

HETEROCHROMATIC TWO-ARM PROBABILITIES FOR METRIC GRAPH GAUSSIAN FREE FIELDS

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ABSTRACT. For the Gaussian free field on the metric graph of $\mathbb{Z}^d$ ($d \geq 3$), we consider the heterochromatic two-arm probability, i.e., the probability that two points v and v' are contained in distinct clusters of opposite signs with diameter at least N . For all $d \geq 3$ except the critical dimension $d_c = 6$, we prove that this probability is asymptotically proportional to $N^{-[(\frac{d}{2}+1)\wedge 4]}$. Furthermore, we prove that conditioned on this two-arm event, the volume growth of each involved cluster is comparable to that of a typical (unconditioned) cluster; precisely, each cluster has a volume of order $M^{(\frac{d}{2}+1)\wedge 4}$ within a box of size M .

1. INTRODUCTION

In this paper, we study the sign clusters of Gaussian free fields (GFF) on metric graphs, with a particular focus on understanding the interactions between them. For clarity, we start with a review of some basic definitions. We denote the edge set of the d -dimensional integer lattice $\mathbb{Z}^d$ by $\mathbb{L}^d := \{\{x, y\} : x, y \in \mathbb{Z}^d, |x - y|_1 = 1\}$, where $|\cdot|_1$ represents the L^1 -norm. Throughout this paper, we assume that $d \geq 3$. For each $e = \{x, y\} \in \mathbb{L}^d$, consider a compact interval I_e of length d , with endpoints identified with x and y respectively. The metric graph of $\mathbb{Z}^d$, denoted by $\tilde{\mathbb{Z}}^d$, is defined as the union of all these intervals. The GFF on $\tilde{\mathbb{Z}}^d$, which we denote by $\{\tilde{\phi}_v\}_{v \in \tilde{\mathbb{Z}}^d}$, was introduced in [28] as a natural extension of the discrete GFF on $\mathbb{Z}^d$. To be precise, $\{\tilde{\phi}_v\}_{v \in \tilde{\mathbb{Z}}^d}$ can be constructed in the following two steps.

- (i) Let $\tilde{\phi}_x$ for $x \in \mathbb{Z}^d$ be a discrete GFF on $\mathbb{Z}^d$, i.e., a family of mean-zero Gaussian random variables whose covariance is given by

$$(1.1) \quad \mathbb{E}[\tilde{\phi}_x \tilde{\phi}_y] = G(x, y), \quad \forall x, y \in \mathbb{Z}^d.$$

Here $G(x, y)$ represents the Green's function on $\mathbb{Z}^d$ —the expected number of times that a simple random walk on $\mathbb{Z}^d$ starting from x visits y .

- (ii) For any $e = \{x, y\} \in \mathbb{L}^d$, $\{\tilde{\phi}_v\}_{v \in I_e}$ is distributed as a Brownian bridge with boundary conditions $\tilde{\phi}_x$ at x and $\tilde{\phi}_y$ at y , generated by a one-dimensional Brownian motion with variance 2 at time 1.

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In fact, $\{\tilde{\phi}_v\}_{v \in \tilde{\mathbb{Z}}^d}$ can be equivalently considered as a mean-zero Gaussian field on $\tilde{\mathbb{Z}}^d$ with covariance $\mathbb{E}[\tilde{\phi}_v \tilde{\phi}_w] = \tilde{G}(v, w)$ for all $v, w \in \tilde{\mathbb{Z}}^d$. Here $\tilde{G}(\cdot, \cdot)$ is the Green's function for the canonical Brownian motion on $\tilde{\mathbb{Z}}^d$ (the precise definition will be provided in Section 3). Moreover, $\tilde{G}(\cdot, \cdot)$ can be expressed as a linear interpolation of the Green's function on the lattice $\mathbb{Z}^d$: suppose that $v \in I_{\{x_1, x_2\}}$ and $w \in I_{\{y_1, y_2\}}$ (where $\{x_1, x_2\}, \{y_1, y_2\} \in \mathbb{L}^d$), then one has

$$(1.2) \quad \tilde{G}(v, w) = d^{-2} \sum_{j, k \in \{1, 2\}} |v - x_{3-j}| \cdot |w - y_{3-j}| \cdot G(x_j, y_k),$$

where $|u - u'|$ denotes the graph distance between u and u' on $\tilde{\mathbb{Z}}^d$, with respect to the Lebesgue measure. In particular, $\tilde{G}(v, w) = G(v, w)$ when $v, w \in \mathbb{Z}^d$.

Many properties of the GFF level-sets $\tilde{E}^{\geq h} := \{v \in \tilde{\mathbb{Z}}^d : \tilde{\phi}_v \geq h\}$ for $h \in \mathbb{R}$ have been established in previous works. Fundamentally, it was proved in [28] that $\tilde{E}^{\geq h}$ percolates (i.e., contains an infinite connected component) if and only if $h < 0$. As in most percolation models, the connecting probabilities at the critical level $\tilde{h}_* = 0$ have attracted major research interest. To be precise, for any disjoint subsets $A_1, A_2 \subset \tilde{\mathbb{Z}}^d$, we denote by $A_1 \xleftrightarrow{\geq 0} A_2$ the event that there exists a path in $\tilde{E}^{\geq 0}$ connecting A_1 and A_2 (we will omit the braces when A_i is a singleton $\{v\}$ with $v \in \tilde{\mathbb{Z}}^d$). [28, Proposition 5.2] showed that the two-point function satisfies

$$(1.3) \quad \mathbb{P}(v \xleftrightarrow{\geq 0} w) = \pi^{-1} \arcsin \left(\frac{\tilde{G}(v, w)}{\sqrt{\tilde{G}(v, v) \tilde{G}(w, w)}} \right) \asymp (|v - w| + 1)^{2-d}$$

for all $v, w \in \tilde{\mathbb{Z}}^d$. Here $f \asymp g$ means that for the functions f and g , there exist constants $C > c > 0$ depending only on d such that $cg \leq f \leq Cg$. On top of that, the one-arm probability $\theta_d(N) := \mathbb{P}(\mathbf{0} \xleftrightarrow{\geq 0} \partial B(N))$ has been estimated rigorously in a series of works [8, 10, 3, 9, 12, 4], where $\mathbf{0} := (0, 0, \dots, 0)$ denotes the origin of $\mathbb{Z}^d$, $B(N) := [-N, N]^d \cap \mathbb{Z}^d$ is the box of side length $[2N]$ in $\mathbb{Z}^d$ centered at $\mathbf{0}$, and $\partial A := \{x \in A : \exists y \in \mathbb{Z}^d \setminus A \text{ such that } \{x, y\} \in \mathbb{L}^d\}$ represents the boundary of $A \subset \mathbb{Z}^d$. Specifically, it has been proved that

$$(1.4) \quad \text{when } 3 \leq d < 6, \quad \theta_d(N) \asymp N^{-\frac{d}{2}+1};$$

$$(1.5) \quad \text{when } d = 6, \quad N^{-2} \lesssim \theta_6(N) \lesssim N^{-2+\varsigma(N)}, \text{ where } \varsigma(N) := \frac{\ln \ln(N)}{\ln^{1/2}(N)} \ll 1;$$

$$(1.6) \quad \text{when } d > 6, \quad \theta_d(N) \asymp N^{-2}.$$

Here $f \lesssim g$ means that $f \leq Cg$ for some constant $C > 0$ depending only on d . The exact order of $\theta_6(N)$ remains open, and it has been conjectured to be $N^{-2}[\ln(N)]^\delta$ for some $\delta > 0$ (see [4, Remark 1.5]).

(P.S. Due to the diverging disparity between the current upper and lower bounds on $\theta_6(N)$, the analysis at the critical dimension $d = 6$ involves additional complexities compared to other dimensions. Nevertheless, the arguments for $d \neq 6$ can usually be extended to $d_c = 6$, albeit yielding weaker estimates with error terms

similar to that in (1.5). Accordingly, we assume $d \neq 6$ throughout the remainder of this paper, unless stated otherwise.)

As a natural extension of the one-arm probability $\theta_d(N)$, the crossing probability $\rho_d(n, N) := \mathbb{P}(B(n) \xleftrightarrow{\geq 0} \partial B(N))$ (where $N > n \geq 1$) was proved in [4] to satisfy

$$(1.7) \quad \text{when } 3 \leq d < 6, \quad \rho_d(n, N) \asymp (n/N)^{\frac{d}{2}+1};$$

$$(1.8) \quad \text{when } d > 6, \quad \rho_d(n, N) \asymp (n^{d-4}N^{-2}) \wedge 1.$$

For the connecting probabilities between general sets, numerous regularity properties have been established in [5]. Among these, the property known as quasi-multiplicativity played a crucial role in [2], which constructed several types of incipient infinite clusters (IIC), including the initial version introduced by [19]—the positive cluster containing $\mathbf{0}$ under the limiting measure

$$(1.9) \quad \mathbb{P}_{\text{IIC}}(\cdot) := \lim_{N \rightarrow \infty} \mathbb{P}(\cdot \mid \mathbf{0} \xleftrightarrow{\geq 0} \partial B(N)),$$

and established the equivalence between them. See also [15, 14] for various properties of the random walk on the IIC in high-dimensional case (i.e., $d \geq 7$). Recently, a powerful switching identity was introduced in [42], which, among many applications, provides an elegant and precise description of the IIC. Furthermore, building on the analysis of connecting probabilities, the volume growth of clusters in the critical level-set $\tilde{E}^{\geq 0}$ has also been extensively studied. To be precise, for any $v \in \tilde{\mathbb{Z}}^d$, we denote by $\mathcal{C}_v^+ := \{w \in \tilde{\mathbb{Z}}^d : w \xleftrightarrow{\geq 0} v\}$ the cluster of $\tilde{E}^{\geq 0}$ containing v . The decay rate of the cluster volume is defined as $\nu_d(M) := \mathbb{P}(\text{vol}(\mathcal{C}_0^+) \geq M)$ for $M \geq 1$, where the volume $\text{vol}(A)$ (for $A \subset \tilde{\mathbb{Z}}^d$) is defined as the cardinality of $A \cap \mathbb{Z}^d$. Through the collective efforts in [3, 5, 11], it has been established that

$$(1.10) \quad \text{when } 3 \leq d < 6, \quad \nu_d(M) \asymp M^{-\frac{d-2}{d+2}};$$

$$(1.11) \quad \text{when } d > 6, \quad \nu_d(M) \asymp M^{-\frac{1}{2}}.$$

In addition, the volume growth rate in large clusters is also well understood. Formally, for any $x \in \tilde{\mathbb{Z}}^d$ and $M \geq 1$, let $\mathcal{V}_x^+(M) := \text{vol}(\mathcal{C}_x^+ \cap B_x(M))$ denote the volume of the cluster $\mathcal{C}_x^+$ inside the box $B_x(M) := x + B(M)$. It was proved in [2, Theorem 1.2] that for some constant $C(d) > 0$, the infimum

$$(1.12) \quad \inf_{M \geq 1, N \geq CM} \mathbb{P}(\lambda^{-1} M^{(\frac{d}{2}+1)\square 4} \leq \mathcal{V}_0^+(M) \leq \lambda M^{(\frac{d}{2}+1)\square 4} \mid \mathbf{0} \xleftrightarrow{\geq 0} \partial B(N))$$

converges to 1 as $\lambda \rightarrow \infty$. Here we employ “ $\square$ ” to denote the operation $f \square g(d) := f(d) \cdot \mathbb{1}_{d \leq 6} + g(d) \cdot \mathbb{1}_{d > 6}$ for any functions f and g that depend on d . Note that the property in (1.12) automatically applies to the IIC. Similar results in terms of the largest cluster in a given box were obtained in [11].

As mentioned above, all existing results on the sign clusters of the GFF $\tilde{\phi}$ (i.e., maximal subgraphs where the GFF values have the same sign) focus on properties of a single cluster. As a complement, in this paper we aim to study the coexistence

of two distinct large sign clusters originating from two given points. To this end, the following two questions arise naturally:

Q1: How likely is this coexistence?

Q2: How does this coexistence alter the geometric properties of the two involved clusters? Do they still behave similarly to a single cluster?

Next, we will explore these two questions in Sections 1.1 and 1.2 respectively.

1.1. Heterochromatic connecting probabilities. We formulate **Q1** by introducing the notion of heterochromatic two-arm probability. Precisely, for any subsets $A_i \subset \tilde{\mathbb{Z}}^d$ for $1 \leq i \leq 4$, we define the event

$$(1.13) \quad H_{A_3, A_4}^{A_1, A_2} := \{A_1 \xleftrightarrow{\geq 0} A_2, A_3 \xleftrightarrow{\leq 0} A_4\},$$

where $A_3 \xleftrightarrow{\leq 0} A_4$ represents the event that a certain path in the negative cluster $\tilde{E}^{\leq 0} := \{v \in \tilde{\mathbb{Z}}^d : \tilde{\phi}_v \leq 0\}$ connects A_3 and A_4 . Here heterochromaticity is imposed to capture the distinction between the two involved sign clusters. Moreover, since the sign of $\tilde{\phi}$ on each sign cluster is independent of the others and is uniformly chosen from $\{+, -\}$ (referring to [28, Theorem 1]), it follows that $\mathbb{P}(H_{A_3, A_4}^{A_1, A_2})$ exactly equals one fourth of the probability A_1 and A_3 are respectively connected to A_2 and A_4 by two distinct sign clusters. In particular, for any $N \geq 1$ and $v, v' \in \tilde{B}(N)$ (where $\tilde{B}(N) := \cup_{e \in \mathbb{L}^d: I_e \cap (-N, N)^d \neq \emptyset} I_e$ denotes the box in $\tilde{\mathbb{Z}}^d$ of radius N centered at $\mathbf{0}$), we define the heterochromatic two-arm event $H_{v'}^v(N) := H_{v', \partial B(N)}^{v, \partial B(N)}$.

Our first result establishes the decay rate of $\mathbb{P}(H_{v'}^v(N))$ as $N \rightarrow \infty$, and its exact orders in different regimes of $|v - v'|$, including the case when $|v - v'|$ is large, so that the macroscopic graph structure of $\tilde{\mathbb{Z}}^d$ plays a role, and the case when $|v - v'|$ is small, with stronger influence from the local, one-dimensional geometry of $\tilde{\mathbb{Z}}^d$.

Theorem 1.1. *For any $d \geq 3$ with $d \neq 6$, there exist constants $C_1(d), c_1(d) > 0$ such that the following hold for all $N \geq C_1$ and $v, v' \in \tilde{B}(c_1 N)$:*

$$(1.14) \quad \text{when } \chi =: |v - v'| \leq 1, \quad \mathbb{P}(H_{v'}^v(N)) \asymp \chi^{\frac{3}{2}} N^{-[(\frac{d}{2}+1)\square 4]},$$

$$(1.15) \quad \text{when } \chi \geq 1, \quad \mathbb{P}(H_{v'}^v(N)) \asymp \chi^{(3-\frac{d}{2})\square 0} N^{-[(\frac{d}{2}+1)\square 4]}.$$

The isomorphism theorem [28, Theorem 1.1] shows that the sign clusters of the GFF $\tilde{\phi}$ have the same distribution as the loop clusters of the loop soup $\tilde{\mathcal{L}}_{1/2}$. Here $\tilde{\mathcal{L}}_{1/2}$ is a Poisson point process consisting of loops—paths that start and end at the same point (the precise definition of $\tilde{\mathcal{L}}_{1/2}$ will be provided in Section 3), and the term “loop clusters” refers to the connected components composed of loops in $\tilde{\mathcal{L}}_{1/2}$. This isomorphism theorem allows us to switch between the two models, which greatly facilitates the proofs. Furthermore, all estimates in Theorem 1.1 also apply to their analogues for the loop clusters of $\tilde{\mathcal{L}}_{1/2}$.

Remark 1.2 (distance between macroscopic clusters). Theorem 1.1 implies that the expected number of edges within a box of size N , whose endpoints are contained in distinct sign clusters with diameter at least N , is of order $N^{(\frac{d}{2}-1)\square(d-4)}$. This suggests that two macroscopic sign clusters may nearly touch at many locations, resulting in a vanishing graph distance between them. In a companion paper [7], we confirm this property through the second moment method. Notably, this contrasts sharply with the two-dimensional case, where the typical distance between macroscopic clusters is comparable to their diameters (see [29, 38]).

Remark 1.3 (pivotal edge). The phenomenon noted in Remark 1.2 suggests that in a sign cluster connecting two distant sets, there are numerous pivotal edges—those whose removal breaks the connectivity between the sets. From the perspective of the loop cluster, this property indicates that removing all loops of length 1 might significantly reduce the connecting probability. In a companion paper [6], we verify this reduction through a multi-scale analysis, and establish that the dimension of clusters in a Brownian loop soup of intensity $\frac{1}{2}$ on $\mathbb{R}^3$ is strictly less than that of loop clusters on $\tilde{\mathbb{Z}}^3$. This shows that already in dimension 3, microscopic randomness (on top of the macroscopic loops) is involved in the construction of loop soup cluster, which corrects the scenario proposed in [41, Section 3].

As a byproduct of proving Theorem 1.1, we also obtain the exact order of the heterochromatic four-point function, which is interesting in its own right.

Theorem 1.4. *For any $d \geq 3$ with $d \neq 6$, there exists constant $C_2(d), c_2(d) > 0$ such that the following hold for all $N \geq C_2$, $v, v' \in \tilde{B}(c_2 N)$, and $w, w' \in \tilde{B}(10N) \setminus \tilde{B}(N)$ with $|w - w'| \geq N$:*

$$(1.16) \quad \text{when } \chi =: |v - v'| \leq 1, \quad \mathbb{P}(\mathbf{H}_{v', w'}^{v, w}) \asymp \chi^{\frac{3}{2}} N^{-[(\frac{3d}{2}-1)\square(2d-4)]},$$

$$(1.17) \quad \text{when } \chi \geq 1, \quad \mathbb{P}(\mathbf{H}_{v', w'}^{v, w}) \asymp \chi^{(3-\frac{d}{2})\square 0} N^{-[(\frac{3d}{2}-1)\square(2d-4)]}.$$

1.2. Volume growth under the coexistence. On the heterochromatic two-arm event $\mathbf{H}_v^v(N)$, the magnitude of the positive cluster $\mathcal{C}_v^+$ imposes a constraint on the growth of the negative cluster $\mathcal{C}_{v'}^- := \{w \in \tilde{\mathbb{Z}}^d : w \xleftrightarrow{\leq 0} v'\}$. To study the interaction described in **Q2**, we aim to analyze how this constraint influences the geometric properties of the two involved clusters, particularly their volume growth. According to (1.12), conditioned on a point being contained in a large cluster, its volume within the M -neighborhood of this point is typically of order $M^{(\frac{d}{2}+1)\square 4}$. Naturally, one would ask whether conditioning on v and v' being contained in two distinct large clusters will impose a constraint that alters their inherent volume growth rates. At the first glance, it may seem plausible that the volume growth will be affected when $\mathbf{H}_v^v(N) \ll [\theta_d(N)]^2$ (which holds except when $d > 6$ or $|v - v'| \asymp N$), as it indicates that the constraint between the two clusters is significant (note that the volume of a macroscopic cluster has fluctuation of the

same order as its typical value, as proved by [2, Theorem 1.3]; this implies that a priori, a significant constraint might change the volume significantly). Somewhat surprisingly, our next result shows that on the heterochromatic two-arm event, the two involved clusters actually maintain their inherent volume growth rates.

Recall $\mathcal{V}_x^+(M)$ above (1.12). For any $v \in \widetilde{\mathbb{Z}}^d$, let $\bar{v}$ denote the lattice point closest to v (with ties broken in a predetermined manner), and define $\mathcal{V}_v^+(M) := \mathcal{V}_{\bar{v}}^+(M)$. We denote by $\mathcal{V}_v^-(M)$ the analogue of $\mathcal{V}_v^+(M)$ obtained by replacing $\widetilde{E}^{\geq 0}$ with $\widetilde{E}^{\leq 0}$.

Theorem 1.5. *Assume that N, v, v' satisfy the conditions in Theorem 1.1. There exist constants $c_3(d), c_4(d) \in (0, 1)$ and $C_3(d) > 1$ such that for any $1 \leq M \leq c_3 N$,*

$$(1.18) \quad \mathbb{P}(\mathcal{V}_v^+(M), \mathcal{V}_{v'}^-(M) \in [C_3^{-1}M^{(\frac{d}{2}+1)\square 4}, C_3M^{(\frac{d}{2}+1)\square 4}] \mid \mathbf{H}_{v'}^v(N)) \geq c_4.$$

Remark 1.6. We expect that the right-hand side of (1.18) can be enhanced to any number arbitrarily close to 1 (as in (1.12)), although the techniques developed in this paper are insufficient to achieve this. More precisely, as presented in [2], the proof of (1.12) relies on a multi-scale argument. The key idea is that the volume growth of a cluster within an increasing sequence of annuli at scale M is asymptotically independent across annuli, while each annulus potentially adds a term of order $cM^{(\frac{d}{2}+1)\square 4}$ to the total volume. In fact, this asymptotic independence is built on the property that large crossing loops (i.e., loops that traverse an annulus with a big ratio of outer to inner diameters) are rare in a loop cluster. Intuitively, this property should persist under the co-existence of two clusters, although the correlation between clusters will significantly complicate the adaptation of the approach in [2]. Nevertheless, we believe that the decomposition argument of loop clusters tailored for two-arm events in the companion paper [7] should be able to resolve this difficulty, albeit requiring significant additional technical efforts.

The key to the proof of Theorem 1.5 lies in what we term the “rigidity lemma” (see Lemma 4.1), which asserts that conditioned on the coexistence of two clusters, the probability of an **increasing** event for one cluster is at most proportional to its conditional probability given only the existence of the corresponding cluster (with a small error term). In other words, an increasing event that is rare for one cluster remains rare when conditioned on the existence of the other cluster. Notably, this property together with (1.12) immediately shows that conditioned on the heterochromatic two-arm event $\mathbf{H}_{v'}^v(N)$, with high probability the volumes $\mathcal{V}_v^+(M)$ and $\mathcal{V}_{v'}^-(M)$ are $O(M^{(\frac{d}{2}+1)\square 4})$. Note that only the upper bounds on the cluster volumes can be obtained in this way because $\{\mathcal{V}_v^+(M) \geq T\}$ is increasing with respect to the corresponding cluster $\mathcal{C}_v^+$, while $\{\mathcal{V}_v^+(M) \leq T\}$ is decreasing (similarly for $\{\mathcal{V}_{v'}^-(M) \geq T\}$ and $\mathcal{C}_{v'}^-$).

As for the other direction, the proof relies on the so-called “separation lemma”, whose statement is quite plausible (and may even appear obvious at first glance)—conditioned on two random sets (in our case, clusters containing different points) being disjoint, their extremities are significantly separated. In our context, this

lemma implies that any region sufficiently far from $\{v, v'\}$ with high probability contains at most one cluster from $\mathcal{C}_v^+$ or $\mathcal{C}_{v'}^-$. This cluster can then expand unimpeded by the other, yielding the desired lower bound on its volume growth under coexistence, as stated in Theorem 1.5. The separation lemma has been extensively applied, primarily in the study of Brownian motion [22, 24, 25], random walk [21, 32, 26, 36] and Bernoulli percolation [20, 39, 34, 37, 18, 13]. See also a recent application to percolation in two-dimensional loop soups [16, 17]. In the existing percolation literature, proofs of the separation lemma typically require multi-scale analysis and, to some extent, rely on the planarity of the underlying graphs (which allows a single curve to separate one set from another). However, in the context of this paper, we circumvent these restrictions and prove the separation lemma (see Lemma 4.5) using the rigidity lemma introduced above. Notably, this separation lemma plays a key role in establishing the stability of $\mathbb{P}(\mathbf{H}_{v'}^v(N))$ under perturbations of the points v and v' , which constitutes a crucial ingredient in the derivation of Theorem 1.1.

Statements about constants. We use the letters C and c to denote constants, which may vary from line to line and depend on the context. Moreover, constants labeled with numerical subscripts (such as $C_1, C_2, c_1, c_2, \dots$) remain fixed. The letter C (with or without additional indices) is reserved for large constants, while c is used for small constants. Unless stated otherwise, constants are assumed to depend only on the dimension d . If a constant depends on other parameters, we will specify these parameters in parentheses.

Organization of this paper. In Section 2, we present a heuristic for Theorem 1.1. In Section 3, we review relevant notations and preliminary results. Section 4 is devoted to the proof of the rigidity lemma and some corollaries. Subsequently, in Section 5 we establish the upper bound in (1.16) and the lower bound in (1.17). In Section 6, we prove Theorem 1.1 and the remaining bounds in Theorem 1.4. Finally, we provide the proof of Theorem 1.5 in Section 7.

