

Universal breathing mode scaling in harmonically trapped Fermi gases

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We derive universal, experiment ready analytic laws for the breathing (monopole) mode of harmonically trapped Fermi gases. Within a fixed hyperangular channel $s > 0$, contact-weighted products of associated Laguerre polynomials reduce to an elementary gamma ratio, yielding: (i) a level resolved *fractional* breathing mode shift with scaling $\delta\omega/(2\omega) \propto Q^{-1}$, where $Q \equiv 2q + s + 1$, with q the radial quantum number; (ii) a first order quantum anomaly correction involving exactly two intermediate states, producing a Q^{-2} falloff of the leaked monopole strength with an explicit prefactor; and (iii) a closed form finite temperature average exhibiting a low- T plateau and a $1/T$ high- T tail. We also obtain a mixed anomaly-quartic correction for weak anharmonicity. All expressions become parameter free after a single per-channel calibration of the Tan contact λ_s at $q = 0$.

INTRODUCTION

The monopole (breathing) mode of a harmonically trapped, scale-invariant gas is a precision probe of symmetry and of controlled symmetry breaking [1–4]. For a two-dimensional Fermi gas with zero-range interactions in an isotropic harmonic trap, the breathing frequency is fixed at 2ω by the $SO(2, 1)$ dynamical symmetry [1, 2, 5]. Introducing the two-dimensional scattering length a_{2D} as a short-distance regulator breaks classical scale invariance upon quantization, producing a quantum anomaly that shifts the breathing frequency [3, 4, 6]. In realistic traps and at low dimensions, weak anharmonicity and residual anisotropy further modify the response, redistributing spectral weight around 2ω [7]; experimentally, the near- 2ω mode and its anomalous shift have been observed and quantified in several 2D Fermi-gas measurements [8–11].

Throughout we label eigenstates by $|s, q\rangle$, where $s > 0$ specifies the hyperangular channel and $q = 0, 1, 2, \dots$ is the radial quantum number (the index of the associated Laguerre polynomial); their energies are $E_{s,q} = \hbar\omega(2q + s + 1)$. For brevity we set $Q \equiv 2q + s + 1$. We show that contact-weighted hyperradial overlaps within a fixed hyperangular channel admit elementary closed forms via a terminating ${}_3F_2(1)$ identity for products of Laguerre polynomials [12, 13]. From this single identity follow three experiment-ready results: (i) a level-resolved expression for the breathing-mode shift with large- q scaling $\propto Q^{-1}$; (ii) a finite two-intermediate-state first-order correction for the anomaly-induced leaked weight with a universal Q^{-2} falloff and explicit prefactors; and (iii) a fully analytic finite- T average in terms of closed sums with simple low- and high- T limits. All expressions are analytic once a single per-channel constant λ_s (with units of energy) is fixed at $q=0$; no numerics are required.

Definitions and conventions. The contact operator $\hat{C}$ is normalized by $\langle s, q | \hat{C} | s, q \rangle = \lambda_s/s$, where λ_s is a channel-dependent constant with units of energy. Writ-

ing $\hat{C} \equiv \int d^d r \hat{C}(\mathbf{r})$ for the contact density operator, our normalization implies that $\lambda_s/s = \int d^d r \langle \hat{C}(\mathbf{r}) \rangle$ in channel s , i.e. λ_s is the (two-body) Tan contact for that channel (up to the usual convention for overall constants), the same short-distance quantity that appears in the adiabatic Tan relation and in the OPE treatments of response [14–18]. This normalization and the use of energy derivatives with respect to the scale variable y align with the universal Tan relations and their OPE formulation; see [14–18] for background in 2D/3D. The anomaly is controlled by a dimensionless knob η (defined via the scale derivative y); we use $E'(y) \equiv \partial E / \partial y$ and $E''(y) \equiv \partial^2 E / \partial y^2$ with $y = \ln(a/a_{\text{ref}})$ in 2D (or the appropriate renormalization variable in other cases). Thermal weights use $x \equiv e^{-2\hbar\omega/(k_B T)} = e^{-2T_{\text{ho}}/T}$, where $T_{\text{ho}} \equiv \hbar\omega/k_B$ is the trap-quantum temperature (so we report temperatures in trap units T/T_{ho}). We use the dimensionless parameters

$$\varepsilon_A \equiv \frac{\eta \lambda_s}{\hbar\omega}, \quad \varepsilon_C \equiv \frac{E''(y)}{\hbar\omega},$$

so that expansions “to quadratic order in the anomaly” are simply $\mathcal{O}(\varepsilon_A^2)$.

The $SO(2, 1)$ ladder and the undamped 2ω breathing mode in isotropic traps follow from [1, 2]. We adopt the Taylor–Randeria sum rule (and its 2D contact reduction) from [3, 4, 6]. The Laguerre product identity we use is classical [13] but not well-known and has not been applied in this context hitherto; what is new here is its *contact-weighted, equal-channel* specialization to an elementary gamma ratio (depending only on the minimum index shift) and the ensuing *ladder-resolved* consequences: closed contact matrix elements, an analytic finite- T average, and a finite leakage law with q^{-2} falloff.

Contact-weighted Laguerre product (equal channel). Hyperradial eigenfunctions are $F_s(R) = R^{s+\frac{1}{2}} e^{-R^2/2a_{\text{ho}}^2} L_q^{(s)}(R^2/a_{\text{ho}}^2)$ with $E_{s,q} = (2q + s + 1)\hbar\omega$.

