SEPARATION AND CUT EDGE IN MACROSCOPIC CLUSTERS FOR METRIC GRAPH GAUSSIAN FREE FIELDS

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ABSTRACT. We prove that for the Gaussian free field (GFF) on the metric graph of $\mathbb{Z}^d$ (for all $d \geq 3$ except the critical dimension $d_c = 6$), with uniformly positive probability there exist two distinct sign clusters of diameter at least cN within a box of size N such that their graph distance is less than $N^{-[(d-2)\vee(2d-8)]}$. This phenomenon contrasts sharply with the two-dimensional case, where the distance between two macroscopic clusters is typically on the order of their diameters, following from the basic property of the scaling limit "conformal loop ensembles" CLE_4 (Sheffield-Werner'2001).

As a byproduct, we derive that the number of pivotal edges for the one-arm event (i.e., the sign cluster containing the origin has diameter at least N) is typically of order $N^{(\frac{d}{2}-1)\wedge 2}$. This immediately implies that for the incipient infinite cluster (IIC) of the metric graph GFF, the dimension of cut edges (i.e., edges whose removal leads to disconnection of the IIC) equals $(\frac{d}{2}-1)\wedge 2$. Translated in the language of critical loop soups (whose clusters by the isomorphism theorem, have the same distribution as GFF sign clusters), this leads to the analogous estimates where the counterpart of a pivotal edge is a pivotal loop at scale 1. This result hints at the new and possibly surprising idea that already in dimension 3, microscopic loops (even those at scale 1) play a crucial role in the construction of macroscopic loop clusters.

1. INTRODUCTION

In this paper, we study the geometric properties of Gaussian free field (GFF) sign clusters on the metric graph $\widetilde{\mathbb{Z}}^d$. For precision, we first recall the definitions of $\widetilde{\mathbb{Z}}^d$ and the GFF on this graph. Unless otherwise specified, we assume that $d \geq 3$. For the d-dimensional integer lattice $\mathbb{Z}^d$, we denote its edge set by $\mathbb{L}^d := \{\{x, y\} : x, y \in \mathbb{Z}^d, \|x - y\| = 1\}$, where $\|\cdot\|$ represents the Euclidean distance. For each edge $e = \{x, y\} \in \mathbb{L}^d$, consider a compact interval I_e of length d, whose endpoints are identical to x and y respectively. The metric graph $\widetilde{\mathbb{Z}}^d$ is defined as the union of I_e for all $e \in \mathbb{L}^d$. The GFF $\{\widetilde{\phi}_v\}_{v\in\widetilde{\mathbb{Z}}^d}$ can be generated as follows:

(i) Sample a discrete GFF $\{\phi_x\}_{x\in\mathbb{Z}^d}$, i.e., mean-zero Gaussian random variables with covariance $\mathbb{E}\big[\phi_x\phi_y\big] = G(x, y)$ for all $x, y \in \mathbb{Z}^d$, where $G(x, y)$ denotes

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the Green's function, representing the expected number of visits to y by a simple random walk on $\mathbb{Z}^d$ starting from x.

(ii) For each $e = \{x, y\} \in \mathbb{L}^d$, independently sample a Brownian bridge on I_e (derived from a Brownian motion with variance 2 at time 1) with boundary conditions ϕ_x at x and ϕ_y at y (sampled by Step (i)). For any $v \in \widetilde{\mathbb{Z}}^d$, define $\widetilde{\phi}_v$ as the value at v of the Brownian bridge on the interval I_e containing v.

The level-set $\widetilde{E}^{\geq h} := \{v \in \widetilde{\mathbb{Z}}^d : \widetilde{\phi}_v \geq h\}$ for $h \in \mathbb{R}$ have been extensively studied. A fundamental property of $\widetilde{E}^{\geq h}$ is that the critical threshold for percolation exactly equals zero. Precisely, it was proved in [26] that $\widetilde{E}^{\geq h}$ percolates (i.e., contains an infinite connected component) if and only if $h < 0$. The key ingredient for proving this critical threshold is the following two-point function estimate:

$$\mathbb{P}\big(x \xleftrightarrow{\geq 0} y\big) = \pi^{-1} \arcsin\left(\tfrac{G(x,y)}{\sqrt{G(x,x)G(y,y)}}\right) \asymp \|x - y\|^{2-d}, \quad \forall x \neq y \in \mathbb{Z}^d. \tag{1.1}$$

Here $A_1 \xleftrightarrow{\geq 0} A_2$ denotes the event that that there exists a path in $\widetilde{E}^{\geq 0}$ connecting A_1 and A_2, and $f \asymp g$ means that the functions f and g satisfy $cg \leq f \leq Cg$ for some constants $C > c > 0$ depending only on d. Based on this estimate, subsequent works rigorously studied the growth of critical level-sets. In particular, the exact order of the (critical) one-arm probability has been established through a series of works [13, 15, 3, 14, 17, 4]. Specifically, for the one-arm probability $\theta_d(N) := \mathbb{P}(\mathbf{0} \xleftrightarrow{\geq 0} \partial B(N))$, where $\mathbf{0} := (0, 0, \cdots, 0)$ is the origin of $\mathbb{Z}^d$, $B(N) := [-N, N]^d \cap \mathbb{Z}^d$ is the box of side length $\lfloor 2N \rfloor$ centered at $\mathbf{0}$, and $\partial A := \{x \in A : \exists y \in \mathbb{Z}^d \setminus A \text{ such that } \{x, y\} \in \mathbb{L}^d\}$ is the boundary of $A \subset \mathbb{Z}^d$, it is known that

$$\text{when } 3 \leq d < 6, \qquad \theta_d(N) \asymp N^{-\frac{d}{2}+1}; \tag{1.2}$$

$$\text{when } d = 6, \qquad N^{-2} \lesssim \theta_6(N) \lesssim N^{-2+\varsigma(N)}, \text{ where } \varsigma(N) := \tfrac{\ln\ln(N)}{\ln^{1/2}(N)} \ll 1; \tag{1.3}$$

$$\text{when } d > 6, \qquad \theta_d(N) \asymp N^{-2}. \tag{1.4}$$

Here $f \lesssim g$ means that the functions f and g satisfy $f \leq Cg$ for some constant $C > 0$ depending only on d. Notably, it has been conjectured in [4] that $\theta_6(N) \asymp N^{-2}[\ln(N)]^{\delta}$ for some $\delta > 0$. To avoid the additional complexity caused by the diverging disparity between the current upper and lower bounds on $\theta_6(N)$, we assume that $d \neq 6$ unless stated otherwise. (The techniques developed for $d \neq 6$ can usually be adapted to $d = 6$, leading to weaker estimates with error terms similar to that in (1.3).) As a natural extension of the one-arm probability $\theta_d(N)$, the crossing probability $\rho_d(n, N) := \mathbb{P}(B(n) \xleftrightarrow{\geq 0} \partial B(N))$ for $N > n \geq 1$ was also computed. Precisely, it has been proved in [4] that

$$\text{when } 3 \leq d < 6, \ \rho_d(n, N) \asymp \big(n/N\big)^{\frac{d}{2}+1}; \tag{1.5}$$

$$\text{when } d > 6, \qquad \rho_d(n, N) \asymp (n^{d-4}N^{-2}) \wedge 1. \tag{1.6}$$

Moreover, other quantities, such as volume exponent [5, 16] and cluster dimension [2, 16], were also studied. Furthermore, the incipient infinite cluster (IIC), which is a key concept introduced by [22] to understand the behavior of large clusters at criticality, was constructed and analyzed in [5, 2, 34]. Specifically, building on the quasi-multiplicativity established in [5], it was proved in [2] that for any $d \geq 3$ with $d \neq 6$, the following four limiting measures all exist and are the same:

$$\mathbb{P}^{(1)}_{d,\mathrm{IIC}}(\cdot) := \lim_{N\to\infty} \mathbb{P}\big(\cdot \mid \mathbf{0} \xleftrightarrow{\geq 0} \partial B(N)\big), \tag{1.7}$$

$$\mathbb{P}^{(2)}_{d,\mathrm{IIC}}(\cdot) := \lim_{h\uparrow 0} \mathbb{P}\big(\cdot \mid \mathbf{0} \xleftrightarrow{\geq h} \infty\big), \tag{1.8}$$

$$\mathbb{P}^{(3)}_{d,\mathrm{IIC}}(\cdot) := \lim_{x\to\infty} \mathbb{P}\big(\cdot \mid \mathbf{0} \xleftrightarrow{\geq 0} x\big), \tag{1.9}$$

$$\mathbb{P}^{(4)}_{d,\mathrm{IIC}}(\cdot) := \lim_{T\to\infty} \mathbb{P}\big(\cdot \mid \mathrm{cap}(\mathcal{C}^+_{\mathbf{0}}) \geq T\big), \tag{1.10}$$

where $\mathrm{cap}(\mathcal{C}^+_{\mathbf{0}})$ represents the capacity of the cluster $\mathcal{C}^+_{\mathbf{0}} := \{v \in \widetilde{\mathbb{Z}}^d : v \xleftrightarrow{\geq 0} \mathbf{0}\}$. Moreover, the switching identity introduced in [34] (see the precise statement in Lemma 2.21) yields the existence of $\mathbb{P}^{(3)}_{d,\mathrm{IIC}}(\cdot)$ for all $d \geq 3$. Although this method is only applicable to $\mathbb{P}^{(3)}_{d,\mathrm{IIC}}(\cdot)$, it remarkably provides an explicit description for the distribution of IIC—the cluster consisting of a Brownian excursion from $\mathbf{0}$ to infinity and a loop soup on $\widetilde{\mathbb{Z}}^d \setminus \{\mathbf{0}\}$ of intensity $\frac{1}{2}$ (the definitions of Brownian excursion and loop soup will be provided below). Random walks on the IIC in high dimensions was studied in [19, 18]. A detailed account can be found in the companion paper [6].

The derivation of many aforementioned results highly rely on the isomorphism theorem, which establishes a rigorous equivalence between the GFF and the loop soup. Specifically, the loop soup $\widetilde{\mathcal{L}}_\alpha$ of intensity $\alpha > 0$ is a Poisson point process with intensity measure $\alpha\widetilde{\mu}$. Here $\widetilde{\mu}$ (referred to as the loop measure) is a σ-finite measure on the space of rooted loops (i.e., continuous paths on $\widetilde{\mathbb{Z}}^d$ that start and end at the same point), defined by

$$\widetilde{\mu}(\cdot) := \int_{v\in\widetilde{\mathbb{Z}}^d} \mathrm{d}m(v) \int_{0<t<\infty} t^{-1}\widetilde{q}_t(v,v)\widetilde{\mathbb{P}}^t_{v,v}(\cdot)\mathrm{d}t, \tag{1.11}$$

where $m(\cdot)$ is the Lebesgue measure on $\widetilde{\mathbb{Z}}^d$, $\widetilde{q}_t(v_1,v_2)$ is the transition density of the Brownian motion on $\widetilde{\mathbb{Z}}^d$ (see Section 2 for the precise definition), and $\widetilde{\mathbb{P}}^t_{v_1,v_2}(\cdot)$ is the law of the Brownian bridge on $\widetilde{\mathbb{Z}}^d$ from v_1 to v_2 with duration t (its transition density is given by $\frac{\widetilde{q}_s(v_1,\cdot)\widetilde{q}_{t-s}(\cdot,v_2)}{\widetilde{q}_t(v_1,v_2)}$ for $0 \leq s \leq t$). The isomorphism theorem (see [26, Proposition 2.1]) constructs a coupling between the GFF $\{\widetilde{\phi}_v\}_{v\in\widetilde{\mathbb{Z}}^d}$ and the critical loop soup $\widetilde{\mathcal{L}}_{1/2}$ (the criticality of the intensity $\frac{1}{2}$ for percolation was verified in [12]) satisfying the following properties:

(a) $\widehat{\mathcal{L}}^v_{1/2} = \frac{1}{2}\widetilde{\phi}^2_v$ for all $v \in \widetilde{\mathbb{Z}}^d$, where $\widehat{\mathcal{L}}^v_{1/2}$ represents the total local time at v of all loops in $\widetilde{\mathcal{L}}_{1/2}$. In particular, each sign cluster (i.e., a maximal connected subgraph where GFF values have the same sign) is exactly a (critical) loop cluster (i.e., a connected component consisting of loops in $\widetilde{\mathcal{L}}_{1/2}$).

(b) Given all loop clusters, the sign of the GFF values on each loop cluster is independent of the others, taking "+" (or "−") with probability $\frac{1}{2}$.

Notably, Property (a) implies that all results for GFF sign clusters equally hold for loop clusters, and vice versa.

1.1. **Graph distance between macroscopic clusters.** As for most discrete stochastic models endowed with spatial structures, constructing the scaling limit usually constitutes a central research objective. Particularly, in the two-dimensional case, the scaling limit of loop clusters (or equivalently, GFF sign clusters) has been established. Precisely, in the celebrated work [31], the conformal loop ensembles CLE_κ (where the parameter $\kappa \in (\frac{8}{3}, 4]$), defined as the boundaries of (continuum) loop clusters in the Brownian loop soup of intensity $\frac{(3\kappa-8)(6-\kappa)}{4\kappa}$, were introduced and shown to consist of loops in the form of Schramm-Loewner Evolution (SLE_κ). Afterwards, it was proved in [27] that the boundaries of (discrete) loop clusters on the two-dimensional graphs (including the discrete half-plane and the corresponding metric graph) converge to CLE_4. In particular, this together with the property of CLE_4 that every two loops do not touch each other (which follows from the non-crossing property and Property (1) of CLE_κ in [31, Section 2.2]) implies that within a box, the minimal distance between all macroscopic (discrete) loop clusters is macroscopic, as their total number is tight.

The main result of this paper, as a sharp contrast, shows that the aforementioned minimal distance is microscopic (and may even be $o(1)$!) when $d > 2$. Precisely, for $A \subset \widetilde{\mathbb{Z}}^d$, its diameter is defined as $\mathrm{diam}(A) := \sup_{v,w\in A} |v-w|$, where $|v-w|$ is the graph distance between v and w on $\widetilde{\mathbb{Z}}^d$. The distance between two sets $D, D' \subset \widetilde{\mathbb{Z}}^d$ is given by $\mathrm{dist}(D, D') := \inf_{v\in D, v'\in D'} |v-v'|$. For any $N \geq 1$ and $\delta > 0$, we denote by $\mathfrak{C}_{N,\delta}$ the collection of sign clusters contained in $\widetilde{B}(N)$ with diameter at least δN, where $\widetilde{B}(N) := \cup_{e\in\mathbb{L}^d: I_e\cap(-N,N)^d\neq\emptyset} I_e$ is the box in $\widetilde{\mathbb{Z}}^d$ of radius N centered at $\mathbf{0}$. We then define the minimal distance

$$\mathcal{D}_{N,\delta} := \min\left\{\mathrm{dist}(\mathcal{C}, \mathcal{C}') : \mathcal{C} \neq \mathcal{C}' \in \mathfrak{C}_{N,\delta}\right\}. \tag{1.12}$$

In addition, for any functions f and g that depend on d, we define the operation

$$f \boxdot g(d) := f(d) \cdot \mathbb{1}_{d\leq 6} + g(d) \cdot \mathbb{1}_{d>6}. \tag{1.13}$$

Theorem 1.1. *For any $d \geq 3$ with $d \neq 6$, there exists a constant $c_1(d) > 0$ such that for any $N \geq 1$ and $\chi > 0$,*

$$\mathbb{P}\left(\mathcal{D}_{N,c_1} \leq \chi\right) \asymp (\chi^{\frac{1}{2}} N^{(\frac{d}{2}-1)\boxdot(d-4)}) \wedge 1. \tag{1.14}$$

The starting point for Theorem 1.1 is the estimates on heterochromatic two-arm probabilities established in the companion paper [6]. Specifically, for any subsets $A_i \subset \widetilde{\mathbb{Z}}^d$ for $1 \leq i \leq 4$, consider the event

$$\mathsf{H}^{A_1,A_2}_{A_3,A_4} := \big\{ A_1 \xleftrightarrow{\geq 0} A_2, A_3 \xleftrightarrow{\leq 0} A_4 \big\}. \tag{1.15}$$

Here $A_3 \xleftrightarrow{\leq 0} A_4$ denotes the event that there exists a path in the negative cluster $\widetilde{E}^{\leq 0} := \{v \in \widetilde{\mathbb{Z}}^d : \widetilde{\phi}_v \leq 0\}$ connecting A_3 and A_4. It was proved in [6] that for any $N \geq 1$ and $v, v' \in \widetilde{B}(cN)$,

$$\text{when } \chi =: |v - v'| \leq 1, \ \mathbb{P}\big(\mathsf{H}^{v,\partial B(N)}_{v',\partial B(N)}\big) \asymp \chi^{\frac{3}{2}} N^{-[(\frac{d}{2}+1)\square 4]}; \tag{1.16}$$

$$\text{when } \chi \geq 1, \qquad \mathbb{P}\big(\mathsf{H}^{v,\partial B(N)}_{v',\partial B(N)}\big) \asymp \chi^{(3-\frac{d}{2})\square 0} N^{-[(\frac{d}{2}+1)\square 4]}. \tag{1.17}$$

As noted in [6, Remark 1.2], these bounds (with $\chi \asymp 1$) imply that the expected number of edges within a box of size N, whose endpoints are contained in distinct sign clusters with diameter at least N, is of order $N^{(\frac{d}{2}-1)\square(d-4)}$. This observation suggests that two macroscopic sign clusters may nearly touch at many locations, leading to an extremely small graph distance between them (as the configurations near different touching points are independent, by the strong Markov property of GFF). Furthermore, since the first moment for touching points is well-estimated, it becomes natural to compute their second moment and thus verify the vanishing of the minimal distance $\mathcal{D}_{N,\delta}$ in (1.12) via the second moment method. More details will be provided in Section 3.

Remark 1.2. As verified by Theorem 1.1, the probability of having two macroscopic clusters in $\mathfrak{C}_{N,c}$ with graph distance of $o(1)$ is uniformly bounded away from zero. In fact, this probability is bounded away from 1 in the low-dimensional case (i.e., $3 \leq d \leq 5$), while approaches 1 in the high-dimensional case (i.e., $d \geq 7$). Specifically, when $3 \leq d \leq 5$, the estimate (1.5) shows that for each box of size $c'N$ (where $c' \in (0, c)$), the number of clusters in $\mathfrak{C}_{N,c}$ intersecting this box is of order 1. This yields that with a uniformly positive probability, there are exactly two clusters in $\mathfrak{C}_{N,c}$ with graph distance of order N. To see this, one may take two boxes inside $B(N)$ of size $c'N$ with graph distance $\frac{N}{10}$, and require that for each of the box, there exists a unique cluster with diameter in $[cN, \frac{N}{100}]$ that intersects it, which occurs with a uniformly positive probability (by (1.5)). Given these two clusters, we further require that there is no other cluster in $\mathfrak{C}_{N,c}$. This event also occurs with uniformly positive probability, which follows from a direct application of the restriction property of the loop soup and the FKG inequality. To sum up, one has $\liminf_{N\geq 1} \mathbb{P}\big(\mathcal{D}_{N,c} \geq c'N\big) > 0$ for $3 \leq d \leq 5$.

When $d \geq 7$, for each box $\mathfrak{B}$ inside $B(N)$ of size $N^{\frac{4}{d}} (\ll N)$, it follows from (1.16) that the expected number of pairs of neighboring lattice points in $\mathfrak{B}$ belonging to two distinct clusters of $\mathfrak{C}_{N,c}$ is of order 1. Therefore, one may employ the second moment method as in the proof of Theorem 1.1 to show that with a uniformly positive probability there exist two clusters in $\mathfrak{C}_{N,c}$ that both intersect $\mathfrak{B}$ and have graph distance less than d within $\mathfrak{B}$ (we denote this event by $\mathsf{A}(\mathfrak{B})$). Note that one may take $c'N^{d-4}$ of these boxes whose mutual distance is of the same order as their sizes (we enumerate them by $\mathfrak{B}_1, ..., \mathfrak{B}_{c'N^{d-4}}$). As shown in [6, Lemma 5.5], although a loop cluster at scale N intersecting one of these boxes may also reach numerous other boxes, imposing this cluster as an absorbing boundary will decrease the two-point function associated to any point in another box by at most a proportion of $O(N^{-\frac{4(d-6)}{d}}) \ll 1$. Moreover, referring to [3], as long as the two-point function is changed by only a constant factor, the estimates for the one-arm probability and volume exponent (and consequently for (1.14)) are preserved. In other words, for any $1 \leq i_1 \neq i_2 \leq c'N^{d-4}$, the events $\mathsf{A}(\mathfrak{B}_{i_1})$ and $\mathsf{A}(\mathfrak{B}_{i_2})$ are asymptotically independent. By applying Chebyshev's inequality, this observation together with $\min_{1\leq i\leq c'N^{d-4}} \mathbb{P}(\mathsf{A}(\mathfrak{B}_i)) \asymp 1$ implies that with high probability, at least a positive proportion of the events in $\{\mathsf{A}(\mathfrak{B}_i)\}_{1\leq i\leq c'N^{d-4}}$ occur. Combined with the strong Markov property of GFF, it yields that $\lim_{N\to\infty} \mathbb{P}(\mathcal{D}_{N,c} \ll 1) = 1$.

Remark 1.3 (Bernoulli percolation). For critical Bernoulli percolation on the triangular lattice, the clusters have been well understood, and in fact exhibit behavior similar to that of the GFF sign clusters considered here. Specifically, the groundbreaking work [32] proved that the boundary of a cluster converges to SLE_6. Based on this result, [10, 11] (see [9] for a comprehensive review) constructed the full scaling limit of all cluster boundaries, consisting of continuum loops which locally behave like SLE_6 curves. Moreover, as stated in [10, Theorem 2], these loops touch each other frequently, which reflects the same behavior for the critical clusters.

Remark 1.4 (construction of the scaling limit). [33, Conjecture A] proposed that the scaling limit of loop clusters on $\widetilde{\mathbb{Z}}^3$ is exactly the clusters of Brownian loop soups of intensity $\frac{1}{2}$ on $\mathbb{R}^3$. As explained in [33, Section 3], this conjecture corresponds to the intuition that the contribution of microscopic loops to the construction of macroscopic clusters is negligible in the scaling limit. Indeed, assuming this intuition, each macroscopic cluster depends essentially only on macroscopic loops in $\widetilde{\mathcal{L}}_{1/2}$. As established in [24], these loops admit a bijective approximation by continuum loops in the corresponding Brownian loop soup, (plausibly) leading to the convergence between the discrete and continuum clusters in [33, Conjecture A]. However, one can notice that in the approximation scheme of [24], the Hausdorff distance between a discrete loop (of diameter N) and its corresponding continuum loop is typically $O(\log(N))$ (this bound is essentially sharp, referring to [23, Theorem 2]). Consequently, when the graph distance between two (discrete) loop clusters of diameters N is substantially smaller than $O(\log(N))$ (which, by Theorem 1.1, occurs rather frequently), the error in the approximation of loops will, theoretically, disrupt the correspondence between discrete and continuum clusters. So, Theorem 1.1 already indicates that there is a problem with this conjecture, as inherent fluctuations in the approximation scheme provide different outcomes for the limits of metric graph loop soups associated to a given continuum loop soup.

In a companion paper [7], we confirm these ideas in more precise terms and show that [33, Conjecture A] does indeed not hold. More precisely, we show that removing microscopic loops (even those at scale 1) does significantly reduce the connecting probabilities for loop soup clusters, and further show that the dimension of clusters in Brownian loop soup in $\mathbb{R}^3$ is actually strictly smaller than the one arising from the scaling limits of metric graph clusters.

1.2. **Pivotal edge and loop.** In percolation models, a bit (vertex in site percolation, or edge in bond percolation) is called pivotal for a connecting event (i.e., an event of the form $A_1 \leftrightarrow A_2$) if flipping its state changes the occurrence of this event. As emphasized in many significant articles (e.g., [21, Section 1]), the pivotal bit is a key concept to understand the near-critical behavior of a percolation model. Particularly, for planar Bernoulli percolation, the establishment of the scaling limit [32] has led to a deep understanding of the distribution of pivotal points [21]. See also [1, 20, 30] for applications of pivotal points on the noise sensitivity and dynamical percolation. For Bernoulli percolation on non-planar graphs and other percolation models, our understanding of the pivotal bit remains rather limited. In the next result, we show that for the one-arm event $\mathbf{0} \xleftrightarrow{\ge 0} \partial B(N)$, with a uniformly positive probability the number of pivotal edges is of order $N^{(\frac{d}{2}-1)\boxdot(d-4)}$. Precisely, we say $e \in \mathbb{L}^d$ is pivotal for the event $\big\{\mathbf{0} \xleftrightarrow{\ge 0} \partial B(N)\big\}$ if $\big\{\mathbf{0} \xleftrightarrow{\ge 0} \partial B(N)\big\}$ occurs but $\big\{\mathbf{0} \xleftrightarrow{\widetilde{E}^{\ge 0}\setminus I_e} \partial B(N)\big\}$ does not. Let $\mathbf{Piv}_N^+$ denote the collection of all these edges. We now present the statement as follows.

Theorem 1.5. *For any $d \ge 3$ with $d \neq 6$, and any $\epsilon \in (0,1)$, there exist constants $c_2(d,\epsilon), C_1(d,\epsilon), C_2(d,\epsilon) > 0$ such that for any $N \ge C_1$,*

$$\mathbb{P}\big(|\mathbf{Piv}_N^+| \in \big[c_2 N^{(\frac{d}{2}-1)\boxdot 2}, C_2 N^{(\frac{d}{2}-1)\boxdot 2}\big] \mid \mathbf{0} \xleftrightarrow{\ge 0} \partial B(N)\big) \ge 1-\epsilon. \tag{1.18}$$

As a direct corollary of Theorem 1.5, the dimension of cut edges in the IIC (as defined by any of the limiting measures in (1.7)-(1.10)) equals $(\frac{d}{2}-1)\boxdot 2$. Here an edge is called a cut edge if removing it disconnects the IIC.

From the perspective of the loop soup, a loop is called pivotal for a connecting event if removing it from the loop soup changes the occurrence of this event. To imitate the pivotal edge in bond percolation, we restrict the discussion to the loops contained in a single interval I_e and intersecting the two trisection points of I_e (let $\mathfrak{L}$ denote the collection of all these loops). For any $A_1, A_2 \subset \widetilde{\mathbb{Z}}^d$, let $A_1 \leftrightarrow A_2$ denote the event that there exists a loop cluster of $\widetilde{\mathcal{L}}_{1/2}$ connecting A_1 and A_2. For any $N \ge 1$, we define the edge collection

$$\widetilde{\mathbf{Piv}}_N := \big\{e \in \mathbb{L}^d : \exists \text{ pivotal } \widetilde{\ell} \in \mathfrak{L} \text{ contained in } I_e \text{ for the event } \mathbf{0} \leftrightarrow \partial B(N)\big\}.$$

Next, we present the analogue of Theorem 1.5 for the loop soup $\widetilde{\mathcal{L}}_{1/2}$.

Theorem 1.6. *For any $d \geq 3$ with $d \neq 6$, and any $\epsilon > 0$, there exist constants $c_3(d,\epsilon), C_3(d,\epsilon), C_4(d,\epsilon) > 0$ such that for any $N \geq C_3$,*

$$\mathbb{P}\big(\big|\widetilde{\mathbf{Piv}}_N\big| \in \big[c_3 N^{(\frac{d}{2}-1)\boxdot 2}, C_4 N^{(\frac{d}{2}-1)\boxdot 2}\big] \mid \mathbf{0} \leftrightarrow \partial B(N)\big) \geq 1-\epsilon. \tag{1.19}$$

We outline the proof of Theorem 1.6 as follows. On the one hand, it can be derived from [6, Theorem 1.4] that conditioned on $\{\mathbf{0} \leftrightarrow \partial B(N)\}$, the expectation of $\big|\widetilde{\mathbf{Piv}}_N\big|$ is at most of order $N^{(\frac{d}{2}-1)\boxdot 2}$. This together with Markov's inequality directly implies that $\big|\widetilde{\mathbf{Piv}}_N\big| \leq CN^{(\frac{d}{2}-1)\boxdot 2}$ occurs with high probability. On the other hand, we show that the conditional probability of $\big|\widetilde{\mathbf{Piv}}_N\big| \leq cN^{(\frac{d}{2}-1)\boxdot 2}$ vanishes uniformly as $c \to 0$ in the following three steps:

(i) Applying the second moment method (where the estimate on the second moment of pivotal edges can be derived from the proof of Theorem 1.1), we show that conditioned on the event $\{\mathbf{0} \leftrightarrow \partial B(N)\}$, with a uniformly positive probability $\delta > 0$ there exist at least $\lambda(a)N^{(\frac{d}{2}-1)\boxdot 2}$ pivotal edges within the annulus $B(CaN) \setminus B(aN)$ for all $a \in (0, C^{-1})$.

(ii) We divide the box $B(N)$ into an increasing sequence of annuli $\{\widetilde{B}(C^{i+1}bN)\setminus \widetilde{B}(C^{i}bN)\}_{1\leq i\leq K}$, where we require $\lambda(b) > c$ and $K \gtrsim \log\big(1/b\big)$. Note that one may further require $b \to 0$ (and hence $K \to \infty$) as $c \to 0$.

(iii) By adapting the decomposition of loop clusters in [2, Section 5.1], we show that for the event $\{\mathbf{0} \leftrightarrow \partial B(N)\}$, the numbers of edges containing a pivotal loop in $\mathfrak{L}$ are asymptotically independent across different annuli in $\{\widetilde{B}(C^{i+1}bN) \setminus \widetilde{B}(C^{i}bN)\}_{1\leq i\leq K}$.

pivotal edges in different annuli in $\{\widetilde{B}(C^{i+1}bN) \setminus \widetilde{B}(C^{i}bN)\}_{1\leq i\leq K}$ are asymptotically independent. Combined with Steps (i) and (ii), it yields that the conditional probability of $\big|\widetilde{\mathbf{Piv}}_N\big| \leq cN^{(\frac{d}{2}-1)\boxdot 2}$ is approximately upper-bounded by $(1-\delta)^K$, and thus vanishes as $c \to 0$ (since $K \to \infty$).

As for Theorem 1.5, note that each $e \in \widetilde{\mathbf{Piv}}_N$ must be a pivotal edge for the event $\{\mathbf{0} \leftrightarrow \partial B(N)\}$, which, according to the isomorphism theorem, shares the same distribution as that in $\mathbf{Piv}_N^+$. In other words, $|\mathbf{Piv}_N^+|$ stochastically dominates $|\widetilde{\mathbf{Piv}}_N|$, and thus exceeds $cN^{(\frac{d}{2}-1)\boxdot 2}$ with high probability. The opposite domination fails because a pivotal edge for $\{\mathbf{0} \leftrightarrow \partial B(N)\}$ is not guaranteed to contain a pivotal loop at scale 1. Instead, such an edge may merely serve as the overlapping region for the two connecting clusters from $\mathbf{0}$ and $\partial B(N)$ respectively. To control the contribution of this scenario to the first moment of $\mathbf{Piv}_N^+$, we need to bound the probability of having two neighboring large loops that are connected to $\mathbf{0}$ and $\partial B(N)$ respectively by two distinct loop clusters. As shown in Section 6, this estimate relies on a decomposition method tailored for two coexisting clusters, which is also a key ingredient in the proof of Theorem 1.1.

Statements about constants. The letters C and c denote positive constants, which are allowed to vary depending on context. Numerical constants (e.g., C_1, c_2)

remain fixed throughout the paper. By convention, C (with or without subscripts) denotes large constants, while c represents small ones. Unless stated otherwise, a constant depends only on the dimension d. Dependence on other parameters must be explicitly indicated (e.g., $C(\lambda)$, $c(\delta)$).

Organization of this paper. In Section 2, we fix the necessary notations and recall some relevant results. In Section 3, we prove Theorem 1.1, assuming a key estimate (see Proposition 3.1) on the probability of two distinct sign clusters nearly touching at two given points. This estimate is then verified in Sections 4, thereby completing the proof of Theorem 1.1. Finally, we provide the proofs of Theorems 1.6 and 1.5 in Sections 5 and 6 respectively.

2. Preliminaries

In this section, we review some primary notations and useful results.

2.1. Basic notations on graphs.

- For any $v \in \widetilde{\mathbb{Z}}^d$, we denote by $\bar{v}$ the closest lattice point to v (we break the tie in a predetermined manner).
- Recall that we have defined the boxes $B(N)$ and $\widetilde{B}(N)$ in Section 1. We also denote the Euclidean ball $\mathcal{B}(N) := \{x \in \mathbb{Z}^d : \|x\| \leq N\}$. For any $v \in \widetilde{\mathbb{Z}}^d$, we define $B_v(N) := \bar{v}+B(N)$, $\widetilde{B}_v(N) := \bar{v}+\widetilde{B}(N)$ and $\mathcal{B}_v(N) := \bar{v}+\mathcal{B}(N)$.
- For any $e \in \mathbb{L}^d$ and $v, w \in I_e$, we denote by $I_{[v,w]}$ the subinterval of I_e with endpoints v and w.
- We use the letter D (possibly with superscripts or subscripts) to denote a subset of $\widetilde{\mathbb{Z}}^d$ composed of finitely many compact connected components.
- The boundary of $D \subset \widetilde{\mathbb{Z}}^d$ is $\widetilde{\partial} D := \{v \in D : \inf_{w \in \widetilde{\mathbb{Z}}^d \setminus D} |v - w| = 0\}$. Since $\widetilde{\mathbb{Z}}^d$ is locally one-dimensional, $\widetilde{\partial} D$ is a countable set.
- A path in $\widetilde{\mathbb{Z}}^d$ is a continuous function $\widetilde{\eta} : [0, T) \to \widetilde{\mathbb{Z}}^d$, where the duration $T \in \mathbb{R}^+ \cup \{\infty\}$. When $T < \infty$, we denote $\widetilde{\eta}(T) = \lim_{t \uparrow T} \widetilde{\eta}(t)$. Recall that a rooted loop $\widetilde{\varrho}$ (of duration $T < \infty$) is a path satisfying $\widetilde{\varrho}(0) = \widetilde{\varrho}(T)$.
- For a path $\widetilde{\eta}$, we denote its range by $\mathrm{ran}(\widetilde{\eta}) := \{\widetilde{\eta}(t) : 0 \leq t < T\}$.
- For any set $D \subset \widetilde{\mathbb{Z}}^d$ and any path $\widetilde{\eta}$, we denote the first time when $\widetilde{\eta}$ hits D by $\tau_D(\widetilde{\eta}) := \inf\{0 \leq t < T : \widetilde{\eta}(t) \in D\}$ (where we set $\tau_\emptyset := +\infty$ for completeness). In particular, we abbreviate $\tau_v := \tau_{\{v\}}$ for $v \in \widetilde{\mathbb{Z}}^d$.

2.2. Brownian motion on $\widetilde{\mathbb{Z}}^d$.

The Brownian motion on $\widetilde{\mathbb{Z}}^d$, denoted by $\{\widetilde{S}_t\}_{t \geq 0}$, is a continuous-time Markov chain evolving as follows. When $\widetilde{S}_\cdot$ is in the interior of an interval I_e ($e \in \mathbb{L}^d$), it behaves as a standard one-dimensional Brownian motion. When $\widetilde{S}_\cdot$ reaches a lattice point $x \in \mathbb{Z}^d$, it uniformly picks an interval I_e incident to x, and then performs a Brownian excursion from x in I_e; in addition, once the excursion hits an endpoint of I_e, the process continues evolving in the same manner from this endpoint. For brevity, we denote by $\tau_D := \tau_D(\widetilde{S}_\cdot)$ the first hitting time at D for the Brownian motion $\widetilde{S}_\cdot$. For each $v \in \widetilde{\mathbb{Z}}^d$, we denote by $\widetilde{\mathbb{P}}_v$ the law of $\{\widetilde{S}_t\}_{t\geq 0}$ starting from v. The expectation under $\widetilde{\mathbb{P}}_v$ is denoted by $\widetilde{\mathbb{E}}_v$.

In what follows, we record several basic properties of Brownian motion. The first one states that the hitting distribution on the boundary of a Euclidean ball is comparable to the uniform distribution. For the proof details, readers may refer to [25, Lemma 6.3.7] and [8, Lemma 3.7]. For any $N \geq 1$ and $\delta > 0$, we denote $\mathfrak{B}^1_{N,\delta} := \widetilde{B}(\delta N)$ and $\mathfrak{B}^{-1}_{N,\delta} := [\widetilde{B}(\delta^{-1}N)]^c$.

Lemma 2.1. *For any $d \geq 3$, $r \geq 1$, $v \in \mathfrak{B}^1_{N,d^{-1}} \cup \mathfrak{B}^{-1}_{N,d^{-1}}$ and $x \in \partial\mathcal{B}(r)$,*

$$\widetilde{\mathbb{P}}_v\big(\tau_{\partial\mathcal{B}(r)} = \tau_x < \infty\big) \lesssim r^{1-d}. \tag{2.1}$$

As shown in the following lemma, the hitting distribution remains stable under the addition of appropriate obstacles. The proof can be found in [5, Lemma 2.1].

Lemma 2.2. *For any $d \geq 3$, $N \geq 1$, $x \in \partial B(N)$, $i \in \{1, -1\}$, $D_i \subset \mathfrak{B}^i_{N,1/2}$ and $v \in \widetilde{\partial} D_i$,*

$$\widetilde{\mathbb{P}}_x\big(\tau_{D_i} = \tau_v < \tau_{D_{-i}}\big) \asymp \widetilde{\mathbb{P}}_x\big(\tau_{D_i} = \tau_v < \infty\big). \tag{2.2}$$

Next, we extend the stability in Lemma 2.2 to more general D_j as follows.

Lemma 2.3. *For any $d \geq 3$, there exists $c > 0$ such that for any $N \geq 1$, $i \in \{1, -1\}$, $D_i \subset \widetilde{\mathbb{Z}}^d$, $v \in (\widetilde{\partial} D_i) \cap \mathfrak{B}^i_{N,1}$ and $D_{-i} \subset \mathfrak{B}^{-i}_{N,c}$,*

$$\sum\nolimits_{x\in\partial\mathcal{B}(d^iN)} \widetilde{\mathbb{P}}_x\big(\tau_{D_i} = \tau_v < \tau_{D_{-i}}\big) \asymp \sum\nolimits_{x\in\partial\mathcal{B}(d^iN)} \widetilde{\mathbb{P}}_x\big(\tau_{D_i} = \tau_v < \infty\big). \tag{2.3}$$

P.S. In fact, it is necessary to take the sum over $x \in \partial\mathcal{B}(d^iN)$ in (2.3). Otherwise, if one considers the hitting probability in D_i for the Brownian motion starting from a fixed point $x \in \partial\mathcal{B}(d^iN)$, then imposing D_{-i} as absorbing boundaries may drastically decrease this probability, as the following scenario illustrates. Let the set D_i form a tunnel such that there exists a path starting from x and first hitting D_i within $\mathfrak{B}^i_{N,1}$, and that all such paths must intersect $\mathfrak{B}^{-i}_{N,C}$. Note that $\widetilde{\mathbb{P}}_x(\tau_{D_i} = \tau_v < \infty) > 0$ for this D_i. However, for $D_{-i} = \mathfrak{B}^{-i}_{N,C}$, the probability $\widetilde{\mathbb{P}}_x(\tau_{D_i} = \tau_v < \tau_{D_{-i}})$ vanishes.

Proof. We only provide the proof for $i = 1$, since the case $i = -1$ follows similarly. By the strong Markov property of Brownian motion, we have: for any $x \in \partial\mathcal{B}(dN)$,

$$\begin{aligned}
0 \leq& \widetilde{\mathbb{P}}_x\big(\tau_{D_1} = \tau_v < \infty\big) - \widetilde{\mathbb{P}}_x\big(\tau_{D_1} = \tau_v < \tau_{D_{-1}}\big)\\
=& \widetilde{\mathbb{P}}_x\big(\tau_{D_{-1}} < \tau_{D_1} = \tau_v < \infty\big)\\
\leq& \max_{z\in D_{-1}} \widetilde{\mathbb{P}}_z\big(\tau_{\partial\mathcal{B}(10dN)} < \infty\big) \cdot \max_{z'\in\partial\mathcal{B}(10dN)} \sum\nolimits_{z''\in\partial\mathcal{B}(dN)}\\
& \qquad \widetilde{\mathbb{P}}_{z'}\big(\tau_{\partial\mathcal{B}(dN)} = \tau_{z''} < \infty\big) \cdot \widetilde{\mathbb{P}}_{z''}\big(\tau_{D_1} = \tau_v < \infty\big).
\end{aligned} \tag{2.4}$$

By the potential theory (see e.g., (2.22) and (2.23) below), one has: for $z \in D_{-1}$,

$$\widetilde{\mathbb{P}}_v\big(\tau_{\partial \mathcal{B}(10dN)} < \infty\big) \lesssim C^{2-d}. \tag{2.5}$$

Meanwhile, Lemma 2.1 implies that for any $z' \in \partial \mathcal{B}(10dN)$,

$$\widetilde{\mathbb{P}}_{z'}\big(\tau_{\partial \mathcal{B}(dN)} = \tau_{z''} < \infty\big) \lesssim N^{1-d}. \tag{2.6}$$

Inserting (2.5) and (2.6) into (2.4), and then summing over $x \in \partial \mathcal{B}(dN)$, we get

$$\begin{aligned} 0 \leq & \sum\nolimits_{x \in \partial \mathcal{B}(dN)} \widetilde{\mathbb{P}}_x\big(\tau_{D_1} = \tau_v < \infty\big) - \sum\nolimits_{x \in \partial \mathcal{B}(dN)} \widetilde{\mathbb{P}}_x\big(\tau_{D_1} = \tau_v < \tau_{D_{-1}}\big) \\ \lesssim & C^{2-d} N^{1-d} |\partial \mathcal{B}(dN)| \cdot \sum\nolimits_{z'' \in \partial \mathcal{B}(dN)} \widetilde{\mathbb{P}}_{z''}\big(\tau_{D_1} = \tau_v < \infty\big). \end{aligned} \tag{2.7}$$

Combined with $|\partial \mathcal{B}(dN)| \asymp N^{d-1}$, it implies that for any sufficiently large $C > 0$, the inquality (2.3) holds. □

Green's function. For any $D \subset \widetilde{\mathbb{Z}}^d$, the Green's function for D is defined by

$$\widetilde{G}_D(v,w) := \int_0^\infty \Big\{ \widetilde{q}_t(v,w) - \widetilde{\mathbb{E}}_v\big[\widetilde{q}_{t-\tau_D}(\widetilde{S}_{\tau_D}, w) \cdot \mathbb{1}_{\tau_D < t}\big] \Big\} dt, \quad \forall v, w \in \widetilde{\mathbb{Z}}^d. \tag{2.8}$$

Note that $\widetilde{G}_D$ is finite, symmetric, continuous, and is decreasing in D. In addition, $\widetilde{G}_D(v,w) = 0$ when $\{v,w\} \cap D \neq \emptyset$. When $D = \emptyset$, we abbreviate $\widetilde{G}(\cdot,\cdot) := \widetilde{G}_\emptyset(\cdot,\cdot)$. Moreover, restricted to $\mathbb{Z}^d$, $\widetilde{G}(\cdot,\cdot)$ is identical to the Green's function $G(\cdot,\cdot)$ on $\mathbb{Z}^d$, which implies that $\widetilde{G}(v,w) \asymp (|v-w|+1)^{2-d}$ for all $v, w \in \widetilde{\mathbb{Z}}^d$. The strong Markov property of Brownian motion yields that

$$\widetilde{G}_D(v,w) = \widetilde{\mathbb{P}}_v\big(\tau_w < \tau_D\big) \cdot \widetilde{G}_D(w,w). \tag{2.9}$$

In addition, using the potential theory, one has

$$\widetilde{G}_D(w,w) \asymp \mathrm{dist}(D,w) \wedge 1. \tag{2.10}$$

For any $D \subset \widetilde{\mathbb{Z}}^d$, we denote by $\mathbb{P}^D$ the law of $\{\widetilde{\phi}_v\}_{v \in \widetilde{\mathbb{Z}}^d}$ conditioned on the event $\{\widetilde{\phi}_v = 0, \forall v \in D\}$. Under $\mathbb{P}^D$, the GFF $\widetilde{\phi}_\cdot$ is mean-zero and has covariance

$$\mathbb{E}^D\big[\widetilde{\phi}_v \widetilde{\phi}_w\big] = \widetilde{G}_D(v,w), \quad \forall v, w \in \widetilde{\mathbb{Z}}^d. \tag{2.11}$$

Let $\mathbb{E}^D$ denote the expectation under $\mathbb{P}^D$. Referring to [26, Proposition 5.2], the formula in (1.1) for two-point functions is valid for $\widetilde{\phi}_\cdot \sim \mathbb{P}^D$, i.e., for any $D \subset \widetilde{\mathbb{Z}}^d$ and $v \neq w \in \widetilde{\mathbb{Z}}^d \setminus D$,

$$\begin{aligned} \mathbb{P}^D\big(v \xleftrightarrow{\geq 0} w\big) = & \pi^{-1} \arcsin\Big(\tfrac{\widetilde{G}_D(v,w)}{\sqrt{\widetilde{G}_D(v,v)\widetilde{G}_D(w,w)}}\Big) \\ \asymp & \tfrac{\widetilde{G}_D(v,w)}{\sqrt{\widetilde{G}_D(v,v)\widetilde{G}_D(w,w)}} \overset{(2.9)}{\asymp} \widetilde{\mathbb{P}}_w\big(\tau_v < \tau_D\big) \cdot \sqrt{\tfrac{\widetilde{G}_D(v,v)}{\widetilde{G}_D(w,w)}}. \end{aligned} \tag{2.12}$$

Noting that $\widetilde{G}_D(w,w) \vee \widetilde{G}_D(v,v) \lesssim 1$, this immediately implies

$$\widetilde{G}_D(v,w) \lesssim \mathbb{P}^D\big(v \xleftrightarrow{\geq 0} w\big). \tag{2.13}$$

Lemma 2.4. *For any $3 \le d \le 5$, there exists $c > 0$ such that for any $D \subset \widetilde{\mathbb{Z}}^d$, $R \ge 1$, and $v_1, v_2 \in \widetilde{\mathbb{Z}}^d$ with $|v_1 - v_2| \ge R$,*

$$\widetilde{G}_D(v_1, v_2) \lesssim R^{2-d} \prod\nolimits_{i\in\{1,2\}} \widetilde{\mathbb{P}}_{v_i}\big(\tau_{\partial\mathcal{B}_{v_i}(cR)} < \tau_D\big). \tag{2.14}$$

Proof. Using the strong Markov property of Brownian motion, we have

$$\widetilde{\mathbb{P}}_{v_1}\big(\tau_{v_2} < \tau_D\big) \le \widetilde{\mathbb{P}}_{v_1}\big(\tau_{\partial\mathcal{B}_{v_1}(cR)} < \tau_D\big) \cdot \max_{y\in\partial\mathcal{B}_{v_1}(cR)} \widetilde{\mathbb{P}}_y\big(\tau_{v_2} < \tau_D\big). \tag{2.15}$$

Moreover, by the last-exit decomposition (see e.g., [29, Section 8.2]),

$$\widetilde{\mathbb{P}}_y\big(\tau_{v_2} < \tau_D\big) \lesssim \sum_{z\in\partial\mathcal{B}_{v_2}(cR)} \widetilde{G}_D(y,z) \sum_{z'\in\mathcal{B}_{v_2}(cR):\{z,z'\}\in\mathbb{L}^d} \widetilde{\mathbb{P}}_{z'}\big(\tau_{v_2} < \tau_{D\cup\partial\mathcal{B}_{v_2}(cR)}\big). \tag{2.16}$$

By reversing the Brownian motion, we have

$$\begin{aligned}&\widetilde{\mathbb{P}}_{z'}\big(\tau_{v_2} < \tau_{D\cup\partial\mathcal{B}_{v_2}(cR)}\big)\\ \lesssim&[\widetilde{G}_D(v_2,v_2)]^{-1}\cdot \sum_{z''\in\partial\mathcal{B}_{v_2}(cR):\{z',z''\}\in\mathbb{L}^d} \widetilde{\mathbb{P}}_{v_2}\big(\tau_{\partial\mathcal{B}_{v_2}(cR)} = \tau_{z''} < \tau_D\big).\end{aligned} \tag{2.17}$$

Note that $\widetilde{G}_D(y,z) \asymp R^{2-d}$ for $y \in \partial\mathcal{B}_{v_1}(cR)$ and $z \in \partial\mathcal{B}_{v_2}(cR)$. Combined with (2.16) and (2.17), it yields that for any $y \in \partial\mathcal{B}_{v_1}(cR)$,

$$\begin{aligned}\widetilde{\mathbb{P}}_y\big(\tau_{v_2} < \tau_D\big) \lesssim& R^{2-d}[\widetilde{G}_D(v_2,v_2)]^{-1}\cdot \sum\nolimits_{z''\in\partial\mathcal{B}_{v_2}(cR)} \widetilde{\mathbb{P}}_{v_2}\big(\tau_{\partial\mathcal{B}_{v_2}(cR)} = \tau_{z''} < \tau_D\big)\\ =&R^{2-d}[\widetilde{G}_D(v_2,v_2)]^{-1}\cdot \widetilde{\mathbb{P}}_{v_2}\big(\tau_{\partial\mathcal{B}_{v_2}(cR)} < \tau_D\big).\end{aligned}$$

Substituting this into (2.15) and using (2.9), we derive (2.14). $\square$

Boundary excursion kernel. For any $D \subset \widetilde{\mathbb{Z}}^d$, the boundary excursion kernel for D is defined as follows: for any $v \neq w \in \widetilde{\partial} D$,

$$\mathbb{K}_D(v,w) := \lim_{\epsilon\downarrow 0} (2\epsilon)^{-1} \sum\nolimits_{v'\in\widetilde{\mathbb{Z}}^d:|v-v'|=\epsilon} \widetilde{\mathbb{P}}_{v'}\big(\tau_D = \tau_w < \infty\big). \tag{2.18}$$

Similar to the Green's function $\widetilde{G}_D$, $\mathbb{K}_D(\cdot,\cdot)$ is symmetry and decreasing in D.

By [6, Lemma 3.2], one has: for any $D \subset \widetilde{\mathbb{Z}}^d$ and $v, w \in \widetilde{\mathbb{Z}}^d \setminus D$ with $|v - w| \ge d$,

$$\mathbb{K}_{D\cup\{v,w\}}(v,w) \asymp \big[\widetilde{G}_D(w,w)\big]^{-1} \cdot \widetilde{\mathbb{P}}_w\big(\tau_v < \tau_D\big). \tag{2.19}$$

Combined with (2.12) and $\widetilde{G}_D(w,w) \vee \widetilde{G}_D(v,v) \lesssim 1$, it implies that

$$\mathbb{P}^D\big(v \xleftrightarrow{\ge 0} w\big) \lesssim \mathbb{K}_{D\cup\{v,w\}}(v,w). \tag{2.20}$$

Capacity. For any $D \subset \widetilde{\mathbb{Z}}^d$, the equilibrium measure for D is defined by

$$\mathbb{Q}_D(v) := \lim_{N\to\infty} \sum\nolimits_{w\in\partial B(N)} \mathbb{K}_{D\cup\partial B(N)}(v,w),\ \forall v \in \widetilde{\partial} D. \tag{2.21}$$

The capacity of D is defined by the total mass of the equilibrium measure, i.e., $\mathrm{cap}(D) := \sum_{v\in\widetilde{\partial}D} \mathbb{Q}_D(v)$. The capacity is closely related to the hitting probability: for any $D \subset \widetilde{B}(N)$, $x \in \partial B(2N)$ and $D' \subset [\widetilde{B}(3N)]^c$,

$$\widetilde{\mathbb{P}}_x\big(\tau_D < \tau_{D'}\big) \asymp N^{2-d} \cdot \mathrm{cap}(D). \tag{2.22}$$

In addition, the capacity of a box is given by

$$\mathrm{cap}(\widetilde{B}(N)) \asymp N^{d-2}, \ \ \forall N \geq 1. \tag{2.23}$$

2.3. **Loop soup.** Recall the definitions of the loop measure $\widetilde{\mu}$ and the loop soup $\widetilde{\mathcal{L}}_\alpha$ in Section 1. By forgetting the roots of all rooted loops, $\widetilde{\mu}$ can be considered as a measure on the space of equivalence classes of rooted loops, where every pair of loops in the same class are equivalent under a time-shift. Here each equivalence class $\widetilde{\ell}$ is called a loop. Note that every two equivalent rooted loops have the same range. In light of this, we define $\mathrm{ran}(\widetilde{\ell})$ as $\mathrm{ran}(\widetilde{\varrho})$ for any $\widetilde{\varrho} \in \widetilde{\ell}$.