2. HEURISTIC FOR THE HETEROCHROMATIC TWO-ARM EXPONENT

In this section, we provide some intuitions for the exponents in Theorems 1.1 and 1.4. We take a large number N , and two points $x_- := (-N, 0, \dots, 0)$ and $x_+ := (N, 0, \dots, 0)$ on $\mathbb{Z}^d$. Suppose that x_- is contained in a sign cluster $\mathcal{C}$ with diameter cN , which occurs with probability $\theta_d(cN) \asymp N^{-[(\frac{d}{2}-1)\square 2]}$. The strong Markov property of the GFF implies that conditioned on the configuration of $\mathcal{C}$, the values of $\tilde{\phi}_v$ for $v \in \mathbb{Z}^d \setminus \mathcal{C}$ are distributed as an independent metric graph GFF with zero boundary condition on $\mathcal{C}$. Therefore, for each edge $\{y, y'\} \in \mathbb{L}^d$ such that $y \in \mathcal{C}$ and $y' \notin \mathcal{C}$ (for convenience, we assume that y is exactly on the boundary on $\mathcal{C}$), according to the two-point function estimate for general graphs (see (3.28)), given the cluster $\mathcal{C}$, the conditional probability of y' and x_+ being connected by a sign cluster is proportional to the Green's function at (y', x_+) with

absorbing boundary on $\mathcal{C}$, denoted by $\tilde{G}_{\mathcal{C}}(y', x_+)$ (see (3.2)). Moreover, using the strong Markov property of random walks and the transience of $\mathbb{Z}^d$ for $d \geq 3$, it follows that $\tilde{G}_{\mathcal{C}}(y', x_+)$ is of the same order as N^{2-d} times the probability that a simple random walk starting from y never returns to the cluster $\mathcal{C}$, denoted by $\text{Esc}(y')$. In conclusion, the total number $\mathbf{M}$ of edges $\{y, y'\}$ such that y and y' are connected to x_- and x_+ respectively by distinct sign clusters typically satisfies

$$(2.1) \quad \mathbf{M} \asymp \theta_d(cN) \cdot N^{2-d} \cdot \sum_{\{y, y'\} \in \mathbb{L}^d: y \in \mathcal{C}, y' \notin \mathcal{C}} \text{Esc}(y).$$

It is well-known that the sum on the right-hand side equals the capacity of $\mathcal{C}$, denoted by $\text{cap}(\mathcal{C})$ (see e.g., [23, Section 6.5]). Moreover, the decay rate of $\text{cap}(\mathcal{C})$ has been computed in [8, 10]: there exists $C > 0$ such that for any $T > 0$,

$$(2.2) \quad \mathbb{P}(\text{cap}(\mathcal{C}) \geq T) \sim CT^{-\frac{1}{2}}.$$

Here “ $f(T) \sim g(T)$ ” means that $\lim_{T \rightarrow \infty} f(T)/g(T) = 1$. Combined with the estimates (1.4) and (1.6) on one-arm probabilities, it shows that $\text{cap}(\mathcal{C})$ is typically of order $N^{(d-2)\square 4}$ (note that $\mathbb{P}(\text{cap}(\mathcal{C}) \geq N^{(d-2)\square 4}) \asymp \theta_d(cN)$). To sum up,

$$(2.3) \quad \mathbf{M} \asymp \theta_d(cN) \cdot N^{2-d} \cdot N^{(d-2)\square 4} = N^{-[(\frac{d}{2}-1)\square(d-4)]}.$$

By Fubini’s theorem, the expectation of $\mathbf{M}$ equals four times the sum of heterochromatic four-point functions $\mathbb{P}(\mathbf{H}_{y', x_-}^{y, x_+})$ over all edges $\{y, y'\} \in \mathbb{L}^d$ (where the factor of four arises because the sign of each sign cluster is chosen uniformly at random). To relate this sum to the probability $\mathbb{P}(\mathbf{H}_{v_-, x_-}^{v_+, x_+})$ of interest (where $v_+ := (1, 0, \dots, 0)$ and $v_- := \mathbf{0}$), we need the following regularity: for any edges $\{y_1, y'_1\}$ and $\{y_2, y'_2\}$ (except those are too close to x_+ or x_-), the probabilities $\mathbb{P}(\mathbf{H}_{y'_1, x_-}^{y_1, x_+})$ and $\mathbb{P}(\mathbf{H}_{y'_2, x_-}^{y_2, x_+})$ are comparable. These two properties together yield

$$(2.4) \quad \mathbb{E}[\mathbf{M}] \asymp N^d \cdot \mathbb{P}(\mathbf{H}_{v_-, x_-}^{v_+, x_+}).$$

We now sketch the proof of the aforementioned regularity. For each $\diamond \in \{+, -\}$, let $\hat{x}_\diamond = \hat{x}_\diamond(y_1, y'_1, y_2, y'_2)$ denote the image of $x_\diamond$ under the unique automorphism of $\mathbb{Z}^d$ that maps y_2 and y'_2 to y_1 and y'_1 respectively. Therefore, $\mathbb{P}(\mathbf{H}_{y'_2, x_-}^{y_2, x_+}) = \mathbb{P}(\mathbf{H}_{y'_1, \hat{x}_-}^{y_1, \hat{x}_+})$. As mentioned in Section 1.2, the proof of this regularity property essentially relies on the “separation lemma”, which asserts that conditioned on the event $\mathbf{H}_{y'_1, x_-}^{y_1, x_+}$, the positive cluster $\mathcal{C}_{y_1}^+$ is typically disjoint from the ϵN -neighborhood of x_- for a sufficiently small $\epsilon > 0$. In fact, when this event occurs, given the cluster $\mathcal{C}_{y_1}^+$ (which serves as zero boundary conditions), the probabilities of $\{y'_1 \xleftrightarrow{0} z\}$ for all z in the ϵN -neighborhood of w_- —approximately the Green’s function at (y'_1, z) with absorbing boundary on $\mathcal{C}_{y_1}^+$ —are comparable, by Harnack’s inequality. The separation lemma ensures that this comparability holds for two-arm probabilities, namely, $\mathbb{P}(\mathbf{H}_{y'_1, x_-}^{y_1, x_+})$ is comparable to $\mathbb{P}(\mathbf{H}_{y'_1, z}^{y_1, x_+})$ for all z in ϵN -neighborhood of w_- . Repeating this argument several times yields that $\mathbb{P}(\mathbf{H}_{y'_1, x_-}^{y_1, x_+}) \asymp \mathbb{P}(\mathbf{H}_{y'_1, \hat{x}_-}^{y_1, \hat{x}_+}) = \mathbb{P}(\mathbf{H}_{y'_2, x_-}^{y_2, x_+})$.

Putting (2.3) and (2.4) together, we immediately obtain that

$$(2.5) \quad \mathbb{P}(\mathbf{H}_{v_-,x_-}^{v_+,x_+}) \asymp N^{-[(\frac{3d}{2}-1)\Box(2d-4)]},$$

which explains the exponent of N in Theorem 1.4 restricted to the case $n \asymp 1$. In fact, this estimate indicates that to accommodate two distinct sign clusters with diameter N from a pair of neighboring points, the energy cost is

$$(2.6) \quad \xi_N := \frac{\mathbb{P}(\mathbf{H}_{v_-,x_-}^{v_+,x_+})}{\mathbb{P}(v_+ \xrightarrow{\geq 0} x_+) \cdot \mathbb{P}(v_- \xrightarrow{\leq 0} x_-)} \asymp N^{(\frac{d}{2}-3)\Box 0}.$$

Combined with the estimates on $\theta_d(\cdot)$ in (1.4) and (1.6), it suggests that

$$(2.7) \quad \mathbb{P}(\mathbf{H}_{v_-}^{v_+}(N)) \asymp \xi_N \cdot [\theta_d(N)]^2 \asymp N^{-[(\frac{d}{2}+1)\Box 4]},$$

which matches the exponent of N in Theorem 1.1 for $\chi \asymp 1$.

Next, we discuss the exponent of χ in the cases $\chi \ll 1$ and $\chi \gg 1$ separately.

When $\chi \ll 1$. Arbitrarily take $v, v' \in I_{\{v_-, v_+\}}$ satisfying $|v - v'| = \chi$. For clarity, we assume that v is the point among v and v' that is closer to v_+ , and that neither v nor v' is too close to the endpoints $\{v_-, v_+\}$, i.e., $|v - v_\diamond| \wedge |v' - v_\diamond| \geq 1$ for $\diamond \in \{+, -\}$. With (2.7) at hand, deriving the exponent of χ in (1.14) reduces to proving that

$$(2.8) \quad \mathbb{P}(\mathbf{H}_{v'}^v(N) \mid \mathbf{H}_{v_-}^{v_+}(N)) \asymp \chi^{\frac{3}{2}}.$$

This bound follows from the following three observations:

- (i) Conditioned on the event $\mathbf{H}_{v_-}^{v_+}(N)$, the GFF values at v_+ and v_- are typically of order 1.
- (ii) Given the values of $\tilde{\phi}_{v_+}$ and $\tilde{\phi}_{v_-}$, which are of order 1 and satisfies $\tilde{\phi}_{v_+} > 0$ and $\tilde{\phi}_{v_-} < 0$, the conditional probability of $\{v \xrightarrow{\geq 0} v_+\} \cap \{v \xrightarrow{\geq 0} v'\}^c$ is proportional to χ .
- (iii) Under the same conditioning as in Item (ii) and given the positive cluster of v_+ (which reaches v and is separated from v' by distance $c\chi$), the conditional probability of $\{v' \xrightarrow{\leq 0} v_-\}$ is of order $\chi^{\frac{1}{2}}$.

Next, we explain Observations (i)-(iii) separately. For Observation (i), we recall from [4, Lemma 3.3] that conditioned on a point being contained in a large sign cluster, the GFF value of this point is typically of order 1. Applying the “rigidity lemma” mentioned below Theorem 1.5 (see Lemma 4.1), we show that this property is preserved under the conditioning of the two-arm event. For Observation (ii), note that given $\{\tilde{\phi}_v = a\}$ (where $a > 0$ is sufficiently small), the diameter of the positive cluster containing v is typically of order a^2 (here the square arises from the Brownian scaling relation, since the GFF restricted to an interval behaves as a Brownian bridge). Hence, the occurrence of $\{v \xrightarrow{\geq 0} v'\}^c$ imposes the constraint $0 < \tilde{\phi}_v \lesssim |v - v'|^{1/2} = \chi^{1/2}$, which occurs with probability $\asymp \chi^{1/2}$. Moreover,

given that $0 < \tilde{\phi}_v \lesssim \chi^{\frac{1}{2}}$ and $\tilde{\phi}_{v_+} \asymp 1$, the probability of $\{v \xleftrightarrow{\geq 0} v_+\}$ is proportional to $\chi^{\frac{1}{2}}$ (see (3.29)). Combining these two estimates, we obtain Observation (ii). For Observation (iii), under the zero boundary conditions imposed by the cluster $\mathcal{C}_{v_+}^+$ (whose distance to v' is $c\chi$), the variance of $\tilde{\phi}_{v'}$ is of order χ (see (3.6)). In other words, $|\tilde{\phi}_{v'}|$ typically takes values of order $\chi^{\frac{1}{2}}$. This together with $|\tilde{\phi}_{v_-}| \asymp 1$ (as ensured by Observation (i)) and the formula (3.29) yields that the conditional probability of $\{v' \xleftrightarrow{\leq 0} v_-\}$ is of order $\chi^{\frac{1}{2}}$.

The exponent of χ in (1.16) arises from the same mechanism, so we skip it.

When $\chi \gg 1$. As noted below [2, Theorem 1.2], the sign cluster is expected to exhibit self-similarity—its geometric properties remain invariant under scaling transformations. This indicates that the energy cost $\hat{\xi}_{\chi,N}$ of accommodating two clusters that cross the annulus $B(N) \setminus B(\chi)$ (where $N \gg \chi$) is a polynomial of N/χ , and has the same exponent as ξ_N . I.e.,

$$(2.9) \quad \hat{\xi}_{\chi,N} \asymp (N/\chi)^{(\frac{d}{2}-3)\square 0}.$$

For the same reasoning as in (2.7), it yields that for $v, v' \in \tilde{B}(cN)$ with $|v - v'| = \chi$,

$$(2.10) \quad \mathbb{P}(\mathbf{H}_{v'}^v(N)) \asymp \hat{\xi}_{\chi,N} \cdot [\theta_d(N)]^2 \asymp \chi^{(3-\frac{d}{2})\square 0} N^{-[(\frac{d}{2}+1)\square 4]},$$

which coincides with (1.15) in Theorem 1.1. For Theorem 1.4, the bound (1.17) can be derived similarly as follows: for any $w, w' \in \tilde{B}(10N) \setminus \tilde{B}(N)$ with $|w - w'| \geq N$,

$$(2.11) \quad \mathbb{P}(\mathbf{H}_{v',w'}^{v,w}) \asymp \hat{\xi}_{\chi,N} \cdot \mathbb{P}(v \xleftrightarrow{\geq 0} w) \cdot \mathbb{P}(v' \xleftrightarrow{\leq 0} w') \asymp \chi^{(3-\frac{d}{2})\square 0} N^{-[(\frac{3d}{2}-1)\square (2d-4)]}.$$

3. PRELIMINARIES

In this section, we review some basic definitions and useful results.

3.1. Brownian motion on $\tilde{\mathbb{Z}}^d$. We recall the definition of the canonical Brownian motion on $\tilde{\mathbb{Z}}^d$ (denoted by $\{\tilde{S}_t\}_{t \geq 0}$) as follows. Precisely, within each edge interval I_e ($e \in \mathbb{L}^d$), the process $\tilde{S}$ behaves as a standard one-dimensional Brownian motion. Upon reaching a lattice point $x \in \mathbb{Z}^d$, it uniformly chooses an edge $e = \{x, y\}$ incident to x and performs a Brownian excursion along the interval I_e away from x . When the excursion hits a lattice point (either returning to x or reaching y), the process resumes from that lattice point, and continues in the same manner. We denote the transition density of this process by $\tilde{q}_t(v, w)$ for $t > 0$ and $v, w \in \tilde{\mathbb{Z}}^d$. When $\tilde{S}$ starts from $v \in \tilde{\mathbb{Z}}^d$, we denote its law by $\tilde{\mathbb{P}}_v$. Let $\tilde{\mathbb{E}}_v$ represent the expectation under $\tilde{\mathbb{P}}_v$.

For any $D \subset \tilde{\mathbb{Z}}^d$, we denote the hitting time $\tau_D := \inf\{t \geq 0 : \tilde{S}_t \in D\}$. For completeness, we set $\tau_\emptyset = \inf \emptyset := +\infty$. When $D = \{v\}$ for some $v \in \tilde{\mathbb{Z}}^d$, we omit the braces and write $\tau_v := \tau_{\{v\}}$. Moreover, we may also use $\tau_D(\tilde{\eta})$ to denote the hitting time for a given path $\tilde{\eta}$ on $\tilde{\mathbb{Z}}^d$.

(P.S. Unless otherwise specified, when we use the letter D to denote a subset of $\tilde{\mathbb{Z}}^d$, we assume that it consists of finitely many compact connected components. Since $\tilde{\mathbb{Z}}^d$ is locally one-dimensional, the boundary of any $D \subset \tilde{\mathbb{Z}}^d$, defined by $\tilde{\partial}D := \{v \in D : \inf_{w \in \tilde{\mathbb{Z}}^d \setminus D} |v - w| = 0\}$, is a countable set.)

Similar to [5, Lemma 2.1], when $v, v', w \in \tilde{\mathbb{Z}}^d$ satisfy $|v - v'| \geq |v - w|$, for the Brownian motion starting from v , imposing an absorbing point at v' reduces the hitting probability of w by a factor depending only on d . I.e.,

$$(3.1) \quad \tilde{\mathbb{P}}_v(\tau_w < \tau_{v'}) \asymp \tilde{\mathbb{P}}_v(\tau_w < \infty) \asymp (|v - w| + 1)^{2-d}.$$

Green's function with absorbing boundary. For any $D \subset \tilde{\mathbb{Z}}^d$, the Green's function for D is defined by

$$(3.2) \quad \tilde{G}_D(v, w) := \int_0^\infty \left\{ \tilde{q}_t(v, w) - \tilde{\mathbb{E}}_v[\tilde{q}_{t-\tau_D}(\tilde{S}_{\tau_D}, w) \cdot \mathbb{1}_{\tau_D < t}] \right\} dt, \quad \forall v, w \in \tilde{\mathbb{Z}}^d.$$

Note that $\tilde{G}_D(v, w)$ is finite, symmetric, continuous, and is decreasing in D . Moreover, it equals zero when v or w belongs to D . When $D = \emptyset$, $\tilde{G}_\emptyset(\cdot, \cdot)$ is identical to $\tilde{G}(\cdot, \cdot)$ defined in (1.2). The strong Markov property of $\tilde{S}$ implies that

$$(3.3) \quad \tilde{G}_D(v, w) = \tilde{\mathbb{P}}_v(\tau_w < \tau_D) \cdot \tilde{G}_D(w, w).$$

Moreover, for any $D_1 \subset D_2 \subset \tilde{\mathbb{Z}}^d$ and $v, w \in \tilde{\mathbb{Z}}^d \setminus D_2$, one has

$$(3.4) \quad \tilde{G}_{D_1}(v, w) - \tilde{G}_{D_2}(v, w) = \sum_{z \in (\tilde{\partial}D_2) \setminus D_1} \tilde{\mathbb{P}}_v(\tau_{D_2} = \tau_z < \infty) \cdot \tilde{G}_{D_1}(z, w).$$

For any $D, D' \subset \tilde{\mathbb{Z}}^d$, let $\text{dist}(D, D') := \inf_{v \in D, v' \in D'} |v - v'|$ denote the graph distance between D and D' (where we may omit the braces when D or D' is a singleton $\{v\}$ for some $v \in \tilde{\mathbb{Z}}^d$). It follows from (3.4) that

$$(3.5) \quad \tilde{G}_{D_1}(v, w) - \tilde{G}_{D_2}(v, w) \lesssim [\text{dist}(w, D_2 \setminus D_1) + 1]^{2-d}.$$

In addition, it can be derived using the potential theory of Brownian motion that

$$(3.6) \quad \tilde{G}_D(v, v) \asymp \text{dist}(v, D) \wedge 1.$$

Boundary excursion kernel. For any $D \subset \tilde{\mathbb{Z}}^d$, the boundary excursion kernel for D is defined by

$$(3.7) \quad \mathbb{K}_D(v, w) := \lim_{\epsilon \downarrow 0} (2\epsilon)^{-1} \sum_{v' \in \tilde{\mathbb{Z}}^d: |v-v'|=\epsilon} \tilde{\mathbb{P}}_{v'}(\tau_D = \tau_w < \infty), \quad \forall v \neq w \in \tilde{\partial}D.$$

It follows from the time-reversal symmetry of Brownian motion that $\mathbb{K}_D(\cdot, \cdot)$ is symmetric, i.e., $\mathbb{K}_D(v, w) = \mathbb{K}_D(w, v)$. In addition, $\mathbb{K}_D(\cdot, \cdot)$ is decreasing in D , i.e., for any $D \subset D' \subset \tilde{\mathbb{Z}}^d$ and any $v, w \in (\tilde{\partial}D) \cap (\tilde{\partial}D')$, one has $\mathbb{K}_D(v, w) \geq \mathbb{K}_{D'}(v, w)$. As pointed out in [30, Section 2], $\mathbb{K}_D(\cdot, \cdot)$ can be viewed as the equivalent effective conductance of the graph $\tilde{\mathbb{Z}}^d \setminus D$. An excellent introduction to effective conductance on graphs can be found in [31, Section 2].

For any $v, w \in I_e$, let $I_{[v,w]}$ denote the subinterval of I_e with endpoints v and w . For any $z \in I_{[v,w]}$, the optional stopping theorem (see e.g., [4, (2.19)]) implies that

$$(3.8) \quad \tilde{\mathbb{P}}_z(\tau_v < \tau_w) = |z - w|/|v - w|.$$

This immediately yields the equivalent effective conductance of a line segment:

$$(3.9) \quad \mathbb{K}_{\tilde{\mathbb{Z}}^d \setminus I_{[v,w]}}(v, w) = \lim_{\epsilon \downarrow 0} (2\epsilon)^{-1} \cdot \frac{\epsilon}{|v-w|} = (2|v-w|)^{-1}.$$

Consequently, for any disjoint subintervals $\{I_{[x,w_i]}\}_{1 \leq i \leq k}$ (all incident to x),

$$(3.10) \quad \tilde{\mathbb{P}}_x(\tau_{\{w_i\}_{1 \leq i \leq k}} = \tau_{w_j}) = \frac{|x-w_j|^{-1}}{\sum_{1 \leq i \leq k} |x-w_i|^{-1}}, \quad \forall 1 \leq j \leq k.$$

Lemma 3.1. *For any $d \geq 3$ and $v \neq w \in \tilde{\mathbb{Z}}^d$ with $|v - w| \geq 10d$,*

$$(3.11) \quad \mathbb{K}_{\{v,w\}}(v, w) \asymp |v - w|^{2-d}.$$

Proof. We divide the proof into two cases, according to whether v is a lattice point.

When $v \in \mathbb{Z}^d$. Using the identity (3.8), we have

$$(3.12) \quad \mathbb{K}_{\{v,w\}}(v, w) = (2d)^{-1} \sum_{y \in \mathbb{Z}^d: \{y,v\} \in \mathbb{L}^d} \tilde{\mathbb{P}}_y(\tau_w < \tau_v) \stackrel{(3.1)}{\asymp} |v - w|^{2-d}.$$

When $v \notin \mathbb{Z}^d$. Let $I_{\{x_1, x_2\}}$ denote the interval that contains v . Without loss of generality, we assume that $|v - x_1| \geq |v - x_2|$ (which implies $|v - x_1| \geq \frac{d}{2}$). Applying the identity (3.8), we have

$$(3.13) \quad \begin{aligned} 0 &\leq \mathbb{K}_{\{v,w\}}(v, w) - (2|v - x_1|)^{-1} \cdot \tilde{\mathbb{P}}_{x_1}(\tau_w < \tau_v) \\ &= (2|v - x_2|)^{-1} \cdot \tilde{\mathbb{P}}_{x_2}(\tau_w < \tau_v) \\ &\stackrel{(3.10)}{=} (2|v - x_2|)^{-1} \cdot \frac{\sum_{z \in \mathbb{Z}^d: \{z, x_2\} \in \mathbb{L}^d, z \neq x_1} d^{-1} \tilde{\mathbb{P}}_z(\tau_w < \tau_v)}{(2d-1) \cdot d^{-1} + |v - x_2|^{-1}} \stackrel{(3.1)}{\asymp} |v - w|^{2-d}. \end{aligned}$$

Moreover, by $|v - x_1| \geq \frac{d}{2}$ and (3.1), one has

$$(3.14) \quad (2|v - x_1|)^{-1} \cdot \tilde{\mathbb{P}}_{x_1}(\tau_w < \tau_v) \stackrel{(3.1)}{\asymp} |v - w|^{2-d}.$$

Combining (3.13) and (3.14), we get (3.11). $\square$

Lemma 3.2. *For any $d \geq 3$, $D \subset \tilde{\mathbb{Z}}^d$ and $v, w \in \tilde{\mathbb{Z}}^d \setminus D$ with $|v - w| \geq d$,*

$$(3.15) \quad \mathbb{K}_{D \cup \{v,w\}}(v, w) \asymp [\tilde{G}_D(w, w)]^{-1} \cdot \tilde{\mathbb{P}}_w(\tau_v < \tau_D).$$

Proof. We denote by F the collection of lattice points $y \in \mathbb{Z}^d$ such that one of the intervals incident to y contains w . Let $\hat{F} := \{y \in F : D \cap I_{[y,w]} = \emptyset\}$. Applying the identity (3.8), we have

$$(3.16) \quad \mathbb{K}_{D \cup \{v,w\}}(v, w) = \sum_{y \in \hat{F}} (2|y - w|)^{-1} \cdot \tilde{\mathbb{P}}_y(\tau_v < \tau_{D \cup \{w\}}).$$

Meanwhile, using the strong Markov property of Brownian motion, one has

$$(3.17) \quad \begin{aligned} \tilde{\mathbb{P}}_y(\tau_v < \tau_D) &= \tilde{\mathbb{P}}_y(\tau_v < \tau_{D \cup \{w\}}) + \tilde{\mathbb{P}}_y(\tau_w < \tau_v < \tau_D) \\ &= \tilde{\mathbb{P}}_y(\tau_v < \tau_{D \cup \{w\}}) + \tilde{\mathbb{P}}_y(\tau_w < \tau_{D \cup \{v\}}) \cdot \tilde{\mathbb{P}}_w(\tau_v < \tau_D), \end{aligned}$$

$$(3.18) \quad \tilde{\mathbb{P}}_w(\tau_v < \tau_D) = \mathbb{J}^{-1} \sum_{y \in \hat{F}} (2|y - w|)^{-1} \cdot \tilde{\mathbb{P}}_y(\tau_v < \tau_D),$$

where the quantity $\mathbb{J}$ is defined by

$$(3.19) \quad \mathbb{J} := \sum_{y \in \hat{F}} (2|y - w|)^{-1} + \sum_{y \in F \setminus \hat{F}} [2 \cdot \text{dist}(w, D \cap I_{[y, w]})]^{-1}.$$

Note that (3.6) implies

$$(3.20) \quad \mathbb{J} \asymp [\text{dist}(w, D) \wedge 1]^{-1} \asymp [\tilde{G}_D(w, w)]^{-1}.$$

Combining (3.16), (3.17) and (3.18), we have

$$(3.21) \quad \begin{aligned} \mathbb{K}_{D \cup \{v, w\}}(v, w) &\stackrel{(3.16), (3.17)}{=} \sum_{y \in \hat{F}} (2|y - w|)^{-1} \cdot \tilde{\mathbb{P}}_y(\tau_v < \tau_D) \\ &\quad - \tilde{\mathbb{P}}_w(\tau_v < \tau_D) \sum_{y \in \hat{F}} \frac{\tilde{\mathbb{P}}_y(\tau_w < \tau_{D \cup \{v\}})}{2|y - w|} \\ &\stackrel{(3.18)}{=} \left[\mathbb{J} - \sum_{y \in \hat{F}} \frac{\tilde{\mathbb{P}}_y(\tau_w < \tau_{D \cup \{v\}})}{2|y - w|} \right] \cdot \tilde{\mathbb{P}}_w(\tau_v < \tau_D). \end{aligned}$$

For the upper bound in (3.15), using (3.20) and (3.21), we obtain

$$(3.22) \quad \mathbb{K}_{D \cup \{v, w\}}(v, w) \stackrel{(3.21)}{\leq} \mathbb{J} \cdot \tilde{\mathbb{P}}_w(\tau_v < \tau_D) \stackrel{(3.20)}{\asymp} [\tilde{G}_D(w, w)]^{-1} \cdot \tilde{\mathbb{P}}_w(\tau_v < \tau_D).$$

