 \ \textbf{(\#3-24).}}$ Let p_i be the projection of Z onto its i-th factor ($i = 1,2,3$). We use the following notation for line bundles on Z: $\mathcal{O}(a,b,c) := p_1^*\mathcal{O}_{\mathbb{P}^1}(a) \otimes p_2^*\mathcal{O}_{\mathbb{P}^2}(b) \otimes p_3^*\mathcal{O}_{\mathbb{P}^2}(c)$.

First of all, notice that if $U_1 = \mathcal{O}(1,1,0)$ and $U_2 = \mathcal{O}(0,1,1)$, then $U_1 \simeq \mathbb{F}_1 \times \mathbb{P}^2$ and $U_2 \simeq \mathbb{P}^1 \times P$, where P is a divisor of degree (1,1) in $\mathbb{P}^2 \times \mathbb{P}^2$, i.e. is the projective bundle $\mathbb{P}_{\mathbb{P}^2}(T_{\mathbb{P}^2})$.

Consider the projection $p_{12} : \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^1 \times \mathbb{P}^2$. Clearly $p_{12}(U_1) = \mathbb{F}_1 \subset \mathbb{P}^1 \times \mathbb{P}^2$ and $p_{12}(U_2) = \mathbb{P}^1 \times p_1(P) = \mathbb{P}^1 \times \mathbb{P}^2$. Hence $p_{12}(Z) = \mathbb{F}_1 \subset \mathbb{P}^1 \times \mathbb{P}^2$, and $p_{12}|_Z : Z \to \mathbb{F}_1$ is a $\mathbb{P}^1$-bundle. It is an extremal contraction of fiber type that contracts all curves in the class of a fiber of the projective bundle $p_1 : P \to \mathbb{P}^2$, i.e. every line of the following form: $\{*\} \times \{*\} \times \mathcal{O}_{\mathbb{P}^2}(1)$ contained in Z.

Consider now the projection $p_{13} : \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^1 \times \mathbb{P}^2$. Clearly $p_{13}|_{U_1} : \mathbb{F}_1 \times \mathbb{P}^2 \to \mathbb{P}^1 \times \mathbb{P}^2$ coincides with the classical $\mathbb{P}^1$-bundle from $\mathbb{F}_1$ to $\mathbb{P}^1$ on the first component, and is the identity of $\mathbb{P}^2$ on the second component. We have $p_{13}(U_1) = \mathbb{P}^1 \times \mathbb{P}^2$, and $p_{13}(U_2) = \mathbb{P}^1 \times p_2(P) = \mathbb{P}^1 \times \mathbb{P}^2$. Therefore $p_{13}(Z) = \mathbb{P}^1 \times \mathbb{P}^2$, and $p_{13}|_Z$ is a birational divisorial extremal contraction that contracts curves in the class of $\{*\} \times \mathcal{O}_{\mathbb{P}^2}(1) \times \{*\}$, contained in Z.

Finally, consider the projection $p_{23} : \mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^2 \times \mathbb{P}^2$. The restriction $p_{23}|_{U_1} : \mathbb{F}_1 \times \mathbb{P}^2 \to \mathbb{P}^2 \times \mathbb{P}^2$ concides with the blowing up $\mathbb{F}_1 \to \mathbb{P}^2$ on the first component and the identity of $\mathbb{P}^2$ on the second component. We have $p_{23}(U_1) = \mathbb{P}^2 \times \mathbb{P}^2$ and $p_{23}(U_2) = P$. Hence $p_{23}(Z) = P$ and $p_{23}|_Z : Z \to P$ is a birational divisorial extremal contraction that contracts all curves in the class of $\mathbb{P}^1 \times \{*\} \times \{*\}$, contained in Z.

So we have that every extremal contraction of Z is the restriction of an extremal contraction of $\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$, hence the restriction $i_* : \mathrm{Nef}(\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2) \to \mathrm{Nef}(Z)$ is an isomorphism. Let

$H_1 := \mathcal{O}(1,0,0)$, $H_2 := \mathcal{O}(0,1,0)$, $H_3 := \mathcal{O}(0,0,1)$ be the generators of the nef cone in $\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$, and H_1^Z, H_2^Z, H_3^Z their restrictions to Z.

Call E_1 and E_2 the exceptional divisor respectively of p_{12} and p_{13}. It is not difficult to prove that $E_1 \equiv -H_1^Z + H_2^Z$ and $E_2 \equiv H_1^Z - H_2^Z + H_3^Z$. We have $\mathrm{Nef}(Z) = \mathbb{R}_{\geq 0} H_1^Z + \mathbb{R}_{\geq 0} H_2^Z + \mathbb{R}_{\geq 0} H_3^Z$ and $\mathrm{Eff}(Z) = \mathbb{R}_{\geq 0} H_1^Z + \mathbb{R}_{\geq 0} E_1 + \mathbb{R}_{\geq 0} E_2$.

Recall that, by the adjunction formula, $-K_Z = -K_{\mathbb{P}^1\times\mathbb{P}^2\times\mathbb{P}^2}|_Z - \det(\mathcal{N}_{Z/\mathbb{P}^1\times\mathbb{P}^2\times\mathbb{P}^2}) = 2H_1^Z + 3H_2^Z + 3H_3^Z - (H_1^Z + 2H_2^Z + H_3^Z) = H_1^Z + H_2^Z + 2H_3^Z$. This is enough to apply Proposition 2.3 and find all possible choices of A, D in Z that yield a Fano 4-fold X. They are listed in Table 5.

A	D	#	c_1^4	$c_1^2c_2$	$h^0(-K_X)$	$h^{1,2}$	$h^{2,2}$	$h^{1,3}$	Toric?	Product?
H_3^Z	$0/H_3^Z$	1	278	164	61	0	9	0	No	No
$2H_3^Z$	H_3^Z	2	220	148	50	0	10	0	No	No

Table 5: Choices of A, D in Z=complete intersection of degree (1,1,0) and (0,1,1) in $\mathbb{P}^1 \times \mathbb{P}^2 \times \mathbb{P}^2$.

3. $\boxed{Z = \mathbb{P}_{\mathbb{P}^1\times\mathbb{P}^1}(\mathcal{O}(-1,0) \oplus \mathcal{O}(0,-1))\ \textbf{(\#3-25).}}$ Observe that Z is isomorphic to the blow up of $\mathbb{P}^3$ along two disjoint lines l_1 and l_2. Call E_1 and E_2 the exceptional divisors over the two blown up lines in $\mathbb{P}^3$, and call H the transform of a general hyperplane in $\mathbb{P}^3$.

In $N_1(Z)$, let $\mathbf{e_1}$ and $\mathbf{e_2}$ be the classes of one dimensional fibers respectively of E_1 and E_2, and let $\mathbf{f}$ be the class of the transform of a generic line intersecting both the blown up lines non trivially. They generate the cone of effective 1-cycles $\mathrm{NE}(Z)$ as well as $N_1(Z)$.

The classes of H, E_1, E_2 generate $N^1(Z)$ and $\mathrm{Nef}(Z) = \mathbb{R}_{\geq 0}[H] + \mathbb{R}_{\geq 0}[H - E_1] + \mathbb{R}_{\geq 0}[H - E_2]$. Moreover the effective cone is $\mathrm{Eff}(Z) = \mathbb{R}_{\geq 0}[H - E_1] + \mathbb{R}_{\geq 0}[H - E_2] + \mathbb{R}_{\geq 0}[E_1] + \mathbb{R}_{\geq 0}[E_2]$ (see [DPU17]). Recall that $-K_Z \equiv 2H + (H - E_1) + (H - E_2)$. This is enough to use Proposition 2.3 and find all possible choices of A and D that yield a Fano 4-fold X. They are listed in Table 6.

A	D	#	c_1^4	$c_1^2c_2$	$h^0(-K_X)$	$h^{1,2}$	$h^{2,2}$	$h^{1,3}$	Toric?	Product?
E_1	$H/-H+E_1$	1	331	178	71	0	8	0	Q_{12}	No
	$E_1/0$	2	310	172	67	0	8	0	Q_{16}	No
H	$H/0$	3	283	166	62	0	9	0	No	No
$2H$	H	4	214	148	49	0	12	0	No	No

Table 6: Choices of A, D in $Z = \mathbb{P}_{\mathbb{P}^1\times\mathbb{P}^1}(\mathcal{O}(-1,0) \oplus \mathcal{O}(0,-1))$.

4. $\boxed{Z = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1\ \textbf{(\#3-27).}}$ Call $H_1 := \{*\} \times \mathbb{P}^1 \times \mathbb{P}^1$, $H_2 := \mathbb{P}^1 \times \{*\} \times \mathbb{P}^1$, $H_3 := \mathbb{P}^1 \times \mathbb{P}^1 \times \{*\}$. Both $\mathrm{Nef}(Z)$ and $\mathrm{Eff}(Z)$ are generated by H_1, H_2, H_3. Recall that $-K_Z \equiv 2H_1 + 2H_2 + 2H_3$. This is enough to apply Proposition 2.3 and find all possible choices of A and D (denoted by their coordinates with respect to H_1, H_2, H_3). They are listed in Table 7.