Isomorphism theorem. For any $D \subset \widetilde{\mathbb{Z}}^d$, we denote $\widetilde{\mathcal{L}}_\alpha^D := \widetilde{\mathcal{L}}_\alpha \cdot \mathbb{1}_{\mathrm{ran}(\widetilde{\ell})\cap D=\emptyset}$. Referring to [26, Proposition 2.1], the isomorphism theorem mentioned in Section 1 is valid for the metric graph $\widetilde{\mathbb{Z}}^d \setminus D$. I.e., there exists a coupling between $\{\widetilde{\phi}_v\}_{v\in\widetilde{\mathbb{Z}}^d} \sim \mathbb{P}^D$ and $\widetilde{\mathcal{L}}_{1/2}^D$ satisfying Properties (a) and (b) described in Section 1 (where $\widehat{\mathcal{L}}_{1/2}^v$ is replaced by $\widehat{\mathcal{L}}_{1/2}^{D,v}$, the total local time at v of all loops in $\widetilde{\mathcal{L}}_{1/2}^D$). To facilitate the use of this isomorphism theorem, we record the following notations on clusters:

- For a point measure $\mathcal{L}$ consisting of paths, we denote by $\cup\mathcal{L}$ the union of ranges of all paths in $\mathcal{L}$. For any $D, A, A' \subset \widetilde{\mathbb{Z}}^d$, we denote by $A \xleftrightarrow{(D)} A'$ the event that there exists a path in $\cup\widetilde{\mathcal{L}}_{1/2}^D$ connecting A and A'. We then define the cluster $\mathcal{C}_A^D := \{v \in \widetilde{\mathbb{Z}}^d : v \xleftrightarrow{(D)} A\}$. When $D = \emptyset$, we write $\mathcal{C}_A^\emptyset$ as $\mathcal{C}_A$ for brevity.
- For any $A \subset \widetilde{\mathbb{Z}}^d$, we denote the positive (resp. negative) cluster containing A by $\mathcal{C}_A^+ := \{v \in \widetilde{\mathbb{Z}}^d : v \xleftrightarrow{\geq 0} A\}$ (resp. $\mathcal{C}_A^- := \{v \in \widetilde{\mathbb{Z}}^d : v \xleftrightarrow{\leq 0} A\}$). The sign cluster containing A is defined as $\mathcal{C}_A^\pm := \mathcal{C}_A^+ \cup \mathcal{C}_A^-$.
- For any loop cluster $\mathcal{C}$, let $\mathcal{F}_\mathcal{C}$ denote the σ-field generated by $\mathcal{C}$ and all loops included in $\mathcal{C}$. Especially, the occupation field restricted to $\mathcal{C}$ is measurable with respect to $\mathcal{F}_\mathcal{C}$. In addition, for any sign cluster $\mathcal{C}'$, we denote by $\mathcal{F}_{\mathcal{C}'}$ the σ-field generated by $\mathcal{C}'$ and all GFF values on $\mathcal{C}'$.

Restriction property. Conditioned on $\{\mathcal{C}_A^D = F\}$ (for any proper realization F of $\mathcal{C}_A^D$), the thinning property of Poisson point processes implies that the point measure $\widetilde{\mathcal{L}}_{1/2}^D \cdot \mathbb{1}_{\mathrm{ran}(\widetilde{\ell})\cap\mathcal{C}_A^D=\emptyset}$ has the same distribution as $\widetilde{\mathcal{L}}_{1/2}^{D\cup F}$. From the perspective of GFF, the corresponding property also holds. I.e., for $\diamond \in \{+, -, \pm\}$, conditioned on $\{\mathcal{C}_A^\diamond = F\}$, the law of GFF values on $\widetilde{\mathbb{Z}}^d \setminus \mathcal{C}_A^\diamond$ is given by $\mathbb{P}^F$.

Connecting probability. For any $A_1, A_2, D \subset \widetilde{\mathbb{Z}}^d$, the event $A_1 \xleftrightarrow{(D)} A_2$ is measurable with respect to $\mathcal{C}^D_{A_1}$. Thus, when $\mathcal{C}^D_{A_1}$ is disjoint from $D' \subset \widetilde{\mathbb{Z}}^d$ (and hence consists only of loops in $\widetilde{\mathcal{L}}^{D\cup D'}_{1/2}$), we have $\mathcal{C}^D_{A_1} = \mathcal{C}^{D\cup D'}_{A_1}$ and $A_1 \xleftrightarrow{(D\cup D')} A_2$. As a result, one has

$$\big\{A_1 \xleftrightarrow{(D)} A_2\big\} \cap \big\{A_1 \xleftrightarrow{(D)} D'\big\}^c \subset \big\{\mathcal{C}^D_{A_1} = \mathcal{C}^{D\cup D'}_{A_1}\big\} \cap \big\{A_1 \xleftrightarrow{(D\cup D')} A_2\big\}. \tag{2.24}$$

This immediately implies that the probability of the left-hand side is less equal to that of the right-hand side. In the next lemma, we show that the reverse inequality may hold (up to a constant factor) under appropriate conditions.

Lemma 2.5. *(1) For any $3 \le d \le 5$, there exists $c > 0$ such that for any $D \subset \widetilde{\mathbb{Z}}^d$, $N \ge 1$, $i \in \{1, -1\}$, and $A_1, A_2 \subset \mathfrak{B}^i_{N,c}$,*

$$\mathbb{P}\big(A_1 \xleftrightarrow{(D\cup\widetilde{\partial}\mathfrak{B}^i_{N,c})} A_2\big) \lesssim \mathbb{P}\big(A_1 \xleftrightarrow{(D)} A_2, \big\{A_1 \xleftrightarrow{(D)} \partial B(N)\big\}^c\big). \tag{2.25}$$

(2) For any $3 \le d \le 5$, there exists $C > 0$ such that for any $D \subset \widetilde{\mathbb{Z}}^d$, $N_2 > N_1 \ge 1$, and $A_1, A_2 \subset \widetilde{B}(N_2) \setminus \widetilde{B}(N_1)$,

$$\begin{aligned}
&\mathbb{P}\big(A_1 \xleftrightarrow{(D\cup\partial B(N_1)\cup\partial B(N_2))} A_2\big) \\
\lesssim&\mathbb{P}\big(A_1 \xleftrightarrow{(D)} A_2, \big\{A_1 \xleftrightarrow{(D)} \partial B(\tfrac{N_1}{C}) \cup \partial B(CN_2))\big\}^c\big).
\end{aligned} \tag{2.26}$$

Proof. (1) Since $\big\{\widetilde{\partial}\mathfrak{B}^i_{N,c} \xleftrightarrow{(D)} \partial B(N)\big\}^c$ depends only on the loops intersecting $[\mathfrak{B}^i_{N,c}]^c$, it is independent of the event $A_1 \xleftrightarrow{(D\cup\widetilde{\partial}\mathfrak{B}^i_{N,c})} A_2$. Therefore,

$$\begin{aligned}
&\mathbb{P}\big(A_1 \xleftrightarrow{(D\cup\widetilde{\partial}\mathfrak{B}^i_{N,c})} A_2, \big\{\widetilde{\partial}\mathfrak{B}^i_{N,c} \xleftrightarrow{(D)} \partial B(N)\big\}^c\big) \\
=&\mathbb{P}\big(A_1 \xleftrightarrow{(D\cup\widetilde{\partial}\mathfrak{B}^i_{N,c})} A_2\big)\mathbb{P}\big(\big\{\widetilde{\partial}\mathfrak{B}^i_{N,c} \xleftrightarrow{(D)} \partial B(N)\big\}^c\big) \overset{(1.5)}{\gtrsim} \mathbb{P}\big(A_1 \xleftrightarrow{(D\cup\widetilde{\partial}\mathfrak{B}^i_{N,c})} A_2\big).
\end{aligned} \tag{2.27}$$

Meanwhile, when $\big\{A_1 \xleftrightarrow{(D\cup\widetilde{\partial}\mathfrak{B}^i_{N,c})} A_2\big\}\cap\big\{\widetilde{\partial}\mathfrak{B}^i_{N,c} \xleftrightarrow{(D)} \partial B(N)\big\}^c$ occurs, by $\widetilde{\mathcal{L}}^{D\cup\widetilde{\partial}\mathfrak{B}^i_{N,c}}_{1/2} \le \widetilde{\mathcal{L}}^D_{1/2}$ we have $A_1 \xleftrightarrow{(D)} A_2$; moreover, since $A_1 \subset \mathfrak{B}^i_{N,c}$, one has $\big\{A_1 \xleftrightarrow{(D)} \partial B(N)\big\}^c$. Consequently, we have

$$\begin{aligned}
&\mathbb{P}\big(A_1 \xleftrightarrow{(D\cup\widetilde{\partial}\mathfrak{B}^i_{N,c})} A_2, \big\{\widetilde{\partial}\mathfrak{B}^i_{N,c} \xleftrightarrow{(D)} \partial B(N)\big\}^c\big) \\
\le&\mathbb{P}\big(A_1 \xleftrightarrow{(D)} A_2, \big\{A_1 \xleftrightarrow{(D)} \partial B(N)\big\}^c\big).
\end{aligned} \tag{2.28}$$

Combined with (2.27), it implies (2.25).

(2) For the same reason as in (2.27), we have

$$\begin{aligned}
&\mathbb{P}\big(\big\{A_1 \xleftrightarrow{(D\cup\partial B(N_1)\cup\partial B(N_2))} A_2\big\} \cap \big\{B(\tfrac{N_1}{C}) \xleftrightarrow{(D)} \partial B(N_1)\big\}^c \\
&\qquad\qquad\qquad\qquad\qquad \cap \big\{B(N_2) \xleftrightarrow{(D)} \partial B(CN_2)\big\}^c\big) \\
=&\mathbb{P}\big(A_1 \xleftrightarrow{(D\cup\partial B(N_1)\cup\partial B(N_2))} A_2\big) \\
&\cdot \mathbb{P}\big(\big\{B(\tfrac{N_1}{C}) \xleftrightarrow{(D)} \partial B(N_1)\big\}^c \cap \big\{B(N_2) \xleftrightarrow{(D)} \partial B(CN_2)\big\}^c\big) \\
\overset{\text{(FKG),(1.5)}}{\gtrsim}& \mathbb{P}\big(A_1 \xleftrightarrow{(D\cup\partial B(N_1)\cup\partial B(N_2))} A_2\big).
\end{aligned} \tag{2.29}$$

In addition, as in (2.28), the event on the left-hand side of (2.29) implies that the cluster $\mathcal{C}_{A_1}^D$ is contained in the annulus $\widetilde{B}(CN_2)\setminus\widetilde{B}(\frac{N_1}{C})$. This observation together with (2.29) yields (2.26). $\square$

As verified in [5], the connecting probability between two general sets is stable under the addition of certain zero boundary conditions. Precisely, referring to the proof of [5, Proposition 1.8] in [5, Section 5], we have the following property:

Lemma 2.6. *For any $d\geq 3$, there exist $c,c'>0$ such that for any $i\in\{1,-1\}$, $N\geq 1$, $A,D\subset \mathfrak{B}_{N,c}^i$ and $D'\subset \mathfrak{B}_{N,c'}^{-i}$,*

$$\mathbb{P}\big(A \xleftrightarrow{(D)} \partial B(N)\big) \asymp \mathbb{P}\big(A \xleftrightarrow{(D\cup D')} \partial B(N)\big). \tag{2.30}$$

In addition to Lemma 2.6, we also present the corresponding stability for point-to-set connecting probabilities as follows.

Lemma 2.7. *For any $3\leq d\leq 5$, there exists $c>0$ such that for any sufficiently large $N\geq 1$, and any $x\in\partial B(N)$, $A,D\subset\mathfrak{B}_{N,c}^1$ and $D'\subset\mathfrak{B}_{N,c}^{-1}$,*

$$\mathbb{P}\big(A \xleftrightarrow{(D)} x\big) \asymp \mathbb{P}\big(A \xleftrightarrow{(D\cup D')} x\big). \tag{2.31}$$

For any $d\geq 7$, there exists $c'>0$ such that (2.31) is valid if the following additional condition holds:

$$\sum\nolimits_{z\in\mathbb{Z}^d:\widetilde{B}_z(1)\cap D'\neq\emptyset} |z|^{2-d} \leq c' N^{6-d}. \tag{2.32}$$

The proof of Lemma 2.7 will be provided in Appendix A.

The next lemma shows that this stability holds for the average of two-point functions on the boundary of a Euclidean ball, with more general absorbing boundaries.

Lemma 2.8. *For any $3\leq d\leq 5$, there exists $c>0$ such that for any $D\subset\widetilde{\mathbb{Z}}^d$, $i\in\{1,-1\}$, $r_{-1}>r_1\geq 1$, $D'\subset\mathfrak{B}_{r_i,c}^i$, and $x\in\partial\mathcal{B}(r_{-i})$,*

$$\sum\nolimits_{y\in\partial\mathcal{B}(r_i)} \mathbb{P}\big(x \xleftrightarrow{(D)} y\big) \asymp \sum\nolimits_{y\in\partial\mathcal{B}(r_i)} \mathbb{P}\big(x \xleftrightarrow{(D\cup D')} y\big). \tag{2.33}$$

P.S. As in Lemma 2.3, it is necessary to take the sum over $y \in \partial\mathcal{B}(r_i)$ in (2.33), which can be explained via an example analogous to the one proposed below (2.3). The same applies to the subsequent corollaries of Lemma 2.8.

Proof. The proofs for $i = 1$ and $i = -1$ employ similar arguments, so we only show the details for the case when $i = 1$. Since $\widetilde{\mathcal{L}}_{1/2}^{D} \geq \widetilde{\mathcal{L}}_{1/2}^{D\cup D'}$, the left-hand side of (2.33) is greater equal to the right-hand side. Thus, by (2.12), it suffices to show

$$\sum_{y\in\partial\mathcal{B}(r_1)} \widetilde{\mathbb{P}}_x\big(\tau_y < \tau_D\big)\sqrt{\tfrac{\widetilde{G}_D(y,y)}{\widetilde{G}_D(x,x)}} \lesssim \sum_{y\in\partial\mathcal{B}(r_1)} \widetilde{\mathbb{P}}_x\big(\tau_y < \tau_{D\cup D'}\big)\sqrt{\tfrac{\widetilde{G}_{D\cup D'}(y,y)}{\widetilde{G}_{D\cup D'}(x,x)}}. \tag{2.34}$$

By Lemma 2.3, we have

$$\sum\nolimits_{y\in\partial\mathcal{B}(r_1)} \mathbb{P}\big(x \xleftrightarrow{(D)} y\big) \lesssim \sum\nolimits_{y\in\partial\mathcal{B}(r_1)} \mathbb{P}\big(x \xleftrightarrow{(D\cup D')} y\big). \tag{2.35}$$

Meanwhile, note that $\mathrm{dist}(D', z) \geq 1$ for all $z \in \partial\mathcal{B}(r_1) \cup \partial\mathcal{B}(r_{-1})$. Thus, for each $w \in \{x, y\}$, it follows from (2.10) that

$$\widetilde{G}_D(w,w) \asymp \mathrm{dist}(D,w) \wedge 1 \asymp \mathrm{dist}(D\cup D', w) \wedge 1 \asymp \widetilde{G}_{D\cup D'}(w,w). \tag{2.36}$$

Combining (2.35) and (2.36), we get (2.34) and thus complete the proof. $\square$

In what follows, we present several corollaries of Lemmas 2.5 and 2.8. We say an event A is a two-point event for $\widetilde{\mathcal{L}}_{1/2}^{D}$ outside D' if there exist $v, w \in \widetilde{\mathbb{Z}}^d$ such that $\mathsf{A} = \big\{v \xleftrightarrow{(D)} w\big\} \cap \big\{v \xleftrightarrow{(D)} D'\big\}^c$. The intersection of finitely many two-point events is called a pairwise connecting event.

Corollary 2.9. *We maintain the same conditions for d, D, i, r_{-1} and r_1 as in Lemma 2.8. Then there exist $c > c' > 0$ such that the following hold:*

(1) For any $x \in \partial\mathcal{B}(r_{-i})$, and any pairwise connecting event A for $\widetilde{\mathcal{L}}_{1/2}^{D}$ outside $[\mathfrak{B}^i_{r_i,c'}]^c$,

$$\sum_{y\in\partial\mathcal{B}(r_i)} \mathbb{P}\big(\mathsf{A} \cap \big\{x \xleftrightarrow{(D)} y\big\} \cap \big\{x \xleftrightarrow{(D)} \widetilde{\partial}\mathfrak{B}^i_{r_i,c}\big\}^c\big) \gtrsim \mathbb{P}(\mathsf{A}) \cdot \sum_{y\in\partial\mathcal{B}(r_i)} \mathbb{P}\big(x \xleftrightarrow{(D)} y\big). \tag{2.37}$$

(2) For any pairwise connecting event A' for $\widetilde{\mathcal{L}}_{1/2}^{D}$ outside $\mathfrak{B}^{-1}_{r_{-1},c'} \setminus \mathfrak{B}^{1}_{r_1,c'}$,

$$\begin{aligned} &\sum_{x\in\partial\mathcal{B}(r_1),y\in\partial\mathcal{B}(r_2)} \mathbb{P}\big(\mathsf{A}' \cap \big\{x \xleftrightarrow{(D)} y\big\} \cap \big\{x \xleftrightarrow{(D)} \widetilde{\partial}\mathfrak{B}^1_{r_1,c} \cup \widetilde{\partial}\mathfrak{B}^{-1}_{r_{-1},c}\big\}^c\big) \\ \gtrsim\, &\mathbb{P}(\mathsf{A}') \cdot \sum_{x\in\partial\mathcal{B}(r_1),y\in\partial\mathcal{B}(r_2)} \mathbb{P}\big(x \xleftrightarrow{(D)} y\big). \end{aligned} \tag{2.38}$$

Proof. (1) Let $\overline{\mathcal{C}}$ denote the union of all clusters involved in the event A. Since A is outside $[\mathfrak{B}^i_{r_i,c'}]^c$, one has $\mathsf{A} \subset \{\overline{\mathcal{C}} \subset \mathfrak{B}^i_{r_i,c}\}$. Thus, using the restriction property and Lemmas 2.5 and 2.8, we obtain (2.37) as follows:

$$\begin{aligned}
&\sum\nolimits_{y\in\partial\mathcal{B}(r_i)} \mathbb{P}\big(\mathsf{A}\cap\{x\xleftrightarrow{(D)} y\}\cap\{x\xleftrightarrow{(D)}\widetilde{\partial}\mathfrak{B}^i_{r_i,c}\}^c\big)\\
=&\mathbb{E}\big[\mathbb{1}_{\mathsf{A}}\cdot\sum\nolimits_{y\in\partial\mathcal{B}(r_i)}\mathbb{P}\big(x\xleftrightarrow{(D\cup\overline{\mathcal{C}})} y,\{x\xleftrightarrow{(D\cup\overline{\mathcal{C}})}\widetilde{\partial}\mathfrak{B}^i_{r_i,c}\}^c\mid\mathcal{F}_{\overline{\mathcal{C}}}\big)\big]\\
\overset{(2.25)}{\gtrsim}&\,\mathbb{P}(\mathsf{A})\cdot\sum\nolimits_{y\in\partial\mathcal{B}(r_i)}\mathbb{P}\big(x\xleftrightarrow{(D\cup\widetilde{\partial}\mathfrak{B}^i_{r_i,\sqrt{c\cdot c'}})} y\big)\overset{(\text{Lemma 2.8})}{\asymp}\mathbb{P}(\mathsf{A})\cdot\sum\nolimits_{y\in\partial\mathcal{B}(r_i)}\mathbb{P}\big(x\xleftrightarrow{(D)} y\big).
\end{aligned}$$

(2) Let $\overline{\mathcal{C}}'$ be the union of all relevant clusters in A', and note that $\mathsf{A}'\subset\{\overline{\mathcal{C}}\subset(\mathfrak{B}^{-1}_{r_{-1},c'}\setminus\mathfrak{B}^1_{r_1,c'})^c\}$. Therefore, similar to the proof in Item (1), we have

$$\begin{aligned}
&\sum_{x\in\partial\mathcal{B}(r_1),y\in\partial\mathcal{B}(r_2)}\mathbb{P}\big(\mathsf{A}'\cap\{x\xleftrightarrow{(D)} y\}\cap\{x\xleftrightarrow{(D)}\widetilde{\partial}\mathfrak{B}^1_{r_1,c}\cup\widetilde{\partial}\mathfrak{B}^{-1}_{r_{-1},c}\}^c\big)\\
=&\mathbb{E}\Big[\mathbb{1}_{\mathsf{A}'}\cdot\sum_{x\in\partial\mathcal{B}(r_1),y\in\partial\mathcal{B}(r_2)}\mathbb{P}\big(x\xleftrightarrow{(D\cup\overline{\mathcal{C}}')} y,\{x\xleftrightarrow{(D\cup\overline{\mathcal{C}}')}\widetilde{\partial}\mathfrak{B}^1_{r_1,c}\cup\widetilde{\partial}\mathfrak{B}^{-1}_{r_{-1},c}\}^c\mid\mathcal{F}_{\overline{\mathcal{C}}'}\big)\Big]\\
\overset{(2.26)}{\gtrsim}&\,\mathbb{P}(\mathsf{A}')\cdot\sum_{x\in\partial\mathcal{B}(r_1),y\in\partial\mathcal{B}(r_2)}\mathbb{P}\big(x\xleftrightarrow{(D\cup\widetilde{\partial}\mathfrak{B}^1_{r_1,\sqrt{c\cdot c'}}\cup\widetilde{\partial}\mathfrak{B}^{-1}_{r_{-1},\sqrt{c\cdot c'}})} y\big).
\end{aligned}$$

Meanwhile, by applying Lemma 2.8 twice, one has

$$\sum_{x\in\partial\mathcal{B}(r_1),y\in\partial\mathcal{B}(r_2)}\mathbb{P}\big(x\xleftrightarrow{(D\cup\widetilde{\partial}\mathfrak{B}^1_{r_1,\sqrt{c\cdot c'}}\cup\widetilde{\partial}\mathfrak{B}^{-1}_{r_{-1},\sqrt{c\cdot c'}})} y\big)\asymp\sum_{x\in\partial\mathcal{B}(r_1),y\in\partial\mathcal{B}(r_2)}\mathbb{P}\big(x\xleftrightarrow{(D)} y\big).$$

Plugging this into the last inequality, we obtain (2.38). □

The following result relates the escape probability to the two-point event.

Corollary 2.10. *For any $3\le d\le 5$, there exist $C,c>0$ such that for any $R\ge 1$, $v\in\widetilde{B}(cR)$ and $D\subset\widetilde{\mathbb{Z}}^d$,*

$$\widetilde{\mathbb{P}}_v\big(\tau_{\partial B(R)}<\tau_D\big)\lesssim R^{-1}\sqrt{\widetilde{G}_D(v,v)}\sum\nolimits_{z\in\partial\mathcal{B}(\frac{R}{10})}\mathbb{P}\big(v\xleftrightarrow{(D)} z,\{v\xleftrightarrow{(D)}\partial B(CR)\}^c\big).$$

Proof. By the last-exit decomposition, we have

$$\begin{aligned}
&\widetilde{\mathbb{P}}_v\big(\tau_{\partial B(R)}<\tau_D\big)\\
\lesssim&\sum\nolimits_{z\in\partial\mathcal{B}(\frac{R}{10})}\widetilde{G}_D(v,z)\cdot\sum\nolimits_{z'\in[\mathcal{B}(\frac{R}{10})]^c:\{z,z'\}\in\mathbb{L}^d}\widetilde{\mathbb{P}}_{z'}\big(\tau_{\partial B(R)}<\tau_{D\cup\mathcal{B}(\frac{R}{10})}\big).
\end{aligned}\tag{2.39}$$

Using the potential theory of Brownian motion, one has

$$\widetilde{\mathbb{P}}_{z'}\big(\tau_{\partial B(R)}<\tau_{D\cup\mathcal{B}(\frac{R}{10})}\big)\le\widetilde{\mathbb{P}}_{z'}\big(\tau_{\partial B(R)}<\tau_{\mathcal{B}(\frac{R}{10})}\big)\lesssim R^{-1}.\tag{2.40}$$

Moreover, by (2.12) and Lemmas 2.5 and 2.8, we have

$$\begin{aligned}
\sum_{z\in\partial\mathcal{B}(\frac{R}{10})} \widetilde{G}_D(v,z) &\overset{(2.12)}{\lesssim} \sqrt{\widetilde{G}_D(v,v)} \sum_{z\in\partial\mathcal{B}(\frac{R}{10})} \mathbb{P}\big(v \xleftrightarrow{(D)} z\big) \\
&\overset{\text{(Lemma 2.8)}}{\lesssim} \sqrt{\widetilde{G}_D(v,v)} \sum_{z\in\partial\mathcal{B}(\frac{R}{10})} \mathbb{P}\big(v \xleftrightarrow{(D\cup\widetilde{\partial}\mathfrak{B}^{-1}_{R,\sqrt{c}})} z\big) \qquad (2.41)\\
&\overset{\text{(Lemma 2.5)}}{\lesssim} \sqrt{\widetilde{G}_D(v,v)} \sum_{z\in\partial\mathcal{B}(\frac{R}{10})} \mathbb{P}\big(v \xleftrightarrow{(D)} z, \big\{v \xleftrightarrow{(D)} \partial B(CR)\big\}^c\big).
\end{aligned}$$

Plugging (2.40) and (2.41) into (2.39), we complete the proof. □

By Lemma 2.4 and Corollary 2.10, we may deduce the following decomposition.

Corollary 2.11. *For any $3 \le d \le 5$, there exists $c > 0$ such that for any $D \subset \widetilde{\mathbb{Z}}^d$, $R \ge 1$, and $v_1, v_2 \in \widetilde{\mathbb{Z}}^d$ with $|v_1 - v_2| \ge R$,*

$$\mathbb{P}\big(v_1 \xleftrightarrow{(D)} v_2\big) \lesssim R^{-d} \prod_{i\in\{1,2\}} \sum_{z_i\in\partial\mathcal{B}_{v_i}(cR)} \mathbb{P}\big(v_i \xleftrightarrow{(D)} z_i, \big\{v_i \xleftrightarrow{(D)} \partial B_{v_i}(\tfrac{R}{100})\big\}^c\big). \qquad (2.42)$$

Proof. Applying (2.12), Lemma 2.4 and Corollary 2.10, we have

$$\begin{aligned}
\mathbb{P}\big(v_1 \xleftrightarrow{(D)} v_2\big) &\overset{(2.12)}{\asymp} \widetilde{G}_D(v_1,v_2) \prod\nolimits_{i\in\{1,2\}} \big[\widetilde{G}_D(v_i,v_i)\big]^{-\frac{1}{2}} \\
&\overset{\text{(Lemma 2.4)}}{\lesssim} R^{2-d} \prod\nolimits_{i\in\{1,2\}} \big[\widetilde{G}_D(v_i,v_i)\big]^{-\frac{1}{2}} \cdot \widetilde{\mathbb{P}}_{v_i}\big(\tau_{\partial\mathcal{B}_{v_i}(c'R)} < \tau_D\big) \\
&\overset{\text{(Corollary 2.10)}}{\lesssim} R^{-d} \prod_{i\in\{1,2\}} \sum_{z_i\in\partial\mathcal{B}_{v_i}(cR)} \mathbb{P}\big(v_i \xleftrightarrow{(D)} z_i, \big\{v_i \xleftrightarrow{(D)} \partial B_{v_i}(\tfrac{R}{100})\big\}^c\big). \qquad \square
\end{aligned}$$

Following similar arguments as in the proof of Lemma 2.3, we may derive the mean value property of connecting probabilities as follows.

Corollary 2.12. *For any $d \ge 3$, there exists $c > 0$ such that for any $D \subset \widetilde{\mathbb{Z}}^d$, $R \ge 1$, $v \in \widetilde{B}^1_{R,c}$ and $w \in \widetilde{B}^{-1}_{R,c}$,*

$$\mathbb{P}\big(v \xleftrightarrow{(D)} w\big) \lesssim |w|^{2-d} R^{-1} \cdot \sum\nolimits_{x\in\partial\mathcal{B}(R)} \mathbb{P}\big(v \xleftrightarrow{(D)} x\big). \qquad (2.43)$$

Proof. Recall from (2.12) that

$$\mathbb{P}\big(v \xleftrightarrow{(D)} w\big) \asymp \widetilde{\mathbb{P}}_w\big(\tau_v < \tau_D\big) \cdot \sqrt{\tfrac{\widetilde{G}_D(v,v)}{\widetilde{G}_D(w,w)}}. \qquad (2.44)$$

By the strong Markov property of Brownian motion, one has

$$
\begin{aligned}
\widetilde{\mathbb{P}}_w\big(\tau_v<\tau_D\big) \leq & \widetilde{\mathbb{P}}_w\big(\tau_{\partial B_w(2)}<\tau_D\big)\cdot \max_{z\in\partial B_w(2)} \widetilde{\mathbb{P}}_z\big(\tau_{B(dR)}<\infty\big) \\
& \cdot \max_{z'\in \partial B(dR)} \sum_{x\in\partial \mathcal{B}(R)} \widetilde{\mathbb{P}}_{z'}\big(\tau_{\partial\mathcal{B}(R)}=\tau_x<\infty\big)\cdot \widetilde{\mathbb{P}}_x\big(\tau_v<\tau_D\big).
\end{aligned} \tag{2.45}
$$

The probabilities on the right-hand side can be estimated as follows. Firstly, using the formulas [6, (3.10)] and (2.10) in turn, one has

$$
\widetilde{\mathbb{P}}_w\big(\tau_{\partial B_w(2)}<\tau_D\big) \lesssim \mathrm{dist}(w,D)\wedge 1 \asymp \widetilde{G}_D(w,w). \tag{2.46}
$$

Secondly, by (2.22) and (2.23), we have: for any $z\in\partial B_w(2)$,

$$
\widetilde{\mathbb{P}}_z\big(\tau_{B(dR)}<\infty\big) \asymp \big(\tfrac{R}{|w|}\big)^{d-2}. \tag{2.47}
$$

Thirdly, it follows from Lemma 2.1 that for any $z'\in\partial B(dR)$ and $x\in\partial\mathcal{B}(R)$,

$$
\widetilde{\mathbb{P}}_{z'}\big(\tau_{\partial\mathcal{B}(R)}=\tau_x<\infty\big) \lesssim R^{1-d}. \tag{2.48}
$$

Moreover, the formula (2.12) together with $\widetilde{G}(x,x)\lesssim 1$ implies that

$$
\widetilde{\mathbb{P}}_x\big(\tau_v<\tau_D\big) \lesssim \mathbb{P}\big(v \xleftrightarrow{(D)} x\big)\cdot \big[\widetilde{G}_D(v,v)\big]^{-\frac{1}{2}}. \tag{2.49}
$$

Plugging (2.46)-(2.49) into (2.45), we get

$$
\widetilde{\mathbb{P}}_w\big(\tau_v<\tau_D\big) \lesssim |w|^{2-d}R^{-1}\cdot \tfrac{\widetilde{G}_D(w,w)}{\sqrt{\widetilde{G}_D(v,v)}}\cdot \textstyle\sum_{x\in\partial\mathcal{B}(R)} \mathbb{P}\big(v \xleftrightarrow{(D)} x\big). \tag{2.50}
$$

Combining (2.44), (2.50) and $\widetilde{G}(w,w)\lesssim 1$, we complete the proof. $\square$

The following relation between the point-to-set and boundary-to-set connecting probabilities has been verified in [5, (5.2)].

Lemma 2.13. *For any $d\geq 3$ with $d\neq 6$, $D\subset\widetilde{\mathbb{Z}}^d$, $R\geq 1$, $i\in\{1,-1\}$, and $A\subset \mathfrak{B}^i_{R,(10d^2)^{-1}}$,*

$$
\mathbb{P}\big(A \xleftrightarrow{(D)} \partial B(R)\big) \lesssim R^{-(\frac{d}{2}\square 3)} \textstyle\sum_{x\in\partial\mathcal{B}(d^{-i}R)} \mathbb{P}\big(A \xleftrightarrow{(D)} x\big). \tag{2.51}
$$

Combining Lemmas 2.5, 2.8 and 2.13, we obtain the following estimate.

Corollary 2.14. *For any $3\leq d\leq 5$, there exist $C,c>0$ such that for any $D\subset\widetilde{\mathbb{Z}}^d$, $R\geq 1$, $v\in[\widetilde{B}(CR)]^c$,*

$$
\mathbb{P}\big(v \xleftrightarrow{(D)} \partial B(R)\big) \lesssim R^{-\frac{d}{2}} \textstyle\sum_{z\in\partial\mathcal{B}(dR)} \mathbb{P}\big(v \xleftrightarrow{(D)} z, \big\{v \xleftrightarrow{(D)} \partial B(cR)\big\}^c\big). \tag{2.52}
$$

Proof. Using Lemmas 2.5, 2.8 and 2.13, we have

$$\begin{aligned}\mathbb{P}\big(v \xleftrightarrow{(D)} \partial B(R)\big) &\overset{\text{(Lemma 2.13)}}{\lesssim} R^{-\frac{d}{2}} \sum\nolimits_{z\in\partial\mathcal{B}(dR)} \mathbb{P}\big(v \xleftrightarrow{(D)} z\big)\\ &\overset{\text{(Lemma 2.8)}}{\lesssim} R^{-\frac{d}{2}} \sum\nolimits_{z\in\partial\mathcal{B}(dR)} \mathbb{P}\big(v \xleftrightarrow{(D\cup\partial B(c'R))} z\big)\\ &\overset{\text{(Lemma 2.5)}}{\lesssim} R^{-\frac{d}{2}} \sum\nolimits_{z\in\partial\mathcal{B}(dR)} \mathbb{P}\big(v \xleftrightarrow{(D)} z, \{v \xleftrightarrow{(D)} \partial B(cR)\}^c\big). \quad \square\end{aligned}$$

We also cite the following result on the scaling of point-to-set connecting probabilities (see [5, Proposition 1.5]).

Lemma 2.15. *For any $d \geq 3$ with $d \neq 6$, there exists $c > 0$ such that for any $R \geq 1$, $A, D \subset \widetilde{B}(cR)$ and $x_1, x_2 \subset [\widetilde{B}(R)]^c$,*

$$\mathbb{P}\big(A \xleftrightarrow{(D)} x_1\big) \cdot |x_1|^{d-2} \asymp \mathbb{P}\big(A \xleftrightarrow{(D)} x_2\big) \cdot |x_2|^{d-2}. \tag{2.53}$$

Combining Lemmas 2.13 and 2.15, one may obtain the following bound:

Lemma 2.16. *For any $d \geq 3$ with $d \neq 6$, there exists $c > 0$ such that for any $R \geq 1$, $A, D \subset \widetilde{B}(cR)$ and $x \in [B(R)]^c$,*

$$\mathbb{P}\big(A \xleftrightarrow{(D)} \partial B(R)\big) \lesssim R^{-[(\frac{d}{2}-1)\square 2]}|x|^{d-2} \cdot \mathbb{P}\big(A \xleftrightarrow{(D)} x\big). \tag{2.54}$$

In the analysis of loop clusters, decomposing connection probabilities is a fundamental technique. A key property in this context is quasi-multiplicativity (see [5, Theorem 1.1]), which was instrumental in the derivation of the IIC.

Lemma 2.17 (quasi-multiplicativity)**.** *For any $d \geq 3$ with $d \neq 6$, there exist $c, C > 0$ such that for any $R \geq 1$, $A_1, D_1 \subset \widetilde{B}(cR^{1\square\frac{2}{d-4}})$ and $A_{-1}, D_{-1} \subset \big[\widetilde{B}(CR^{1\square\frac{d-4}{2}})\big]^c$,*

$$\mathbb{P}\big(A_1 \xleftrightarrow{(D_1\cup D_{-1})} A_{-1}\big) \asymp R^{0\square(6-d)} \prod\nolimits_{i\in\{1,-1\}} \mathbb{P}\big(A_i \xleftrightarrow{(D_i)} \partial B(N)\big). \tag{2.55}$$

As a supplement, we present the following decomposition on point-to-set connecting probabilities. To maintain the flow, we defer its proof to Appendix B.

Lemma 2.18. *For any $d \geq 3$ with $d \neq 6$, there exist $c, C > 0$ such that for any $R \geq 1$, $x \in \partial B(R)$, $y \in \mathfrak{B}^{-1}_{R,c}$, and any $A \subset \mathfrak{B}^1_{R,c}$ and $D \subset \mathfrak{B}^1_{R,c} \cup \mathfrak{B}^{-1}_{R,c}$,*

$$\mathbb{P}\big(A \xleftrightarrow{(D)} y\big) \gtrsim R^{d-2} \cdot \mathbb{P}\big(A \xleftrightarrow{(D)} x\big) \cdot \Big[\mathbb{P}\big(x \xleftrightarrow{(D)} y\big) - \tfrac{C[\mathrm{diam}(A)]^{d-4}}{R^{d-4}|y|^{2-d}} \cdot \mathbb{1}_{d\geq 7}\Big]. \tag{2.56}$$

The subsequent corollary provides a decomposition for two-point events. Unlike in quasi-multiplicativity, it bounds the two-point probability with that of a pairwise connecting event formed from two two-point events.

Corollary 2.19. *For any $3 \le d \le 5$, there exist $C > 0$ and $c > c' > c'' > 0$ such that for any $D \subset \widetilde{\mathbb{Z}}^d$, $R \ge C$, $R_1 \le c''R$, $R_{-1} \ge \frac{R}{c''}$, and $w_{-1} \in \partial \mathcal{B}(R_{-1})$,*

$$
\begin{aligned}
&\sum\nolimits_{w_1 \in \partial \mathcal{B}(R_1)} \mathbb{P}\big(w_1 \xleftrightarrow{(D)} w_{-1}\big) \\
\lesssim & R^{-d} \sum_{w_1 \in \partial \mathcal{B}(R_1), v_1 \in \widetilde{\partial} \mathfrak{B}^1_{R,c'}, v_{-1} \in \widetilde{\partial} \mathfrak{B}^{-1}_{R,c'}} \mathbb{P}\big(v_1 \xleftrightarrow{(D)} w_1, v_{-1} \xleftrightarrow{(D)} w_{-1}, \\
& \qquad \big\{v_1 \xleftrightarrow{(D)} \widetilde{\partial} \mathfrak{B}^1_{R,c} \cup \widetilde{\partial} \mathfrak{B}^1_{R_1,c}\big\}^c, \big\{v_{-1} \xleftrightarrow{(D)} \widetilde{\partial} \mathfrak{B}^{-1}_{R,c}\big\}^c\big).
\end{aligned}
\tag{2.57}
$$

Proof. For any $w_1 \in \partial \mathcal{B}(R_1)$ and $w_{-1} \in \partial \mathcal{B}(R_{-1})$, by (2.12) one has

$$
\mathbb{P}\big(w_1 \xleftrightarrow{(D)} w_{-1}\big) \asymp \widetilde{\mathbb{P}}_{w_1}\big(\tau_{w_{-1}} < \tau_D\big) \cdot \sqrt{\tfrac{\widetilde{G}_D(w_{-1}, w_{-1})}{\widetilde{G}_D(w_1, w_1)}}. \tag{2.58}
$$

By the last-exit decomposition and the strong Markov property of Brownian motion, the probability $\widetilde{\mathbb{P}}_{w_1}\big(\tau_{w_{-1}} < \tau_D\big)$ is upper-bounded by

$$
\begin{aligned}
& C \sum_{v_1 \in \widetilde{\partial} \mathfrak{B}^1_{R,c'}} \widetilde{G}_D(w_1, v_1) \sum_{v_1' \in [\mathfrak{B}^1_{R,c'}]^c : \{v_1, v_1'\} \in \mathbb{L}^d} \widetilde{\mathbb{P}}_{v_1'}\big(\tau_{\partial B(R)} < \infty\big) \\
& \cdot \max_{z \in \partial B(R)} \sum\nolimits_{v_{-1} \in \widetilde{\partial} \mathfrak{B}^{-1}_{R,c'}} \widetilde{\mathbb{P}}_z\big(\tau_{\widetilde{\partial} \mathfrak{B}^{-1}_{R,c'}} = \tau_{v_{-1}} < \infty\big) \cdot \widetilde{\mathbb{P}}_{v_{-1}}\big(\tau_{w_{-1}} < \tau_D\big).
\end{aligned}
\tag{2.59}
$$

Moreover, using the potential theory, we have

$$
\widetilde{\mathbb{P}}_{v_1'}\big(\tau_{\widetilde{\partial} \mathfrak{B}^1_{R,4c'}} < \infty\big) \lesssim R^{-1}. \tag{2.60}
$$

In addition, Lemma 2.1 implies that for any $z \in \partial B(R)$ and $v_{-1} \in \widetilde{\partial} \mathfrak{B}^{-1}_{R,c'}$,

$$
\widetilde{\mathbb{P}}_z\big(\tau_{\widetilde{\partial} \mathfrak{B}^{-1}_{R,c'}} = \tau_{v_{-1}} < \infty\big) \lesssim R^{1-d}. \tag{2.61}
$$

Plugging (2.60) and (2.61) into (2.59), and then combining with (2.58), we have

$$
\begin{aligned}
\mathbb{P}\big(w_1 \xleftrightarrow{(D)} w_{-1}\big) \lesssim & R^{-d} \sum_{v_1 \in \widetilde{\partial} \mathfrak{B}^1_{R,c'}, v_{-1} \in \widetilde{\partial} \mathfrak{B}^{-1}_{R,c'}} \widetilde{G}_D(w_1, v_1) \widetilde{\mathbb{P}}_{v_{-1}}\big(\tau_{w_{-1}} < \tau_D\big) \sqrt{\tfrac{\widetilde{G}_D(w_{-1}, w_{-1})}{\widetilde{G}_D(w_1, w_1)}} \\
& \overset{(2.58)}{\lesssim} R^{-d} \prod\nolimits_{i \in \{1,-1\}} \sum\nolimits_{v_1 \in \widetilde{\partial} \mathfrak{B}^i_{R,c'}} \mathbb{P}\big(w_i \xleftrightarrow{(D)} v_i\big).
\end{aligned}
$$

Summing over $w_1 \in \partial \mathcal{B}(R_1)$, and using Lemmas 2.5 and 2.8, we obtain

$$
\begin{aligned}
&\sum\nolimits_{w_1 \in \partial \mathcal{B}(R_1)} \mathbb{P}\big(w_1 \xleftrightarrow{(D)} w_{-1}\big) \\
\lesssim & R^{-d} \sum\nolimits_{w_1 \in \partial \mathcal{B}(R_1), v_1 \in \widetilde{\partial} \mathfrak{B}^1_{R,c'}} \mathbb{P}\big(w_1 \xleftrightarrow{(D)} v_1\big) \cdot \sum\nolimits_{v_{-1} \in \widetilde{\partial} \mathfrak{B}^{-1}_{R,c'}} \mathbb{P}\big(w_{-1} \xleftrightarrow{(D)} v_{-1}\big) \\
\overset{\text{(Lemma 2.8)}}{\lesssim} & R^{-d} \sum\nolimits_{w_1 \in \partial \mathcal{B}(R_1), v_1 \in \widetilde{\partial} \mathfrak{B}^1_{R,c'}} \mathbb{P}\big(w_1 \xleftrightarrow{(D \cup \widetilde{\partial} \mathfrak{B}^1_{R,\sqrt{c \cdot c'}} \cup \widetilde{\partial} \mathfrak{B}^1_{R_1,\sqrt{c \cdot c'}})} v_1\big) \\
& \cdot \sum\nolimits_{v_{-1} \in \widetilde{\partial} \mathfrak{B}^{-1}_{R,c'}} \mathbb{P}\big(w_{-1} \xleftrightarrow{(D \cup \widetilde{\partial} \mathfrak{B}^{-1}_{R,\sqrt{c \cdot c'}})} v_{-1}\big) \\
\overset{\text{(Lemma 2.5)}}{\lesssim} & R^{-d} \sum\nolimits_{w_1 \in \partial \mathcal{B}(R_1), v_1 \in \widetilde{\partial} \mathfrak{B}^1_{R,c'}} \mathbb{P}\big(v_1 \xleftrightarrow{(D)} w_1, \big\{v_1 \xleftrightarrow{(D)} \widetilde{\partial} \mathfrak{B}^1_{R,c} \cup \widetilde{\partial} \mathfrak{B}^1_{R_1,c}\big\}^c\big) \\
& \cdot \sum\nolimits_{v_{-1} \in \widetilde{\partial} \mathfrak{B}^{-1}_{R,c'}} \mathbb{P}\big(v_{-1} \xleftrightarrow{(D)} w_{-1}, \big\{v_{-1} \xleftrightarrow{(D)} \widetilde{\partial} \mathfrak{B}^{-1}_{R,c}\big\}^c\big).
\end{aligned}
$$

Combined with Corollary 2.9, it implies the desired bound (2.57). □

2.4. **Brownian excursion measure.** We arbitrarily take $D \subset \widetilde{\mathbb{Z}}^d$. As pointed out in [34], the loops of $\widetilde{\mathcal{L}}^D_{1/2}$ intersecting a fixed point can be equivalently described by a Poisson point process of excursions. Specifically, for any $v \in \widetilde{\mathbb{Z}}^d$, rooted loop $\widetilde{\varrho}$ with $\widetilde{\varrho}(0) = \widetilde{\varrho}(T) = v$ (where T is the duration of $\widetilde{\ell}$) and $0 < t < T$, define

$$\delta^-_t = \delta^-_t(\widetilde{\varrho}, v) := \sup\{s \le t : \widetilde{\varrho}(s) = v\}, \tag{2.62}$$

$$\delta^+_t = \delta^+_t(\widetilde{\varrho}, v) := \inf\{s \ge t : \widetilde{\varrho}(s) = v\}. \tag{2.63}$$

Clearly, $\widetilde{\varrho}$ consists of the excursions $\{\widetilde{\varrho}[\delta^-_t, \delta^+_t] : 0 < t < T : \delta^-_t < \delta^+_t\}$. Let $\mathscr{E}_v(\widetilde{\varrho})$ denote the point measure supported on these excursions. Note that $\mathscr{E}_v(\widetilde{\varrho}) = \mathscr{E}_v(\widetilde{\varrho}')$ when $\widetilde{\varrho}$ and $\widetilde{\varrho}'$ are equivalent. Hence, for any loop $\widetilde{\ell}$ intersecting v, we define $\mathscr{E}_v(\widetilde{\ell})$ as $\mathscr{E}_v(\widetilde{\varrho})$ for any $\widetilde{\varrho} \in \widetilde{\ell}$ with $\widetilde{\varrho}(0) = \widetilde{\varrho}(T) = v$. For completeness, we set $\mathscr{E}_v(\widetilde{\ell}) := 0$ when $\widetilde{\ell}$ is disjoint from v.

For any $D \subset \widetilde{\mathbb{Z}}^d$, $v \in \widetilde{\mathbb{Z}}^d \setminus D$ and $a > 0$, conditioned on the event $\big\{\widehat{\mathcal{L}}^{D,v}_{1/2} = a\big\}$, the point measure $\widetilde{\mathcal{E}}^D_v := \sum_{\widetilde{\ell} \in \widetilde{\mathcal{L}}^D_{1/2}} \mathscr{E}_v(\widetilde{\ell})$ is a Poisson point process, whose intensity measure $\mathbf{e}^D_{v,v}$ (commonly referred to as the "Brownian excursion measure") is a σ-finite measure on the space of excursions from v. (**P.S.** As in the case of "Itô's excursion measure", $\mathbf{e}^D_{v,v}$ lacks a direct and simple definition.) In addition, for any $v \neq w \in \widetilde{\mathbb{Z}}^d$, we may restrict $\mathbf{e}^D_{v,v}$ to excursions that hit w and for each excursion we only keep the part up to the first time it reaches w, we then obtain the measure $\mathbf{e}^D_{v,w}$ on the space of excursions from v to w. According to [6, Lemma 3.14], the total mass of $\mathbf{e}^D_{v,w}$ exactly equals the boundary excursion kernel $\mathbb{K}_{D \cup \{v,w\}}(v, w)$.

The Brownian excursion measure $\mathbf{e}^D_{v,w}$ (for both $v = w$ and $v \neq w$) possess the strong Markov property, which greatly facilitates related computations. Precisely, for any $A \subset \widetilde{\mathbb{Z}}^d \setminus (D \cup \{v, w\})$, under the measure $\mathbf{e}^D_{v,w}$, given the configuration of an excursion $\widetilde{\eta}$ up to the first time it hits A (suppose that the first hitting point is $z \in \widetilde{\partial} A$), the remaining part of $\widetilde{\eta}$ is distributed by $\widetilde{\mathbb{P}}_z\big(\{\widetilde{S}_t\}_{0\le t\le \tau_w} \in \cdot \mid \tau_w < \tau_{D\cup\{v\}}\big)$.

The following lemma, as a strengthened version of [6, Lemma 3.17], provides an upper bound on the total mass of excursions that reach a distant location.