For the lower bound, since $1 - \tilde{\mathbb{P}}_y(\tau_w < \tau_{D \cup \{v\}}) \geq \tilde{\mathbb{P}}_y(\tau_w = \infty) \asymp |y - w|$, it follows from (3.20) and (3.21) that

$$(3.23) \quad \begin{aligned} \mathbb{K}_{D \cup \{v, w\}}(v, w) &\stackrel{(3.21)}{=} \left(\sum_{y \in \hat{F}} (2|y - w|)^{-1} \cdot [1 - \tilde{\mathbb{P}}_y(\tau_w < \tau_{D \cup \{v\}})] \right. \\ &\quad \left. + \sum_{y \in F \setminus \hat{F}} [2 \cdot \text{dist}(w, D \cap I_{[y, w]})]^{-1} \right) \cdot \tilde{\mathbb{P}}_w(\tau_v < \tau_D) \\ &\geq \left(c|\hat{F}| + \sum_{y \in F \setminus \hat{F}} [2 \cdot \text{dist}(w, D \cap I_{[y, w]})]^{-1} \right) \cdot \tilde{\mathbb{P}}_w(\tau_v < \tau_D) \\ &\gtrsim [\text{dist}(w, D) \wedge 1]^{-1} \cdot \tilde{\mathbb{P}}_w(\tau_v < \tau_D) \\ &\stackrel{(3.6)}{\asymp} [\tilde{G}_D(w, w)]^{-1} \cdot \tilde{\mathbb{P}}_w(\tau_v < \tau_D). \end{aligned}$$

By (3.22) and (3.23), we complete the proof of this lemma. $\square$

Capacity. The equilibrium measure for D is given by

$$(3.24) \quad \mathbb{Q}_D(v) := \lim_{N \rightarrow \infty} \sum_{w \in \partial B(N)} \mathbb{K}_{D \cup \partial B(N)}(v, w), \quad \forall v \in \tilde{\partial} D.$$

The capacity of D is defined as the total mass of the equilibrium measure, i.e.,

$$(3.25) \quad \text{cap}(D) := \sum_{v \in \tilde{\partial} D} \mathbb{Q}_D(v).$$

It is well-known that $\text{cap}(D)$ is increasing in D (i.e., $\text{cap}(D_1) \leq \text{cap}(D_2)$ if $D_1 \subset D_2$), sub-additive (i.e., $\text{cap}(D_1 \cup D_2) \leq \text{cap}(D_1) + \text{cap}(D_2)$ for all $D_1, D_2 \subset \tilde{\mathbb{Z}}^d$), and satisfies $\text{cap}(\tilde{B}(N)) \asymp N^{d-2}$ for $N \geq 1$. Another key property is its relation to the hitting probability: for any $D \subset \tilde{B}(N)$, $x \in \partial B(2N)$ and $D' \subset [\tilde{B}(3N)]^c$,

$$(3.26) \quad \tilde{\mathbb{P}}_x(\tau_D < \tau_{D'}) \asymp N^{2-d} \cdot \text{cap}(D).$$

3.2. Properties of GFFs. For any $D \subset \tilde{\mathbb{Z}}^d$, we denote by $\mathbb{P}^D$ the law of $\{\tilde{\phi}_v\}_{v \in \tilde{\mathbb{Z}}^d}$ with zero boundary conditions on D , i.e., conditioned on $\tilde{\phi}_v = 0$ for all $v \in D$. We denote by $\mathbb{E}^D$ the expectation under $\mathbb{P}^D$. Clearly, the GFF $\tilde{\phi} \sim \mathbb{P}^D$ is mean-zero. In addition, its covariance is given by the Green's function with absorbing boundary on D , i.e.,

$$(3.27) \quad \mathbb{E}^D[\tilde{\phi}_v \tilde{\phi}_w] = \tilde{G}_D(v, w), \quad \forall v, w \in \tilde{\mathbb{Z}}^d.$$

The formula in (1.3) for the two-point function also works under zero boundary conditions. To be precise, for any $D \subset \tilde{\mathbb{Z}}^d$ and any distinct $v, w \in \tilde{\mathbb{Z}}^d \setminus D$,

$$(3.28) \quad \begin{aligned} \mathbb{P}^D(v \xleftrightarrow{\geq 0} w) &= \pi^{-1} \arcsin \left(\frac{\tilde{G}_D(v, w)}{\sqrt{\tilde{G}_D(v, v) \tilde{G}_D(w, w)}} \right) \\ &\asymp \frac{\tilde{G}_D(v, w)}{\sqrt{\tilde{G}_D(v, v) \tilde{G}_D(w, w)}} = \tilde{\mathbb{P}}_v(\tau_w < \tau_D) \sqrt{\frac{\tilde{G}_D(w, w)}{\tilde{G}_D(v, v)}}. \end{aligned}$$

The powerful formula of the conditional two-point function in [30, (18)] states that

$$(3.29) \quad \mathbb{P}^D(v \xleftrightarrow{\geq 0} w \mid \tilde{\phi}_v = a, \tilde{\phi}_w = b) = 1 - e^{-2ab \cdot \mathbb{K}_{D \cup \{v, w\}}(v, w)}, \quad \forall a, b \geq 0.$$

As shown in the following lemma, the difference between the two-point functions under different zero boundary conditions can be approximated by the corresponding difference of Green's functions.

Lemma 3.3. *For $d \geq 3$, $n \geq 1$, $D \subset \tilde{\mathbb{Z}}^d$, and $v, w \in \tilde{\mathbb{Z}}^d$ with $\text{dist}(D, \{v, w\}) \geq n$,*

$$(3.30) \quad |\mathbb{P}(v \xleftrightarrow{\geq 0} w) - \mathbb{P}^D(v \xleftrightarrow{\geq 0} w)| \lesssim \tilde{G}(v, w) - \tilde{G}_D(v, w) + n^{2-d} \cdot \mathbb{P}(v \xleftrightarrow{\geq 0} w).$$

Proof. By applying (3.28), we have

$$(3.31) \quad \begin{aligned} |\mathbb{P}(v \xleftrightarrow{\geq 0} w) - \mathbb{P}^D(v \xleftrightarrow{\geq 0} w)| &\asymp \left| \frac{\tilde{G}(v, w)}{\sqrt{\tilde{G}(v, v) \tilde{G}(w, w)}} - \frac{\tilde{G}_D(v, w)}{\sqrt{\tilde{G}_D(v, v) \tilde{G}_D(w, w)}} \right| \\ &\leq \mathbb{I}_1 + \mathbb{I}_2 + \mathbb{I}_3 := \frac{\tilde{G}(v, w) - \tilde{G}_D(v, w)}{\sqrt{\tilde{G}(v, v) \tilde{G}(w, w)}} + \frac{\tilde{G}_D(v, w)}{\sqrt{\tilde{G}(w, w)}} \cdot \left(\frac{1}{\sqrt{\tilde{G}_D(v, v)}} - \frac{1}{\sqrt{\tilde{G}(v, v)}} \right) \\ &\quad + \frac{\tilde{G}_D(v, w)}{\sqrt{\tilde{G}_D(v, v)}} \cdot \left(\frac{1}{\sqrt{\tilde{G}_D(w, w)}} - \frac{1}{\sqrt{\tilde{G}(w, w)}} \right). \end{aligned}$$

Note that $\tilde{G}(v, v)$, $\tilde{G}_D(v, v)$, $\tilde{G}(w, w)$ and $\tilde{G}_D(w, w)$ are all of order 1. Thus,

$$(3.32) \quad \mathbb{I}_1 \asymp \tilde{G}(v, w) - \tilde{G}_D(v, w).$$

Meanwhile, since $\tilde{G}_D(v, w) \leq \tilde{G}(v, w)$, we have

$$(3.33) \quad \mathbb{I}_2 = \frac{\tilde{G}_D(v, w) \cdot [\tilde{G}(v, v) - \tilde{G}_D(v, v)]}{\sqrt{\tilde{G}(w, w) \tilde{G}_D(v, v) \tilde{G}(v, v)} \cdot [\sqrt{\tilde{G}_D(v, v)} + \sqrt{\tilde{G}(v, v)}]} \stackrel{(3.5), (3.28)}{\lesssim} n^{2-d} \cdot \mathbb{P}(v \stackrel{\geq 0}{\longleftrightarrow} w).$$

For the same reason, this bound also holds for $\mathbb{I}_3$. Inserting these estimates on $\mathbb{I}_1$, $\mathbb{I}_2$ and $\mathbb{I}_3$ into (3.31), we get the desired bound (3.30). $\square$

Sign clusters. For any $A \subset \tilde{\mathbb{Z}}^d$, let $\mathcal{C}_A^+ := \{v \in \tilde{\mathbb{Z}}^d : v \stackrel{\geq 0}{\longleftrightarrow} A\}$ (resp. $\mathcal{C}_A^- := \{v \in \tilde{\mathbb{Z}}^d : v \stackrel{\leq 0}{\longleftrightarrow} A\}$) denote the positive (resp. negative) cluster containing A . We then define the sign cluster $\mathcal{C}_A^\pm := \mathcal{C}_A^+ \cup \mathcal{C}_A^-$. Note that $\mathcal{C}_A^+ \cap \mathcal{C}_A^- = A$. For $\diamond \in \{+, -, \pm\}$, we denote by $\mathcal{F}_{\mathcal{C}_A^\diamond}$ the σ -field generated by $\mathcal{C}_A^\diamond$ and all GFF values on $\mathcal{C}_A^\diamond$.

We record the following corollary of the strong Markov property of $\tilde{\phi} \sim \mathbb{P}^D$ (see e.g., [8, Theorem 8]), which we will use multiple times. Suppose that $\diamond \in \{+, -, \pm\}$ and $A \subset D' \subset \tilde{\mathbb{Z}}^d \setminus D$. Conditioned on $\mathcal{F}_{\mathcal{C}_A^\diamond}$, on the event $\{\mathcal{C}_A^\diamond = D'\}$, one has

$$(3.34) \quad \{\tilde{\phi}_v\}_{v \in \tilde{\mathbb{Z}}^d \setminus (D \cup D')} \stackrel{d}{=} \{\tilde{\phi}'_v + \mathcal{H}_v(D \cup D')\}_{v \in \tilde{\mathbb{Z}}^d \setminus (D \cup D')}.$$

Here $\tilde{\phi}'$ is distributed under $\mathbb{P}^{D \cup D'}$, and $\mathcal{H}_v(D \cup D')$ is the harmonic function on $\tilde{\mathbb{Z}}^d \setminus (D \cup D')$ with boundary conditions $\{\tilde{\phi}_v\}_{v \in D \cup D'}$. Moreover, by the continuity of $\tilde{\phi}$, one has $\tilde{\phi}_v = 0$ for all $v \in (\partial D') \setminus A$, which yields that

$$(3.35) \quad \mathcal{H}_v(D \cup D') := \sum_{w \in (\partial D') \cap A} \tilde{\mathbb{P}}_v(\tau_{D \cup D'} = \tau_w < \infty) \cdot \tilde{\phi}_w.$$

In particular, when A is a singleton or $\diamond = \pm$, the continuity of $\tilde{\phi}$ also implies that $\tilde{\phi}$ vanishes on $\partial \mathcal{C}_A^\diamond$, and thus $\mathcal{H}_v(D \cup D') = 0$ for all $v \in \tilde{\mathbb{Z}}^d \setminus (D \cup D')$.

Regularity of connecting probabilities. [5] established a series of regularity properties for the connecting probabilities of $\tilde{E}^{\geq 0}$. In what follows, we review some of these properties that are useful for later proofs. The first property states that Harnack's inequality holds for the point-to-set connecting probabilities.

Lemma 3.4 ([5, Proposition 1.6]). *For any $d \geq 3$ with $d \neq 6$, there exists $C > 0$ such that for any $N \geq 1$, $v, w \in \tilde{B}(N)$ and $A, D \subset [\tilde{B}(CN)]^c$,*

$$(3.36) \quad \mathbb{P}^D(v \stackrel{\geq 0}{\longleftrightarrow} A) \asymp \mathbb{P}^D(w \stackrel{\geq 0}{\longleftrightarrow} A).$$

Remark 3.5. In the original statement of [5, Proposition 1.6], the points v, w were required to lie in the lattice. In Lemma 3.4, however, we allow v, w to be located in the interior of some interval of $\tilde{\mathbb{Z}}^d$. This extension is valid since for any $v \in I_{\{x, y\}}$, it follows from a union bound that $\mathbb{P}^D(v \stackrel{\geq 0}{\longleftrightarrow} A)$ is bounded from above by a linear combination of $\mathbb{P}^D(x \stackrel{\geq 0}{\longleftrightarrow} A)$ and $\mathbb{P}^D(y \stackrel{\geq 0}{\longleftrightarrow} A)$; meanwhile, a direct application of

the FKG inequality yields that $\mathbb{P}^D(v \xrightarrow{\geq 0} A)$ is at least proportional to $\mathbb{P}^D(x \xrightarrow{\geq 0} A)$ (see e.g., [5, (2.26)]). Similarly, such an extension also applies to [5, Lemma 5.3].

Let $\|\cdot\|$ denote the Euclidean norm on $\mathbb{Z}^d$. For any $v \in \tilde{\mathbb{Z}}^d$ and $R \geq 1$, we denote the Euclidean ball $\mathcal{B}_v(R) := \{y \in \mathbb{Z}^d : \|\bar{v} - y\| \leq R\}$ (recalling that $\bar{v}$ represents the closest lattice point to v). Similarly, we denote $B_v(R) := \bar{v} + B(R)$ and $\tilde{B}_v(R) := \bar{v} + \tilde{B}(R)$. When $x = \mathbf{0}$, we may omit the subscript $\mathbf{0}$. Referring to [5, (5.2)], we have the following quantitative relation between the point-to-set and boundary-to-set connection probabilities.

Lemma 3.6. *We denote $F_1 := [\tilde{B}(10d^2N)]^c$ and $F_{-1} := \tilde{B}(\frac{N}{10d^2})$. For any $d \geq 3$ with $d \neq 6$, $D \subset \tilde{\mathbb{Z}}^d$, $N \geq 1$, $i \in \{1, -1\}$, and $v \in (\tilde{\mathbb{Z}}^d \setminus D) \cap F_i$,*

$$(3.37) \quad \mathbb{P}^D(v \xrightarrow{\geq 0} \partial B(N)) \lesssim N^{-(\frac{d}{2} \square 3)} \sum_{x \in \partial \mathcal{B}(d^i N)} \mathbb{P}^D(v \xrightarrow{\geq 0} x).$$

The loop decomposition argument developed in [5, Section 5.2] provides a series of upper bounds on the connecting probability between multiple sets. We recall two basic applications of these bounds as follows.

Lemma 3.7 ([5, Lemma 5.3]). *For any $3 \leq d \leq 5$, there exists $C > 0$ such that for any $M \geq 1$, $v, w \in \tilde{B}(M)$ and $A, D \subset [\tilde{B}(CM)]^c$,*

$$(3.38) \quad \mathbb{P}^D(A \xrightarrow{\geq 0} v, A \xrightarrow{\geq 0} w) \lesssim (|v - w| + 1)^{-\frac{d}{2}+1} \mathbb{P}^D(A \xrightarrow{\geq 0} v).$$

Lemma 3.8. *For any $3 \leq d \leq 5$, there exists $C > C' > 0$ such that for any $M \geq 1$, $N \geq CM$, $v \in \tilde{B}(\frac{N}{C'}) \setminus \tilde{B}(M)$ and $D \subset [\tilde{B}(M)]^c$,*

$$(3.39) \quad \mathbb{P}^D(\mathbf{0} \xrightarrow{\geq 0} \partial B(N), v \xrightarrow{\geq 0} \partial B(N)) \lesssim M^{-\frac{d}{2}+1} N^{-\frac{d}{2}} \cdot \sum_{x \in \partial \mathcal{B}(dN)} \mathbb{P}^D(v \xrightarrow{\geq 0} x).$$

Proof. Applying [5, Lemma 5.2], one has

$$(3.40) \quad \begin{aligned} & \mathbb{P}^D(\mathbf{0} \xrightarrow{\geq 0} \partial B(N), v \xrightarrow{\geq 0} \partial B(N)) \\ & \lesssim \mathbb{P}^D(\mathbf{0} \xrightarrow{\geq 0} \partial B(cM)) \left[\mathbb{P}^D(v \xrightarrow{\geq 0} \partial B(N)) \right. \\ & \quad \left. + \mathbb{P}^D(\partial B(cM) \xrightarrow{\geq 0} \partial B(N)) \mathbb{P}^D(v \xrightarrow{\geq 0} \partial B(cM)) \right]. \end{aligned}$$

Referring to [5, Proposition 1.8], imposing zero boundary conditions on D changes the probability of $\{\mathbf{0} \xrightarrow{\geq 0} \partial B(cM)\}$ by only a constant factor. Hence,

$$(3.41) \quad \mathbb{P}^D(\mathbf{0} \xrightarrow{\geq 0} \partial B(cM)) \asymp \theta_d(cM) \stackrel{(1.4)}{\asymp} M^{-\frac{d}{2}+1}.$$

Meanwhile, it follows from Lemma 3.6 that for any $z \in \tilde{B}(\frac{N}{C'})$,

$$(3.42) \quad \mathbb{P}^D(z \xrightarrow{\geq 0} \partial B(N)) \lesssim N^{-\frac{d}{2}} \cdot \sum_{x \in \partial \mathcal{B}(dN)} \mathbb{P}^D(z \xrightarrow{\geq 0} x).$$

By (3.40), (3.41) and (3.42) (with $z = v$), it remains to show that

$$(3.43) \quad \begin{aligned} & \mathbb{P}^D(\partial B(cM) \xrightarrow{\geq 0} \partial B(N)) \mathbb{P}^D(v \xrightarrow{\geq 0} \partial B(cM)) \\ & \lesssim N^{-\frac{d}{2}} \cdot \sum_{x \in \partial \mathcal{B}(dN)} \mathbb{P}^D(v \xrightarrow{\geq 0} x). \end{aligned}$$

By the relation between the point-to-set and boundary-to-set connecting probabilities in [5, Proposition 1.9], one has

$$(3.44) \quad \mathbb{P}^D(\partial B(cM) \xrightarrow{\geq 0} \partial B(N)) \asymp M^{\frac{d}{2}-1} \cdot \mathbb{P}^D(\mathbf{0} \xrightarrow{\geq 0} \partial B(N)),$$

$$(3.45) \quad \mathbb{P}^D(v \xrightarrow{\geq 0} \partial B(cM)) \asymp M^{\frac{d}{2}-1} \cdot \mathbb{P}^D(\mathbf{0} \xrightarrow{\geq 0} v).$$

By (3.42) (with $z = \mathbf{0}$), (3.44) and (3.45), we have

$$(3.46) \quad \begin{aligned} & \mathbb{P}^D(\partial B(cM) \xrightarrow{\geq 0} \partial B(N)) \mathbb{P}^D(v \xrightarrow{\geq 0} \partial B(cM)) \\ & \lesssim M^{d-2} N^{-\frac{d}{2}} \cdot \sum_{x \in \partial \mathcal{B}(dN)} \mathbb{P}^D(\mathbf{0} \xrightarrow{\geq 0} x) \cdot \mathbb{P}^D(\mathbf{0} \xrightarrow{\geq 0} v). \end{aligned}$$

For each $x \in \partial \mathcal{B}(dN)$, using (3.28) and the fact that $\tilde{G}_D(\mathbf{0}, \mathbf{0}) \asymp 1$, one has

$$(3.47) \quad \mathbb{P}^D(\mathbf{0} \xrightarrow{\geq 0} x) \cdot \mathbb{P}^D(\mathbf{0} \xrightarrow{\geq 0} v) \asymp \frac{\tilde{G}_D(\mathbf{0}, x) \tilde{G}_D(\mathbf{0}, v)}{\sqrt{\tilde{G}_D(x, x) \tilde{G}_D(v, v)}}.$$

In addition, by the strong Markov property of Brownian motion, we have

$$(3.48) \quad \begin{aligned} \tilde{\mathbb{P}}_x(\tau_v < \tau_D) & \geq \tilde{\mathbb{P}}_x(\tau_{B(M/2)} < \tau_D) \cdot \min_{w \in \partial B(M/2)} \tilde{\mathbb{P}}_w(\tau_v < \tau_D) \\ & \geq \frac{\tilde{\mathbb{P}}_x(\tau_0 < \tau_D)}{\max_{w' \in \partial B(M/2)} \tilde{\mathbb{P}}_{w'}(\tau_0 < \infty)} \cdot \min_{w \in \partial B(M/2)} \tilde{\mathbb{P}}_w(\tau_v < \tau_D) \\ & \asymp M^{d-2} \cdot \tilde{\mathbb{P}}_x(\tau_0 < \tau_D) \cdot \tilde{\mathbb{P}}_{\mathbf{0}}(\tau_v < \tau_D), \end{aligned}$$

where in the last line we used (3.26) and Harnack's inequality. Therefore,

$$(3.49) \quad \begin{aligned} \tilde{G}_D(\mathbf{0}, x) \tilde{G}_D(\mathbf{0}, v) & \lesssim \tilde{\mathbb{P}}_x(\tau_0 < \tau_D) \cdot \tilde{\mathbb{P}}_{\mathbf{0}}(\tau_v < \tau_D) \cdot \tilde{G}_D(v, v) \\ & \stackrel{(3.48)}{\lesssim} M^{2-d} \cdot \tilde{\mathbb{P}}_x(\tau_v < \tau_D) \cdot \tilde{G}_D(v, v) = M^{2-d} \cdot \tilde{G}_D(x, v). \end{aligned}$$

Plugging (3.49) into (3.47), and then using (3.28), we get

$$(3.50) \quad \mathbb{P}^D(\mathbf{0} \xrightarrow{\geq 0} x) \cdot \mathbb{P}^D(\mathbf{0} \xrightarrow{\geq 0} v) \lesssim M^{2-d} \cdot \mathbb{P}^D(v \xrightarrow{\geq 0} x).$$

Combined with (3.46), it yields (3.43), and hence completes the proof. $\square$

The following two technical lemmas will be used several times in our application of the second moment method. Since their proofs involve numerous computations and are relatively standard, we defer them to Appendices A and B.

Lemma 3.9. For $d \geq 3$ with $d \neq 6$, there exists $C > 0$ such that for any $M \geq 1$, $N \geq CM$, $y \in \partial B(N)$ and $z \in \partial B_y(4M)$,

$$(3.51) \quad \sum_{x_1, x_2 \in B_y(M)} \tilde{\mathbb{P}}_z(\tau_{x_1} < \infty, \tau_{x_2} < \infty) \cdot \mathbb{P}(\mathbf{0} \xrightarrow{\geq 0} x_1, \mathbf{0} \xrightarrow{\geq 0} x_2) \lesssim N^{2-d} M^{(5-\frac{d}{2})\square^2}.$$

Lemma 3.10. For $d \geq 7$, there exists $C > 0$ such that for any $M \geq 1$, $N \geq CM$ and $y \in \partial B(N)$,

$$(3.52) \quad \max_{x_1 \in B_y(M)} \sum_{x_2 \in B_y(M)} \mathbb{P}(\mathbf{0} \xrightarrow{\geq 0} x_1, \mathbf{0} \xrightarrow{\geq 0} x_2) \lesssim M^4 N^{2-d}.$$

In the high-dimensional case (i.e., $d \geq 7$), it follows from (1.12) that with high probability, the volume of a large sign cluster within a box of size M is at least cM^4 . Next, we present an analogue under a different conditioning, where $\{\mathbf{0} \xrightarrow{\geq 0} \partial B(N)\}$ is replaced by the event that two distant points are connected. As the proof closely follows the argument in [2, Section 5.1], we defer it to Appendix C.

Lemma 3.11. Assume that d, M, N, y satisfy all conditions in Lemma 3.10. For any $\epsilon > 0$, there exists $c(d, \epsilon) > 0$ such that

$$(3.53) \quad \mathbb{P}(\mathcal{V}_y^+(M) \leq cM^4 \mid \mathbf{0} \xrightarrow{\geq 0} y) \leq \epsilon.$$

For any $v \in \tilde{\mathbb{Z}}^d$ and $r > 0$, let $\mathbf{B}_v(r) := \{w \in \tilde{\mathbb{Z}}^d : |v - w| \leq r\}$ denote the ball of radius r centered at v under the graph distance on $\tilde{\mathbb{Z}}^d$.

The subsequent result shows that the two-point function satisfies the maximum principle, up to certain correction factors.