A	D	#	c_1^4	$c_1^2c_2$	$h^0(-K_X)$	$h^{1,2}$	$h^{2,2}$	$h^{1,3}$	Toric?	Product?
(0,0,1)	(-1,-1,0)/(1,1,1)	1	394	196	83	0	8	0	Q_3	No
	(-1,0,0)/(1,0,1)	2	352	184	75	0	8	0	Q_8	(#4-11)$\times\mathbb{P}^1$
	(0,0,0)/(0,0,1)	3	336	180	72	0	8	0	Q_{11}	(#4-10)$\times\mathbb{P}^1$
	(-1, -1, 1)/(1,1,0)	4	330	180	71	0	8	0	Q_{13}	No
	(-1,0,1)/(1,0,0)	5	320	176	69	0	8	0	Q_{15}	(#4-9)$\times\mathbb{P}^1$
	(0,1,0)/(0,-1,1)	6	320	176	69	0	8	0	Q_{15}	(#4-9)$\times\mathbb{P}^1$
	(-1,1,0)/(1,-1,1)	7	310	172	67	0	8	0	Q_{16}	No
(0,1,1)	(-1,0,0)/(1,1,1)	8	330	180	71	0	8	0	No	No
	(0,0,0)/(0,1,1)	9	304	172	66	0	8	0	No	(#4-8)$\times\mathbb{P}^1$
	(0,0,1)/(0,1,0)	10	288	168	63	0	8	0	No	(#4-7)$\times\mathbb{P}^1$
	(-1,0,1)/(1,1,0)	11	288	168	63	0	8	0	No	No
	(-1,1,1)/(1,0,0)	12	278	164	61	0	8	0	No	No
(0,1,2)	(-1,0,1)/(1,1,1)	13	266	164	59	0	8	0	No	No
	(0,0,1)/(0,1,1)	14	256	160	57	0	8	0	No	(#4-5)$\times\mathbb{P}^1$
	(-1,1,1)/(1,0,1)	15	246	156	55	0	8	0	No	No
(0,2,2)	(-1,1,1)/(1,1,1)	16	224	152	51	1	8	0	No	No
	(0,1,1)	17	224	152	51	1	8	0	No	(#4-2)$\times\mathbb{P}^1$
(1,1,1)	(0,0,0)/(1,1,1)	18	282	168	62	0	10	0	No	No
	(0,0,1)/(1,1,0)	19	256	160	57	0	10	0	No	No
(1,1,2)	(0,0,1)/(1,1,1)	20	234	156	53	0	12	0	No	No
	(0,1,1)/(1,0,1)	21	224	152	51	0	12	0	No	No
(1,2,2)	(0,1,1)/(1,1,1)	22	202	148	47	0	16	0	No	No
(2,2,2)	(1,1,1)	23	180	144	43	0	26	1	No	No

Table 7: Choices of A, D in $Z = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

5. $\boxed{Z = \mathbb{F}_1 \times \mathbb{P}^1 \text{ (\#3-28).}}$ Call $U_1 := \mathbf{f} \times \mathbb{P}^1$, $U_2 := \mathbf{e} \times \mathbb{P}^1$, $U_3 := \mathbb{F}_1 \times \{*\}$, so that $N^1(Z)$ is generated by U_1, U_2, U_3. Call $u_1 := \mathbf{f} \times \{*\}$, $u_2 := \mathbf{e} \times \{*\}$, $u_3 := \{*\} \times \mathbb{P}^1$. Then $N_1(Z)$ is generated by u_1, u_2, u_3. We have $\mathrm{Eff}(Z) = \mathbb{R}_{\geq 0}[U_1] + \mathbb{R}_{\geq 0}[U_2] + \mathbb{R}_{\geq 0}[U_3]$.

It can be seen that $\mathrm{NE}(Z)$ is generated by u_1, u_2, u_3 and $\mathrm{Nef}(Z) = \mathbb{R}_{\geq 0}[U_1] + \mathbb{R}_{\geq 0}[U_1 + U_2] + \mathbb{R}_{\geq 0}[U_3]$. Moreover, we have $-K_Z = p_1^*(-K_{\mathbb{F}_1}) + p_2^*(-K_{\mathbb{P}^1}) \equiv 3U_1 + 2U_2 + 2U_3$. This is enough to apply Proposition 2.3, which yields the choices for A and D (denoted by their coordinates with respect to the the base of $N^1(Z)$ given by U_1, U_2, U_3) listed in Table 8.

A	D	#	c_1^4	$c_1^2c_2$	$h^0(-K_X)$	$h^{1,2}$	$h^{2,2}$	$h^{1,3}$	Toric?	Product?
(0,0,1)	(-1,-1,0)/(1,1,1)	1	373	190	79	0	8	0	Q_5	No
	(0,0,0)/(0,0,1)	2	336	180	72	0	8	0	Q_{10}	$\mathbb{F}_1 \times \mathrm{Bl}_2\mathbb{P}^2$
	(-1, -1, 1)/(1,1,0)	3	325	178	70	0	8	0	Q_{14}	No
(0,1,0)	(-1,0,-1)/(1,1,1)	4	405	198	85	0	8	0	Q_4	No
	(-1,0,0)/(1,1,0)	5	368	188	78	0	8	0	Q_6	(#4-12)$\times\mathbb{P}^1$
	(0,1,1)/(0,0,-1)	6	363	186	77	0	8	0	Q_7	No
	(0,1,0)/(0,0,0)	7	352	184	75	0	8	0	Q_8	(#4-11)$\times\mathbb{P}^1$

	(0,1,-1)/(0,0,1)	8	341	182	73	0	8	0	Q_9	No
	(-1,0,1)/(1,1,-1)	9	331	178	71	0	8	0	Q_{12}	No
(1,1,0)	(0,0,-1)/(1,1,1)	10	341	182	85	0	8	0	Q_9	No
	(0,0,0)/(1,1,0)	11	320	176	78	0	8	0	Q_{15}	(#4-9)$\times\mathbb{P}^1$
	(0,0,1)/(1,1,-1)	12	299	170	71	0	8	0	Q_{17}	No
(1,1,1)	(0,0,0)/(1,1,1)	13	293	170	64	0	9	0	No	No
(1,1,2)	(0,0,1)/(1,1,1)	14	245	158	55	0	10	0	No	No
(2,2,0)	(1,1,-1)/(1,1,1)	15	256	160	57	0	8	0	No	No
	(1,1,0)	16	256	160	57	0	8	0	No	(#4-4)$\times\mathbb{P}^1$
(2,2,1)	(1,1,0)/(1,1,1)	17	229	154	52	0	12	0	No	No
(2,2,2)	(1,1,1)	18	202	148	47	0	16	0	No	No

Table 8: Choices of A, D in $Z = \mathbb{F}_1 \times \mathbb{P}^1$.

6. $\boxed{Z = \mathbb{P}_{\mathbb{F}_1}(\mathcal{O} \oplus \mathcal{O}(-\mathbf{l}))\ \textbf{(\#3-30).}}$ Call $\pi\colon Z \to \mathbb{F}_1$ the projection. Call $H := \pi^*\mathbf{l}$, $\tilde{E} := \pi^*\mathbf{e}$ and E the negative section of π. Call $\mathbf{f}$ the class in $N_1(Z)$ of the generic fiber of π, $\mathbf{l}$ the class of $\pi^{-1}(\mathbf{l}) \cap E$ and $\mathbf{e}$ the class of $\pi^{-1}(\mathbf{e}) \cap E$. We have that $N^1(Z)$ is generated by the classes of H, $\tilde{E}$ and E. Moreover $\mathrm{NE}(Z)$ is generated by $\mathbf{f}$, $\mathbf{l} - \mathbf{e}$ and $\mathbf{e}$. The following intersection table holds:

$\cdot$	$\mathbf{f}$	$\mathbf{l}$	$\mathbf{e}$
H	0	1	0
$\tilde{E}$	0	0	-1
E	1	-1	0

Therefore we have $\mathrm{Nef}(Z) = \mathbb{R}_{\geq 0}[H] + \mathbb{R}_{\geq 0}[H - \tilde{E}] + \mathbb{R}_{\geq 0}[H + E]$. Moreover, the effective cone is $\mathrm{Eff}(Z) = \mathbb{R}_{\geq 0}[E] + \mathbb{R}_{\geq 0}[H - \tilde{E}] + \mathbb{R}_{\geq 0}[\tilde{E}]$.

Recall that $-K_Z = 2c_1(\mathcal{O}_Z(1)) + \pi^*(-K_{\mathbb{F}_1} - \det(\mathcal{O} \oplus \mathcal{O}(-\mathbf{l}))) \equiv 4H - \tilde{E} + 2E$. This is enough to apply Proposition 2.3 and obtain the list of admissible choices of A and D in Z contained in Table 9.

A	D	#	c_1^4	$c_1^2c_2$	$h^0(-K_X)$	$h^{1,2}$	$h^{2,2}$	$h^{1,3}$	Toric?	Product?
E	$-H/H+E$	1	405	198	85	0	8	0	Q_2	No
	$E/0$	2	373	190	79	0	8	0	Q_5	No
$H+E$	$H+E/0$	3	325	178	70	0	8	0	Q_{14}	No
$2H+2E$	$H+E$	4	250	160	56	0	10	0	No	No

Table 9: Choices of A, D in Z=$\mathbb{P}_{\mathbb{F}_1}(\mathcal{O} \oplus \mathcal{O}(-l))$.

7. $\boxed{Z = \mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-1,-1))\ \textbf{(\#3-31).}}$ Call $\pi\colon Z \to \mathbb{P}^1 \times \mathbb{P}^1$ the projection, $H_1 := \pi^{-1}(\{*\} \times \mathbb{P}^1)$, $H_2 := \pi^{-1}(\mathbb{P}^1 \times \{*\})$ and E the section corresponding to the quotient $\mathcal{O} \oplus \mathcal{O}(-1,-1) \twoheadrightarrow \mathcal{O}(-1,-1)$. Observe that $\mathcal{O}_Z(E) = \mathcal{O}_Z(1)$, and H_1, H_2, E generate $N^1(Z)$.

In $N_1(Z)$, let $\mathbf{f}$ be the class of a one dimensional fiber of π, and let $\mathbf{e_1}$ and $\mathbf{e_2}$ be the copies in E of the two rulings in $\mathbb{P}^1 \times \mathbb{P}^1$. They generate $N_1(Z)$.