Lemma 2.20. *For any $3 \le d \le 5$, there exist $C > 0$, $C' > C^2$ and $c > 0$ such that the following holds. Suppose that $r \ge 1$, $R \ge C'r$, $\{\{x_i, y_i\}\}_{i\in\{1,2\}} \subset \mathbb{L}^d$, $v_i \in I_{\{x_i,y_i\}}$ satisfying $|v_i - x_i| \wedge |v_i - y_i| \ge \frac{1}{10}$ for $i \in \{1, 2\}$, $D \subset \widetilde{\mathbb{Z}}^d$ satisfying $x_i \in D$ and $D \cap I_{[v_i,y_i]} = \emptyset$ for $i \in \{1, 2\}$, $v_i' \in I_{[v_i,y_i]}$ satisfying $|v_i' - v_i| \wedge |v_i' - y_i| \ge \frac{1}{100}$ for $i \in \{1, 2\}$, and $A \subset [\widetilde{B}(R)]^c$. Then we have*

$$\begin{aligned}
&\mathbf{e}^D_{v_1,v_2}\big(\{\widetilde{\eta} : \mathrm{ran}(\widetilde{\eta}) \cap A \neq \emptyset\}\big)\\
\lesssim& r^{1-d} R^{1-d} \cdot \sum\nolimits_{z\in\partial\mathcal{B}(Cr),z'\in\partial\mathcal{B}(\frac{R}{C})} \mathbb{P}\big(z \xleftrightarrow{(D)} z', \{z \xleftrightarrow{(D)} \partial B(10r) \cup \partial B(\tfrac{R}{10})\}^c\big) \qquad (2.64)\\
&\cdot \mathbf{Q}(v_1', v_2') \cdot \max_{z''\in\partial\mathcal{B}(\frac{R}{C})} \widetilde{\mathbb{P}}_{z''}\big(\tau_A < \tau_D\big).
\end{aligned}$$

Here the quantity $\mathbf{Q}(v_1', v_2')$ is defined as follows:

- *when $\chi := |v_1' - v_2'| \le 10c^{-1}$, $\mathbf{Q}(v_1', v_2') := \widetilde{\mathbb{P}}_{v_1'}\big(\tau_{\partial B(r)} < \tau_D\big)$;*
- *when $\chi > 10c^{-1}$, we define*

$$\mathbf{Q}(v_1', v_2') := \chi^{-2} \prod\nolimits_{i\in\{1,2\}} \sum\nolimits_{z_i\in\partial\mathcal{B}_{v_i'}(c\chi)} \mathbb{P}\big(v_i' \xleftrightarrow{(D)} z_i, \{v_i' \xleftrightarrow{(D)} \partial B_{v_i'}(\tfrac{\chi}{100})\}^c\big). \qquad (2.65)$$

Proof. For $i \in \{1, 2\}$, since $x_i \in D$, all excursions $\widetilde{\eta}$ starting from v_i and reaching A before D must intersect v_i'. Thus, by the strong Markov property of $\mathbf{e}^D_{v_1,v_2}$,

$$\mathbf{e}^D_{v_1,v_2}\big(\{\widetilde{\eta} : \mathrm{ran}(\widetilde{\eta}) \cap A \neq \emptyset\}\big) \asymp \widetilde{\mathbb{P}}_{v_1'}\big(\tau_A < \tau_{v_2'} < \tau_D\big). \qquad (2.66)$$

Define $\mathbf{R}_1 := \widetilde{\mathbb{P}}_{v_1'}\big(\tau_{\partial B(r)} < \tau_D\big)$ and $\mathbf{R}_2 := 1$ when $\chi \le 10c^{-1}$, and define $\mathbf{R}_i := \widetilde{\mathbb{P}}_{v_i'}\big(\tau_{\partial\mathcal{B}_{v_i'}(10c\chi)} < \tau_D\big)$ for $i \in \{1, 2\}$ when $\chi \le 10c^{-1}$. Corollary 2.10 implies that

$$\mathbf{R}_1\mathbf{R}_2 \lesssim \mathbf{Q}(v_1', v_2'). \qquad (2.67)$$

Using the strong Markov property of Brownian motion repeatedly, we have

$$\begin{aligned}
&\widetilde{\mathbb{P}}_{v_1'}\big(\tau_A < \tau_{v_2} < \tau_D\big)\\
\le& \mathbf{R}_1 \cdot \max_{w\in B(r)} \sum\nolimits_{z\in\partial B(Cr)} \widetilde{\mathbb{P}}_w\big(\tau_{\partial B(Cr)} = \tau_z < \tau_D\big) \cdot \widetilde{\mathbb{P}}_z\big(\tau_{\partial B(\frac{10R}{C})} < \tau_D\big) \qquad (2.68)\\
&\cdot \max_{w'\in\partial B(\frac{10R}{C})} \widetilde{\mathbb{P}}_{w'}\big(\tau_A < \tau_D\big) \cdot \max_{w''\in A} \widetilde{\mathbb{P}}_{w''}\big(\tau_{v_2} < \tau_D\big).
\end{aligned}$$

Next, we estimate the probabilities on the right-hand side separately. Firstly, by Lemma 2.1 one has: for any $w \in B(r)$ and $z \in \partial B(Cr)$,

$$\widetilde{\mathbb{P}}_w\big(\tau_{\partial B(Cr)} = \tau_z < \tau_D\big) \le \widetilde{\mathbb{P}}_w\big(\tau_{\partial B(Cr)} = \tau_z < \infty\big) \asymp r^{1-d}. \tag{2.69}$$

Secondly, applying Lemma 2.8 and Corollary 2.10, we have

$$\begin{aligned}
&\sum\nolimits_{z\in\partial B(Cr)} \widetilde{\mathbb{P}}_z\big(\tau_{\partial B(\frac{10R}{C})} < \tau_D\big)\\
\overset{\text{(Corollary 2.10)}}{\lesssim}\ & \frac{1}{R} \sum_{z\in\partial B(Cr), z'\in\partial\mathcal{B}(\frac{R}{C})} \mathbb{P}\big(z \xleftrightarrow{(D)} z', \{z \xleftrightarrow{(D)} \partial B(\tfrac{R}{10})\}^c\big)\\
\overset{\text{(Lemma 2.8)}}{\lesssim}\ & \frac{1}{R} \sum_{z\in\partial B(Cr), z'\in\partial\mathcal{B}(\frac{R}{C})} \mathbb{P}\big(z \xleftrightarrow{(D)} z', \{z \xleftrightarrow{(D)} \partial B(10r)\cup\partial B(\tfrac{R}{10})\}^c\big).
\end{aligned} \tag{2.70}$$

Thirdly, for any $w'' \in A \subset [\widetilde{B}(R)]^c$, in the case when $\chi \le 10c^{-1}$, one has

$$\widetilde{\mathbb{P}}_{w''}\big(\tau_{v_2'} < \tau_D\big) \lesssim |w'' - v_2'|^{2-d} \lesssim R^{2-d}. \tag{2.71}$$

In the case when $\chi > 10c^{-1}$, by the last-exit decomposition, one has

$$\begin{aligned}
\widetilde{\mathbb{P}}_{w''}\big(\tau_{v_2'} < \tau_D\big) \lesssim & \sum\nolimits_{y\in\partial\mathcal{B}_{v_2'}(10c\chi)} \widetilde{G}_D(w'', y)\\
& \cdot \sum\nolimits_{y'\in\mathcal{B}_{v_2'}(10c\chi):\{y,y'\}\in\mathbb{L}^d} \widetilde{\mathbb{P}}_{y'}\big(\tau_{v_2'} < \tau_{\partial\mathcal{B}_{v_2'}(10c\chi)} < \tau_D\big).
\end{aligned} \tag{2.72}$$

Note that for any $w'' \in A$ and $y \in \partial\mathcal{B}_{v_2'}(10c\chi)$, one has $|w'' - y| \gtrsim R$ and hence,

$$\widetilde{G}_D(w'', y) \lesssim R^{2-d}. \tag{2.73}$$

Moreover, by reversing the Brownian motion and using $\widetilde{G}_D(v_2', v_2') \asymp 1$ (since $|v_2' - v_2| \wedge |v_2' - y_2| \ge \frac{1}{100}$), we have: for any $y' \in \mathcal{B}_{v_2'}(10c\chi)$,

$$\begin{aligned}
&\widetilde{\mathbb{P}}_{y'}\big(\tau_{v_2'} < \partial\mathcal{B}_{v_2'}(10c\chi) < \tau_D\big)\\
\lesssim & \sum\nolimits_{y''\in\mathcal{B}_{v_2'}(10c\chi):\{y',y''\}\in\mathbb{L}^d} \widetilde{\mathbb{P}}_{v_2'}\big(\tau_{\partial\mathcal{B}_{v_2'}(10c\chi)} = \tau_{y''} < \tau_D\big).
\end{aligned} \tag{2.74}$$

Combining (2.71), (2.72), (2.73) and (2.74), we get

$$\widetilde{\mathbb{P}}_{w''}\big(\tau_{v_2'} < \tau_D\big) \lesssim R^{2-d} \cdot \mathbf{R}_2. \tag{2.75}$$

Plugging (2.69), (2.70) and (2.75) into (2.68), and then using (2.67), we obtain the desired bound (2.64). □

Switching identity. In what follows, we review the powerful tool "switching identity", which was introduced in [34]. Roughly speaking, it provides an explicit description for the loop cluster connecting two given points. We state the following version tailored to our needs in this paper.

Lemma 2.21 (switching identity). *For any $d \geq 3$, $D \subset \widetilde{\mathbb{Z}}^d$, $v \neq w \in \widetilde{\mathbb{Z}}^d \setminus D$, and $a, b > 0$, conditioned on the event $\big\{v \xleftrightarrow{(D)} w, \widehat{\mathcal{L}}^{D,v}_{1/2} = a, \widehat{\mathcal{L}}^{D,w}_{1/2} = b\big\}$, the occupation field $\big\{\widehat{\mathcal{L}}^{D,z}_{1/2}\big\}_{z \in \widetilde{\mathbb{Z}}^d}$ has the same distribution as the total local time of the following four independent components:*

(1) *loops in the loop soup $\widetilde{\mathcal{L}}^{D \cup \{v,w\}}_{1/2}$;*
(2) *a Poisson point process with intensity measure $a \cdot \mathbf{e}^{D \cup \{w\}}_{v,v}$;*
(3) *a Poisson point process with intensity measure $b \cdot \mathbf{e}^{D \cup \{v\}}_{w,w}$;*
(4) *a Poisson point process with intensity measure $\sqrt{ab} \cdot \mathbf{e}^{D}_{v,w}$, where the number of excursions is conditioned to be odd.*

In order to use this lemma, we fix the following notations as in [6].

- For each $i \in \{1, 2, 3, 4\}$, we denote by $\mathcal{P}^{(i)}$ the Poisson point process in Component (i).
- We denote by $\check{\mathbb{P}}^{D}_{v \leftrightarrow w, a, b}$ the probability measure generated by Components (1)-(4). The expectation under $\check{\mathbb{P}}^{D}_{v \leftrightarrow w, a, b}$ is denoted by $\check{\mathbb{E}}^{D}_{v \leftrightarrow w, a, b}$.
- We define $\check{\mathcal{C}} := \{z \in \widetilde{\mathbb{Z}}^d : z \xleftrightarrow{\cup(\sum_{1 \leq i \leq 4} \mathcal{P}^{(i)})} v\}$ as the cluster containing v and consisting of all loops and excursions in $\sum_{1 \leq i \leq 4} \mathcal{P}^{(i)}$.
- Let $\mathfrak{p}^{D}_{v \leftrightarrow w, a, b}$ denote the density of the event $\big\{v \xleftrightarrow{(D)} w, \widehat{\mathcal{L}}^{D,v}_{1/2} = a, \widehat{\mathcal{L}}^{D,w}_{1/2} = b\big\}$.

To facilitate the analysis of $\mathcal{P}^{(4)}$, we record the following fact. Let $\mathbf{X}$ be a Poisson random variable with parameter $0 < \lambda \lesssim 1$. In addition, assume that $\{\mathbf{q}_j\}_{j \in \mathbb{N}}$ is an independent family of i.i.d. Bernoulli random variables with expectation q. Let $\mathbf{Y} := \sum_{1 \leq j \leq \mathbf{X}} \mathbf{q}_j$. Then one has

$$
\begin{aligned}
\mathbb{P}\big(\mathbf{Y} > 0 \mid \mathbf{X} \text{ is odd}\big) =& 1 - \frac{\sum_{m \geq 1 : m \text{ odd}} e^{-\lambda} \cdot \frac{\lambda^m}{m!} \cdot (1-q)^m}{\sum_{m \geq 1 : m \text{ odd}} e^{-\lambda} \cdot \frac{\lambda^m}{m!}} \\
=& \frac{\sinh(\lambda) - \sinh\big(\lambda(1-q)\big)}{\sinh(\lambda)} \asymp q.
\end{aligned}
\tag{2.76}
$$

3. Proof of Theorem 1.1

In this section, we present the proof of Theorem 1.1 assuming Proposition 3.1, which, as introduced later, serves as a crucial estimate for controlling the second moment of touching points between two large clusters.

We begin with the upper bound in (1.14). When $\chi^{\frac{1}{2}} N^{(\frac{d}{2}-1) \boxdot (d-4)} \geq 1$ (i.e., $\chi \geq N^{-[(d-2) \boxdot (2d-8)]}$), this upper bound holds trivially. Therefore, it suffices to consider the case when $\chi \leq N^{-[(d-2) \boxdot (2d-8)]}$. By a standard covering argument, since $\widetilde{\mathbb{Z}}^d$ is locally one-dimensional, we may take a family of subsets $A_1, ..., A_K \subset \widetilde{B}(N)$ with diameter 2χ such that $K \lesssim \chi^{-1} N^d$, and that for any $v, w \in \widetilde{B}(N)$ with $|v - w| \leq \chi$, there exists $1 \leq i \leq K$ such that $v, w \in A_i$. Thus, on the event $\{\mathcal{D}_{N,c} \leq \chi\}$, some

A_i intersects two distinct sign clusters of diameter at least cN and hence, there exist two distinct points $v, v' \in \widetilde{\partial} A_i$ (note that $|\widetilde{\partial} A_i| \leq 2d$) that are connected to $\partial B_v(c'N)$ by two different sign clusters. By the isomorphism theorem and (1.16), the probability of this event is at most $C\chi^{\frac{3}{2}} N^{-[(\frac{d}{2}+1)\boxdot 4]}$. In conclusion, we obtain the upper bound in (1.14):

$$\mathbb{P}\big(\mathcal{D}_{N,c} \leq \chi\big) \lesssim \sum\nolimits_{1\leq i\leq K} |\widetilde{\partial} A_i|^2 \cdot \chi^{\frac{3}{2}} N^{-[(\frac{d}{2}+1)\boxdot 4]} \lesssim \chi^{\frac{1}{2}} N^{(\frac{d}{2}-1)\boxdot (d-4)}. \tag{3.1}$$

It remains to prove the lower bound in (1.14). Since the right-hand side of (1.14) equals 1 for all $\chi \geq N^{-[(d-2)\boxdot(2d-8)]}$, and $\mathbb{P}(\mathcal{D}_{N,c} \leq \chi)$ is increasing in χ, it suffices to consider the case $\chi \leq N^{-[(d-2)\boxdot(2d-8)]}$. We take a small constant $c_4 > 0$, and denote the boxes

$$\mathfrak{B}_j := B_{(2j\cdot c_4 N, 0, \ldots, 0)}(c_4^2 N), \quad \forall j \in \{-1, 0, 1\}. \tag{3.2}$$

We define $\mathfrak{I}$ as the collection of edges $e = \{x, y\} \in \mathbb{L}^d$ with $x, y \in \mathfrak{B}_0$. Note that $|\mathfrak{I}| \asymp N^d$. For any edge $e \in \mathbb{L}^d$, we denote its two trisection points by v_e^- and v_e^+, ordered such that the sum of the coordinates of v_e^- is smaller than that of v_e^+. For each $e \in \mathfrak{I}$, we define the event

$$\begin{aligned} \mathsf{A}_e := &\big\{v_e^+ \xleftrightarrow{\geq 0} \partial B(c_4 N), v_e^- \xleftrightarrow{\leq 0} \partial B(c_4 N)\big\} \cap \big\{|\widetilde{\phi}_{v_e^+}|, |\widetilde{\phi}_{v_e^-}| \in [c_6, C_5]\big\} \\ &\cap \big\{v_e^+ \xleftrightarrow{\geq 0} \partial B(N)\big\}^c \cap \big\{v_e^- \xleftrightarrow{\leq 0} \partial B(N)\big\}^c. \end{aligned} \tag{3.3}$$

Let $\mathfrak{I}_{\mathsf{A}}$ denote the collection of edges $e \in \mathfrak{I}$ such that A_e occurs.

We claim that there exists $c_5 > 0$ such that

$$\mathbb{P}\big(|\mathfrak{I}_{\mathsf{A}}| \geq c_5 N^{(\frac{d}{2}-1)\boxdot(d-4)}\big) \gtrsim 1. \tag{3.4}$$

Before verifying this claim, we first prove Theorem 1.1 using it. Precisely, on the event $\{\mathfrak{I}_{\mathsf{A}} = \omega\}$ (for any realization ω of $\mathfrak{I}_{\mathsf{A}}$), we explore the sign cluster $\mathcal{C}^{\pm}_{\partial B(c_4 N)}$ from $\partial B(c_4 N)$, locally pausing upon encountering any v_e^- or v_e^+ for $e \in \omega$, and resuming the exploration elsewhere. After the exploration is finished, by the strong Markov property of $\widetilde{\phi}_\cdot$, the GFF configurations restricted to the subinterval $I_{[v_e^-, v_e^+]}$ are independent across different $e \in \omega$. In addition, for each $e \in \omega$,

$$\mathsf{G}_e := \big\{\mathrm{dist}(\mathcal{C}^+_{v_e^+} \cap I_{[v_e^-, v_e^+]}, \mathcal{C}^-_{v_e^-} \cap I_{[v_e^-, v_e^+]}) \leq \chi\big\} \tag{3.5}$$

occurs with probability at least $c\chi^{\frac{1}{2}}$. To see this, we take $c'\chi^{-1}$ disjoint subintervals $I_{[z_1^-, z_1^+]}, \ldots, I_{[z_K^-, z_K^+]} \subset I_{[v_e^-, v_e^+]}$ such that for all $1 \leq j \leq K$, $\mathrm{dist}(I_{[z_j^-, z_j^+]}, \{v_e^-, v_e^+\}) \geq \frac{d}{9}$, $|z_j^- - v_e^-| < |z_j^- - v_e^+|$ and $|z_j^- - z_j^+| = \chi$. Note that the events

$$\mathsf{G}_e^j := \big\{z_j^+ \xleftrightarrow{\geq 0} v_e^+, z_j^- \xleftrightarrow{\leq 0} v_e^-\big\} \tag{3.6}$$

for $1 \leq j \leq K$ are disjoint, and that $\mathsf{G}_e \supset \cup_{1\leq j\leq K} \mathsf{G}_e^j$. Moreover, by [6, (6.27)], the conditional probability (given the exploration) of each G_e^j is at least of order $\chi^{\frac{3}{2}}$.

Thus, we obtain that G_e occurs with probability at least of order $\chi^{-1}\cdot\chi^{\frac{3}{2}}=\chi^{\frac{1}{2}}$, as claimed above. To sum up, we have

$$
\begin{aligned}
\mathbb{P}\big(\mathcal{D}_{N,\frac{1}{2}c_4}\leq\chi\big) \geq & \mathbb{P}\big(|\mathfrak{I}_{\mathsf{A}}|\geq c_5N^{(\frac{d}{2}-1)\square(d-4)},\cup_{e\in\mathfrak{I}_{\mathsf{A}}}\mathsf{G}_e\big)\\
\geq & \sum\nolimits_{\omega\subset\mathfrak{I}:|\omega|\geq c_5N^{(\frac{d}{2}-1)\square(d-4)}}\mathbb{P}\big(\mathfrak{I}_{\mathsf{A}}=\omega\big)\cdot\mathbb{P}\big(\mathbf{Z}>0\big),
\end{aligned}
\tag{3.7}
$$

where $\mathbf{Z}\sim\mathrm{Binomial}(\lfloor c_5N^{(\frac{d}{2}-1)\square(d-4)}\rfloor,c\chi^{\frac{1}{2}})$, i.e., the sum of $\lfloor c_5N^{(\frac{d}{2}-1)\square(d-4)}\rfloor$ i.i.d. Bernoulli random variables with expectation $c\chi^{\frac{1}{2}}$. Note that a direct application of the second moment method yields $\mathbb{P}\big(\mathbf{Z}>0\big)\gtrsim\chi^{\frac{1}{2}}N^{(\frac{d}{2}-1)\square(d-4)}$. Plugging this into (3.7), and using the claim (3.4), we obtain the lower bound in (1.14):

$$
\mathbb{P}\big(\mathcal{D}_{N,\frac{1}{2}c_4}\leq\chi\big)\gtrsim\chi^{\frac{1}{2}}N^{(\frac{d}{2}-1)\square(d-4)}\cdot\mathbb{P}\big(|\mathfrak{I}_{\mathsf{A}}|\geq c_5N^{\frac{d}{2}-1}\big)\gtrsim\chi^{\frac{1}{2}}N^{(\frac{d}{2}-1)\square(d-4)}. \tag{3.8}
$$

Next, we prove (3.4) for the low- and high- dimensional cases separately.

When $3\leq d\leq5$**.** According to [6, Lemma 6.2], there exist $C_5>c_6>0$ such that for any $w_+\in\mathfrak{B}_1$, $w_-\in\mathfrak{B}_{-1}$ and $e\in\mathfrak{I}$,

$$
\mathbb{P}\big(\mathsf{A}_e^{w_+,w_-}\big)\gtrsim N^{-\frac{3d}{2}+1}, \tag{3.9}
$$

where $\mathsf{A}_e^{w_+,w_-}$ is defined as $\mathsf{H}_{v_e^+,w_+}^{v_e^-,w_-}\cap\{|\widetilde{\phi}_{v_e^+}|,|\widetilde{\phi}_{v_e^-}|\in[c_6,C_5]\}\cap\big\{v_e^+\xleftrightarrow{\geq0}\partial B(N)\big\}^c\cap\big\{v_e^-\xleftrightarrow{\leq0}\partial B(N)\big\}^c$. As a result, we obtain that for any $w_+\in\mathfrak{B}_1$ and $w_-\in\mathfrak{B}_{-1}$,

$$
\mathbb{E}\big[|\mathfrak{I}_{\mathsf{A}}^{w_+,w_-}|\big]:=\mathbb{E}\Big[\sum\nolimits_{e\in\mathfrak{I}}\mathbb{1}_{\mathsf{A}_e^{w_+,w_-}}\Big]\gtrsim N^{-\frac{d}{2}+1}. \tag{3.10}
$$

To bound the second moment of $|\mathfrak{I}_{\mathsf{A}}^{w_+,w_-}|$, we need the following proposition. For any events A_1 and A_2 measurable with respect to $\widetilde{\mathcal{L}}_{1/2}$, let $\mathsf{A}_1\|\mathsf{A}_2$ denote the event that A_1 and A_2 both occur and are certified by two disjoint collections of clusters in $\cup\widetilde{\mathcal{L}}_{1/2}$. For any $z_1,z_2,z_3\in\widetilde{\mathbb{Z}}^d$ and $D\subset\widetilde{\mathbb{Z}}^d$, we denote $\big\{z_1\xleftrightarrow{(D)}z_2,z_3\big\}:=\big\{z_1\xleftrightarrow{(D)}z_2\big\}\cap\big\{z_1\xleftrightarrow{(D)}z_3\big\}$ for brevity. We then define the event (letting $C_6:=\frac{1}{2}(C_5)^2$)

$$
\begin{aligned}
\mathsf{F}(e,e';w_+,w_-):= & \big\{w_+\leftrightarrow v_e^+,v_{e'}^+\|w_-\leftrightarrow v_e^-,v_{e'}^-\big\}\\
& \cap\big\{\widehat{\mathcal{L}}_{1/2}^{v_e^+},\widehat{\mathcal{L}}_{1/2}^{v_{e'}^+},\widehat{\mathcal{L}}_{1/2}^{v_e^-},\widehat{\mathcal{L}}_{1/2}^{v_{e'}^-}\in[0,C_6]\big\}.
\end{aligned}
\tag{3.11}
$$

Proposition 3.1. *When* $3\leq d\leq5$*, for any* $w_+\in\mathfrak{B}_1$*,* $w_-\in\mathfrak{B}_{-1}$ *and* $e,e'\in\mathfrak{I}$*,*

$$
\mathbb{P}\big(\mathsf{F}(e,e';w_+,w_-)\big)\lesssim(|v_e^+-v_{e'}^+|+1)^{-\frac{d}{2}-1}N^{-\frac{3d}{2}+1}. \tag{3.12}
$$

The proof of Proposition 3.1 will be provided in Section 4. Under the coupling given by the isomorphism theorem [26, Proposition 2.1], Property (a) (recall it below (1.11)) implies that on the event $\mathsf{A}_e^{w_+,w_-}\cap\mathsf{A}_{e'}^{w_+,w_-}$, for each $\diamond\in\{+,-\}$, the loop cluster containing $w_\diamond$ intersects both $v_e^\diamond$ and $v_{e'}^\diamond$, and satisfies $\widehat{\mathcal{L}}_{1/2}^{v_e^+},\widehat{\mathcal{L}}_{1/2}^{v_{e'}^+}\in[0,C_6]$. I.e., $\mathsf{F}(e,e';w_+,w_-)$ occurs. Thus, applying Proposition 3.1, we have

$$
\mathbb{P}\big(\mathsf{A}_e^{w_+,w_-}\cap\mathsf{A}_{e'}^{w_+,w_-}\big)\leq\mathbb{P}\big(\mathsf{F}(e,e';w_+,w_-)\big)\lesssim(|v_e^+-v_{e'}^+|+1)^{-\frac{d}{2}-1}N^{-\frac{3d}{2}+1}. \tag{3.13}
$$

This further yields that

$$\begin{aligned}\mathbb{E}\big[|\mathfrak{I}_{\mathsf{A}}^{w_+,w_-}|^2\big] =&\sum\nolimits_{e,e'\in\mathfrak{I}}\mathbb{P}\big(\mathsf{A}_e^{w_+,w_-}\cap \mathsf{A}_{e'}^{w_+,w_-}\big)\\ \lesssim& N^{-\frac{3d}{2}+1}\sum\nolimits_{w,w'\in\mathfrak{B}_0}(|w-w'|+1)^{-\frac{d}{2}-1}\\ \lesssim& N^{-\frac{3d}{2}+1}\cdot|\mathfrak{B}_0|\cdot N^{\frac{d}{2}-1}\asymp 1,\end{aligned}\tag{3.14}$$

where in the second inequality we used the following basic fact (see e.g., [3]): for any $d\geq 3$, $a\neq d$, $x\in\mathbb{Z}^d$ and $M\geq 1$,

$$\sum\nolimits_{y\in B(M)}(|x-y|+1)^{-a}\lesssim M^{(d-a)\vee 0}.\tag{3.15}$$

By the Paley-Zygmund inequality, (3.10) and (3.14), we have

$$\mathbb{P}\big(\mathfrak{I}_{\mathsf{A}}^{w_+,w_-}\neq\emptyset\big)\gtrsim\frac{\big(\mathbb{E}\big[|\mathfrak{I}_{\mathsf{A}}^{w_+,w_-}|\big]\big)^2}{\mathbb{E}\big[|\mathfrak{I}_{\mathsf{A}}^{w_+,w_-}|^2\big]}\gtrsim N^{2-d}\tag{3.16}$$

for all $w_+\in\mathfrak{B}_1$ and $w_-\in\mathfrak{B}_{-1}$. In fact, this estimate is sharp: since $\{\mathfrak{I}_{\mathsf{A}}^{w_+,w_-}\neq\emptyset\}$ implies both $\{w_+\xleftrightarrow{\geq 0}\partial B_{w_+}(c_4N)\}$ and $\{w_-\xleftrightarrow{\leq 0}\partial B_{w_-}(c_4N)\}$, it follows from the FKG inequality that

$$\mathbb{P}\big(\mathfrak{I}_{\mathsf{A}}^{w_+,w_-}\neq\emptyset\big)\leq\big[\theta_d(c_4N)\big]^2\lesssim N^{2-d}.\tag{3.17}$$

Combining $\mathbb{P}\big(\mathfrak{I}_{\mathsf{A}}^{w_+,w_-}\neq\emptyset\big)\asymp N^{2-d}$ with (3.10) and (3.14) respectively, one has

$$\mathbb{E}\big[|\mathfrak{I}_{\mathsf{A}}^{w_+,w_-}|\mid \mathfrak{I}_{\mathsf{A}}^{w_+,w_-}\neq\emptyset\big]\gtrsim N^{\frac{d}{2}-1},\tag{3.18}$$

$$\mathbb{E}\big[|\mathfrak{I}_{\mathsf{A}}^{w_+,w_-}|^2\mid \mathfrak{I}_{\mathsf{A}}^{w_+,w_-}\neq\emptyset\big]\lesssim N^{d-2}.\tag{3.19}$$

These two estimates together with the Paley-Zygmund inequality imply that for some constant $c_5>0$,

$$\mathbb{P}\big(|\mathfrak{I}_{\mathsf{A}}^{w_+,w_-}|\geq c_5N^{\frac{d}{2}-1}\mid \mathfrak{I}_{\mathsf{A}}^{w_+,w_-}\neq\emptyset\big)\gtrsim\frac{\big(\mathbb{E}\big[|\mathfrak{I}_{\mathsf{A}}^{w_+,w_-}|\mid \mathfrak{I}_{\mathsf{A}}^{w_+,w_-}\neq\emptyset\big]\big)^2}{\mathbb{E}\big[|\mathfrak{I}_{\mathsf{A}}^{w_+,w_-}|^2\mid \mathfrak{I}_{\mathsf{A}}^{w_+,w_-}\neq\emptyset\big]}\gtrsim 1.\tag{3.20}$$

Combined with (3.16), it yields that for any $w_+\in\mathfrak{B}_1$ and $w_-\in\mathfrak{B}_{-1}$,

$$\mathbb{P}\big(|\mathfrak{I}_{\mathsf{A}}^{w_+,w_-}|\geq c_5N^{\frac{d}{2}-1}\big)\gtrsim N^{2-d}.\tag{3.21}$$

Next, we consider the quantity

$$\mathbf{X}:=\sum\nolimits_{w_+\in\mathfrak{B}_1,w_-\in\mathfrak{B}_{-1}}\mathbb{1}_{|\mathfrak{I}_{\mathsf{A}}^{w_+,w_-}|\geq c_5N^{\frac{d}{2}-1}}.\tag{3.22}$$

For the first moment of $\mathbf{X}$, it follows from (3.21) that

$$\mathbb{E}[\mathbf{X}]\gtrsim|\mathfrak{B}_1|\cdot|\mathfrak{B}_{-1}|\cdot N^{2-d}\asymp N^{d+2}.\tag{3.23}$$

For the second moment of $\mathbf{X}$, since $|\mathfrak{I}_{\mathsf{A}}^{w_+,w_-}| \geq c_5 N^{\frac{d}{2}-1} \geq 1$ implies both $w_+ \xleftrightarrow{\geq 0} \partial B_{w_+}(c_4 N)$ and $w_- \xleftrightarrow{\leq 0} \partial B_{w_-}(c_4 N)$, we have

$$
\begin{aligned}
\mathbb{E}\big[\mathbf{X}^2\big] \leq & \sum_{w_+,w_+'\in\mathfrak{B}_1,w_-,w_-'\in\mathfrak{B}_{-1}} \mathbb{P}\big(w_+ \xleftrightarrow{\geq 0} \partial B_{w_+}(c_4N), w_+' \xleftrightarrow{\geq 0} \partial B_{w_+'}(c_4N), \\
& \qquad\qquad w_- \xleftrightarrow{\leq 0} \partial B_{w_-}(c_4N), w_-' \xleftrightarrow{\leq 0} \partial B_{w_-'}(c_4N)\big) \\
\leq & \prod_{j\in\{1,-1\}} \sum_{w_j,w_j'\in\mathfrak{B}_j} \mathbb{P}\big(w_j \xleftrightarrow{\geq 0} \partial B_{w_j}(c_4N), w_j' \xleftrightarrow{\geq 0} \partial B_{w_j'}(c_4N)\big),
\end{aligned} \tag{3.24}
$$

where in the last line we used the FKG inequality and the symmetry of GFF. In addition, when $w_j \xleftrightarrow{\geq 0} \partial B_{w_j}(c_4N)$ and $w_j' \xleftrightarrow{\geq 0} \partial B_{w_j'}(c_4N)$ both occur, their occurrences are certified either by a single cluster or by two distinct clusters. We denote the corresponding events by $\mathsf{C}_{w_j,w_j'}^{(1),+}$ and $\mathsf{C}_{w_j,w_j'}^{(2),+}$ respectively. According to [2, Theorem 1.4], we have

$$
\mathbb{P}\big(\mathsf{C}_{w_j,w_j'}^{(1),+}\big) \lesssim (|w_j - w_j'|+1)^{-\frac{d}{2}+1} N^{-\frac{d}{2}+1}. \tag{3.25}
$$

For $\mathsf{C}_{w_j,w_j'}^{(2),+}$, by the symmetry of GFF and the FKG inequality, we have

$$
\mathbb{P}\big(\mathsf{C}_{w_j,w_j'}^{(2),+}\big) = \mathbb{P}\Big(\mathsf{H}_{w_j',\partial B_{w_j'}(c_4N)}^{w_j,\partial B_{w_j}(c_4N)}\Big) \overset{\text{(FKG)}}{\leq} \big[\theta_d(c_4N)\big]^2 \overset{(1.2)}{\lesssim} N^{2-d}. \tag{3.26}
$$

By (3.25) and (3.26), one has

$$
\begin{aligned}
&\mathbb{P}\big(w_j \xleftrightarrow{\geq 0} \partial B_{w_j}(c_4N), w_j' \xleftrightarrow{\geq 0} \partial B_{w_j'}(c_4N)\big) \\
\lesssim &(|w_j - w_j'|+1)^{-\frac{d}{2}+1} N^{-\frac{d}{2}+1} + N^{2-d}.
\end{aligned} \tag{3.27}
$$

Plugging (3.27) into (3.24), we obtain

$$
\begin{aligned}
\mathbb{E}\big[\mathbf{X}^2\big] \lesssim & \prod_{j\in\{1,-1\}} \sum_{w_j,w_j'\in\mathfrak{B}_j} \big[(|w_j-w_j'|+1)^{-\frac{d}{2}+1}\cdot N^{-\frac{d}{2}+1} + N^{2-d}\big] \\
\overset{(3.15)}{\lesssim} & \prod_{j\in\{1,-1\}} \big(|\mathfrak{B}_j|\cdot N^{\frac{d}{2}+1}\cdot N^{-\frac{d}{2}+1} + |\mathfrak{B}_j|^2\cdot N^{2-d}\big) \lesssim N^{2d+4}.
\end{aligned} \tag{3.28}
$$

By the Paley-Zygmund inequality, (3.23) and (3.28), we get

$$
\mathbb{P}\big(\mathbf{X}>0\big) \gtrsim \frac{\big(\mathbb{E}[\mathbf{X}]\big)^2}{\mathbb{E}\big[\mathbf{X}^2\big]} \gtrsim 1. \tag{3.29}
$$

On the event $\{\mathbf{X}>0\}$, there exist $w_+\in\mathfrak{B}_1$ and $w_-\in\mathfrak{B}_{-1}$ such that the event $\mathsf{A}_e^{w_+,w_-}$ occurs for at least $c_5N^{\frac{d}{2}-1}$ edges in $\mathfrak{I}$. Combined with the inclusion that

$\mathsf{A}_e^{w_+,w_-} \subset \mathsf{A}_e$ for all $e \in \mathfrak{I}$, $w_+ \in \mathfrak{B}_1$ and $w_- \in \mathfrak{B}_{-1}$, it yields the claim (3.4):

$$\mathbb{P}\big(|\mathfrak{I}_{\mathsf{A}}| \geq c_5 N^{\frac{d}{2}-1}\big) \geq \mathbb{P}\big(\mathbf{X} > 0\big) \gtrsim 1. \tag{3.30}$$

When $d \geq 7$. The proof in this case is relatively straightforward. Recall the event A_e in (3.3). According to [6, Lemma 6.1], we have

$$\mathbb{P}\big(\mathsf{A}_e\big) \gtrsim N^{-4}, \quad \forall e \in \mathfrak{I}, \tag{3.31}$$

which implies that the first moment of $\mathfrak{I}_{\mathsf{A}}$ satisfies

$$\mathbb{E}[|\mathfrak{I}_{\mathsf{A}}|] \gtrsim |\mathfrak{I}| \cdot N^{-4} \asymp N^{d-4}. \tag{3.32}$$

For the second moment, as in (3.24), by the FKG inequality we have

$$\begin{aligned}\mathbb{E}\big[|\mathfrak{I}_{\mathsf{A}}|^2\big] =& \sum\nolimits_{e,e' \in \mathfrak{I}} \mathbb{P}\big(\mathsf{A}_e \cap \mathsf{A}_{e'}\big) \\ \leq& \sum\nolimits_{e,e' \in \mathfrak{I}} \mathbb{P}\big(v_e^+ \xleftrightarrow{\geq 0} \partial B(c_4N), v_{e'}^+ \xleftrightarrow{\geq 0} \partial B(c_4N)\big) \\ &\cdot \mathbb{P}\big(v_e^- \xleftrightarrow{\leq 0} \partial B(c_4N), v_{e'}^- \xleftrightarrow{\leq 0} \partial B(c_4N)\big).\end{aligned} \tag{3.33}$$

For the same reason as in (3.27), it follows from [2, Theorem 1.4] and (1.4) that

$$\mathbb{P}\big(v_e^+ \xleftrightarrow{\geq 0} \partial B(c_4N), v_{e'}^+ \xleftrightarrow{\geq 0} \partial B(c_4N)\big) \lesssim (|v_e^+ - v_{e'}^+| + 1)^{4-d} N^{-2} + N^{-4}. \tag{3.34}$$

The same bound also holds for $\{v_e^- \xleftrightarrow{\leq 0} \partial B(c_4N), v_{e'}^- \xleftrightarrow{\leq 0} \partial B(c_4N)\}$. Thus,

$$\begin{aligned}\mathbb{E}\big[|\mathfrak{I}_{\mathsf{A}}|^2\big] \lesssim& \sum\nolimits_{e,e' \in \mathfrak{I}} \big[(|v_e^+ - v_{e'}^-| + 1)^{4-d} N^{-2} + N^{-4}\big] \\ &\cdot \big[(|v_e^+ - v_{e'}^-| + 1)^{4-d} N^{-2} + N^{-4}\big] \\ \lesssim& N^{-4} \sum\nolimits_{x,x' \in \mathfrak{B}_0} (|x - x'| + 1)^{8-2d} + N^{-8} |\mathfrak{B}_0|^2 \\ &+ N^{-6} \cdot |\mathfrak{B}_0| \cdot \max_{x \in \mathfrak{B}_0} \sum\nolimits_{x' \in \mathfrak{B}_0} (|x - x'| + 1)^{4-d} \\ \overset{|x-x'| \lesssim N, |\mathfrak{B}_0| \asymp N^d}{\lesssim}& N^{d-2} \cdot \max_{x \in \mathfrak{B}_0} \sum\nolimits_{x' \in \mathfrak{B}_0} (|x - x'| + 1)^{6-2d} + N^{2d-8} \\ &+ N^{d-6} \cdot \max_{x \in \mathfrak{B}_0} \sum\nolimits_{x' \in \mathfrak{B}_0} (|x - x'| + 1)^{4-d} \\ \overset{(3.15)}{\lesssim}& N^{d-2} + N^{2d-8} \overset{(d \geq 7)}{\lesssim} N^{2d-8}.\end{aligned} \tag{3.35}$$

By the Paley-Zygmund inequality, (3.32) and (3.35), we obtain the claim (3.4):

$$\mathbb{P}\big(|\mathfrak{I}_{\mathsf{A}}| \geq c_5 N^{d-4}\big) \gtrsim \frac{(\mathbb{E}[|\mathfrak{I}_{\mathsf{A}}|])^2}{\mathbb{E}\big[|\mathfrak{I}_{\mathsf{A}}|^2\big]} \gtrsim 1. \tag{3.36}$$

In conclusion, we have established Theorem 1.1 assuming Proposition 3.1. $\square$

4. Proof of Proposition 3.1

The proof of Proposition 3.1 mainly relies on the following lemma. To state this lemma precisely, we first introduce some notations as follows:

- Let $C_7, C_8 > 10^4$ be constants that will be determined later. Here C_8 is much larger than C_7.
- For any $D \subset \widetilde{\mathbb{Z}}^d$, $r \geq 1$, $v, v' \in \widetilde{\mathbb{Z}}^d$ with $r \leq |v - v'| \leq 10r$,

$$\mathbf{Q}_{\mathrm{I}}^D(v, v'; r) := \sum\nolimits_{z \in \partial \mathcal{B}_v(\frac{r}{C_7}), z' \in \partial \mathcal{B}_{v'}(\frac{r}{C_7})} \mathbb{P}\big(\mathsf{A}_{\mathrm{I}}^D(v, z; r)\big) \cdot \mathbb{P}\big(\mathsf{A}_{\mathrm{I}}^D(v', z'; r)\big), \tag{4.1}$$

  where $\mathsf{A}_{\mathrm{I}}^D(v, z; r) := \big\{v \xleftrightarrow{(D)} z\big\} \cap \big\{v \xleftrightarrow{(D)} \partial B_v(\frac{r}{100})\big\}^c$.
- For any $D \subset \widetilde{\mathbb{Z}}^d$, $v \in \widetilde{\mathbb{Z}}^d$, $r \geq 1$ and $r' > r$,

$$\mathbf{Q}_{\mathrm{II}}^D(v; r, r') := \sum_{z \in \partial \mathcal{B}_v(r), z' \in \partial \mathcal{B}_v(r')} \mathbb{P}\big(\mathsf{A}_{\mathrm{II}}^D(v, z, z'; r, r')\big), \tag{4.2}$$

  where $\mathsf{A}_{\mathrm{II}}^D(v, z, z'; r, r') := \big\{z \xleftrightarrow{(D)} z'\big\} \cap \big\{z \xleftrightarrow{(D)} \partial B_v(\frac{r}{C_7}) \cup \partial B_v(C_7 r')\big\}^c$. For convenience, we set $\mathbf{Q}_{\mathrm{II}}^D(v; r, r') := r^d$ when $r \leq r'$.
- For any $r \geq 1$ and $k \in \mathbb{N}$, we denote $r_k := (C_8)^k \cdot r$. For any $R \geq r_5$, let

$$k_\star(r, R) := \min\{k \geq 1 : r_{k+3} \geq R\}. \tag{4.3}$$

  For any $D \subset \widetilde{\mathbb{Z}}^d$, $v \in \widetilde{\mathbb{Z}}^d$, $r' \in [r_1, 2r_{k_\star}]$ and $w \in [\widetilde{B}_v(2R)]^c$,

$$\mathbf{Q}_{\mathrm{III}}^D(v, w; r') := \sum\nolimits_{z \in \partial \mathcal{B}_v(\frac{r'}{C_7})} \mathbb{P}\big(\mathsf{A}_{\mathrm{III}}^D(v, w, z; r')\big), \tag{4.4}$$

  where $\mathsf{A}_{\mathrm{III}}^D(v, w, z; r') := \big\{w \xleftrightarrow{(D)} z\big\} \cap \big\{w \xleftrightarrow{(D)} \partial B_v(\frac{r'}{C_7^2})\big\}^c$. In addition, we define $\mathsf{A}_{\mathrm{III}}^D(v, w, z; r') := \{\mathbf{A}_\star^z = 1\}$ for all $r' > 2r_{k_\star}$, where $\{\mathbf{A}_\star^z\}_{z \in \mathbb{Z}^d}$ is a family of i.i.d. Bernoulli random variables independent of $\widetilde{\mathcal{L}}_{1/2}$ with expectation R^{2-d}.

Lemma 4.1. *For any $3 \leq d \leq 5$, there exist $C_7, C_8 > 0$ that the following holds. Suppose that $r \geq 1$, $R \geq r_5$, $\{\{x_i, y_i\}\}_{i \in \{1,2\}} \subset \mathbb{L}^d$ satisfying $r \leq |x_1 - x_2| \leq 10r$, $v_i \in I_{e_i}$ satisfying $|v_i - x_i| \wedge |v_i - y_i| \geq \frac{1}{10}$ for $i \in \{1, 2\}$, $D \subset \widetilde{\mathbb{Z}}^d$ satisfying $x_i \in D$ and $D \cap I_{[v_i, y_i]} = \emptyset$ for $i \in \{1, 2\}$, $v_i' \in I_{[v_i, y_i]}$ satisfying $|v_i' - v_i| \wedge |v_i' - y_i| \geq \frac{1}{100}$ for $i \in \{1, 2\}$, and $w \in [\widetilde{B}_{v_1}(2R)]^c$. Then we have*

$$\begin{aligned} &\mathbb{P}\big(\big\{w \xleftrightarrow{(D)} v_1, v_2\big\} \cap \big\{\widehat{\mathcal{L}}_{1/2}^{D, v_1}, \widehat{\mathcal{L}}_{1/2}^{D, v_2} \in [0, C_6]\big\}\big) \\ \lesssim\, & r^{-d-1} \mathbf{Q}_{\mathrm{I}}^D(v_1', v_2'; r) \sum\nolimits_{0 \leq k \leq k_\star(r, R)} r_k^{-\frac{3d}{2}+1} \mathbf{Q}_{\mathrm{II}}^D(v_1; r_1, r_k) \mathbf{Q}_{\mathrm{III}}^D(v_1, w; r_{k+1}). \end{aligned} \tag{4.5}$$

The rest of this section is organized as follows: in Section 4.1, we prove Proposition 3.1 assuming Lemma 4.1; in Section 4.2, we prove Lemma 4.1.

4.1. **Proof of Proposition 3.1.** We arbitrarily take $w_+ \in \mathfrak{B}_1$, $w_- \in \mathfrak{B}_{-1}$ and $e, e' \in \mathfrak{I}$. Since $\mathbb{P}(v_e^+ \xleftrightarrow{\geq 0} w_+, v_e^- \xleftrightarrow{\leq 0} w_-) \lesssim N^{-\frac{3d}{2}+1}$ for all $e \in \mathfrak{I}$ (by [6, Theorem 1.4]), it suffices to consider the case when $r := |v_e^+ - v_{e'}^+|$ is sufficiently large. For each $\diamond \in \{+,-\}$, we define

$$\mathsf{F}_\diamond^D := \big\{w_\diamond \xleftrightarrow{(D)} v_e^\diamond, v_{e'}^\diamond\big\} \cap \big\{\widehat{\mathcal{L}}_{1/2}^{D,v_e^\diamond}, \widehat{\mathcal{L}}_{1/2}^{D,v_{e'}^\diamond} \in [0, C_6]\big\}. \tag{4.6}$$

When $D = \emptyset$, we abbreviate $\mathsf{F}_\diamond := \mathsf{F}_\diamond^\emptyset$. Note that $\mathsf{F}(e,e';w_+,w_-) = \{\mathsf{F}_+ \| \mathsf{F}_-\}$. Therefore, by the restriction property and Lemma 4.1, one has

$$\begin{aligned}
&\mathbb{P}\big(\mathsf{F}(e,e';w_+,w_-)\big) = \mathbb{E}\big[\mathbb{1}_{\mathsf{F}_-} \cdot \mathbb{P}\big(\mathsf{F}_+^{\mathcal{C}_{w_-}}\big)\big] \\
\lesssim& r^{-d-1}\mathbb{E}\Big[\mathbb{1}_{\mathsf{F}_-} \cdot \mathbb{E}\big[\mathbf{Q}_{\mathrm{I}}^{\mathcal{C}_{w_-}}(\hat{v}_e^+, \hat{v}_{e'}^+; r) \sum\nolimits_{0\leq k_+ \leq k_\star(r,R)} r_k^{-\frac{d}{2}-1} \mathbf{Q}_{\mathrm{II}}^{\mathcal{C}_{w_-}}(v_e^+; r, r_k) \\
&\qquad\qquad\qquad\qquad \cdot \mathbf{Q}_{\mathrm{III}}^{\mathcal{C}_{w_-}}(v_e^+, w_+; r_{k+1}) \mid \mathcal{F}_{\mathcal{C}_{w_-}}\big]\Big] \\
=& r^{-d-1} \sum\nolimits_{1\leq k \leq k_\star(r,N)} r_k^{-\frac{3d}{2}+1} \sum\nolimits_{\bar{z} \in \Upsilon_+(r,r_k)} \mathbb{E}\big[\mathbb{1}_{\mathsf{F}_-} \cdot \mathfrak{U}_+^{\mathcal{C}_{w_-}}(\bar{z}; r, r_k)\big],
\end{aligned} \tag{4.7}$$

where $\bar{z} := (z_1, z_2, z_3, z_4, z_5)$, and for each $\diamond \in \{+,-\}$, $\hat{v}_e^\diamond$ denotes the midpoint between $v_e^\diamond$ and the endpoint of e that is closer to $v_e^\diamond$, $\Upsilon_\diamond(R,R') := \partial\mathcal{B}_{\hat{v}_e^\diamond}(\frac{R}{C_7}) \times \partial\mathcal{B}_{\hat{v}_{e'}^\diamond}(\frac{R}{C_7}) \times \partial\mathcal{B}_{v_e^\diamond}(dR) \times \partial\mathcal{B}_{v_e^\diamond}(R') \times \partial\mathcal{B}_{v_e^\diamond}(\frac{C_8 R'}{C_7})$, and $\mathfrak{U}_\diamond^D(\bar{z}; R, R')$ represents the product of the probabilities of $\mathsf{A}_{\mathrm{I}}^D(\hat{v}_e^\diamond, z_1; R)$, $\mathsf{A}_{\mathrm{I}}^D(\hat{v}_{e'}^\diamond, z_2; R)$, $\mathsf{A}_{\mathrm{II}}^D(v_e^\diamond, z_3, z_4; R, R')$ and $\mathsf{A}_{\mathrm{III}}^D(v_e^\diamond, w_\diamond, z_5; C_8 R')$. In addition, we denote the intersection of these four events by $\overline{\mathsf{A}}_\diamond^D(\bar{z}; R, R')$, and we abbreviate $\overline{\mathsf{A}}_\diamond := \overline{\mathsf{A}}_\diamond^\emptyset$. (**P.S.** Here, $v_e^\diamond$ is not necessarily the midpoint; it can be any point satisfying the conditions for v_i' in Lemma 4.1, where $v_i = v_e^\diamond$ and y_i is the endpoint of e closer to $v_e^\diamond$.)

By using Corollary 2.9 repeatedly, we have

$$\mathfrak{U}_+^{\mathcal{C}_{w_-}}(\bar{z}; r, r_k) \lesssim \mathbb{P}\big(\overline{\mathsf{A}}_+^{\mathcal{C}_{w_-}}(\bar{z}; r, r_k) \mid \mathcal{F}_{\mathcal{C}_{w_-}}\big). \tag{4.8}$$

Let $\overline{\mathcal{C}}_{\bar{z}} := \cup_{j\in\{1,2,3,5\}} \mathcal{C}_{z_j}$ (note that $z_4 \in \mathcal{C}_{z_3}$). See Figure 1 for an illustration of the event $\overline{\mathsf{A}}_+^{\mathcal{C}_{w_-}}(\bar{z}; r, r_k)$. Plugging (4.8) into (4.7), and then applying Lemma 4.1 and Corollary 2.9 as in (4.7) and (4.8), we get

$$\begin{aligned}
&\mathbb{P}\big(\mathsf{F}(e,e';w_+,w_-)\big) \\
\lesssim& r^{-d-1} \sum\nolimits_{1\leq k\leq k_\star(r,N)} r_k^{-\frac{3d}{2}+1} \sum\nolimits_{\bar{z}\in\Upsilon_+(r,r_k)} \mathbb{E}\big[\mathbb{1}_{\overline{\mathsf{A}}_+(\bar{z};r,r_k)} \cdot \mathbb{P}\big(\mathsf{F}_-^{\overline{\mathcal{C}}_{\bar{z}}} \mid \mathcal{F}_{\overline{\mathcal{C}}_{\bar{z}}}\big)\big] \\
\lesssim& r^{-2d-2} \sum\nolimits_{1\leq k,k'\leq k_\star(r,N)} r_k^{-\frac{3d}{2}+1} r_{k'}^{-\frac{3d}{2}+1} \sum\nolimits_{\bar{z}\in\Upsilon_+(r,r_{k'})} \\
&\qquad \sum\nolimits_{\bar{z}' := (z_1',z_2',z_3',z_4',z_5') \in \Upsilon_-(2r,2r_{k'})} \mathbb{P}\big(\overline{\mathsf{A}}_+(\bar{z}; r, r_k) \| \overline{\mathsf{A}}_-(\bar{z}'; 2r, 2r_{k'})\big).
\end{aligned} \tag{4.9}$$

See Figure 2 for an illustration for the event $\overline{\mathsf{A}}_+(\bar{z}; r, r_k) \| \overline{\mathsf{A}}_-(\bar{z}'; 2r, 2r_{k'})$.