Lemma 3.12. For any $d \geq 3$, $D \subset \tilde{\mathbb{Z}}^d$, $n \geq 10$ and $v, w \in \tilde{\mathbb{Z}}^d \setminus D$ with $|v - w| \geq 2n$,

$$(3.54) \quad \mathbb{P}^D(v \xrightarrow{\geq 0} w) \lesssim (\text{dist}(v, D) \wedge 1)^{\frac{1}{2}} \cdot |\partial \mathcal{B}_v(n)|^{-1} \sum_{z \in \partial \mathcal{B}_v(n)} \mathbb{P}^D(z \xrightarrow{\geq 0} w).$$

Proof. By the strong Markov property of $\tilde{S}$, $\tilde{\mathbb{P}}_v(\tau_w < \tau_D)$ is upper-bounded by

$$(3.55) \quad \tilde{\mathbb{P}}_v(\tau_{\tilde{\partial} \mathbf{B}_v(\frac{1}{2})} < \tau_D) \max_{y_1 \in \tilde{\partial} \mathbf{B}_v(\frac{1}{2})} \sum_{y_2 \in \partial \mathcal{B}_v(n)} \tilde{\mathbb{P}}_{y_1}(\tau_{\partial \mathcal{B}_v(n)} = \tau_{y_2}) \cdot \tilde{\mathbb{P}}_{y_2}(\tau_w < \tau_D).$$

Since D is compact, there exists $v_{\dagger} \in D$ such that $|v - v_{\dagger}| = \text{dist}(v, D)$. Thus, by $\tau_D \leq \tau_{v_{\dagger}}$ and the potential theory of Brownian motion, we have

$$(3.56) \quad \tilde{\mathbb{P}}_v(\tau_{\tilde{\partial} \mathbf{B}_v(\frac{1}{2})} < \tau_D) \leq \tilde{\mathbb{P}}_v(\tau_{\tilde{\partial} \mathbf{B}_v(\frac{1}{2})} < \tau_{v_{\dagger}}) \asymp \text{dist}(v, D) \wedge 1.$$

Meanwhile, since the hitting probability on $\partial \mathcal{B}_v(n)$ for a Brownian motion starting from $\tilde{\partial} \mathbf{B}_v(\frac{1}{2})$ is proportional to the uniform distribution on $\partial \mathcal{B}_v(n)$ (see e.g. [23, Lemma 6.3.7]), we have: for any $y_1 \in \tilde{\partial} \mathbf{B}_v(\frac{1}{2})$ and $y_2 \in \partial \mathcal{B}_v(n)$,

$$(3.57) \quad \tilde{\mathbb{P}}_{y_1}(\tau_{\partial \mathcal{B}_v(n)} = \tau_{y_2}) \asymp |\partial \mathcal{B}_v(n)|^{-1}.$$

Plugging (3.56) and (3.57) into (3.55), we obtain

$$(3.58) \quad \tilde{\mathbb{P}}_v(\tau_w < \tau_D) \lesssim (\text{dist}(v, D) \wedge 1) \cdot |\partial\mathcal{B}_v(n)|^{-1} \sum_{z \in \partial\mathcal{B}_v(n)} \tilde{\mathbb{P}}_z(\tau_w < \tau_D).$$

Combined with (3.6) and (3.28), it implies the desired bound (3.54):

$$\begin{aligned} \mathbb{P}^D(v \xrightarrow{\geq 0} w) &\stackrel{(3.6), (3.28)}{\lesssim} \tilde{\mathbb{P}}_v(\tau_w < \tau_D) \sqrt{\tilde{G}_D(w, w)} \cdot (\text{dist}(v, D) \wedge 1)^{-\frac{1}{2}} \\ &\stackrel{(3.58), \tilde{G}_D(z, z) \lesssim 1}{\lesssim} (\text{dist}(v, D) \wedge 1)^{\frac{1}{2}} \cdot |\partial\mathcal{B}_v(n)|^{-1} \\ &\quad \cdot \sum_{z \in \partial\mathcal{B}_v(n)} \tilde{\mathbb{P}}_z(\tau_w < \tau_D) \sqrt{\frac{\tilde{G}_D(w, w)}{\tilde{G}_D(z, z)}} \\ &\stackrel{(3.28)}{\lesssim} (\text{dist}(v, D) \wedge 1)^{\frac{1}{2}} \cdot |\partial\mathcal{B}_v(n)|^{-1} \sum_{z \in \partial\mathcal{B}_v(n)} \mathbb{P}^D(z \xrightarrow{\geq 0} w). \quad \square \end{aligned}$$

Subsequently, we provide an upper bound on the conditional connecting probability between two points, given the GFF value at one of them. The proof needs the following basic facts:

- For any random variable $\mathbf{X}$, by Jensen's inequality one has

$$(3.59) \quad \mathbb{E}[\mathbf{X} \wedge 1] \leq \mathbb{E}[\mathbf{X}] \wedge 1.$$

- For any $u_1 > 0$ and $0 < u_2 \leq C$, we have

$$(3.60) \quad \left(\frac{u_1}{u_2} \cdot (u_1 + u_2)\right) \wedge 1 \lesssim \frac{u_1}{\sqrt{u_2}}.$$

For (3.60), we consider the following two cases separately. When $u_1 \leq u_2$,

$$(3.61) \quad \left(\frac{u_1}{u_2} \cdot (u_1 + u_2)\right) \wedge 1 \stackrel{(u_1+u_2 \leq 2u_2)}{\leq} 2u_1 \stackrel{(u_2 \leq C)}{\lesssim} \frac{u_1}{\sqrt{u_2}}.$$

When $u_1 \geq u_2$, since $a \wedge 1 \leq \sqrt{a}$ for $a > 0$, one has

$$(3.62) \quad \left(\frac{u_1}{u_2} \cdot (u_1 + u_2)\right) \wedge 1 \stackrel{(u_1+u_2 \leq 2u_1)}{\leq} \frac{2u_1^2}{u_2} \wedge 1 \lesssim \frac{u_1}{\sqrt{u_2}}.$$

To sum up, we obtain (3.60).

- For any $\mu, \sigma \geq 0$ and $\mathbf{Y} \sim N(\mu, \sigma^2)$, one has (see e.g., [27, (3)])

$$(3.63) \quad \mathbb{E}[\mathbf{Y} \cdot \mathbb{1}_{\mathbf{Y} \geq 0}] = \frac{\sigma}{\sqrt{2\pi}} e^{-\frac{\mu^2}{2\sigma^2}} + \frac{\mu}{2} \cdot (1 - 2\Phi(-\frac{\mu}{\sigma})) \leq \sigma + \mu,$$

where $\Phi(t) = \mathbb{P}(\mathbf{Z} \leq t)$ for $\mathbf{Z} \sim N(0, 1)$.

Lemma 3.13. *For any $a > 0$, $d \geq 3$, $D \subset \tilde{\mathbb{Z}}^d$ and $v, w \in \tilde{\mathbb{Z}}^d \setminus D$ with $|v - w| \geq d$,*

$$(3.64) \quad \mathbb{P}^D(v \xrightarrow{\geq 0} w \mid \tilde{\phi}_v = a) \lesssim \frac{a \cdot \mathbb{P}^D(v \xrightarrow{\geq 0} w)}{\sqrt{\tilde{G}_D(v, v)}}.$$

Proof. Using (3.29), (3.59) and Lemma 3.2, we have

$$\begin{aligned}
(3.65) \quad & \mathbb{P}^D(v \xleftrightarrow{\geq 0} w \mid \tilde{\phi}_v = a) \\
& \stackrel{(3.29)}{\asymp} \mathbb{E}^D[(a \cdot \tilde{\phi}_w \cdot \mathbb{K}_{D \cup \{v, w\}}(v, w) \cdot \mathbb{1}_{\tilde{\phi}_w \geq 0}) \wedge 1 \mid \tilde{\phi}_v = a] \\
& \stackrel{(3.59), \text{Lemma 3.2}}{\lesssim} \left(\frac{a \cdot \tilde{\mathbb{P}}_w(\tau_v < \tau_D)}{\tilde{G}_D(w, w)} \cdot \mathbb{E}^D[\tilde{\phi}_w \cdot \mathbb{1}_{\tilde{\phi}_w \geq 0} \mid \tilde{\phi}_v = a] \right) \wedge 1.
\end{aligned}$$

By the strong Markov property of $\tilde{\phi}$, the distribution of $\tilde{\phi}_w$ under $\mathbb{P}^D(\cdot \mid \tilde{\phi}_v = a)$ is given by $N(a \cdot \tilde{\mathbb{P}}_w(\tau_v < \tau_D), \tilde{G}_{D \cup \{v\}}(w, w))$. Thus, by (3.63) and $\tilde{G}_{D \cup \{v\}}(w, w) \leq \tilde{G}_D(w, w)$, we have

$$(3.66) \quad \mathbb{E}^D[\tilde{\phi}_w \cdot \mathbb{1}_{\tilde{\phi}_w \geq 0} \mid \tilde{\phi}_v = a] \leq a \cdot \tilde{\mathbb{P}}_w(\tau_v < \tau_D) + \tilde{G}_D(w, w).$$

By plugging (3.66) into (3.65) and then applying (3.60), we get

$$\mathbb{P}^D(v \xleftrightarrow{\geq 0} w \mid \tilde{\phi}_v = a) \lesssim \frac{a \cdot \tilde{\mathbb{P}}_w(\tau_v < \tau_D)}{\sqrt{\tilde{G}_D(w, w)}} \stackrel{(3.28)}{\asymp} \frac{a \cdot \mathbb{P}^D(v \xleftrightarrow{\geq 0} w)}{\sqrt{\tilde{G}_D(v, v)}}. \quad \square$$

3.3. Loop soup. A rooted loop on $\tilde{\mathbb{Z}}^d$ is a path $\tilde{\varrho} : [0, T) \rightarrow \tilde{\mathbb{Z}}^d$ (where $T \in \mathbb{R}^+$) satisfying $\tilde{\varrho}(0) = \tilde{\varrho}(T)$. For any $\alpha > 0$, the loop soup of intensity α is the Poisson point process with intensity measure $\alpha \cdot \tilde{\mu}$. Here the measure $\tilde{\mu}(\cdot)$ is supported on the space of rooted loops and is defined by

$$(3.67) \quad \tilde{\mu}(\cdot) = \int_{v \in \tilde{\mathbb{Z}}^d} dm(v) \int_{0 < t < \infty} t^{-1} \tilde{q}_t(v, v) \tilde{\mathbb{P}}_{v, v}^t(\cdot) dt,$$

where $m(\cdot)$ denotes the Lebesgue measure on $\tilde{\mathbb{Z}}^d$, and $\tilde{\mathbb{P}}_{v, v}^t(\cdot)$ denotes the law of the Brownian bridge on $\tilde{\mathbb{Z}}^d$ with duration t , starting and ending at v , i.e., the Markov process with transition density $\frac{\tilde{q}_s(v, \cdot) \tilde{q}_{t-s}(\cdot, v)}{\tilde{q}_t(v, v)}$ for $0 \leq s \leq t$.

We say that two rooted loops are equivalent if one can be transformed into the other by a certain time shift. We refer each equivalence class of rooted loops under this relation as a loop (usually denoted by $\tilde{\ell}$). Since the densities of two equivalent rooted loops in $\tilde{\mu}$ are identical, $\tilde{\mu}$ can be viewed as a measure on the space of loops by forgetting the root of each rooted loop. We use $\text{ran}(\cdot)$ to denote the range of a path, i.e., the collection of points visited by the path. With a slight abuse of notation, we define $\text{ran}(\tilde{\ell})$ as $\text{ran}(\tilde{\varrho})$ for any $\tilde{\varrho} \in \tilde{\ell}$ (the choice of $\tilde{\varrho}$ is irrelevant). For a point measure $\mathcal{L}$ consisting of paths, we denote by $\cup \mathcal{L}$ the union of the ranges of all paths in $\mathcal{L}$.

For any $D \subset \tilde{\mathbb{Z}}^d$, let $\tilde{\mu}^D$ denote the restriction of $\tilde{\mu}$ to the set of loops that are disjoint from D . It follows that the truncated point measure $\tilde{\mathcal{L}}_\alpha^D := \tilde{\mathcal{L}}_\alpha \cdot \mathbb{1}_{\text{ran}(\tilde{\ell}) \cap D = \emptyset}$ is a Poisson point process with intensity measure $\alpha \tilde{\mu}^D$. For any $A, A' \subset \tilde{\mathbb{Z}}^d$, we denote by $A \xleftrightarrow{(D)} A'$ (resp. $A \leftrightarrow A'$) the event that there exists a path within $\cup \tilde{\mathcal{L}}_{1/2}^D$ (resp. $\cup \tilde{\mathcal{L}}_{1/2}$) connecting A and A' . Moreover, we denote by $\mathcal{C}_A^D := \{v \in \tilde{\mathbb{Z}}^d :$

$v \xleftrightarrow{(D)} A\}$ the union of clusters of $\cup \tilde{\mathcal{L}}_{1/2}^D$ intersecting A . By the thinning property of Poisson point processes, conditioned on the event $\{\mathcal{C}_A^D = F\}$ (for any proper configuration F of $\mathcal{C}_A^D$), the point measure $\tilde{\mathcal{L}}_{1/2}^D \cdot \mathbb{1}_{\text{ran}(\tilde{\ell}) \cap \mathcal{C}_A^D = \emptyset}$, consisting of loops that are not used in the construction of $\mathcal{C}_A^D$, has the same distribution as $\tilde{\mathcal{L}}_{1/2}^{D \cup F}$. This is commonly known as the “restriction property” of the loop soup. (P.S. When the superscript D is $\emptyset$, it may be omitted. The same convention applies to notations introduced later.)

Isomorphism theorem. [28, Proposition 2.1] shows that for any $D \subset \tilde{\mathbb{Z}}^d$, there exist a coupling between $\tilde{\mathcal{L}}_{1/2}^D$ and $\tilde{\phi} \sim \mathbb{P}^D$ satisfying the following properties:

- (i) For any $v \in \tilde{\mathbb{Z}}^d \setminus D$, the total local time at v of loops in $\tilde{\mathcal{L}}_{1/2}^D$, denoted by $\hat{\mathcal{L}}_{1/2}^{D,v}$, equals to $\frac{1}{2}\tilde{\phi}_v^2$;
- (ii) Each sign cluster of $\tilde{\phi}$ is exactly a loop cluster of $\tilde{\mathcal{L}}_{1/2}^D$.
- (iii) On each loop cluster of $\tilde{\mathcal{L}}_{1/2}^D$, the sign of $\tilde{\phi}$ is independently and uniformly chosen from $\{+, -\}$.

Here we list some useful corollaries of this isomorphism theorem:

- (1) For any $D \subset \tilde{\mathbb{Z}}^d$ and any $A_i^{(1)}, A_i^{(2)} \subset \tilde{\mathbb{Z}}^d$ for $1 \leq i \leq k$ ($k \in \mathbb{N}^+$), we denote the events $\mathbf{A} := \cap_{1 \leq i \leq k} \{A_i^{(1)} \xleftrightarrow{\geq 0} A_i^{(2)}\}$ and $\bar{\mathbf{A}}^D := \cap_{1 \leq i \leq k} \{A_i^{(1)} \xleftrightarrow{(D)} A_i^{(2)}\}$. The isomorphism theorem then implies that

$$(3.68) \quad 2^{-k} \cdot \mathbb{P}(\bar{\mathbf{A}}^D) \leq \mathbb{P}^D(\mathbf{A}) \leq \mathbb{P}(\bar{\mathbf{A}}^D).$$

Here the second bound follows immediately from Property (ii). To see the first bound, observe that on $\bar{\mathbf{A}}^D$, there exists a collection of at most k loop clusters of $\tilde{\mathcal{L}}_{1/2}^D$ that can certify $\bar{\mathbf{A}}^D$. Thus, for any realization of $\tilde{\mathcal{L}}_{1/2}^D$ under which $\bar{\mathbf{A}}^D$ occurs, one can force the signs of $\tilde{\phi}$ on at most k loop cluster to be $+$ (whose probability is at least 2^{-k} by Property (iii)) so that $\mathbf{A}$ holds.

- (2) For any $D_1 \subset D_2 \subset \tilde{\mathbb{Z}}^d$ and $A, A' \subset \tilde{\mathbb{Z}}^d$, since $\tilde{\mathcal{L}}_{1/2}^D$ is decreasing in D ,

$$(3.69) \quad \mathbb{P}(A \xleftrightarrow{(D_2)} A') \leq \mathbb{P}(A \xleftrightarrow{(D_1)} A').$$

Combined with (3.68), it implies that

$$(3.70) \quad \mathbb{P}^{D_2}(A \xleftrightarrow{\geq 0} A') \leq 2\mathbb{P}^{D_1}(A \xleftrightarrow{\geq 0} A').$$

Excursion decomposition of loops. For any path $\tilde{\eta} : [0, T] \rightarrow \tilde{\mathbb{Z}}^d$ ($T \in \mathbb{R}^+$) and any $0 \leq t_1 < t_2 \leq T$, we denote by $\tilde{\eta}[t_1, t_2]$ the sub-path satisfying

$$(3.71) \quad \tilde{\eta}[t_1, t_2](s) := \tilde{\eta}(t_1 + s), \quad \forall 0 \leq s \leq t_2 - t_1.$$

For any $v \in \tilde{\mathbb{Z}}^d$ and any loop $\tilde{\ell}$ intersecting v , arbitrarily take a rooted loop $\tilde{\varrho} \in \tilde{\ell}$ with $\tilde{\varrho}(0) = \tilde{\varrho}(T) = v$ (where T is the duration of $\tilde{\ell}$). For any $t \in (0, T)$, let

$\delta_t^- = \delta_t^-(\tilde{\varrho}, v) := \sup\{s \leq t : \tilde{\varrho}(s) = v\}$ and $\delta_t^+ = \delta_t^+(\tilde{\varrho}, v) := \inf\{s \geq t : \tilde{\varrho}(s) = v\}$. We define $\mathcal{E}_v(\tilde{\varrho})$ as the point measure supported on $\{\tilde{\varrho}[\delta_t^-, \delta_t^+] : 0 < t < T, \delta_t^- < \delta_t^+\}$. Since $\mathcal{E}_v(\tilde{\varrho})$ is invariant under the selection of $\tilde{\varrho} \in \tilde{\ell}$, we also denote $\mathcal{E}_v(\tilde{\varrho})$ by $\mathcal{E}_v(\tilde{\ell})$. For completeness, we set $\mathcal{E}_v(\tilde{\ell}) := 0$ when $\tilde{\ell}$ does not intersect v .

Referring to [42, Section 1], for any $D \subset \tilde{\mathbb{Z}}^d$, $v \in \tilde{\mathbb{Z}}^d \setminus D$ and $a > 0$, conditioned on the event $\{\hat{\mathcal{L}}_{1/2}^{D,v} = a\}$, the point measure $\tilde{\mathcal{E}}_v^D := \sum_{\tilde{\ell} \in \tilde{\mathcal{L}}_{1/2}^D} \mathcal{E}_v(\tilde{\ell})$ is a Poisson point process consisting of excursions from v . (This property of the loop soup is derived from the rewiring property [40], which is specific to the intensity $\frac{1}{2}$.) Moreover, its intensity measure can be written as $a \cdot \mathbf{e}_{v,v}^D$, where the ‘‘Brownian excursion measure’’ $\mathbf{e}_{v,v}^D$ is an infinite but σ -finite measure. Notably, similar to the well-known ‘‘Itô’s excursion measure’’, it is not convenient to construct $\mathbf{e}_{v,v}^D$ directly. For any distinct $v, w \in \tilde{\mathbb{Z}}^d$, the measure $\mathbf{e}_{v,w}^D$ on the space of excursions from v to w can be obtained by first restricting $\mathbf{e}_{v,v}^D$ to the paths that hits w , and then keeping only the part of each path until the first time it hits w . Note that $\mathbf{e}_{v,w}^D$ is a finite measure when $v \neq w$. In addition, $\mathbf{e}_{v,w}^D$ (for both the cases $v = w$ and $v \neq w$) satisfies the strong Markov property. I.e., for an excursion $\tilde{\eta}$ sampled from $\mathbf{e}_{v,w}^D$ and for any $A \subset \tilde{\mathbb{Z}}^d \setminus (D \cup \{v, w\})$, given $\{\tilde{\eta}(t)\}_{0 \leq t \leq \tau_A(\tilde{\eta})}$ (suppose that $\tilde{\eta}(\tau_A(\tilde{\eta})) = z$), the law of the remaining part of $\tilde{\eta}$ is given by

$$(3.72) \quad \tilde{\mathbb{P}}_z(\{\tilde{S}_t\}_{0 \leq t \leq \tau_w} \in \cdot \mid \tau_{D \cup \{v, w\}} = \tau_w < \infty).$$

Moreover, according to the restriction property, $\mathbf{e}_{v,w}^D$ can be obtained by restricting $\mathbf{e}_{v,w} := \mathbf{e}_{v,w}^\emptyset$ to excursions that stay disjoint from D .

For more details on excursion measures, readers may refer to [35, Chapter XII] or [1, Chapters III and IV]. Subsequently, we present a relation between the total mass of the Brownian excursion measure and the boundary excursion kernel. Although its analogue for one-dimensional Brownian motions is a classical result, we present a proof in the context of metric graphs for the sake of completeness.

Lemma 3.14. *For any $d \geq 3$, $D \subset \tilde{\mathbb{Z}}^d$ and $v \neq w \in \tilde{\mathbb{Z}}^d \setminus D$, the total mass of the measure $\mathbf{e}_{v,w}^D$ exactly equals to $\mathbb{K}_{D \cup \{v, w\}}(v, w)$.*

Proof. By the strong Markov property of $\mathbf{e}_{v,v}^D$, the measure $\mathbf{e}_{v,v}^D(\{\tilde{\eta} : w \in \text{ran}(\tilde{\eta})\})$ equals the product of the total mass of $\mathbf{e}_{v,w}^D$ and the probability $\tilde{\mathbb{P}}_w(\tau_v < \tau_D)$. Thus, recalling (3.7), it suffices to show that for all sufficiently small $\epsilon > 0$,

$$(3.73) \quad \begin{aligned} & \mathbf{e}_{v,v}^D(\{\tilde{\eta} : w \in \text{ran}(\tilde{\eta})\}) \\ &= \tilde{\mathbb{P}}_w(\tau_v < \tau_D) \cdot \left(\frac{1}{2\epsilon} + o(\epsilon^{-1})\right) \sum_{z \in \tilde{\partial}\mathbf{B}_v(\epsilon)} \tilde{\mathbb{P}}_z(\tau_{D \cup \{v, w\}} = \tau_w < \infty). \end{aligned}$$

To see this, for any sufficiently small $\epsilon > 0$ satisfying $\mathbf{B}_v(\epsilon) \cap D = \emptyset$, and any $z \in \tilde{\partial}\mathbf{B}_v(\epsilon)$, it follows from the strong Markov property of $\mathbf{e}_{v,v}^D$ that

$$(3.74) \quad \frac{\mathbf{e}_{v,v}^D(\{\tilde{\eta} : w \in \text{ran}(\tilde{\eta}), \tau_{\tilde{\partial}\mathbf{B}_v(\epsilon)}(\tilde{\eta}) = \tau_z(\tilde{\eta})\})}{\mathbf{e}_{v,v}^D(\{\tilde{\eta} : \tau_{\tilde{\partial}\mathbf{B}_v(\epsilon)}(\tilde{\eta}) = \tau_z(\tilde{\eta})\})} = \frac{\tilde{\mathbb{P}}_z(\tau_w < \tau_v < \tau_D)}{\tilde{\mathbb{P}}_z(\tau_v < \tau_D)}.$$

Note that $\tilde{\mathbb{P}}_z(\tau_v < \tau_D)$ converges to 1 as $\epsilon \rightarrow 0$. Meanwhile, by the strong Markov property of Brownian motion, one has $\tilde{\mathbb{P}}_z(\tau_w < \tau_v < \tau_D) = \tilde{\mathbb{P}}_z(\tau_{D \cup \{v,w\}} = \tau_w < \infty) \cdot \tilde{\mathbb{P}}_w(\tau_v < \tau_D)$. Plugging these two facts into (3.74), we obtain that

$$(3.75) \quad \begin{aligned} \mathbf{e}_{v,v}^D(\{\tilde{\eta} : w \in \text{ran}(\tilde{\eta})\}) &= \tilde{\mathbb{P}}_w(\tau_v < \tau_D) \cdot (1 + o(1)) \\ &\cdot \sum_{z \in \tilde{\partial}\mathbf{B}_v(\epsilon)} \tilde{\mathbb{P}}_z(\tau_{D \cup \{v,w\}} = \tau_w < \infty) \cdot \mathbf{e}_{v,v}^D(\{\tilde{\eta} : \tau_{\tilde{\partial}\mathbf{B}_v(\epsilon)}(\tilde{\eta}) = \tau_z(\tilde{\eta})\}), \end{aligned}$$

where the $o(1)$ term depends only on ϵ . Moreover, [35, Theorem (1.10) in Chapter VI] shows that ϵ times the number of Brownian excursions from v to z converges to half of the total local time at v in the L_p norm for all $p \geq 1$, as $\epsilon \rightarrow 0$. Meanwhile, recall that conditioned on $\{\hat{\mathcal{L}}_{1/2}^{D,v} = 1\}$, the number of excursions in $\tilde{\mathcal{E}}_v^D$ hitting z is a Poisson random variable with parameter $\mathbf{e}_{v,v}^D(\{\tilde{\eta} : \tau_{\tilde{\partial}\mathbf{B}_v(\epsilon)}(\tilde{\eta}) = \tau_z(\tilde{\eta})\})$. Putting these two observations together, we conclude that

$$(3.76) \quad \mathbf{e}_{v,v}^D(\{\tilde{\eta} : \tau_{\tilde{\partial}\mathbf{B}_v(\epsilon)}(\tilde{\eta}) = \tau_z(\tilde{\eta})\}) = \frac{1}{2\epsilon} + o(\epsilon^{-1}).$$

Plugging (3.76) into (3.75), we obtain (3.73) and thus complete the proof. $\square$

Based on Lemmas 3.1 and 3.14, we derive the following bound on the conditional probability that two nearby points are not connected by the loop cluster.