We have $\mathrm{NE}(Z) = \mathbb{R}_{\geq 0}\mathbf{f} + \mathbb{R}_{\geq 0}\mathbf{e_1} + \mathbb{R}_{\geq 0}\mathbf{e_2}$, $\mathrm{Nef}(Z) = \mathbb{R}_{\geq 0}[H_1] + \mathbb{R}_{\geq 0}[H_2] + \mathbb{R}_{\geq 0}[H_1 + H_2 + E]$. Moreover the effective cone is $\mathrm{Eff}(Z) = \mathbb{R}_{\geq 0}[H_1] + \mathbb{R}_{\geq 0}[H_2] + \mathbb{R}_{\geq 0}[E]$.

Recall that

$$-K_Z = 2\mathcal{O}_Z(1) + \pi^*(-K_{\mathbb{P}^1\times\mathbb{P}^1} - \det(\mathcal{O} \oplus \mathcal{O}(-1,-1))) \equiv 3H_1 + 3H_2 + 2E.$$

This is enough to be able to apply Proposition 2.3, from which it follows that the only (up to symmetries) admissible choices of A and D in Z are the ones listed in Table 10.

A	D	#	c_1^4	$c_1^2c_2$	$h^0(-K_X)$	$h^{1,2}$	$h^{2,2}$	$h^{1,3}$	Toric?	Product?
E	$-H_1-H_2/$ H_1+H_2+E	1	442	208	92	0	8	0	Q_1	No
	$-H_1/H_1+E$	2	405	198	85	0	8	0	Q_4	No
	$0/E$	3	394	196	83	0	8	0	Q_3	No
H_1+H_2+E	$0/H_1+H_2+E$	4	330	180	71	0	8	0	Q_{13}	No
$2H_1+2H_2+2E$	H_1+H_2+E	5	244	160	55	0	12	0	No	No

Table 10: Choices of A, D in Z=$\mathbb{P}_{\mathbb{P}^1\times\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O}(-1,-1))$

Ambiguities In Tables 4-10 there are some families of Fano 4-folds arising from different constructions of type A that have the same invariants. We are going to prove that all non-toric Fano 4-folds listed in said tables are distinct. The study of toric 4-folds can be done by comparing their descriptions with the ones in [Bat99].

We begin by making some general observations that will be useful in solving some of the ambiguities.

Remark 10.2. Let X be a smooth Fano 4-fold arising from two distinct constructions of type A, $\mathscr{C}_A(Z; A, D) \simeq X \simeq \mathscr{C}_A(Z'; A', D')$, with $Z \not\simeq Z'$. Suppose that in Z there are no effective divisors that have negative intersection with every curve contained in their support. Then both D' and $A' - D'$ are not ample.

Proof. Suppose by contradiction that D' is ample. In X there exists an effective divisor $G' \simeq Z'$ such that $\mathcal{N}_{G'/X} \simeq \mathcal{O}_{Z'}(-D')$. In X there are also two disjoint copies of Z, which are G and $\hat{G}$. Now, since $Z \not\simeq Z'$, G' does not coincide with G nor with $\hat{G}$. If $G' \cap G \neq \emptyset$, $G'|_G$ would be an effective divisor in $G \simeq Z$ that has negative intersection with every curve contained in its support, which is impossible by our assumptions. Therefore $G' \cap G = \emptyset$ and similarly it can be proved that $G' \cap \hat{G} = \emptyset$. Now, consider the blow up $\sigma\colon X \to Y$ and the $\mathbb{P}^1$-bundle $\pi\colon Y \to Z$ given by $\mathscr{C}_A(Z; A, D)$. Since $G' \cap \hat{G} = \emptyset$, G' does not coincide with the exceptional divisor of σ and $\sigma(G')$ is an effective divisor in Y, disjoint from G_Y and $\hat{G}_Y$. However, from the fact that $\mathrm{Pic}(Y) = \mathbb{Z}\hat{G}_Y + \pi^*(\mathrm{Pic}(Z))$, it follows that this is impossible. Similar arguments can be made under the assumption that $A' - D'$ is ample. ☐

Remark 10.3. Let $X \simeq \mathscr{C}_A(Z; A, D)$ be a smooth n-dimensional variety ($n \geq 3$). Suppose $X \simeq \mathbb{P}^1 \times X_0$, where X_0 is some smooth $(n-1)$-dimensional variety. Then the blow up $\sigma\colon X \to Y$ has to be of the following form: $Y \simeq \mathbb{P}^1 \times Y_0$ for some smooth $(n-1)$-dimensional variety Y_0, and the blown-up locus A_Y must be isomorphic to $\mathbb{P}^1 \times A_0$, where A_0 is a smooth $(n-3)$-dimensional variety lying in Y_0, so that $X_0 \simeq \mathrm{Bl}_{A_0}(Y_0)$.

We now list all ambiguities.

There are pairs of families with identical numerical invariants, where one family consists of varieties that admit a clear product structure $W \times \mathbb{P}^1$, with W a Fano 3-fold, while the other family does not. This is the case for the following couples:

$$(\#15 \text{ Table } 8, \#16 \text{ Table } 8) \; ; (\#10 \text{ Table } 7, \#11 \text{ Table } 7) \; ; (\#16 \text{ Table } 7, \#17 \text{ Table } 7).$$

Families $\#16$ in Table 8, $\#10$ in Table 7 and $\#17$ in Table 7 are clearly products, as it emerges from the data of Construction A that yield them. If X is a member of one of the remaining families, we see that in all their constructions, Y and A_Y are not in the right form for X to be a product (see Remark 10.3). Thus in all three pairs listed above, each family is distinct.

The last ambiguous case, is that of the couple ($\#22$ Table 7 , $\#18$ Table 8). Let X be a member of family $\#18$ in Table 8. We have $Z = \mathbb{F}_1 \times \mathbb{P}^1$, $A \equiv 2U_1 + 2U_2 + 2U_3$, and $D \equiv U_1 + U_2 + U_3$. Let X' be a member of family $\#22$ in Table 7. We have $Z' = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, $A' \equiv H_1 + 2H_2 + 2H_3$, and $D' \equiv H_1 + H_2 + H_3$. We see that in Z there are no effective divisors having negative intersections with all curves contained in their support and D' is ample in Z'. Therefore we can apply Remark 10.2 and conclude that the two families are distinct.

10.2 Case (B.2)

In this section, we are going to list every Fano 4-fold V obtained through the iteration of two constructions of type A, starting from $\mathbb{P}^2$ (case (B.2) of Theorem 8.2). Observe that by the proof of Theorem 8.2, we may just look for all choices of A and D in #3-14, #3-19, #3-26 that satisfy the conditions in Proposition 2.3, similarly to what we did for case (B.1). However it is much more straightforward to use the iterative Construction A with base $\mathbb{P}^2$, since all Fano 4-folds we are looking for arise from it. We refer to Section 5 for the notation used in this section. By Lemma 5.1, recall that V is completely determined by the choice of four Cartier divisors in $Z := \mathbb{P}^2$: A, A' (two smooth irreducible surfaces in Z) and D, D'. Now, let a, d, a', d' be integers such that $A = \mathcal{O}_{\mathbb{P}^2}(a)$, $A' = \mathcal{O}_{\mathbb{P}^2}(a')$ $D = \mathcal{O}_{\mathbb{P}^2}(d)$ $D' = \mathcal{O}_{\mathbb{P}^2}(d')$. Observe that, being A and A' effective, we need $a, a' \geq 1$. In this setting, the conditions for V to be Fano listed in Proposition 5.3 read

$$\begin{cases} 3 - (a' - d') - (a - d) \geq 1 \\ 3 - (a' - d') - d \geq 1 \\ 3 - d' - (a - d) \geq 1 \\ 3 - d' - d \geq 1 \\ 3 - (a' - d') \geq 1 \\ 3 - d' \geq 1 \end{cases} \tag{9}$$

Up to exchanging d with $a - d$ or d' with $a' - d'$, the only values of (a, d, a', d') that satisfy (9) are the following: $\{(1, 1, 1, 1), (1, 1, 2, 1), (2, 1, 1, 1), (2, 1, 2, 1)\}$. Observe that the choices of $(a, d, a', d') = (1, 1, 2, 1)$ and $(a, d, a', d') = (2, 1, 1, 1)$ yield the same family of Fano 4-folds by Proposition 5.2. We thus obtain three distinct families, listed in Table 11.

$\mathcal{O}(A)$	$\mathcal{O}(D)$	$\mathcal{O}(A')$	$\mathcal{O}(D')$	#	c_1^4	$c_1^2c_2$	$h^0(-K_X)$	$h^{2,2}$	Toric?	Product?
$\mathcal{O}_{\mathbb{P}^2}(1)$	$\mathcal{O}_{\mathbb{P}^2}(1)/\mathcal{O}_{\mathbb{P}^2}$	$\mathcal{O}_{\mathbb{P}^2}(1)$	$\mathcal{O}_{\mathbb{P}^2}(1)/\mathcal{O}_{\mathbb{P}^2}$	1	310	172	67	9	Yes[2]	No
		$\mathcal{O}_{\mathbb{P}^2}(2)$	$\mathcal{O}_{\mathbb{P}^2}(1)$	2	252	156	56	10	No	No
$\mathcal{O}_{\mathbb{P}^2}(2)$	$\mathcal{O}_{\mathbb{P}^2}(1)$	$\mathcal{O}_{\mathbb{P}^2}(2)$	$\mathcal{O}_{\mathbb{P}^2}(1)$	3	200	140	46	12	No	No

Table 11: Choices of A, D, A', D' in $\mathbb{P}^2$ that yield a Fano variety through a composition of two constructions of type A. All resulting Fano 4-folds have $h^{1,2}$=$h^{1,3} = 0$.

11 Fano 4-folds arising from Construction A with $\rho = 4$ and $\delta = 2$

According to Theorem 8.2, there are four possibilities for a Fano 4-fold X arising from Construction A with $\rho_X = 4$.