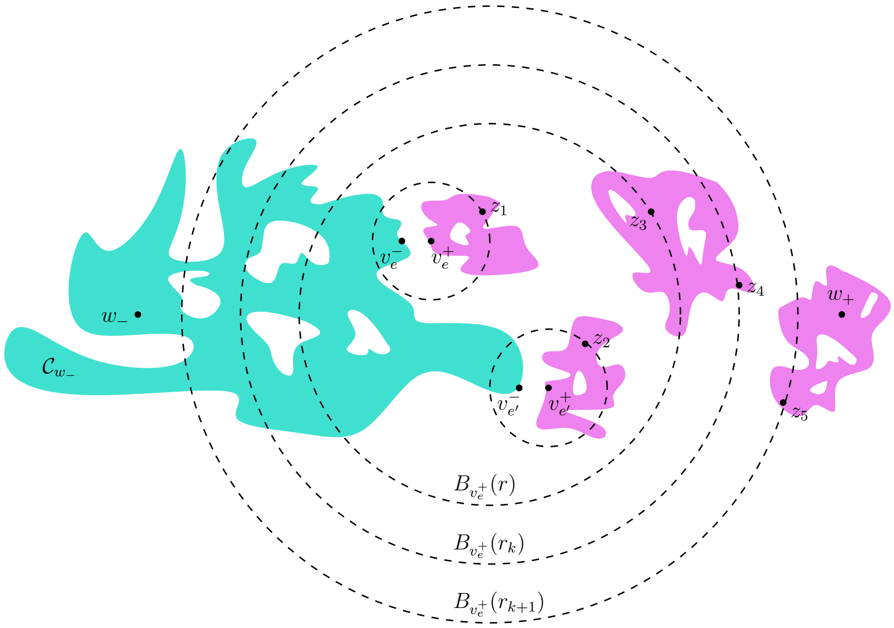

FIGURE 1. In this illustration, the cyan region is the cluster $\mathcal{C}_{w_1}$, and the pink regions represent the clusters $\{\mathcal{C}_{z_j}\}_{j\in\{1,2,3,5\}}$. Note that on $\overline{\mathsf{A}}_+^{\mathcal{C}_{w_-}}(\bar{z};r,r_k)$, the latter four clusters are restricted in different areas, and thus are constructed by disjoint collection of loops. This implies that given $\mathcal{C}_{w_1}$, the four sub-events involved in $\overline{\mathsf{A}}_+^{\mathcal{C}_{w_-}}(\bar{z};r,r_k)$ are conditionally independent. In the next step (4.9), we fix $\{\mathcal{C}_{z_j}\}_{j\in\{1,2,3,5\}}$ and then decompose the cluster $\mathcal{C}_{w_1}$ via Lemma 4.1.

For any $\diamond\in\{+,-\}$, $R\geq 1$ and $R'>R$, we denote $\widehat{\Upsilon}_\diamond(R,R'):=\partial\mathcal{B}_{v_e^\diamond}(dR)\times\partial\mathcal{B}_{v_e^\diamond}(R')\times\partial\mathcal{B}_{v_e^\diamond}(\frac{C_8R'}{C_7})$. We claim that for any $1\leq k,k'\leq k_\star(r,N)$, $z_1\in\partial\mathcal{B}_{\hat{v}_e^+}(\frac{r}{C_7})$, $z_2\in\partial\mathcal{B}_{\hat{v}_{e'}^+}(\frac{r}{C_7})$, $z_1'\in\partial\mathcal{B}_{\hat{v}_e^-}(\frac{2r}{C_7})$ and $z_2'\in\partial\mathcal{B}_{\hat{v}_{e'}^-}(\frac{2r}{C_7})$,

$$\begin{aligned}&\sum\nolimits_{\hat{z}\in\widehat{\Upsilon}_+(r,r_k),\hat{z}'\in\widehat{\Upsilon}_-(2r,2r_{k'})}\mathbb{P}\big(\overline{\mathsf{A}}_+(z_1,z_2,\hat{z};r,r_k)\|\overline{\mathsf{A}}_-(z_1',z_2',\hat{z}';2r,2r_{k'})\big)\\ \lesssim\, & r^{-\frac{3d}{2}+3}r_k^dr_{k'}^dN^{-\frac{3d}{2}+1}.\end{aligned}\tag{4.10}$$

Note that substituting (4.10) into (4.9) immediately yields Proposition 3.1:

$$\begin{aligned}&\mathbb{P}\big(\mathsf{F}(e,e';w_+,w_-)\big)\\ \overset{|\partial B(R)|\asymp R^{d-1}}{\lesssim}\, & r^{-2d-2}\sum\nolimits_{1\leq k,k'\leq k_\star(r,N)}r_k^{-\frac{3d}{2}+1}r_{k'}^{-\frac{3d}{2}+1}\cdot(r^{d-1})^4\cdot r^{-\frac{3d}{2}+3}r_k^dr_{k'}^dN^{-\frac{3d}{2}+1}\\ =\, & r^{\frac{d}{2}-3}N^{-\frac{3d}{2}+1}\sum\nolimits_{1\leq k,k'\leq k_\star(r,N)}r_k^{-\frac{d}{2}+1}r_{k'}^{-\frac{d}{2}+1}\asymp r^{-\frac{d}{2}-1}N^{-\frac{3d}{2}+1}.\end{aligned}$$

Next, we verify the claim (4.10). Without loss of generality, we only consider the case $k \geq k'$. For simplicity, we denote the events

$$\widehat{\mathsf{A}}_{-}(\bar{z}'; R, R') := \mathsf{A}_{\mathrm{I}}(\hat{v}_e^-, z_1'; R) \cap \mathsf{A}_{\mathrm{I}}(\hat{v}_{e'}^-, z_2'; R) \cap \mathsf{A}_{\mathrm{II}}(v_e^-, z_3', z_4'; R, R'), \tag{4.11}$$

and the cluster $\widehat{\mathcal{C}}_- = \widehat{\mathcal{C}}_-(\bar{z}, \bar{z}') := (\cup_{i \in \{1,2,3,5\}} \mathcal{C}_{z_i}) \cup (\cup_{1 \leq i \leq 3} \mathcal{C}_{z_i'})$.

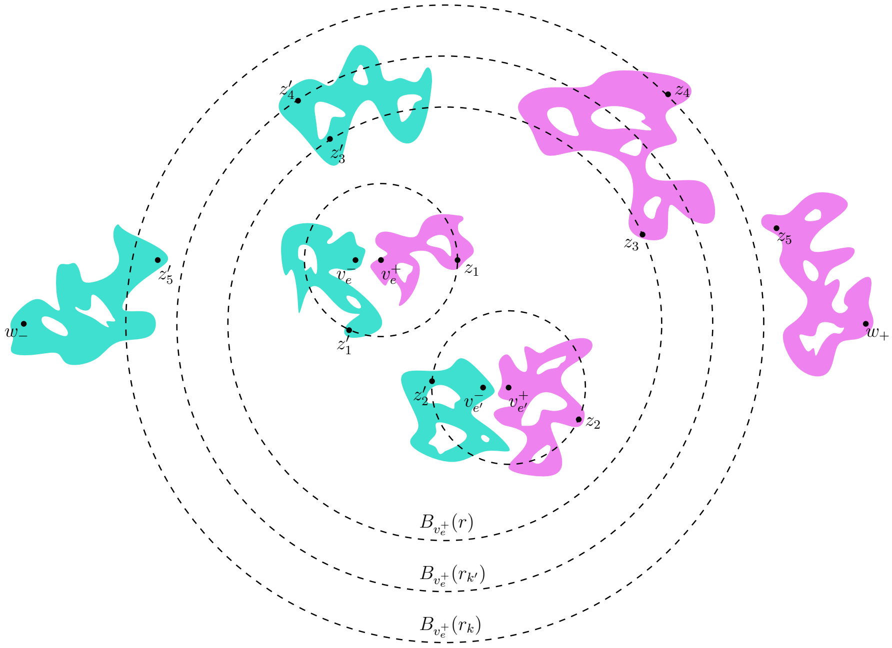

FIGURE 2. In this illustration, each colored region represents a loop cluster. The clusters $\mathcal{C}_{v_e^+}$ and $\mathcal{C}_{v_e^-}$ certify the four-point event $\{v_e^+ \leftrightarrow z_1 \| v_e^- \leftrightarrow z_1'\}$, whose probability has been estimated in the companion paper [6] (see (4.25)). The same applies to $\mathcal{C}_{v_{e'}^+}$ and $\mathcal{C}_{v_{e'}^-}$. To ensure that the remaining clusters certify four-point events restricted to disjoint annuli, we use Corollary 2.19 to decompose the clusters $\mathcal{C}_{w_-}$ and $\mathcal{C}_{z_3}$ around the boundaries $\partial B(r_k)$ and $\partial B(r_{k'})$ respectively (see (4.13) and (4.14)). In the special case when $k = k'$, this decomposition is not needed since the clusters $\mathcal{C}_{z_3}$, $\mathcal{C}_{z_3'}$, $\mathcal{C}_{w_+}$ and $\mathcal{C}_{w_-}$ already meet the requirements. This necessitates the distinct definitions of $\widehat{\mathsf{A}}_{\mathrm{III}}^D$ for the cases $k = k'$ and $k > k'$.

Let $r_k^{\mathrm{out}} := C_7^{3/2} C_8^{1/2} r_k$ and $r_k^{\mathrm{in}} := C_7^{-5/2} C_8^{1/2} r_k$. In addition, for any $v \in \widetilde{\mathbb{Z}}^d$, $w \in [\widetilde{B}_{v_e^-}(c_4 N)]^c$, $w' \in \partial \mathcal{B}_{v_e^-}(\frac{2 r_{k'+1}}{C_7})$, $z \in \partial \mathcal{B}_{v_e^-}(r_k^{\mathrm{out}})$ and $z' \in \partial \mathcal{B}_{v_e^-}(r_k^{\mathrm{in}})$, we define the event $\widehat{\mathsf{A}}_{\mathrm{III}}^D = \widehat{\mathsf{A}}_{\mathrm{III}}^D(v, w, w', z, z'; k, k')$ separately for each case as follows:

- When $k = k'$, we define $\widehat{\mathsf{A}}_{\mathrm{III}}^D := \mathsf{A}_{\mathrm{III}}^D(v, w, z; 2 r_{k+1})$;

- When $k > k'$, we define

$$\widehat{\mathsf{A}}^D_{\mathrm{III}} := \mathsf{A}^D_{\mathrm{III}}(v,w,z;r^{\mathrm{out}}_k) \cap \big\{w' \xleftrightarrow{(D)} z'\big\} \cap \big\{w' \xleftrightarrow{(D)} \partial B_{v_e^-}(\tfrac{2r_{k'+1}}{C_7^2}) \cup \partial B_{v_e^-}(C_7 r^{\mathrm{in}}_k)\big\}^c. \tag{4.12}$$

Note that $\mathsf{A}^D_{\mathrm{III}}(v,w,z;2r_{k_\star+1}) = \mathsf{A}^D_{\mathrm{III}}(v,w,z;2r_{k_\star+1}) = \{\mathbf{A}^z_\star = 1\}$.

Similarly, we denote $\widehat{\mathsf{A}}_{\mathrm{III}} := \widehat{\mathsf{A}}^{\emptyset}_{\mathrm{III}}$ when $D=\emptyset$.

By the restriction property and Corollary 2.19, one has

$$\begin{aligned}
&\sum\nolimits_{z_5'\in\partial\mathcal{B}_{v_e^-}(\frac{2r_{k'+1}}{C_7})} \mathbb{P}\big(\overline{\mathsf{A}}_+(\bar z;r,r_k)\|\overline{\mathsf{A}}_-(\bar z';2r,2r_{k'})\big)\\
\le &\sum_{z_5'\in\partial\mathcal{B}_{v_e^-}(\frac{2r_{k'+1}}{C_7})} \mathbb{E}\big[\mathbb{1}_{\{\overline{\mathsf{A}}_+(\bar z;r,r_k)\|\widehat{\mathsf{A}}_-(\bar z';2r,2r_{k'})\}}\cdot \mathbb{P}\big(w_- \xleftrightarrow{(\widehat{\mathcal{C}}_-)} z_5' \mid \mathcal{F}_{\widehat{\mathcal{C}}_-}\big)\big]\\
\overset{(2.57)}{\lesssim} & \; r_k^{-d} \sum_{z_5'\in\partial\mathcal{B}_{v_e^-}(\frac{2r_{k'+1}}{C_7}),z_6'\in\partial\mathcal{B}_{v_e^-}(2r^{\mathrm{out}}_k),z_7'\in\partial\mathcal{B}_{v_e^-}(2r^{\mathrm{in}}_k)} \mathbb{E}\Big[\mathbb{1}_{\{\overline{\mathsf{A}}_+(\bar z;r,r_k)\|\widehat{\mathsf{A}}_-(\bar z';2r,2r_{k'})\}}\\
&\qquad\qquad \cdot \mathbb{P}\big(\widehat{\mathsf{A}}^{\widehat{\mathcal{C}}_-}_{\mathrm{III}}(v_e^-,w_-,z_5',z_6',z_7';k,k') \mid \mathcal{F}_{\widehat{\mathcal{C}}_-}\big)\Big]\\
= & \; r_k^{-d} \sum_{z_5'\in\partial\mathcal{B}_{v_e^-}(\frac{2r_{k'+1}}{C_7}),z_6'\in\partial\mathcal{B}_{v_e^-}(2r^{\mathrm{out}}_k),z_7'\in\partial\mathcal{B}_{v_e^-}(2r^{\mathrm{in}}_k)} \mathbb{P}\big(\overline{\mathsf{A}}_+(\bar z;r,r_k)\|\widetilde{\mathsf{A}}_-\big),
\end{aligned} \tag{4.13}$$

where $\widetilde{\mathsf{A}}_- := \widehat{\mathsf{A}}_-(\bar z';2r,2r_{k'}) \cap \widehat{\mathsf{A}}_{\mathrm{III}}(v_e^-,w_-,z_5',z_6',z_7';k,k')$. Following the same arguments as in (4.13), we have

$$\begin{aligned}
&\sum\nolimits_{z_3\in\partial\mathcal{B}_{v_e^+}(dr),z_4\in\partial\mathcal{B}_{v_e^+}(r_k)} \mathbb{P}\big(\overline{\mathsf{A}}_+(\bar z;r,r_k)\|\widetilde{\mathsf{A}}_-\big)\\
\lesssim & \; r_{k'}^{-d} \sum_{z_3\in\partial\mathcal{B}_{v_e^+}(dr),z_4\in\partial\mathcal{B}_{v_e^+}(r_k),z_6\in\partial\mathcal{B}_{v_e^+}(r^{\mathrm{out}}_{k'}),z_7\in\partial\mathcal{B}_{v_e^+}(r^{\mathrm{in}}_{k'})} \mathbb{P}\big(\widetilde{\mathsf{A}}_+\|\widetilde{\mathsf{A}}_-\big).
\end{aligned} \tag{4.14}$$

Here the event $\widetilde{\mathsf{A}}_+$ is defined by

$$\widetilde{\mathsf{A}}_+ := \mathsf{A}_{\mathrm{I}}(\hat v_e^+,z_1;r)\cap \mathsf{A}_{\mathrm{I}}(\hat v_{e'}^+,z_2;r)\cap \mathsf{A}_{\mathrm{III}}(v_e^+,w_+,z_5;r_{k+1})\cap \widehat{\mathsf{A}}_{\mathrm{II}}, \tag{4.15}$$

where $\widehat{\mathsf{A}}_{\mathrm{II}} = \widehat{\mathsf{A}}_{\mathrm{II}}(v_e^+,z_3,z_4,z_6,z_7;k,k')$ equals $\big\{z_3 \xleftrightarrow{(D)} z_4\big\} \cap \big\{z_3 \xleftrightarrow{(D)} \partial B_{v_e^+}(\frac{r_1}{C_7}) \cup \partial B_{v_e^+}(C_7 r_k)\big\}^c$ for $k=k'$, and equals

$$\begin{aligned}
&\big\{z_3 \xleftrightarrow{(D)} z_6, z_4 \xleftrightarrow{(D)} z_7\big\} \cap \big\{z_3 \xleftrightarrow{(D)} \partial B_{v_e^+}(C_7 r_1) \cup \partial B_{v_e^+}(\tfrac{r^{\mathrm{in}}_{k'}}{C_7})\big\}^c\\
&\cap \big\{z_3 \xleftrightarrow{(D)} \partial B_{v_e^+}(\tfrac{r_k}{C_7}) \cup \partial B_{v_e^+}(\tfrac{r^{\mathrm{out}}_{k'}}{C_7})\big\}^c
\end{aligned} \tag{4.16}$$

for $k > k'$. Plugging (4.14) into (4.13), we obtain

$$\begin{aligned}&\sum_{\hat z\in\widehat{\Upsilon}_+(r,r_k),\hat z'\in\widehat{\Upsilon}_-(2r,2r_{k'})} \mathbb{P}\big(\overline{\mathsf{A}}_+(z_1,z_2,\hat z;r,r_k)\|\overline{\mathsf{A}}_-(z_1',z_2',\hat z';2r,2r_{k'})\big)\\ \lesssim\, & r_k^{-d}r_{k'}^{-d}\sum_{\tilde z:=(z_3,z_4,z_5,z_6,z_7)\in\widetilde{\Upsilon}_+(k,k'),\tilde z':=(z_3',z_4',z_5',z_6',z_7')\in\widetilde{\Upsilon}_-(k,k')} \mathbb{P}\big(\widetilde{\mathsf{A}}_+\|\widetilde{\mathsf{A}}_-\big),\end{aligned} \tag{4.17}$$

where we denote $\widetilde{\Upsilon}_+(k,k') := \widehat{\Upsilon}_+(r,r_k)\times\partial\mathcal{B}_{v_e^+}(r_{k'}^{\mathrm{out}})\times\partial\mathcal{B}_{v_e^+}(r_{k'}^{\mathrm{in}})$ and $\widetilde{\Upsilon}_-(2r,2r_{k'}) := \widehat{\Upsilon}_-(2r,2r_{k'})\times\partial\mathcal{B}_{v_e^-}(2r_k^{\mathrm{out}})\times\partial\mathcal{B}_{v_e^-}(2r_k^{\mathrm{in}})$. See Figure 3 for an illustration of $\big\{\widetilde{\mathsf{A}}_+\|\widetilde{\mathsf{A}}_-\big\}$.

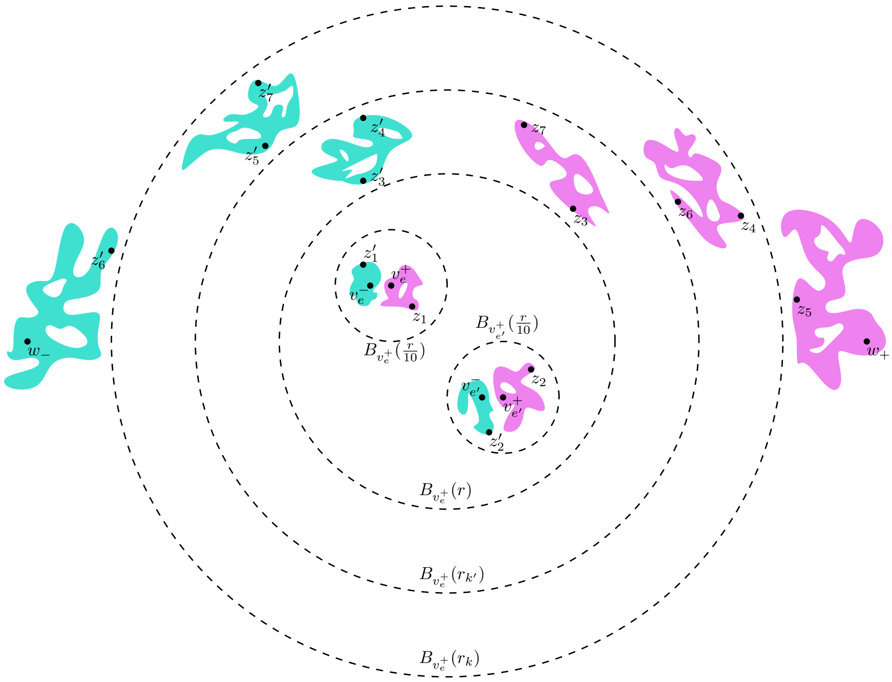

FIGURE 3. In this illustration, each colored region is a loop cluster. These clusters are arranged in disjoint balls and annuli in pairs such that each pair of clusters certifies a four-point event in $\{\mathsf{C}_j\}_{1\le j\le 5}$. In particular, when $k = k_\star$, as verified in the companion paper [6], imposing $\mathcal{C}_{w_-}$ (resp. $\mathcal{C}_{w_+}$) as an absorbing boundary typically does not alter the order of the probability of $z_5 \leftrightarrow w_+$ (resp. $z_6' \leftrightarrow w_-$), i.e., R^{2-d}. In light of this, we substitute the four-point event $\{z_5 \leftrightarrow w_+\|z_6' \leftrightarrow w_-\}$ with the event $\{\mathbf{A}_\star^{z_5} = \mathbf{A}_\star^{z_6'} = 1\}$ to simplify the analysis, while changing the probability by only a constant factor.

By (2.24), on $\big\{\widetilde{\mathsf{A}}_+\|\widetilde{\mathsf{A}}_-\big\}$, the events $\{\mathsf{C}_j\}_{1\le j\le 5}$ (as defined below) occur.

- $\mathsf{C}_1 := \big\{\hat v_e^+ \xleftrightarrow{(\partial B_{v_e^+}(\frac{r}{10}))} z_1\|\hat v_e^- \xleftrightarrow{(\partial B_{v_e^+}(\frac{r}{10}))} z_1'\big\}$;

- $\mathsf{C}_2 := \big\{\hat{v}^-_{e'} \xleftrightarrow{(\partial B_{v^+_{e'}}(\frac{r}{10}))} z_2 \| \hat{v}^-_{e'} \xleftrightarrow{(\partial B_{v^+_{e'}}(\frac{r}{10}))} z'_2\big\}$;
- C_3, C_4 and C_5, which are defined separately for the cases $k = k'$ and $k > k'$:
  - When $k = k'$,

$$\mathsf{C}_3 := \big\{z_3 \xleftrightarrow{(\partial B_{v^+_e}(\frac{r_1}{10C_7}) \cup \partial B_{v^+_e}(10C_7 r_k))} z_4 \| z'_3 \xleftrightarrow{(\partial B_{v^+_e}(\frac{r_1}{10C_7}) \cup \partial B_{v^+_e}(10C_7 r_k))} z'_4\big\}, \tag{4.18}$$

C_4 is the sure event, and

$$\text{when } k < k_\star,\ \mathsf{C}_5 := \big\{z_5 \xleftrightarrow{(\partial B_{v^+_e}(\frac{r_{k+1}}{10C_7^2}))} w_+ \| z'_5 \xleftrightarrow{(\partial B_{v^+_e}(\frac{r_{k+1}}{10C_7^2}))} w_-\big\}; \tag{4.19}$$

$$\text{when } k = k_\star,\ \mathsf{C}_5 := \big\{\mathbf{A}^{z_5}_\star = \mathbf{A}^{z'_5}_\star = 1\big\}. \tag{4.20}$$

  - When $k > k'$,

$$\mathsf{C}_3 := \big\{z_3 \xleftrightarrow{(\partial B_{v^+_e}(\frac{r_1}{10C_7}) \cup \partial B_{v^+_e}(10C_7 r^{\mathrm{in}}_{k'}))} z_7 \| z'_3 \xleftrightarrow{(\partial B_{v^+_e}(\frac{r_1}{10C_7}) \cup \partial B_{v^+_e}(10C_7 r^{\mathrm{in}}_{k'}))} z'_4\big\}, \tag{4.21}$$

$$\mathsf{C}_4 := \big\{z_6 \xleftrightarrow{(\partial B_{v^+_e}(\frac{r^{\mathrm{out}}_{k'}}{10C_7}) \cup \partial B_{v^+_e}(10C_7 r^{\mathrm{in}}_{k}))} z_4 \| z'_5 \xleftrightarrow{(\partial B_{v^+_e}(\frac{r^{\mathrm{out}}_{k'}}{10C_7}) \cup \partial B_{v^+_e}(10C_7 r^{\mathrm{in}}_{k}))} z'_7\big\}, \tag{4.22}$$

$$\text{when } k < k_\star,\ \mathsf{C}_5 := \big\{z_5 \xleftrightarrow{(\partial B_{v^+_e}(\frac{r^{\mathrm{out}}_{k}}{10C_7}))} w_+ \| z'_6 \xleftrightarrow{(\partial B_{v^+_e}(\frac{r^{\mathrm{out}}_{k}}{10C_7}))} w_-\big\}; \tag{4.23}$$

$$\text{when } k = k_\star,\ \mathsf{C}_5 := \big\{\mathbf{A}^{z_5}_\star = \mathbf{A}^{z'_6}_\star = 1\big\}. \tag{4.24}$$

Furthermore, the sub-events $\{\mathsf{C}_j\}_{1\le j\le 5}$ are independent since they depend on disjoint collections of loops in $\widetilde{\mathcal{L}}_{1/2}$. Meanwhile, by the isomorphism theorem and the estimates on heterochromatic four-point functions in [6, Lemma 6.4], one has: for any $d \ge 3$ with $d \neq 6$, sufficiently large $m \ge 1$, $M \ge 1$, $v, v' \in \widetilde{B}(10m) \setminus \widetilde{B}(m)$ with $|v-v'| \ge 1$, $w, w' \in \widetilde{B}(10M) \setminus \widetilde{B}(M)$ with $|w-w'| \ge M$ and $D \subset \widetilde{B}(\frac{m}{C}) \cup [\widetilde{B}(CM)]^c$,

$$\mathbb{P}\big(v \xleftrightarrow{(D)} v' \| w \xleftrightarrow{(D)} w'\big) \asymp |v-v'|^{(3-\frac{d}{2})\boxdot 0} M^{-[(\frac{3d}{2}-1)\boxdot(2d-4)]}. \tag{4.25}$$

This estimate directly implies that

$$\mathbb{P}(\mathsf{C}_1) \vee \mathbb{P}(\mathsf{C}_2) \lesssim r^{-\frac{3d}{2}+1}, \tag{4.26}$$

$$\mathbb{P}(\mathsf{C}_3) \cdot \mathbb{P}(\mathsf{C}_4) \lesssim r^{3-\frac{d}{2}} r_{k'}^{-\frac{3d}{2}+1} \cdot r_{k'}^{3-\frac{d}{2}} r_k^{-\frac{3d}{2}+1} = r^{3-\frac{d}{2}} r_{k'}^{-2d+4} r_k^{-\frac{3d}{2}+1}, \tag{4.27}$$

$$\mathbb{P}(\mathsf{C}_5) \lesssim r_k^{3-\frac{d}{2}} N^{-\frac{3d}{2}+1}. \tag{4.28}$$

Note that in (4.28) we used $\mathbb{P}(\mathbf{A}^z_\star = 1) = R^{2-d} \asymp r_{k_\star}^{2-d}$ when $k = k_\star$. Consequently,

$$\mathbb{P}\big(\widetilde{\mathsf{A}}_+ \| \widetilde{\mathsf{A}}_-\big) \le \prod\nolimits_{1\le j\le 5} \mathbb{P}(\mathsf{C}_j) \lesssim r^{-\frac{7d}{2}+5} r_{k'}^{-2d+4} r_k^{-2d+4} N^{-\frac{3d}{2}+1}. \tag{4.29}$$

Inserting (4.29) into (4.17), we derive the claim (4.10) as follows:

$$\begin{aligned}
&\sum_{\hat{z}\in\widehat{\Upsilon}_+(r,r_k),\hat{z}'\in\widehat{\Upsilon}_-(2r,2r_{k'})} \mathbb{P}\big(\overline{\mathsf{A}}_+(z_1,z_2,\hat{z};r,r_k)\|\overline{\mathsf{A}}_-(z_1',z_2',\hat{z}';2r,2r_{k'})\big)\\
\lesssim& r_k^{-d}r_{k'}^{-d}\cdot|\widehat{\Upsilon}_+(r,r_k)|\cdot|\widehat{\Upsilon}_-(2r,2r_{k'})|\cdot r^{-\frac{7d}{2}+5}r_{k'}^{-2d+4}r_k^{-2d+4}N^{-\frac{3d}{2}+1}\\
\lesssim& r^{-\frac{3d}{2}+3}r_k^d r_{k'}^d N^{-\frac{3d}{2}+1}.
\end{aligned}\tag{4.30}$$

In conclusion, we have confirmed Proposition 3.1 assuming Lemma 4.1. $\square$

4.2. **Proof of Lemma 4.1.** Using the switching identity, we have (recalling the notations $\mathcal{P}^{(i)}$ and $\check{\mathcal{C}}$ below Lemma 2.21)

$$\begin{aligned}
&\mathbb{P}\big(\{w\xleftrightarrow{(D)}v_1,v_2\}\cap\{\widehat{\mathcal{L}}^{D,v_1}_{1/2},\widehat{\mathcal{L}}^{D,v_2}_{1/2}\in[0,C_6]\}\big)\\
=&\int_{0\le a\le b\le C_6}\check{\mathbb{P}}^D_{v_1\leftrightarrow v_2,a,b}\big(w\in\check{\mathcal{C}}\big)\mathfrak{p}^D_{v_1\leftrightarrow v_2,a,b}\mathrm{d}a\mathrm{d}b.
\end{aligned}\tag{4.31}$$

For each integer $k\in[1,k_\star-1]$, we define G_k as the event that $\cup(\sum_{i\in\{2,3,4\}}\mathcal{P}^{(i)})$ (under the measure $\check{\mathbb{P}}^D_{v_1\leftrightarrow v_2,a,b}(\cdot)$) intersects $\partial B_{v_1}(r_k)$ and is contained in $\widetilde{B}_{v_1}(r_{k+1})$. In addition, we denote the events

$$\mathsf{G}_0:=\big\{\cup\big(\sum\nolimits_{i\in\{2,3,4\}}\mathcal{P}^{(i)}\big)\subset\partial\widetilde{B}_{v_1}(r_1)\big\}\text{ and }\mathsf{G}_{k_\star}:=\big(\cup_{0\le k\le k_\star-1}\mathsf{G}_k\big)^c.\tag{4.32}$$

Therefore, it follows from (4.31) that

$$\begin{aligned}
&\mathbb{P}\big(\{w\xleftrightarrow{(D)}v_1,v_2\}\cap\{\widehat{\mathcal{L}}^{D,v_1}_{1/2},\widehat{\mathcal{L}}^{D,v_2}_{1/2}\in[0,C_6]\}\big)\\
\lesssim&\mathbb{P}(v_1\xleftrightarrow{(D)}v_2)\cdot\sum\nolimits_{0\le k\le k_\star}\max_{0\le a\le b\le C_6}\check{\mathbb{P}}^D_{v_1\leftrightarrow v_2,a,b}(w\in\check{\mathcal{C}},\mathsf{G}_k).
\end{aligned}\tag{4.33}$$

For convenience, we denote the k-th term on the right-hand side of (4.5) by

$$\mathbf{T}_k:=r^{-d-1}r_k^{-\frac{3d}{2}+1}\mathbf{Q}^D_{\mathrm{I}}(v_1',v_2';r)\mathbf{Q}^D_{\mathrm{II}}(v_1;r_1,r_k)\mathbf{Q}^D_{\mathrm{III}}(v_1,w;r_{k+1}).\tag{4.34}$$

Thus, it suffices to prove that for any $0\le k\le k_\star$,

$$\mathbb{P}(v_1\xleftrightarrow{(D)}v_2)\cdot\max_{0\le a\le b\le C_6}\check{\mathbb{P}}^D_{v_1\leftrightarrow v_2,a,b}(w\in\check{\mathcal{C}},\mathsf{G}_k)\lesssim\mathbf{T}_{k\vee 1}.\tag{4.35}$$

In what follows, we arbitrarily fix numbers a,b with $0\le a\le b\le C_6$.

When $k=0$**.** Recall $\mathbf{Q}_\cdot(\cdot)$ in (4.1), (4.2) and (4.4). Since $\mathbf{Q}^D_{\mathrm{II}}(v_1;r_1,r)=r_1^d$,

$$\mathbf{T}_0=r^{-\frac{3d}{2}}\mathbf{Q}^D_{\mathrm{I}}(v_1',v_2';r)\mathbf{Q}^D_{\mathrm{III}}(v_1,w;r_1).\tag{4.36}$$

Note that Corollary 2.11 directly implies

$$\mathbb{P}(v_1\xleftrightarrow{(D)}v_2)\lesssim r^{-d}\mathbf{Q}^D_{\mathrm{I}}(v_1,v_2;r).\tag{4.37}$$

Recall from the conditions that the endpoint x_i of e_i is contained in D, and that on the interval I_{e_i}, points x_i and v_i' lie on opposite sides of v_i. As a result, any subset of $\widetilde{\mathbb{Z}}^d \setminus D$ connecting v_i and $z \in \partial \mathcal{B}_{v_i}(\frac{r}{C_7})$ must hit v_i', which yields $\mathbf{Q}_{\mathrm{I}}^D(v_1, v_2; r) \le \mathbf{Q}_{\mathrm{I}}^D(v_1', v_2'; r)$. Combined with (4.37), it implies that

$$\mathbb{P}(v_1 \xleftrightarrow{(D)} v_2) \lesssim r^{-d} \mathbf{Q}_{\mathrm{I}}^D(v_1', v_2'; r). \tag{4.38}$$

Meanwhile, on the event $\{w \in \check{\mathcal{C}}\} = \{w \xleftrightarrow{\cup \mathcal{P}^{(1)}} \cup\big(\sum_{i\in\{2,3,4\}} \mathcal{P}^{(i)}\big)\}$, if G_0 occurs, then one has $w \xleftrightarrow{\cup \mathcal{P}^{(1)}} \widetilde{B}_{v_1}(r_1)$. Thus, since $\mathcal{P}^{(1)} = \widetilde{\mathcal{L}}_{1/2}^{D\cup\{v_1,v_2\}} \le \widetilde{\mathcal{L}}_{1/2}^{D}$,

$$\check{\mathbb{P}}^D_{v_1\leftrightarrow v_2,a,b}\big(w \in \check{\mathcal{C}}, \mathsf{G}_0\big) \le \mathbb{P}\big(w \xleftrightarrow{(D)} \widetilde{B}_{v_1}(r_1)\big). \tag{4.39}$$

Moreover, Corollary 2.14 implies that for any $0 \le k \le k_\star$,

$$\mathbb{P}\big(w \xleftrightarrow{(D)} \widetilde{B}_{v_1}(r_{k+1})\big) \lesssim r_k^{-\frac{d}{2}} \mathbf{Q}_{\mathrm{III}}^D(v_1, w; r_k). \tag{4.40}$$

Combining (4.38), (4.39) and (4.40), we derive (4.35) for $k = 0$.

When $1 \le k \le k_\star - 1$. Similarly, when $\{w \in \check{\mathcal{C}}, \mathsf{G}_k\}$ occurs, one has $w \xleftrightarrow{\cup \mathcal{P}^{(1)}} \widetilde{B}_{v_1}(r_{k+1})$, which implies that

$$\begin{aligned} &\check{\mathbb{P}}^D_{v_1\leftrightarrow v_2,a,b}\big(w \in \check{\mathcal{C}}, \mathsf{G}_k\big) \\ \le &\check{\mathbb{P}}^D_{v_1\leftrightarrow v_2,a,b}\Big(\bigcup_{i\in\{2,3,4\}} \big\{(\cup \mathcal{P}^{(i)}) \cap \partial B_{v_1}(r_k) \neq \emptyset\big\}\Big) \cdot \mathbb{P}\big(w \xleftrightarrow{(D)} \widetilde{B}_{v_1}(r_{k+1})\big) \\ \le &\sum\nolimits_{i\in\{2,3,4\}} \check{\mathbb{P}}^D_{v_1\leftrightarrow v_2,a,b}\big((\cup \mathcal{P}^{(i)}) \cap \partial B_{v_1}(r_k) \neq \emptyset\big) \cdot \mathbb{P}\big(w \xleftrightarrow{(D)} \widetilde{B}_{v_1}(r_{k+1})\big). \end{aligned} \tag{4.41}$$

For each $i \in \{2, 3\}$, note that

$$\begin{aligned} &\check{\mathbb{P}}^D_{v_1\leftrightarrow v_2,a,b}\big((\cup \mathcal{P}^{(i)}) \cap \partial B_{v_1}(r_k) \neq \emptyset\big) \\ \lesssim &\mathbf{e}^D_{v_{i-1},v_{i-1}}\big(\{\widetilde{\eta} : \mathrm{ran}(\widetilde{\eta}) \cap \partial B_{v_1}(r_k) \neq \emptyset\}\big). \end{aligned} \tag{4.42}$$

Moreover, applying Lemma 2.20 (where we drop both two terms in the third line of (2.64)), we derive that

$$\mathbf{e}^D_{v_{i-1},v_{i-1}}\big(\{\widetilde{\eta} : \mathrm{ran}(\widetilde{\eta}) \cap \partial B_{v_1}(r_k) \neq \emptyset\}\big) \lesssim r^{1-d} r_k^{1-d} \mathbf{Q}_{\mathrm{II}}^D(v_1; r_1, r_k). \tag{4.43}$$

Combined with (4.42), it implies that for $i \in \{2, 3\}$,

$$\check{\mathbb{P}}^D_{v_1\leftrightarrow v_2,a,b}\big((\cup \mathcal{P}^{(i)}) \cap \partial B_{v_1}(r_k) \neq \emptyset\big) \lesssim r^{1-d} r_k^{1-d} \mathbf{Q}_{\mathrm{II}}^D(v_1; r_1, r_k). \tag{4.44}$$

For $i = 4$, according to (2.76), $\check{\mathbb{P}}^D_{v_1\leftrightarrow v_2,a,b}\big((\cup \mathcal{P}^{(4)}) \cap \partial B_{v_1}(r_k) \neq \emptyset\big)$ is at most of the same order as the probability of an excursion hitting $\partial B_{v_1}(r_k)$ under the measure $\mathbf{e}^D_{v_1,v_2}$. Combined with the fact that the total mass of $\mathbf{e}^D_{v_1,v_2}$ equals $\mathbb{K}_{D\cup\{v_1,v_2\}}(v_1, v_2)$, it yields that

$$\begin{aligned}
&\breve{\mathbb{P}}^D_{v_1\leftrightarrow v_2,a,b}\big((\cup\mathcal{P}^{(4)})\cap\partial B_{v_1}(r_k)\neq\emptyset\big)\\
\lesssim&\big[\mathbb{K}_{D\cup\{v_1,v_2\}}(v_1,v_2)\big]^{-1}\cdot\mathbf{e}^D_{v_1,v_2}\big(\{\widetilde{\eta}:\mathrm{ran}(\widetilde{\eta})\cap\partial B_{v_1}(r_k)\neq\emptyset\}\big)\\
\overset{\text{(Lemma 2.20)}}{\lesssim}&\big[\mathbb{K}_{D\cup\{v_1,v_2\}}(v_1,v_2)\big]^{-1}\cdot r^{-1-d}r_k^{1-d}\mathbf{Q}^D_{\mathrm{I}}(v_1',v_2';r)\mathbf{Q}^D_{\mathrm{II}}(v_1;r_1,r_k).
\end{aligned}\tag{4.45}$$

Here in the application of Lemma 2.20, we used the case $\chi>10c^{-1}$ and dropped the second term in the third line of (2.64). Meanwhile, by (2.20) and (4.38),

$$\mathbb{P}(v_1\xleftrightarrow{(D)}v_2)\lesssim\big[\mathbb{K}_{D\cup\{v_1,v_2\}}(v_1,v_2)\big]\wedge\big[r^{-d}\mathbf{Q}^D_{\mathrm{I}}(v_1',v_2';r)\big].\tag{4.46}$$

Combining (4.44), (4.45) and (4.46), we obtain that

$$\begin{aligned}
&\sum\nolimits_{i\in\{2,3,4\}}\mathbb{P}(v_1\xleftrightarrow{(D)}v_2)\cdot\breve{\mathbb{P}}^D_{v_1\leftrightarrow v_2,a,b}\big((\cup\mathcal{P}^{(i)})\cap\partial B_{v_1}(r_k)\neq\emptyset\big)\\
\lesssim&r^{-1-d}r_k^{1-d}\mathbf{Q}^D_{\mathrm{I}}(v_1',v_2';r)\mathbf{Q}^D_{\mathrm{II}}(v_1;r_1,r_k).
\end{aligned}\tag{4.47}$$

Putting (4.40), (4.44) and (4.47) together, we derive (4.35) for $1\le k\le k_\star-1$.

When $k=k_\star$. Since $\mathbf{Q}^D_{\mathrm{III}}(v_1,w;r_{k_\star+1})\asymp R$, it suffices to prove

$$\begin{aligned}
&\mathbb{P}(v_1\xleftrightarrow{(D)}v_2)\cdot\breve{\mathbb{P}}^D_{v_1\leftrightarrow v_2,a,b}(w\in\breve{\mathcal{C}},\mathsf{G}_{k_\star})\\
\lesssim&r^{-d-1}R^{-\frac{3d}{2}+2}\mathbf{Q}^D_{\mathrm{I}}(v_1',v_2';r)\mathbf{Q}^D_{\mathrm{II}}(v_1;r_1,r_k).
\end{aligned}\tag{4.48}$$

We denote $l_\dagger:=\min\big\{l\in\mathbb{N}:2^{l+3}\ge R\big\}$. For each $0\le l\le l_\dagger-1$, we define the box $\mathfrak{B}_l:=\widetilde{B}_w(2^l)$. We also denote $\mathfrak{B}_{-1}:=\emptyset$ and $\mathfrak{B}_{l_\dagger}:=[\widetilde{B}_{v_1}(r_{k_\star})]^c$. For each $i\in\{2,3,4\}$ and $0\le l\le l_\dagger$, we define the event

$$\mathsf{G}^{i,l}_\star:=\big\{\big(\cup\mathcal{P}^{(i)}\big)\cap\mathfrak{B}_l\neq\emptyset,\big(\cup\mathcal{P}^{(i)}\big)\cap\mathfrak{B}_{l-1}=\emptyset\big\}.\tag{4.49}$$

Since $\mathsf{G}_{k_\star}\subset\cup_{i\in\{2,3,4\},0\le l\le l_\dagger}\mathsf{G}^{i,l}_\star$, it follows from the union bound that

$$\breve{\mathbb{P}}^D_{v_1\leftrightarrow v_2,a,b}\big(w\in\breve{\mathcal{C}},\mathsf{G}_{k_\star}\big)\lesssim\sum\nolimits_{i\in\{2,3,4\}}\sum\nolimits_{0\le l\le l_\dagger}\breve{\mathbb{P}}^D_{v_1\leftrightarrow v_2,a,b}\big(w\in\breve{\mathcal{C}},\mathsf{G}^{i,l}_\star\big).\tag{4.50}$$

Moreover, for $0\le l\le l_\dagger$, on the event $\{w\in\breve{\mathcal{C}}\}\cap\mathsf{G}_{k_\star}$, the cluster of $\cup\mathcal{P}^{(1)}$ containing w must intersect $\widetilde{\partial}\mathfrak{B}_{l-1}$ (where we set $\widetilde{\partial}\emptyset:=\widetilde{\mathbb{Z}}^d$ for completeness). Thus,

$$\begin{aligned}
&\breve{\mathbb{P}}^D_{v_1\leftrightarrow v_2,a,b}\big(w\in\breve{\mathcal{C}},\mathsf{G}^{i,l}_\star\big)\\
\le&\breve{\mathbb{P}}^D_{v_1\leftrightarrow v_2,a,b}\big(\big(\cup\mathcal{P}^{(i)}\big)\cap\mathfrak{B}_l\neq\emptyset\big)\cdot\mathbb{P}\big(w\xleftrightarrow{(D)}\widetilde{\partial}\mathfrak{B}_{l-1}\big).
\end{aligned}\tag{4.51}$$

Since $\widetilde{\mathcal{L}}^D_{1/2}\le\widetilde{\mathcal{L}}_{1/2}$, it follows from (1.2) that

$$\mathbb{P}\big(w\xleftrightarrow{(D)}\widetilde{\partial}\mathfrak{B}_{l-1}\big)\lesssim\theta_d(2^l)\lesssim2^{l(-\frac{d}{2}+1)}.\tag{4.52}$$

Next, we estimate the probability $\check{\mathbb{P}}^D_{v_1\leftrightarrow v_2,a,b}\big((\cup\mathcal{P}^{(i)})\cap\mathfrak{B}_l\neq\emptyset\big)$ for $i\in\{2,3,4\}$. When $i\in\{2,3\}$, as in (4.42), one has

$$\check{\mathbb{P}}^D_{v_1\leftrightarrow v_2,a,b}\big((\cup\mathcal{P}^{(i)})\cap\mathfrak{B}_l\neq\emptyset\big)\lesssim \mathbf{e}^D_{v_{i-1},v_{i-1}}\big(\{\widetilde{\eta}:\mathrm{ran}(\widetilde{\eta})\cap\mathfrak{B}_l\neq\emptyset\}\big). \tag{4.53}$$

Moreover, by applying Lemma 2.20, we have

$$\mathbf{e}^D_{v_{i-1},v_{i-1}}\big(\{\widetilde{\eta}:\mathrm{ran}(\widetilde{\eta})\cap\mathfrak{B}_l\neq\emptyset\}\big)\lesssim r^{1-d}R^{3-2d}\mathbf{Q}^D_{\mathrm{II}}(v_1;r_1,r_{k_\star})\cdot 2^{l(d-2)}. \tag{4.54}$$

Here we dropped the first term in the third line of (2.64), and we bound the second term using the fact that

$$\widetilde{\mathbb{P}}_z\big(\tau_{\mathfrak{B}_l}<\infty\big)\lesssim \big(2^l/R\big)^{d-2},\ \ \forall z\in B(\tfrac{R}{2}). \tag{4.55}$$

Combining (4.53) and (4.54), we obtain that

$$\check{\mathbb{P}}^D_{v_1\leftrightarrow v_2,a,b}\big((\cup\mathcal{P}^{(i)})\cap\mathfrak{B}_l\neq\emptyset\big)\lesssim r^{1-d}R^{3-2d}\mathbf{Q}^D_{\mathrm{II}}(v_1;r_1,r_{k_\star})\cdot 2^{l(d-2)}. \tag{4.56}$$

When $i=4$, note that

$$\begin{aligned}&\check{\mathbb{P}}^D_{v_1\leftrightarrow v_2,a,b}\big((\cup\mathcal{P}^{(4)})\cap\mathfrak{B}_l\neq\emptyset\big)\\ \lesssim&\big[\mathbb{K}_{D\cup\{v_1,v_2\}}(v_1,v_2)\big]^{-1}\cdot\mathbf{e}^D_{v_1,v_2}\big(\{\widetilde{\eta}:\mathrm{ran}(\widetilde{\eta})\cap\mathfrak{B}_l\neq\emptyset\}\big).\end{aligned} \tag{4.57}$$

Moreover, Lemma 2.20 implies that

$$\begin{aligned}&\mathbf{e}^D_{v_1,v_2}\big(\{\widetilde{\eta}:\mathrm{ran}(\widetilde{\eta})\cap\mathfrak{B}_l\neq\emptyset\}\big)\\ \lesssim& r^{-d-1}R^{3-2d}\mathbf{Q}^D_{\mathrm{I}}(v_1',v_2';r)\mathbf{Q}^D_{\mathrm{II}}(v_1;r_1,r_{k_\star})\cdot 2^{l(d-2)}.\end{aligned} \tag{4.58}$$

Here we used the case $\chi=r>10c^{-1}$, and bounded the second term in the third line of (2.64) through (4.55). Plugging (4.58) into (4.57), one has

$$\begin{aligned}&\check{\mathbb{P}}^D_{v_1\leftrightarrow v_2,a,b}\big((\cup\mathcal{P}^{(4)})\cap\mathfrak{B}_l\neq\emptyset\big)\\ \lesssim&\big[\mathbb{K}_{D\cup\{v_1,v_2\}}(v_1,v_2)\big]^{-1}r^{-d-1}R^{3-2d}\mathbf{Q}^D_{\mathrm{I}}(v_1',v_2';r)\mathbf{Q}^D_{\mathrm{II}}(v_1;r_1,r_{k_\star})\cdot 2^{l(d-2)}.\end{aligned} \tag{4.59}$$

Putting (4.46), (4.52), (4.56) and (4.59) together, we obtain that

$$\begin{aligned}&\mathbb{P}(v_1\xleftrightarrow{(D)}v_2)\cdot\sum_{i\in\{2,3,4\}}\sum_{0\le l\le l_\dagger}\check{\mathbb{P}}^D_{v_1\leftrightarrow v_2,a,b}\big((\cup\mathcal{P}^{(i)})\cap\mathfrak{B}_l\neq\emptyset\big)\mathbb{P}\big(w\xleftrightarrow{(D)}\widetilde{\partial}\mathfrak{B}_{l-1}\big)\\ \lesssim& r^{-d-1}R^{3-2d}\mathbf{Q}^D_{\mathrm{I}}(v_1',v_2';r)\mathbf{Q}^D_{\mathrm{II}}(v_1;r_1,r_{k_\star})\cdot\sum\nolimits_{0\le l\le l_\dagger}2^{l(\frac{d}{2}-1)}\\ \lesssim& r^{-d-1}R^{-\frac{3d}{2}+2}\mathbf{Q}^D_{\mathrm{I}}(v_1',v_2';r)\mathbf{Q}^D_{\mathrm{II}}(v_1;r_1,r_{k_\star}).\end{aligned} \tag{4.60}$$

Combined with (4.50) and (4.51), it yields (4.48).

To sum up, we have verified Lemma 4.1, thereby completing the proof of Proposition 3.1. □

5. Typical number of pivotal loops at scale 1

In this section, we aim to prove Theorem 1.6.

5.1. **Proof of Theorem 1.6: upper bound.** To achieve this, we first estimate the first moment of $\widetilde{\mathbf{Piv}}_N$ as follows. Recall that for each $e \in \mathbb{L}^d$, we denote the two trisection points of I_e by v_e^+ and v_e^-. In addition, we denote the point measure

$$\mathfrak{L}_e := \widetilde{\mathcal{L}}_{1/2} \cdot \mathbb{1}_{\mathrm{ran}(\widetilde{\ell}) \subset I_e, \{v_e^+, v_e^-\} \subset \mathrm{ran}(\widetilde{\ell})}. \tag{5.1}$$

For any edge $e \in \mathbb{L}^d$ and event A, we denote by $\mathsf{L}_e(\mathsf{A})$ the event that there exists a loop $\widetilde{\ell} \in \mathfrak{L}_e$ which is pivotal for A. In addition, we define the set

$$\widetilde{\mathbf{Piv}}(\mathsf{A}) := \big\{ e \in \mathbb{L}^d : \mathsf{L}_e(\mathsf{A}) \text{ occurs} \big\}. \tag{5.2}$$

Recall that we denote $\widetilde{\mathbf{Piv}}_N = \widetilde{\mathbf{Piv}}(\mathbf{0} \leftrightarrow \partial B(N))$.