Lemma 3.15. *For any $d \geq 3$, there exists $c > 0$ such that for any $a > 0$, $v \neq w \in \tilde{\mathbb{Z}}^d$ with $|v - w| \leq 1$, and $D \subset \tilde{\mathbb{Z}}^d$ with $\text{dist}(D, w) \geq |v - w|$,*

$$(3.77) \quad \mathbb{P}^D(\{v \xleftrightarrow{\geq 0} w\}^c \mid \tilde{\phi}_v = a) \leq e^{-ca^2|v-w|^{-1}}.$$

Proof. By the isomorphism theorem, it suffices to show that for any $T > 0$

$$(3.78) \quad \mathbb{P}(\{v \xleftrightarrow{(D)} w\}^c \mid \tilde{\mathcal{L}}_{1/2}^{D,v} = T) \leq Ce^{-cT|v-w|^{-1}}.$$

To see this, note that on the event $\{v \xleftrightarrow{(D)} w\}^c$, no excursion in $\tilde{\mathcal{E}}_v^D$ reaches w . In addition, conditioned on $\{\tilde{\mathcal{L}}_{1/2}^{D,v} = T\}$, the number of excursions in $\tilde{\mathcal{E}}_v^D$ intersecting w is a Poisson random variable with parameter

$$(3.79) \quad \begin{aligned} &T \cdot \mathbf{e}_{v,v}^D(\{\tilde{\eta} : w \in \text{ran}(\tilde{\eta})\}) \\ &\stackrel{(\text{Lemma 3.14})}{=} T \cdot \mathbb{K}_{D \cup \{v,w\}}(v, w) \cdot \tilde{\mathbb{P}}_w(\tau_v < \tau_D) \stackrel{(3.1), (3.9)}{\gtrsim} T \cdot |v - w|^{-1}. \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} & \mathbb{P}(\{v \xleftrightarrow{(D)} w\}^c \mid \tilde{\mathcal{L}}_{1/2}^{D,v} = T) \\ & \leq \mathbb{P}(\{\tilde{\eta} \in \tilde{\mathcal{E}}_v^D : w \in \text{ran}(\tilde{\eta})\} = \emptyset \mid \tilde{\mathcal{L}}_{1/2}^{D,v} = T) \leq e^{-cT \cdot |v-w|^{-1}}. \end{aligned} \quad \square$$

Next, we present two estimates on the total mass of certain excursions.

Lemma 3.16. *For any $d \geq 3$ and $v_1, v_2, v_3 \in \tilde{\mathbb{Z}}^d$ with $|v_i - v_j| \geq d$ for all $i \neq j$, under the measure $\mathbf{e}_{v_1, v_2}(\cdot)$, the probability of an excursion hitting v_3 is at most of order $\frac{|v_1 - v_3|^{2-d} |v_3 - v_2|^{2-d}}{|v_1 - v_2|^{2-d}}$.*

Proof. Similar to (3.74), we have: for any $z \in \partial \mathbf{B}_{v_1}(1)$,

$$(3.80) \quad \frac{\mathbf{e}_{v_1, v_2}(\{\tilde{\eta} : v_3 \in \text{ran}(\tilde{\eta}), \tau_{\tilde{\partial} \mathbf{B}_{v_1}(1)}(\tilde{\eta}) = \tau_z(\tilde{\eta})\})}{\mathbf{e}_{v_1, v_2}(\{\tilde{\eta} : \tau_{\tilde{\partial} \mathbf{B}_{v_1}(1)}(\tilde{\eta}) = \tau_z(\tilde{\eta})\})} = \frac{\tilde{\mathbb{P}}_z(\tau_{v_3} < \tau_{v_2} < \tau_{v_1})}{\tilde{\mathbb{P}}_z(\tau_{v_2} < \tau_{v_1})}.$$

Note that $\mathbf{e}_{v_1, v_2}(\{\tilde{\eta} : \tau_{\tilde{\partial} \mathbf{B}_{v_1}(1)}(\tilde{\eta}) = \tau_z(\tilde{\eta})\}) \asymp 1$ and $\tilde{\mathbb{P}}_z(\tau_{v_2} < \tau_{v_1}) \asymp |v_1 - v_2|^{2-d}$.

In addition, by the strong Markov property of $\tilde{S} \sim \tilde{\mathbb{P}}_z$, one has

$$\tilde{\mathbb{P}}_z(\tau_{v_3} < \tau_{v_2} < \tau_{v_1}) \leq \tilde{\mathbb{P}}_z(\tau_{v_3} < \infty) \cdot \tilde{\mathbb{P}}_{v_3}(\tau_{v_2} < \infty) \asymp |v_1 - v_3|^{2-d} |v_3 - v_2|^{2-d}.$$

Combining these estimates, we obtain this lemma. $\square$

Lemma 3.17. *For any $d \geq 3$, $v \in \tilde{\mathbb{Z}}^d$ and $N \geq 1$,*

$$(3.81) \quad \mathbf{e}_{v, v}(\{\tilde{\eta} : \text{ran}(\tilde{\eta}) \cap \partial B(N) \neq \emptyset\}) \lesssim N^{2-d}.$$

Proof. As in (3.74), for any $z \in \partial \mathbf{B}_v(1)$, we have

$$\frac{\mathbf{e}_{v, v}(\{\tilde{\eta} : \text{ran}(\tilde{\eta}) \cap \partial B(N) \neq \emptyset, \tau_{\tilde{\partial} \mathbf{B}_v(1)}(\tilde{\eta}) = \tau_z(\tilde{\eta})\})}{\mathbf{e}_{v, v}(\{\tilde{\eta} : \tau_{\tilde{\partial} \mathbf{B}_v(1)}(\tilde{\eta}) = \tau_z(\tilde{\eta})\})} = \frac{\tilde{\mathbb{P}}_z(\tau_{\partial B(N)} < \tau_v < \infty)}{\tilde{\mathbb{P}}_z(\tau_v < \infty)}.$$

Here $\mathbf{e}_{v, v}(\{\tilde{\eta} : \tau_{\tilde{\partial} \mathbf{B}_v(1)}(\tilde{\eta}) = \tau_z(\tilde{\eta})\})$ and $\tilde{\mathbb{P}}_z(\tau_v < \infty)$ are both of order 1. Moreover, by the strong Markov property of $\tilde{S} \sim \tilde{\mathbb{P}}_z$, one has

$$(3.82) \quad \tilde{\mathbb{P}}_z(\tau_{\partial B(N)} < \tau_v < \infty) \leq \max_{x \in \partial B(N)} \tilde{\mathbb{P}}_x(\tau_v < \infty) \lesssim N^{2-d}.$$

Putting these estimates together, we obtain the desired bound (3.81). $\square$

3.4. Werner's switching identity. In this subsection, we review a recent major progress in the study of the loop soup clusters. Precisely, the main result in [42] (referred to as the “switching identity”) provides a workable description of the loop cluster that connects two given points. When restricted to $\tilde{\mathbb{Z}}^d$, it yields:

Lemma 3.18 (switching identity). *For any $d \geq 3$, $v \neq w \in \tilde{\mathbb{Z}}^d$, and $a, b > 0$, conditioned on $\{v \leftrightarrow w, \hat{\mathcal{L}}_{1/2}^v = a, \hat{\mathcal{L}}_{1/2}^w = b\}$, the occupation field $\{\hat{\mathcal{L}}_{1/2}^z\}_{z \in \tilde{\mathbb{Z}}^d}$ has the same distribution as the total local time of the following four independent components:*

- (1) loops in the loop soup $\tilde{\mathcal{L}}_{1/2}^{\{v,w\}}$;
- (2) a Poisson point process with intensity measure $a \cdot \mathbf{e}_{v,v}^{\{w\}}$;
- (3) a Poisson point process with intensity measure $b \cdot \mathbf{e}_{w,w}^{\{v\}}$;
- (4) a Poisson point process with intensity measure $\sqrt{ab} \cdot \mathbf{e}_{v,w}$, where the number of excursions is conditioned to be odd.

To facilitate the use of this switching identity, we record the following notations.

- We denote the conditional probability measure

$$(3.83) \quad \hat{\mathbb{P}}_{v \leftrightarrow w, a, b}(\cdot) := \mathbb{P}(\cdot \mid v \leftrightarrow w, \hat{\mathcal{L}}_{1/2}^v = a, \hat{\mathcal{L}}_{1/2}^w = b).$$

Let $\mathbf{p}_{v \leftrightarrow w, a, b}$ denote the density (at the point (a, b)) of the event $\{v \leftrightarrow w, \hat{\mathcal{L}}_{1/2}^v = a, \hat{\mathcal{L}}_{1/2}^w = b\}$. By the isomorphism theorem and (3.29), one has

$$(3.84) \quad \mathbf{p}_{v \leftrightarrow w, a, b} = (ab)^{-\frac{1}{2}} \cdot \mathbf{g}_{v,w}(\sqrt{2a}, \sqrt{2b}) \cdot (1 - e^{-4\sqrt{ab} \cdot \mathbb{K}_{\{v,w\}}(v,w)}),$$

where $\mathbf{g}_{v,w}(\cdot, \cdot)$ is the density of $(\tilde{\phi}_v, \tilde{\phi}_w)$ (see e.g., [43, Page 329]), i.e.,

$$(3.85) \quad \mathbf{g}_{v,w}(s, t) = \frac{\exp\left(-\frac{1}{2(1-\delta^2)} \cdot \left[\frac{s^2}{\tilde{G}(v,v)} - \frac{2\delta st}{\sqrt{\tilde{G}(v,v)\tilde{G}(w,w)}} + \frac{t^2}{\tilde{G}(w,w)}\right]\right)}{2\pi\sqrt{\tilde{G}(v,v)\tilde{G}(w,w)(1-\delta^2)}}, \quad \forall s, t \geq 0.$$

Here we denote $\delta := \frac{\tilde{G}(v,w)}{\sqrt{\tilde{G}(v,v)\tilde{G}(w,w)}}$.

- We define $\check{\mathbb{P}}_{v \leftrightarrow w, a, b}(\cdot)$ as the probability measure generated by Components (1)-(4) in Lemma 3.18 (we also denote its expectation by $\check{\mathbb{E}}_{v \leftrightarrow w, a, b}[\cdot]$). Under the measure $\check{\mathbb{P}}_{v \leftrightarrow w, a, b}(\cdot)$, for each $i \in \{1, 2, 3, 4\}$, we denote the Poisson point process in Component (i) by $\mathcal{P}^{(i)}$. We define

$$(3.86) \quad \check{\mathcal{C}} := \{z \in \tilde{\mathbb{Z}}^d : z \xleftarrow{\cup(\sum_{1 \leq i \leq 4} \mathcal{P}^{(i)})} v\}$$

as the cluster containing v , formed by all loops and excursions in $\sum_{1 \leq i \leq 4} \mathcal{P}^{(i)}$.

4. RIGIDITY LEMMA

This section is devoted to proving the rigidity lemma and some useful corollaries. For convenience, we formulate it in terms of the loop soup to avoid dealing with signs. Roughly speaking, the rigidity lemma asserts that a rare increasing event for a large loop cluster remains rare when conditioned on the existence of another loop cluster. The necessity of the “increasing” condition is not difficult to justify—the presence of an additional cluster constrains the space available for the target cluster and may substantially increase the probability of certain decreasing events. However, this same reasoning may be used to “challenge” the rigidity lemma itself. For instance, the additional cluster, as an obstacle, is possible to force the target cluster to grow through a winding “tunnel”, potentially leading to an atypically large volume (which is increasing and rare for the target cluster). In a word, it is

not a priori clear whether the interaction between the two clusters is positive or negative (and even the event of having two distinct clusters is not monotonic). As shown in the next subsection, Werner's switching identity (see Lemma 3.18) plays a key role in resolving this dilemma. The underlying idea can be summarized as follows. Under the description of the target cluster given by the switching identity, the increasing event under consideration and the presence of the additional cluster exhibit opposite monotonicity, which allows an FKG-type inequality to be applied, thereby showing that the interaction is indeed negative.

For any events A_1 and A_2 measurable with respect to $\tilde{\mathcal{L}}_{1/2}$, we define $A_1 \parallel A_2$ as the event that A_1 and A_2 are certified by different clusters of $\cup \tilde{\mathcal{L}}_{1/2}$. For any $v \in \tilde{\mathbb{Z}}^d$, we say an event $A \in \mathcal{F}_{\mathcal{C}_v}$ is increasing with respect to the cluster $\mathcal{C}_v$ if for any realizations D_1, D_2 of $\mathcal{C}_v$, the inclusion $\{\mathcal{C}_v = D_1\} \subset A$ implies $\{\mathcal{C}_v = D_2\} \subset A$ whenever $D_1 \subset D_2$. Now we present the formal statement of the rigidity lemma.

Lemma 4.1 (rigidity lemma). *For any $d \geq 3$, there exist constants $C_4, c_5 > 0$ such that for any distinct $v, v', w \in \tilde{\mathbb{Z}}^d$ with $|v - w| \geq d$, and any events A and A' increasing with respect to $\mathcal{C}_v$ and $\mathcal{C}_{v'}$ respectively,*

$$(4.1) \quad \mathbb{P}(A \mid \{v \leftrightarrow w \parallel A'\}) \leq C_4 \left[\mathbb{P}(A \mid v \leftrightarrow w) + \exp(-c_5 |v - w|^{\frac{d}{2}-1}) \right].$$

The proof of this rigidity lemma relies on several preliminary results. We begin by estimating the decay rate of the local time on a large loop cluster.

Lemma 4.2. *For any $d \geq 3$, there exist $C, c > 0$ such that for any $T > 0$, $D \subset \tilde{\mathbb{Z}}^d$ and $v, w \in \tilde{\mathbb{Z}}^d \setminus D$ with $|v - w| \geq d$,*

$$(4.2) \quad \mathbb{P}(\hat{\mathcal{L}}_{1/2}^{D,v} \geq T \mid v \overset{(D)}{\longleftrightarrow} w) \leq C e^{-cT}.$$

Proof. By the isomorphism theorem, it suffices to prove that for any $K > 0$,

$$(4.3) \quad \mathbb{P}^D(\tilde{\phi}_v \geq K \mid v \overset{\geq 0}{\longleftrightarrow} w) \leq C e^{-cK^2}.$$

In fact, since $\tilde{\phi}_v \sim N(0, \tilde{G}_D(v, v))$, $\mathbb{P}^D(\tilde{\phi}_v \geq K, v \overset{\geq 0}{\longleftrightarrow} w)$ can be written as

$$(4.4) \quad \begin{aligned} & \int_{a \geq K} \mathbb{P}^D(v \overset{\geq 0}{\longleftrightarrow} w \mid \tilde{\phi}_v = a) \cdot \frac{1}{\sqrt{2\pi\tilde{G}_D(v, v)}} e^{-\frac{a^2}{2\tilde{G}_D(v, v)}} da \\ & \stackrel{(\text{Lemma 3.13})}{\lesssim} \mathbb{P}^D(v \overset{\geq 0}{\longleftrightarrow} w) \int_{a \geq K} \frac{a}{\tilde{G}_D(v, v)} \cdot e^{-\frac{a^2}{2\tilde{G}_D(v, v)}} da \\ & \lesssim \mathbb{P}^D(v \overset{\geq 0}{\longleftrightarrow} w) \cdot e^{-\frac{K^2}{2\tilde{G}_D(v, v)}} \stackrel{(\tilde{G}_D(v, v) \lesssim 1)}{\lesssim} \mathbb{P}^D(v \overset{\geq 0}{\longleftrightarrow} w) \cdot e^{-cK^2}. \end{aligned}$$

This yields (4.3) and thus completes the proof. $\square$

Lemma 4.2 directly implies the following property, which will be used repeatedly.

Corollary 4.3. *For any $d \geq 3$, there exist $C, c > 0$ such that for any $T > 0$, distinct $v, v', w \in \tilde{\mathbb{Z}}^d$ with $|v - w| \geq d$, and event $\mathbf{A}$ measurable with respect to $\mathcal{C}_{v'}$,*

$$(4.5) \quad \mathbb{P}(\hat{\mathcal{L}}_{1/2}^v \geq T \mid \{v \leftrightarrow w \mid \mathbf{A}\}) \leq Ce^{-cT}.$$

Proof. By applying the restriction property and Lemma 4.2, we have

$$(4.6) \quad \begin{aligned} & \mathbb{P}(\hat{\mathcal{L}}_{1/2}^v \geq T, \{v \leftrightarrow w \mid \mathbf{A}\}) \\ & \stackrel{(\text{restriction property})}{=} \mathbb{E}[\mathbb{1}_{\mathbf{A}} \cdot \mathbb{P}(\hat{\mathcal{L}}_{1/2}^{\mathcal{C}_{v'}, v} \geq T, v \xleftrightarrow{(\mathcal{C}_{v'})} w \mid \mathcal{F}_{\mathcal{C}_{v'}})] \\ & \stackrel{(\text{Lemma 4.2})}{\leq} Ce^{-cT} \cdot \mathbb{E}[\mathbb{1}_{\mathbf{A}} \cdot \mathbb{P}(v \xleftrightarrow{(\mathcal{C}_{v'})} w \mid \mathcal{F}_{\mathcal{C}_{v'}})] \\ & \stackrel{(\text{restriction property})}{=} Ce^{-cT} \cdot \mathbb{P}(v \leftrightarrow w \mid \mathbf{A}). \end{aligned}$$

This gives the desired bound (4.5). $\square$

Recall $\mathbf{p}_{v \leftrightarrow w, a, b}$ below (3.83). The subsequent lemma shows that $\mathbf{p}_{v \leftrightarrow w, a, b}$ can be approximated by the density of a product measure. For any $z \in \tilde{\mathbb{Z}}^d$ and $u \geq 0$, we denote $\mathbf{q}_z(u) := \sqrt{\frac{2}{\pi \tilde{G}(z, z)}} \cdot e^{-\frac{u}{\tilde{G}(z, z)}}$.

Lemma 4.4. *For any $d \geq 3$, $v, w \in \tilde{\mathbb{Z}}^d$ with $|v - w| \geq d$, and $0 \leq a, b \leq |v - w|^{d-2}$,*

$$(4.7) \quad \left| \frac{\mathbf{p}_{v \leftrightarrow w, a, b}}{\bar{\mathbf{p}}_{v \leftrightarrow w, a, b}} - 1 \right| \lesssim (ab + 1) \cdot |v - w|^{2-d},$$

where $\bar{\mathbf{p}}_{v \leftrightarrow w, a, b} := \mathbb{K}_{\{v, w\}}(v, w) \cdot \mathbf{q}_v(a) \cdot \mathbf{q}_w(b)$.

Proof. Recalling the formula for $\mathbf{p}_{v \leftrightarrow w, a, b}(\cdot)$ in (3.84) and letting $\delta = \frac{\tilde{G}(v, w)}{\sqrt{\tilde{G}(v, v)\tilde{G}(w, w)}}$ as in (3.85), we have

$$(4.8) \quad \mathbf{p}_{v \leftrightarrow w, a, b} = \prod_{1 \leq i \leq 5} \mathbf{p}^{(i)},$$

where $\mathbf{p}^{(1)} = 1 - e^{-4\sqrt{ab} \cdot \mathbb{K}_{\{v, w\}}(v, w)}$, $\mathbf{p}^{(2)} = (2\pi \sqrt{ab \tilde{G}(v, v) \tilde{G}(w, w) (1 - \delta^2)})^{-1}$, $\mathbf{p}^{(3)} = \exp(-\frac{a}{(1 - \delta^2) \tilde{G}(v, v)})$, $\mathbf{p}^{(4)} = \exp(-\frac{b}{(1 - \delta^2) \tilde{G}(w, w)})$ and $\mathbf{p}^{(5)} = \exp(\frac{2\delta \sqrt{ab}}{(1 - \delta^2) \sqrt{\tilde{G}(v, v) \tilde{G}(w, w)}})$.

In addition, we define $\bar{\mathbf{p}}^{(1)} = 4\sqrt{ab} \cdot \mathbb{K}_{\{v, w\}}(v, w)$, and define $\bar{\mathbf{p}}^{(i)}$ for $2 \leq i \leq 5$ by replacing δ with 0 in $\mathbf{p}^{(i)}$. Clearly, one has

$$(4.9) \quad \bar{\mathbf{p}}_{v \leftrightarrow w, a, b} = \prod_{1 \leq i \leq 5} \bar{\mathbf{p}}^{(i)}.$$

Next, we estimate $\left| \frac{\mathbf{p}^{(i)} - \bar{\mathbf{p}}^{(i)}}{\bar{\mathbf{p}}^{(i)}} \right|$ for $1 \leq i \leq 5$ separately.

For $i = 1$. Since $|1 - e^{-u} - u| \leq \frac{1}{2}u^2$ for $u \geq 0$ and $\mathbb{K}_{\{v, w\}}(v, w) \asymp |v - w|^{2-d}$,

$$(4.10) \quad \left| \frac{\mathbf{p}^{(1)} - \bar{\mathbf{p}}^{(1)}}{\bar{\mathbf{p}}^{(1)}} \right| \leq \frac{1}{2} \bar{\mathbf{p}}^{(1)} \asymp \sqrt{ab} \cdot |v - w|^{2-d}.$$

For $i = 2$. Since $\delta = \frac{\tilde{G}(v,w)}{\sqrt{\tilde{G}(v,v)\tilde{G}(w,w)}} \asymp |v - w|^{2-d}$, one has

$$(4.11) \quad \left| \frac{\mathbf{p}^{(2)} - \bar{\mathbf{p}}^{(2)}}{\bar{\mathbf{p}}^{(2)}} \right| = \frac{\delta^2}{1 - \delta^2} \lesssim |v - w|^{2(2-d)}.$$

For $i = 3$. Since $|e^{-u} - 1| \leq u$ for $u \geq 0$, one has

$$(4.12) \quad \left| \frac{\mathbf{p}^{(3)} - \bar{\mathbf{p}}^{(3)}}{\bar{\mathbf{p}}^{(3)}} \right| = \left| e^{-\frac{a\delta^2}{(1-\delta^2)\tilde{G}(v,v)}} - 1 \right| \lesssim a \cdot |v - w|^{2(2-d)}.$$

For $i = 4$. For the same reason, we have

$$(4.13) \quad \left| \frac{\mathbf{p}^{(4)} - \bar{\mathbf{p}}^{(4)}}{\bar{\mathbf{p}}^{(4)}} \right| = \left| e^{-\frac{b\delta^2}{(1-\delta^2)\tilde{G}(w,w)}} - 1 \right| \lesssim b \cdot |v - w|^{2(2-d)}.$$

For $i = 5$. By $|e^u - 1| \lesssim u$ for $0 \leq u \leq C$, we have

$$(4.14) \quad \left| \frac{\mathbf{p}^{(5)} - \bar{\mathbf{p}}^{(5)}}{\bar{\mathbf{p}}^{(5)}} \right| = \left| \exp\left(\frac{2\delta\sqrt{ab}}{(1-\delta^2)\sqrt{\tilde{G}(v,v)\tilde{G}(w,w)}}\right) - 1 \right| \lesssim \sqrt{ab} \cdot |v - w|^{2-d}.$$

Note that the right-hand sides of (4.10)-(4.14) are all bounded from above by $C(ab+1) \cdot |v - w|^{2-d}$, and that $\mathbf{p}^{(i)} \lesssim \bar{\mathbf{p}}^{(i)}$ for all $2 \leq i \leq 5$. Putting these estimates together, we complete the proof of this lemma:

$$\left| \frac{\mathbf{p}_{v \leftrightarrow w, a, b}}{\bar{\mathbf{p}}_{v \leftrightarrow w, a, b}} - 1 \right| = \left| \sum_{1 \leq i \leq 5} \frac{\mathbf{p}^{(i)} - \bar{\mathbf{p}}^{(i)}}{\bar{\mathbf{p}}^{(i)}} \cdot \prod_{i < j \leq 5} \frac{\mathbf{p}^{(j)}}{\bar{\mathbf{p}}^{(j)}} \right| \lesssim (ab+1) \cdot |v - w|^{2-d}. \quad \square$$

Now we are ready to establish the rigidity lemma.