(A.1) There exists a $\mathbb{P}^1$-bundle $Z \to \mathbb{P}^2$.

(A.2) There exists a fibration $Z \to \mathbb{P}^1$ whose generic fiber is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and whose singular fibers are isomorphic to a quadric cone $\mathcal{Q} \subset \mathbb{P}^3$.

(A.3) There exists a $\mathbb{P}^2$-bundle $Z \to \mathbb{P}^1$.

(A.4) $Z \simeq \mathrm{Bl}_p\mathcal{Q}_3$, where $\mathcal{Q}_3$ is a smooth quadric threefold in $\mathbb{P}^4$ and p is a point in $\mathcal{Q}_3$.

By studying the classification of Fano 3-folds with $\rho = 2$ by Mori and Mukai and the work of Szurek-Wiśniewski ([SW90]), we get that the families satisfying conditions (A.1)-(A.4) are the following:

- (A.1) #2-24, #2-27, #2-31, #2-32, #2-34, #2-35, #2.36.
- (A.2) #2-18, #2-25, #2-29.
- (A.3) #2-33, #2-34.
- (A.4) #2-30.

In the following paragraphs, we will find all choices of A and D in Fano 3-folds Z belonging to the families above that yield a Fano 4-fold through Construction A.

For every Z, we call $\gamma_1 \colon Z \to W_1$ and $\gamma_2 \colon Z \to W_2$ its elementary contractions, and we call H_i the pull back through γ_i of the ample generator of $\mathrm{Pic}(W_i)$ for $i = 1, 2$, so that $\mathrm{Nef}(Z) = \mathbb{R}_{\geq 0}H_1 + \mathbb{R}_{\geq 0}H_2$. We know that $\{H_1, H_2\}$ is also a $\mathbb{Z}$-basis for $\mathrm{Pic}(Z)$, and $-K_Z \equiv \mu_2 H_1 + \mu_1 H_2$, where μ_i is the length of the extremal ray associated to γ_i (see [MM81]). We also know that smooth irreducible hypersurfaces in Z must be equivalent to a linear combination of H_1 and H_2 with non-negative integer coefficients, or to one of the exceptional divisors in Z, if there is any. This is enough to apply Proposition 2.3 and obtain Tables 12-23.

Observe that every Fano 4-fold X constructed this way has $\delta_X = 2$. Indeed, $\rho_Z = 2$ implies that $\delta_Z \leq 1$, and then by Lemma 2.4 we get $\delta_X \in \{2, 3\}$. However, there cannot be Fano 4-folds with $\rho_X = 4$ and $\delta_X = 3$ (see [CRS22]), so $\delta_X = 2$.

We ultimately obtain the following theorem.

[2] number 108 in [Bat99], §4.

Theorem 11.1. *There are 96 distinct families of Fano 4-folds X with $\rho_X = 4$ arising from Construction A, and they are all listed in Table 24. All of them have $\delta_X = 2$. Among them, 68 are not toric nor products of lower dimensional varieties.*

As a side comment, it is worth noticing that the Fano polytopes of the toric Fano 4-folds arising from Construction A with $\rho = 4$ are all of type H_n or I_n, for some $n \in \mathbb{N}$ (see [Bat99]).

1. $\boxed{Z = \textbf{Double cover of } \mathbb{P}^1 \times \mathbb{P}^2 \textbf{ with branch locus a divisor of degree } (2,2)\ \textbf{(\#2-18).}}$
The elementary contractions of Z are $\gamma_1\colon Z \to \mathbb{P}^1$ and $\gamma_2\colon Z \to \mathbb{P}^2$, where γ_1 is a quadric bundle, and γ_2 is a conic bundle.

A	D	#	c_1^4	$c_1^2c_2$	$h^0(-K_X)$	$h^{1,2}$	$h^{2,2}$	$h^{1,3}$	Toric?	Product?
H_2	$H_2/0$	1	152	128	37	2	10	0	No	No
$2H_2$	H_2	2	112	112	29	2	14	0	No	No

Table 12: Choices of A, D in Z = double cover of $\mathbb{P}^1 \times \mathbb{P}^2$ with branch locus a $(2,2)$-divisor.

2. $\boxed{Z = (1,2)\textbf{-divisor in } \mathbb{P}^2 \times \mathbb{P}^2\ \textbf{(\# 2-24).}}$ The elementary contractions of Z are $\gamma_1\colon Z \to W$ and $\gamma_2\colon Z \to \mathbb{P}^2$, where γ_1 is a conic bundle onto a surface W, and γ_2 is a $\mathbb{P}^1$-bundle.

A	D	#	c_1^4	$c_1^2c_2$	$h^0(-K_X)$	$h^{1,2}$	$h^{2,2}$	$h^{1,3}$	Toric?	Product?
H_1	$H_1/0$	1	194	140	45	0	9	0	No	No
$2H_1$	H_1	2	148	124	36	0	12	0	No	No

Table 13: Choices of A, D in $Z = (1,2)$-divisor in $\mathbb{P}^2 \times \mathbb{P}^2$.

3. $\boxed{Z = (1,2)\textbf{-divisor in } \mathbb{P}^1 \times \mathbb{P}^3\ \textbf{(\#2-25).}}$ The elementary contractions of Z are $\gamma_1\colon Z \to \mathbb{P}^1$ and $\gamma_2\colon Z \to \mathbb{P}^3$, where γ_1 is a quadric bundle, γ_2 is the blow up of the intersection of two quadrics in $\mathbb{P}^3$.

A	D	#	c_1^4	$c_1^2c_2$	$h^0(-K_X)$	$h^{1,2}$	$h^{2,2}$	$h^{1,3}$	Toric?	Product?
$-H_1+2H_2$	$-H_1/2H_2$	1	216	144	49	2	6	0	No	No
	$-H_1+H_2/H_2$	2	164	128	39	2	6	0	No	No
	$-H_1+2H_2/0$	3	176	128	41	2	6	0	No	No
H_2	$H_2/0$	4	199	142	46	1	9	0	No	No
$4H_2$	H_2	5	142	124	35	1	14	0	No	No

Table 14: Choices of A, D in $Z = (1,2)$ divisor in $\mathbb{P}^1 \times \mathbb{P}^3$.

4. $\boxed{Z = \textbf{Blow up of a twisted cubic in } \mathbb{P}^3 \textbf{ (\#2-27).}}$ The elementary contractions of Z are $\gamma_1 \colon Z \to \mathbb{P}^2$ and $\gamma_2 \colon Z \to \mathbb{P}^3$, where γ_1 is a $\mathbb{P}^1$-bundle and γ_2 is the blow up of a twisted cubic in $\mathbb{P}^3$. [H]

A	D	#	c_1^4	$c_1^2c_2$	$h^0(-K_X)$	$h^{1,2}$	$h^{2,2}$	$h^{1,3}$	Toric?	Product?
H_2	$H_2/0$	1	241	154	50	0	6	0	No	No
$-H_1+2H_2$	$H_2/-H_1+H_2$	2	221	146	54	0	8	0	No	No
$2H_2$	H_2	3	178	136	42	0	12	0	No	No

Table 15: Choices of A, D in $Z =$ Blow up of $\mathbb{P}^3$ along a twisted cubic.

5. $\boxed{Z = \textbf{Blow up of a conic in a quadric in } \mathbb{P}^4 \textbf{ (\#2-29).}}$ The elementary contractions of Z are $\gamma_1 \colon Z \to \mathbb{P}^1$ and $\gamma_2 \colon Z \to \mathcal{Q} \subset \mathbb{P}^4$, where γ_1 is a quadric bundle over $\mathbb{P}^1$ and γ_2 is the blow up of a conic in a smooth quadric $\mathcal{Q} \subset \mathbb{P}^4$.

A	D	#	c_1^4	$c_1^2c_2$	$h^0(-K_X)$	$h^{1,2}$	$h^{2,2}$	$h^{1,3}$	Toric?	Product?
$-H_1+H_2$	$-H_1/H_2$	1	414	204	87	1	6	0	No	No
	$-H_1+H_2/0$	2	356	188	76	1	6	0	No	No
H_2	$H_2/0$	3	246	156	55	0	8	0	No	No
$2H_2$	H_2	4	172	136	41	0	14	0	No	No

Table 16: Choices of A, D in $Z =$ Blow up of a quadric in $\mathbb{P}^3$ along a conic.

6. $\boxed{Z = \textbf{Blow up of a conic in } \mathbb{P}^3 \textbf{ (\#2-30).}}$ The elementary contractions of Z are $\gamma_1 \colon Z \to \mathbb{P}^3$ and $\gamma_2 \colon Z \to \mathcal{Q} \subset \mathbb{P}^4$, where γ_1 is a the blow up of a plane conic in $\mathbb{P}^3$, and γ_2 is the blow up of a point in a smooth quadric $\mathcal{Q} \subset \mathbb{P}^4$.

A	D	#	c_1^4	$c_1^2c_2$	$h^0(-K_X)$	$h^{1,2}$	$h^{2,2}$	$h^{1,3}$	Toric?	Product?
$2H_1-H_2/0$	H_1/H_1-H_2	1	316	172	68	0	6	0	No	No
H_1	$H_1/0$	2	299	170	65	0	7	0	No	No
$2H_1$	H_1	3	230	152	52	0	10	0	No	No

Table 17: Choices of A, D in $Z =$ Blow up of $\mathbb{P}^3$ along a conic.

7. $\boxed{Z = \textbf{Blow up of a quadric in } \mathbb{P}^4 \textbf{ along a line (\# 2-31).}}$ The elementary contractions of Z are $\gamma_1\colon Z \to \mathbb{P}^2$ and $\gamma_2\colon Z \to \mathcal{Q}$, where γ_1 is a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$ and γ_2 is the blow up of a smooth quadric $\mathcal{Q}$ in $\mathbb{P}^4$ along a line.