Lemma 5.1. *For any $d \geq 3$ with $d \neq 6$, and any $N \geq 1$,*

$$\mathbb{E}\big[\big| \widetilde{\mathbf{Piv}}_N \big| \mid \mathbf{0} \leftrightarrow \partial B(N) \big] \lesssim N^{(\frac{d}{2}-1) \boxdot 2}. \tag{5.3}$$

Proof. In this proof, we abbreviate $\mathsf{L}_e\big(\mathbf{0} \leftrightarrow \partial B(N)\big)$ as L_e. For each $e = \{x_e^1, x_e^2\} \in \mathbb{L}^d$, on the event L_e, we know that $\mathsf{F}_e := \big\{ \mathfrak{L}_e \neq 0 \big\}$ and

$$\mathsf{C}_e := \cup_{i \in \{1,2\}} \big\{ x_e^i \xleftrightarrow{\cup(\widetilde{\mathcal{L}}_{1/2} - \mathfrak{L}_e)} \mathbf{0} \| x_e^{3-i} \xleftrightarrow{\cup(\widetilde{\mathcal{L}}_{1/2} - \mathfrak{L}_e)} \partial B(N) \big\} \tag{5.4}$$

both occur. Note that F_e and C_e are independent (since they rely on disjoint collections of loops in $\widetilde{\mathcal{L}}_{1/2}$), and that $\mathbb{P}(\mathsf{F}_e)$ is bounded away from 0 and 1. Therefore,

$$\mathbb{P}\big(\mathsf{L}_e\big) \leq \mathbb{P}\big(\mathsf{C}_e \cap \mathsf{F}_e\big) = \mathbb{P}\big(\mathsf{C}_e \cap \mathsf{F}_e^c\big) \cdot \frac{\mathbb{P}(\mathsf{F}_e)}{\mathbb{P}(\mathsf{F}_e^c)} \asymp \mathbb{P}\big(\mathsf{C}_e \cap \mathsf{F}_e^c\big). \tag{5.5}$$

Moreover, on $\mathsf{F}_e^c = \{\mathfrak{L}_e = 0\}$, the event C_e is equivalent to

$$\overline{\mathsf{C}}_e := \cup_{i \in \{1,2\}} \overline{\mathsf{C}}_e^i := \cup_{i \in \{1,2\}} \big\{ x_e^i \leftrightarrow \mathbf{0} \| x_e^{3-i} \leftrightarrow \partial B(N) \big\}. \tag{5.6}$$

Combining (5.5) and (5.6), and using the union bound, we have

$$\begin{aligned} \mathbb{E}\big[\big| \widetilde{\mathbf{Piv}}_N \big| \cdot \mathbb{1}_{\mathbf{0} \leftrightarrow \partial B(N)} \big] &= \sum\nolimits_{e \in \mathbb{L}^d : I_e \subset \widetilde{B}(N)} \mathbb{P}\big(\mathsf{L}_e\big) \\ &\lesssim \sum\nolimits_{e \in \mathbb{L}^d : I_e \subset \widetilde{B}(N)} \sum\nolimits_{i \in \{1,2\}} \mathbb{P}\big(\overline{\mathsf{C}}_e^i\big). \end{aligned} \tag{5.7}$$

We claim that for any $e = \{x, y\} \in \mathbb{L}^d$ with $x, y \in B(N)$,

$$\mathbb{P}\big(x \leftrightarrow \mathbf{0} \| y \leftrightarrow \partial B(N)\big) \lesssim \begin{cases} (|x|+1)^{-[(\frac{d}{2}+1) \boxdot (d-2)]} N^{-[(\frac{d}{2}-1) \boxdot 2]} & x \in B(\frac{N}{2}); \\ \big(\mathrm{dist}(x, \partial B(N)) + 1\big)^{-2} N^{2-d} & x \notin B(\frac{N}{2}). \end{cases} \tag{5.8}$$

In fact, this claim together with (5.7) implies the desired bound (5.3). Specifically, by (5.7) and (5.8), we have

$$\mathbb{E}\big[\big| \widetilde{\mathbf{Piv}}_N \big| \cdot \mathbb{1}_{\mathbf{0} \leftrightarrow \partial B(N)} \big] \lesssim \mathbb{I}^{\mathrm{in}} + \mathbb{I}^{\mathrm{out}}, \tag{5.9}$$

where the quantities $\mathbb{I}^{\mathrm{in}}$ and $\mathbb{I}^{\mathrm{out}}$ are define by

$$\mathbb{I}^{\mathrm{in}} := N^{-[(\frac{d}{2}-1)\square 2]} \sum\nolimits_{x\in B(\frac{N}{2})} (|x|+1)^{-[(\frac{d}{2}+1)\square(d-2)]}, \tag{5.10}$$

$$\mathbb{I}^{\mathrm{out}} := N^{2-d} \sum\nolimits_{x\in B(N)\setminus B(\frac{N}{2})} \big(\mathrm{dist}(x,\partial B(N))+1\big)^{-2}. \tag{5.11}$$

Using the fact that $|\partial B(k)| \asymp k^{d-1}$ for $k\ge 1$, we have

$$\mathbb{I}^{\mathrm{in}} \lesssim N^{-[(\frac{d}{2}-1)\square 2]} \sum\nolimits_{0\le k\le \frac{N}{2}} (k+1)^{(\frac{d}{2}-2)\square 1} \lesssim 1, \tag{5.12}$$

$$\mathbb{I}^{\mathrm{out}} \lesssim N^{2-d} \sum\nolimits_{\frac{N}{2}\le k\le N} (N-k+1)^{d-3} \lesssim 1. \tag{5.13}$$

Plugging (5.12) and (5.13) into (5.9), and using (1.2) and (1.4), we obtain (5.3).

Next, we prove the claim (5.8) for the cases $x\in B(\frac{N}{2})$ and $x\notin B(\frac{N}{2})$ separately.

Case 1: when $x\in B(\frac{N}{2})$**.** Applying Lemma 2.13, one has

$$\mathbb{P}\big(y \xleftrightarrow{(D)} \partial B(N)\big) \lesssim N^{-(\frac{d}{2}\square 3)} \sum\nolimits_{z\in \partial\mathcal{B}(d^{-1}N)} \mathbb{P}\big(y \xleftrightarrow{(D)} z\big). \tag{5.14}$$

Combined with the restriction property, it yields that

$$\begin{aligned}\mathbb{P}\big(x\leftrightarrow \mathbf{0} \| y\leftrightarrow \partial B(N)\big) =& \mathbb{E}\big[\mathbb{1}_{x\leftrightarrow\mathbf{0}}\cdot \mathbb{P}\big(y \xleftrightarrow{(\mathcal{C}_{\mathbf{0}})} \partial B(N) \mid \mathcal{F}_{\mathcal{C}_{\mathbf{0}}}\big)\big]\\ \lesssim& N^{-(\frac{d}{2}\square 3)} \sum\nolimits_{z\in\partial\mathcal{B}(d^{-1}N)} \mathbb{P}\big(x\leftrightarrow\mathbf{0}\|y\leftrightarrow z\big).\end{aligned} \tag{5.15}$$

Moreover, for any $z\in\partial\mathcal{B}(d^{-1}N)$, by the restriction property and Corollary 2.12,

$$\begin{aligned}&\mathbb{P}\big(x\leftrightarrow\mathbf{0}\|y\leftrightarrow z\big) = \mathbb{E}\big[\mathbb{1}_{x\leftrightarrow\mathbf{0}}\cdot\mathbb{P}\big(y\xleftrightarrow{(\mathcal{C}_{\mathbf{0}})} z \mid \mathcal{F}_{\mathcal{C}_{\mathbf{0}}}\big)\big]\\ \lesssim& N^{2-d}(|x|+1)^{-1}\sum\nolimits_{z'\in\partial\mathcal{B}(d^{-1}|x|)} \mathbb{E}\big[\mathbb{1}_{x\leftrightarrow\mathbf{0}}\cdot\mathbb{P}\big(y\xleftrightarrow{(\mathcal{C}_{\mathbf{0}})} z' \mid \mathcal{F}_{\mathcal{C}_{\mathbf{0}}}\big)\big]\\ =& N^{2-d}(|x|+1)^{-1}\sum\nolimits_{z'\in\partial\mathcal{B}(d^{-1}|x|)} \mathbb{P}\big(x\leftrightarrow\mathbf{0}\|y\leftrightarrow z'\big)\\ \overset{(4.25)}{\lesssim}& (|x|+1)^{-[(\frac{d}{2}+1)\square(2-d)]}N^{2-d}.\end{aligned} \tag{5.16}$$

This together with (5.15) yields that

$$\mathbb{P}\big(x\leftrightarrow\mathbf{0}\|y\leftrightarrow\partial B(N)\big) \lesssim (|x|+1)^{-[(\frac{d}{2}+1)\square(2-d)]} N^{-[(\frac{d}{2}-1)\square 2]}. \tag{5.17}$$

Case 2: when $x\notin B(\frac{N}{2})$**.** Assume that $x\in\partial B(N-k)$ with $0\le k\le\frac{N}{2}$. Thus,

$$\mathbb{P}\big(x\leftrightarrow\mathbf{0}\|y\leftrightarrow\partial B(N)\big) \le \mathbb{P}\big(x\leftrightarrow\mathbf{0}\|y\leftrightarrow\partial B_y(k)\big). \tag{5.18}$$

Moreover, similar to (5.15), by the restriction property and Lemma 2.13, one has

$$\mathbb{P}\big(x\leftrightarrow\mathbf{0}\|y\leftrightarrow\partial B_y(k)\big) \lesssim (k+1)^{-(\frac{d}{2}\square 3)} \sum\nolimits_{z\in\partial\mathcal{B}_y(d^{-1}k)} \mathbb{P}\big(x\leftrightarrow\mathbf{0}\|y\leftrightarrow z\big). \tag{5.19}$$

In addition, as in (5.16), it follows from Corollary 2.12 that for any $z \in \partial \mathcal{B}_y(d^{-1}k)$,

$$
\begin{aligned}
\mathbb{P}\big(x \leftrightarrow \mathbf{0} \| y \leftrightarrow z\big) \lesssim & N^{2-d}(k+1)^{-1} \sum\nolimits_{z' \in \partial \mathcal{B}_y(k)} \mathbb{P}\big(x \leftrightarrow z' \| y \leftrightarrow z\big) \\
\overset{(4.25)}{\lesssim} & N^{2-d}(k+1)^{-[(\frac{d}{2}+1)\square(d-2)]}.
\end{aligned} \tag{5.20}
$$

Substituting (5.20) into (5.19), we get

$$
\mathbb{P}\big(x \leftrightarrow \mathbf{0} \| y \leftrightarrow \partial B_y(k)\big) \lesssim N^{2-d}(k+1)^{-2}. \tag{5.21}
$$

To sum up, we have verified the claim (5.8), thereby completing the proof. $\square$

By Lemma 5.1 and Markov's inequality, we have: for any $\epsilon > 0$, there exists $C_4(d,\epsilon) > 0$ such that

$$
\mathbb{P}\big(\big|\widetilde{\mathbf{Piv}}_N\big| \geq C_4 N^{(\frac{d}{2}-1)\square 2} \mid \mathbf{0} \leftrightarrow \partial B(N)\big) \leq \tfrac{1}{2}\epsilon. \tag{5.22}
$$

5.2. **Proof of Theorem 1.6: lower bound.** We now show that for any $\epsilon > 0$, there exists $c_3(d,\epsilon) > 0$ such that

$$
\mathbb{P}\big(\big|\widetilde{\mathbf{Piv}}_N\big| \leq c_3 N^{(\frac{d}{2}-1)\square 2} \mid \mathbf{0} \leftrightarrow \partial B(N)\big) \leq \tfrac{1}{2}\epsilon. \tag{5.23}
$$

To prove (5.23), we first introduce some notations as follows.

- Let $n_0, \lambda > 1$ be two large numbers that will be determined later. For each $i \in \mathbb{N}^+$, we denote

$$
n_i = n_i(n_0, \lambda) := \lambda^2 i^4 n_{i-1}. \tag{5.24}
$$

- For sufficiently large $N \geq 1$, we define

$$
i_\star = i_\star(n_0, \lambda, N) := \min\big\{i \geq 1 : n_{2i+3} \geq N\big\}. \tag{5.25}
$$

- For each $i \in \mathbb{N}$, we employ the partial cluster $\widehat{\mathfrak{C}}_i$ introduced in [2, Section 3.2]. Roughly speaking, $\widehat{\mathfrak{C}}_i$ is the loop cluster where only the intersections inside $\widetilde{B}(n_{2i})$ are taken into account. To be precise, one may define $\widehat{\mathfrak{C}}_i$ via the following inductive exploration process:
  - Step 0: We set $\mathcal{C}_0 := \{\mathbf{0}\}$;
  - Step j ($j \geq 1$): Given $\mathcal{C}_{j-1}$, we define $\mathcal{C}_j$ as the union of $\mathcal{C}_{j-1}$ and the ranges of all loops in $\widetilde{\mathcal{L}}_{1/2}$ that intersect $\mathcal{C}_{j-1} \cap \widetilde{B}(n_{2i})$. If $\mathcal{C}_j = \mathcal{C}_{j-1}$, we stop the construction and define $\widehat{\mathfrak{C}}_i := \mathcal{C}_j$; otherwise (i.e., $\mathcal{C}_{j-1} \subsetneq \mathcal{C}_j$), we proceed to the $(j+1)$-th step.

According to the construction of $\widehat{\mathfrak{C}}_i$, we have the following observations:

(i) Conditioned on $\{\widehat{\mathfrak{C}}_i = D\}$ (for any proper realization D of $\widehat{\mathfrak{C}}_i$), the point measure composed of all remaining loops (i.e., loops that are not used in the construction of $\widehat{\mathfrak{C}}_i$) is distributed by $\widetilde{\mathcal{L}}_{1/2}^{D \cap \widetilde{B}(n_{2i})}$. Consequently, the cluster

$\mathcal{C}_{\mathbf{0}}$ can be equivalently written as $\big\{v \in \widetilde{\mathbb{Z}}^d : v \xleftrightarrow{(\widehat{\mathfrak{C}}_i \cap \widetilde{B}(n_{2i}))} \widehat{\mathfrak{C}}_i\big\}$. Therefore, for any $A \subset \widetilde{\mathbb{Z}}^d$, one has $\big\{\mathbf{0} \leftrightarrow A\big\} = \big\{\widehat{\mathfrak{C}}_i \xleftrightarrow{(\widehat{\mathfrak{C}}_i \cap \widetilde{B}(n_{2i}))} A\big\}$.

(ii) For each $0 \le i \le i_\star$, on the event $\{\mathbf{0} \leftrightarrow \partial B(N)\}$, given $\widehat{\mathfrak{C}}_i$, for any $e \in \mathbb{L}^d$, if there exists $\widetilde{\ell} \in \mathfrak{L}_e$ which is pivotal for the event $\{\widehat{\mathfrak{C}}_i \leftrightarrow \partial B(n_{2i+2})\}$ (we denote by $\widetilde{\mathbf{Piv}}_{N,i}$ the collection of all these edges), then $\widetilde{\ell}$ is also pivotal for $\{\mathbf{0} \leftrightarrow \partial B(N)\}$. (**P.S.** To see this, if we remove the loop $\widetilde{\ell}$ from $\widetilde{\mathcal{L}}_{1/2}$, then $\widehat{\mathfrak{C}}_i$ and $\partial B(n_{2i+2})$ are not connected by $\cup \widetilde{\mathcal{L}}_{1/2}^{\widehat{\mathfrak{C}}_i \cap \widetilde{B}(n_{2i})}$. Combined with Observation (i), it yields that $\{\mathbf{0} \leftrightarrow \partial B(N)\}$ does not occur.) Furthermore, since $\widetilde{\ell}$ is pivotal for $\{\widehat{\mathfrak{C}}_i \leftrightarrow \partial B(n_{2i+2})\}$, it must be connected to $\widehat{\mathfrak{C}}_i$ by $\cup \widetilde{\mathcal{L}}_{1/2}^{\partial B(n_{2i+2})}$, and thus is included in $\widehat{\mathfrak{C}}_{i+1}$. As a result, $\widetilde{\mathbf{Piv}}_{N,i}$ is measurable with respect to $\mathcal{F}_{\widehat{\mathfrak{C}}_{i+1}}$.

Next, we present two auxiliary lemmas, and establish (5.23) using them. The first lemma states that conditioned on the event $\{\mathbf{0} \leftrightarrow \partial B(N)\}$, it is unlikely to have a loop that crosses an annulus with a large ratio between the outer and inner radii. For any $M > m \ge 1$, we denote the point measure

$$\mathfrak{L}[m, M] := \widetilde{\mathcal{L}}_{1/2} \cdot \mathbb{1}_{\mathrm{ran}(\widetilde{\ell}) \cap \partial B(m) \neq \emptyset, \mathrm{ran}(\widetilde{\ell}) \cap \partial B(M) \neq \emptyset}. \tag{5.26}$$

Lemma 5.2. *For any $d \ge 3$ with $d \neq 6$, there exist $C, c > 0$ such that for any $n \ge 1$, $A, D \subset \widetilde{B}(n)$, $m \ge Cn$, $M \ge Cm$ and $N \ge (CM) \vee n^{1\square\frac{d-4}{2}}$,*

$$\mathbb{P}\big(\mathfrak{L}[m, M] \neq 0 \mid A \xleftrightarrow{(D)} \partial B(N)\big) \lesssim \sqrt{\tfrac{m}{M}}. \tag{5.27}$$

Since the proof of Lemma 5.2 is relatively standard, we postpone it to Appendix C. By the construction of $\widehat{\mathcal{C}}_i$, we know that every loop included in $\widehat{\mathcal{C}}_i$ must intersect $\widetilde{B}(n_{2i})$. As a result, on $\{\widehat{\mathcal{C}}_i \cap \partial B(n_{2i+1}) \neq \emptyset\}$, there exists a loop $\widetilde{\ell} \in \widetilde{\mathcal{L}}_{1/2}$ intersecting both $\partial B(n_{2i})$ and $\partial B(n_{2i+1})$, i.e., $\mathfrak{L}[n_{2i}, n_{2i+1}] \neq 0$. Thus, for $0 \le i \le i_\star$,

$$\begin{aligned} &\mathbb{P}\big(\widehat{\mathcal{C}}_i \cap \partial B(n_{2i+1}) \neq \emptyset \mid \mathbf{0} \leftrightarrow \partial B(N)\big) \\ \le &\mathbb{P}\big(\mathfrak{L}[n_{2i}, n_{2i+1}] \neq 0 \mid \mathbf{0} \leftrightarrow \partial B(N)\big) \overset{\text{(Lemma 5.2)}}{\lesssim} \lambda^{-1}(i+1)^{-2}. \end{aligned} \tag{5.28}$$

By taking $\lambda = \epsilon^{-2}$, it follows from (5.28) that

$$\mathbb{P}\big(\mathsf{F}_i^c \mid \mathbf{0} \leftrightarrow \partial B(N)\big) \le \tfrac{1}{4}\epsilon, \ \ \forall 0 \le i \le i_\star. \tag{5.29}$$

Here the event F_i is defined by $\cap_{0 \le j \le i}\big\{\widehat{\mathcal{C}}_j \subset \widetilde{B}(n_{2j+1})\big\}$.

(**P.S.** Although in (5.28) we only applied Lemma 5.2 for the case when $A = \{\mathbf{0}\}$ and $D = \emptyset$, its general case will play a crucial role in the companion paper [7].)

The second lemma shows that conditioned on a general set being connected to a distant box boundary, with a uniformly positive probability the number of pivotal loops at scale 1 for this event is at least of order $N^{(\frac{d}{2}-1)\square 2}$.

Lemma 5.3. *For any $d \geq 3$ with $d \neq 6$, there exist $C > 0$ and $c, c', c'' > 0$ such that for any $M \geq 1$, $N \geq CM$, and $A, D \subset \widetilde{B}(cM)$,*

$$\mathbb{P}\big(\big|\widetilde{\mathbf{Piv}}\big(A \xleftrightarrow{(D)} \partial B(M)\big)\big| \geq c' M^{(\frac{d}{2}-1)\boxdot 2} \mid A \xleftrightarrow{(D)} \partial B(N)\big) \geq c''. \tag{5.30}$$

The proof of Lemma 5.3 will be provided in Section 5.3.

Proof of (5.23). Without loss of generality, we assume that $\epsilon > 0$ is sufficiently small. We set $\lambda = \epsilon^{-2}$, $n_0 = \exp(-\epsilon^{-10})N$ and $c_3 = \exp(-\epsilon^{-100})$. Note that $i_\star \geq \epsilon^{-1}$. For each $0 \leq i \leq i_\star$, we define the event

$$\mathsf{G}_i := \cap_{0 \leq j \leq i} \big\{\big|\widetilde{\mathbf{Piv}}_{N,j}\big| \leq c_3 N^{(\frac{d}{2}-1)\boxdot 2}\big\}. \tag{5.31}$$

Using Observation (ii), we have $\cup_{0 \leq i \leq i_\star} \widetilde{\mathbf{Piv}}_{N,i} \subset \widetilde{\mathbf{Piv}}_N$, which implies that

$$\big\{\big|\widetilde{\mathbf{Piv}}_N\big| \leq c_3 N^{(\frac{d}{2}-1)\boxdot 2}\big\} \subset \mathsf{G}_{i_\star}. \tag{5.32}$$

For each $1 \leq i \leq i_\star$, recall from Observation (i) below (5.25) that conditioned on $\mathcal{F}_{\widehat{\mathfrak{C}}_i}$ (note that $\widehat{\mathfrak{C}}_i$ is now determined), the point measure consisting of the remaining loops has the same distribution as $\widetilde{\mathcal{L}}_{1/2}^{\widehat{\mathfrak{C}}_i \cap \widetilde{B}(n_{2i})}$. As a result, we have

$$\begin{aligned} &\mathbb{P}\big(\mathbf{0} \leftrightarrow \partial B(N), \mathsf{F}_i, \mathsf{G}_i\big) \\ =&\mathbb{E}\big[\mathbb{1}_{\mathsf{F}_i \cap \mathsf{G}_{i-1}} \cdot \mathbb{P}\big(\widehat{\mathfrak{C}}_i \xleftrightarrow{(\widehat{\mathfrak{C}}_i \cap \widetilde{B}(n_{2i}))} \partial B(N), \big|\widetilde{\mathbf{Piv}}_{N,i}\big| \leq c_3 N^{(\frac{d}{2}-1)\boxdot 2} \mid \mathcal{F}_{\widehat{\mathfrak{C}}_i}\big)\big]. \end{aligned} \tag{5.33}$$

In addition, since $\mathsf{F}_i \subset \big\{\widehat{\mathfrak{C}}_i \subset \widetilde{B}(n_{2i+1})\big\}$, it follows from Lemma 5.3 (with $A = \widehat{\mathfrak{C}}_i$ and $D = \widehat{\mathfrak{C}}_i \cap \widetilde{B}(n_{2i})$) that there exists $c_\dagger \in (0,1)$ such that on F_i,

$$\begin{aligned} &\mathbb{P}\big(\widehat{\mathfrak{C}}_i \xleftrightarrow{(\widehat{\mathfrak{C}}_i \cap \widetilde{B}(n_{2i}))} \partial B(N), \big|\widetilde{\mathbf{Piv}}_{N,i}\big| \leq c_3 N^{(\frac{d}{2}-1)\boxdot 2} \mid \mathcal{F}_{\widehat{\mathfrak{C}}_i}\big) \\ \leq&(1-c_\dagger) \cdot \mathbb{P}\big(\widehat{\mathfrak{C}}_i \xleftrightarrow{(\widehat{\mathfrak{C}}_i \cap \widetilde{B}(n_{2i}))} \partial B(N) \mid \mathcal{F}_{\widehat{\mathfrak{C}}_i}\big). \end{aligned} \tag{5.34}$$

Plugging (5.34) into (5.33), and using the inclusion $\mathsf{F}_i \subset \mathsf{F}_{i-1}$, we get

$$\begin{aligned} &\mathbb{P}\big(\mathbf{0} \leftrightarrow \partial B(N), \mathsf{F}_i, \mathsf{G}_i\big) \\ \leq&(1-c_\dagger) \cdot \mathbb{E}\big[\mathbb{1}_{\mathsf{F}_{i-1} \cap \mathsf{G}_{i-1}} \cdot \mathbb{P}\big(\widehat{\mathfrak{C}}_i \xleftrightarrow{(\widehat{\mathfrak{C}}_i \cap \widetilde{B}(n_{2i}))} \partial B(N) \mid \mathcal{F}_{\widehat{\mathfrak{C}}_i}\big)\big] \\ =&(1-c_\dagger) \cdot \mathbb{P}\big(\mathbf{0} \leftrightarrow \partial B(N), \mathsf{F}_{i-1}, \mathsf{G}_{i-1}\big), \end{aligned} \tag{5.35}$$

where the transition in the last line follows from Observation (i). By repeating this step, we obtain that

$$\begin{aligned} &\mathbb{P}\big(\mathbf{0} \leftrightarrow \partial B(N), \mathsf{F}_{i_\star}, \mathsf{G}_{i_\star}\big) \\ \leq&(1-c_\dagger)^{i_\star} \cdot \mathbb{P}\big(\mathbf{0} \leftrightarrow \partial B(N), \mathsf{F}_0, \mathsf{G}_0\big) \\ \leq&(1-c_\dagger)^{i_\star} \cdot \mathbb{P}\big(\mathbf{0} \leftrightarrow \partial B(N)\big) \leq \tfrac{1}{4}\epsilon \cdot \mathbb{P}\big(\mathbf{0} \leftrightarrow \partial B(N)\big). \end{aligned} \tag{5.36}$$

Combined with (5.32), it implies that

$$\begin{aligned}\mathbb{P}\big(\mathbf{0} \leftrightarrow \partial B(N), \mathsf{F}_{i_\star}, \big|\widetilde{\mathbf{Piv}}_N\big| \le c_3 N^{(\frac{d}{2}-1)\square 2}\big) \\ \overset{(5.32)}{\le} \mathbb{P}\big(\mathbf{0} \leftrightarrow \partial B(N), \mathsf{F}_{i_\star}, \mathsf{G}_{i_\star}\big) \overset{(5.36)}{\le} \tfrac{1}{4}\epsilon \cdot \mathbb{P}\big(\mathbf{0} \leftrightarrow \partial B(N)\big).\end{aligned} \tag{5.37}$$

Putting (5.29) and (5.37) together, we derive (5.23). □

Combining (5.22) and (5.23) completes the proof of Theorem 1.6. □

5.3. **Proof of Lemma 5.3.** We arbitrarily take a point $w_\dagger \in \partial B(\frac{M}{2})$, and denote by $\mathfrak{D}$ the collection of $e \in \mathbb{L}^d$ with $I_e \subset \widetilde{B}_{w_\dagger}(\frac{M}{100})$. Recall the notations $\mathfrak{L}_e$ and $\mathsf{L}_e(\cdot)$ at the beginning of Section 5.1. In this proof, we write $\mathsf{L}_e(A \xleftrightarrow{(D)} \partial B(M))$ as L_e for brevity. On the event L_e, after removing all loops in $\mathfrak{L}_e$, the endpoints of e belong to two distinct loop clusters that intersect A and $\partial B(M)$ respectively (we denote them by $\mathcal{C}_{\mathrm{in}}$ and $\mathcal{C}_{\mathrm{out}}$). In addition, if $\mathcal{C}_{\mathrm{out}}$ reaches $\partial B(N)$, then $A \xleftrightarrow{(D)} \partial B(N)$ also occurs. We say a cluster $\mathcal{C}$ is good if it satisfies the following conditions:

- When $3 \le d \le 5$, $\mathcal{C} \cap B(c_\dagger M) = \emptyset$, where $c_\dagger > 0$ is a constant that is much larger than c;
- When $d \ge 7$, $\mathcal{C} \cap B(c_\dagger M) = \emptyset$ and

$$\sum\nolimits_{z \in \mathbb{Z}^d : \mathcal{C} \cap \widetilde{B}_z(1) \neq \emptyset} |z|^{2-d} \le C_\ddagger N^{6-d}. \tag{5.38}$$

  Here the constant $C_\ddagger > 0$ will be determined later.

Note that when $d \ge 7$, one has $C_\ddagger N^{6-d} \ll M^{6-d}$ since $N \gg M$. Let $\widehat{\mathsf{L}}_e$ denote the subevent of L_e with the additional requirements that the cluster $\mathcal{C}_{\mathrm{out}}$ reaches $\partial B(N)$ and is good. Let $\mathbf{X} := \sum_{e \in \mathfrak{D}} \mathbb{1}_{\widehat{\mathsf{L}}_e}$. Since $\mathbf{X} \le \big|\widetilde{\mathbf{Piv}}\big(A \xleftrightarrow{(D)} \partial B(M)\big)\big|$, it is sufficient to prove that

$$\mathbb{P}\big(\mathbf{X} \ge c' M^{(\frac{d}{2}-1)\square 2} \mid A \xleftrightarrow{(D)} \partial B(N)\big) \ge c''. \tag{5.39}$$

We derive (5.39) through the following estimates. Arbitrarily take $x_\star \in \partial B(N)$.

(1) For any $e \in \mathfrak{D}$,

$$\mathbb{P}\big(\widehat{\mathsf{L}}_e\big) \gtrsim M^{-[(\frac{d}{2}+1)\square(d-2)]} N^{(\frac{d}{2}-1)\square(d-4)} \cdot \mathbb{P}\big(A \xleftrightarrow{(D)} x_\star\big); \tag{5.40}$$

(2) For any $e = \{x, y\}, e' = \{x', y'\} \in \mathfrak{D}$,

$$\mathbb{P}\big(\widehat{\mathsf{L}}_e \cap \widehat{\mathsf{L}}_{e'}\big) \lesssim [M \cdot (|x-y|+1)]^{-[(\frac{d}{2}+1)\square(d-2)]} N^{(\frac{d}{2}-1)\square(d-4)} \cdot \mathbb{P}\big(A \xleftrightarrow{(D)} x_\star\big). \tag{5.41}$$

We first present the proof of (5.39) assuming (5.40) and (5.41). By (5.40) one has

$$\mathbb{E}\big[\mathbf{X}\big] \gtrsim M^{(\frac{d}{2}-1)\square 2} N^{(\frac{d}{2}-1)\square(d-4)} \cdot \mathbb{P}\big(A \xleftrightarrow{(D)} x_\star\big). \tag{5.42}$$

Meanwhile, it follows from (5.41) that

$$
\begin{aligned}
\mathbb{E}\big[\mathbf{X}^2\big] \lesssim & M^{-[(\frac{d}{2}+1)\boxdot(d-2)]}N^{(\frac{d}{2}-1)\boxdot(d-4)}\cdot \mathbb{P}\big(A \xleftrightarrow{(D)} x_\star\big)\\
&\cdot \sum\nolimits_{x,y\in B_{w_\dagger}(\frac{M}{5})} (|x-y|+1)^{-[(\frac{d}{2}+1)\boxdot(d-2)]}\\
\overset{(3.15)}{\lesssim}& M^{(d-2)\boxdot 4}N^{(\frac{d}{2}-1)\boxdot(d-4)}\cdot \mathbb{P}\big(A \xleftrightarrow{(D)} x_\star\big).
\end{aligned}
\tag{5.43}
$$

Applying the Paley-Zygmund inequality, (5.42) and (5.43), we have

$$
\mathbb{P}\big(\mathbf{X}>0\big) \gtrsim \frac{\big(\mathbb{E}[\mathbf{X}]\big)^2}{\mathbb{E}\big[\mathbf{X}^2\big]} \gtrsim N^{(\frac{d}{2}-1)\boxdot(d-4)}\cdot \mathbb{P}\big(A \xleftrightarrow{(D)} x_\star\big).
\tag{5.44}
$$

Combined with (5.43), it implies that

$$
\mathbb{E}\big[\mathbf{X}^2 \mid \mathbf{X}>0\big] \lesssim M^{(d-2)\boxdot 4}.
\tag{5.45}
$$

Meanwhile, since $\widehat{\mathsf{L}}_e \subset \{A \xleftrightarrow{(D)} \partial B(N)\}$ holds for each $e\in\mathfrak{D}$, one has

$$
\mathbb{P}\big(\mathbf{X}>0\big) \le \mathbb{P}\big(A \xleftrightarrow{(D)} \partial B(N)\big) \overset{(\text{Lemma 2.16})}{\lesssim} N^{(\frac{d}{2}-1)\boxdot(d-4)}\cdot \mathbb{P}\big(A \xleftrightarrow{(D)} x_\star\big).
\tag{5.46}
$$

This together with (5.42) yields that

$$
\mathbb{E}\big[\mathbf{X} \mid \mathbf{X}>0\big] \gtrsim M^{(\frac{d}{2}-1)\boxdot 2}.
\tag{5.47}
$$

By the Paley-Zygmund inequality, (5.45) and (5.47), we have

$$
\mathbb{P}\big(\mathbf{X}\ge c' M^{(\frac{d}{2}-1)\boxdot 2} \mid \mathbf{X}>0\big) \gtrsim \frac{\big(\mathbb{E}[\mathbf{X}\mid \mathbf{X}>0]\big)^2}{\mathbb{E}\big[\mathbf{X}^2\mid \mathbf{X}>0\big]} \gtrsim 1.
\tag{5.48}
$$

Combining (5.44) and (5.48), and using Lemma 2.16, we obtain (5.39):

$$
\mathbb{P}\big(\mathbf{X}\ge c' M^{(\frac{d}{2}-1)\boxdot 2}\big) \gtrsim N^{(\frac{d}{2}-1)\boxdot(d-4)}\cdot \mathbb{P}\big(A \xleftrightarrow{(D)} x_\star\big) \gtrsim \mathbb{P}\big(A \xleftrightarrow{(D)} \partial B(N)\big).
\tag{5.49}
$$

In what follows, we establish (5.40) and (5.41).

Proof of (5.40). Recall that v_e^+ and v_e^- denote the two trisection points of I_e. For any $e\in\mathfrak{D}$, we define the event

$$
\widehat{\mathsf{C}}_e := \big\{v_e^+ \xleftrightarrow{(D)} A \| v_e^- \xleftrightarrow{(D)} \partial B(N)\big\} \cap \big\{\mathcal{C}^D_{v_e^-} \text{ is good}\big\}.
\tag{5.50}
$$

Note that on the event $\widehat{\mathsf{C}}_e$, one has $\mathfrak{L}_e=0$; in addition, if adding a loop into $\mathfrak{L}_e$, then $\widehat{\mathsf{L}}_e$ occurs. Consequently,

$$
\mathbb{P}\big(\widehat{\mathsf{L}}_e\big) \ge \mathbb{P}\big(\widehat{\mathsf{C}}_e\big)\cdot \frac{\mathbb{P}\big(\mathfrak{L}_e\neq 0\big)}{\mathbb{P}\big(\mathfrak{L}_e= 0\big)} \gtrsim \mathbb{P}\big(\widehat{\mathsf{C}}_e\big).
\tag{5.51}
$$

By the restriction property, and Lemmas 2.7 and 2.18, we have

$$
\begin{aligned}
\mathbb{P}\big(\widehat{\mathsf{C}}_e\big) =& \mathbb{E}\Big[\mathbb{1}_{v_e^- \xleftrightarrow{(D)} \partial B(N), \mathcal{C}^D_{v_e^-} \text{ is good}} \cdot \mathbb{P}\big(v_e^+ \xleftrightarrow{(D\cup\mathcal{C}_{v_e^-})} A\big)\Big] \\
\overset{\text{(Lemma 2.18)}}{\gtrsim} & M^{d-2}\cdot \mathbb{E}\Big[\mathbb{1}_{v_e^- \xleftrightarrow{(D)} \partial B(N), \mathcal{C}^D_{v_e^-} \text{ is good}} \cdot \mathbb{P}\big(A \xleftrightarrow{(D\cup\mathcal{C}_{v_e^-})} x_\spadesuit\big) \\
& \qquad \cdot \Big(\mathbb{P}\big(v_e^+ \xleftrightarrow{(D\cup\mathcal{C}_{v_e^-})} x_\spadesuit\big) - \tfrac{C'[\mathrm{diam}(A)]^{d-4}}{M^{d-4}|v_e^+|^{d-2}}\cdot \mathbb{1}_{d\geq 7}\Big)\Big] \\
\overset{\text{(Lemma 2.7),(1.4)}}{\gtrsim} & M^{d-2}\cdot \mathbb{P}\big(A \xleftrightarrow{(D)} x_\spadesuit\big)\cdot \Big[\mathbb{P}\big(\widehat{\mathsf{C}}'_e\big) - \tfrac{C''[\mathrm{diam}(A)]^{d-4}}{N^2M^{2d-6}}\cdot \mathbb{1}_{d\geq 7}\Big],
\end{aligned}
\tag{5.52}
$$

where $c_\spadesuit \in (c, c_\dagger)$ (recall from the condition that $A, D\subset \widetilde{B}(cM)$), $x_\spadesuit$ is an arbitrary point in $\partial B(c_\spadesuit M)$, and $\widehat{\mathsf{C}}'_e$ is defined by

$$
\widehat{\mathsf{C}}'_e := \big\{v_e^+ \xleftrightarrow{(D)} x_\spadesuit \| v_e^- \xleftrightarrow{(D)} \partial B(N)\big\} \cap \big\{\mathcal{C}^D_{v_e^-} \text{ is good}\big\}. \tag{5.53}
$$

Note that in (5.52), the occurrence of $\{\mathcal{C}^D_{v_e^-}$ is good$\}$ is necessary for the application of Lemmas 2.7 and 2.18.

Next, we estimate the probability of $\widehat{\mathsf{C}}'_e$ through the second moment method. Consider the quantity

$$
\mathbf{Y} := \sum\nolimits_{y\in\partial B(N)} \mathbb{1}_{\widetilde{\mathsf{C}}''_{e,y}}, \tag{5.54}
$$

where $\widetilde{\mathsf{C}}''_{e,y} := \big\{v_e^+ \xleftrightarrow{(D)} x_\spadesuit \| v_e^- \xleftrightarrow{(D)} y\big\} \cap \big\{\mathcal{C}^D_{v_e^-} \text{ is good}\big\}$. In fact, conditioned on $\big\{v_e^- \xleftrightarrow{(D)} y\big\}$, the event $\big\{\mathcal{C}^D_{v_e^-} \text{ is good}\big\}$ occurs with high probability. Precisely, since $v_e^- \in \widetilde{B}_{w_\dagger}(\frac{M}{100})$ with $w_\dagger \in \partial B(\frac{M}{2})$, one has $v_e^- \in [\widetilde{B}(\frac{M}{3})]$. Thus, by [6, Lemma 4.6], under this conditioning, the probability of $\big\{v_e^- \xleftrightarrow{(D)} B(c_\dagger M)\big\}$ uniformly converges to zero as $c_\dagger \to 0$. Meanwhile, the probability of the absence of the additional condition (5.38) for $d\geq 7$ also uniformly converges to zero as $C_\ddagger \to \infty$, following from Markov's inequality and the lemma below:

Lemma 5.4. *For any $d\geq 7$, for any $y\in\partial B(N)$,*

$$
\mathbb{E}\Big[\sum\nolimits_{z\in\mathbb{Z}^d} \mathbb{1}_{z \xleftrightarrow{(D)} v_e^-} (|z|+1)^{2-d} \mid v_e^- \xleftrightarrow{(D)} y\Big] \lesssim N^{6-d}. \tag{5.55}
$$

The proof of Lemma 5.4 will be provided in Appendix D.

Combining the aforementioned observation (i.e., $\big\{\mathcal{C}^D_{v_e^-} \text{ is good}\big\}$ occurs with high probability given $\big\{v_e^- \xleftrightarrow{(D)} y\big\}$) with the rigidity lemma (see [6, Lemma 4.1]), we obtain that conditioned on $\big\{v_e^+ \xleftrightarrow{(D)} x_\spadesuit \| v_e^- \xleftrightarrow{(D)} y\big\}$, the probability of $\big\{\mathcal{C}^D_{v_e^-} \text{ is good}\big\}$

is bounded away from zero (note that this application of the rigidity lemma relies on the fact that $\{\mathcal{C}^D_{v_e^-}$ is good$\}$ is decreasing with respect to the range of $\mathcal{C}^D_{v_e^-}$). I.e.,

$$\mathbb{P}\big(\widetilde{\mathsf{C}}''_{e,y}\big) \asymp \mathbb{P}\big(v_e^+ \xleftrightarrow{(D)} x_\spadesuit \| v_e^- \xleftrightarrow{(D)} y\big). \tag{5.56}$$

We take a large constant $C_\clubsuit > 0$. By the restriction property, the right-hand side of (5.56) can be bounded from below by

$$\begin{aligned}
&\mathbb{E}\Big[\mathbb{1}_{v_e^+ \xleftrightarrow{(D)} x_\spadesuit, \{v_e^+ \xleftrightarrow{(D)} \partial B(C_\clubsuit M)\}^c} \cdot \mathbb{P}\big(v_e^- \xleftrightarrow{(D\cup\mathcal{C}_{v_e^+})} y\big)\Big] \\
\overset{\text{(Lemma 2.15)}}{\asymp} &\big(\tfrac{M}{N}\big)^{d-2} \cdot \mathbb{E}\Big[\mathbb{1}_{v_e^+ \xleftrightarrow{(D)} x_\spadesuit, \{v_e^+ \xleftrightarrow{(D)} \partial B(C_\clubsuit M)\}^c} \cdot \mathbb{P}\big(v_e^- \xleftrightarrow{(D\cup\mathcal{C}_{v_e^+})} z_\spadesuit\big)\Big] \\
\asymp &\big(\tfrac{M}{N}\big)^{d-2} \cdot \mathbb{P}\big(\{v_e^+ \xleftrightarrow{(D)} x_\spadesuit \| v_e^- \xleftrightarrow{(D)} z_\spadesuit\} \cap \{v_e^+ \xleftrightarrow{(D)} \partial B(C_\clubsuit M)\}^c\big).
\end{aligned} \tag{5.57}$$

Here $z_\spadesuit$ is an arbitrarily point in $\partial B(2c_\spadesuit M)$. Employing the same argument as in (5.56), and using (4.25), one has

$$\begin{aligned}
&\mathbb{P}\big(\{v_e^+ \xleftrightarrow{(D)} x_\spadesuit \| v_e^- \xleftrightarrow{(D)} z_\spadesuit\} \cap \{v_e^+ \xleftrightarrow{(D)} \partial B(C_\clubsuit M)\}^c\big) \\
\asymp &\mathbb{P}\big(\{v_e^+ \xleftrightarrow{(D)} x_\spadesuit \| v_e^- \xleftrightarrow{(D)} z_\spadesuit\}\big) \asymp M^{-[(\frac{3d}{2}-1)\boxdot(2d-4)]}.
\end{aligned} \tag{5.58}$$

Combining (5.56), (5.57) and (5.58), we obtain

$$\mathbb{E}\big[\mathbf{Y}\big] = \sum\nolimits_{y\in\partial B(N)} \mathbb{P}\big(\widetilde{\mathsf{C}}''_{e,y}\big) \asymp \ M^{-[(\frac{d}{2}+1)\boxdot(d-2)]}N. \tag{5.59}$$

Next, we estimate the second moment of $\mathbf{Y}$ for the cases $3 \le d \le 5$ and $d \ge 7$ separately. Note that $\mathbb{E}[\mathbf{Y}^2] = \sum_{y_1,y_2\in\partial B(N)} \mathbb{P}\big(\widetilde{\mathsf{C}}''_{e,y_1} \cap \widetilde{\mathsf{C}}''_{e,y_2}\big)$, and that

$$\mathbb{P}\big(\widetilde{\mathsf{C}}''_{e,y_1} \cap \widetilde{\mathsf{C}}''_{e,y_2}\big) \lesssim \mathbb{I}_{y_1,y_2} := \mathbb{P}\big(\{v_e^+ \xleftrightarrow{(D)} x_\spadesuit\} \| \{v_e^- \xleftrightarrow{(D)} y_1, v_e^- \xleftrightarrow{(D)} y_2\}\big). \tag{5.60}$$

When $3 \le d \le 5$**.** For any $y_1, y_2 \in \partial B(N)$, for the same reason as in [6, (4.104)],

$$\begin{aligned}
\mathbb{I}_{y_1,y_2} \lesssim &N^{d-2} \cdot \mathbb{P}\big(v_e^- \leftrightarrow y_1, v_e^- \leftrightarrow y_2\big) \cdot \mathbb{P}\big(v_e^+ \xleftrightarrow{(D)} x_\spadesuit \| v_e^- \xleftrightarrow{(D)} y_1\big) \\
\lesssim &(|y_1 - y_2| + 1)^{-\frac{d}{2}+1} \cdot \mathbb{P}\big(v_e^+ \xleftrightarrow{(D)} x_\spadesuit \| v_e^- \xleftrightarrow{(D)} y_1\big),
\end{aligned} \tag{5.61}$$

where in the last inequality we used [5, Lemma 5.3]. Meanwhile, by the restriction property, Corollary 2.12 and (4.25), we have

$$\begin{aligned}
&\mathbb{P}\big(v_e^+ \xleftrightarrow{(D)} x_\spadesuit \| v_e^- \xleftrightarrow{(D)} y_1\big) = \mathbb{E}\Big[\mathbb{1}_{v_e^+ \xleftrightarrow{(D)} x_\spadesuit} \cdot \mathbb{P}\big(v_e^- \xleftrightarrow{(D\cup\mathcal{C}_{v_e^+})} y_1\big)\Big] \\
\overset{\text{(Corollary 2.12)}}{\lesssim} &N^{2-d}M^{-1} \sum_{z\in\partial\mathcal{B}(M)} \mathbb{P}\big(v_e^+ \xleftrightarrow{(D)} x_\spadesuit \| v_e^- \xleftrightarrow{(D)} z\big) \overset{(4.25)}{\lesssim} M^{-\frac{d}{2}-1}N^{2-d}.
\end{aligned} \tag{5.62}$$

Combining (5.61) and (5.62), we obtain

$$\mathbb{E}[\mathbf{Y}^2] \lesssim M^{-\frac{d}{2}-1}N^{2-d} \sum\nolimits_{y_1,y_2\in\partial B(N)} (|y_1 - y_2| + 1)^{-\frac{d}{2}+1} \overset{(3.15)}{\lesssim} M^{-\frac{d}{2}-1}N^{\frac{d}{2}+1}. \tag{5.63}$$

When $d \geq 7$. By the BKR inequality (see e.g., [3, Lemma 3.3]) and (1.1),

$$\mathbb{I}_{y_1,y_2} \leq M^{2-d} \cdot \mathbb{P}\big(v_e^- \leftrightarrow y_1, v_e^- \leftrightarrow y_2\big). \tag{5.64}$$

Summing over all $y_1, y_2 \in \partial B(N)$, and using [3, Lemma 4.1], we get

$$\mathbb{E}[\mathbf{Y}^2] \lesssim M^{2-d} N^4. \tag{5.65}$$

We are now ready to apply the Paley-Zygmund inequality for all $d \geq 3$ with $d \neq 6$. Since $\{\mathbf{Y} > 0\} \subset \widehat{\mathsf{C}}_e'$, we have

$$\mathbb{P}\big(\widehat{\mathsf{C}}_e'\big) \geq \mathbb{P}\big(\mathbf{Y} > 0\big) \gtrsim \big(\mathbb{E}\big[\mathbf{Y}\big]\big)^2 / \mathbb{E}\big[\mathbf{Y}^2\big] \gtrsim M^{-[(\frac{d}{2}+1)\square(d-2)]} N^{-[(\frac{d}{2}-1)\square 2]}. \tag{5.66}$$

Consequently, when $d \geq 7$, since $A \subset \widetilde{B}(cN)$, one may choose a sufficiently small $c > 0$ such that

$$\tfrac{C''[\mathrm{diam}(A)]^{d-4}}{N^2 M^{2d-6}} \leq \tfrac{1}{2} \cdot \mathbb{P}\big(\widehat{\mathsf{C}}_e'\big). \tag{5.67}$$

By plugging (5.66) and (5.67) into (5.52), we have

$$\mathbb{P}\big(\widehat{\mathsf{C}}_e\big) \gtrsim M^{-[(\frac{d}{2}+1)\square(d-2)]} N^{(\frac{d}{2}-1)\square(d-4)} \cdot \mathbb{P}\big(A \xleftrightarrow{(D)} x_\star\big). \tag{5.68}$$

This together with (5.51) implies the estimate (5.40).

Proof of (5.41). We denote by $\Psi_{e,e'}$ the collection of all quadruples (x_1, y_1, x_2, y_2) such that x_1, y_1 and x_2, y_2 are distinct endpoints of e and e' respectively. On $\widehat{\mathsf{L}}_e \cap \widehat{\mathsf{L}}_{e'}$, after removing all loops in $\mathfrak{L}_e + \mathfrak{L}_{e'}$, there exists $\psi = (x_1, y_1, x_2, y_2) \in \Psi_{e,e'}$ such that the event

$$\overline{\mathsf{C}}_\psi := \big\{x_1 \xleftrightarrow{(D)} A \| x_2 \xleftrightarrow{(D)} \partial B(N) \| y_1 \xleftrightarrow{(D)} y_2\big\} \cap \big\{\{x_2, y_2\} \xleftrightarrow{(D)} B(c_\dagger M)\big\}^c \tag{5.69}$$

occurs. Therefore, applying the union bound and following the same reasoning as in (5.5), one obtains

$$\mathbb{P}\big(\widehat{\mathsf{L}}_e \cap \widehat{\mathsf{L}}_{e'}\big) \lesssim \sum\nolimits_{\psi \in \Psi_{e,e'}} \mathbb{P}\big(\overline{\mathsf{C}}_\psi\big) \cdot \frac{\mathbb{P}(\mathfrak{L}_e \neq 0, \mathfrak{L}_{e'} \neq 0)}{\mathbb{P}(\mathfrak{L}_e = \mathfrak{L}_{e'} = 0)} \asymp \sum\nolimits_{\psi \in \Psi_{e,e'}} \mathbb{P}\big(\overline{\mathsf{C}}_\psi\big). \tag{5.70}$$

Subsequently, we estimate the probability of $\overline{\mathsf{C}}_\psi$ for $3 \leq d \leq 5$ and $d \geq 7$ separately.

When $3 \leq d \leq 5$. For any subset $D' \subset \widetilde{\mathbb{Z}}^d$ satisfying $D \subset D'$ and $(D' \setminus D) \cap B(c_\dagger M) = \emptyset$, by Lemmas 2.6, 2.16 and 2.17, we have (recalling that $x_\star \in \partial B(N)$)

$$\begin{aligned} \mathbb{P}\big(x_1 \xleftrightarrow{(D')} A\big) &\overset{\text{(Lemma 2.17)}}{\asymp} \mathbb{P}\big(A \xleftrightarrow{(D)} \partial B(c_\spadesuit M)\big) \cdot \mathbb{P}\big(x_1 \xleftrightarrow{(D' \setminus D)} \partial B(c_\spadesuit M)\big) \\ &\overset{\text{(Lemmas 2.6 and 2.16)}}{\lesssim} N^{d-2} \cdot \mathbb{P}\big(A \xleftrightarrow{(D)} x_\star\big) \cdot \mathbb{P}\big(x_1 \xleftrightarrow{(D')} x_\spadesuit\big). \end{aligned} \tag{5.71}$$

By the restriction property and (5.71), one has

$$\begin{aligned} \mathbb{P}\big(\overline{\mathsf{C}}_\psi\big) =& \mathbb{E}\Big[\mathbb{1}_{\{x_2 \xleftrightarrow{(D)} \partial B(N) \| y_1 \xleftrightarrow{(D)} y_2\} \cap \{\{x_2,y_2\} \xleftrightarrow{(D)} B(c_\dagger M)\}^c} \mathbb{P}\big(x_1 \xleftrightarrow{(D \cup \mathcal{C}_{\{x_2,y_2\}})} A\big)\Big] \\ &\overset{(5.71)}{\lesssim} N^{d-2} \cdot \mathbb{P}\big(A \xleftrightarrow{(D)} x_\star\big) \cdot \mathbb{P}\big(\overline{\mathsf{C}}_\psi'\big), \end{aligned} \tag{5.72}$$

where the event $\overline{\mathsf{C}}'_\psi$ is defined by $\{x_2 \overset{(D)}{\longleftrightarrow} \partial B(N) \| x_1 \overset{(D)}{\longleftrightarrow} x_\spadesuit \| y_1 \overset{(D)}{\longleftrightarrow} y_2\}$. Moreover, by applying Corollary 2.12 and Lemma 2.13, we have

$$
\begin{aligned}
\mathbb{P}\big(\overline{\mathsf{C}}'_\psi\big) =& \mathbb{E}\Big[\mathbb{1}_{x_1 \overset{(D)}{\longleftrightarrow} x_\spadesuit \| y_1 \overset{(D)}{\longleftrightarrow} y_2} \cdot \mathbb{P}\big(x_2 \xleftrightarrow{(D\cup \mathcal{C}_{\{x_1,y_1\}})} \partial B(N)\big)\Big] \\
\overset{(\text{Lemma 2.13})}{\lesssim} & N^{-\frac{d}{2}} \sum\nolimits_{v\in \partial \mathcal{B}(d^{-1}N)} \mathbb{E}\Big[\mathbb{1}_{x_1 \overset{(D)}{\longleftrightarrow} x_\spadesuit \| y_1 \overset{(D)}{\longleftrightarrow} y_2} \cdot \mathbb{P}\big(x_2 \xleftrightarrow{(D\cup \mathcal{C}_{\{x_1,y_1\}})} v\big)\Big] \\
\overset{(\text{Corollary 2.12})}{\lesssim} & M^{-1} N^{-\frac{d}{2}+1} \sum\nolimits_{w\in \partial \mathcal{B}(M)} \mathbb{P}\big(\overline{\mathsf{C}}''_{\psi,w}\big),
\end{aligned}
\tag{5.73}
$$

where we denote $\overline{\mathsf{C}}''_{\psi,w} := \big\{x_1 \overset{(D)}{\longleftrightarrow} x_\spadesuit \| x_2 \overset{(D)}{\longleftrightarrow} w \| y_1 \overset{(D)}{\longleftrightarrow} y_2\big\}$.