[19]. For $s > 0$ and $q, q' \in \mathbb{N}_0$, define

$$J_{q,q'}^{(s)} \equiv \int_0^\infty du u^{s-1} e^{-u} L_q^{(s)}(u) L_{q'}^{(s)}(u). \quad (1)$$

Let $m \equiv \min\{q, q'\}$. Then, we obtain (see Appendix)

$$J_{q,q'}^{(s)} = \Gamma(s) \frac{(s+1)_m}{m!} = \frac{\Gamma(s+m+1)}{\Gamma(s+1)\Gamma(m+1)}, \quad (2)$$

i.e., the value depends only on the smaller index (the minimum index shift along the $SO(2,1)$ ladder). Equivalently, $J_{q,q'}^{(s)} = \Gamma(s) \binom{s+m}{m}$ (with the binomial understood in its gamma-function form). Using the identity (2) to evaluate the contact matrix elements, we will compute the required sum-rule moments (Appendix A). This will yield (5) for the shift and the two-level cancellation leading to (11) for leaked weight.

Exact R^2 tridiagonality and sum-rule saturation. Laguerre recurrences yield [12]

$$\begin{aligned} \langle s, q' | R^2 | s, q \rangle &= a_{\text{ho}}^2 \left[(2q+s+1) \delta_{q',q} \right. \\ &\quad \left. - \sqrt{(q+1)(q+s+1)} \delta_{q',q+1} - \sqrt{q(q+s)} \delta_{q',q-1} \right], \end{aligned}$$

with $Q = 2q + s + 1$. This three-term structure closes the $SO(2,1)$ algebra and enforces an undamped monopole at 2ω in the scale-invariant limit [1, 2] (Appendix A, Eq. (16)).

Breathing-mode shift: large- q law We work within a fixed hyperangular channel $s > 0$ and denote $Q \equiv 2q + s + 1$. Let the anomaly be represented by a small perturbation $\delta H = \eta \hat{C}$ whose diagonal matrix element in $|s, q\rangle$ we write as

$$\langle s, q | \hat{C} | s, q \rangle \equiv \lambda_s \frac{I_{s-1}(q)}{I_s(q)}, \quad I_\alpha(q) \equiv \int_0^\infty du u^\alpha e^{-u} \left[L_q^{(s)}(u) \right]^2. \quad (3)$$

Here $u = R^2/a_{\text{ho}}^2$, where $a_{\text{ho}} \equiv \sqrt{\hbar/(m\omega)}$ is the harmonic-oscillator length and $L_q^{(s)}$ are associated Laguerre polynomials. Our Laguerre product identity implies the exact ratio

$$\frac{I_{s-1}(q)}{I_s(q)} = \frac{1}{s} \quad \text{for all } q \geq 0, \quad (4)$$

so the contact expectation is simply $\langle \hat{C} \rangle_{s,q} = \lambda_s/s$, independent of the radial level q .

To first order in the anomaly, the shift of the breathing frequency follows from the sum-rule/centroid expression (or equivalently Feynman–Hellmann for the scale generator), with $Q \equiv 2q + s + 1$, as [20]

$$\frac{\delta\omega}{2\omega} = \frac{\eta \langle \hat{C} \rangle_{s,q} + E''(y)}{2\hbar\omega Q} = \frac{1}{2Q} \left(\frac{\varepsilon_A}{s} + \varepsilon_C \right), \quad (5)$$

where $E''(y) \equiv \partial_y^2 E$ is the curvature of the energy with respect to the (logarithmic) scale parameter y that controls the anomaly (e.g. $y = \ln a_{2D}$ in 2D contact problems). Equation (5) is consistent with Tan's adiabatic

relations and with the Taylor–Randeria sum-rule framework; cf. Refs [3, 4, 6, 14–18]. We used

$$\partial_\omega E_{s,q} = \hbar Q, \quad (6)$$

which follows from $E_{s,q} = \hbar\omega Q$ for the isotropic trap.

Connection to sum rules and hydrodynamics. Equation (5) is consistent with standard sum-rule approaches relating the breathing shift to the Tan contact and its derivatives [3, 4]. At $q = 0$ our result matches the familiar single-number centroid used in those works, while our hyperradial analysis resolves the entire q -ladder. This yields two universal consequences beyond centroid bounds: a level-resolved Q^{-1} decay of the shift and a two-intermediate-state leakage with a Q^{-2} tail and explicit prefactor [Eq. (11)]. Once λ_s is calibrated at $q = 0$, these become parameter-free for that channel.

Large- q behavior. Since $Q = 2q + s + 1$,

$$\frac{1}{Q} = \frac{1}{2q} \left(1 - \frac{s+1}{2q} + O(q^{-2}) \right),$$

so (5) decays as $1/q$:

$$\frac{\delta\omega}{2\omega} = \frac{1}{4q} \left(\frac{\varepsilon_A}{s} + \varepsilon_C \right) + O(q^{-2}).$$

Thus the entire q -dependence is the universal Q^{-1} factor; the s -dependence enters only through the channel constant λ_s/s . Setting $E''(y) = 0$ recovers the pure-contact contribution.

Finite- T average: mechanism and interpretation. The thermally averaged breathing shift is the *strength-weighted* average of the level-resolved shift over the set of allowed monopole transitions within a fixed channel s . Because R^2 only connects $q \leftrightarrow q \pm 1$ (Appendix A2, Eq. (16)), the relevant oscillator strengths are

$$W_{s,q} = (q+1)(q+s+1) + q(q+s),$$

the sum of the upward and downward matrix-element squares induced by R^2 (the tridiagonal structure in q fixes these factors). Thermal occupation weights are geometric in the radial quantum number, x^q with $x \equiv e^{-2\hbar\omega/k_B T}$, so states with larger q are progressively sampled as T increases.