A	D	#	c_1^4	$c_1^2c_2$	$h^0(-K_X)$	$h^{1,2}$	$h^{2,2}$	$h^{1,3}$	Toric?	Product?
$-H_1+H_2$	$H_2/-H_1$	1	368	188	78	0	6	0	No	No
	$-H_1+H_2/0$	2	331	178	71	0	6	0	No	No
H_2	$H_2/0$	3	288	168	63	0	7	0	No	No
$2H_2$	$H_2/0$	4	208	148	48	0	12	0	No	No

Table 18: Choices of A, D in $Z =$ Blow up of a quadric in $\mathbb{P}^4$ along a line.

8. $\boxed{Z = (1,1)\textbf{-divisor in } \mathbb{P}^2 \times \mathbb{P}^2 \textbf{ (\# 2-32).}}$ The elementary contractions $\gamma_1\colon Z \to \mathbb{P}^2$ and $\gamma_2\colon Z \to \mathbb{P}^2$ of Z are the restrictions of the projections of $\mathbb{P}^2 \times \mathbb{P}^2$ onto its factors. Both realize Z as $\mathbb{P}(T_{\mathbb{P}^2})$.

A	D	#	c_1^4	$c_1^2c_2$	$h^0(-K_X)$	$h^{1,2}$	$h^{2,2}$	$h^{1,3}$	Toric?	Product?
H_2	$H_1+H_2/-H_1$	1	362	188	77	0	6	0	No	No
	$H_2/0$	2	320	176	69	0	6	0	No	No
	$-H_1+H_2/H_1$	3	304	172	66	0	6	0	No	No
$2H_2$	$H_1+H_2/-H_1+H_2$	4	266	164	59	0	6	0	No	No
	H_2	5	256	160	57	0	6	0	No	No
H_1+H_2	$H_1+H_2/0$	6	282	168	62	0	8	0	No	No
	H_2/H_1	7	256	160	57	0	8	0	No	No
H_1+2H_2	H_2/H_1+H_2	8	218	152	50	0	12	0	No	No
$2H_1+2H_2$	H_1+H_2	9	180	144	43	0	24	1	No	No

Table 19: Choices of A, D in $Z = (1,1)$-divisor in $\mathbb{P}^2 \times \mathbb{P}^2$.

9. $\boxed{Z = \textbf{Blow up of a line in } \mathbb{P}^3 \textbf{ (\#2-33).}}$ The elementary contractions of Z are $\gamma_1\colon Z \to \mathbb{P}^1$ and $\gamma_2\colon Z \to \mathbb{P}^3$, where γ_1 is a $\mathbb{P}^2$-bundle that realizes Z as $\mathbb{P}_{\mathbb{P}^1}(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(1))$, and γ_2 is the blow up of a line in $\mathbb{P}^3$.

A	D	#	c_1^4	$c_1^2c_2$	$h^0(-K_X)$	$h^{1,2}$	$h^{2,2}$	$h^{1,3}$	Toric?	Product?
$-H_1+H_2$	$-H_1-H_2/2H_2$	1	496	220	102	0	6	0	I_1	No
	$-H_2/-H_1+2H_2$	2	433	202	90	0	6	0	I_4	No
	$-H_1/H_2$	3	411	198	86	0	6	0	I_6	No
	$-H_1+H_2/0$	4	390	192	82	0	6	0	I_9	No

H_2	$2H_2/-H_2$	5	415	202	87	0	6	0	I_5	No
	$H_2/0$	6	357	186	76	0	6	0	I_{14}	No
$2H_2$	$2H_2/0$	7	308	176	67	0	8	0	No	No
	H_2	8	282	168	62	0	8	0	No	No
$3H_2$	$H_2/2H_2$	9	233	158	53	0	14	0	No	No
$4H_2$	$2H_2$	10	184	148	44	0	6	1	No	No

Table 20: Choices of A, D in $Z =$ Blow up of a line in $\mathbb{P}^3$.

10. $\boxed{Z = \mathbb{P}^1 \times \mathbb{P}^2\ (\#\ \textbf{2-34}).}$ The elementary contractions of Z are the projections $\gamma_1 \colon Z \to \mathbb{P}^1$ and $\gamma_2 \colon Z \to \mathbb{P}^2$.

A	D	#	c_1^4	$c_1^2 c_2$	$h^0(-K_X)$	$h^{1,2}$	$h^{2,2}$	$h^{1,3}$	Toric?	Product?
H_1	$-2H_2/H_1+2H_2$	1	478	220	99	0	5	0	H_3	No
	$-H_2/H_1+H_2$	2	415	202	87	0	5	0	H_5	No
	$2H_2/H_1-2H_2$	3	382	196	81	0	5	0	H_7	No
	$H_1/0$	4	378	192	80	0	5	0	H_8	$\mathrm{Bl}_2\mathbb{P}^2 \times \mathbb{P}^2$
	H_1/H_1-H_2	5	367	190	78	0	5	0	H_9	No
H_2	$-H_1-H_2/H_1+2H_2$	6	463	214	96	0	6	0	I_2	No
	$-H_2/2H_2$	7	400	196	84	0	6	0	I_7	$\mathbb{P}^1\times$(#3-29)
	$-H_1/H_1+H_2$	8	389	194	82	0	6	0	I_{10}	No
	$H_2/0$	9	368	188	78	0	6	0	I_{13}	$\mathbb{P}^1\times$(#3-26)
	$-H_1+H_2/H_1$	10	347	196	84	0	6	0	I_{12}	No
	$-H_1+2H_2/H_1-H_2$	11	337	178	72	0	6	0	I_{15}	No
$2H_2$	$-H_1/H_1+2H_2$	12	356	188	76	0	6	0	No	No
	$2H_2/0$	13	320	176	69	0	6	0	No	$\mathbb{P}^1\times$(#3-22)
	H_2	14	304	172	66	0	6	0	No	$\mathbb{P}^1\times$(#3-19)
	$-H_1+H_2/H_1+H_2$	15	304	172	66	0	6	0	No	No
	$-H_1+2H_2/H_1$	16	284	164	62	0	6	0	No	No
$3H_2$	$-H_1+H_2/H_1+2H_2$	17	271	166	60	1	6	0	No	No
	$H_2/2H_2$	18	256	160	57	1	6	0	No	$\mathbb{P}^1\times$(#3-14)
	$-H_1+2H_2/H_1+H_2$	19	241	154	54	1	6	0	No	No
$4H_2$	$-H_1+2H_2/H_1+2H_2$	20	208	148	48	3	6	0	No	No
	$2H_2$	21	208	148	48	3	6	0	No	$\mathbb{P}^1\times$(#3-9)
H_1+H_2	$-H_2/H_1+2H_2$	22	382	196	81	0	6	0	No	No
	$H_1+H_2/0$	23	335	182	72	0	6	0	No	No
	$2H_2/H_1-H_2$	24	319	178	69	0	6	0	No	No
	H_2/H_1	25	314	176	68	0	6	0	No	No
H_1+2H_2	$H_1+2H_2/0$	26	302	176	66	0	9	0	No	No
	H_1/H_1+H_2	27	271	166	60	0	9	0	No	No
	$2H_2/H_1$	28	266	164	59	0	9	0	No	No

H_1+3H_2	H_2/H_1+2H_2	29	238	160	54	0	14	0	No	No
	$2H_2/H_1+2H_2$	30	223	154	51	0	14	0	No	No
H_1+4H_2	$2H_2/H_1+2H_2$	31	190	148	45	0	21	0	No	No
$2H_1+H_2$	H_1-H_2/H_1+2H_2	32	301	178	66	0	6	0	No	No
	H_1/H_1+H_2	33	281	170	62	0	6	0	No	No
$2H_1+2H_2$	H_1/H_1+2H_2	34	248	164	56	0	12	0	No	No
	H_1+H_2	35	238	160	54	0	12	0	No	No
$2H_1+3H_2$	H_1+H_2/H_1+2H_2	36	205	154	48	0	24	1	No	No
$2H_1+4H_2$	H_1+2H_2	37	172	148	42	0	42	3	No	No

Table 21: Choices of A, D in $Z=\mathbb{P}^1\times\mathbb{P}^2$.

11. $\boxed{Z=\mathbb{P}_{\mathbb{P}^2}(\mathcal{O}\oplus\mathcal{O}(1))\ (\#\ \textbf{2-35}).}$ The elementary contractions of Z are $\gamma_1\colon Z\to\mathbb{P}^2$ and $\gamma_2\colon Z\to\mathbb{P}^3$, where γ_1 realizes Z as $\mathbb{P}_{\mathbb{P}^2}(\mathcal{O}\oplus\mathcal{O}(1))$, while γ_2 is the blow up of a point in $\mathbb{P}^3$.

A	D	#	c_1^4	$c_1^2c_2$	$h^0(-K_X)$	$h^{1,2}$	$h^{2,2}$	$h^{1,3}$	Toric?	Product?
$-H_1+H_2$	$-H_1/H_2$	1	447	210	93	0	5	0	H_4	No
	$-H_1+H_2/0$	2	415	202	87	0	5	0	H_5	No
H_1	$H_1+H_2/-H_2$	3	442	208	92	0	6	0	I_3	No
	$H_1/0$	4	384	192	81	0	6	0	I_8	No
	H_1-H_2/H_2	5	384	192	81	0	6	0	I_8	No
$2H_1$	H_1-H_2/H_1+H_2	6	346	184	74	0	6	0	No	No
	H_1	7	320	176	69	0	6	0	No	No
H_2	$-H_1/H_1+H_2$	8	409	202	86	0	5	0	H_6	No
	$H_2/0$	9	367	190	78	0	5	0	H_9	No
	$-H_1+H_2/H_1$	10	351	186	75	0	5	0	H_{10}	No
H_1+H_2	$H_1+H_2/0$	11	329	182	71	0	7	0	No	No
	H_1/H_2	12	303	174	66	0	7	0	No	No
$2H_1+H_2$	H_1+H_2/H_1	13	265	166	59	0	11	0	No	No
$2H_2$	$-H_1+H_2/H_1+H_2$	14	296	176	65	0	6	0	No	No
	H_2	15	286	172	63	0	6	0	No	No
H_1+2H_2	H_2/H_1+H_2	16	248	164	56	0	12	0	No	No
$2H_1+2H_2$	H_1+H_2	17	210	156	49	0	24	1	No	No

Table 22: Choices of A, D in $Z=\mathbb{P}_{\mathbb{P}^2}(\mathcal{O}\oplus\mathcal{O}(1))$.