Recall the events $\mathsf{A}^{\cdot}_\diamond(\cdot)$ for $\diamond \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}\}$ at the beginning of Section 4. For brevity, we denote $r := |y_1 - y_2| + 1$. For any $D' \subset \widetilde{\mathbb{Z}}^d$, Corollary 2.11 implies that

$$
\begin{aligned}
&\mathbb{P}\big(y_1 \overset{(D')}{\longleftrightarrow} y_2\big) \\
\lesssim & r^{-d} \sum_{z_1 \in \partial \mathcal{B}_{y_1}(\frac{r}{100C_7}), z_2 \in \partial \mathcal{B}_{y_2}(\frac{r}{100C_7})} \mathbb{P}\big(\mathsf{A}^{D'}_{\mathrm{I}}(y_1, z_1; \tfrac{r}{100}), \mathsf{A}^{D'}_{\mathrm{I}}(y_2, z_2; \tfrac{r}{100})\big).
\end{aligned}
\tag{5.74}
$$

Meanwhile, it follows from Corollary 2.19 that

$$
\begin{aligned}
\mathbb{P}\big(x_1 \overset{(D')}{\longleftrightarrow} x_\spadesuit\big) \lesssim r^{-d} & \sum\nolimits_{z_3 \in \partial \mathcal{B}_{x_1}(10C_7 r), z_4 \in \partial \mathcal{B}_{x_1}(100C_7^3 r)} \\
& \mathbb{P}\big(\mathsf{A}^{D'}_{\mathrm{II}}(x_1, x_1, z_3; 10C_7 r, 20C_7 r), \mathsf{A}^{D'}_{\mathrm{III}}(x_1, x_\spadesuit, z_4; 100C_7^3 r)\big),
\end{aligned}
\tag{5.75}
$$

$$
\begin{aligned}
\mathbb{P}\big(x_2 \overset{(D')}{\longleftrightarrow} w\big) \lesssim r^{-d} & \sum\nolimits_{z_5 \in \partial \mathcal{B}_{x_2}(20C_7 r), z_6 \in \partial \mathcal{B}_{x_2}(200C_7^3 r)} \\
& \mathbb{P}\big(\mathsf{A}^{D'}_{\mathrm{II}}(x_2, x_2, z_5; 10C_7 r, 20C_7 r), \mathsf{A}^{D'}_{\mathrm{III}}(x_2, w, z_6; 100C_7^3 r)\big).
\end{aligned}
\tag{5.76}
$$

We write $\bar{z} := (z_1, z_2, z_3, z_4, z_5, z_6)$, and let $\overline{\mathsf{A}}^{D'}_{\bar{z}}$ denote the intersection of all the six events on the right-hand sides of (5.74), (5.75) and (5.76). Combining these three inequalities with the restriction property, we have

$$
\mathbb{P}\big(\overline{\mathsf{C}}''_{\psi,w}\big) \lesssim r^{-3d} \sum\nolimits_{\bar{z} := (z_1,z_2,z_3,z_4,z_5,z_6) \in \Upsilon} \mathbb{P}\big(\overline{\mathsf{A}}^{D}_{\bar{z}}\big),
\tag{5.77}
$$

where the set Υ is defined by

$$
\begin{aligned}
\Upsilon :=& \partial \mathcal{B}_{y_1}(\tfrac{r}{100C_7}) \times \partial \mathcal{B}_{y_2}(\tfrac{r}{100C_7}) \times \partial \mathcal{B}_{x_1}(10C_7 r) \\
& \times \partial \mathcal{B}_{x_1}(100C_7^3 r) \times \partial \mathcal{B}_{x_2}(20C_7 r) \times \partial \mathcal{B}_{x_2}(200C_7^3 r).
\end{aligned}
\tag{5.78}
$$

For each $\bar{z} \in \Upsilon$, on $\overline{\mathsf{A}}^{D}_{\bar{z}}$, the following three independent events occur:

$$
\mathsf{C}_1 := \big\{y_1 \xleftrightarrow{(\partial B_{x_1}(\frac{r}{10}))} z_1 \| x_1 \xleftrightarrow{(\partial B_{x_1}(\frac{r}{10}))} z_3\big\},
\tag{5.79}
$$

$$
\mathsf{C}_2 := \big\{y_2 \xleftrightarrow{(\partial B_{x_2}(\frac{r}{10}))} z_2 \| x_2 \xleftrightarrow{(\partial B_{x_2}(\frac{r}{10}))} z_5\big\},
\tag{5.80}
$$

$$\mathsf{C}_3 := \big\{x_{\spadesuit} \xleftrightarrow{(D\cup\partial B_{w_1}(10r))} z_4 \| w \xleftrightarrow{(D\cup\partial B_{w_1}(10r))} z_6\big\}. \tag{5.81}$$

Combined with (4.25) and $|\Upsilon| \asymp r^{6d-6}$, it yields that

$$\mathbb{P}\big(\overline{\mathsf{C}}''_{\psi,z}\big) \lesssim r^{-\frac{d}{2}-1}M^{-\frac{3d}{2}+1}. \tag{5.82}$$

Putting (5.72), (5.73) and (5.82) together, we obtain

$$\mathbb{P}\big(\overline{\mathsf{C}}_\psi\big) \lesssim (rM)^{-\frac{d}{2}-1}N^{\frac{d}{2}-1} \cdot \mathbb{P}\big(A \xleftrightarrow{(D)} x_\star\big). \tag{5.83}$$

When $d \geq 7$. Using the BKR inequality, we have

$$\mathbb{P}\big(\overline{\mathsf{C}}_\psi\big) \leq \mathbb{P}\big(x_1 \xleftrightarrow{(D)} A\big) \cdot \mathbb{P}\big(x_2 \xleftrightarrow{(D)} \partial B(N)\big) \cdot \mathbb{P}\big(y_1 \xleftrightarrow{(D)} y_2\big). \tag{5.84}$$

Note that Lemma 2.16 implies

$$\mathbb{P}\big(x_1 \xleftrightarrow{(D)} A\big) \lesssim M^{2-d}N^{d-2} \cdot \mathbb{P}\big(A \xleftrightarrow{(D)} x_\star\big). \tag{5.85}$$

Meanwhile, using (1.4) and (1.1) respectively, one has

$$\mathbb{P}\big(x_2 \xleftrightarrow{(D)} \partial B(N)\big) \lesssim N^{-2} \ \text{ and } \ \mathbb{P}\big(y_1 \xleftrightarrow{(D)} y_2\big) \lesssim r^{2-d}. \tag{5.86}$$

Plugging (5.84) and (5.85) into (5.86), we get

$$\mathbb{P}\big(\overline{\mathsf{C}}_\psi\big) \lesssim (rM)^{2-d}N^{d-4} \cdot \mathbb{P}\big(A \xleftrightarrow{(D)} x_\star\big). \tag{5.87}$$

Combining (5.70), $|\Psi_{e,e'}| \asymp 1$, (5.83) and (5.87), we obtain (5.41).

To sum up, we have completed the proof of Lemma 5.3. $\square$

6. Typical number of pivotal edges

This section is devoted to proving Theorem 1.5. We denote by $\mathbf{Piv}_N$ the analogue of $\mathbf{Piv}_N^+$ obtained by replacing $\widetilde{E}^{\geq 0}$ with $\cup\widetilde{\mathcal{L}}_{1/2}$ (i.e., replacing connectivity via GFF level sets to connectivity via loop clusters). Applying the isomorphism theorem, we have

$$\mathbb{P}\big(\mathbf{0} \leftrightarrow \partial B(N)\big) = 2\mathbb{P}\big(\mathbf{0} \xleftrightarrow{\geq 0} \partial B(N)\big), \tag{6.1}$$

$$\mathbb{P}\big(|\mathbf{Piv}_N| \in F\big) = 2\mathbb{P}\big(|\mathbf{Piv}_N^+| \in F\big) \tag{6.2}$$

for any non-empty subset $F \subset \mathbb{N}$. Thus, it suffices to show that

$$\mathbb{P}\big(|\mathbf{Piv}_N| \in \big[c_2N^{(\frac{d}{2}-1)\boxdot 2}, C_2N^{(\frac{d}{2}-1)\boxdot 2}\big] \mid \mathbf{0} \leftrightarrow \partial B(N)\big) \geq 1-\epsilon. \tag{6.3}$$

To see this, for any $e \in \mathbb{L}^d$, when $e \in \widetilde{\mathbf{Piv}}_N$ occurs, removing all loops in $\mathfrak{L}_e$ (recalling (5.1)) from $\widetilde{\mathcal{L}}_{1/2}$ violates the event $\{\mathbf{0} \leftrightarrow \partial B(N)\}$. As a result, removing the edge e also has the same effect. Thus, one has $\widetilde{\mathbf{Piv}}_N \subset \mathbf{Piv}_N$. Combined with (5.23), it implies that for $c_2 := c_3$,

$$\begin{aligned}&\mathbb{P}\big(|\mathbf{Piv}_N| \leq c_2N^{(\frac{d}{2}-1)\boxdot 2} \mid \mathbf{0} \leftrightarrow \partial B(N)\big)\\ \leq&\mathbb{P}\big(|\widetilde{\mathbf{Piv}}_N| \leq c_2N^{(\frac{d}{2}-1)\boxdot 2} \mid \mathbf{0} \leftrightarrow \partial B(N)\big) \leq \tfrac{1}{2}\epsilon.\end{aligned} \tag{6.4}$$

As in the proof of Theorem 1.6, we need to bound the first moment of $|\mathbf{Piv}_N|$:

Lemma 6.1. *For any $d \geq 3$ with $d \neq 6$, and any $N \geq 1$,*

$$\mathbb{E}\big[\big|\mathbf{Piv}_N\big| \mid \mathbf{0} \leftrightarrow \partial B(N)\big] \lesssim N^{(\frac{d}{2}-1)\square 2}. \tag{6.5}$$

By Markov's inequality and Lemma 6.1, there exists $C_2(d,\epsilon) > 0$ such that

$$\mathbb{P}\big(|\mathbf{Piv}_N| \geq C_2 N^{(\frac{d}{2}-1)\square 2} \mid \mathbf{0} \leftrightarrow \partial B(N)\big) \leq \tfrac{1}{2}\epsilon. \tag{6.6}$$

Combining (6.4) and (6.6), we obtain Theorem 1.6 (assuming Lemma 6.1). $\square$

The rest of this section is devoted to the proof of Lemma 6.1. To this end, we denote by L_e the event that the edge e is pivotal for $\{\mathbf{0} \leftrightarrow \partial B(N)\}$. Like in the proof of Lemma 5.1, it is sufficient to prove that for any $e = \{x_1, x_2\} \in \mathbb{L}^d$ with $I_e \subset \widetilde{B}(N)$,

$$\mathbb{P}\big(\mathsf{L}_e\big) \lesssim \mathbb{I}_e := \begin{cases} (|x_1|+1)^{-[(\frac{d}{2}+1)\square(d-2)]} N^{-[(\frac{d}{2}-1)\square 2]} & x_1 \in B(\frac{N}{2}); \\ \big(\mathrm{dist}(x_1, \partial B(N)) + 1\big)^{-2} N^{2-d} & x_1 \notin B(\frac{N}{2}). \end{cases} \tag{6.7}$$

Note that replacing x_1 with x_2 will change the value of $\mathbb{I}_e$ by only a constant factor.

We arbitrarily fix an edge e. For each loop $\widetilde{\ell} \in \widetilde{\mathcal{L}}_{1/2}$, it can be decomposed into the following parts: for each $j \in \{1,2\}$,

(i) the excursions that start from x_j, end at x_j, and do not touch I_e;
(ii) the excursions that start from x_j, end at x_{3-j}, and do not touch I_e;
(iii) the excursions that start from x_j, end at $\{x_1, x_2\}$, and are contained in I_e.

For $\diamond \in \{\mathrm{i}, \mathrm{ii}, \mathrm{iii}\}$, we define $\mathfrak{E}_j^{\diamond}$ as the union of ranges of excursions in Part $(\diamond)$ for all loops $\widetilde{\ell} \in \widetilde{\mathcal{L}}_{1/2}$. In addition, let $\mathcal{C}_1$ (resp. $\mathcal{C}_2$) denote the cluster of $\cup \widetilde{\mathcal{L}}_{1/2}^{\{x_1,x_2\}}$ intersecting $\mathbf{0}$ (resp. $\partial B(N)$). Observe that on the event L_e, one has $\mathfrak{E}_1^{\mathrm{ii}} = \mathfrak{E}_2^{\mathrm{ii}} = \emptyset$; otherwise, removing the edge e will not change the connectivity between x_1 and x_2 in the cluster $\mathcal{C}_{\mathbf{0}}$, and thus e is not pivotal. Moreover, the event L_e is independent of the excursions in $\mathfrak{E}_1^{\mathrm{iii}}$ and $\mathfrak{E}_2^{\mathrm{iii}}$. Therefore, when L_e occurs, there exists $j \in \{1,2\}$ such that $\mathfrak{E}_j^{\mathrm{i}}$ intersects $\mathcal{C}_1$, $\mathfrak{E}_{3-j}^{\mathrm{i}}$ intersects $\mathcal{C}_2$, and that $\mathfrak{E}_j^{\mathrm{i}} \cup \mathcal{C}_1$ is disjoint from $\mathfrak{E}_{3-j}^{\mathrm{i}} \cup \mathcal{C}_2$ (we denote this event by F_j). Consequently, in order to verify (6.7), it suffices to show that

$$\mathbb{P}\big(\mathsf{F}_j\big) \lesssim \mathbb{I}_e, \quad \forall j \in \{1,2\}. \tag{6.8}$$

Without loss of generality, we only consider the case $j = 1$, and write F_1 as F.

We first present the proof of (6.8) for $d \geq 7$, since it is relatively simpler than that of the low-dimensional case $3 \leq d \leq 5$. Let $R_1 := |x_1| + (C_9)^{10}$ and $R_2 := \mathrm{dist}(x_1, \partial B(N)) + (C_9)^{10}$, where $C_9 > 0$ is a sufficiently large constant. On the event F, there exist $z_1, z_2 \in B(N)$ such that $\widetilde{B}_{z_j}(1) \cap \mathfrak{E}_j^{\mathrm{i}} \cap \mathcal{C}_j \neq \emptyset$ holds for $j \in \{1,2\}$; moreover, $\mathcal{C}_1$ is disjoint from $\mathcal{C}_2$. In addition, note that $\mathfrak{E}_1^{\mathrm{i}}$ and $\mathfrak{E}_2^{\mathrm{i}}$ are independent of $\mathcal{C}_1$ and $\mathcal{C}_2$. Thus, by the union bound, we have

$$\mathbb{P}(\mathsf{F}) \lesssim \sum\nolimits_{z_1,z_2\in B(N)} \mathbb{P}\big(\widetilde{B}_{z_j}(1)\cap \mathfrak{C}_j^{\mathrm{i}}\neq \emptyset, \forall j\in\{1,2\}\big) \cdot \mathbb{P}\big(\widetilde{B}_{z_1}(1) \xleftrightarrow{(\{x,y\})} \mathbf{0} \| \widetilde{B}_{z_2}(1) \xleftrightarrow{(\{x,y\})} \partial B(N)\big). \tag{6.9}$$

By the strong Markov property of Brownian excursion measure, we have

$$\begin{aligned} &\mathbb{P}\big(\widetilde{B}_{z_j}(1)\cap \mathfrak{C}_j^{\mathrm{i}}\neq \emptyset, \forall j\in\{1,2\}\big) \\ \lesssim&(|x_1-z_1|+1)^{4-2d}\cdot(|x_2-z_2|+1)^{4-2d}. \end{aligned} \tag{6.10}$$

Meanwhile, by the BKR inequality, (1.1) and (1.4), we have

$$\begin{aligned} &\mathbb{P}\big(\widetilde{B}_{z_1}(1) \xleftrightarrow{(\{x,y\})} \mathbf{0} \| \widetilde{B}_{z_2}(1) \xleftrightarrow{(\{x,y\})} \partial B(N)\big) \\ \overset{\text{(BKR)}}{\lesssim}& \mathbb{P}\big(\widetilde{B}_{z_1}(1) \xleftrightarrow{(\{x,y\})} \mathbf{0}\big) \cdot \mathbb{P}\big(\widetilde{B}_{z_2}(1) \xleftrightarrow{(\{x,y\})} \partial B(N)\big) \\ \overset{(1.1),(1.4)}{\lesssim}& (|z_1|+1)^{2-d}\big(\mathrm{dist}(z_2,\partial B(N))+1\big)^{-2}. \end{aligned} \tag{6.11}$$

Plugging (6.10) and (6.11) into (6.9), we get

$$\mathbb{P}(\mathsf{F}) \lesssim \mathbb{U}_1\cdot\mathbb{U}_2, \tag{6.12}$$

where $\mathbb{U}_1$ and $\mathbb{U}_2$ are defined by

$$\mathbb{U}_1 := \sum\nolimits_{z_1\in B(N)} (|x_1-z_1|+1)^{4-2d}(|z_1|+1)^{2-d}, \tag{6.13}$$

$$\mathbb{U}_2 := \sum\nolimits_{z_2\in B(N)} (|x_2-z_2|+1)^{4-2d}\big(\mathrm{dist}(z_2,\partial B(N))+1\big)^{-2}. \tag{6.14}$$

For $\mathbb{U}_1$, recall the basic fact that (see e.g., [3, Lemma 4.4])

$$\sum\nolimits_{z\in\mathbb{Z}^d} (|z-v_1|+1)^{6-2d}(|z-v_2|+1)^{2-d} \lesssim (|v_1-v_2|+1)^{2-d},\ \forall v_1,v_2\in\widetilde{\mathbb{Z}}^d. \tag{6.15}$$

This immediately implies that

$$\mathbb{U}_1 \leq \sum\nolimits_{z\in B(N)} (|x_1-z|+1)^{6-2d}(|z|+1)^{2-d} \lesssim (|x_1|+1)^{2-d} \asymp R_1^{2-d}. \tag{6.16}$$

For $\mathbb{U}_2$, since $\mathrm{dist}(z_2,\partial B(N)) \gtrsim R_2$ for $z\in B_{x_1}(\frac{R_2}{10})$, one has

$$\begin{aligned} \mathbb{U}_2 \lesssim& R_2^{-2} \sum\nolimits_{z_2\in B_{x_1}(\frac{R_2}{10})} (|x_2-z_2|+1)^{4-2d} \\ &+ \sum_{z_2\in B(N)\setminus B_{x_1}(\frac{R_2}{10})} (|x_2-z_2|+1)^{4-2d}\big(\mathrm{dist}(z_2,\partial B(N))+1\big)^{-2} \\ \overset{(3.15)}{\lesssim}& R_2^{-2} + \sum\nolimits_{1\leq k\leq N} k^{d-1}\cdot(k+R_2)^{4-2d}\cdot(N-k+1)^{-2}. \end{aligned} \tag{6.17}$$

Moreover, restricted to $1\leq k\leq R_2$, the sum on the right-hand side is at most

$$CR_2^{4-2d}N^{-2}\sum\nolimits_{1\leq k\leq R_2} k^{d-1} \lesssim R_2^{2-d}N^{-2}. \tag{6.18}$$

Meanwhile, restricted to $R_2 \le k \le N$, this sum is upper-bounded by

$$CR_2^{3-d} \sum\nolimits_{R_2 \le k \le N} (N-k+1)^{-2} \lesssim R_2^{3-d}. \tag{6.19}$$

Combining (6.17), (6.18) and (6.19), we get

$$\mathbb{U}_2 \lesssim R_2^{-2} + R_2^{2-d} N^{-2} + R_2^{3-d} \lesssim R_2^{-2}. \tag{6.20}$$

Inserting (6.16) and (6.20) into (6.12), we derive the bound (6.8) for $d \ge 7$.

We now turn to the case when $3 \le d \le 5$. For each $j \in \{1,2\}$, let

$$k_j := \min\{k \ge 1 : r_{k+3}^{(j)} \ge R_j\}, \tag{6.21}$$

where $r_i^{(j)} := (C_9)^{i+\frac{j}{2}}$ for $i \in \mathbb{N}$. Moreover, for any $1 \le k \le k_j - 1$, we define

$$\mathsf{G}_k^{(j)} := \big\{\mathcal{C}_j \cap B_{x_1}(r_{k+1}^{(j)}) \neq \emptyset, \mathcal{C}_j \cap B_{x_1}(r_k^{(j)}) = \emptyset\big\}. \tag{6.22}$$

For completeness, we also define

$$\mathsf{G}_0^{(j)} := \big\{\mathcal{C}_j \cap B_{x_1}(r_1^{(j)}) \neq \emptyset\big\} \ \text{ and } \ \mathsf{G}_{k_j}^{(j)} := \big\{\mathcal{C}_j \cap B_{x_1}(r_{k_j}^{(j)}) = \emptyset\big\}. \tag{6.23}$$

For any $0 \le m_1 \le k_1$ and $0 \le m_2 \le k_2$, we define the event

$$\mathsf{F}_{m_1,m_2} := \mathsf{F} \cap \mathsf{G}_{m_1}^{(1)} \cap \mathsf{G}_{m_2}^{(2)}. \tag{6.24}$$

Since the union of $\mathsf{G}_m^{(j)}$ for $0 \le m \le k_j$ is the sure event, one has

$$\mathsf{F} = \cup_{0 \le m_1 \le k_1, 0 \le m_2 \le k_2} \mathsf{F}_{m_1,m_2}. \tag{6.25}$$

Thus, to obtain (6.7), it is sufficient to establish that for any $\delta \in (0, \frac{1}{2})$,

$$\mathbb{P}\big(\mathsf{F}_{m_1,m_2}\big) \lesssim \mathbb{J}_{m_1,m_2} := \begin{cases} (r_{m_1 \vee m_2}^{(1)})^{-\frac{d}{2}+1+\delta} R_1^{-\frac{d}{2}-1} N^{-\frac{d}{2}+1} & x_1 \in B(\frac{N}{2}); \\ (r_{m_1 \vee m_2}^{(1)})^{-\frac{d}{2}+1+\delta} R_2^{-2} N^{2-d} & x_1 \notin B(\frac{N}{2}). \end{cases} \tag{6.26}$$

We only provide its proof for the case $m_1 \le m_2$, since that of $m_1 \ge m_2$ is similar.

Case 1: When $m_1 \le k_1 - 1$**.** On the event F_{m_1,m_2}, we know that $\mathfrak{C}_1^{\mathrm{i}}$ must intersect $\partial B_{x_1}(r_{m_1}^{(1)})$, and that $\mathfrak{C}_2^{\mathrm{i}}$ must intersect $\partial B_{x_1}(r_{m_2}^{(2)})$ and is disjoint from both $\mathfrak{C}_1^{\mathrm{i}}$ and $\mathcal{C}^{\mathrm{in}}$. Similar to [6, Lemma 3.17], the following lemma provides the order of the probability of $\{\mathfrak{C}_1^{\mathrm{i}} \cap \partial B_{x_1}(r_{m_1}^{(1)}) \neq \emptyset\}$:

Lemma 6.2. *For any* $d \ge 3$ *and* $R \ge 1$*,*

$$\mathbb{P}\big(\mathfrak{C}_1^{\mathrm{i}} \cap \partial B_{x_1}(R) \neq \emptyset\big) \asymp R^{2-d}. \tag{6.27}$$

Proof. By the strong Markov property of Brownian excursion measure, one has

$$\mathbb{P}\big(\mathfrak{C}_1^{\mathrm{i}} \cap \partial B_{x_1}(R) \neq \emptyset\big) \asymp \sum\nolimits_{z \in F} \widetilde{\mathbb{P}}_z\big(\tau_{\partial B_{x_1}(R)} < \tau_{\{x_1,x_2\}} = \tau_{x_1} < \infty\big), \tag{6.28}$$

where $F := \{z \in \widetilde{\mathbb{Z}}^d : |z - x_1| = 1, z \notin I_{\{x_1,x_2\}}\}$. Note that $|F| = 2d - 1$. Moreover, by the strong Markov property of Brownian motion, we have

$$\begin{aligned} &\frac{\widetilde{\mathbb{P}}_z\big(\tau_{\partial B_{x_1}(R)} < \tau_{\{x_1,x_2\}} = \tau_{x_1} < \infty\big)}{\widetilde{\mathbb{P}}_z\big(\tau_{\partial B_{x_1}(R)} < \tau_{\{x_1,x_2\}}\big)} \\ \in &\Big[\min_{w\in\partial B_{x_1}(R)} \widetilde{\mathbb{P}}_w\big(\tau_{\{x_1,x_2\}} = \tau_{x_1} < \infty\big), \max_{w\in\partial B_{x_1}(R)} \widetilde{\mathbb{P}}_w\big(\tau_{\{x_1,x_2\}} = \tau_{x_1} < \infty\big)\Big]. \end{aligned} \tag{6.29}$$

For each $z \in F$, it follows from the transience of $\widetilde{\mathbb{Z}}^d$ that

$$\widetilde{\mathbb{P}}_z\big(\tau_{\partial B_{x_1}(R)} < \tau_{\{x_1,x_2\}}\big) \asymp 1. \tag{6.30}$$

Meanwhile, for any $w \in \partial B_{x_1}(R)$, by (2.22) one has

$$\widetilde{\mathbb{P}}_w\big(\tau_{\{x_1,x_2\}} < \infty\big) \asymp R^{2-d}. \tag{6.31}$$

Moreover, given that a Brownian motion starting from $w \in \partial B_{x_1}(R)$ hits $\{x_1, x_2\}$, [25, Proposition 6.5.4] implies that the conditional probability of reaching x_1 before x_2 is proportional to the harmonic measure of $\{x_1, x_2\}$ at x_1, which exactly equals $\frac{1}{2}$ by the symmetry. Combined with (6.31), it yields that

$$\widetilde{\mathbb{P}}_w\big(\tau_{\{x_1,x_2\}} = \tau_{x_1} < \infty\big) \asymp R^{2-d}, \quad \forall w \in \partial B_{x_1}(R). \tag{6.32}$$

Putting (6.29), (6.30) and (6.32) together, we get

$$\widetilde{\mathbb{P}}_z\big(\tau_{\partial B_{x_1}(R)} < \tau_{\{x_1,x_2\}} = \tau_{x_1} < \infty\big) \asymp R^{2-d}, \quad \forall z \in F. \tag{6.33}$$

This together with (6.28) and $|F| = 2d - 1$ yields (6.27). □

Additionally, given that $\{\mathfrak{C}_1^{\mathrm{i}} \cap \partial B_{x_1}(r_{m_1}^{(1)}) \neq \emptyset\}$ occurs, we have the following stochastic domination for $\mathfrak{C}_1^{\mathrm{i}}$. For any path $\widetilde{\eta}$ on $\widetilde{\mathbb{Z}}^d$, let $\mathrm{proj}(\widetilde{\eta})$ denote its projection onto the lattice $\mathbb{Z}^d$. For any path $\overline{\eta}$ on $\mathbb{Z}^d$, we denote its length (i.e., the number of jumps in $\overline{\eta}$) by $\mathrm{len}(\overline{\eta})$. Recall that $\widetilde{\partial}\widetilde{B}_{x_1}(1) = \{z \in \mathbb{Z}^d : \{z, x_1\} \in \mathbb{L}^d\}$.

Lemma 6.3. *Conditioned on the event $\big\{\mathfrak{C}_1^{\mathrm{i}} \cap \partial B_{x_1}(r_{m_1}^{(1)}) \neq \emptyset\big\}$, there exists two (random) paths $\widehat{\eta}_1$ and $\widehat{\eta}_2$ satisfying the following properties:*

- *(i)* $\cup_{j\in\{1,2\}}\mathrm{ran}(\widehat{\eta}_i) \subset \mathfrak{C}_1^{\mathrm{i}}$;
- *(ii) For each $j \in \{1, 2\}$, $\widehat{\eta}_j$ starts from $\widetilde{\partial}\widetilde{B}_{x_1}(1) \setminus \{x_2\}$, stops when reaching $\partial B_{x_1}(r_{m_1-1}^{(1)})$, and never hits $\{x_1, x_2\}$. We denote by Γ the collection of all possible projections onto $\mathbb{Z}^d$ of such paths.*
- *(iii) For any $\overline{\eta}_1, \overline{\eta}_2 \in \Gamma$,*

$$\mathbb{P}\big(\mathrm{proj}(\widehat{\eta}_j) = \overline{\eta}_j, \forall j \in \{1, 2\}\big) \lesssim (2d)^{-\mathrm{len}(\overline{\eta}_1)-\mathrm{len}(\overline{\eta}_2)}. \tag{6.34}$$

Proof. Let η be an excursion in $\mathfrak{C}_1^{\mathrm{i}}$ that reaches $\partial B_{x_1}(r_{m_1}^{(1)})$. We define $\widehat{\eta}_1$ as its subpath from the first time it hits $\widetilde{\partial}\widetilde{B}_{x_1}(1)$ until it reaches $\partial B_{x_1}(r_{m_1}^{(1)})$, and define $\widehat{\eta}_2$ as the reversed path of its subpath from the last time it stays outside $\widetilde{B}_{x_1}(r_{m_1-1}^{(1)})$

until the last time it visits $\widetilde{\partial}\widetilde{B}_{x_1}(1)$. By the construction, $\widehat{\eta}_1$ and $\widehat{\eta}_2$ satisfy both Properties (i) and (ii).

For Property (iii), we denote the starting and ending point of the path $\overline{\eta}_j$ by y_j and z_j respectively. Let $\overline{\eta}'_2$ be the path on $\mathbb{Z}^d$ that starts from x_1, jumps to y_1 in the first step, and then moves along $\overline{\eta}_2$. Note that $\mathrm{len}(\overline{\eta}'_2) = \mathrm{len}(\overline{\eta}_2) + 1$. By the strong Markov property of Brownian excursion measure and the last-exit decomposition, we have

$$\begin{aligned} &\mathbb{P}\big(\mathfrak{E}^{\mathrm{i}}_1 \cap \partial B_{x_1}(r^{(1)}_{m_1}) \neq \emptyset, \mathrm{proj}(\widehat{\eta}_j) = \overline{\eta}_j, \forall j \in \{1,2\}\big) \\ \lesssim &\widetilde{\mathbb{P}}_{y_1}\big(\mathrm{proj}\big(\{\widetilde{S}_t : 0 \le t \le \tau_{\partial B_{x_1}(r^{(1)}_{m_1-1})}\}\big) = \overline{\eta}_1\big) \cdot \max_{w \in \partial B_{x_1}(r^{(1)}_{m_1})} G(w, z_2) \\ &\cdot \widetilde{\mathbb{P}}_{z_2}\big(\mathrm{proj}\big(\{\widetilde{S}_{\tau_{\{x_1,x_2\}}-s} : 0 \le s \le \tau_{\{x_1,x_2\}}\}\big) = \overline{\eta}'_2\big). \end{aligned} \tag{6.35}$$

Recalling from Section 2.2 that the projection of $\widetilde{S}_\cdot \sim \widetilde{\mathbb{P}}_{y_1}$ is a simple random walk starting from y_1, we have

$$\widetilde{\mathbb{P}}_{y_1}\big(\mathrm{proj}\big(\{\widetilde{S}_t : 0 \le t \le \tau_{\partial B_{x_1}(r^{(1)}_{m_1-1})}\}\big) = \overline{\eta}_1\big) = (2d)^{-\mathrm{len}(\overline{\eta}_1)}, \tag{6.36}$$

$$\widetilde{\mathbb{P}}_{z_2}\big(\mathrm{proj}\big(\{\widetilde{S}_{\tau_{\{x_1,x_2\}}-s} : 0 \le s \le \tau_{\{x_1,x_2\}}\}\big) = \overline{\eta}'_2\big) = (2d)^{-\mathrm{len}(\overline{\eta}_2)+1}. \tag{6.37}$$

Plugging (6.36), (6.37) and $\max_{w \in \partial B_{x_1}(r^{(1)}_{m_1})} G(w, z_2) \asymp (r^{(1)}_{m_1-1})^{2-d}$ into (6.35), we get

$$\begin{aligned} &\mathbb{P}\big(\mathfrak{E}^{\mathrm{i}}_1 \cap \partial B_{x_1}(r^{(1)}_{m_1}) \neq \emptyset, \mathrm{proj}(\widehat{\eta}_j) = \overline{\eta}_j, \forall j \in \{1,2\}\big) \\ \lesssim &(r^{(1)}_{m_1})^{2-d} \cdot (2d)^{-\mathrm{len}(\overline{\eta}_1)-\mathrm{len}(\overline{\eta}_2)}. \end{aligned} \tag{6.38}$$

Combined with $\mathbb{P}(\mathfrak{E}^{\mathrm{i}}_1 \cap \partial B_{x_1}(r^{(1)}_{m_1}) \neq \emptyset) \asymp (r^{(1)}_{m_1})^{2-d}$ (by Lemma 6.2), it completes the proof of this lemma. $\square$

Given $\mathfrak{E}^{\mathrm{i}}_1$ and $\mathcal{C}^{\mathrm{in}}$, by applying Lemma 2.20 (where we use the case $\chi \le 10c^{-1}$ and drop the second term in the third line of (2.64)), we derive that the probability of $\{\mathfrak{E}^{\mathrm{i}}_2 \cap \partial B_{x_1}(r^{(2)}_{m_2}) \neq \emptyset, \mathfrak{E}^{\mathrm{i}}_2 \cap (\mathfrak{E}^{\mathrm{i}}_1 \cup \mathcal{C}_1) = \emptyset\}$ is upper-bounded by

$$\begin{aligned} &C(r^{(1)}_{m_1} \cdot r^{(2)}_{m_2})^{1-d} \cdot \widetilde{\mathbb{P}}_{x_2}\big(\tau_{\partial B_{x_1}(r^{(1)}_{m_1-1})} < \tau_{\mathfrak{E}^{\mathrm{i}}_1}\big) \\ &\cdot \sum\nolimits_{z_1 \in \partial \mathcal{B}_{x_1}(C_{10} r^{(1)}_{m_1}), z_2 \in \partial \mathcal{B}_{x_1}(\frac{r^{(2)}_{m_2}}{C_{10}})} \mathbb{P}\big(\mathsf{A}^{\mathcal{C}_1 \cup \{x_1,x_2\}}(z_1, z_2; C_{10} r^{(1)}_{m_1}, \tfrac{r^{(2)}_{m_2}}{C_{10}})\big)\big). \end{aligned} \tag{6.39}$$

Here $C_{10} > 0$ is a constant that is much smaller than C_9, and the event $\mathsf{A}^D(z, z'; r, r')$ is defined by $\big\{z \xleftrightarrow{(D)} z'\big\} \cap \big\{z \xleftrightarrow{(D)} \partial B_{x_1}(10r) \cup \partial B_{x_1}(10r')\big\}^c$. Consequently,

$$\begin{aligned} \mathbb{P}\big(\mathsf{F}_{m_1,m_2}\big) \lesssim &(r^{(1)}_{m_1} r^{(2)}_{m_2})^{1-d} \mathbb{E}_{\mathfrak{E}^{\mathrm{i}}_1}\Big[\widetilde{\mathbb{P}}_{x_2}\big(\mathfrak{E}^{\mathrm{i}}_1 \cap \partial B_{x_1}(r^{(1)}_{m_1}) \neq \emptyset, \tau_{\partial B_{x_1}(r^{(1)}_{m_1-1})} < \tau_{\mathfrak{E}^{\mathrm{i}}_1}\big)\Big] \\ &\cdot \sum\nolimits_{z_1 \in \partial \mathcal{B}_{x_1}(C_{10} r^{(1)}_{m_1}), z_2 \in \partial \mathcal{B}_{x_2}(\frac{r^{(2)}_{m_2}}{C_{10}})} \mathbb{P}\big(\overline{\mathsf{C}}^{m_1,m_2}_{z_1,z_2}\big). \end{aligned} \tag{6.40}$$

Here the event $\overline{\mathsf{C}}^{m_1,m_2}_{z_1,z_2}$ is defined by

$$\overline{\mathsf{C}}^{m_1,m_2}_{z_1,z_2} := \big\{\mathbf{0} \xleftrightarrow{(\{x_1,x_2\})} B_{x_1}(r^{(1)}_{m_1}) \| \mathsf{A}^{\{x_1,x_2\}}(z_1, z_2; C_{10} r^{(1)}_{m_1}, \tfrac{r^{(2)}_{m_2}}{C_{10}})) \cap \mathsf{D}_{m_2}\big\}, \tag{6.41}$$

where $\mathsf{D}_{m_2} := \{B_{x_1}(r^{(2)}_{m_2}) \xleftrightarrow{(\{x_1,x_2\})} B_{x_1}(R_2)\}$ for $0 \le m_2 \le k_2 - 1$, and D_{m_2} is the sure event for $m_2 = k_2$.

We bound the expectation on the right-hand side in the following lemma. With a slight abuse of notation, for any path $\overline{\eta}$ on $\mathbb{Z}^d$, we denote its range (i.e., the collection of lattice points visited by $\overline{\eta}$) by $\mathrm{ran}(\overline{\eta})$. For any $x \in \mathbb{Z}^d$, we denote by $\mathbb{P}_x$ the law of the simple random walk on $\mathbb{Z}^d$ starting from x.

Lemma 6.4. *For any $d \ge 3$ and $\delta \in (0, \frac{1}{2})$,*

$$\mathbb{E}_{\mathfrak{C}^{\mathrm{i}}_1}\Big[\widetilde{\mathbb{P}}_{x_2}\big(\mathfrak{C}^{\mathrm{i}}_1 \cap \partial B_{x_1}(r^{(1)}_{m_1}) \neq \emptyset, \tau_{\partial B_{x_1}(r^{(1)}_{m_1-1})} < \tau_{\mathfrak{C}^{\mathrm{i}}_1}\big)\Big] \lesssim (r^{(1)}_{m_1})^{2-d-(1-\delta)\cdot \mathbb{1}_{d=3}}. \tag{6.42}$$

Proof. Note that Lemma 6.2 implies

$$\mathbb{P}\big(\mathfrak{C}^{\mathrm{i}}_1 \cap \partial B_{x_1}(r^{(1)}_{m_1}) \neq \emptyset\big) \asymp (r^{(1)}_{m_1})^{2-d}. \tag{6.43}$$

This already gives the bound (6.42) for $d \ge 4$. Hence, we focus on the case $d = 3$ for the remainder of the proof.

Moreover, by Lemma 6.3, conditioned on $\big\{\mathfrak{C}^{\mathrm{i}}_1 \cap \partial B_{x_1}(r^{(1)}_{m_1}) \neq \emptyset\big\}$, the expectation of the probability of $\widetilde{\mathbb{P}}_{x_2}\big(\tau_{\partial B_{x_1}(r^{(1)}_{m_1-1})} < \tau_{\mathfrak{C}^{\mathrm{i}}_1}\big)$ is upper-bounded by

$$\begin{aligned} &\mathbb{E}_{\widehat{\eta}_1,\widehat{\eta}_2}\Big[\widetilde{\mathbb{P}}_{x_2}\big(\tau_{\partial B_{x_1}(r^{(1)}_{m_1-1})} < \tau_{\cup_{j\in\{1,2\}}\mathrm{ran}(\widehat{\eta}_j)}\big)\Big] \\ =&\sum\nolimits_{\overline{\eta}_1,\overline{\eta}_2\in\Gamma} \mathbb{P}\big(\mathrm{proj}(\widehat{\eta}_j) = \overline{\eta}_j, \forall j \in \{1,2\}\big)\widetilde{\mathbb{P}}_{x_2}\big(\tau_{\partial B_{x_1}(r^{(1)}_{m_1-1})} < \tau_{\cup_{j\in\{1,2\}}\mathrm{ran}(\overline{\eta}_j)}\big) \\ \le&\sum\nolimits_{\overline{\eta}_1,\overline{\eta}_2\in\Gamma} (2d)^{-\mathrm{len}(\overline{\eta}_1)-\mathrm{len}(\overline{\eta}_2)} \cdot \widetilde{\mathbb{P}}_{x_2}\big(\tau_{\partial B_{x_1}(r^{(1)}_{m_1-1})} < \tau_{\cup_{j\in\{1,2\}}\mathrm{ran}(\overline{\eta}_j)}\big). \end{aligned} \tag{6.44}$$

For any $y \in \widetilde{B}_{x_1}(1) \setminus \{x_2\}$, we denote by Γ_y the subset of Γ restricted to the paths that start from y. We arbitrarily take $y_1, y_2 \in \widetilde{B}_{x_1}(1) \setminus \{x_2\}$, and take three independent random walks $S^1_\cdot \sim \mathbb{P}_{y_1}$, $S^2_\cdot \sim \mathbb{P}_{y_2}$ and $S^3_\cdot \sim \mathbb{P}_{x_2}$. For each $1 \le i \le 3$, we denote by $\mathring{\tau}_i$ the first time when $S^i_\cdot$ hits $\partial B_{x_1}(r^{(1)}_{m_1-1})$, and we define the path $\mathring{\eta}_i$ (resp. $\mathring{\eta}^*_i$) as $\big\{\overline{S}^i_n : 0 \le n \le \overline{\tau}_i\big\}$ (resp. $\big\{\overline{S}^i_n : 0 \le n \le (r^{(1)}_{m_1})^{2-2\delta}\big\}$). Note that

$$\begin{aligned} &\sum\nolimits_{\overline{\eta}_1\in\Gamma_{y_1},\overline{\eta}_2\in\Gamma_{y_2}} (2d)^{-\mathrm{len}(\overline{\eta}_1)-\mathrm{len}(\overline{\eta}_2)} \cdot \widetilde{\mathbb{P}}_{x_2}\big(\tau_{\partial B_{x_1}(r^{(1)}_{m_1-1})} < \tau_{\cup_{j\in\{1,2\}}\mathrm{ran}(\overline{\eta}_j)}\big) \\ \le&\mathbb{P}\big(\big[\cup_{i=1,2}\mathrm{ran}(\mathring{\eta}_i)\big] \cap \mathrm{ran}(\mathring{\eta}_3) = \emptyset\big). \end{aligned} \tag{6.45}$$

Moreover, since $\{\mathring{\tau}_i \ge (r^{(1)}_{m_1})^{2-2\delta}\}$ implies $\mathrm{ran}(\mathring{\eta}^*_i) \subset \mathrm{ran}(\mathring{\eta}_i)$, we have

$$\begin{aligned} &\mathbb{P}\big(\big[\cup_{i=1,2}\mathrm{ran}(\mathring{\eta}_i)\big] \cap \mathrm{ran}(\mathring{\eta}_3) = \emptyset\big) \\ \le&\mathbb{P}\big(\big[\cup_{i=1,2}\mathrm{ran}(\mathring{\eta}^*_i)\big] \cap \mathrm{ran}(\mathring{\eta}^*_3) = \emptyset\big) + \sum\nolimits_{1\le i\le 3}\mathbb{P}\big(\mathring{\tau}_i \le (r^{(1)}_{m_1})^{2-2\delta}\big) \\ \lesssim&(r^{(1)}_{m_1})^{-1+\delta} + \exp(-c(r^{(1)}_{m_1})^{2\delta}), \end{aligned} \tag{6.46}$$

where in the last inequality we used [25, Proposition 2.1.2 and Corollary 10.2.2].

Inserting (6.46) into (6.45), and then summing over all $y_1, y_2 \in \widetilde{B}_{x_1}(1) \setminus \{x_2\}$, we obtain that the right-hand side of (6.44) is at most

$$C\big[(r^{(1)}_{m_1})^{-1+\delta} + \exp(-c(r^{(1)}_{m_1})^{2\delta})\big] \lesssim (r^{(1)}_{m_1})^{-1+\delta}. \tag{6.47}$$

Combined with (6.43) and (6.44), it yields the desired bound (6.42) for $d = 3$, and thus completes the proof. □

Remark 6.5. Following the same argument as in the proof of Lemma 6.4, one may derive the following result for the discrete loop soup on $\mathbb{Z}^3$ (denoted by $\mathcal{L}_\alpha$, where α is the intensity). For any $N \gg n \geq 1$, let $\widehat{\mathfrak{L}}[n, N]$ denote the collection of loops in $\mathcal{L}_{1/2}$ that cross the annulus $B(N) \setminus B(n)$. Suppose that ℓ is a loop in $\widehat{\mathfrak{L}}[n, N]$, sampled from the loop measure. Then for any $\delta \in (0, \frac{1}{2})$ and $x \in \partial B(n)$,

$$\mathbb{E}_\ell\big[\mathbb{1}_{\widehat{\mathfrak{L}}[n,N]\neq\emptyset} \cdot \mathbb{P}_x\big(\tau_{\partial B(cN)} < \tau_{\mathrm{ran}(\ell)}\big)\big] \lesssim \big(n/N\big)^{2-\delta}. \tag{6.48}$$

First of all, the probability of $\{\widehat{\mathfrak{L}}[n, N] \neq \emptyset\}$ is $O(n/N)$. In addition, as shown in (6.45), for the simple random walk starting from x, the probability of escaping ℓ can be upper-bounded by that of escaping two independent random walks starting from $B(n)$ and stopped upon hitting $\partial B(cN)$. Parallel to (6.46), the latter probability is $O\big((n/N)^{1-\delta}\big)$. These observations together yield the bound (6.48).

We now estimate the probability of $\overline{\mathsf{C}}^{m_1,m_2}_{z_1,z_2}$. For any $D' \subset \widetilde{\mathbb{Z}}^d$, by Corollary 2.11 and Lemma 2.13,

$$\begin{aligned}&\mathbb{P}\big(B_{x_1}(r^{(2)}_{m_2}) \xleftrightarrow{(D')} B_{x_1}(R_2)\big)\\ \lesssim&(r^{(2)}_{m_2} \cdot R_2)^{-\frac{d}{2}} \sum_{z_3\in\partial\mathcal{B}_{x_1}(dr^{(2)}_{m_2}),z_4\in\partial\mathcal{B}_{x_1}(d^{-1}R_2)} \mathbb{P}\big(\mathsf{A}^{D'}(z_3, z_4; dr^{(2)}_{m_2}, d^{-1}R_2)\big).\end{aligned} \tag{6.49}$$

Meanwhile, it follows from Corollary 2.11 and Lemma 2.13 that

$$\begin{aligned}&\mathbb{P}\big(\mathbf{0} \xleftrightarrow{(D')} B_{x_1}(r^{(1)}_{m_1})\big)\\ \lesssim&(r^{(1)}_{m_1})^{-\frac{d}{2}}(r^{(2)}_{m_2})^{-\frac{d}{2}-1} \sum\nolimits_{z_5\in\partial\mathcal{B}_{x_1}(\frac{1}{2}C_{10}r^{(1)}_{m_1}),z_6\in\partial\mathcal{B}_{x_1}(\frac{2r^{(2)}_{m_2}}{C_{10}}),z_7\in\partial\mathcal{B}_{x_1}(2dr^{(2)}_{m_2})}\\ &\mathbb{P}\big(\mathsf{A}^{D'}(z_5, z_6; \tfrac{1}{2}C_{10}r^{(1)}_{m_1}, \tfrac{2r^{(2)}_{m_2}}{C_{10}}) \cap \mathsf{A}^{D'}(z_7, \mathbf{0}; 2dr^{(2)}_{m_2}, r_1)\big).\end{aligned} \tag{6.50}$$

Using the restriction property, (6.49) and (6.50), we get

$$\mathbb{P}\big(\overline{\mathsf{C}}^{m_1,m_2}_{\bar{z}}\big) \lesssim (r^{(1)}_{m_1})^{-\frac{d}{2}}(r^{(2)}_{m_2})^{-d-1}R_2^{-\frac{d}{2}} \cdot \sum_{\bar{z}:=(z_3,z_4,z_5,z_6,z_7)\in\Psi_{m_1,m_2}} \mathbb{P}\big(\widetilde{\mathsf{C}}^{m_1,m_2}_{\bar{z}}\big), \tag{6.51}$$

where we denote $\Psi_{m_1,m_2} := \partial\mathcal{B}_{x_1}(dr^{(2)}_{m_2})\times\partial\mathcal{B}_{x_1}(d^{-1}R_2)\times\partial\mathcal{B}_{x_1}(\frac{1}{2}C_{10}r^{(1)}_{m_1})\times\partial\mathcal{B}_{x_1}(\frac{2r^{(2)}_{m_2}}{C_{10}})\times \partial\mathcal{B}_{x_1}(2dr^{(2)}_{m_2})$, and define $\widetilde{\mathsf{C}}^{m_1,m_2}_{\bar{z}}$ as the intersection of $\mathsf{A}^{\{x_1,x_2\}}(z_1, z_2; C_{10}r^{(1)}_{m_1}, \frac{r^{(2)}_{m_2}}{C_{10}}))$,

$\mathsf{A}^{\{x_1,x_2\}}(z_3,z_4;dr^{(2)}_{m_2},d^{-1}R_2)$, $\mathsf{A}^{\{x_1,x_2\}}(z_5,z_6;\frac{1}{2}C_{10}r^{(1)}_{m_1},\frac{2r^{(2)}_{m_2}}{C_{10}})$ and $\mathsf{A}^{\{x_1,x_2\}}(z_7,\mathbf{0};2dr^{(2)}_{m_2},r_1)$. Moreover, on the event $\widetilde{\mathsf{A}}^{m_1,m_2}_{\bar{z}}$, the following independent events occur:

$$\mathsf{C}_1 := \big\{ z_1 \xleftrightarrow{(\{x,y\}\cup \partial B_{x_1}(100C_{10}r^{(1)}_{m_1})\cup \partial B_{x_1}(\frac{100r^{(2)}_{m_2}}{C_{10}}))} z_2 \| z_5 \xleftrightarrow{(\{x,y\}\cup \partial B_{x_1}(100C_{10}r^{(1)}_{m_1})\cup \partial B_{x_1}(\frac{100r^{(2)}_{m_2}}{C_{10}}))} z_6 \big\},$$

$$\mathsf{C}_2 := \big\{ z_3 \xleftrightarrow{(\{x,y\}\cup \partial B_{x_1}(\frac{r^{(2)}_{m_2}}{10}))} z_4 \| z_7 \xleftrightarrow{(\{x,y\}\cup \partial B_{x_1}(\frac{r^{(2)}_{m_2}}{10})))} \mathbf{0} \big\}.$$

By the estimate (4.25), one has

$$\mathbb{P}\big(\mathsf{C}_1\big) \lesssim (r^{(1)}_{m_1})^{3-\frac{d}{2}}(r^{(2)}_{m_2})^{-\frac{3d}{2}+1}. \tag{6.52}$$

Moreover, it has been verified in (5.16) and (5.20) that

$$\mathbb{P}\big(\mathsf{C}_2\big) \lesssim (R_1R_2)^{2-d}\cdot \big(\tfrac{R_1\wedge R_2}{r^{(2)}_{m_2}}\big)^{\frac{d}{2}-3}. \tag{6.53}$$

Putting (6.51), (6.52) and (6.53) together, we get

$$\begin{aligned}\mathbb{P}\big(\overline{\mathsf{C}}^{m_1,m_2}_{\bar{z}}\big) \lesssim & |\Psi_{m_1,m_2}|\cdot (r^{(1)}_{m_1})^{3-d}(r^{(2)}_{m_2})^{-3d+3}R_1^{2-d}R_2^{-\frac{3d}{2}+2}(R_1\wedge R_2)^{\frac{d}{2}-3}\\ \lesssim & (r^{(1)}_{m_1})^{2}R_1^{2-d}R_2^{-\frac{d}{2}+1}(R_1\wedge R_2)^{\frac{d}{2}-3}.\end{aligned} \tag{6.54}$$

Plugging (6.42) and (6.54) into (6.40), we obtain (6.26):

$$\mathbb{P}\big(\mathsf{F}_{m_1,m_2}\big) \lesssim (r^{(1)}_{m_1})^{4-d-(1-\delta)\cdot \mathbb{1}_{d=3}}R_1^{2-d}R_2^{-\frac{d}{2}+1}(R_1\wedge R_2)^{\frac{d}{2}-3} \lesssim \mathbb{J}_{m_1,m_2}. \tag{6.55}$$

When $m_1=k_1$. We first decompose the event F_{k_1,m_2} as follows. We define

$$l_\star := \min\{l\ge 1: 2^{l+3}\ge R_1\}. \tag{6.56}$$

For $1\le l\le l_\star-1$, we denote the event

$$\mathsf{U}_l := \big\{\mathfrak{C}^{\mathrm{i}}_1\cap \partial B(2^l)\neq\emptyset, \mathfrak{C}^{\mathrm{i}}_1\cap \partial B(2^{l-1})=\emptyset\big\}. \tag{6.57}$$

In addition, we also denote $\mathsf{U}_0 := \big\{\mathfrak{C}^{\mathrm{i}}_1\cap \partial B(1)\neq\emptyset\big\}$ and

$$\mathsf{U}_{l_\star} := \big\{\mathfrak{C}^{\mathrm{i}}_1\cap \partial B(2^{l_\star-1})=\emptyset, \mathfrak{C}^{\mathrm{i}}_1\cap \partial B_{x_1}(r^{(1)}_{k_1})\neq\emptyset\big\}. \tag{6.58}$$

Clearly, one has $\{\mathfrak{C}^{\mathrm{i}}_1\cap \partial B_{x_1}(r^{(1)}_{k_1})\neq\emptyset\}\subset \cup_{0\le l\le l_\star}\mathsf{U}_l$. Moreover, for each $1\le l\le l_\star$, on the event U_l, if $\mathfrak{C}^{\mathrm{i}}_1$ intersects $\mathcal{C}_1$, then $\mathsf{V}_l := \big\{\mathbf{0}\xleftrightarrow{\{x_1,x_2\}}\partial B(2^{l-1})\big\}$ must occur. For convenience, we also let V_0 be the sure event.