Proof of Lemma 4.1. Let $n := |v - w|$. It follows from Corollary 4.3 that

$$(4.15) \quad \begin{aligned} & \mathbb{P}(\hat{\mathcal{L}}_{1/2}^v \vee \hat{\mathcal{L}}_{1/2}^w \geq n^{\frac{d}{2}-1}, \{v \leftrightarrow w\} | \mathbf{A}') \\ & \leq C \cdot \exp(-cn^{-\frac{d}{2}+1}) \cdot \mathbb{P}(v \leftrightarrow w | \mathbf{A}'). \end{aligned}$$

Note that on $\{v \leftrightarrow w\} | \mathbf{A}'$, $\hat{\mathcal{L}}_{1/2}^v$ and $\hat{\mathcal{L}}_{1/2}^w$ are both measurable with respect to the cluster $\mathcal{C}_v$. Thus, it remains to show that

$$(4.16) \quad \begin{aligned} \mathbb{I} &:= \mathbb{P}(\{\hat{\mathcal{L}}_{1/2}^v \vee \hat{\mathcal{L}}_{1/2}^w \leq n^{\frac{d}{2}-1}, \mathbf{A}, v \leftrightarrow w\} | \mathbf{A}') \\ &\lesssim \mathbb{P}(\mathbf{A} \mid v \leftrightarrow w) \cdot \mathbb{P}(v \leftrightarrow w | \mathbf{A}'). \end{aligned}$$

Recall the notations $\check{\mathbb{P}}$ and $\check{\mathbb{E}}$ after (3.85). Using the switching identity, we have

$$(4.17) \quad \mathbb{I} = \int_{0 \leq a, b \leq n^{\frac{d}{2}-1}} \check{\mathbb{E}}_{v \leftrightarrow w, a, b} [\mathbb{1}_{\check{\mathbf{A}}} \cdot \mathbb{P}(\check{\mathbf{A}}')] \mathbf{p}_{v \leftrightarrow w, a, b} da db,$$

where $\check{\mathbf{A}}$ (resp. $\check{\mathbf{A}}'$) is the analogue of the event $\mathbf{A}$ (resp. $\mathbf{A}'$) obtained by replacing $\mathcal{C}_v$ (resp. $\mathcal{C}_{v'}$) with $\check{\mathcal{C}}$ (resp. the cluster of $\check{\mathcal{L}}_{1/2}^{\check{\mathcal{C}}}$ containing v'). Since $\mathbb{1}_{\check{\mathbf{A}}}$ is increasing

and $\mathbb{P}(\check{\mathbf{A}}')$ is decreasing (with respect to $\{\mathcal{P}^{(i)}\}_{i=1}^4$), it follows from (4.17) and the FKG inequality (which applies as $\check{\mathbb{P}}_{v \leftrightarrow w, a, b}$ is essentially a product measure) that

$$(4.18) \quad \mathbb{I} \leq \int_{0 \leq a, b \leq n^{\frac{d}{2}-1}} \check{\mathbb{P}}_{v \leftrightarrow w, a, b}(\check{\mathbf{A}}) \check{\mathbb{E}}_{v \leftrightarrow w, a, b}[\mathbb{P}(\check{\mathbf{A}}')] \mathfrak{p}_{v \leftrightarrow w, a, b} da db.$$

Recall $\mathbf{q}$ and $\bar{\mathbf{p}}$ in Lemma 4.4. For any $z \in \mathbb{Z}^d$, we define the normalization factor $\lambda_{z, n} := \int_{0 \leq u \leq n^{\frac{d}{2}-1}} \mathbf{q}_z(u) du$, and the probability measure $\bar{\mathbf{q}}_{z, n}(\cdot) := \lambda_{z, n}^{-1} \cdot \mathbf{q}_z(\cdot)$. Note that $\lambda_{z, n} \asymp 1$. Moreover, for any $0 \leq a, b \leq n^{\frac{d}{2}-1}$, it follows from Lemma 4.4 that

$$(4.19) \quad \mathfrak{p}_{v \leftrightarrow w, a, b} \lesssim \bar{\mathfrak{p}}_{v \leftrightarrow w, a, b} = \mathbb{K}_{\{v, w\}}(v, w) \lambda_{v, n} \lambda_{w, n} \cdot \bar{\mathbf{q}}_{v, n}(a) \bar{\mathbf{q}}_{w, n}(b).$$

Plugging (4.19) into (4.18), we obtain

$$(4.20) \quad \begin{aligned} \mathbb{I} &\lesssim \mathbb{K}_{\{v, w\}}(v, w) \lambda_{v, n} \lambda_{w, n} \\ &\quad \cdot \int_{0 \leq a, b \leq n^{\frac{d}{2}-1}} \check{\mathbb{P}}_{v \leftrightarrow w, a, b}(\check{\mathbf{A}}) \check{\mathbb{E}}_{v \leftrightarrow w, a, b}[\mathbb{P}(\check{\mathbf{A}}')] \bar{\mathbf{q}}_{v, n}(a) \bar{\mathbf{q}}_{w, n}(b) da db \\ &\lesssim \mathbb{K}_{\{v, w\}}(v, w) \lambda_{v, n} \lambda_{w, n} \cdot \int_{0 \leq a, b \leq n^{\frac{d}{2}-1}} \check{\mathbb{P}}_{v \leftrightarrow w, a, b}(\check{\mathbf{A}}) \bar{\mathbf{q}}_{v, n}(a) \bar{\mathbf{q}}_{w, n}(b) da db \\ &\quad \cdot \int_{0 \leq a, b \leq n^{\frac{d}{2}-1}} \check{\mathbb{E}}_{v \leftrightarrow w, a, b}[\mathbb{P}(\check{\mathbf{A}}')] \bar{\mathbf{q}}_{v, n}(a) \bar{\mathbf{q}}_{w, n}(b) da db \\ &= [\mathbb{K}_{\{v, w\}}(v, w) \lambda_{v, n} \lambda_{w, n}]^{-1} \cdot \int_{0 \leq a, b \leq n^{\frac{d}{2}-1}} \check{\mathbb{P}}_{v \leftrightarrow w, a, b}(\check{\mathbf{A}}) \bar{\mathfrak{p}}_{v \leftrightarrow w, a, b} da db \\ &\quad \cdot \int_{0 \leq a, b \leq n^{\frac{d}{2}-1}} \check{\mathbb{E}}_{v \leftrightarrow w, a, b}[\mathbb{P}(\check{\mathbf{A}}')] \bar{\mathfrak{p}}_{v \leftrightarrow w, a, b} da db, \end{aligned}$$

where in the second inequality we used the FKG inequality for product measures (note that $\check{\mathbb{P}}_{v \leftrightarrow w, a, b}(\check{\mathbf{A}})$ is increasing and $\check{\mathbb{E}}_{v \leftrightarrow w, a, b}[\mathbb{P}(\check{\mathbf{A}}')]$ is decreasing in both a and b). By Lemma 4.4 and the switching identity, the first integral on the right-hand side of (4.20) is bounded from above by

$$C \int_{0 \leq a, b \leq n^{\frac{d}{2}-1}} \check{\mathbb{P}}_{v \leftrightarrow w, a, b}(\check{\mathbf{A}}) \mathfrak{p}_{v \leftrightarrow w, a, b} da db \lesssim \mathbb{P}(\mathbf{A}, v \leftrightarrow w).$$

For the same reason, the second integral is upper-bounded by a constant multiple of $\mathbb{P}(v \leftrightarrow w \| \mathbf{A}')$. Putting these estimates together, we obtain (4.16):

$$(4.21) \quad \mathbb{I} \lesssim \frac{\mathbb{P}(\mathbf{A}, v \leftrightarrow w) \cdot \mathbb{P}(v \leftrightarrow w \| \mathbf{A}')}{\mathbb{K}_{\{v, w\}}(v, w) \lambda_{v, n} \lambda_{w, n}} \asymp \mathbb{P}(\mathbf{A} \mid v \leftrightarrow w) \cdot \mathbb{P}(v \leftrightarrow w \| \mathbf{A}'),$$

where in the last inequality we used $\lambda_{v, n} \asymp \lambda_{w, n} \asymp 1$ and

$$(4.22) \quad \mathbb{K}_{\{v, w\}}(v, w) \stackrel{(\text{Lemma 3.1})}{\asymp} |v - w|^{2-d} \stackrel{(1.3)}{\asymp} \mathbb{P}(v \leftrightarrow w).$$

Combining (4.15) and (4.21), we obtain (4.1). $\square$

4.1. Separation Lemma. As promised above, in this subsection we aim to derive the separation lemma, which asserts that the extremities of two coexisting clusters are likely to be separated. To state the lemma precisely, we first introduce some notations as follows.

- For any $n \geq 1$ and $N \geq 10n$, let $\Psi(n, N)$ denote the collection of quadruples (v_1, v_2, w_1, w_2) such that $v_1, v_2 \in \tilde{B}(10n) \setminus \tilde{B}(n)$ with $|v_1 - v_2| \geq n$, and $w_1, w_2 \in \tilde{B}(10N) \setminus \tilde{B}(N)$ with $|w_1 - w_2| \geq N$.
- For any $\psi = (v_1, v_2, w_1, w_2)$, we define the event

$$(4.23) \quad \mathcal{C}[\psi] := \{v_1 \leftrightarrow w_1 \mid v_2 \leftrightarrow w_2\}.$$

In addition, for any $\delta \in (0, 1)$, we define $\mathcal{S}[\psi, \delta] := \cup_{i \in \{1, 2\}} \mathcal{S}_i[\psi, \delta]$, where

$$(4.24) \quad \mathcal{S}_i[\psi, \delta] := \{v_i \leftrightarrow B_{v_{3-i}}(\delta n) \cup B_{w_{3-i}}(\delta N) \cup B(\delta n) \cup [B(\delta^{-1}N)]^c\}.$$

We then denote $\hat{\mathcal{C}}[\psi, \delta] := \mathcal{C}[\psi] \cap (\mathcal{S}[\psi, \delta])^c$.

Lemma 4.5 (separation lemma). *For any $d \geq 3$ with $d \neq 6$, and $\epsilon > 0$, there exist $C_5(d, \epsilon), c_6(d, \epsilon) > 0$ such that for any $n \geq C_5$, $N \geq C_5 n$ and $\psi \in \Psi(n, N)$,*

$$(4.25) \quad \mathbb{P}(\mathcal{S}[\psi, c_6] \mid \mathcal{C}[\psi]) \leq \epsilon.$$

Consequently, there exists $c_7(d, \lambda) > 0$ such that

$$(4.26) \quad \mathbb{P}(\hat{\mathcal{C}}[\psi, c_7]) \asymp \mathbb{P}(\mathcal{C}[\psi]), \quad \forall \psi \in \Psi(n, N).$$

According to the rigidity lemma (see Lemma 4.1), it suffices to prove the corresponding property in terms of a single cluster. To this end, we present the following lemma, which also plays an important role in later sections.

Lemma 4.6. *For any $d \geq 3$ with $d \neq 6$, and $\epsilon > 0$, there exist $C_6(d, \epsilon), C_7(d, \epsilon)$ and $c_8(d, \epsilon) > 0$ such that the following holds: for any $N \geq C_6$, $y_1 \in \tilde{B}(100N)$, $y_2 \in [\tilde{B}_{y_1}(N)]^c$, $z_1, z_2 \in \tilde{B}(100N)$ with $\text{dist}(\{y_1, y_2\}, \{z_1, z_2\}) \geq \frac{N}{100}$, and any path $\tilde{\eta}$ that consists of at most $2d$ line segments on $\tilde{\mathbb{Z}}^d$, connects z_1 and z_2 , and satisfies $\text{vol}(\text{ran}(\tilde{\eta})) \leq 200dN$ and $\text{dist}(\text{ran}(\tilde{\eta}), v) \geq \frac{N}{100}$,*

$$(4.27) \quad \mathbb{P}(y_1 \leftrightarrow \cup_{v \in \text{ran}(\tilde{\eta})} B_v(c_8 N) \cup \partial B(C_7 N) \mid y_1 \leftrightarrow y_2) \leq \epsilon.$$

Particularly, when $z_1 = z_2 = z$, the path $\tilde{\eta}$ degenerates to the single point z . Thus,

$$(4.28) \quad \mathbb{P}(y_1 \leftrightarrow B_z(c_8 N) \cup \partial B(C_7 N) \mid y_1 \leftrightarrow y_2) \leq \epsilon.$$

To achieve this, we need the following technical lemma:

Lemma 4.7. *For any $3 \leq d \leq 5$ and $\epsilon > 0$, there exist $C_8(\epsilon), c_9(\epsilon) > 0$ such that for any $N \geq C_8$, $v \in \tilde{B}(100N)$, and any line segment $\tilde{\eta} \subset \tilde{B}(200dN)$ of length at most N satisfying $\text{dist}(\text{ran}(\tilde{\eta}), v) \geq \frac{N}{100}$,*

$$(4.29) \quad \tilde{\mathbb{P}}_v \times \mathbb{P}(\tau_{\tilde{\mathcal{C}}_{\tilde{D}}} < \infty) \leq \epsilon,$$

where the set $\tilde{D}$ is defined as $\cup_{w \in \text{ran}(\tilde{\eta})} B_w(c_9 N)$.

Proof. Without loss of generality, we assume that ϵ is sufficiently small. Using the isomorphism theorem, we have

$$\begin{aligned}
& \tilde{\mathbb{P}}_v \times \mathbb{P}(\tau_{\mathcal{C}_{\tilde{D}}} < \infty) \asymp \tilde{\mathbb{P}}_v \times \mathbb{P}(\tau_{\mathcal{C}_{\tilde{D}}^+} < \infty) \\
& \leq \tilde{\mathbb{P}}_v \times \mathbb{P}(\tau_{\mathcal{C}_{\tilde{D}}^+} < \infty, \mathcal{C}_{\tilde{D}}^+ \subset \tilde{B}(\epsilon^{-3}N), v \notin \mathcal{C}_{\tilde{D}}^+) \\
(4.30) \quad & + \mathbb{P}(\tilde{D} \xrightarrow{\geq 0} \partial B(\epsilon^{-3}N)) + \mathbb{P}(v \xrightarrow{\geq 0} \tilde{D}) \\
& \stackrel{(1.4), (1.7)}{\leq} \tilde{\mathbb{P}}_v \times \mathbb{P}(\tau_{\mathcal{C}_{\tilde{D}}^+} < \infty, \mathcal{C}_{\tilde{D}}^+ \subset \tilde{B}(\epsilon^{-3}N), v \notin \mathcal{C}_{\tilde{D}}^+) + C(\epsilon^{\frac{3d}{2}-3} + N^{-\frac{d}{2}+1}).
\end{aligned}$$

Thus, by taking a sufficiently large $C_8(d, \epsilon)$ (such that $C(\epsilon^{\frac{3d}{2}-3} + N^{-\frac{d}{2}+1}) < \frac{1}{2}\epsilon$), it suffices to show that

$$(4.31) \quad \tilde{\mathbb{P}}_v \times \mathbb{P}(\tau_{\mathcal{C}_{\tilde{D}}^+} < \infty, \mathcal{C}_{\tilde{D}}^+ \subset \tilde{B}(\epsilon^{-3}N), v \notin \mathcal{C}_{\tilde{D}}^+) \leq \frac{1}{2}\epsilon.$$

In what follows, we establish (4.31) using the exploration martingale argument, which was originally introduced in [30] and later developed in [8, 4]. Precisely, for any $t \geq 0$, we define $\mathcal{C}_t$ as the collection of points $z \in \tilde{\mathbb{Z}}^d$ such that there exists a path of length at most t in $\tilde{E}^{\geq 0}$ connecting z and $\tilde{D}$. Note that $\mathcal{C}_0 = \tilde{D}$ and $\mathcal{C}_\infty = \mathcal{C}_{\tilde{D}}^+$. In addition, since $\mathcal{C}_t$ is increasing in t , we have that

$$(4.32) \quad \mathcal{M}_t := \mathbb{E}[\tilde{\phi}_v \mid \mathcal{F}_{\mathcal{C}_t}]$$

for $t \geq 0$ is a martingale. In particular,

$$(4.33) \quad \mathcal{M}_0 = \sum_{z \in \tilde{\partial}\tilde{D}} \tilde{\mathbb{P}}_v(\tau_{\tilde{D}} = \tau_z < \infty) \cdot \tilde{\phi}_z.$$

On the event $\{v \notin \mathcal{C}_{\tilde{D}}^+\}$, one has $\mathcal{M}_\infty = 0$. Moreover, referring to [8, Corollary 10], the quadratic variation of $\mathcal{M}_t$ can be written as

$$(4.34) \quad \langle \mathcal{M} \rangle_t = \sum_{z \in \tilde{\partial}\mathcal{C}_t} \tilde{\mathbb{P}}_v(\tau_{\mathcal{C}_t} = \tau_z < \infty) \cdot \tilde{G}(z, v).$$

In addition, when $\{\mathcal{C}_{\tilde{D}}^+ \subset \tilde{B}(\epsilon^{-3}N)\}$ occurs, one has $\tilde{G}(z, v) \gtrsim (\epsilon^{-3}N)^{2-d}$ and thus,

$$(4.35) \quad \langle \mathcal{M} \rangle_t \gtrsim (\epsilon^{-3}N)^{2-d} \cdot \tilde{\mathbb{P}}_v(\tau_{\mathcal{C}_t} < \infty), \quad \forall t \geq 0.$$

As a result, there exists $c_\dagger > 0$ such that for any $\delta > 0$,

$$\begin{aligned}
& \mathbb{P}(\mathcal{C}_{\tilde{D}}^+ \subset \tilde{B}(\epsilon^{-3}N), v \notin \mathcal{C}_{\tilde{D}}^+, \tilde{\mathbb{P}}_v(\tau_{\mathcal{C}_{\tilde{D}}^+} < \infty) \geq \delta) \\
(4.36) \quad & \leq \mathbb{P}(\mathcal{C}_{\tilde{D}}^+ \subset \tilde{B}(\epsilon^{-3}N), v \notin \mathcal{C}_{\tilde{D}}^+, \langle \mathcal{M} \rangle_\infty \geq c_\dagger \delta (\epsilon^{-3}N)^{2-d}) \\
& \leq \mathbb{P}(|\mathcal{M}_0| \geq [\ln(c_9^{-1})]^{-1} \epsilon^{-1} N^{-\frac{d}{2}+1}) \\
& + \mathbb{P}(|\mathcal{M}_0| \leq [\ln(c_9^{-1})]^{-1} \epsilon^{-1} N^{-\frac{d}{2}+1}, \langle \mathcal{M} \rangle_\infty \geq c_\dagger \delta (\epsilon^{-3}N)^{2-d}).
\end{aligned}$$

Note that $\mathcal{M}_0$ is a mean-zero Gaussian random variable with variance

$$(4.37) \quad \text{Var}(\mathcal{M}_0) = \sum_{z_1, z_2 \in \tilde{\partial}\tilde{D}} \tilde{\mathbb{P}}_v(\tau_{\tilde{D}} = \tau_{z_1} < \infty) \cdot \tilde{\mathbb{P}}_v(\tau_{\tilde{D}} = \tau_{z_2} < \infty) \cdot \tilde{G}(z_1, z_2).$$

We claim that

$$(4.38) \quad \text{Var}(\mathcal{M}_0) \lesssim [\ln(c_9^{-1})]^{-2} N^{2-d}.$$

Before proving this claim, we first derive the bound (4.31) using it. For the first term on the right-hand side of (4.36), by (4.38) and the tail estimate for the normal distribution (see e.g., [4, Lemma 2.11]), we have

$$(4.39) \quad \mathbb{P}(|\mathcal{M}_0| \geq [\ln(c_9^{-1})]^{-1} \epsilon^{-1} N^{-\frac{d}{2}+1}) \lesssim e^{-c\epsilon^{-2}}.$$

For the second term, by the martingale representation theorem, the following process is a standard Brownian motion stopped at time $\langle \mathcal{M} \rangle_\infty$ (see [8, Theorem 11]):

$$(4.40) \quad W_t := \begin{cases} \mathcal{M}_{T_t} - \mathcal{M}_0 & \forall 0 \leq t \leq \langle \mathcal{M} \rangle_\infty; \\ \mathcal{M}_\infty - \mathcal{M}_0 & \forall t \geq \langle \mathcal{M} \rangle_\infty, \end{cases}$$

where $T_t := \inf\{s > 0 : \langle \mathcal{M} \rangle_s \geq t\}$. Therefore, for any $\delta \in [0, \frac{1}{4}\epsilon]$, we have

$$(4.41) \quad \begin{aligned} & \mathbb{P}(\langle \mathcal{M} \rangle_\infty \geq c_{\dagger} \delta (\epsilon^{-3} N)^{2-d} \mid \mathcal{F}_{\mathcal{C}_0}) \cdot \mathbb{1}_{|\mathcal{M}_0| \leq [\ln(c_9^{-1})]^{-1} \epsilon^{-1} N^{-\frac{d}{2}+1}} \\ & \stackrel{(\text{symmetry})}{\leq} \mathbb{P}(\sup_{0 \leq t \leq \frac{1}{4}\epsilon \cdot c_{\dagger} (\epsilon^{-3} N)^{2-d}} W_t \leq [\ln(c_9^{-1})]^{-1} \epsilon^{-1} N^{-\frac{d}{2}+1}) \\ & \stackrel{(\text{reflection principle})}{=} 1 - 2\mathbb{P}(W_{\frac{1}{4}c_{\dagger}\epsilon^{3d-5}N^{2-d}} \geq [\ln(c_9^{-1})]^{-1} \epsilon^{-1} N^{-\frac{d}{2}+1}) \\ & \lesssim [\ln(c_9^{-1})]^{-2} \epsilon^{3-3d}. \end{aligned}$$

Putting (4.36), (4.39) and (4.41) together, we get: for any $\delta \geq \frac{1}{4}\epsilon$,

$$(4.42) \quad \mathbb{P}(\mathcal{C}_{\tilde{D}}^+ \subset \tilde{B}(\epsilon^{-3}N), v \notin \mathcal{C}_{\tilde{D}}^+, \tilde{\mathbb{P}}_v(\tau_{\mathcal{C}_{\tilde{D}}^+} < \infty) \geq \delta) \lesssim e^{-c\epsilon^{-2}} + [\ln(c_9^{-1})]^{-2} \epsilon^{3-3d}.$$

Thus, by taking $c_9 = e^{-\epsilon^{-2d}}$, we obtain the desired bound (4.31):

$$(4.43) \quad \begin{aligned} & \tilde{\mathbb{P}}_v \times \mathbb{P}(\tau_{\mathcal{C}_{\tilde{D}}^+} < \infty, \mathcal{C}_{\tilde{D}}^+ \subset \tilde{B}(\epsilon^{-3}N), v \notin \mathcal{C}_{\tilde{D}}^+) \\ & \leq \frac{1}{4}\epsilon + \int_{\frac{1}{4}\epsilon \leq \delta \leq 1} \mathbb{P}(\mathcal{C}_{\tilde{D}}^+ \subset \tilde{B}(\epsilon^{-3}N), v \notin \mathcal{C}_{\tilde{D}}^+, \tilde{\mathbb{P}}_v(\tau_{\mathcal{C}_{\tilde{D}}^+} < \infty) \geq \delta) d\delta \\ & \leq \frac{1}{4}\epsilon + C(e^{-c\epsilon^{-2}} + [\ln(c_9^{-1})]^{-2} \epsilon^{3-3d}) \leq \frac{1}{2}\epsilon. \end{aligned}$$

It remains to verify the claim (4.38). To achieve this, by reversing the Brownian motion and using the strong Markov property, we have: for any $z \in \tilde{\partial}\tilde{D}$,

$$\begin{aligned} \tilde{\mathbb{P}}_v(\tau_{\tilde{D}} = \tau_z < \infty) & \lesssim \sum_{x \in \mathbb{Z}^d: d \leq |x-z| \leq 2d} \tilde{\mathbb{P}}_x(\tau_v < \tau_{\tilde{D}}) \\ & \leq \sum_{x \in \mathbb{Z}^d: d \leq |x-z| \leq 2d} \tilde{\mathbb{P}}_x(\tau_{\partial B_z(c_9 N)} < \tau_{\tilde{D}}) \cdot \max_{w \in \partial B_z(c_9 N)} \tilde{\mathbb{P}}_w(\tau_{\partial B_z(\frac{N}{200})} < \tau_{\tilde{D}}) \\ & \quad \cdot \max_{w' \in \partial B_z(\frac{N}{200})} \tilde{\mathbb{P}}_{w'}(\tau_v < \infty). \end{aligned}$$

Next, we bound the probabilities on the right-hand side individually. For the first probability, by the potential theory we have

$$(4.44) \quad \tilde{\mathbb{P}}_x(\tau_{\partial B_z(c_9 N)} < \tau_{\tilde{D}}) \lesssim (c_9 N)^{-1}.$$

For the second one, recall that $\tilde{D}$ is an orthotope consisting of $O(c_9^{-1})$ boxes of size $c_9 N$. Thus, by the invariance principle, $\tilde{\mathbb{P}}_w(\tau_{\partial B_z(\frac{N}{200})} < \tau_{\tilde{D}})$ (where $|w - z| \asymp c_9 N$) is bounded from above by a constant multiple of the probability that a Brownian motion on $\mathbb{R}^d$, starting at $O(1)$ distance from a cylinder of diameter 1, reaches distance c_9^{-1} without intersecting this cylinder. By projecting the Brownian motion onto the hyperplane orthogonal to the cylinder's axis, the latter probability corresponds to that of a Brownian motion on $\mathbb{R}^{d-1}$, starting at $O(1)$ distance from a unit ball, reaches distance c_9^{-1} without hitting this ball, which is of order $[\ln(c_9^{-1})]^{-1}$ for $d = 3$, and of order 1 for $d \geq 4$ (see e.g., [33, Theorem 3.18]). To sum up,

$$(4.45) \quad \tilde{\mathbb{P}}_w(\tau_{\partial B_z(\frac{N}{200})} < \tau_{\tilde{D}}) \lesssim \mathbb{1}_{d=3} \cdot [\ln(c_9^{-1})]^{-1} + \mathbb{1}_{d \geq 4}.$$

For the third probability, one has

$$(4.46) \quad \max_{w' \in \partial B_z(\frac{N}{200})} \tilde{\mathbb{P}}_{w'}(\tau_v < \infty) \lesssim [\text{dist}(B_z(\frac{N}{200}), v)]^{2-d} \asymp N^{2-d}.$$

Putting these estimates together, we have: for any $z \in \tilde{\partial\tilde{D}}$,

$$(4.47) \quad \tilde{\mathbb{P}}_v(\tau_{\tilde{D}} = \tau_z < \infty) \lesssim c_9^{-1} (\mathbb{1}_{d=3} \cdot [\ln(c_9^{-1})]^{-1} + \mathbb{1}_{d \geq 4}) \cdot N^{1-d}.$$

Plugging (4.47) into (4.37), we obtain that

$$(4.48) \quad \text{Var}(\mathcal{M}_0) \lesssim c_9^{-2} (\mathbb{1}_{d=3} \cdot [\ln(c_9^{-1})]^{-1} + \mathbb{1}_{d \geq 4})^2 N^{2-2d} \cdot \sum_{z_1, z_2 \in \tilde{\partial\tilde{D}}} (|z_1 - z_2| + 1)^{2-d}.$$

Since $\tilde{\eta}$ is a line segment, there exist $O(c_9^{-1})$ points $y_1, \dots, y_K$ in $\tilde{\eta}$ such that $\tilde{D} = \cup_{1 \leq j \leq K} B_{y_j}(c_9 N)$ and $|y_j - y_{j'}| \geq \frac{|j-j'|}{10} \cdot c_9 N$ for all $j \neq j'$. Therefore,

$$(4.49) \quad \begin{aligned} & \sum_{z_1, z_2 \in \tilde{\partial\tilde{D}}} (|z_1 - z_2| + 1)^{2-d} \\ & \leq \sum_{1 \leq j \leq K, z \in \partial B_{y_j}(c_9 N)} \sum_{1 \leq j' \leq K: |j-j'| \leq 100} \sum_{z' \in \partial B_{y_{j'}}(c_9 N)} (|z - z'| + 1)^{2-d} \\ & \quad + \sum_{1 \leq j, j' \leq K: |j-j'| > 100} \sum_{z \in \partial B_{y_j}(c_9 N), z' \in \partial B_{y_{j'}}(c_9 N)} (|z - z'| + 1)^{2-d}. \end{aligned}$$

In addition, the first sum on the right-hand side is at most

$$(4.50) \quad \begin{aligned} & CK \cdot |\partial B_{y_1}(c_9 N)| \cdot \max_{z \in \partial B_{y_j}(c_9 N), |j-j'| \leq 100} \sum_{z' \in \partial B_{y_{j'}}(c_9 N)} (|z - z'| + 1)^{2-d} \\ & \lesssim c_9^{-1} \cdot (c_9 N)^{d-1} \cdot c_9 N = c_9^{d-1} N^d, \end{aligned}$$

where the inequality in the second line follows from a direct calculation (see e.g., [3, (4.4)]). Meanwhile, since $|z - z'| \gtrsim |j - j'| \cdot c_9 N$ for all $|j - j'| \geq 100$, $z \in \partial B_{y_j}(c_9 N)$ and $z' \in \partial B_{y_{j'}}(c_9 N)$, the second sum is bounded from above by

$$(4.51) \quad C |\partial B_{y_1}(c_9 N)|^2 \cdot (c_9 N)^{2-d} \cdot \sum_{1 \leq j < j' \leq K} |j - j'|^{2-d} \lesssim c_9^d \ln(c_9^{-1}) N^d \lesssim c_9^{d-1} N^d.$$

Combining (4.48)-(4.51), we derive the claim (4.38):

$$(4.52) \quad \text{Var}(\mathcal{M}_0) \lesssim c_9^{d-3} (\mathbb{1}_{d=3} \cdot [\ln(c_9^{-1})]^{-1} + \mathbb{1}_{d \geq 4})^2 N^{2-d} \lesssim [\ln(c_9^{-1})]^{-2} N^{2-d}.$$

In conclusion, we have completed the proof of the lemma. $\square$

We now prove Lemma 4.6 using Lemma 4.7.