12. $\boxed{Z=\mathbb{P}_{\mathbb{P}^2}(\mathcal{O}\oplus\mathcal{O}(2))\ (\#\ \textbf{2-36}).}$ The elementary contractions of Z are $\gamma_1\colon Z\to\mathbb{P}^2$ and $\gamma_2\colon Z\to W$, where γ_1 realizes Z as $\mathbb{P}_{\mathbb{P}^2}(\mathcal{O}\oplus\mathcal{O}(2))$, while γ_2 is the blow up of a quadruple non-Gorenstein point in a threefold W.

A	D	#	c_1^4	$c_1^2c_2$	$h^0(-K_X)$	$h^{1,2}$	$h^{2,2}$	$h^{1,3}$	Toric?	Product?
$-2H_1+H_2$	$-2H_1/H_2$	1	558	240	114	0	5	0	H_1	No
	$-H_1+H_2/-H_1$	2	505	226	104	0	5	0	H_2	No
	$-2H_1+H_2/0$	3	478	220	99	0	5	0	H_3	No
H_2	$H_2/0$	4	382	196	81	0	5	0	H_7	No
$2H_2$	H_2	5	268	172	60	0	12	0	No	No

Table 23: Choices of A, D in $Z = \mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(2))$.

Ambiguities In Tables 12-23 there are some families of Fano 4-folds arising from different constructions of type A that have the same invariants. We are going to prove that all non-toric Fano 4-folds listed in said tables are distinct. The study of the toric 4-folds can be done by comparing their descriptions with the ones in [Bat99].

We now list all ambiguities, namely all couples or triples of families sharing the same invariants.

- (#6 in Table 19 , #8 in Table 20). We prove that they are two distinct families. Let $X = \mathscr{C}_A(Z; A, D)$ be a member of the first family, and $X' = \mathscr{C}_A(Z'; A', D')$ be a member of the second one. We have $Z = (1,1)$-divisor in $\mathbb{P}^2 \times \mathbb{P}^2$, $A = H_1 + H_2$, $D = H_1 + H_2$; $Z' =$ blow-up of a line in $\mathbb{P}^3$, $A' = 2H_2'$, $D' = H_2'$. Since Z' does not contain any effective divisor having negative intersection with all curves contained in its support, and D is ample, it follows that X is not isomorphic to X', by Remark 10.2.

- (#3 Table 19 , #14 Table 21, #15 in Table 21) Let X_1, X_2, X_3 be each a member of one of those families. We show that they are not isomorphic to each other.

  First of all, observe that $X_2 \simeq \mathbb{P}^1 \times$(#3-19), while the other two varieties are not products by Remark 10.3.

  Now, let $X_3 = \mathscr{C}_A(Z; A, D)$ be the description of X_3 given in Table 21. We have that $D = H_1 + H_2$ is an ample divisor in $Z = \mathbb{P}^1 \times \mathbb{P}^2$. This implies that X cannot have an alternative construction of type A with base $\mathbb{P}_{\mathbb{P}^2}(T_{\mathbb{P}^2})$, which does not have any divisor with negative intersection with every curve contained in its support (see Remark 10.2). Hence X_3 is not isomorphic to X_1.

- (#20 Table 21 , #21 Table 21). We now show that they are distinct families. Indeed, #21 is the product $\mathbb{P}^1 \times$(#3-9), while members of family #20 cannot be products of that kind, as we can see by applying Remark 10.3.

- (#34 in Table 20 , #16 in Table 21). We now prove that those families are distinct. Let X be a Fano 4-fold in the latter family. By the description in Table 21, $X = \mathscr{C}_A(A; Z, D)$ with $Z = \mathbb{P}^1 \times \mathbb{P}^2$, $A = 2H_1 + 2H_2$, $D = H_1 + 2H_2$. Observe that D is ample in Z and in the blow up of a line in $\mathbb{P}^3$ there are no effective divisors with negative intersection with every curve contained in their support. Therefore by Remark 10.2, X does not belong to the first family.

- (#2 Table 19 , #13 Table 21 , #7 Table 22). Let X_1, X_2, X_3 be each a member of one of those families. We now prove that they are not isomorphic.

First of all, observe that X_2 can be expressed as the product $\mathbb{P}^1\times(\#3\text{-}22)$, while the other two cannot, as we can see by applying Remark 10.3.

Now, consider the construction of type A that yields $X_3 = \mathscr{C}_A(Z; A, D)$. We have $Z = \mathbb{P}_{\mathbb{P}^2}(\mathcal{O}\oplus\mathcal{O}(1))$, $A = 2H_2$, $D = 2H_2$. On the other hand, X_1 arises from a construction of type A with base $Z' = (1,1)$-divisor in $\mathbb{P}^2\times\mathbb{P}^2$. Suppose by contradiction that $X_1 \simeq X_3$. By similar arguments as in the proof of Remark 10.2, using the fact that $D = 2H_2$ is nef one can conclude that Z' must contain an effective divisor with non-positive intersection with every curve contained in its support. Let U be the class of such divisor. Since $\mathrm{Eff}(Z)_{\mathbb{Z}} = \mathbb{Z}_{\geq 0}H_1 + \mathbb{Z}_{\geq 0}H_2$, there exist $u_1, u_2 \in \mathbb{Z}_{\geq 0}$ such that $U = u_1H_1 + u_2H_2$. Let l_1, l_2 in $N_1(Z)$ be the classes of irreducible curves generating $\mathrm{NE}(Z)$. We have $H_i \cdot l_j = 1-\delta_{i,j}$ (see [MM81]). It follows that necessarily $U\cdot C = 0$ for all irreducible curves C contained in the support of U. But then the only possibility for U is to be equivalent to a multiple of H_1 or H_2. Suppose $U = aH_1$ for some $a > 0$, so that $U = \gamma_1^*(\mathcal{O}_{\mathbb{P}^2}(a)) = \mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(1,1)\cap\mathcal{O}_{\mathbb{P}^2\times\mathbb{P}^2}(a,0)$. But then, one can easily check that there exists a curve C in the support of U that is not a fiber for γ_1 (i.e. not numerically equivalent to l_1), which implies $U\cdot C > 0$, a contradiction.

12 Numerical invariants

In this section we show the tools we used to compute the numerical invariants of the Fano 4-folds constructed in Sections 10 and 11. Every such Fano variety X arises from Construction A. The objects that are part of the structure of Construction A will be denoted by the same notation as in Section 2.1.

First we want to compute the Hodge numbers. To do so, we will use the Hodge polynomial of a smooth variety W. The Hodge polynomial $e(W)(u,v)$ is given by:

$$e(W)(u,v) := \sum_{p,q} h^{p,q}(W)u^p v^q$$

where $h^{p,q}(W)$ are the Hodge numbers, corresponding to the dimensions of the Dolbeault cohomology groups $H^{p,q}(W)$.

Lemma 12.1. *Let W be a smooth variety, let $\mathbb{P}(\mathcal{E})$ be a $\mathbb{P}^n$-bundle over W, and let W' be the blow-up of W along a subvariety V of codimension c. Then the Hodge polynomials satisfy the following relationships:*

$$e(\mathbb{P}(\mathcal{E})) = e(W)\cdot e(\mathbb{P}^n) \qquad \text{and} \qquad e(W') = e(W) + e(V)\cdot\left[e(\mathbb{P}^{c-1}) - 1\right]$$

Proof. It follows from [Che96, Introduction 0.1] and simple computations. □

Corollary 12.2. *Let $Y = \mathbb{P}(\mathcal{O}\oplus\mathcal{O}(D))$ a $\mathbb{P}^1$-bundle over a Fano threefold Z as in Section 2.1. Then $h^{0,1}(Y) = h^{0,2}(Y) = h^{0,3}(Y) = 0$, $h^{1,2}(Y) = h^{1,2}(Z)$, $h^{2,2}(Y) = 2h^{1,1}(Z)$ and $h^{1,3}(Y) = 0$.*

Let $X \to Y$ be the blow up of a codimension 2 subvariety $A_Y \subset Y$ as in Section 2.1. Then $h^{0,1}(X) = h^{0,2}(X) = h^{0,3}(X) = 0$, $h^{1,2}(X) = h^{1,2}(Z) + h^{0,1}(A)$, $h^{2,2}(X) = 2h^{1,1}(Z) + h^{1,1}(A)$, and $h^{1,3}(X) = h^{0,2}(A)$.

Proof. After noticing that, by Kodaira vanishing, $h^{0,i}(Z) = 0$ for all $i > 0$, it is enough to apply Lemma 12.1 to the $\mathbb{P}^1$-bundle $Y \to Z$ and then to the blow up $X \to Y$. $\square$

Therefore, in order to be able to compute the Hodge numbers of X, we need to compute the Hodge numbers of A. The following Lemma provides some tools in that direction.

Lemma 12.3. *Let A be a smooth irreducible surface in a Fano 3-fold Z. Then $h^{0,1}(A) = h^1(\mathcal{O}_Z(K_Z + A))$, and $h^{0,2}(A) = h^0(\mathcal{O}_Z(K_Z + A))$.*

Moreover $h^{1,1}(A) = \chi_{top}(A) + 4h^{0,1}(A) - 2h^{0,2}(A) - 2$, where $\chi_{top}(A) := \sum_i (-1)^i b_i(A)$ is the topological characteristic of A.