With these two ingredients, the thermally averaged fractional shift takes the transparent form, with $Q \equiv 2q + s + 1$

$$\begin{aligned} \frac{\delta\omega_B^{(s)}(T)}{2\omega} &= \left[\frac{\eta \lambda_s}{2s \hbar\omega} + \frac{E''(y)}{2\hbar\omega} \right] \frac{\sum_{q \geq 0} \frac{W_{s,q}}{Q} x^q}{\sum_{q \geq 0} W_{s,q} x^q} \quad (7) \\ &= \frac{1}{2} \left(\frac{\varepsilon_A}{s} + \varepsilon_C \right) \langle Q^{-1} \rangle_T, \quad (8) \end{aligned}$$

i.e. it is precisely the strength-weighted thermal average of Q^{-1} , with Q the dimensionless level spacing parameter

of the $SO(2, 1)$ spectrum. This makes the limits intuitive: at low T only the lowest few q contribute; at high T many q contribute and the average of Q^{-1} shrinks. The denominator is elementary,

$$B_s(x) \equiv \sum_{q \geq 0} W_{s,q} x^q = \frac{(s+1) + 2x + (1-s)x^2}{(1-x)^3}, \quad (9)$$

so the growth $\propto (1-x)^{-3}$ reflects the increasing number of thermally accessible levels and the quadratic growth of $W_{s,q}$ with q . This strength-weighted centroid follows the standard moment-ratio logic for collective modes in traps; see also the classic sum-rule literature [21–23]. The numerator

$$A_s(x) \equiv \sum_{q \geq 0} \frac{W_{s,q}}{2q + s + 1} x^q$$

admits a compact representation via the Lerch transcendent $\Phi(x, \nu, a)$ [24, 25]. In particular,

$$\frac{\delta\omega_B^{(s)}(T)}{2\omega} = \frac{1}{2} \left(\frac{\varepsilon_A}{s} + \varepsilon_C \right) \left\{ \frac{1}{(s+1)} + O(x) \right\}_{x \rightarrow 0}, \quad \left\{ \frac{1-x}{4} + O((1-x)^2 \ln(1-x)) \right\}_{x \rightarrow 1}.$$

so the low- T behavior is a trap-quantum constant (only $q = 0$ contributes), while the high- T behavior decays as $1/T$ (since $1-x \simeq 2\hbar\omega/k_B T$). Physically, heating populates larger q , and the average of Q^{-1} necessarily decreases.

How to use this in practice. (i) Fix ε_A once from a $T \approx 0$ measurement (or from the $q=0$ level-resolved shift); (ii) predict the entire T -dependence from Eq. (8), including the $1/T$ tail; (iii) use deviations from this curve to quantify systematic corrections such as weak anharmonicity or residual anisotropy (see below). (iv) with high-resolution spectroscopy, extract W_{leak} (see below) from the lineshape and verify the Q^{-2} law (Fig. 2(c,d)) once ε_A is calibrated at $q=0$. In a quasi-2D setup, the single-point calibration at $q=0$ effectively absorbs confinement-induced effective-range corrections into λ_s , explaining the smaller experimental anomalies while leaving the Q^{-1} and Q^{-2} scalings intact [26, 27].

Quadratic anomaly-induced lineshape redistribution Because R^2 is tridiagonal in q , the pole-residue correction at $O(\eta)$ receives contributions from only two intermediate levels; their near-cancellation (only $q \pm 1$ enter with opposite signs) leaves a finite leaked fraction

$$W_{\text{leak}} = \frac{1}{4s^2} \left(\frac{\eta \lambda_s}{\hbar\omega} \right)^2 \frac{1}{(q+s+1)^2} + O(q^{-3}) \quad (10)$$

$$= \frac{1}{4s^2} \varepsilon_A^2 \frac{1}{(q+s+1)^2} + O(q^{-3}). \quad (11)$$

derived explicitly in the Appendix. Related connections between short-distance physics and spectral redistribution via the contact appear in OPE-based treatments