Recall that on F_{k_1,m_2}, the events $\mathfrak{C}^{\mathrm{i}}_1\cap \partial B_{x_1}(r^{(1)}_{k_1})\neq\emptyset$, $\mathfrak{C}^{\mathrm{i}}_1\cap\mathcal{C}_1\neq\emptyset$ and D_{m_2} all occur (recalling D_{m_2} below (6.41)); moreover, $\mathfrak{C}^{\mathrm{i}}_2$ intersects $B_{x_1}(r^{(2)}_{m_2})$ and is disjoint from $\mathfrak{C}^{\mathrm{i}}_1$. By the observation in the last paragraph, there exists $0\le l\le l_\star$ such that $\mathsf{U}_l\cap\mathsf{V}_l$ occurs. Note that the estimate (1.2) implies

$$\mathbb{P}\big(\mathsf{V}_l\big) \lesssim 2^{(-\frac{d}{2}+1)l},\ \ \forall 0\le l\le l_\star. \tag{6.59}$$

In addition, on the event U_l, there exists an excursion in $\mathfrak{E}_1^{\mathrm{i}}$ that contains a trajectory of Brownian motion from $\partial B_{x_1}(r_{k_1}^{(1)})$ to $B(2^l)$, and one from $\partial B(2^l)$ to x_1. Thus, by the strong Markov property of Brownian excursion measure, we have

$$\mathbb{P}(\mathsf{U}_l) \lesssim \big(\tfrac{R_1}{2^l}\big)^{2-d} \cdot R_1^{2-d} = 2^{(d-2)l} R_1^{4-2d}, \quad \forall 0 \le l \le l_\star. \tag{6.60}$$

Moreover, conditioned on the event U_l, when $\{\mathfrak{E}_2^{\mathrm{i}} \cap B_{x_1}(r_{m_2}^{(2)}) \neq \emptyset, \mathfrak{E}_2^{\mathrm{i}} \cap \mathfrak{E}_1^{\mathrm{i}} = \emptyset\}$ occurs, there exists an excursion $\mathfrak{E}_2^{\mathrm{i}}$ that contains a trajectory from x_2 to $\partial B_{x_1}(r_{k_1}^{(1)})$ without hitting $\mathfrak{E}_1^{\mathrm{i}}$ (whose probability, for the same reason as in proving (6.47), is at most $CR_1^{-(1-\delta)\cdot \mathbb{1}_{d=3}}$), and one from $B_{x_1}(r_{m_2}^{(2)})$ to x_2 (which happens with probability $O((r_{m_2}^{(2)})^{2-d})$). Furthermore, it follows from (1.5) that arbitrarily given $\mathfrak{E}_1^{\mathrm{i}}$ and $\mathcal{C}_1$, the probability of D_{m_2} is upper-bounded by $C(R_2/r_{m_2}^{(2)})^{-\frac{d}{2}+1}$. To sum up,

$$\begin{aligned}\mathsf{F}_{k_1,m_2} \lesssim & \sum\nolimits_{0\le l\le l_\star} 2^{(-\frac{d}{2}+1)l} \cdot 2^{(d-2)l} R_1^{4-2d} \cdot R_1^{-(1-\delta)\cdot\mathbb{1}_{d=3}} (r_{m_2}^{(2)})^{2-d} \cdot \big(\tfrac{R_2}{r_{m_2}^{(2)}}\big)^{-\frac{d}{2}+1} \\ =&(r_{m_2}^{(2)})^{-\frac{d}{2}+1} R_1^{4-2d-(1-\delta)\cdot\mathbb{1}_{d=3}} R_2^{-\frac{d}{2}+1} \sum\nolimits_{0\le l\le l_\star} 2^{(\frac{d}{2}-1)l} \\ \lesssim&(r_{m_2}^{(2)})^{-\frac{d}{2}+1} R_1^{-\frac{3}{2}d+3-(1-\delta)\cdot\mathbb{1}_{d=3}} R_2^{-\frac{d}{2}+1} \lesssim \mathbb{J}_{k_1,m_2}.\end{aligned} \tag{6.61}$$

By (6.55) and (6.61), we have confirmed the desired bound (6.26), thereby completing the proof of Lemma 6.1. □

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Appendix A. Proof of Lemma 2.7

When $3 \le d \le 5$, the desired bound (2.31) can be derived as follows:

$$\begin{aligned}\mathbb{P}\big(A \xleftrightarrow{(D)} x\big) \asymp & N^{-\frac{d}{2}+1} \cdot \mathbb{P}\big(A \xleftrightarrow{(D)} \partial B(N)\big) \\ \overset{\text{(Lemma 2.6)}}{\asymp} & N^{-\frac{d}{2}+1} \cdot \mathbb{P}\big(A \xleftrightarrow{(D\cup D')} \partial B(N)\big) \asymp \mathbb{P}\big(A \xleftrightarrow{(D\cup D')} x\big).\end{aligned} \tag{A.1}$$

Here the first and last inequalities both follow from [5, Proposition 1.9].

Now we focus on the case when $d \ge 7$. Note that

$$\begin{aligned}0 \le & \mathbb{P}\big(A \xleftrightarrow{(D)} x\big) - \mathbb{P}\big(A \xleftrightarrow{(D\cup D')} x\big) \\ =&\mathbb{P}(\mathsf{F}) := \mathbb{P}\big(A \xleftrightarrow{(D)} x, \big\{\mathbb{P}\big(A \xleftrightarrow{(D\cup D')} x\big\}^c\big).\end{aligned} \tag{A.2}$$

On F, the event $\{A \xleftrightarrow{(D)} x\}$ occurs, and removing all loops that intersect D' must violate it. As a result, there exists a loop $\widetilde{\ell}_\dagger \in \widetilde{\mathcal{L}}_{1/2}$ with $\mathrm{ran}(\widetilde{\ell}_\dagger) \cap D' \neq \emptyset$ such that

$\mathrm{ran}(\widetilde{\ell}_\dagger) \xleftrightarrow{(D)} A$ and $\mathrm{ran}(\widetilde{\ell}_\dagger) \xleftrightarrow{(D)} x$ occurs disjointly (to see this, one may remove the loops intersecting D' one by one, stopping upon the absence of $\{A \xleftrightarrow{(D)} x\}$; then the last removed loop meets the requirements). Here "A and A' occur disjointly" (denoted by $\mathsf{A} \circ \mathsf{A}'$) means that A and A' are certified by two disjoint collections of loops (see [3, Section 3.3] for further details). Thus, there exist $w_1, w_2, w_3 \in \mathbb{Z}^d$ such that $\widetilde{B}_{w_3}(1) \cap D' \neq \emptyset$, $\widetilde{\ell}_\dagger$ intersects $\widetilde{B}_{w_j}(1)$ for all $j \in \{1,2,3\}$, and that $w_1 \xleftrightarrow{(D)} A$ and $w_2 \xleftrightarrow{(D)} x$ occur disjointly. By [3, Lemma 2.3], the probability of having a loop in $\widetilde{\mathcal{L}}^D_{1/2}$ intersecting $\widetilde{B}_{w_j}(1)$ for all $j \in \{1,2,3\}$ is at most

$$C|w_1 - w_2|^{2-d}|w_2 - w_3|^{2-d}|w_3 - w_1|^{2-d}. \tag{A.3}$$

(**P.S.** In this proof and the next two sections, we denote $0^{-a} := 1$ for $a > 0$.)

In what follows, we estimate the probability of F, under different restrictions on the locations of w_1 and w_2. Recall that $A, D \subset \widetilde{B}(cN)$ and $D' \subset [\widetilde{B}(c^{-1}N)]^c$. We also take two small constants $c_\spadesuit, c_\clubsuit > 0$, and require that $c \ll c_\spadesuit \ll c_\clubsuit$.

Case 1: $w_1 \in B(N) \setminus B(c_\spadesuit N)$. Recall that on the event F, $\{w_1 \xleftrightarrow{(D)} A\}$ and $\{w_2 \xleftrightarrow{(D)} x\}$ occur disjointly, whose probabilities are bounded by

$$C|w_1|^{2-d}N^{d-2} \cdot \mathbb{P}\big(A \xleftrightarrow{(D)} x\big) \ \text{ and } \ C|w_2 - x|^{2-d} \tag{A.4}$$

respectively (by Lemma 2.15 and (1.1)). Meanwhile, the loop $\widetilde{\ell}_\dagger$ contains a trajectory of Brownian motion from w_1 to w_2, and one from D' to w_1. Therefore, the probability of having such a loop is at most

$$C|w_1 - w_2|^{2-d}\big[\mathrm{dist}(D, w_1)\big]^{2-d} \lesssim |w_1 - w_2|^{2-d}N^{2-d}. \tag{A.5}$$

Thus, by the BKR inequality, the probability of this case is upper-bounded by

$$C\mathbb{P}(A \xleftrightarrow{(D)} x) \sum_{w_1, w_2 \in \mathbb{Z}^d} |w_1 - w_2|^{2-d}|w_1|^{2-d}|w_2 - x|^{2-d} \lesssim N^{6-d}\mathbb{P}(A \xleftrightarrow{(D)} x), \tag{A.6}$$

where the last inequality can be derived from the following two basic facts (see e.g., [5, (2.79) and (2.80)]): for any $v_1, v_2 \in \mathbb{Z}^d$,

$$\textstyle\sum_{z \in \mathbb{Z}^d} |v_1 - z|^{2-d}|z - v_2|^{2-d} \lesssim |v_1 - v_2|^{4-d}, \tag{A.7}$$

$$\textstyle\sum_{z \in \mathbb{Z}^d} |v_1 - z|^{4-d}|z - v_2|^{2-d} \lesssim |v_1 - v_2|^{6-d}. \tag{A.8}$$

Case 2: $w_1 \in [B(N)]^c$. Applying the BKR inequality, and using the estimates (A.3) and (A.4), the probability of this case is at most

$$
\begin{aligned}
&CN^{d-2}\mathbb{P}\big(A \xleftrightarrow{(D)} x\big) \sum\nolimits_{w_1,w_2,w_3\in\mathbb{Z}^d:\widetilde{B}_{w_3}(1)\cap D'\neq\emptyset} |w_1-w_2|^{2-d} \\
&\qquad\qquad \cdot |w_2-w_3|^{2-d}|w_3-w_1|^{2-d}|w_1|^{2-d}|w_2-x|^{2-d} \\
&\overset{|w_1|\gtrsim N}{\lesssim} \mathbb{P}\big(A \xleftrightarrow{(D)} x\big) \sum\nolimits_{w_1,w_2,w_3\in\mathbb{Z}^d:\widetilde{B}_{w_3}(1)\cap D'\neq\emptyset} |w_1-w_2|^{2-d}|w_2-w_3|^{2-d} \\
&\qquad\qquad \cdot |w_3-w_1|^{2-d}|w_1|^{2-d}|w_2-x|^{2-d} \\
&\lesssim \mathbb{P}\big(A \xleftrightarrow{(D)} x\big) \sum\nolimits_{w_3\in\mathbb{Z}^d:\widetilde{B}_{w_3}(1)\cap D'\neq\emptyset} |w_3|^{2-d} \lesssim N^{6-d}\mathbb{P}(A \xleftrightarrow{(D)} x),
\end{aligned}
\tag{A.9}
$$

where in the last line we used [3, Corollary 4.5] and the condition (2.32) in turn.

It remains to consider the case when $w_1 \in B(c_\spadesuit N)$, for which we need to employ the decomposition argument of loops introduced in [3, Section 2.6.3]. Specifically, for $M > m \geq 1$, each loop in $\widetilde{\mathcal{L}}_{1/2}$ that crosses the annulus $B(M) \setminus B(m)$ can be divided into forward and backward crossing paths, where each forward (resp. backward) crossing path is a Brownian motion starting from $\partial B(m)$ (resp. $\partial B(M)$) and stopping upon hitting $\partial B(M)$ (resp. $\partial B(m)$). Note that the starting point of a forward crossing path is the ending point of certain backward crossing path, and vice versa. We enumerate the forward (resp. backward) crossing paths of all loops in $\widetilde{\mathcal{L}}^D_{1/2}$ crossing $B(M)\setminus B(m)$ as $\{\widetilde{\eta}^{\mathrm{F}}_i\}_{1\leq i\leq\kappa}$ (resp. $\{\widetilde{\eta}^{\mathrm{B}}_i\}_{1\leq i\leq\kappa}$). According to [5, Lemma 2.11], the total number κ of crossings decays exponentially. I.e.,

$$
\mathbb{P}\big(\kappa \geq l\big) \leq \big(\tfrac{Cm}{M}\big)^{(d-2)l}, \ \ \forall l \geq 1. \tag{A.10}
$$

We denote the starting and ending points of $\widetilde{\eta}^{\mathrm{F}}_i$ by $\mathbf{y}_i$ and $\mathbf{z}_i$ respectively. Let $\mathcal{F}_{\mathbf{yz}}$ denote the σ-field generated by $\{(\mathbf{y}_i,\mathbf{z}_i)\}_{1\leq i\leq\kappa}$. Referring to [3, Lemma 2.4], conditioned on $\mathcal{F}_{\mathbf{yz}}$, the crossing paths $\{\widetilde{\eta}^{\mathrm{F}}_i\}_{1\leq i\leq\kappa}\cup\{\widetilde{\eta}^{\mathrm{B}}_i\}_{1\leq i\leq\kappa}$ are independent. In addition, for any $1\leq i\leq\kappa$, $y\in\partial B(m)$ and $z\in\partial B(M)$, when $\widetilde{\eta}^{\mathrm{F}}_i$ starts from y and ends at z, it is distributed as

$$
\widetilde{\eta}^{\mathrm{F}}_i \sim \widetilde{\mathbb{P}}_y\big(\{\widetilde{S}_t\}_{0\leq t\leq\tau_{\partial B(M)}} \in \cdot \mid \tau_{\partial B(M)} = \tau_z < \tau_D\big). \tag{A.11}
$$

Similarly, when $\widetilde{\eta}^{\mathrm{B}}_i$ starts from z and ends at y, it has the distribution

$$
\widetilde{\eta}^{\mathrm{B}}_i \sim \widetilde{\mathbb{P}}_z\big(\{\widetilde{S}_t\}_{0\leq t\leq\tau_{\partial B(m)}} \in \cdot \mid \tau_{\partial B(m)} = \tau_y < \tau_D\big). \tag{A.12}
$$

Now we return to the proof. Recall that $c_\spadesuit \ll c_\clubsuit$.

Case 3: $w_1 \in B(c_\spadesuit N)$, $w_2 \in [B(c_\clubsuit N)]^c$. Let $c_\ddagger^- := c_\spadesuit^{2/3}c_\clubsuit^{1/3}$ and $c_\ddagger^+ := c_\spadesuit^{1/3}c_\clubsuit^{2/3}$. Note that $c_\spadesuit \ll c_\ddagger^- \ll c_\ddagger^+ \ll c_\clubsuit \ll 1$. We consider the crossing paths for $m = c_\ddagger^- N$ and $M = c_\ddagger^+ N$. Since $\widetilde{\ell}_\dagger$ intersects both $\widetilde{B}_{w_1}(1)$ and $\widetilde{B}_{w_2}(1)$, it crosses the annulus $B(c_\ddagger^+ N)\setminus B(c_\ddagger^- N)$. Moreover, there exist $1 \leq i_1, i_2, i_3 \leq \kappa$ such that $\widetilde{\eta}^{\mathrm{B}}_{i_1}$ intersects $\widetilde{B}_{w_2}(1)$, $\widetilde{\eta}^{\mathrm{B}}_{i_2}$ intersects $\widetilde{B}_{w_3}(1)$, and that $\big\{\mathrm{ran}(\widetilde{\eta}^{\mathrm{F}}_{i_3}) \xleftrightarrow{(D)} A\big\}$ and $\big\{w_2 \xleftrightarrow{(D)} x\big\}$ occur disjointly. As shown in [5, (5.37) and (5.38)], conditioned on $\mathcal{F}_{\mathbf{yz}}$, the probability of $\{\mathrm{ran}(\widetilde{\eta}^{\mathrm{B}}_{i_1})\cap \widetilde{B}_{w_2}(1)\neq\emptyset, \mathrm{ran}(\widetilde{\eta}^{\mathrm{B}}_{i_2})\cap \widetilde{B}_{w_3}(1)\neq\emptyset\}$ is at most $C|w_2|^{2-d}|w_3|^{2-d}|w_2-w_3|^{2-d}N^{d-2}$ when $i_1\neq i_2$, and is at most $C|w_2|^{4-2d}|w_3|^{4-2d}N^{2d-4}$ when $i_1=i_2$. By [5, Lemma 4.3], given $\mathcal{F}_{\mathbf{yz}}$, the conditional probability of $\big\{\mathrm{ran}(\widetilde{\eta}^{\mathrm{F}}_{i_3})\xleftrightarrow{(D)} A\big\}$ is at most proportional to $\mathbb{P}\big(A\xleftrightarrow{(D)}\partial B(c_{\spadesuit}N)\big)$. Combining these observations with the BKR inequality and (A.10), we obtain that the probability of this case is at most

$$\begin{aligned}&CN^{d-2}\cdot\mathbb{P}\big(A\xleftrightarrow{(D)}\partial B(c_{\spadesuit}N)\big)\cdot(\mathbb{I}_1+\mathbb{I}_2+\mathbb{I}_3)\\ \overset{\text{(Lemma 2.16)}}{\lesssim}\ &N^{2d-6}\cdot\mathbb{P}\big(A\xleftrightarrow{(D)}x\big)\cdot(\mathbb{I}_1+\mathbb{I}_2+\mathbb{I}_3),\end{aligned}\tag{A.13}$$

where $\mathbb{I}_1$, $\mathbb{I}_2$ and $\mathbb{I}_3$ are defined by

$$\mathbb{I}_1:=\sum\nolimits_{w_2\in B(2N)\setminus B(c_{\clubsuit}N),w_3\in\mathbb{Z}^d:\widetilde{B}_{w_3}(1)\cap D'\neq\emptyset}|w_2|^{2-d}|w_3|^{2-d}|w_2-w_3|^{2-d}|w_2-x|^{2-d},$$

$$\mathbb{I}_2:=\sum\nolimits_{w_2\in[B(2N)]^c,w_3\in\mathbb{Z}^d:\widetilde{B}_{w_3}(1)\cap D'\neq\emptyset}|w_2|^{2-d}|w_3|^{2-d}|w_2-w_3|^{2-d}|w_2-x|^{2-d},$$

$$\mathbb{I}_3:=N^{d-2}\sum\nolimits_{w_2\in[B(c_{\clubsuit}N)]^c,w_3\in\mathbb{Z}^d:\widetilde{B}_{w_3}(1)\cap D'\neq\emptyset}|w_2|^{4-2d}|w_3|^{4-2d}|w_2-x|^{2-d}.$$

We now estimate $\mathbb{I}_1$, $\mathbb{I}_2$ and $\mathbb{I}_3$ individually. For $\mathbb{I}_1$, Since $|w_2|\gtrsim N$ and $|w_2-w_3|\asymp|w_3|$ for all $w_2\in B(2N)\setminus B(c_{\clubsuit}N)$, we have

$$\begin{aligned}\mathbb{I}_1\lesssim&N^{2-d}\sum\nolimits_{w_2\in B(2N)\setminus B(c_{\clubsuit}N),w_3\in\mathbb{Z}^d:\widetilde{B}_{w_3}(1)\cap D'\neq\emptyset}|w_2-x|^{2-d}|w_3|^{4-2d}\\ \overset{(3.15),|w_3|\gtrsim N}{\lesssim}&N^{6-2d}\sum\nolimits_{w_3\in\mathbb{Z}^d:\widetilde{B}_{w_3}(1)\cap D'\neq\emptyset}|w_3|^{2-d}\overset{(2.32)}{\lesssim}N^{12-3d}.\end{aligned}\tag{A.14}$$

For $\mathbb{I}_2$, since $|w_2-x|\asymp|w_2|\gtrsim N$ for all $w_2\in[B(2N)]^c$, we have

$$\begin{aligned}\mathbb{I}_2\lesssim&N^{-2}\sum\nolimits_{w_2\in[B(2N)]^c,w_3\in\mathbb{Z}^d:\widetilde{B}_{w_3}(1)\cap D'\neq\emptyset}|w_2|^{6-2d}|w_3|^{2-d}|w_2-w_3|^{2-d}\\ \overset{(A.7)}{\lesssim}&N^{-2}\sum\nolimits_{w_3\in\mathbb{Z}^d:\widetilde{B}_{w_3}(1)\cap D'\neq\emptyset}|w_3|^{4-2d}\\ \overset{|w_3|\gtrsim N}{\lesssim}&N^{-d}\sum\nolimits_{w_3\in\mathbb{Z}^d:\widetilde{B}_{w_3}(1)\cap D'\neq\emptyset}|w_3|^{2-d}\overset{(2.32)}{\lesssim}c'N^{6-2d}.\end{aligned}\tag{A.15}$$

For $\mathbb{I}_3$, noting that $|w_2|\gtrsim N$ for all $w_2\in[B(c_{\clubsuit}N)]^c$ and that $|w_3|\gtrsim N$ for all w_3 with $\widetilde{B}_{w_3}(1)\cap D'\neq\emptyset$, we have

$$\begin{aligned}\mathbb{I}_3\lesssim&N^{-2}\sum\nolimits_{w_2\in[B(c_{\clubsuit}N)]^c,w_3\in\mathbb{Z}^d:\widetilde{B}_{w_3}(1)\cap D'\neq\emptyset}|w_2|^{6-2d}|x-w_2|^{2-d}|w_3|^{2-d}\\ \overset{(A.7)}{\lesssim}&N^{-d}\sum\nolimits_{w_3\in\mathbb{Z}^d:\widetilde{B}_{w_3}(1)\cap D'\neq\emptyset}|w_3|^{2-d}\overset{(2.32)}{\lesssim}c'N^{6-2d}.\end{aligned}\tag{A.16}$$

Plugging (A.14), (A.15), (A.16) into (A.13), we derive that the probability of this case is upper-bounded by

$$C(N^{6-d}+c')\cdot \mathbb{P}\big(A \xleftrightarrow{(D)} x\big) \lesssim c'\cdot \mathbb{P}\big(A \xleftrightarrow{(D)} x\big). \tag{A.17}$$

Case 4: $w_1\in B(c_\spadesuit N)$, $w_2\in B(c_\clubsuit N)$. Recall that $\widetilde{\ell}_\dagger$ intersects $\widetilde{B}_{w_j}(1)$ for all $j\in\{1,2,3\}$. Therefore, in this case, it must cross the annulus $B(c_\spadesuit^{-2}N)\setminus B(c_\spadesuit^{-1}N)$. We consider the crossing paths for $m=c_\spadesuit^{-1}N$ and $M=c_\spadesuit^{-2}N$. Similar to **Case 3**, there exist $1\le i_1,i_2,i_3\le\kappa$ such that $\widetilde{\eta}^{\mathrm{F}}_{i_1}$ intersects $\widetilde{B}_{w_2}(1)$, $\widetilde{\eta}^{\mathrm{B}}_{i_2}$ intersects $\widetilde{B}_{w_3}(1)$, and that $\big\{\mathrm{ran}(\widetilde{\eta}^{\mathrm{F}}_{i_3})\xleftrightarrow{(D)} A\big\}$ and $\big\{w_2\xleftrightarrow{(D)} x\big\}$ occur disjointly.

Case 4(a): $i_1\neq i_3$. In this case, $\widetilde{\eta}^{\mathrm{F}}_{i_1}$, $\widetilde{\eta}^{\mathrm{B}}_{i_2}$ and $\widetilde{\eta}^{\mathrm{F}}_{i_3}$ are independent given $\mathcal{F}_{\mathbf{yz}}$. Recall from **Case 3** that conditioned on $\mathcal{F}_{\mathbf{yz}}$, the following estimates hold: (i) the probability of $\{\mathrm{ran}(\widetilde{\eta}^{\mathrm{B}}_{i_2})\cap\widetilde{B}_{w_3}(1)\neq\emptyset\}$ is at most $C|w_3|^{4-2d}N^{d-2}$; (ii) the probability of $\big\{\mathrm{ran}(\widetilde{\eta}^{\mathrm{F}}_{i_3})\xleftrightarrow{(D)} A\big\}$ is at most proportional to $\mathbb{P}\big(A\xleftrightarrow{(D)}\partial B(N)\big)$. Moreover, as shown below [5, (5.50)], the conditional probability of $\big\{\mathrm{ran}(\widetilde{\eta}^{\mathrm{F}}_{i_1})\cap\widetilde{B}_{w_2}(1)\neq\emptyset\big\}$ given $\mathcal{F}_{\mathbf{yz}}$ is at most CN^{2-d}. Thus, similar to (A.13), the probability of this case is bounded from above by

$$\begin{aligned} &CN^{d-4}\mathbb{P}\big(A \xleftrightarrow{(D)} x\big)\sum_{w_2\in B(c_\clubsuit N),w_3\in\mathbb{Z}^d:\widetilde{B}_{w_3}(1)\cap D'\neq\emptyset}|w_2-x|^{2-d}|w_3|^{4-2d}\\ \overset{(3.15)}{\lesssim}\ &\mathbb{P}\big(A \xleftrightarrow{(D)} x\big)\sum\nolimits_{w_3\in\mathbb{Z}^d:\widetilde{B}_{w_3}(1)\cap D'\neq\emptyset}|w_3|^{2-d}\overset{(2.32)}{\lesssim} N^{6-d}\cdot\mathbb{P}\big(A \xleftrightarrow{(D)} x\big).\end{aligned}\tag{A.18}$$

Case 4(b): $i_1=i_3$. In this case, there exists $1\le i\le\kappa$ such that $\widetilde{\eta}^{\mathrm{F}}_i$ is connected to both A and x by $\widetilde{\mathcal{L}}^D_{1/2}$ without using any loop crossing the annulus $B(c_\spadesuit^{-2}N)\setminus B(c_\spadesuit^{-1}N)$. According to [5, Lemma 4.2], the conditional probability of this event given $\mathcal{F}_{\mathbf{yz}}$ is at most $\mathbb{P}\big(A\xleftrightarrow{(D)} x\big)$. Meanwhile, there also exists $1\le i'\le\kappa$ such that $\widetilde{\eta}^{\mathrm{B}}_{i'}$ intersects D', whose probability is upper-bounded by $CN^{d-2}\sum_{w_3\in\mathbb{Z}^d:\widetilde{B}_{w_3}(1)\cap D'\neq\emptyset}|w_3|^{4-2d}$. Combined with (A.10), these two observations imply that the probability of this case is bounded from above by

$$\begin{aligned} &CN^{d-2}\mathbb{P}\big(A \xleftrightarrow{(D)} x\big)\cdot\sum\nolimits_{w_3\in\mathbb{Z}^d:\widetilde{B}_{w_3}(1)\cap D'\neq\emptyset}|w_3|^{4-2d}\\ \overset{|w_3|\gtrsim N}{\lesssim}\ &\mathbb{P}\big(A \xleftrightarrow{(D)} x\big)\cdot\sum\nolimits_{w_3\in\mathbb{Z}^d:\widetilde{B}_{w_3}(1)\cap D'\neq\emptyset}|w_3|^{2-d}\overset{(2.32)}{\lesssim} N^{6-d}\cdot\mathbb{P}\big(A \xleftrightarrow{(D)} x\big).\end{aligned}\tag{A.19}$$

Combining (A.6), (A.9), (A.17) and (A.19), we obtain

$$\mathbb{P}(\mathsf{F})\lesssim(N^{6-d}+c')\cdot\mathbb{P}\big(A \xleftrightarrow{(D)} x\big)\lesssim c'\cdot\mathbb{P}\big(A \xleftrightarrow{(D)} x\big). \tag{A.20}$$

This together with (A.2) yields that for a sufficiently small $c'>0$,

$$0\le\mathbb{P}\big(A \xleftrightarrow{(D)} x\big)-\mathbb{P}\big(A \xleftrightarrow{(D\cup D')} x\big)\le\tfrac12\cdot\mathbb{P}\big(A \xleftrightarrow{(D)} x\big), \tag{A.21}$$

which immediately gives the desired bound (2.31). $\square$

Appendix B. Proof of Lemma 2.18

As a preparation, we present the following extension of [2, Lemma 2.1].

Lemma B.1. *For any $d \geq 3$, $D \subset \widetilde{\mathbb{Z}}^d$ and $v \in \widetilde{\mathbb{Z}}^d \setminus D$, suppose that all GFF values on D are given and are all non-negative. Then the probability of $\{v \xleftrightarrow{\geq 0} D\}$ is of the same order as $([\widetilde{G}_D(v,v)]^{-\frac{1}{2}}\mathcal{H}_v) \wedge 1$, where*

$$\mathcal{H}_v := \sum\nolimits_{w \in \widetilde{\partial} D} \widetilde{\mathbb{P}}_v(\tau_D = \tau_w < \infty)\widetilde{\phi}_w. \tag{B.1}$$

Proof. By [28, (18)], the probability of $\{v \xleftrightarrow{\geq 0} D\}$ can be equivalently written as

$$\mathbb{I} := \mathbb{E}\Big[\mathbb{1}_{\widetilde{\phi}_v \geq 0} \cdot \big(1 - e^{-2\widetilde{\phi}_v \sum_{w \in \widetilde{\partial} D} \mathbb{K}_{D \cup \{v\}}(v,w)\widetilde{\phi}_w}\big)\Big], \tag{B.2}$$

where the values of $\{\widetilde{\phi}_w\}_{w \in \widetilde{\partial} D}$ are given by the boundary condition on D, and $\widetilde{\phi}_v$ is distributed by $N(\mu, \sigma^2)$ with $\mu = \mathcal{H}_v$ and $\sigma^2 = \widetilde{G}_D(v,v)$. Moreover, referring to [6, Lemma 3.2], there exist $C_\star > c_\star > 0$ such that

$$c_\star \leq \tfrac{\mathbb{K}_{D \cup \{v\}}(v,w)}{[\widetilde{G}_D(v,v)]^{-1} \cdot \widetilde{\mathbb{P}}_v(\tau_D = \tau_w < \infty)} \leq C_\star. \tag{B.3}$$

Plugging (B.3) into (B.2), we obtain that

$$\mathbb{J}(c_\star) \leq \mathbb{I} \leq \mathbb{J}(C_\star), \tag{B.4}$$

where the quantity $\mathbb{J}(a)$ for $a > 0$ is defined by

$$\begin{aligned} \mathbb{J}(a) :=& \int_0^\infty \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-\mu)^2}{2\sigma^2}} \big(1 - e^{-a\sigma^{-2}\mu t}\big) \mathrm{d}t \\ \overset{(t=\sigma s)}{=}& \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{(s-\frac{\mu}{\sigma})^2}{2}} \big(1 - e^{-a\sigma^{-1}\mu s}\big) \mathrm{d}s. \end{aligned} \tag{B.5}$$

Next, we estimate $\mathbb{J}(c_\star)$ and $\mathbb{J}(C_\star)$ separately in two cases.

When $0 \leq \sigma^{-1}\mu \leq \frac{1}{10}$. On the one hand, one has

$$\begin{aligned} \mathbb{J}(c_\star) \gtrsim& \int_0^{\frac{\sigma}{\mu}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(s-\frac{\mu}{\sigma})^2}{2}} \big(1 - e^{-c_\star\sigma^{-1}\mu s}\big) \mathrm{d}s \\ \gtrsim& \sigma^{-1}\mu \int_0^{\frac{\sigma}{\mu}} \frac{s}{\sqrt{2\pi}} e^{-\frac{(s-\frac{\mu}{\sigma})^2}{2}} \mathrm{d}s \\ \overset{(s=r+\frac{\mu}{\sigma})}{\geq}& \sigma^{-1}\mu \int_{-\frac{\mu}{\sigma}}^{\frac{\sigma}{\mu}-\frac{\mu}{\sigma}} \frac{r}{\sqrt{2\pi}} e^{-\frac{r^2}{2}} \mathrm{d}s \gtrsim \sigma^{-1}\mu, \end{aligned} \tag{B.6}$$

where in the last inequality we used the fact that $\frac{\sigma}{\mu} \geq \frac{3\mu}{\sigma} \vee 10$ for all $0 \leq \mu \leq \frac{\sigma}{10}$. On the other hand, since $1 - e^{-b} \leq b$ for $b \geq 0$, one has

$$
\begin{aligned}
\mathbb{J}(C_\star) \lesssim & \sigma^{-1}\mu \int_0^\infty \frac{s}{\sqrt{2\pi}} e^{-\frac{(s-\frac{\mu}{\sigma})^2}{2}} \mathrm{d}s \\
\overset{(r=s-\frac{\mu}{\sigma})}{=} & \sigma^{-1}\mu \Big(\int_0^\infty \frac{r}{\sqrt{2\pi}} e^{-\frac{r^2}{2}} \mathrm{d}r + \sigma^{-1}\mu \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{r^2}{2}} \mathrm{d}r \Big) \\
\lesssim & \sigma^{-1}\mu \big(1 + \sigma^{-1}\mu\big) \lesssim \sigma^{-1}\mu.
\end{aligned} \tag{B.7}
$$

Combining (B.4), (B.6) and (B.7), we derive $\mathbb{I} \asymp \sigma^{-1}\mu$ in this case.

When $\sigma^{-1}\mu \geq \frac{1}{10}$. Note that $\mathbb{J}(C_\star) \leq 1$ holds trivially. Moreover, since $1 - e^{-c_\star\sigma^{-1}\mu s} \gtrsim 1$ for all $s \geq 1$ in this case, we have

$$
\mathbb{J}(c_\star) \gtrsim \int_1^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{(s-\frac{\mu}{\sigma})^2}{2}} \mathrm{d}s \gtrsim 1. \tag{B.8}
$$

Combined with (B.4), these two estimates yield that $\mathbb{I} \asymp 1$, and thus complete the proof of this lemma. $\square$

Next, we establish Lemma 2.18 for $3 \leq d \leq 5$ and $d \geq 7$ separately.

When $3 \leq d \leq 5$. By the isomorphism theorem, it suffices to prove that

$$
\mathbb{P}^D\big(A \overset{\geq 0}{\longleftrightarrow} y\big) \gtrsim R^{d-2} \cdot \mathbb{P}^D\big(A \overset{\geq 0}{\longleftrightarrow} x\big) \cdot \mathbb{P}^D\big(x \overset{\geq 0}{\longleftrightarrow} y\big). \tag{B.9}
$$

Since $A \subset \mathfrak{B}^1_{R,c}$, the isomorphism theorem together with (1.5) implies that

$$
\begin{aligned}
\mathbb{P}^D\big(A \overset{\leq 0}{\longleftrightarrow} \widetilde{\partial}\mathfrak{B}^1_{R,(10d)^{-1}}\big) \leq & \mathbb{P}\big(\mathfrak{B}^1_{R,c} \leftrightarrow \widetilde{\partial}\mathfrak{B}^1_{R,(10d)^{-1}}\big) \\
\lesssim & \rho_d(cR, (10d)^{-1}R) \lesssim c^{\frac{d}{2}-1}.
\end{aligned} \tag{B.10}
$$

Therefore, for all sufficiently small $c > 0$, one has

$$
\mathbb{P}^D\big(\mathsf{F}\big) := \mathbb{P}^D\big(\{A \overset{\leq 0}{\longleftrightarrow} \widetilde{\partial}\mathfrak{B}^1_{R,(10d)^{-1}}\}^c\big) \leq \tfrac{1}{2}. \tag{B.11}
$$

(**P.S.** When $d \geq 7$, the estimate (B.11) fails for $A = \mathfrak{B}^1_{R,c}$ and $D = \emptyset$ (see (1.6)). This is why we need to treat the high-dimensional case individually.)

By the FKG inequality and (B.11), we have

$$
\mathbb{P}^D\big(\mathsf{F} \mid A \overset{\geq 0}{\longleftrightarrow} x\big) \overset{\text{(FKG)}}{\leq} \mathbb{P}^D\big(\mathsf{F}\big) \overset{\text{(B.11)}}{\leq} \tfrac{1}{2}, \tag{B.12}
$$

which further implies that

$$
\mathbb{P}^D(A \overset{\geq 0}{\longleftrightarrow} x, \mathsf{F}) \asymp \mathbb{P}^D\big(A \overset{\geq 0}{\longleftrightarrow} x\big). \tag{B.13}
$$

Recall that $\mathcal{C}_A^- := \{v \in \widetilde{\mathbb{Z}}^d : v \overset{\leq 0}{\longleftrightarrow} A\}$. For each $v \in \widetilde{\mathbb{Z}}^d$, we define

$$
\mathcal{H}_v := \begin{cases} \sum_{w \in \widetilde{\partial}\mathcal{C}_A^-} \widetilde{\mathbb{P}}_v\big(\tau_{\widetilde{\partial}\mathcal{C}_A^-} = \tau_w < \tau_D\big) \widetilde{\phi}_w & \text{if } v \in \mathcal{C}_A^- \\ 0 & \text{otherwise.} \end{cases} \tag{B.14}
$$

Note that conditioned on $\mathcal{F}_{\mathcal{C}_A^-}$, the boundary condition is non-negative, and only takes positive values on A. By the strong Markov property of Brownian motion and Harnack's inequality, we have that on the event F,

$$\mathcal{H}_y \gtrsim \widetilde{\mathbb{P}}_y\big(\tau_{\partial B(10R)} < \tau_D\big) \cdot \mathcal{H}_x \gtrsim R^{d-2} \cdot \tfrac{\widetilde{G}_D(y,x)}{\sqrt{\widetilde{G}_D(x,x)}} \cdot \mathcal{H}_x. \tag{B.15}$$

Moreover, F also implies the following two bounds:

$$\widetilde{G}_{D\cup\mathcal{C}_A^-}(y,y) \asymp \widetilde{G}_D(y,y), \tag{B.16}$$

$$\widetilde{G}_{D\cup\mathcal{C}_A^-}(x,x) \asymp 1. \tag{B.17}$$

Thus, by applying Lemma B.1, we have

$$\begin{aligned}\mathbb{P}^D\big(A \xleftrightarrow{\ge 0} y\big) \gtrsim & \mathbb{E}^D\big[\mathbb{1}_{\mathsf{F}} \cdot \big([\widetilde{G}_{D\cup\mathcal{C}_A^-}(y,y)]^{-\frac{1}{2}}\mathcal{H}_y\big) \wedge 1\big] \\ \overset{(B.15),(B.16)}{\gtrsim} & \mathbb{E}^D\big[\mathbb{1}_{\mathsf{F}} \cdot \big(R^{d-2} \cdot \tfrac{\widetilde{G}_D(y,x)}{\sqrt{\widetilde{G}_D(x,x)\widetilde{G}_D(y,y)}} \cdot \mathcal{H}_x\big) \wedge 1\big] \\ \gtrsim & R^{d-2} \cdot \mathbb{P}(x \xleftrightarrow{(D)} y) \cdot \mathbb{E}\big[\mathbb{1}_{\mathsf{F}} \cdot \big(\mathcal{H}_x \wedge 1\big)\big],\end{aligned} \tag{B.18}$$

where in the last inequality we used the fact that

$$\frac{\widetilde{G}_D(y,x)}{\sqrt{\widetilde{G}_D(x,x)\widetilde{G}_D(y,y)}} \overset{(2.12)}{\gtrsim} \mathbb{P}(x \xleftrightarrow{(D)} y) \overset{(1.1)}{\gtrsim} R^{2-d}. \tag{B.19}$$

Meanwhile, using (B.13), (B.17) and Lemma B.1, one has

$$\begin{aligned}\mathbb{E}^D\big[\mathbb{1}_{\mathsf{F}} \cdot \big(\mathcal{H}_x \wedge 1\big)\big] \overset{(B.17)}{\asymp} & \mathbb{E}^D\big[\mathbb{1}_{\mathsf{F}} \cdot \big([\widetilde{G}_{D\cup\mathcal{C}_A^-}(x,x)]^{-\frac{1}{2}}\mathcal{H}_x\big) \wedge 1\big] \\ \overset{(\text{Lemma B.1})}{\asymp} & \mathbb{P}^D\big(A \xleftrightarrow{\ge 0} x, \mathsf{F}\big) \overset{(B.13)}{\asymp} \mathbb{P}^D\big(A \xleftrightarrow{\ge 0} x\big).\end{aligned} \tag{B.20}$$

Combined with (B.18), it implies the desired bound (B.9) for $3 \le d \le 5$.

When $d \ge 7$. In this case, by the isomorphism theorem, it suffices to show that

$$\mathbb{P}^D\big(A \xleftrightarrow{\ge 0} y\big) \gtrsim R^{d-2} \cdot \mathbb{P}^D\big(A \xleftrightarrow{\ge 0} x\big) \cdot \big[\mathbb{P}^D\big(x \xleftrightarrow{\ge 0} y\big) - \tfrac{C[\mathrm{diam}(A)]^{d-4}}{R^{d-4}|y|^{d-2}}\big]. \tag{B.21}$$

We denote $\chi := \mathrm{dist}(y, D) \wedge 1$, and define $\mathsf{G} := \big\{\mathrm{dist}(\mathcal{C}_A^-, y) \ge \frac{1}{2}\chi\big\}$. Note that on the event G, it follows from (2.10) that

$$\widetilde{G}_{D\cup\mathcal{C}_A^-}(y,y) \asymp \chi \asymp \widetilde{G}_D(y,y) \tag{B.22}$$

Using Lemma B.1, we have

$$\mathbb{P}^D\big(A \xleftrightarrow{\ge 0} y\big) \gtrsim \mathbb{E}^D\big[\mathbb{1}_{\mathsf{G}} \cdot \big([\widetilde{G}_{D\cup\mathcal{C}_A^-}(y,y)]^{-\frac{1}{2}}\mathcal{H}_y\big) \wedge 1\big]. \tag{B.23}$$

Next, we aim to give a lower bound for $\mathcal{H}_y$. Recall that $A \subset \mathfrak{B}_{R,c}^1$. For simplicity, we denote $\mathcal{C} := \mathcal{C}_A^-$ and $\widehat{\mathcal{C}} := \mathcal{C} \cap \widetilde{B}(\frac{R}{10})$. For any $w \in A$, one has

$$\widetilde{\mathbb{P}}_y\big(\tau_{\mathcal{C}} = \tau_w < \tau_D\big) = \widetilde{\mathbb{P}}_y\big(\tau_{\widehat{\mathcal{C}}} = \tau_w < \tau_D\big) - \widetilde{\mathbb{P}}_y\big(\tau_{\mathcal{C}\setminus\widetilde{B}(\frac{R}{10})} < \tau_{\widehat{\mathcal{C}}} = \tau_w < \tau_D\big). \tag{B.24}$$

The first probability on the right-hand side is bounded from below by

$$\widetilde{\mathbb{P}}_y\big(\tau_{\partial B(10dR)} < \tau_D\big) \cdot \min_{z\in \partial B(10dR)} \sum\nolimits_{z'\in \partial \mathcal{B}(R)} \widetilde{\mathbb{P}}_z\big(\tau_{\partial \mathcal{B}(R)} = \tau_{z'} < \tau_D\big) \cdot \widetilde{\mathbb{P}}_{z'}\big(\tau_{\widehat{\mathcal{C}}} = \tau_w < \tau_D\big). \tag{B.25}$$

By the strong Markov property of Brownian motion, one has

$$\begin{aligned}\widetilde{\mathbb{P}}_y\big(\tau_x < \tau_D\big) \le & \widetilde{\mathbb{P}}_y\big(\tau_{\partial B(10dR)} < \tau_D\big) \cdot \max_{z\in \partial B(10dR)} \widetilde{\mathbb{P}}_z\big(\tau_x < \tau_D\big) \\ \lesssim & R^{2-d} \cdot \widetilde{\mathbb{P}}_y\big(\tau_{\partial B(10dR)} < \tau_D\big).\end{aligned} \tag{B.26}$$

Meanwhile, for any $z \in \partial B(10dR)$ and $z' \in \partial \mathcal{B}(R)$, we have

$$\widetilde{\mathbb{P}}_z\big(\tau_{\partial \mathcal{B}(R)} = \tau_{z'} < \tau_D\big) \overset{\text{(Lemma 2.2)}}{\gtrsim} \widetilde{\mathbb{P}}_z\big(\tau_{\partial \mathcal{B}(R)} = \tau_{z'} < \infty\big) \overset{\text{(Lemma 2.1)}}{\gtrsim} R^{1-d}. \tag{B.27}$$

In addition, using Harnack's inequality, we have

$$\widetilde{\mathbb{P}}_{z'}\big(\tau_{\widehat{\mathcal{C}}} = \tau_w < \tau_D\big) \asymp \widetilde{\mathbb{P}}_x\big(\tau_{\widehat{\mathcal{C}}} = \tau_w < \tau_D\big) \ge \widetilde{\mathbb{P}}_x\big(\tau_{\mathcal{C}} = \tau_w < \tau_D\big). \tag{B.28}$$

Substituting (B.26), (B.27) and (B.28) into (B.25), we get

$$\widetilde{\mathbb{P}}_y\big(\tau_{\widehat{\mathcal{C}}} = \tau_w < \tau_D\big) \gtrsim R^{d-2} \cdot \widetilde{\mathbb{P}}_y\big(\tau_x < \tau_D\big) \cdot \widetilde{\mathbb{P}}_x\big(\tau_{\mathcal{C}} = \tau_w < \tau_D\big). \tag{B.29}$$

For the second probability on the right-hand side of (B.24), by the union bound,

$$\begin{aligned}&\widetilde{\mathbb{P}}_y\big(\tau_{\mathcal{C}\setminus \widetilde{B}(\frac{R}{10})} < \tau_{\widehat{\mathcal{C}}} = \tau_w < \tau_D\big) \\ \le & \sum\nolimits_{z\in [B(\frac{R}{10})]^c} \mathbb{1}_{A \xleftrightarrow{\le 0} \widetilde{B}_z(1)} \cdot \widetilde{\mathbb{P}}_y\big(\tau_z < \tau_{\widehat{\mathcal{C}}} = \tau_w < \tau_D\big).\end{aligned} \tag{B.30}$$

Moreover, for each $z \in [B(\frac{R}{10})]^c$, one has

$$\widetilde{\mathbb{P}}_y\big(\tau_z < \tau_{\widehat{\mathcal{C}}} = \tau_w < \tau_D\big) \le \widetilde{\mathbb{P}}_y\big(\tau_z < \tau_D\big) \cdot \widetilde{\mathbb{P}}_z\big(\tau_{\widehat{\mathcal{C}}} = \tau_w < \tau_D\big). \tag{B.31}$$

By the formula (2.9) and the fact that $\widetilde{G}_D(z,z) \asymp 1$ for all $z \in [B(\frac{R}{10})]^c$, we have

$$\widetilde{\mathbb{P}}_y\big(\tau_z < \tau_D\big) = \tfrac{\widetilde{\mathbb{P}}_z(\tau_y<\tau_D)\widetilde{G}_D(y,y)}{\widetilde{G}_D(z,z)} \lesssim |y-z|^{2-d} \cdot \widetilde{G}_D(y,y). \tag{B.32}$$

In addition, by the strong Markov property of Brownian motion,

$$\begin{aligned}\widetilde{\mathbb{P}}_z\big(\tau_{\widehat{\mathcal{C}}} = \tau_w < \tau_D\big) \lesssim & \widetilde{\mathbb{P}}_z\big(\tau_{\partial B(\frac{R}{100})} < \infty\big) \cdot \max_{z'\in \partial B(\frac{R}{100})} \sum\nolimits_{z''\in \partial \mathcal{B}(\frac{R}{100d})} \\ & \widetilde{\mathbb{P}}_{z'}\big(\tau_{\partial \mathcal{B}(\frac{R}{100d})} = \tau_{z''} < \infty\big) \cdot \widetilde{\mathbb{P}}_{z''}\big(\tau_{\widehat{\mathcal{C}}} = \tau_w < \tau_D\big).\end{aligned} \tag{B.33}$$

Here one may derive from (2.22) and (2.23) that

$$\widetilde{\mathbb{P}}_z\big(\tau_{\partial B(\frac{R}{100})} < \infty\big) \lesssim |z|^{2-d} R^{d-2}. \tag{B.34}$$

Meanwhile, Lemma 2.1 implies that

$$\widetilde{\mathbb{P}}_{z'}\big(\tau_{\partial \mathcal{B}(\frac{R}{100d})} = \tau_{z''} < \infty\big) \lesssim R^{1-d}. \tag{B.35}$$