Proof of Lemma 4.6. For brevity, we denote $D_\dagger := \cup_{v \in \text{ran}(\tilde{\eta})} \tilde{B}_v(c_8 N)$. By Lemma 4.2, there exists $T = T(d, \epsilon) > 0$ such that

$$(4.53) \quad \mathbb{P}(\hat{\mathcal{L}}_{1/2}^{y_1} \vee \hat{\mathcal{L}}_{1/2}^{y_2} \geq T \mid y_1 \leftrightarrow y_2) \leq \frac{1}{2} \epsilon.$$

Meanwhile, using the switching identity, we have

$$(4.54) \quad \begin{aligned} & \mathbb{P}(y_1 \leftrightarrow D_\dagger \cup \partial B(C_7 N), \hat{\mathcal{L}}_{1/2}^{y_1} \vee \hat{\mathcal{L}}_{1/2}^{y_2} \leq T \mid y_1 \leftrightarrow y_2) \\ &= \int_{0 \leq a, b \leq T} \check{\mathbb{P}}_{y_1 \leftrightarrow y_2, a, b}(\check{\mathcal{C}} \cap [D_\dagger \cup \partial B(C_7 N)] \neq \emptyset) \mathbf{p}_{y_1 \leftrightarrow y_2, a, b} da db. \end{aligned}$$

When $\{\check{\mathcal{C}} \cap [D_\dagger \cup \partial B(C_7 N)] \neq \emptyset\}$ occurs, one of the following events happens:

$$(4.55) \quad \mathbf{A}^{(i)} := \left\{ \cup \mathcal{P}^{(i)} \xrightarrow{\cup \mathcal{P}^{(1)}} D_\dagger \cup \partial B(C_7 N) \right\}, \quad \forall i \in \{2, 3, 4\}.$$

Next, we estimate the probability of $\mathbf{A}^{(i)}$ for $i = 2, 3, 4$ separately. For brevity, we denote $\check{\mathbb{P}} := \check{\mathbb{P}}_{y_1 \leftrightarrow y_2, a, b}$.

For $i = 2$. On the event $\mathbf{A}^{(2)}$, either $\{(\cup \mathcal{P}^{(2)}) \cap \partial B_{y_1}(\epsilon^3 N^{1 \square \frac{2}{d-4}}) \neq \emptyset\}$ or $\{B_{y_1}(\epsilon^3 N^{1 \square \frac{2}{d-4}}) \xrightarrow{\cup \mathcal{P}^{(1)}} D_\dagger \cup \partial B(C_7 N)\}$ occurs. Since $a \leq T$, one has

$$(4.56) \quad \begin{aligned} & \check{\mathbb{P}}((\cup \mathcal{P}^{(2)}) \cap \partial B_{y_1}(\epsilon^3 N^{1 \square \frac{2}{d-4}}) \neq \emptyset) \\ & \lesssim T \cdot \mathbf{e}_{y_1, y_1}^{\{y_2\}}(\{\tilde{\eta} : \text{ran}(\tilde{\eta}) \cap \partial B_{y_1}(\epsilon^3 N^{1 \square \frac{2}{d-4}}) \neq \emptyset\}), \end{aligned}$$

where the right-hand side converges to zero as $N \rightarrow \infty$ (by Lemma 3.17). Therefore, by taking a sufficiently large $C_6(d, \epsilon) > 0$ (recall that $N \geq C_6$), we have

$$(4.57) \quad \check{\mathbb{P}}((\cup \mathcal{P}^{(2)}) \cap \partial B_{y_1}(\epsilon^3 N^{1 \square \frac{2}{d-4}}) \neq \emptyset) \leq \frac{1}{12} \epsilon.$$

Meanwhile, since $\mathcal{P}^{(1)} \leq \tilde{\mathcal{L}}_{1/2}$, it follows from (1.7) and (1.8) that

$$(4.58) \quad \check{\mathbb{P}}(B_{y_1}(\epsilon^3 N^{1 \square \frac{2}{d-4}}) \xrightarrow{\cup \mathcal{P}^{(1)}} D_\dagger \cup \partial B(C_7 N)) \leq C \epsilon^{(\frac{3d}{2} - 3) \square (3d - 12)} \leq \frac{1}{12} \epsilon.$$

Thus, by (4.57) and (4.58), we obtain $\check{\mathbb{P}}(\mathbf{A}^{(2)}) \leq \frac{1}{6} \epsilon$.

For $i = 3$. The same argument also implies that $\check{\mathbb{P}}(\mathbf{A}^{(3)}) \leq \frac{1}{6} \epsilon$.

For $i = 4$. By the union bound, one has

$$\begin{aligned}
(4.59) \quad \check{\mathbb{P}}(\mathbf{A}^{(4)}) &\leq \check{\mathbb{P}}(\cup \mathcal{P}^{(4)} \xleftrightarrow{\cup \mathcal{P}^{(1)}} \partial B(C_7 N)) + \check{\mathbb{P}}(\cup \mathcal{P}^{(4)} \xleftrightarrow{\cup \mathcal{P}^{(1)}} D_{\dagger}) \\
&\leq \check{\mathbb{P}}((\cup \mathcal{P}^{(4)}) \cap \partial B(C_7^{\frac{1}{2}} N) \neq \emptyset) \\
&\quad + \check{\mathbb{P}}(\cup \mathcal{P}^{(4)} \subset \tilde{B}(C_7^{\frac{1}{2}} N), \cup \mathcal{P}^{(4)} \xleftrightarrow{\cup \mathcal{P}^{(1)}} \partial B(C_7 N)) \\
&\quad + \check{\mathbb{P}}(\cup \mathcal{P}^{(4)} \xleftrightarrow{\cup \mathcal{P}^{(1)}} D_{\dagger}).
\end{aligned}$$

We denote the three terms on the right-hand side by $\mathbb{I}_1$, $\mathbb{I}_2$ and $\mathbb{I}_3$ respectively.

For $\mathbb{I}_1$, under the measure $\mathbf{e}_{y_1, y_2}(\cdot)$, it follows from its strong Markov property that the probability of an excursion hitting $\partial B(C_7^{\frac{1}{2}} N)$ is at most

$$(4.60) \quad q := \max_{z \in \partial B_v(1)} \tilde{\mathbb{P}}_z(\tau_{\partial B(C_7^{\frac{1}{2}} N)} < \tau_{\{y_1, y_2\}} \mid \tau_{\{y_1, y_2\}} = \tau_{y_2} < \infty),$$

which, by the invariance principle, uniformly converges to zero as $C_7 \rightarrow \infty$. Recall that the number of excursions in $\mathcal{P}^{(4)}$ is a Poisson random variable (we denote its parameter by λ) conditioned to be odd; in addition, λ equals ab times the total mass of $\mathbf{e}_{v, w}$, and is therefore at most $T^2 N^{2-d} \ll 1$ (for a large enough C_6). Thus,

$$(4.61) \quad \mathbb{I}_1 \leq 1 - \frac{\sum_{k \geq 1: k \text{ odd}} e^{-\lambda} \cdot \frac{\lambda^k}{k!} \cdot (1-q)^k}{\sum_{k \geq 1: k \text{ odd}} e^{-\lambda} \cdot \frac{\lambda^k}{k!}} = \frac{\sinh(\lambda) - \sinh(\lambda(1-q))}{\sinh(\lambda)} \asymp q.$$

Consequently, by taking a sufficiently large $C_7(d, \epsilon) > 0$, we have $\mathbb{I}_1 \leq \frac{1}{18}\epsilon$.

We now estimate $\mathbb{I}_2$ separately for $3 \leq d \leq 5$ and $d \geq 7$. When $3 \leq d \leq 5$, by $\mathcal{P}^{(1)} \leq \tilde{\mathcal{L}}_{1/2}$ and (1.7), one has

$$(4.62) \quad \mathbb{I}_2 \lesssim \rho_d(C_7^{\frac{1}{2}} N, C_7 N) \lesssim C_7^{-\frac{d}{4} + \frac{1}{2}}.$$

When $d \geq 7$, by the union bound, we have

$$\begin{aligned}
(4.63) \quad \mathbb{I}_2 &\leq \sum_{x \in B(C_7^{\frac{1}{2}} N + 1)} \check{\mathbb{P}}((\cup \mathcal{P}^{(4)}) \cap \tilde{B}_x(1) \neq \emptyset) \check{\mathbb{P}}(x \xleftrightarrow{\cup \mathcal{P}^{(1)}} \partial B(C_7 N)) \\
&\lesssim_{\mathcal{P}^{(1)} \leq \tilde{\mathcal{L}}_{1/2}, (1.6)} (C_7 N)^{-2} \sum_{x \in B(C_7^{\frac{1}{2}} N + 1)} \check{\mathbb{P}}((\cup \mathcal{P}^{(4)}) \cap \tilde{B}_x(1) \neq \emptyset).
\end{aligned}$$

It follows from Lemma 3.16 that under $\mathbf{e}_{y_1, y_2}(\cdot)$, the probability of an excursion intersecting $\tilde{B}_x(1)$ is at most $C N^{d-2}(|y_1 - x| + 1)^{2-d}(|x - y_2| + 1)^{2-d}$. Therefore, similar to (4.61), this implies

$$(4.64) \quad \check{\mathbb{P}}((\cup \mathcal{P}^{(4)}) \cap \tilde{B}_x(1) \neq \emptyset) \lesssim N^{d-2}(|y_1 - x| + 1)^{2-d}(|x - y_2| + 1)^{2-d}.$$

Combined with (4.63), it yields that

$$(4.65) \quad \mathbb{I}_2 \lesssim C_7^{-2} N^{d-4} \sum_{x \in \mathbb{Z}^d} (|y_1 - x| + 1)^{2-d}(|x - y_2| + 1)^{2-d} \lesssim C_7^{-2}.$$

Here the second inequality relies on the following fact (see, e.g., [3, Lemma 4.3]), which we will use multiple times: for any $v, w \in \widetilde{\mathbb{Z}}^d$,

$$(4.66) \quad \sum_{x \in \mathbb{Z}^d} (|v - x| + 1)^{2-d} (|x - w| + 1)^{2-d} \lesssim (|v - w| + 1)^{4-d}.$$

By (4.62) and (4.65), we obtain $\mathbb{I}_2 \leq \frac{1}{18}\epsilon$ for all sufficiently large $C_7(d, \epsilon) > 0$.

Next, we estimate $\mathbb{I}_3$ in the cases when $3 \leq d \leq 5$ and $d \geq 7$ separately.

When $3 \leq d \leq 5$. Recalling that $\mathcal{P}^{(1)} = \widetilde{\mathcal{L}}_{1/2}^{\{y_1, y_2\}}$, we have

$$(4.67) \quad \mathbb{I}_3 \leq \check{\mathbb{P}}\left(\bigcup \mathcal{P}^{(4)} \xleftrightarrow{\cup \mathcal{P}^{(1)}} D_{\dagger}\right) = \check{\mathbb{P}}(\exists \tilde{\eta} \in \mathcal{P}^{(4)} \text{ intersecting } \mathcal{C}_{D_{\dagger}}^{\{y_1, y_2\}}).$$

Since $\text{dist}(D_{\dagger}, \{y_1, y_2\}) \gtrsim N$, it follows from (1.7) that

$$(4.68) \quad \check{\mathbb{P}}(\mathbf{F}_{\spadesuit}) := \check{\mathbb{P}}(\mathcal{C}_{D_{\dagger}}^{\{y_1, y_2\}} \cap [B_{y_1}(\epsilon^3 N) \cup B_{y_2}(\epsilon^3 N)] \neq \emptyset) \lesssim \epsilon^{\frac{3d}{2}-3}.$$

Moreover, Lemma 4.7 implies that for a sufficiently small $c_8(d, \epsilon) > 0$,

$$(4.69) \quad \widetilde{\mathbb{P}}_{y_1} \times \mathbb{P}(\tau_{\mathcal{C}_{D_{\dagger}}} < \infty) \leq \epsilon^{3d-2}.$$

Combined with Markov's inequality and $\mathcal{C}_{D_{\dagger}}^{\{y_1, y_2\}} \subset \mathcal{C}_{D_{\dagger}}$, it yields that

$$(4.70) \quad \check{\mathbb{P}}(\mathbf{F}_{\clubsuit}) := \check{\mathbb{P}}(\widetilde{\mathbb{P}}_{y_1}(\tau_{\mathcal{C}_{D_{\dagger}}^{\{y_1, y_2\}}} < \infty) \geq \epsilon^{3d-4}) \leq \epsilon^2.$$

It remain to estimate the probability of $\{\exists \tilde{\eta} \in \mathcal{P}^{(4)} \text{ intersecting } \mathcal{C}_{D_{\dagger}}^{\{y_1, y_2\}}\} \cap \mathbf{F}_{\clubsuit}^c \cap \mathbf{F}_{\clubsuit}^c$. On the event $\mathbf{F}_{\clubsuit}^c \cap \mathbf{F}_{\clubsuit}^c$, by the strong Markov property of $\mathbf{e}_{y_1, y_2}$, one has

$$\begin{aligned} & \mathbf{e}_{y_1, y_2}(\{\tilde{\eta} : \text{ran}(\tilde{\eta}) \cap \mathcal{C}_{D_{\dagger}}^{\{y_1, y_2\}} \neq \emptyset\}) \\ & \lesssim \max_{z \in \partial B_{y_1}(1)} \sum_{v \in \partial \mathcal{C}_{D_{\dagger}}} \widetilde{\mathbb{P}}_z(\tau_{\mathcal{C}_{D_{\dagger}}^{\{y_1, y_2\}}} = \tau_v < \infty) \widetilde{\mathbb{P}}_v(\tau_{y_2} < \infty) \\ & \stackrel{\mathcal{C}_{D_{\dagger}}^{\{y_1, y_2\}} \cap B_{y_2}(\epsilon^3 N) = \emptyset}{\lesssim} (\epsilon^3 N)^{2-d} \max_{z \in \partial B_{y_1}(1)} \widetilde{\mathbb{P}}_z(\tau_{\mathcal{C}_{D_{\dagger}}^{\{y_1, y_2\}}} < \infty) \\ & \stackrel{\mathcal{C}_{D_{\dagger}}^{\{y_1, y_2\}} \cap B_{y_1}(\epsilon^3 N) = \emptyset, \text{Harnack's ineq}}{\asymp} (\epsilon^3 N)^{2-d} \widetilde{\mathbb{P}}_{y_1}(\tau_{\mathcal{C}_{D_{\dagger}}^{\{y_1, y_2\}}} < \infty) \lesssim \epsilon^2 N^{2-d}. \end{aligned}$$

Combined with the fact that the total mass of $\mathbf{e}_{y_1, y_2}(\cdot)$ is of order N^{2-d} (by Lemmas 3.1 and 3.14), it implies that under $\mathbf{e}_{y_1, y_2}(\cdot)$, the probability of an excursion hitting $\mathcal{C}_{D_{\dagger}}$ is at most of order ϵ^2 . Thus, similar to (4.61), this yields

$$(4.71) \quad \check{\mathbb{P}}\left(\bigcup \mathcal{P}^{(4)} \xleftrightarrow{\cup \mathcal{P}^{(1)}} D_{\dagger}, \mathbf{F}_{\clubsuit}^c, \mathbf{F}_{\clubsuit}^c\right) \lesssim \epsilon^2.$$

Combining (4.67), (4.68), (4.70) and (4.71), one has

$$(4.72) \quad \mathbb{I}_3 \leq C(\epsilon^{\frac{3d}{2}-3} + \epsilon^2) \leq \frac{1}{18}\epsilon.$$

When $d \geq 7$. We denote $\hat{D}_\dagger := \cup_{w \in \text{ran}(\tilde{\eta}_\dagger)} B_w(c_8^{\frac{1}{d-2}} N)$. Note that $\text{dist}(\hat{D}_\dagger^c, D_\dagger) \geq \frac{1}{2} c_8^{\frac{1}{d-2}} N$. For the same reason as in proving (4.61), there exists a sufficiently small $c_8(d, \epsilon) > 0$ such that

$$(4.73) \quad \check{\mathbb{P}}((\cup \mathcal{P}^{(4)}) \cap \hat{D}_\dagger \neq \emptyset) \leq \frac{1}{36} \epsilon.$$

Meanwhile, by the union bound, we have

$$(4.74) \quad \begin{aligned} & \check{\mathbb{P}}((\cup \mathcal{P}^{(4)}) \cap \hat{D}_\dagger = \emptyset, \cup \mathcal{P}^{(4)} \xrightarrow{\cup \mathcal{P}^{(1)}} D_\dagger) \\ & \leq \sum_{x \in \mathbb{Z}^d: \text{dist}(x, D_\dagger) \geq \frac{1}{4} c_8^{\frac{1}{d-2}} N} \check{\mathbb{P}}((\cup \mathcal{P}^{(4)}) \cap \tilde{B}_x(1) \neq \emptyset) \cdot \mathbb{P}(x \leftrightarrow D_\dagger). \end{aligned}$$

Moreover, recall from (4.64) that

$$(4.75) \quad \check{\mathbb{P}}((\cup \mathcal{P}^{(4)}) \cap \tilde{B}_x(1) \neq \emptyset) \lesssim N^{d-2} (|y_1 - x| + 1)^{2-d} (|x - y_2| + 1)^{2-d}.$$

We take $O(c_8^{-1})$ points $w_1, \dots, w_K$ in $\tilde{\eta}$ such that $D_\dagger \subset \cup_{1 \leq j \leq K} B_{w_j}(c_8 N)$. Thus, for any $x \in \mathbb{Z}^d$ with $\text{dist}(x, D_\dagger) \geq \frac{1}{4} c_8^{\frac{1}{d-2}} N$, one has

$$(4.76) \quad \begin{aligned} \mathbb{P}(x \leftrightarrow D_\dagger) & \leq \sum_{1 \leq j \leq K} \mathbb{P}(x \leftrightarrow B_{w_j}(c_8 N)) \\ & \lesssim \sum_{1 \leq j \leq K} \left(\frac{|x - w_j|}{c_8 N} \right)^{2-d} \cdot \theta_d(c_8 N) \\ & \stackrel{(1.6), \text{dist}(x, D_\dagger) \leq |x - w_j|}{\lesssim} c_8^{d-6} N^{-2}, \end{aligned}$$

where in the second line we used [5, Proposition 1.5]. Plugging (4.75) and (4.76) into (4.74), we have

$$(4.77) \quad \begin{aligned} & \check{\mathbb{P}}((\cup \mathcal{P}^{(4)}) \cap \hat{D}_\dagger = \emptyset, \cup \mathcal{P}^{(4)} \xrightarrow{\cup \mathcal{P}^{(1)}} D_\dagger) \\ & \lesssim c_8^{d-6} N^{d-4} \sum_{x \in \mathbb{Z}^d} (|y_1 - x| + 1)^{2-d} (|x - y_2| + 1)^{2-d} \lesssim c_8^{d-6}, \end{aligned}$$

where the last inequality follows from the following basic fact (see e.g. [3, (4.5)]):

$$(4.78) \quad \max_{z \in \mathbb{Z}^d} \sum_{z' \in B(M)} (|z - z'| + 1)^{-a} \lesssim M^{(d-a) \vee 0}, \quad \forall a \neq d.$$

It directly follows from (4.77) that for a sufficiently small $c_8(d, \epsilon) > 0$,

$$(4.79) \quad \check{\mathbb{P}}((\cup \mathcal{P}^{(4)}) \cap \hat{D}_\dagger = \emptyset, \cup \mathcal{P}^{(4)} \xrightarrow{\cup \mathcal{P}^{(1)}} D_\dagger) \leq \frac{1}{36} \epsilon.$$

Combined with (4.73), it implies that $\mathbb{I}_3 \leq \frac{1}{18} \epsilon$.

Combining (4.59) and these estimates on $\mathbb{I}_1, \mathbb{I}_2$ and $\mathbb{I}_3$, we obtain $\check{\mathbb{P}}(\mathbf{A}^{(4)}) \leq \frac{1}{6} \epsilon$. Putting this together with the bounds on $\check{\mathbb{P}}(\mathbf{A}^{(2)})$ and $\check{\mathbb{P}}(\mathbf{A}^{(3)})$, one has

$$(4.80) \quad \check{\mathbb{P}}_{y_1 \leftrightarrow y_2, a, b}(\check{\mathcal{C}} \cap [D_\dagger \cup \partial B(C_7 N)] \neq \emptyset) \leq \frac{1}{2} \epsilon, \quad \forall 0 \leq a, b \leq T.$$

Inserting this into (4.54) and then combining with (4.53), we obtain (4.27). $\square$

Now we are ready to prove Lemma 4.5.

Proof of Lemma 4.5. For each $i \in \{1, 2\}$, since the event $S_i[\psi, \delta]$ is increasing with respect to $\mathcal{C}_{v_i}$, it follows from Lemma 4.1 that

$$(4.81) \quad \begin{aligned} & \mathbb{P}(S_i[\psi, \delta] \mid \{v_1 \leftrightarrow w_1 \parallel v_2 \leftrightarrow w_2\}) \\ & \lesssim \mathbb{P}(S_i[\psi, \delta] \mid v_i \leftrightarrow w_i) + \exp(-c_5|v_i - w_i|^{\frac{d}{2}-1}). \end{aligned}$$

By the union bound and (4.28), one can choose a small $c_6(d, \epsilon) > 0$ such that

$$(4.82) \quad \mathbb{P}(S_i[\psi, c_6] \mid v_i \leftrightarrow w_i) \leq \frac{1}{4}\epsilon.$$

Moreover, by taking a sufficiently large $C_5(d, \epsilon)$ (recall that $|v_i - w_i| \asymp N > C_5$),

$$(4.83) \quad \exp(-c_5|v_i - w_i|^{\frac{d}{2}-1}) \leq \frac{1}{4}\epsilon.$$

Combining (4.81), (4.82) and (4.83), we get

$$(4.84) \quad \mathbb{P}(S_i[\psi, c_6] \mid \{v_1 \leftrightarrow w_1 \parallel v_2 \leftrightarrow w_2\}) \leq \frac{1}{2}\epsilon$$

for all $i \in \{1, 2\}$ and $\psi \in \Psi(n, N)$, and thus conclude the proof. $\square$

4.2. Regularity of coexistence. Using the aforementioned separation lemma, we can prove a series of regularity properties of the connecting probability involving two clusters. The first property shows that the probability of $\mathbf{C}[(v_1, v_2, w_1, w_2)]$ is stable under perturbations of v_1, v_2, w_1 and w_2 .

Lemma 4.8. *For any $d \geq 3$ with $d \neq 6$, there exists a constant $C_9 > 0$ such that for any $n \geq C_9$, $N \geq C_9 n$ and $\psi, \psi' \in \Psi(n, N)$,*

$$(4.85) \quad \mathbb{P}(\mathbf{C}[\psi]) \asymp \mathbb{P}(\mathbf{C}[\psi']).$$

Proof. We say two quadruples $(v_1, v_2, w_1, w_2), (v'_1, v'_2, w'_1, w'_2) \in \Psi[n, N]$ are δ -close if there exists $i \in \{1, 2\}$ such that one of the following holds:

- (i) $|v_i - v'_i| \leq \delta n$, and $v_{3-i} = v'_{3-i}, w_1 = w'_1, w_2 = w'_2$;
- (ii) $|w_i - w'_i| \leq \delta N$, and $w_{3-i} = w'_{3-i}, v_1 = v'_1, v_2 = v'_2$.