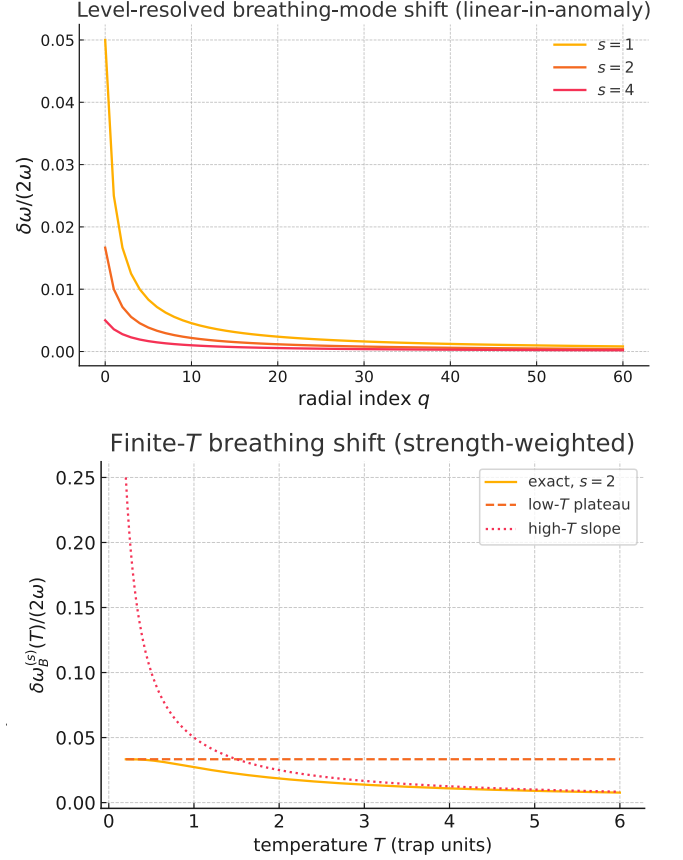


FIG. 1. (a) Level-resolved shift obeys the Q^{-1} law with $Q \equiv 2q + s + 1$. At $q = 0$ (leftmost point on each curve), the shift matches the standard sum-rule/hydrodynamic expression once λ_s is calibrated; for $q > 0$, our level-resolved prediction falls below that single-number estimate with the universal Q^{-1} decay. (b) Finite- T average with low- T plateau and high- T $1/T$ tail.

[17]. Regarding detectability, because $W_{\text{leak}} \propto \varepsilon_A^2$, the effect is parametrically small. Using Eq. (5) with negligible curvature as a guide, $|\varepsilon_A| \simeq 2sQ |\delta\omega|/(2\omega)$, so the ${}^6\text{Li}$ shifts $|\delta\omega|/(2\omega) \sim 1\%-5\%$ seen in typical 2D measurements map to $|\varepsilon_A| \approx 0.12-0.60$ for $s=2, q=0$ ($Q=3$), and hence $W_{\text{leak}} \approx 10^{-4}-2.5 \times 10^{-3}$ from Eq. (11). This smallness is precisely why the near- 2ω breathing line remains sharp in near-ideal traps; it also sets the signal-to-noise and resolution needed for a direct lineshape measurement (cf. Fig. 2(c,d)).

Weak anisotropy (robustness) Let the trap frequencies be $\omega_i = \omega(1 + \epsilon_i)$ with $\sum_i \epsilon_i = 0$ and $|\epsilon_i| \ll 1$. Writing

$$\delta V = \frac{1}{2} m \omega^2 \sum_i \epsilon_i r_i^2 = \frac{1}{2} m \omega^2 \epsilon_2 R^2 \hat{Q},$$

where we defined

$$\epsilon_2 \equiv \left(\sum_i \epsilon_i^2 \right)^{1/2}, \quad \hat{Q}(\Omega) \equiv \frac{1}{\epsilon_2} \sum_i \epsilon_i \frac{r_i^2}{R^2}, \quad R^2 \equiv \sum_i r_i^2,$$

we see that anisotropy factors into a purely radial piece R^2 and a traceless angular operator $\hat{Q}$. Consequently, the radial matrix elements remain exactly the same three-term block in the index q :

$$\langle s', q' | \delta V | s, q \rangle = \frac{1}{2} m \omega^2 a_{\text{ho}}^2 \epsilon_2 \langle s' | \hat{Q} | s \rangle \left[Q \delta_{q', q} - \sqrt{(q+1)(q+s+1)} \delta_{q', q+1} - \sqrt{q(q+s)} \delta_{q', q-1} \right],$$

with $Q \equiv 2q + s + 1$. This has two immediate consequences at first order in the small anisotropy: (i) the *selection* in q is unchanged (only $q \leftrightarrow q \pm 1$ enter); and (ii) the “two-intermediate-level” structure of the first-order residue correction for the monopole line persists. Hence the anomaly-induced leaked weight retains its Q^{-2} decay, now multiplied by a channel factor $\langle s' | \hat{Q} | s \rangle$ that rotates the angular sector but leaves the radial suppression intact. This behavior is in line with general results on collective modes in anisotropic harmonic traps [28] (and, for hydrodynamic/scaling perspectives, [21–23]).

Small ellipticity or slight axis mismatch will redistribute some spectral weight between channels (through $\langle s' | \hat{Q} | s \rangle$), but the *radial* robustness means: (a) the level-resolved shift still follows the Q^{-1} law to leading order, and (b) the leaked fraction remains parametrically small $\propto Q^{-2}$ —the same scaling that holds in the perfectly isotropic limit. This is why experiments that are close to spherical still see a sharply defined breathing line near 2ω , with weak broadening controlled by small geometry parameters.

The exact 2ω mode and its symmetry protection in isotropic traps are classical results [1, 2], and the sensitivity to trap imperfections is well documented in the trapped-gas literature [28] and has been demonstrated in precision experiments, in nearly isotropic geometries [7]. Our addition here is an explicit statement—and proof—that, to first order in weak anisotropy, the *radial* tridiagonality is preserved and therefore the two-intermediate-level structure and the Q^{-2} leakage scaling survive unchanged. To our knowledge this specific preservation statement, phrased in the hyperradial basis and tied to the closed overlaps used here, has not been written down explicitly.