Applying Lemma 2.3, one has

$$\sum_{z''\in\partial\mathcal{B}(\frac{R}{100d})}\widetilde{\mathbb{P}}_{z''}\big(\tau_{\widehat{\mathcal{C}}}=\tau_w<\tau_D\big)\lesssim\sum_{z''\in\partial\mathcal{B}(\frac{R}{100d})}\widetilde{\mathbb{P}}_{z''}\big(\tau_{\mathcal{C}}=\tau_w<\tau_D\big). \tag{B.36}$$

Inserting (B.34), (B.35) and (B.36) into (B.33), we get

$$\widetilde{\mathbb{P}}_z\big(\tau_{\widehat{\mathcal{C}}}=\tau_w<\tau_D\big)\lesssim|z|^{2-d}R^{-1}\sum_{z''\in\partial\mathcal{B}(\frac{R}{100d})}\widetilde{\mathbb{P}}_{z''}\big(\tau_{\mathcal{C}}=\tau_w<\tau_D\big). \tag{B.37}$$

This together with (B.30), (B.31) and (B.32) yields that

$$\begin{aligned}\widetilde{\mathbb{P}}_y\big(\tau_{\mathcal{C}\setminus\widetilde{B}(\frac{R}{10})}<\tau_{\widehat{\mathcal{C}}}=\tau_w<\tau_D\big)\lesssim&R^{-1}\widetilde{G}_D(y,y)\sum_{z\in[B(\frac{R}{10})]^c,z''\in\partial\mathcal{B}(\frac{R}{100d})}\mathbb{1}_{A\xleftrightarrow{\le 0}\widetilde{B}_z(1)}\\&\cdot|y-z|^{2-d}|z|^{2-d}\widetilde{\mathbb{P}}_{z''}\big(\tau_{\mathcal{C}}=\tau_w<\tau_D\big).\end{aligned} \tag{B.38}$$

Recall $\mathcal{H}_\cdot$ in (B.14). Plugging (B.29) and (B.38) into (B.24), we obtain that for some constants $C',c'>0$,

$$\begin{aligned}\mathcal{H}_y\ge&c'R^{d-2}\widetilde{\mathbb{P}}_y\big(\tau_x<\tau_D\big)\mathcal{H}_x\\&-C'R^{-1}\sum_{z\in[B(\frac{R}{10})]^c,z''\in\partial\mathcal{B}(\frac{R}{100d})}\mathbb{1}_{A\xleftrightarrow{\le 0}\widetilde{B}_z(1)}|y-z|^{2-d}|z|^{2-d}\mathcal{H}_{z''}.\end{aligned} \tag{B.39}$$

Combined with (B.23), it implies that

$$\mathbb{P}^D\big(A\xleftrightarrow{\ge 0}y\big)\ge\mathbb{V}_1-\mathbb{V}_2, \tag{B.40}$$

where the quantities $\mathbb{V}_1$ and $\mathbb{V}_2$ satisfy that

$$\mathbb{V}_1\gtrsim\mathbb{E}^D\big[\mathbb{1}_{\mathsf{G}}\cdot\big([\widetilde{G}_{D\cup\mathcal{C}}(y,y)]^{-\frac12}R^{d-2}\widetilde{\mathbb{P}}_y\big(\tau_x<\tau_D\big)\mathcal{H}_x\big)\wedge1\big],$$

$$\begin{aligned}\mathbb{V}_2\lesssim&\mathbb{E}^D\big[\mathbb{1}_{\mathsf{G}}\cdot\big(\widetilde{G}_D(y,y)[\widetilde{G}_{D\cup\mathcal{C}}(y,y)]^{-\frac12}R^{-1}\sum_{z\in[B(\frac{R}{10})]^c,z''\in\partial\mathcal{B}(\frac{R}{100d})}\mathbb{1}_{A\xleftrightarrow{\le 0}\widetilde{B}_z(1)}\\&\cdot|y-z|^{2-d}|z|^{2-d}\mathcal{H}_z\big)\wedge1\big].\end{aligned}$$

For $\mathbb{V}_1$, by $\widetilde{G}_{D\cup\mathcal{C}}(y,y)\le\widetilde{G}_D(y,y)$, $\widetilde{G}_D(x,x)\asymp1$ and (2.12), we have

$$\begin{aligned}\mathbb{V}_1\gtrsim&\mathbb{E}^D\big[\mathbb{1}_{\mathsf{G}}\cdot\big(R^{d-2}\mathbb{P}^D\big(x\xleftrightarrow{\ge 0}y\big)\cdot\mathcal{H}_x\big)\wedge1\big]\\\gtrsim&R^{d-2}\mathbb{P}^D\big(x\xleftrightarrow{\ge 0}y\big)\cdot\mathbb{E}^D\big[\mathbb{1}_{\mathsf{G}}\cdot\mathcal{H}_x\wedge1\big],\end{aligned} \tag{B.41}$$

where in the last line we used the facts that $R^{d-2}\mathbb{P}^{D\cup\mathcal{C}}\big(y\xleftrightarrow{\ge 0}x\big)\lesssim1$ (by (1.1)), and that $(ab)\wedge1\ge a\cdot(b\wedge1)$ for $a\in(0,1)$ and $b>0$. Let $\mathsf{G}':=\big\{\mathrm{dist}(\mathcal{C},x)\ge1\big\}$. Note that G' implies $\widetilde{G}_{D\cup\mathcal{C}}(x,x)\asymp1$. Therefore, one has

$$\mathbb{E}^D\big[\mathbb{1}_{\mathsf{G}}\cdot\big(\mathcal{H}_x\wedge1\big)\big]\gtrsim\mathbb{E}^D\big[\mathbb{1}_{\mathsf{G}\cap\mathsf{G}'}\cdot\big([\widetilde{G}_{D\cup\mathcal{C}}(x,x)]^{-\frac12}\mathcal{H}_x\big)\wedge1\big]. \tag{B.42}$$

Note that G, G', $[\widetilde{G}_{D\cup\mathcal{C}}(x,x)]^{-\frac{1}{2}}$ and $\mathcal{H}_x$ are measurable with respect to $\mathcal{F}_{\mathcal{C}}$, and are decreasing with respect to the GFF. Thus, by the FKG inequality and (B.41),

$$\begin{aligned}\mathbb{V}_1 \gtrsim & R^{d-2}\mathbb{P}^D\big(x \xleftrightarrow{\ge 0} y\big)\mathbb{E}^D\big[\big([\widetilde{G}_{D\cup\mathcal{C}}(x,x)]^{-\frac{1}{2}}\mathcal{H}_x\big)\wedge 1\big]\cdot \mathbb{P}^D\big(\mathsf{G}\big)\mathbb{P}^D\big(\mathsf{G}'\big)\\ \overset{\text{(Lemma B.1)}}{\gtrsim} & R^{d-2}\mathbb{P}^D\big(A \xleftrightarrow{\ge 0} x\big)\mathbb{P}^D\big(x \xleftrightarrow{\ge 0} y\big)\cdot \mathbb{P}^D\big(\mathsf{G}\big)\mathbb{P}^D\big(\mathsf{G}'\big).\end{aligned} \tag{B.43}$$

For $\mathbb{P}^D\big(\mathsf{G}\big)$, by the FKG inequality, (2.10) and (2.12), we have

$$\begin{aligned}\mathbb{P}^D\big(\mathsf{H}\big) :=& \mathbb{P}^D\big(\cap_{z\in\widetilde{\mathbb{Z}}^d:|z-y|=\frac{1}{2}\chi}\,\{z \xleftrightarrow{\le 0} y\}\big)\\ \overset{\text{(FKG)}}{\ge} & \prod_{z\in\widetilde{\mathbb{Z}}^d:|z-y|=\frac{1}{2}\chi}\mathbb{P}^D\big(z \xleftrightarrow{\le 0} y\big)\\ \overset{(2.12)}{\asymp} & \prod_{z\in\widetilde{\mathbb{Z}}^d:|z-y|=\frac{1}{2}\chi}\widetilde{\mathbb{P}}_y\big(\tau_z<\tau_D\big)\cdot\sqrt{\tfrac{\widetilde{G}_D(z,z)}{\widetilde{G}_D(y,y)}}\gtrsim 1,\end{aligned} \tag{B.44}$$

where the last inequality follows from the facts that $\widetilde{\mathbb{P}}_y\big(\tau_z<\tau_D\big)\asymp 1$ (by $|y-z|\le\frac{1}{2}\mathrm{dist}(y,D)$), and that $\widetilde{G}_D(y,y)\asymp\widetilde{G}_D(z,z)$ (using (2.10)). In addition, since $\mathsf{H}\cap\mathsf{G}^c\subset\{A \xleftrightarrow{\le 0} y\}$, one has

$$\mathbb{P}^D\big(A \xleftrightarrow{\ge 0} y\big) \overset{\text{(symmetry)}}{=} \mathbb{P}^D\big(A \xleftrightarrow{\le 0} y\big) \overset{\text{(FKG)}}{\ge} \mathbb{P}^D\big(\mathsf{G}^c\big)\cdot\mathbb{P}^D\big(\mathsf{H}\big) \overset{\text{(B.44)}}{\gtrsim} \mathbb{P}^D\big(\mathsf{G}^c\big). \tag{B.45}$$

Meanwhile, it follows from (1.4) that $\mathbb{P}^D\big(\mathsf{G}'\big)$ is uniformly bounded away from zero. Combined with (B.43) and (B.45), it implies that

$$\mathbb{V}_1 \gtrsim \big[1-\mathbb{P}^D\big(A \xleftrightarrow{\ge 0} y\big)\big]\cdot R^{d-2}\mathbb{P}^D\big(A \xleftrightarrow{\ge 0} x\big)\mathbb{P}^D\big(x \xleftrightarrow{\ge 0} y\big). \tag{B.46}$$

We now bound the quantity $\mathbb{V}_2$. By (B.22) and $\widetilde{G}_D(y,y)\lesssim 1$ one has

$$\mathbb{V}_2 \lesssim R^{-1}\sum_{z\in[B(\frac{R}{10})]^c,z''\in\partial\mathcal{B}(\frac{R}{100d})}|y-z|^{2-d}|z|^{2-d}\cdot\mathbb{E}^D\big[\mathbb{1}_{A\xleftrightarrow{\le 0}\widetilde{B}_z(1)}\cdot\mathcal{H}_{z''}\big]. \tag{B.47}$$

In addition, since $\big\{A \xleftrightarrow{\le 0} \widetilde{B}_z(1)\big\}$ (resp. $\mathcal{H}_{z'}$) is decreasing (resp. increasing) with respect to the GFF, using the FKG inequality, we have

$$\mathbb{E}^D\big[\mathbb{1}_{A\xleftrightarrow{\le 0}\widetilde{B}_z(1)}\cdot\mathcal{H}_{z''}\big]\le\mathbb{P}^D\big(A \xleftrightarrow{\le 0} \widetilde{B}_z(1)\big)\cdot\mathbb{E}^D\big[\mathcal{H}_{z''}\big]. \tag{B.48}$$

Recall that $A\subset\widetilde{B}(cR)$. By the isomorphism theorem, one has: for $z\in[B(\frac{R}{10})]^c$,

$$\mathbb{P}^D\big(A \xleftrightarrow{\le 0} \widetilde{B}_z(1)\big)\le\sum_{z'\in\widetilde{\partial}\widetilde{B}_z(1)}\mathbb{P}\big(A\leftrightarrow z'\big)\lesssim|z|^{2-d}[\mathrm{diam}(A)]^{d-4}, \tag{B.49}$$

where the last inequality is derived from [5, Corollary 2.18]. Meanwhile, it follows from [4, Lemma 3.4] that

$$\mathbb{E}^D\big[\mathcal{H}_{z''}\big]\lesssim\mathbb{P}^D\big(A \xleftrightarrow{\ge 0} z''\big)\overset{\text{(Lemma 2.15)}}{\asymp}\mathbb{P}^D\big(A \xleftrightarrow{\ge 0} x\big). \tag{B.50}$$

Plugging (B.49) and (B.50) into (B.48), and combining it with (B.47), we get

$$\mathbb{V}_2 \lesssim R^{d-2}[\mathrm{diam}(A)]^{d-4}\mathbb{P}^D\big(A \xleftrightarrow{\geq 0} x\big) \sum\nolimits_{z\in[B(\frac{R}{10})]^c} |y-z|^{2-d}|z|^{4-2d}. \tag{B.51}$$

For the sum on the right-hand side, since $|z| \gtrsim |y|$ for all $z \in B_y(\frac{|y|}{10})$, we have

$$\sum\nolimits_{z\in B_y(\frac{|y|}{10})} |y-z|^{2-d}|z|^{4-2d} \lesssim |y|^{4-2d} \sum\nolimits_{z\in B_y(\frac{|y|}{10})} |y-z|^{2-d} \overset{(3.15)}{\lesssim} |y|^{6-2d}. \tag{B.52}$$

In addition, since $|z-y| \gtrsim |y|$ for all $z \in [B(\frac{R}{10})]^c \setminus B_y(\frac{|y|}{10})$, one has

$$\begin{aligned} &\sum\nolimits_{z\in[B(\frac{R}{10})]^c\setminus B_y(\frac{|y|}{10})} |y-z|^{2-d}|z|^{4-2d} \\ \lesssim&|y|^{2-d} \sum\nolimits_{z\in[B(\frac{R}{10})]^c} |z|^{4-2d} \lesssim |y|^{2-d} \sum\nolimits_{k\geq \frac{R}{10}} k^{d-1}\cdot k^{4-2d} \lesssim |y|^{2-d}R^{4-d}. \end{aligned} \tag{B.53}$$

Note that $|y| \gtrsim R$. Thus, combining (B.51), (B.52) and (B.53), we obtain

$$\mathbb{V}_2 \lesssim R^2[\mathrm{diam}(A)]^{d-4}|y|^{2-d}\mathbb{P}^D\big(A \xleftrightarrow{\geq 0} x\big). \tag{B.54}$$

By (B.40), (B.46) and (B.54), there exists $C'' > 0$ such that

$$\begin{aligned} \mathbb{P}^D\big(A \xleftrightarrow{\geq 0} y\big) \gtrsim& R^{d-2}\cdot \mathbb{P}^D\big(A \xleftrightarrow{\geq 0} x\big)\cdot \\ &\cdot \Big(\mathbb{P}^D\big(x \xleftrightarrow{\geq 0} y\big)\cdot \big[1-\mathbb{P}^D\big(A \xleftrightarrow{\geq 0} y\big)\big] - \tfrac{C''[\mathrm{diam}(A)]^{d-4}}{R^{d-4}|y|^{d-2}}\Big). \end{aligned} \tag{B.55}$$

If $\mathbb{P}^D\big(A \xleftrightarrow{\geq 0} y\big) \geq \frac{1}{2}$, the bound (B.21) holds trivially; otherwise, we derive (B.21) from (B.55). In conclusion, we have completed the proof of Lemma 2.18. □

Appendix C. Proof of Lemma 5.2

For $3 \leq d \leq 5$, the desired bound (5.2) has been proved in [2, (2.37)]. Thus, it suffices to consider the high-dimensional case $d \geq 7$. We present the following lemma as a preparation.

Lemma C.1. *For any $d \geq 7$, there exists $C > 0$ such that the following inequalities hold for all $r \geq 1$, $A, D \subset \widetilde{B}(r)$ and $R \geq r^{\frac{d-4}{2}}$:*

$$\mathbb{P}\big(A \xleftrightarrow{(D)} \partial B(R)\big) \asymp |v|^{d-2}R^{-2}\cdot \mathbb{P}\big(A \xleftrightarrow{(D)} v\big), \quad \forall v \in \widetilde{\mathbb{Z}}^d \text{ with } |v| \geq Cr; \tag{C.1}$$

$$\mathbb{P}\big(A \xleftrightarrow{(D)} \partial B(R)\big) \gtrsim \big(\tfrac{r'}{R}\big)^2 \cdot \mathbb{P}\big(A \xleftrightarrow{(D)} \partial B(r')\big), \quad \forall r' \geq Cr. \tag{C.2}$$

Proof. According to [5, Proposition 1.9], one has: for any $z \in \partial B(R)$,

$$\mathbb{P}\big(A \xleftrightarrow{(D)} \partial B(R)\big) \asymp R^{d-4}\cdot \mathbb{P}\big(A \xleftrightarrow{(D)} z\big). \tag{C.3}$$

Combined with Lemma 2.15, it yields (C.1). Moreover, the bound (C.1) together with Lemma 2.16 implies (C.2). □

We now turn to the proof of Lemma 5.2. By [3, Corollary 2.2], one has

$$\mathbb{P}\big(\mathfrak{L}[m,M]\neq 0\big) \lesssim \big(\tfrac{m}{M}\big)^{d-2}. \tag{C.4}$$

By the BKR inequality and (C.4), we have

$$\begin{aligned}\mathbb{P}\big(\mathsf{F}\big) :=&\mathbb{P}\big(\big\{\mathfrak{L}[m,M]\neq 0\big\}\circ\big\{A \xleftrightarrow{(D)} \partial B(\tfrac{N}{10})\big\}\big)\\ \overset{\text{(BKR)}}{\leq}& \mathbb{P}\big(\mathfrak{L}[m,M]\neq 0\big)\cdot \mathbb{P}\big(A \xleftrightarrow{(D)} \partial B(\tfrac{N}{10})\big)\\ \overset{\text{(C.2),(C.4)}}{\lesssim}& \big(\tfrac{m}{M}\big)^{d-2}\cdot \mathbb{P}\big(A \xleftrightarrow{(D)} \partial B(N)\big).\end{aligned} \tag{C.5}$$

Thus, it remains to show that for $d\geq 7$,

$$\mathbb{P}\big(\mathsf{F}'\big) := \mathbb{P}\big(\mathfrak{L}[m,M]\neq 0, A \xleftrightarrow{(D)} \partial B(N), \mathsf{F}^c\big) \lesssim \sqrt{\tfrac{m}{M}}\cdot \mathbb{P}\big(A \xleftrightarrow{(D)} \partial B(N)\big). \tag{C.6}$$

On the event F', there exists a loop $\widetilde{\ell}_\dagger\in\mathfrak{L}[m,M]$ that is pivotal for $\{A \xleftrightarrow{(D)} \partial B(N)\}$. Otherwise, the loop $\widetilde{\ell}_\dagger$ certifies that $\mathfrak{L}[m,M]\neq 0$, while the collection of other loops certifies $A \xleftrightarrow{(D)} \partial B(N)$; consequently, the event F occurs, which is a contradiction. The pivotality of $\widetilde{\ell}_\dagger$ implies that $\big\{\mathrm{ran}(\widetilde{\ell}_\dagger) \xleftrightarrow{(D)} A\big\}$ and $\big\{\mathrm{ran}(\widetilde{\ell}_\dagger) \xleftrightarrow{(D)} \partial B(N)\big\}$ occur disjointly. As a result, there exist $w_1,w_2\in\mathbb{Z}^d$ and loop clusters $\mathcal{C}_1,\mathcal{C}_2$ consisting of disjoint collections of loops such that $\widetilde{\ell}_\dagger$ intersects both $\widetilde{B}_{w_1}(1)$ and $\widetilde{B}_{w_2}(1)$, and that $\mathcal{C}_1$ connects A and w_1, and $\mathcal{C}_2$ connects $\partial B(N)$ and w_2. Note that $w_1\in B(\frac{N}{10})$ (otherwise, the event F occurs since $\mathfrak{L}[m,M]\neq 0$ and $A \xleftrightarrow{(D)} \partial B(\frac{N}{10})$ are certified by $\widetilde{\ell}_\dagger$ and $\mathcal{C}_1$ respectively).

In what follows, we estimate the probability of F' under different restrictions on w_1,w_2 and the pivotal loop $\widetilde{\ell}_\dagger$.

When $\mathrm{ran}(\widetilde{\ell}_\dagger)\cap\partial B(\frac{N}{2})\neq\emptyset$ **and** $w_1\in B(\frac{N}{10})\setminus B(\frac{M}{2})$**.** Since $\widetilde{\ell}_\dagger$ contains a trajectory of Brownian motion from $\partial B(\frac{N}{2})$ to w_1, and one from w_1 to $\partial B(m)$, the probability of having such a loop is at most $CN^{2-d}\cdot|w_1|^{2-d}m^{d-2}$. Note that the event $\{A \xleftrightarrow{(D)} w_1\}$ occurs (without using the loop $\widetilde{\ell}_\dagger$), whose probability is at most $C|w_1|^{2-d}N^2\mathbb{P}\big(A \xleftrightarrow{(D)} \partial B(N)\big)$ (by Lemma C.1). Thus, the probability of this case is upper-bounded by

$$\begin{aligned}&CN^{4-d}m^{d-2}\mathbb{P}\big(A \xleftrightarrow{(D)} \partial B(N)\big)\sum\nolimits_{w_1\in B(\frac{N}{10})\setminus B(\frac{M}{2})}|w_1|^{4-2d}\\ \overset{|\partial B(k)|\asymp k^{d-2}}{\lesssim}& N^{4-d}m^{d-2}\mathbb{P}\big(A \xleftrightarrow{(D)} \partial B(N)\big)\sum\nolimits_{\frac{M}{2}\leq k\leq\frac{N}{10}}k^{3-d}\\ \lesssim& N^{4-d}m^{d-2}M^{4-d}\mathbb{P}\big(A \xleftrightarrow{(D)} \partial B(N)\big)\lesssim\sqrt{\tfrac{m}{M}}\cdot\mathbb{P}\big(A \xleftrightarrow{(D)} \partial B(N)\big).\end{aligned} \tag{C.7}$$

Recall that $A, D \subset \widetilde{B}(n)$ and $m \geq Cn$. In addition, we choose a sufficiently large constant $C_\spadesuit > 0$, and require that $C \gg C_\spadesuit$. Note that by Lemma C.1, one has: for any $w \in [B(C_\spadesuit n)]^c$,

$$\mathbb{P}\big(A \xleftrightarrow{(D)} w\big) \asymp |w|^{2-d} N^{-2} \cdot \mathbb{P}\big(A \xleftrightarrow{(D)} \partial B(N)\big). \tag{C.8}$$

When $\mathrm{ran}(\widetilde{\ell}_\dagger) \cap \partial B(\frac{N}{2}) \neq \emptyset$ **and** $w_1 \in B(\frac{M}{2}) \setminus B(C_\spadesuit n)$**.** Since $\widetilde{\ell}_\dagger$ contains a trajectory of Brownian motion from $\partial B(\frac{N}{2})$ to $\partial B(M)$, and one from $\partial B(M)$ to w_1, the probability of having this loop is at most $C(\frac{N}{M})^{2-d} \cdot M^{2-d} = CN^{2-d}$. Combined with (C.8), it implies that the probability of this case is at most

$$\begin{aligned} &CN^{4-d}\mathbb{P}\big(A \xleftrightarrow{(D)} \partial B(N)\big) \sum\nolimits_{w_1 \in B(\frac{M}{2})} |w_1|^{2-d} \\ \overset{(3.15)}{\lesssim}\ &N^{4-d}M^2 \cdot \mathbb{P}\big(A \xleftrightarrow{(D)} \partial B(N)\big) \lesssim \sqrt{\tfrac{m}{M}} \cdot \mathbb{P}\big(A \xleftrightarrow{(D)} \partial B(N)\big). \end{aligned} \tag{C.9}$$

When $\mathrm{ran}(\widetilde{\ell}_\dagger) \cap \partial B(\frac{N}{2}) \neq \emptyset$ **and** $w_1 \in B(C_\spadesuit n)$**.** Recall the notations for crossing paths from Appendix A. Here we employ these notations for the crossing paths for the annulus $B(C_\spadesuit m) \setminus B(m)$. Here we employ these notations for the crossing paths for the annulus $B(C_\spadesuit m) \setminus B(m)$. In this case, since $\widetilde{\ell}$ intersects both $\partial B(\frac{N}{2})$ and $\widetilde{B}_{w_1}(1)$, it must cross the annulus $B(C_\spadesuit m) \setminus B(m)$. As a result, there exist $1 \leq i_1, i_2 \leq \kappa$ such that $\mathrm{ran}(\widetilde{\eta}^{\mathrm{F}}_{i_1}) \xleftrightarrow{(D)} A$ and that $\widetilde{\eta}^{\mathrm{B}}_{i_2}$ intersects $\partial B(\frac{N}{2})$. Given $\mathcal{F}_{\mathbf{yz}}$, applying [5, Lemma 4.3], the conditional probability of $\big\{\mathrm{ran}(\widetilde{\eta}^{\mathrm{F}}_{i_1}) \xleftrightarrow{(D)} A\big\}$ is bounded from above by

$$C\mathbb{P}\big(A \xleftrightarrow{(D)} \partial B(m)\big) \overset{(C.2)}{\lesssim} m^2 N^{-2} \cdot \mathbb{P}\big(A \xleftrightarrow{(D)} \partial B(N)\big). \tag{C.10}$$

Meanwhile, by [5, Lemma 4.2], restricted to $[\widetilde{B}(\frac{N}{2})]^c$, the range of each path $\widetilde{\eta}^{\mathrm{B}}_i$ is stochastically dominated by $\cup\mathfrak{L}[C_\spadesuit^2 m, C_\spadesuit^3 m]$. Consequently, the probability of $\big\{\mathrm{ran}(\widetilde{\eta}^{\mathrm{B}}_{i_2}) \cap \partial B(\frac{N}{2}) \neq \emptyset\big\}$ is upper-bounded by

$$\begin{aligned} &\mathbb{P}\big(\exists \widetilde{\ell} \in \mathfrak{L}[C_\spadesuit^2 m, C_\spadesuit^3 m] \text{ intersecting } \partial B(\tfrac{N}{2})\big) \\ \leq &\mathbb{P}\big(\mathfrak{L}[C_\spadesuit^2 m, \tfrac{N}{2}] \neq 0\big) \overset{(C.4)}{\lesssim} \big(\tfrac{m}{N}\big)^{d-2}. \end{aligned} \tag{C.11}$$

Combining (A.10), (C.10) and (C.11), we derive that the probability of this case is bounded from above by

$$Cm^d N^{-d} \cdot \mathbb{P}\big(A \xleftrightarrow{(D)} \partial B(N)\big) \lesssim \sqrt{\tfrac{m}{M}} \cdot \mathbb{P}\big(A \xleftrightarrow{(D)} \partial B(N)\big). \tag{C.12}$$

Next, it remains to consider the case when $\mathrm{ran}(\widetilde{\ell}_\dagger) \cap \partial B(\frac{N}{2}) = \emptyset$. We record the following bound for the subsequent analysis. When $w_1 \in [B(C_\spadesuit n)]^c$, the event

$\{\mathrm{ran}(\widetilde{\ell}_\dagger)\cap\partial B(\frac{N}{2})=\emptyset\}$ implies that $A\xleftrightarrow{(D)} w_1$ and $w_2\leftrightarrow\partial B(N)$ occur disjointly, whose probability, by the BKR inequality, is upper-bounded by

$$\begin{aligned}&\mathbb{P}\big(A\xleftrightarrow{(D)} w_1\big)\cdot\mathbb{P}\big(w_2\leftrightarrow\partial B(N)\big)\\ \overset{\text{(Lemma C.1),(1.4)}}{\lesssim}&|w_1|^{2-d}\cdot\mathbb{P}\big(A\xleftrightarrow{(D)}\partial B(N)\big).\end{aligned}\tag{C.13}$$

When $\mathrm{ran}(\widetilde{\ell}_\dagger)\cap\partial B(\frac{N}{2})=\emptyset$**,** $w_1\in B(\frac{M}{2})\setminus B(C_\spadesuit n)$ **and** $w_2\in B(\frac{M}{2})$**.** The probability of having $\widetilde{\ell}_\dagger$ is at most $C|w_1-w_2|^{2-d}\cdot M^{2-d}$, since it contains a trajectory from w_i to w_{3-i}, and one from $\partial B(M)$ to w_i for some $i\in\{1,2\}$. Combined with (C.13), it implies that the probability of this case is bounded from above by

$$\begin{aligned}&CM^{2-d}\mathbb{P}\big(A\xleftrightarrow{(D)}\partial B(N)\big)\sum\nolimits_{w_1,w_2\in B(\frac{M}{2})}|w_1|^{2-d}|w_1-w_2|^{2-d}\\ \overset{(A.7)}{\lesssim}&M^{2-d}\mathbb{P}\big(A\xleftrightarrow{(D)}\partial B(N)\big)\sum\nolimits_{w_2\in B(\frac{M}{2})}|w_2|^{4-d}\\ \overset{(3.15)}{\lesssim}&M^{6-d}\mathbb{P}\big(A\xleftrightarrow{(D)}\partial B(N)\big)\lesssim\sqrt{\tfrac{m}{M}}\cdot N^{-2}.\end{aligned}\tag{C.14}$$

When $\mathrm{ran}(\widetilde{\ell}_\dagger)\cap\partial B(\frac{N}{2})=\emptyset$ **and** $w_1,w_2\in B(\frac{N}{10})\setminus B(\frac{M}{4})$**.** In this case, the probability of having $\widetilde{\ell}_\dagger$ is at most $Cm^{d-2}|w_1-w_2|^{2-d}|w_1|^{2-d}|w_2|^{2-d}$, since there exists $i\in\{1,2\}$ such that $\widetilde{\ell}_\dagger$ contains trajectories from w_i to $\partial B(m)$, from $\partial B(m)$ to w_{3-i}, and from w_{3-i} to w_i (whose corresponding masses are given by $C|w_i|^{2-d}m^{d-2}$, $C|w_{3-i}|$ and $C|w_1-w_2|^{2-d}$ respectively). Therefore, the probability of this case is bounded from above by

$$\begin{aligned}&Cm^{d-2}\mathbb{P}\big(A\xleftrightarrow{(D)}\partial B(N)\big)\sum_{w_1,w_2\in B(\frac{N}{10})\setminus B(\frac{M}{4})}|w_1-w_2|^{2-d}|w_1|^{4-2d}|w_2|^{2-d}\\ \overset{(A.7)}{\lesssim}&m^{d-2}\mathbb{P}\big(A\xleftrightarrow{(D)}\partial B(N)\big)\sum\nolimits_{w_1\in B(\frac{N}{10})\setminus B(\frac{M}{4})}|w_1|^{8-3d}\\ \lesssim&m^{d-2}\mathbb{P}\big(A\xleftrightarrow{(D)}\partial B(N)\big)\sum\nolimits_{k\geq\frac{M}{4}}k^{7-2d}\\ \lesssim&m^{d-2}M^{8-2d}\mathbb{P}\big(A\xleftrightarrow{(D)}\partial B(N)\big)\lesssim\sqrt{\tfrac{m}{M}}\cdot\mathbb{P}\big(A\xleftrightarrow{(D)}\partial B(N)\big).\end{aligned}\tag{C.15}$$

Subsequently, we consider the case when $w_1\in[B(C_\spadesuit n)]^c$, and w_1 and w_2 are on different sides of the annulus $B(\frac{M}{2})\setminus B(\frac{M}{4})$.

When $\mathrm{ran}(\widetilde{\ell}_\dagger)\cap\partial B(\frac{N}{2})=\emptyset$**,** $w_1\in B(\frac{M}{4})\setminus B(C_\spadesuit n)$ **and** $w_2\in B(\frac{N}{10})\setminus B(\frac{M}{2})$**.** Note that $|w_1-w_2|\asymp|w_2|$ in this case. Hence, it follows from (A.3) that the probability of having $\widetilde{\ell}_\dagger$ is at most $C|w_2|^{4-2d}$. Thus, the probability of this case is bounded from above by

$$
\begin{aligned}
&C\mathbb{P}\big(A \xleftrightarrow{(D)} \partial B(N)\big) \sum\nolimits_{w_1 \in B(\frac{M}{4}), w_2 \in B(\frac{N}{10}) \setminus B(\frac{M}{2})} |w_1|^{2-d} |w_2|^{4-2d} \\
\overset{(3.15)}{\lesssim}\ & M^2 \mathbb{P}\big(A \xleftrightarrow{(D)} \partial B(N)\big) \sum\nolimits_{k \geq \frac{M}{2}} k^{3-d} \\
\lesssim & M^{6-d} \cdot \mathbb{P}\big(A \xleftrightarrow{(D)} \partial B(N)\big) \lesssim \sqrt{\tfrac{m}{M}} \cdot \mathbb{P}\big(A \xleftrightarrow{(D)} \partial B(N)\big).
\end{aligned}
\tag{C.16}
$$

When $\mathrm{ran}(\widetilde{\ell}_\dagger) \cap \partial B(\frac{N}{2}) = \emptyset$**,** $w_1 \in B(\frac{N}{10}) \setminus B(\frac{M}{2})$ **and** $w_2 \in B(\frac{M}{4})$**.** Similar to the last case, the probability of having $\widetilde{\ell}_\dagger$ is at most $|w_1|^{4-2d}$. Thus, the probability of this case is bounded from above by

$$
\begin{aligned}
&C|B(\tfrac{M}{4})| \mathbb{P}\big(A \xleftrightarrow{(D)} \partial B(N)\big) \sum\nolimits_{w_1 \in B(\frac{N}{10}) \setminus B(\frac{M}{2})} |w_1|^{6-3d} \\
\lesssim & M^d \mathbb{P}\big(A \xleftrightarrow{(D)} \partial B(N)\big) \sum\nolimits_{k \geq \frac{M}{2}} k^{5-2d} \\
\lesssim & M^{6-d} \cdot \mathbb{P}\big(A \xleftrightarrow{(D)} \partial B(N)\big) \lesssim \sqrt{\tfrac{m}{M}} \cdot \mathbb{P}\big(A \xleftrightarrow{(D)} \partial B(N)\big).
\end{aligned}
\tag{C.17}
$$

It remains to consider the case when $\mathrm{ran}(\widetilde{\ell}_\dagger) \cap \partial B(\frac{N}{2}) = \emptyset$ and $w_1 \in B(C_\spadesuit n)$. Since $\widetilde{\ell}_\dagger$ intersects both $\widetilde{B}_{w_1}(1)$ and $\partial B(M)$, it must cross the annulus $B(M) \setminus B(m)$.

When $w_1 \in B(C_\spadesuit n)$ **and** $w_2 \in B(C_\spadesuit m)$**.** Here we consider the crossing paths for the annulus $B(C_\spadesuit^3 m) \setminus B(C_\spadesuit^2 m)$. In this case, there exist $1 \leq i_1, i_2, i_3 \leq \kappa$ such that $\widetilde{\eta}^{\mathrm{F}}_{i_1}$ intersects $\widetilde{B}_{w_2}(1)$, $\widetilde{\eta}^{\mathrm{B}}_{i_2}$ intersects $\partial B(M)$, and that $\big\{\mathrm{ran}(\widetilde{\eta}^{\mathrm{F}}_{i_3}) \xleftrightarrow{(D)} A\big\}$ and $\big\{w_2 \xleftrightarrow{(D)} \partial B(N)\big\}$ occur disjointly. Like in Appendix A, we divide the subsequent analysis into two cases, according to whether $i_1 = i_3$.

Case (i): when $i_1 \neq i_3$**.** As mentioned above, given $\mathcal{F}_{\mathrm{yz}}$, the probabilities of $\big\{\mathrm{ran}(\widetilde{\eta}^{\mathrm{F}}_{i_1}) \cap \widetilde{B}_{w_2}(1) \neq \emptyset\big\}$, $\big\{\mathrm{ran}(\widetilde{\eta}^{\mathrm{B}}_{i_2}) \cap \partial B(M) \neq \emptyset\big\}$, $\big\{\mathrm{ran}(\widetilde{\eta}^{\mathrm{F}}_{i_3}) \xleftrightarrow{(D)} A\big\}$ and $\big\{w_2 \xleftrightarrow{(D)} \partial B(N)\big\}$ are bounded from above by Cm^{2-d}, $C\big(\frac{m}{M}\big)^2$, $\mathbb{P}\big(A \xleftrightarrow{(D)} \partial B(N)\big)$ and CN^{-2} respectively. Thus, the probabilities of this case is at most

$$
\begin{aligned}
&C|B(C_\spadesuit m)| m^{4-d} M^{-2} N^{-2} \mathbb{P}\big(A \xleftrightarrow{(D)} \partial B(C_\spadesuit m)\big) \\
\overset{(C.2)}{\lesssim}\ & \big(\tfrac{m}{M}\big)^2 \cdot \mathbb{P}\big(A \xleftrightarrow{(D)} \partial B(N)\big) \lesssim \sqrt{\tfrac{m}{M}} \cdot \mathbb{P}\big(A \xleftrightarrow{(D)} \partial B(N)\big).
\end{aligned}
\tag{C.18}
$$

Case (ii): when $i_1 = i_3$**.** In this case, $A \xleftrightarrow{(D)} \partial B(N)$ and $\mathrm{ran}(\widetilde{\eta}^{\mathrm{B}}_{i_2}) \cap \partial B(M) \neq \emptyset$ are certified by two disjoint collections of loops and crossing paths. Thus, by the generalized BKR inequality in [5, Remark 2.15], its probability is at most

$$
\begin{aligned}
&\mathbb{P}\big(A \xleftrightarrow{(D)} \partial B(N) \mid \mathcal{F}_{\mathrm{yz}}\big) \cdot \mathbb{P}\big(\mathrm{ran}(\widetilde{\eta}^{\mathrm{B}}_{i_2}) \cap \partial B(M) \neq \emptyset \mid \mathcal{F}_{\mathrm{yz}}\big) \\
\lesssim & \big(\tfrac{m}{M}\big)^2 \cdot \mathbb{P}\big(A \xleftrightarrow{(D)} \partial B(N) \mid \mathcal{F}_{\mathrm{yz}}\big).
\end{aligned}
\tag{C.19}
$$

Combined with (A.10), it yields that the probability of this case is at most

$$C\big(\tfrac{m}{M}\big)^2\cdot\mathbb{P}\big(A\xleftrightarrow{(D)}\partial B(N)\big)\lesssim\sqrt{\tfrac{m}{M}}\cdot\mathbb{P}\big(A\xleftrightarrow{(D)}\partial B(N)\big). \tag{C.20}$$

When $w_1\in B(C_\spadesuit n)$ and $w_2\in B(\frac{N}{10})\setminus B(C_\spadesuit m)$. Here we consider the crossing paths for the annulus $B(C_\spadesuit^{1/4}m)\setminus B(m)$. In this case, there exist $1\le i_1,i_2,i_3\le\kappa$ such that $\widetilde{\eta}^{\mathrm{B}}_{i_1}$ intersects $\widetilde{B}_{w_2}(1)$, $\widetilde{\eta}^{\mathrm{B}}_{i_2}$ intersects $\partial B(M)$, and that $\mathrm{ran}(\eta^{\mathrm{F}}_{i_3})\xleftrightarrow{(D)}A$ and $w_2\xleftrightarrow{(D)}\partial B(N)$ occur disjointly. Similarly, we consider the following two cases separately, according to whether $i_1=i_2$.

Case (a): when $i_1\neq i_2$. Since the conditional probabilities of $\big\{\mathrm{ran}(\widetilde{\eta}^{\mathrm{B}}_{i_1})\cap\widetilde{B}_{w_2}(1)\neq\emptyset\big\}$, $\big\{\mathrm{ran}(\widetilde{\eta}^{\mathrm{B}}_{i_2})\cap\partial B(M)\neq\emptyset\big\}$, $\big\{\mathrm{ran}(\eta^{\mathrm{F}}_{i_3})\xleftrightarrow{(D)}A\big\}$ and $\big\{w_2\xleftrightarrow{(D)}\partial B(N)\big\}$ are bounded from above by $C|w_2|^{4-2d}m^{d-2}$, $C(\frac{m}{M})^2$, $\mathbb{P}\big(A\xleftrightarrow{(D)}\partial B(m)\big)$ and CN^{-2} respectively. Thus, the probability of this case is at most

$$\begin{aligned}&Cm^dM^{-2}N^{-2}\mathbb{P}\big(A\xleftrightarrow{(D)}\partial B(m)\big)\sum\nolimits_{w_2\in B(\frac{N}{10})\setminus B(C_\spadesuit m)}|w_2|^{4-2d}\\ \overset{\text{(C.2)}}{\lesssim}\;&m^{d+2}M^{-2}\mathbb{P}\big(A\xleftrightarrow{(D)}\partial B(N)\big)\sum\nolimits_{k\ge m}k^{d-1}\cdot k^{4-2d}\\ \lesssim\;&m^2M^{-2}\cdot\mathbb{P}\big(A\xleftrightarrow{(D)}\partial B(N)\big)\lesssim\sqrt{\tfrac{m}{M}}\cdot\mathbb{P}\big(A\xleftrightarrow{(D)}\partial B(N)\big).\end{aligned}\tag{C.21}$$

Case (b): when $i_1=i_2$. We first estimate the conditional probability of $\big\{\mathrm{ran}(\widetilde{\eta}^{\mathrm{B}}_{i_1})\cap\widetilde{B}_{w_2}(1)\neq\emptyset,\mathrm{ran}(\widetilde{\eta}^{\mathrm{B}}_{i_1})\cap\partial B(M)\neq\emptyset\big\}$. When $w_2\in B(\frac{M}{2})\setminus B(C_\spadesuit m)$, for the same reason as in (C.11), this probability is upper-bounded by

$$\begin{aligned}&\mathbb{P}\big(\exists\widetilde{\ell}\in\mathfrak{L}[C_\spadesuit^{1/2}m,C_\spadesuit^{3/4}m]\text{ intersecting }\widetilde{B}_{w_2}(1)\text{ and }\partial B(M)\big)\\ \le&\mathbb{P}\big(\exists\widetilde{\ell}\in\widetilde{\mathcal{L}}_{1/2}\text{ intersecting }\widetilde{B}_{w_2}(1)\text{ and }\partial B_{w_2}(\tfrac{M}{10})\big)\overset{\text{(C.4)}}{\lesssim}M^{2-d}.\end{aligned}\tag{C.22}$$

When $w_2\in B(\frac{N}{10})\setminus B(\frac{M}{2})$, the probability of $\big\{\mathrm{ran}(\widetilde{\eta}^{\mathrm{B}}_{i_1})\cap\widetilde{B}_{w_2}(1)\neq\emptyset\big\}$ is at most $C|w_2|^{4-2d}m^{d-2}$. Recall that $\big\{\mathrm{ran}(\eta^{\mathrm{F}}_{i_3})\xleftrightarrow{(D)}A\big\}$ and $\big\{w_2\xleftrightarrow{(D)}\partial B(N)\big\}$ both occur. Thus, the probability of this case is bounded from above by

$$\begin{aligned}&C\big(\tfrac{m}{M}\big)^{d-2}N^{-2}\mathbb{P}\big(A\xleftrightarrow{(D)}\partial B(m)\big)\cdot\big(|B(\tfrac{M}{2})|\cdot M^{2-d}\\ &\qquad\qquad+m^{d-2}\sum\nolimits_{w_2\in B(\frac{N}{10})\setminus B(\frac{M}{2})}|w_2|^{4-2d}\big)\\ \lesssim&m^{d-4}M^{2-d}\mathbb{P}\big(A\xleftrightarrow{(D)}\partial B(N)\big)\cdot\big(M^2+\sum\nolimits_{k\ge\frac{M}{2}}k^{3-d}\big)\\ \lesssim&m^{d-2}M^{2-d}\cdot\mathbb{P}\big(A\xleftrightarrow{(D)}\partial B(N)\big)\lesssim\sqrt{\tfrac{m}{M}}\cdot\mathbb{P}\big(A\xleftrightarrow{(D)}\partial B(N)\big).\end{aligned}\tag{C.23}$$

In conclusion, by (C.7), (C.9), (C.12), (C.14), (C.15), (C.16), (C.17), (C.18), (C.20), (C.21) and (C.23), we obtain the desired inequality (C.6), which together with (C.5) yields (5.27) for $d \geq 7$, thereby completing the proof of Lemma 5.2. $\square$

Appendix D. Proof of Lemma 5.4

Recall that $N \geq CM$, $D \subset \widetilde{B}(cM)$, $y \in \partial B(N)$ and $v_e^- \in \widetilde{B}_{w_\dagger}(\frac{N}{50})$ with $w_\dagger \in \partial B(\frac{N}{2})$. Therefore, by (2.12) one has

$$\mathbb{P}\big(v_e^- \xleftrightarrow{(D)} y\big) \asymp \widetilde{G}_D(v_e^-, y) \asymp N^{2-d}. \tag{D.1}$$

Thus, by $\widetilde{\mathcal{L}}_{1/2}^D \leq \widetilde{\mathcal{L}}_{1/2}$, it is sufficient to show that

$$\mathbb{I} := \sum\nolimits_{z\in\mathbb{Z}^d} \mathbb{P}\big(y \leftrightarrow z, y \leftrightarrow v_e^-\big) \cdot |z|^{2-d} \lesssim N^{8-2d}. \tag{D.2}$$

On the event $\{y \leftrightarrow z, y \leftrightarrow v_e^-\}$, by the tree expansion argument (see [3, Section 3.4]), there exists a loop $\widetilde{\ell}_\dagger \in \widetilde{\mathcal{L}}_{1/2}$ such that $\mathrm{ran}(\widetilde{\ell}_\dagger) \leftrightarrow y$, $\mathrm{ran}(\widetilde{\ell}_\dagger) \leftrightarrow z$ and $\mathrm{ran}(\widetilde{\ell}_\dagger) \leftrightarrow v_e^-$ occur disjointly. In other words, there exist $w_1, w_2, w_3 \in \mathbb{Z}^d$ such that $\widetilde{\ell}_\dagger$ intersects $\widetilde{B}_j(1)$ for $j \in \{1,2,3\}$, and that $w_1 \leftrightarrow y$, $w_2 \leftrightarrow z$ and $w_3 \leftrightarrow v_e^-$ occur disjointly. Thus, by the BKR inequality, (1.1) and (A.3), we have

$$\begin{aligned}
\mathbb{I} \lesssim & \sum\nolimits_{z,w_1,w_2,w_3\in\mathbb{Z}^d} |w_1-w_2|^{2-d}|w_2-w_3|^{2-d}|w_3-w_1|^{2-d} \\
& \qquad\qquad \cdot |z|^{2-d}|w_1-y|^{2-d}|w_2-z|^{2-d}|w_3-v_e^-|^{2-d} \\
\overset{\text{(A.7)}}{\lesssim} & \sum\nolimits_{w_1,w_2,w_3\in\mathbb{Z}^d} |w_1-w_2|^{2-d}|w_2-w_3|^{2-d}|w_3-w_1|^{2-d} \\
& \qquad\qquad \cdot |w_1-y|^{2-d}|w_2|^{4-d}|w_3-v_e^-|^{2-d}.
\end{aligned} \tag{D.3}$$

We decompose the sum on the right-hand side of (D.3) into three parts, denoted by $\widehat{\mathbb{I}}_1$, $\widehat{\mathbb{I}}_2$ and $\widehat{\mathbb{I}}_3$, according to the following regions:

- $\widehat{\mathbb{I}}_1$: restricted to $w_1 \in [B_y(\frac{N}{10})]^c$;
- $\widehat{\mathbb{I}}_2$: restricted to $w_3 \in [B_{v_e^-}(\frac{N}{10})]^c$;
- $\widehat{\mathbb{I}}_3$: restricted to $w_1 \in B_y(\frac{N}{10})$ and $w_3 \in B_{v_e^-}(\frac{N}{10})$.

In what follows, we estimate $\widehat{\mathbb{I}}_1$, $\widehat{\mathbb{I}}_2$ and $\widehat{\mathbb{I}}_3$ separately.

For $\widehat{\mathbb{I}}_1$. Since $|w_2 - y| \gtrsim N$, we have

$$\begin{aligned}
\widehat{\mathbb{I}}_1 \lesssim & N^{2-d} \sum\nolimits_{w_1,w_2,w_3\in\mathbb{Z}^d} |w_1-w_2|^{2-d}|w_2-w_3|^{2-d}|w_3-w_1|^{2-d} \\
& \qquad\qquad \cdot |w_2|^{4-d}|w_3-v_e^-|^{2-d} \\
\overset{\text{(A.7),(6.15)}}{\lesssim} & N^{2-d} \sum\nolimits_{w_2\in\mathbb{Z}^d} |w_2|^{4-d}|w_2-v_e^-|^{2-d} \overset{\text{(A.8)}}{\lesssim} N^{8-2d}.
\end{aligned} \tag{D.4}$$

For $\widehat{\mathbb{I}}_2$. Similarly, since $|w_3 - v_e^-| \gtrsim N$, we have

$$
\begin{aligned}
\widehat{\mathbb{I}}_2 \lesssim & N^{2-d} \sum\nolimits_{w_1,w_2,w_3 \in \mathbb{Z}^d} |w_1 - w_2|^{2-d} |w_2 - w_3|^{2-d} |w_3 - w_1|^{2-d} \\
& \cdot |w_1 - y|^{2-d} |w_2|^{4-d} \\
\overset{(A.7),(6.15)}{\lesssim} & N^{2-d} \sum\nolimits_{w_2 \in \mathbb{Z}^d} |w_2|^{4-d} |w_2 - y|^{2-d} \overset{(A.8)}{\lesssim} N^{8-2d}.
\end{aligned}
\tag{D.5}
$$

For $\widehat{\mathbb{I}}_3$. In this case, one has $|w_1 - w_3| \gtrsim N$. Thus, we have

$$
\begin{aligned}
\widehat{\mathbb{I}}_3 \lesssim & N^{2-d} \sum\nolimits_{w_1,w_2,w_3 \in \mathbb{Z}^d} |w_1 - w_2|^{2-d} |w_2 - w_3|^{2-d} |w_1 - y|^{2-d} \\
& \cdot |w_2|^{4-d} |w_3 - v_e^-|^{2-d} \\
\overset{(A.7)}{\lesssim} & N^{2-d} \cdot \left[\mathbb{J}(B_y(\tfrac{N}{10})) + \mathbb{J}(B(2N) \setminus B_y(\tfrac{N}{10})) + \mathbb{J}([B(2N)]^c)\right],
\end{aligned}
\tag{D.6}
$$

where we denote $\mathbb{J}(A) := \sum_{w \in A} |w|^{4-d} |w - y|^{4-d} |w - v_e^-|^{4-d}$ for $A \subset \mathbb{Z}^d$.

Firstly, since $|w - v_e^-| \gtrsim N$ and $|w| \lesssim N$ for all $w \in B_y(\frac{N}{10})$, one has

$$
\mathbb{J}(B_y(\tfrac{N}{10})) \lesssim N^{6-d} \sum\nolimits_{w \in \mathbb{Z}^d} |w|^{2-d} |w - y|^{4-d} \overset{(A.8)}{\lesssim} N^{12-2d}.
\tag{D.7}
$$

Secondly, since $|w - y| \gtrsim N$ and $|w| \lesssim N$ for all $w \in B(2N) \setminus B_y(\frac{N}{10})$, we have

$$
\mathbb{J}(B(2N) \setminus B_y(\tfrac{N}{10})) \lesssim N^{6-d} \sum\nolimits_{w \in \mathbb{Z}^d} |w|^{2-d} |w - v_e^-|^{4-d} \overset{(A.8)}{\lesssim} N^{12-2d}.
\tag{D.8}
$$

Thirdly, since $|w| \asymp |w - y| \asymp |w - v_e^-|$ for all $w \in [B(2N)]^c$, one has

$$
\mathbb{J}([B(2N)]^c) \lesssim \sum\nolimits_{w \in \mathbb{Z}^d} |w|^{12-3d} \overset{|\partial B(k)| \asymp k^{d-1}}{\lesssim} \sum\nolimits_{k \geq N} k^{11-2d} \lesssim N^{12-2d}.
\tag{D.9}
$$

Plugging (D.7), (D.8) and (D.9) into (D.6), we get

$$
\widehat{\mathbb{I}}_3 \lesssim N^{14-3d} \lesssim N^{8-2d}.
\tag{D.10}
$$

Combining (D.3), (D.4), (D.5) and (D.10), we obtain (D.2), and thus complete the proof of Lemma 5.4. □

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