Proof. Consider the short exact sequence

$$0 \to \mathcal{O}_Z(-A) \to \mathcal{O}_Z \to \mathcal{O}_A \to 0. \tag{10}$$

Since Z is Fano, by Kodaira vanishing we have $H^i(Z, \mathcal{O}_Z) = 0$ for all $i > 0$, so that the induced long exact secquence in cohomology yields $H^1(\mathcal{O}_A) \simeq H^2(\mathcal{O}_Z(-A))$ and $H^2(\mathcal{O}_A) \simeq H^3(\mathcal{O}_Z(-A))$. By Serre duality, $H^i(Z, \mathcal{O}_Z(-A)) \simeq H^{3-i}(Z, \mathcal{O}_Z(K_Z + A))^*$.

Finally, using the fact that $b_i(A) = b_{4-i}(A)$, $h^{p,q}(A) = h^{q,p}(A)$, and $b_i(A) = \sum_k h^{k,i-k}(A)$, we get

$$\chi_{top}(A) = 2b_0(A) - 2b_1(A) + b_2(A) = 2 - 4h^{0,1}(A) + 2h^{0,2}(A) + h^{1,1}(A).$$

$\square$

Corollary 12.4. *Let A be a smooth irreducible surface in a Fano threefold Z.*

If $K_Z + A$ is not effective, then

$$h^{0,2}(A) = 0 \qquad h^{0,1}(A) = 1 - \frac{1}{12}(K_A^2 + \chi_{top}(A)) \qquad h^{1,1}(A) = \chi_{top}(A) + 4h^{0,1}(A) - 2$$

If $A \in |-K_Z|$, then

$$h^{0,2}(A) = 1 \qquad h^{0,1}(A) = 0 \qquad h^{1,1}(A) = \chi_{top}(A) - 4$$

Proof. It is enough to apply Lemma 12.3 and notice that if $K_Z + A$ is not effective, then $h^0(K_Z + A) = 0$. For computing $h^{0,1}(A)$, observe that, by the Noether formula, $\chi(\mathcal{O}_A) = \frac{1}{12}(K_A^2 + \chi_{top}(A))$. After recalling that $\chi(\mathcal{O}_A) = h^{0,0}(A) - h^{0,1}(A) + h^{0,2}(A)$, we get the formula for $h^{0,1}$.

On the other hand, if $A \in |-K_Z|$, then (again by Lemma 12.3), we get $h^{0,1}(A) = h^1(Z, \mathcal{O}_Z) = 0$ because Z is Fano, and $h^{0,2}(A) = h^0(Z, \mathcal{O}_Z) = 1$. $\square$

The following facts provide us all necessary tools for computing the invariants $c_1(X)^4$, $c_1(X)^2c_2(X)$ and $\chi(-K_X)$. Recall that, if X is Fano, by Kodaira vanishing we have $\chi(-K_X) = h^0(-K_X)$.

Proposition 12.5 ([CR22], Lemma 3.2)**.** *Let W be a smooth projective variety, $\dim W = 4$, and let $\alpha : \tilde{W} \to W$ be the blow-up of W along a smooth irreducible surface V. Then*

(i) $K_{\tilde{W}}^4 = K_W^4 - 3(K_W|_V)^2 - 2K_V \cdot K_W|_V + c_2(\mathcal{N}_{V/W}) - K_V^2$;

(ii) $K_{\tilde{W}}^2 \cdot c_2(\tilde{W}) = K_W^2 \cdot c_2(W) - 12\chi(\mathcal{O}_V) + 2K_V^2 - 2K_V \cdot K_W|_V - 2c_2(\mathcal{N}_{V/W})$;

(iii) $\chi(O_{\tilde{W}}(-K_{\tilde{W}})) = \chi(O_W(-K_W)) - \chi(O_V) - \frac{1}{2}((K_W|_V)^2 + K_V \cdot K_W|_V)$.

Proposition 12.6 ([Sec23], Proposition 3.4)**.** *Let W be a smooth projective variety of dimension 3, and $\mathcal{E}$ a rank 2 vector bundle on W. Let $\beta : \mathbb{P}(\mathcal{E}) \to W$ be a $\mathbb{P}^1$-bundle over W. Then*

(i) $K^4_{\mathbb{P}(\mathcal{E})} = -8K_W \cdot c_1(\mathcal{E})^2 + 32K_W \cdot c_2(\mathcal{E}) - 8K^3_W$;

(ii) $K^2_{\mathbb{P}(\mathcal{E})} \cdot c_2(\mathbb{P}(\mathcal{E})) = -2K_W \cdot c_1(\mathcal{E})^2 + 8K_W \cdot c_2(\mathcal{E}) - 2K^3_W - 4K_W \cdot c_2(W)$;

(iii) $\chi(O_{\mathbb{P}(\mathcal{E})}(-K_{\mathbb{P}(\mathcal{E})})) = \chi(O_{\mathbb{P}(\mathcal{E})}) + 6K_W \cdot c_2(\mathcal{E}) - \frac{1}{2}(3K^3_W + 3K_W \cdot c_1(\mathcal{E})^2) - \frac{1}{3}K_W \cdot c_2(W)$.

Appendix A. Final tables

In Tables 24 and 25 we list every family of Fano 4-folds X (respectively with $\rho = 4$ and $\rho = 5$) that arise from Construction A. They all have $\delta = 2$. Under the column "Z", we make explicit the Fano 3-fold Z from which X arises through Construction A, by using the classical notation. Under the column "Description of X", we refer to the detailed description of the construction of type A that yields X, for example "#7-23" means that the description of X can be found in Table 7, row number 23.

Table 24: List of all families of Fano 4-folds arising from Construction A with $\rho = 4$

	ρ	c_1^4	$c_1^2c_2$	$h^0(-K_X)$	$h^{1,2}$	$h^{1,3}$	$h^{2,2}$	Toric?	Product?	Z	Description of X
1	4	112	112	29	2	0	14	No	No	#2-18	#12-2
2	4	142	124	35	1	0	14	No	No	#2-25	#14-5
3	4	148	124	36	0	0	12	No	No	#2-24	#13-2
4	4	152	128	37	2	0	10	No	No	#2-18	#12-1
5	4	164	128	39	2	0	6	No	No	#2-25	#14-2
6	4	172	136	41	0	0	14	No	No	#2-29	#16-4
7	4	172	148	42	0	3	42	No	No	#2-34	#21-37
8	4	176	128	41	2	0	6	No	No	#2-25	#14-3
9	4	178	136	42	0	0	12	No	No	#2-27	#15-3
10	4	180	144	43	0	1	24	No	No	#2-32	#19-9
11	4	184	148	44	0	1	6	No	No	#2-33	#20-10
12	4	190	148	45	0	0	21	No	No	#2-34	#21-31
13	4	194	140	45	0	0	9	No	No	#2-24	#13-1
14	4	199	142	46	1	0	9	No	No	#2-25	#14-4
15	4	205	154	48	0	1	24	No	No	#2-34	#21-36
16	4	208	148	48	0	0	12	No	No	#2-31	#18-4
17	4	208	148	48	3	0	6	No	No	#3-24	#21-20
18	4	208	148	48	3	0	6	No	(#3-9)$\times\mathbb{P}^1$	#3-24	#21-21
19	4	210	156	49	0	1	24	No	No	#2-35	#22-17
20	4	216	144	49	2	0	6	No	No	#2-25	#14-1
21	4	218	152	50	0	0	12	No	No	#2-32	#19-8
22	4	221	146	54	0	0	8	No	No	#2-27	#15-2
23	4	223	154	51	0	0	14	No	No	#2-34	#21-30
24	4	230	152	52	0	0	10	No	No	#2-30	#17-3

Table 24: List of all families of Fano 4-folds arising from Construction A with $\rho = 4$

	ρ	c_1^4	$c_1^2c_2$	$h^0(-K_X)$	$h^{1,2}$	$h^{1,3}$	$h^{2,2}$	Toric?	Product?	Z	Description of X
25	4	233	158	53	0	0	14	No	No	#2-33	#20-9
26	4	238	160	54	0	0	14	No	No	#2-34	#21-29
27	4	238	160	54	0	0	12	No	No	#2-34	#21-35
28	4	241	154	50	0	0	6	No	No	#2-27	#15-1
29	4	241	154	54	1	0	6	No	No	#2-34	#21-19
30	4	246	156	55	0	0	8	No	No	#2-29	#16-3
31	4	248	164	56	0	0	12	No	No	#2-34	#21-34
32	4	248	164	56	0	0	12	No	No	#2-35	#22-16
33	4	256	160	57	0	0	6	No	No	#2-32	#19-5
34	4	256	160	57	0	0	8	No	No	#2-32	#19-7
35	4	256	160	57	1	0	6	No	(#3-14)$\times\mathbb{P}^1$	#2-34	#21-18
36	4	265	166	59	0	0	11	No	No	#2-35	#22-13
37	4	266	164	59	0	0	6	No	No	#2-32	#19-4
38	4	266	164	59	0	0	9	No	No	#2-34	#21-28
39	4	268	172	60	0	0	12	No	No	#2-36	#23-5
40	4	271	166	60	1	0	6	No	No	#2-34	#21-17
41	4	271	166	60	0	0	9	No	No	#2-34	#21-27
42	4	281	170	62	0	0	6	No	No	#2-34	#21-33
43	4	282	168	62	0	0	8	No	No	#2-32	#19-6
44	4	282	168	62	0	0	8	No	No	#2-33	#20-8
45	4	284	164	62	0	0	6	No	No	#2-34	#21-16
46	4	286	172	63	0	0	6	No	No	#2-35	#22-15
47	4	288	168	63	0	0	7	No	No	#2-31	#18-3
48	4	296	176	65	0	0	6	No	No	#2-35	#22-14
49	4	299	170	65	0	0	7	No	No	#2-30	#17-2
50	4	301	178	66	0	0	6	No	No	#2-34	#21-32
51	4	302	176	66	0	0	9	No	No	#2-34	#21-26
52	4	303	174	66	0	0	7	No	No	#2-35	#22-12
53	4	304	172	66	0	0	6	No	No	#2-32	#19-3
54	4	304	172	66	0	0	6	No	(#3-19)$\times\mathbb{P}^1$	#2-34	#21-14
55	4	304	172	66	0	0	6	No	No	#2-34	#21-15
56	4	308	176	67	0	0	8	No	No	#2-33	#20-7
57	4	314	176	68	0	0	6	No	No	#2-34	#21-25
58	4	316	172	68	0	0	6	No	No	#2-30	#17-1
59	4	319	178	69	0	0	6	No	No	#2-34	#21-24
60	4	320	176	69	0	0	6	No	No	#2-32	#19-2
61	4	320	176	69	0	0	6	No	(#3-22)$\times\mathbb{P}^1$	#2-34	#21-13
62	4	320	176	69	0	0	6	No	No	#2-35	#22-7
63	4	329	182	71	0	0	7	No	No	#2-35	#22-11
64	4	331	178	71	0	0	6	No	No	#2-31	#18-2
65	4	335	182	72	0	0	6	No	No	#2-34	#21-23
66	4	337	178	72	0	0	6	I_{15}	No	#2-34	#21-11
67	4	346	184	74	0	0	6	No	No	#2-35	#22-6
68	4	347	196	84	0	0	6	I_{12}	No	#2-34	#21-10