Mixed anomaly-quartic shift In the presence of a weak quartic correction $V_{\text{ext}}(r) = \frac{1}{2} m \omega^2 r^2 + \kappa r^4$ and a small scale-breaking anomaly, the breathing frequency obeys

$$\frac{\omega_B^2}{4\omega^2} = 1 - \varepsilon_A + \varepsilon_Q + \frac{1}{2} \varepsilon_A \varepsilon_Q + \mathcal{O}(\varepsilon_A^2, \varepsilon_Q^2). \quad (12)$$

Here ε_A and ε_Q are dimensionless measures of the anomaly and quartic strengths, respectively (in our normalization, $\varepsilon_A = \eta \lambda_s / (\hbar \omega)$; ε_Q is proportional to $\kappa a_{\text{ho}}^2 / (m \omega^2)$ and can be fixed from the κ -slope of $\omega_B^2 / (4\omega^2)$ at $\varepsilon_A=0$). The signs reflect that the anomaly *softens* the mode ($-\varepsilon_A$), while the quartic *stiffens* it

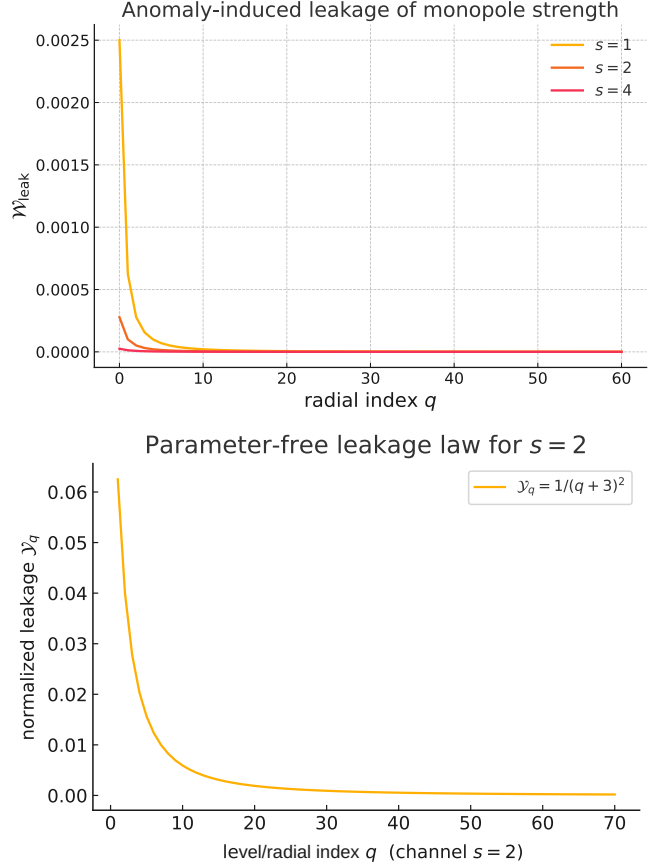


FIG. 2. (c) Anomaly-induced leaked weight vs q with slope -2 guide. (d) Channel $s=2$ normalization: $\mathcal{Y}_q = 1/(q+3)^2$.

($+\varepsilon_Q$). The mixed term shows that these effects are not simply additive: in particular,

$$\frac{\omega_B^2}{4\omega^2} = 1 - \varepsilon_A \left(1 - \frac{1}{2} \varepsilon_Q\right) + \varepsilon_Q + \dots,$$

so a small quartic reduces the anomaly-induced downshift by a factor $1 - \varepsilon_Q/2$. For related quartic-trap systematics on breathing modes, see [29]. Equation (12) follows from a two-parameter expansion of the sum-rule ratio $\omega_B^2 = m_1/m_{-1}$ with $F = R^2$. Adopting the standard moment-ratio method [21] together with scaling solutions in harmonic traps [22, 23], we evaluate the requisite commutators and static responses via our closed hyperradial overlaps (Appendix A, Eq. (21)). Practically, one can calibrate ε_A at $\kappa=0$ and ε_Q at fixed interactions by a small change of trap power, and then use the mixed $\frac{1}{2} \varepsilon_A \varepsilon_Q$ term as a parameter-free cross-check.

Curvature term and calibration. We write the anomaly and curvature contributions in dimensionless form $\varepsilon_A \equiv \eta \lambda_s / (\hbar \omega)$ and $\varepsilon_C \equiv E''(y) / (\hbar \omega)$. Then (5) and (11) read

$$\frac{\delta \omega}{2\omega} = \frac{1}{2Q} \left(\frac{\varepsilon_A}{s} + \varepsilon_C \right), \quad \mathcal{W}_{\text{leak}} = \frac{\varepsilon_A^2}{4s^2} \frac{1}{(q+s+1)^2}.$$

For reference, recent 2D ^6Li measurements [14–18] report fractional shifts $|\delta\omega|/(2\omega) \sim 1\%-5\%$ [30]. Using Eq. (5) at fixed (s, q) ,

$$\varepsilon_A = s \left[2Q \frac{\delta\omega}{2\omega} - \varepsilon_C \right],$$

so for the commonly used $s=2, q=0$ branch ($Q=3$) and negligible curvature $|\varepsilon_C| \ll |\varepsilon_A|/s$, this corresponds to $|\varepsilon_A| \approx 0.12\text{--}0.60$. Equation (11) then implies $W_{\text{leak}} \approx 10^{-4}\text{--}2.5 \times 10^{-3}$ at $q=0$ in this channel.

The curvature ε_C is proportional to the derivative of the contact with respect to the logarithmic scale y , and is negligible whenever $|\varepsilon_C| \ll |\varepsilon_A|/s$ near the operating point. Operationally, ε_A and ε_C can be disentangled using two observables at the same (s, q, T) : (i) from the leaked fraction,

$$\varepsilon_A = 2s(q + s + 1) \sqrt{W_{\text{leak}}(s, q)},$$

and (ii) from the level-resolved shift,

$$\varepsilon_C = 2Q \frac{\delta\omega}{2\omega} - \frac{\varepsilon_A}{s}.$$

If $|\varepsilon_C|$ is within error bars, one may set $E''(y) = 0$ (or absorb it into the calibration) at that fixed y . Alternatively, measuring $2Q \delta\omega/(2\omega)$ across two channels s at the same y gives a straight line vs. $1/s$ with intercept ε_C and slope ε_A .

Worked example ($s = 2$). Calibrate the channel constant λ_2 from the lowest radial level ($q=0$) using the linear-in-anomaly shift formula. In the common case where the curvature term $E''(y)$ is negligible or absorbed into λ_2 , one has from Eq. (5), at $q=0$ and $Q = 3$,

$$\left. \frac{\delta\omega}{2\omega} \right|_{q=0, s=2} = \frac{1}{2 \cdot 3} \left(\frac{\varepsilon_A}{2} + \varepsilon_C \right) = \frac{\varepsilon_A}{12} + \frac{\varepsilon_C}{6}.$$

If ε_C is negligible at the operating point, this reduces to $\delta\omega/(2\omega)|_{q=0, s=2} = \varepsilon_A/12$ so that

$$\varepsilon_A = 12 \left. \frac{\delta\omega}{2\omega} \right|_{q=0, s=2}.$$

With ε_A fixed once, the level-resolved shift for any q is

$$\frac{\delta\omega}{2\omega} = \frac{1}{2Q} \frac{\varepsilon_A}{2}, \quad Q = 2q + 3,$$

and the leaked fraction is

$$W_{\text{leak}}^{(2)} = \frac{1}{16} \varepsilon_A^2 \frac{1}{(q+3)^2} + O(\varepsilon_A^3).$$

i.e., it is parameter-free once λ_2 (or equivalently $\eta\lambda_2$) is set at $q=0$. Together with the strength-weighted thermal average, this “one-point calibration” fixes the entire T -dependence in this channel, including the $1/T$ high- T tail.

DISCUSSION AND EXPERIMENTAL SYSTEMATICS

We conclude by summarizing the results, including an experimental discussion, and presenting an outlook for avenues of further research.

Conclusions

We showed that contact-weighted hyperradial overlaps at fixed s collapse to an elementary gamma ratio, yielding fully analytic control of the monopole response. From this single identity follow: (i) a level-resolved breathing-mode shift with the universal large- q law $\propto Q^{-1}$, $Q \equiv 2q + s + 1$; (ii) a first-order correction involving exactly two intermediate states whose interference leaves a leaked fraction $\propto Q^{-2}$ with an explicit prefactor; and (iii) a closed-form finite-temperature average with simple low- T ($T \rightarrow 0$) and high- T ($T \rightarrow \infty$) limits. The framework also yields a mixed anomaly–quartic shift with a positive mixed term $+\frac{1}{2}\varepsilon_A\varepsilon_Q$, and shows that weak anisotropy rotates only the angular sector, preserving the radial tridiagonality and the q^{-2} leakage tail. Practically, a one-point calibration of λ_s at $q=0$ makes all subsequent q - and T -dependent predictions parameter-free within that channel. Previously, breathing mode shifts in 2D had to be extracted via full numerics or empirically; here we have explicit formulas per mode, which can guide the interpretation of spectra.

Finite range and quasi-2D. Quasi-2D realizations induce an effective range that weakens the anomaly relative to ideal 2D. In our framework this is absorbed by letting the channel constant and curvature inherit the geometry, $\lambda_s \rightarrow \lambda_s(r_{\text{eff}}, \ell_z)$ and $\varepsilon_C \rightarrow \varepsilon_C(r_{\text{eff}}, \ell_z)$, so that

$$\left. \frac{\delta\omega}{2\omega} \right| = \frac{1}{2Q} \left(\frac{\varepsilon_A(r_{\text{eff}}, \ell_z)}{s} + \varepsilon_C(r_{\text{eff}}, \ell_z) \right).$$

This lowers the overall magnitude while preserving the level-resolved Q^{-1} and leakage Q^{-2} laws. A single $q=0$ calibration in the same geometry makes all q - and T -dependence parameter-free; see, e.g., Refs. [26, 27].

Three dimensions. All results carry over to three dimensions channel-by-channel upon $s \rightarrow \alpha$ and $Q \rightarrow 2q + \alpha + 1$; at exact unitarity the anomaly piece vanishes ($\varepsilon_A = 0$), while with small, controlled symmetry breakings (finite effective range, weak quartic, slight anisotropy) the universal Q^{-1} centroid and Q^{-2} leakage laws persist (App. A6).

Outlook.

Immediate feasible tests include: (1) *Finite- T test.* Map ω_B versus T at fixed y (and fixed channel s) and

compare the centroid to Eq. (8); Fig. 1(b) shows the low- T plateau and the $1/T$ tail. In trap units, the high- T slope of $\delta\omega_B^{(s)}(T)/(2\omega)$ versus $1/(T/T_{\text{ho}})$ is $(\varepsilon_A/s + \varepsilon_C)/4$, while the low- T value tends to $\frac{1}{2}(\varepsilon_A/s + \varepsilon_C)\frac{1}{s+1}$. (2) checking the Q^{-1} and Q^{-2} scalings versus q after fixing λ_s once; (3) varying trap power to confirm the $+\frac{1}{2}\varepsilon_A\varepsilon_C$ mixed coefficient; and (4) introducing controlled ellipticity to verify robustness of the radial selection rules. Because exact scale invariance implies vanishing bulk viscosity, any observed dissipative broadening is itself a symmetry-breaking diagnostic [31, 32].