Table 24: List of all families of Fano 4-folds arising from Construction A with $\rho = 4$

	ρ	c_1^4	$c_1^2c_2$	$h^0(-K_X)$	$h^{1,2}$	$h^{1,3}$	$h^{2,2}$	Toric?	Product?	Z	Description of X
69	4	351	186	75	0	0	5	H_{10}	No	#2-35	#22-10
70	4	356	188	76	1	0	6	No	No	#2-29	#16-2
71	4	356	188	76	0	0	6	No	No	#2-34	#21-12
72	4	357	186	76	0	0	6	I_{14}	No	#2-33	#20-6
73	4	362	188	77	0	0	6	No	No	#2-33	#19-1
74	4	367	190	78	0	0	5	H_9	No	#2-34, #2-35	#21-5, #22-9
75	4	368	188	78	0	0	6	No	No	#2-31	#18-1
76	4	368	188	78	0	0	6	I_{13}	(#3-26)$\times\mathbb{P}^1$	#2-34	#21-9
77	4	378	192	80	0	0	5	H_8	$\mathbb{P}^2 \times \mathrm{Bl}_2\,\mathbb{P}^2$	#2-34	#21-4
78	4	382	196	81	0	0	5	H_7	No	#2-34, #2-36	#21-3, #23-4
79	4	382	196	81	0	0	6	No	No	#2-34	#21-22
80	4	384	192	81	0	0	6	I_8	No	#2-35	#22-4, #22-5
81	4	389	194	82	0	0	6	I_{10}	No	#2-34	#21-8
82	4	390	192	62	0	0	6	I_9	No	#2-33	#20-4
83	4	400	196	84	0	0	6	I_7	(#3-29)$\times\mathbb{P}^1$	#2-34	#21-7
84	4	409	202	86	0	0	5	H_6	No	#2-35	#22-8
85	4	411	198	86	0	0	6	I_6	No	#2-33	#20-3
86	4	414	204	87	1	0	6	No	No	#2-29	#16-1
87	4	415	202	87	0	0	6	I_5	No	#2-33	#20-5
88	4	415	202	87	0	0	5	H_5	No	#2-34, #2-35	#21-2, #22-2
89	4	433	202	90	0	0	6	I_4	No	#2-33	#20-2
90	4	442	208	92	0	0	6	I_3	No	#2-35	#22-3
91	4	447	210	93	0	0	5	H_4	No	#2-35	#22-1
92	4	463	214	96	0	0	6	I_2	No	#2-34	#21-6
93	4	478	220	99	0	0	5	H_3	No	#2-34, #2-36	#21-1, #23-3
94	4	496	220	102	0	0	6	I_1	No	#2-33	#20-1
95	4	505	226	104	0	0	5	H_2	No	#2-36	#23-2
96	4	558	240	114	0	0	5	H_1	No	#2-36	#23-1

Table 25: List of all families of Fano 4-folds arising from Construction A with $\rho = 5$

	ρ	c_1^4	$c_1^2c_2$	$h^0(-K_X)$	$h^{1,2}$	$h^{1,3}$	$h^{2,2}$	Toric?	Product?	Z	Description of X
1	5	180	144	43	0	1	26	No	No	#3-27	#7-23
2	5	184	136	43	0	0	12	No	No	#3-17	#4-2
3	5	200	140	46	0	0	12	No	No	#3-19	#11-3
4	5	202	148	47	0	0	16	No	No	#3-28	#8-18
5	5	202	148	47	0	0	16	No	No	#3-27	#7-23
6	5	214	148	49	0	0	12	No	No	#3-25	#6-4
7	5	220	148	50	0	0	10	No	No	#3-24	#5-2
8	5	224	152	51	1	0	8	No	No	#3-27	#7-16
9	5	224	152	51	1	0	8	No	(#4-2)$\times\mathbb{P}^1$	#3-27	#7-17
10	5	224	152	51	0	0	12	No	No	#3-27	#7-21

Table 25: List of all families of Fano 4-folds arising from Construction A with $\rho = 5$

	ρ	c_1^4	$c_1^2c_2$	$h^0(-K_X)$	$h^{1,2}$	$h^{1,3}$	$h^{2,2}$	Toric?	Product?	Z	Description of X
11	5	229	154	52	0	0	12	No	No	#3-28	#8-17
12	5	234	156	53	0	0	12	No	No	#3-27	#7-20
13	5	236	152	53	0	0	10	No	No	#3-17	#4-1
14	5	244	160	55	0	0	12	No	No	#3-31	#10-5
15	5	245	158	55	0	0	10	No	No	#3-28	#8-14
16	5	246	156	55	0	0	8	No	No	#3-27	#7-15
17	5	250	160	56	0	0	10	No	No	#3-30	#9-4
18	5	252	156	56	0	0	10	No	No	#3-26, #3-19	#11-2
19	5	256	160	57	0	0	8	No	No	#3-28	#8-15
20	5	256	160	57	0	0	8	No	(#4-4)$\times\mathbb{P}^1$	#3-28	#8-16
21	5	256	160	57	0	0	8	No	(#4-5)$\times\mathbb{P}^1$	#3-27	#7-14
22	5	256	160	57	0	0	10	No	No	#3-27	#7-19
23	5	266	164	59	0	0	8	No	No	#3-27	#7-13
24	5	278	164	61	0	0	8	No	No	#3-27	#7-12
25	5	278	164	61	0	0	9	No	No	#3-24	#5-1
26	5	282	168	62	0	0	10	No	No	#3-27	#7-18
27	5	283	166	62	0	0	9	No	No	#3-25	#6-3
28	5	288	168	63	0	0	8	No	(#4-7)$\times\mathbb{P}^1$	#3-27	#7-10
29	5	288	168	63	0	0	8	No	No	#3-27	#7-11
30	5	293	170	64	0	0	9	No	No	#3-28	#8-13
31	5	299	170	71	0	0	8	Q_{17}	No	#3-28	#8-12
32	5	304	172	66	0	0	8	No	(#4-8)$\times\mathbb{P}^1$	#3-27	#7-9
33	5	310	172	67	0	0	8	Q_{16}	No	#3-27, #3-25	#7-7, #6-2
34	5	310	172	67	0	0	9	Yes[3]	No	#3-26	#11-1
35	5	320	176	78	0	0	8	Q_{15}	(#4-9)$\times\mathbb{P}^1$	#3-28, #3-27	#8-11, #7-5, #7-6
36	5	325	178	70	0	0	8	Q_{14}	No	#3-30, #3-28	#9-3, #8-3
37	5	330	180	71	0	0	8	Q_{13}	No	#3-31, #3-27	#10-4, #7-4
38	5	330	180	71	0	0	8	No	No	#3-27	#7-8
39	5	331	178	71	0	0	8	Q_{12}	No	#3-28, #3-25	#8-9, #6-1
40	5	336	180	72	0	0	8	Q_{10}	$\mathbb{F}_1 \times \mathrm{Bl}_2\,\mathbb{P}^2$	#3-28	#8-2
41	5	336	180	72	0	0	8	Q_{11}	(#4-10)$\times\mathbb{P}^1$	#3-27	#7-3
42	5	341	182	73	0	0	8	Q_9	No	#3-28	#8-8, #8-10
43	5	352	184	75	0	0	8	Q_8	(#4-11)$\times\mathbb{P}^1$	#3-28, #3-27	#8-7, #7-2
44	5	363	186	77	0	0	8	Q_7	No	#3-28	#8-6
45	5	368	188	78	0	0	8	Q_6	(#4-12)$\times\mathbb{P}^1$	#3-28	#8-5
46	5	373	190	79	0	0	8	Q_5	No	#3-30, #3-28	#9-2, #8-1
47	5	394	196	83	0	0	8	Q_3	No	#3-31, #3-27	#10-3, #7-1
48	5	405	198	85	0	0	8	Q_4	No	#3-31, #3-28	#10-2, #8-4
49	5	405	198	85	0	0	8	Q_2	No	#3-30	#9-1
50	5	442	208	92	0	0	8	Q_1	No	#3-31	#10-1

[3] number 108 in [Bat99], §4.

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Università di Torino, Dipartimento di Matematica, via Carlo Alberto 10, 10123 Torino - Italy
Email address: pierroberto.pastorino@unito.it