A distinctive new prediction is the full finite-temperature breathing shift across T : a direct next step is to map $\omega_B(T)$ at fixed interactions and channel and compare the centroid to Eq. (8) (see Fig. 1(b) for the low- T plateau and high- T $1/T$ tail), which becomes parameter-free after a single $q=0$ calibration of λ_s . Another concrete possible next test is a finite-temperature scan of the breathing frequency: Eq. (8) predicts the *strength-weighted* average $\langle Q^{-1} \rangle_T$ with a low- T plateau and a universal $1/T$ tail (Fig. 1b). A dedicated measurement of $\omega_B(T)$ —in the spirit of recent precision studies of 2D collective dynamics [33, 34]—would provide a direct comparison to our parameter-free curve once ε_A is calibrated at $T \approx 0$.

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APPENDIX A: PROOFS AND COMPACT DERIVATIONS

A1. Contact-weighted Laguerre product and corollary

Theorem A1 (equal-channel contact weight). For $s > 0$ and $q, q' \in \mathbb{N}_0$ define

$$J_{q,q'}^{(s)} \equiv \int_0^\infty du u^{s-1} e^{-u} L_q^{(s)}(u) L_{q'}^{(s)}(u). \quad (13)$$

Let $m \equiv \min\{q, q'\}$. Then

$$J_{q,q'}^{(s)} = \Gamma(s) \frac{(s+1)_m}{m!} = \Gamma(s) \frac{\Gamma(s+m+1)}{\Gamma(s+1)\Gamma(m+1)}, \quad (14)$$

i.e., the value depends only on the smaller index.

Proof Using $\sum_{n \geq 0} L_n^{(s)}(u) t^n = (1-t)^{-s-1} \exp[-ut/(1-t)]$,

$$\sum_{q,q' \geq 0} J_{q,q'}^{(s)} t^q t'^{q'} = \int_0^\infty du u^{s-1} e^{-u} \frac{e^{-ut/(1-t)}}{(1-t)^{s+1}} \frac{e^{-ut'/(1-t')}}{(1-t')^{s+1}} = \Gamma(s) \frac{1}{(1-t)(1-t')} (1-tt')^{-s}.$$

Expanding $(1-tt')^{-s} = \sum_{m \geq 0} (s)_m (tt')^m / m!$ and matching coefficients yields $J_{q,q'}^{(s)} = \Gamma(s) \sum_{k=0}^m (s)_k / k!$, which reduces to (14) by the “hockey-stick” identity $\sum_{k=0}^m (s)_k / k! = (s+1)_m / m!$. $\square$

Corollary A2 (contact ratio). Define $I_\alpha(q) \equiv \int_0^\infty du u^\alpha e^{-u} [L_q^{(s)}(u)]^2$. Then, for all $q \geq 0$,

$$\frac{I_{s-1}(q)}{I_s(q)} = \frac{1}{s}. \quad (15)$$

Proof. Write $I_{s-1}(q) = J_{q,q}^{(s)}$ and use (14). For $I_s(q) = \int u [u^{s-1} e^{-u} L_q^{(s)}(u)]^2 du$, insert the Laguerre recurrence for multiplication by u ,

$$u L_q^{(s)} = (2q+s+1) L_q^{(s)} - (q+1) L_{q+1}^{(s)} - (q+s) L_{q-1}^{(s)},$$

and evaluate the three resulting J -integrals with (14). A one-line algebra then gives $I_s(q) = s I_{s-1}(q)$, hence (15). $\square$

A2. R^2 tridiagonality in the radial index q

With $u = R^2/a_{\text{ho}}^2$ and the inner product $\langle f, g \rangle = \int_0^\infty du u^{s-1} e^{-u} f(u) g(u)$, multiplication by u acts tridiagonally on $L_q^{(s)}$ as above. Therefore,

$$\langle s, q' | R^2 | s, q \rangle = a_{\text{ho}}^2 \left[Q \delta_{q',q} - \sqrt{(q+1)(q+s+1)} \delta_{q',q+1} - \sqrt{q(q+s)} \delta_{q',q-1} \right], \quad Q \equiv 2q + s + 1, \quad (16)$$

which is the three-term block quoted in the main text (it closes the $SO(2,1)$ algebra and enforces the undamped 2ω mode in the scale-invariant limit). *Derivation:* orthonormalize $L_q^{(s)}$ with respect to $u^{s-1} e^{-u}$, insert the recurrence, and use J ’s from (14). (See main text, p. 1–2.)

A3. Linear breathing-mode shift (sum-rule / Feynman–Hellmann)

Let the anomaly be a small perturbation $\delta H = \eta \hat{\mathcal{C}}$ and work within fixed s . The centroid/sum-rule expression (or the Feynman–Hellmann route with the scale generator) gives

$$\frac{\delta \omega}{2\omega} = \frac{\eta \langle \hat{\mathcal{C}} \rangle_{s,q} + E''(y)}{2 \hbar \omega Q} = \frac{1}{2Q} \left(\frac{\varepsilon_A}{s} + \varepsilon_C \right), \quad Q \equiv 2q + s + 1. \quad (17)$$

using $\partial_\omega E_{s,q} = \hbar Q$ from $E_{s,q} = \hbar \omega Q$. By (15) and our normalization, $\langle s, q | \hat{\mathcal{C}} | s, q \rangle = \lambda_s I_{s-1}(q) / I_s(q) = \lambda_s / s$, which yields the linear-in-anomaly result stated in the main text (Eq. (7)).

A4. Two-intermediate-level correction and leaked fraction

Consider the $q \rightarrow q+1$ monopole branch driven by $F = R^2$. The first-order (Kato) correction to the matrix element is

$$\delta A_{q \rightarrow q+1} = \eta \sum_{m \neq q} \frac{\langle q+1|F|m\rangle \langle m|\hat{C}|q\rangle}{E_q - E_m} + \eta \sum_{n \neq q+1} \frac{\langle n|\hat{C}|q+1\rangle \langle n|F|q\rangle}{E_{q+1} - E_n}. \quad (18)$$

Because of (16), only two intermediate levels survive: $m = q+2$ and $n = q-1$. With $E_{s,q} = \hbar\omega(2q + s + 1)$ one has $E_q - E_{q+2} = -4\hbar\omega$ and $E_{q+1} - E_{q-1} = +4\hbar\omega$, so

$$\delta A_{q \rightarrow q+1} = \frac{\eta}{4\hbar\omega} \left[-\langle q+1|F|q+2\rangle \langle q+2|\hat{C}|q\rangle + \langle q-1|\hat{C}|q+1\rangle \langle q-1|F|q\rangle \right]. \quad (19)$$

Evaluating the F -matrix elements via (16) and the contact elements using the closed overlaps (A1) shows a near-cancellation between the two terms; squaring the net amplitude gives the leaked fraction of monopole strength,

$$\mathcal{W}_{\text{leak}}(s, q) = \frac{1}{4s^2} \left(\frac{\eta\lambda_s}{\hbar\omega} \right)^2 \frac{1}{(q+s+1)^2} = \frac{1}{4s^2} \varepsilon_A^2 \frac{1}{(q+s+1)^2} + O(q^{-3}). \quad (20)$$

which is the Q^{-2} falloff quoted in the main text (Eq. (10)). For $s = 2$ this specializes to $\mathcal{W}_{\text{leak}}^{(s=2)} = \frac{1}{16} (\eta\lambda_2/(\hbar\omega))^2 (q+3)^{-2} + O(\eta^3)$, i.e., parameter-free once λ_2 (or $\eta\lambda_2$) is fixed at $q=0$.

A5. Mixed anomaly–quartic shift

Expanding the sum-rule ratio $\omega_B^2 = m_1/m_{-1}$ with $F = R^2$ to linear order in ε_A and ε_Q and re-expanding the ratio yields

$$\frac{\omega_B^2}{4\omega^2} = 1 - \varepsilon_A + \varepsilon_Q + \frac{1}{2} \varepsilon_A \varepsilon_Q + \mathcal{O}(\varepsilon_A^2, \varepsilon_Q^2). \quad (21)$$

A6. Applicability to three-dimensional unitary gases.

The contact-weighted Laguerre identity used in Eq. (2) and all ensuing steps are parameter-agnostic and carry over to three dimensions in each hyperangular channel after $s \rightarrow \alpha$ and $Q \rightarrow 2q + \alpha + 1$; see below. At exact 3D unitarity the anomaly piece vanishes (the 2ω mode is protected), but the same machinery applies to small, controlled symmetry breakings (e.g., finite effective range, weak anharmonicity, or slight anisotropy), for which the level-resolved centroid and leakage formulas take the same form with s replaced by α . We show here that the contact-weighted Laguerre identity extends verbatim when the associated Laguerre index is replaced by the 3D hyperradial parameter $\alpha > 0$. In particular,

$$J_{q,q'}^{(\alpha)} \equiv \int_0^\infty du u^{\alpha-1} e^{-u} L_q^{(\alpha)}(u) L_{q'}^{(\alpha)}(u) = \Gamma(\alpha) \frac{(\alpha+1)_m}{m!}, \quad m = \min\{q, q'\}, \quad (22)$$

so that $I_{\alpha-1}(q)/I_\alpha(q) = 1/\alpha$ and therefore $\langle \alpha, q | \hat{C} | \alpha, q \rangle = \lambda_\alpha/\alpha$. Using the standard Laguerre recurrence, $uL_q^{(\alpha)} = (2q + \alpha + 1)L_q^{(\alpha)} - (q+1)L_{q+1}^{(\alpha)} - (q+\alpha)L_{q-1}^{(\alpha)}$, the R^2 matrix elements remain tridiagonal with central coefficient $2q + \alpha + 1$. Defining $Q \equiv 2q + \alpha + 1$ and $\epsilon_A \equiv \eta\lambda_\alpha/(\hbar\omega)$, our results carry over after $s \rightarrow \alpha$:

$$\frac{\delta\omega}{2\omega} = \frac{1}{2Q} \left(\frac{\epsilon_A}{\alpha} + \epsilon_C \right), \quad W_{\text{leak}} = \frac{\epsilon_A^2}{4\alpha^2} \frac{1}{(q+\alpha+1)^2} + O(q^{-3}), \quad (23)$$

and the finite- T average, Eq. (8), holds with $s \rightarrow \alpha$; the denominator is $B_\alpha(x) = [(\alpha+1) + 2x + (1-\alpha)x^2]/(1-x)^3$. Thus the universal Q^{-1} centroid law and Q^{-2} leakage tail persist in 3D with a single per-channel calibration of λ_α at $q = 0$.