Clearly, for any $\delta > 0$, there exists $K_{\dagger}(d, \delta) \in \mathbb{N}^+$ such that any two $\psi, \psi' \in \Psi(n, N)$ can be connected by a δ -close sequence of length at most $K_{\dagger}$, i.e., $\exists K \leq K_{\dagger}$ and $\{\psi_i\}_{1 \leq i \leq K} \subset \Psi[n, N]$ satisfying $\psi_1 = \psi$, $\psi_K = \psi'$, and ψ_j is δ -close to ψ_{j+1} for all $1 \leq j \leq K-1$. Consequently, it suffices to show that for some $c_{\star} > 0$, all pairs of $c_{\star}$ -close quadruples $\tilde{\psi}, \hat{\psi} \in \Psi[n, N]$ satisfy

$$(4.86) \quad \mathbb{P}(\mathbf{C}[\tilde{\psi}]) \asymp \mathbb{P}(\mathbf{C}[\hat{\psi}]).$$

Without loss of generality, we consider the case when $\tilde{\psi}$ and $\hat{\psi}$ differ only in their first coordinates. Lemma 4.5 implies that for some $c_{\Delta} > 0$,

$$(4.87) \quad \mathbb{P}(\hat{\mathbf{C}}[\tilde{\psi}, c_{\Delta}]) \asymp \mathbb{P}(\mathbf{C}[\tilde{\psi}]).$$

Assume that $\tilde{\psi} = (\tilde{v}_1, v_2, w_1, w_2)$ and $\hat{\psi} = (\hat{v}_1, v_2, w_1, w_2)$. On the event $\hat{\mathbf{C}}[\tilde{\psi}, c_{\Delta}]$, since $\mathcal{C}_{v_2}$ is disjoint from the $(c_{\Delta}n)$ -neighborhood of $\tilde{v}_1$, it follows from Lemma

3.4 that conditioned on $\mathcal{C}_{v_2}$, the connecting probability between $\tilde{v}_1$ and w_1 is proportional to that between $\hat{v}_1$ and w_1 (since $|\tilde{v}_1 - \hat{v}_1| \leq c_\star n$, it suffices to choose $c_\star \ll c_\Delta$ to ensure the conditions in Lemma 3.4). As a result, we obtain

$$(4.88) \quad \mathbb{P}(\mathbf{C}[\tilde{\psi}]) \stackrel{(4.87)}{\asymp} \mathbb{P}(\hat{\mathbf{C}}[\tilde{\psi}, c_\Delta]) \lesssim \mathbb{P}(\mathbf{C}[\hat{\psi}]).$$

By the same reasoning, we also have $\mathbb{P}(\mathbf{C}[\hat{\psi}]) \lesssim \mathbb{P}(\mathbf{C}[\tilde{\psi}])$. In conclusion, we confirm (4.86) and thus complete the proof. $\square$

The next property gives a quantitative relation between the probability of $\mathbf{C}[\psi]$ and that of a modified event where certain points are replaced by boxes and some restrictions are imposed on the volume and capacity of the cluster. Before showing the result, we record some notations as follows.

- For simplicity, when referring to $\psi = (v_1, v_2, w_1, w_2) \in \Psi(n, N)$, we denote

$$(4.89) \quad z_i^{\text{in}} := v_i, \quad z_i^{\text{out}} := w_i, \quad m_{\text{in}} := n \text{ and } m_{\text{out}} := N.$$

For example, we may denote the event $\mathbf{C}[\psi]$ by $\{z_1^{\text{in}} \leftrightarrow z_1^{\text{out}} \parallel z_2^{\text{in}} \leftrightarrow z_2^{\text{out}}\}$. Moreover, we also use “-in” to represent “out”, and vice versa.

- For $\psi = (v_1, v_2, w_1, w_2) \in \Psi(n, N)$, $\beta \in (0, 1)$ and $\diamond \in \{\text{in}, \text{out}\}$, we define

$$(4.90) \quad \mathbf{C}^\diamond[\psi, \beta] := \{B_{z_1^\diamond}(\beta m_\diamond) \leftrightarrow z_1^{-\diamond} \parallel B_{z_2^\diamond}(\beta m_\diamond) \leftrightarrow z_2^{-\diamond}\}.$$

- For any $\delta \in (0, 1)$, $i \in \{1, 2\}$ and $\diamond \in \{\text{in}, \text{out}\}$, we denote

$$(4.91) \quad \mathbf{S}_i^\diamond[\psi, \delta] := \{z_i^{-\diamond} \leftrightarrow B_{z_{3-i}^\diamond}(\delta m_\diamond) \cup B_{z_{3-i}^{-\diamond}}(\delta m_{-\diamond}) \cup B(\delta n) \cup [B(\delta^{-1}N)]^c\}.$$

Note that $\mathbf{S}_i^{\text{out}}[\psi, \delta]$ is identical to $\mathbf{S}_i[\psi, \delta]$, as defined in (4.24). In addition, we denote $\mathbf{S}^\diamond[\psi, \delta] := \cup_{i \in \{1, 2\}} \mathbf{S}_i^\diamond[\psi, \delta]$

- For any $z, z' \in \tilde{\mathbb{Z}}^d$ and $M \geq 1$, we define

$$(4.92) \quad \mathcal{V}_{z, z'}(M) := \text{vol}(\mathcal{C}_z \cap \tilde{B}_{z'}(M)), \quad \mathcal{T}_{z, z'}(M) := \text{cap}(\mathcal{C}_z \cap \tilde{B}_{z'}(M)).$$

Especially, when $z = z'$, we denote $\mathcal{V}_z := \mathcal{V}_{z, z}$ and $\mathcal{T}_z := \mathcal{T}_{z, z}$ for brevity.

- For $\diamond \in \{\text{in}, \text{out}\}$, we define the truncated event $\hat{\mathbf{C}}^\diamond$ separately in low- and high-dimensional cases as follows:

(i) When $3 \leq d \leq 5$, for any $\gamma > 0$, define

$$\hat{\mathbf{C}}^\diamond[\psi, \beta, \delta, \gamma] := \bigcap_{i \in \{1, 2\}} \{\mathcal{V}_{z_i^{-\diamond}, z_i^\diamond}(\beta m_\diamond) \geq \gamma m_\diamond^{\frac{d}{2}+1}, \mathcal{T}_{z_i^{-\diamond}, z_i^\diamond}(\beta m_\diamond) \geq \gamma m_\diamond^{d-2}\} \cap (\mathbf{S}^\diamond[\psi, \delta])^c;$$

In particular, for any $n \geq 1$, $m \geq Cn$, $N \geq Cm$, $\psi = (v_1, v_2, w_1, w_2) \in \Psi(n, m)$ and $\psi' = (v'_1, v'_2, w'_1, w'_2) \in \Psi(10m, N)$, when $\hat{\mathbf{C}}^{\text{out}}[\psi, \beta, \delta, \gamma]$ and $\hat{\mathbf{C}}^{\text{in}}[\psi', \beta, \delta, \gamma]$ both occur, since the four involved clusters all have capacity cm^{d-2} inside $\tilde{B}(100m)$, one may add two loops to $\tilde{\mathcal{L}}_{1/2}$ that connect $\mathcal{C}_{v_1}$ to $\mathcal{C}_{v'_1}$, and $\mathcal{C}_{v_2}$ to $\mathcal{C}_{v'_2}$ respectively, while changing the probability by only a constant factor; in addition, after this modification, the event $\mathbf{C}^{\text{out}}[(v_1, v_2, w'_1, w'_2), \beta] \cap \{\mathcal{V}_{v_1}(100m) \wedge \mathcal{V}_{v_2}(100m) \geq \gamma m^{\frac{d}{2}+1}\}$

occurs. As shown in later sections, this observation plays a key role in proving the upper bounds in (1.15) and (1.17), and Theorem 1.5 in the low-dimensional case.

(ii) When $d \geq 7$, define

$$(4.93) \quad \widehat{\mathbf{C}}^\diamond[\psi, \beta, \delta] := \mathbf{C}^\diamond[\psi, \beta] \cap (\mathbf{S}^\diamond[\psi, \delta])^c.$$

Here we do not impose the volume and capacity constraints as above because the proofs in high dimensions follow a different approach from the one described in Item (i).

Lemma 4.9. *For any $d \geq 3$ with $d \neq 6$, there exist $C_{10}, c_{10}, c_{11}, c_{12} > 0$ such that for any $n \geq C_{10}$, $N \geq C_{10}n$, $\psi \in \Psi(n, N)$ and $\diamond \in \{\text{in}, \text{out}\}$, the following hold:*

$$(4.94) \quad \text{when } 3 \leq d \leq 5, \quad \mathbb{P}(\mathbf{C}^\diamond[\psi, c_{10}]) \asymp m_\diamond^{d-2} \cdot \mathbb{P}(\mathbf{C}[\psi]) \asymp \mathbb{P}(\widehat{\mathbf{C}}^\diamond[\psi, c_{10}, c_{11}, c_{12}]);$$

$$(4.95) \quad \text{when } d \geq 7, \quad \mathbb{P}(\mathbf{C}^\diamond[\psi, c_{10}]) \asymp m_\diamond^{2d-8} \cdot \mathbb{P}(\mathbf{C}[\psi]) \asymp \mathbb{P}(\widehat{\mathbf{C}}^\diamond[\psi, c_{10}, c_{11}]).$$

Proof. We provide the proof details only for the case $\diamond = \text{in}$, as the case $\diamond = \text{out}$ is similar. Throughout this proof, we assume that $\psi = (v_1, v_2, w_1, w_2)$.

Using the restriction property and Lemma 3.6, we have

$$(4.96) \quad \begin{aligned} \mathbb{P}(\mathbf{C}^{\text{in}}[\psi, c_{10}]) &= \mathbb{E} \left[\mathbb{1}_{B_{v_1}(c_{10}n) \leftrightarrow w_1} \mathbb{P}(B_{v_2}(c_{10}n) \xrightarrow{(c_{w_1})} w_2 \mid \mathcal{F}_{c_{w_1}}) \right] \\ &\stackrel{(\text{Lemma 3.6})}{\lesssim} n^{-(\frac{d}{2} \sqcup 3)} \sum_{x \in \partial \mathcal{B}_{v_2}(c_{10}dn)} \mathbb{E} \left[\mathbb{1}_{B_{v_1}(c_{10}n) \leftrightarrow w_1} \mathbb{P}(x \xrightarrow{(c_{w_1})} w_2 \mid \mathcal{F}_{c_{w_1}}) \right] \\ &= n^{-(\frac{d}{2} \sqcup 3)} \sum_{x \in \partial \mathcal{B}_{v_2}(c_{10}dn)} \mathbb{P}(B_{v_1}(c_{10}n) \leftrightarrow w_1 \parallel x \leftrightarrow w_2). \end{aligned}$$

Similarly, for any $x \in \partial \mathcal{B}_{v_2}(c_{10}dn)$, we have

$$(4.97) \quad \begin{aligned} &\mathbb{P}(B_{v_1}(c_{10}n) \leftrightarrow w_1 \parallel x \leftrightarrow w_2) \\ &= \mathbb{E} \left[\mathbb{1}_{x \leftrightarrow w_2} \mathbb{P}(B_{v_1}(c_{10}n) \xrightarrow{(c_{w_2})} w_1) \right] \\ &\stackrel{(\text{Lemma 3.6})}{\lesssim} n^{-(\frac{d}{2} \sqcup 3)} \sum_{y \in \partial \mathcal{B}_{v_1}(c_{10}dn)} \mathbb{P}(x \leftrightarrow w_2 \parallel y \leftrightarrow w_1) \\ &\stackrel{(\text{Lemma 4.8})}{\asymp} n^{(\frac{d}{2}-1) \sqcup (d-4)} \cdot \mathbb{P}(\mathbf{C}[\psi]). \end{aligned}$$

Plugging (4.97) into (4.96), we obtain

$$(4.98) \quad \mathbb{P}(\mathbf{C}^{\text{in}}[\psi, c_{10}]) \lesssim n^{(d-2) \sqcup (2d-8)} \cdot \mathbb{P}(\mathbf{C}[\psi]).$$

Thus, since $\widehat{\mathbf{C}}^{\text{in}}$ is a sub-event of $\mathbf{C}^{\text{in}}$, it remains to show that the following hold:

$$(4.99) \quad \text{when } 3 \leq d \leq 5, \quad \mathbb{P}(\widehat{\mathbf{C}}^{\text{in}}[\psi, c_{10}, c_{11}, c_{12}]) \gtrsim n^{d-2} \cdot \mathbb{P}(\mathbf{C}[\psi]);$$

$$(4.100) \quad \text{when } d \geq 7, \quad \mathbb{P}(\widehat{\mathbf{C}}^{\text{in}}[\psi, c_{10}, c_{11}]) \gtrsim n^{2d-8} \cdot \mathbb{P}(\mathbf{C}[\psi]).$$

We achieve these estimates through the second moment method, as shown below.

When $3 \leq d \leq 5$. In previous works, the volume and capacity of the sign cluster can be estimated via the second moment method [5] and the exploration martingale [8] respectively. However, due to the correlation between clusters, these methods are no longer valid for the analysis of the involved clusters in two-arm events. To resolve this, we introduce an independent Brownian motion and calculate its hitting probability and expected occupation time in the target cluster, which in turn yields the required lower bounds on the volume and capacity. To be precise, we arbitrarily take a point $z \in \partial B_{v_1}(4c_{10}n)$ and let $\tilde{S}^z \sim \tilde{\mathbb{P}}_z$ be a Brownian motion starting from z , independent of the loop soup. Recall the event $\hat{\mathbf{C}}[\cdot]$ below (4.24). Consider the quantity

$$(4.101) \quad \mathbf{X} := \sum_{x \in B_{v_1}(c_{10}n)} \mathbb{1}_{\{\tilde{S}^z \text{ hits } x\} \cap \hat{\mathbf{C}}[(x, v_2, w_1, w_2), c_7]}$$

Using Lemmas 4.5 and 4.8, one has

$$(4.102) \quad \mathbb{E}[\mathbf{X}] = \sum_{x \in B_{v_1}(c_{10}n)} \tilde{\mathbb{P}}_z(\tau_x < \infty) \cdot \mathbb{P}(\hat{\mathbf{C}}[(x, v_2, w_1, w_2), c_7]) \asymp n^2 \cdot \mathbb{P}(\mathbf{C}[\psi]).$$

For the second moment of $\mathbf{X}$, we have

$$(4.103) \quad \begin{aligned} \mathbb{E}[\mathbf{X}^2] &\leq \sum_{x_1, x_2 \in B_{v_1}(c_{10}n)} \tilde{\mathbb{P}}_z(\tau_{x_1} < \infty, \tau_{x_2} < \infty) \\ &\quad \cdot \mathbb{P}(\{x_1 \leftrightarrow w_1, x_2 \leftrightarrow w_1\} \| v_2 \leftrightarrow w_2). \end{aligned}$$

For the second probability on the right-hand side, it follows from Lemma 4.1 that

$$(4.104) \quad \begin{aligned} &\mathbb{P}(\{x_1 \leftrightarrow w_1, x_2 \leftrightarrow w_1\} \| v_2 \leftrightarrow w_2) \\ &\lesssim [\mathbb{P}(x_1 \leftrightarrow w_1 \mid x_2 \leftrightarrow w_1) + \exp(-c_5 |x_2 \leftrightarrow w_1|^{\frac{d}{2}-1})] \\ &\quad \cdot \mathbb{P}(x_2 \leftrightarrow w_1 \| v_2 \leftrightarrow w_2) \\ &\lesssim N^{d-2} \cdot \mathbb{P}(x_1 \leftrightarrow w_1, x_2 \leftrightarrow w_1) \cdot \mathbb{P}(\mathbf{C}[\psi]), \end{aligned}$$

where the last inequality relies on Lemma 4.8, and the facts that $\mathbb{P}(x_2 \leftrightarrow w_1) \asymp |x_2 - w_1|^{2-d} \asymp N^{2-d}$ and $\mathbb{P}(x_1 \leftrightarrow w_1, x_2 \leftrightarrow w_1) \gtrsim \mathbb{P}(x_1 \leftrightarrow w_1) \cdot \mathbb{P}(x_2 \leftrightarrow w_1) \gtrsim N^{4-2d}$ (by the FKG inequality). Plugging (4.104) into (4.103), and then applying Lemma 3.9, we have

$$(4.105) \quad \begin{aligned} \mathbb{E}[\mathbf{X}^2] &\lesssim N^{d-2} \cdot \mathbb{P}(\mathbf{C}[\psi]) \sum_{x_1, x_2 \in B_{v_1}(c_{10}n)} \tilde{\mathbb{P}}_z(\tau_{x_1} < \infty, \tau_{x_2} < \infty) \\ &\quad \cdot \mathbb{P}(x_1 \leftrightarrow w_1, x_2 \leftrightarrow w_1) \\ &\lesssim n^{5-\frac{d}{2}} \cdot \mathbb{P}(\mathbf{C}[\psi]). \end{aligned}$$

By applying the Paley-Zygmund inequality and using (4.102) and (4.105), we get

$$(4.106) \quad \mathbb{P}(\mathbf{X} > 0) \geq \frac{(\mathbb{E}[\mathbf{X}])^2}{\mathbb{E}[\mathbf{X}^2]} \gtrsim n^{\frac{d}{2}-1} \cdot \mathbb{P}(\mathbf{C}[\psi]).$$

Meanwhile, since $\{\mathbf{X} > 0\} \subset \mathring{\mathbf{C}}[\psi] := \{B_{v_1}(c_{10}n) \xrightarrow{(C_{w_2})} w_1 \| v_2 \leftrightarrow w_2\} \cap (\text{Sin}[\psi, c_7])^c$, it follows from (4.97) that

$$(4.107) \quad \mathbb{P}(\mathbf{X} > 0) \leq \mathbb{P}(\mathring{\mathbf{C}}[\psi]) \stackrel{(4.97)}{\lesssim} n^{\frac{d}{2}-1} \cdot \mathbb{P}(\mathbf{C}[\psi]),$$

Combining (4.106) and (4.107), we obtain that

$$(4.108) \quad \mathbb{P}(\mathring{\mathbf{C}}[\psi]) \asymp n^{\frac{d}{2}-1} \cdot \mathbb{P}(\mathbf{C}[\psi]),$$

which in turn, together with (4.102) and (4.105), yields that

$$(4.109) \quad \mathbb{E}[\mathbf{X} \mid \mathring{\mathbf{C}}[\psi]] \asymp n^{3-\frac{d}{2}} \quad \text{and} \quad \mathbb{E}[\mathbf{X}^2 \mid \mathring{\mathbf{C}}[\psi]] \lesssim n^{6-d}.$$

By the Paley-Zygmund inequality and (4.109), we derive that for some $c_{\dagger}, c_{\clubsuit} > 0$,

$$(4.110) \quad \mathbb{P}(\mathbf{X} \geq c_{\dagger} n^{3-\frac{d}{2}} \mid \mathring{\mathbf{C}}[\psi]) \geq c_{\clubsuit}.$$

Note that given $\mathcal{C}_{\{w_1, w_2\}}$, the quantity $\mathbf{X}$ depends only on $\tilde{S}^z$. Hence, the event $\mathbf{G} := \{\mathbb{P}(\mathbf{X} \geq c_{\dagger} n^{3-\frac{d}{2}} \mid \mathcal{F}_{\mathcal{C}_{\{w_1, w_2\}}}) \geq \frac{1}{2} c_{\clubsuit}\}$ is measurable with respect to the cluster $\mathcal{C}_{\{w_1, w_2\}}$. Therefore, it follows from (4.110) that

$$(4.111) \quad \begin{aligned} c_{\clubsuit} &\leq \mathbb{P}(\mathbf{G} \mid \mathring{\mathbf{C}}[\psi]) + \mathbb{P}(\mathbf{X} \geq c_{\dagger} n^{3-\frac{d}{2}}, \mathbf{G}^c \mid \mathring{\mathbf{C}}[\psi]) \\ &\leq \mathbb{P}(\mathbf{G} \mid \mathring{\mathbf{C}}[\psi]) + \frac{\mathbb{E}[\mathbb{1}_{\mathbf{G}^c, \mathring{\mathbf{C}}[\psi]} \cdot \mathbb{P}(\mathbf{X} \geq c_{\dagger} n^{3-\frac{d}{2}} \mid \mathcal{F}_{\mathcal{C}_{\{w_1, w_2\}}})]}{\mathbb{P}(\mathring{\mathbf{C}}[\psi])} \\ &\leq \mathbb{P}(\mathbf{G} \mid \mathring{\mathbf{C}}[\psi]) + \frac{1}{2} c_{\clubsuit}, \end{aligned}$$

which further implies

$$(4.112) \quad \mathbb{P}(\mathbf{G} \mid \mathring{\mathbf{C}}[\psi]) \geq \frac{1}{2} c_{\clubsuit}.$$

We claim that for some $c_{\clubsuit} > 0$,

$$(4.113) \quad \mathbf{G} \subset \overline{\mathbf{G}} := \{\mathcal{V}_{w_1, v_1}(c_{10}n) \geq c_{\clubsuit} n^{\frac{d}{2}+1}\} \cap \{\mathcal{T}_{w_1, v_1}(c_{10}n) \geq c_{\clubsuit} n^{d-2}\}.$$

To see this, since $\mathring{\mathbf{C}}[\psi]$ is independent from $\tilde{S}^z$, one has

$$(4.114) \quad \mathbb{E}(\mathbf{X} \mid \mathcal{F}_{\mathcal{C}_{\{w_1, w_2\}}}) = \sum_{x \in \mathcal{C}_{w_1} \cap B_{v_1}(c_{10}n)} \tilde{\mathbb{P}}_z(\tau_x < \infty) \asymp n^{2-d} \cdot \mathcal{V}_{w_1, v_1}(c_{10}n),$$

$$(4.115) \quad \mathbb{P}(\mathbf{X} > 0 \mid \mathcal{F}_{\mathcal{C}_{\{w_1, w_2\}}}) \asymp n^{2-d} \cdot \mathcal{T}_{w_1, v_1}(c_{10}n).$$

Meanwhile, on the event $\mathbf{G}$, we have

$$(4.116) \quad \mathbb{E}(\mathbf{X} \mid \mathcal{F}_{\mathcal{C}_{\{w_1, w_2\}}}) \geq c_{\dagger} n^{3-\frac{d}{2}} \cdot \mathbb{P}(\mathbf{X} \geq c_{\dagger} n^{3-\frac{d}{2}} \mid \mathcal{F}_{\mathcal{C}_{\{w_1, w_2\}}}) \geq \frac{1}{2} c_{\clubsuit} c_{\dagger} n^{3-\frac{d}{2}},$$

$$(4.117) \quad \mathbb{P}(\mathbf{X} > 0 \mid \mathcal{F}_{\mathcal{C}_{\{w_1, w_2\}}}) \geq \mathbb{P}(\mathbf{X} \geq c_{\dagger} n^{3-\frac{d}{2}} \mid \mathcal{F}_{\mathcal{C}_{\{w_1, w_2\}}}) \geq \frac{1}{2} c_{\spadesuit}.$$

Combining (4.114) with (4.116), and (4.115) with (4.117), we obtain (4.113).

By (4.108), (4.112) and (4.113), we have

$$(4.118) \quad \mathbb{P}(\{\bar{\mathbf{G}}\|v_2 \leftrightarrow w_2\} \cap (\mathbf{S}^{\text{in}}[\psi, c_7])^c) \gtrsim n^{\frac{d}{2}-1} \cdot \mathbb{P}(\mathbf{C}[\psi]).$$

We repeat the second moment method used above for the quantity

$$(4.119) \quad \mathbf{X}' := \sum_{x \in B_{v_2}(c_{10}n)} \mathbb{1}_{\{\tilde{S}^{z'} \text{ hits } x\} \cap \{\bar{\mathbf{G}}\|x \leftrightarrow w_2\} \cap (\mathbf{S}^{\text{in}}[\psi, c_7])^c},$$

where $z' \in B_{v_2}(4c_{10}n)$, and $\tilde{S}^{z'} \sim \tilde{\mathbb{P}}_{z'}$ is an independent Brownian motion starting from z' . This yields that for some constants $c_{10}, c_{11}, c_{12} > 0$,

$$(4.120) \quad \mathbb{P}(\hat{\mathbf{C}}^{\text{in}}[\psi, c_{10}, c_{11}, c_{12}]) \gtrsim n^{\frac{d}{2}-1} \cdot \mathbb{P}(\{\bar{\mathbf{G}}\|v_2 \leftrightarrow w_2\} \cap (\mathbf{S}^{\text{in}}[\psi, c_7])^c).$$

The only difference from the previous case is the need for the following inequality, which plays the role of Lemma 4.8 during the proof:

$$(4.121) \quad \mathbb{P}(\{\bar{\mathbf{G}}\|v \leftrightarrow w_2\} \cap (\mathbf{S}^{\text{in}}[\psi, c_7])^c) \asymp \mathbb{P}(\{\bar{\mathbf{G}}\|v' \leftrightarrow w_2\} \cap (\mathbf{S}^{\text{in}}[\psi, c_7])^c)$$

for all $v, v' \in \tilde{B}_{v_2}(c_{10}n)$. It can be derived as follows:

$$\begin{aligned} & \mathbb{P}(\{\bar{\mathbf{G}}\|v \leftrightarrow w_2\} \cap (\mathbf{S}^{\text{in}}[\psi, c_7])^c) \\ & \stackrel{(4.118)}{\gtrsim} n^{\frac{d}{2}-1} \cdot \mathbb{P}(\mathbf{C}[(v, v_2, w_1, w_2)]) \\ & \stackrel{(\text{Lemma 4.8})}{\gtrsim} n^{\frac{d}{2}-1} \cdot \mathbb{P}(\mathbf{C}[(v', v_2, w_1, w_2)]) \stackrel{(4.107)}{\gtrsim} \mathbb{P}(\{\bar{\mathbf{G}}\|v' \leftrightarrow w_2\} \cap (\mathbf{S}^{\text{in}}[\psi, c_7])^c). \end{aligned}$$

By plugging (4.118) into (4.120), we obtain (4.99).

When $d \geq 7$. Compared to the proof in low dimensions, this one follows the same spirit and is more straightforward. We consider the quantity

$$(4.122) \quad \mathbf{Y} := \sum_{x \in B_{v_1}(c_{10}n)} \mathbb{1}_{\hat{\mathbf{C}}[(x, v_2, w_1, w_2)]}.$$

For the first moment of $\mathbf{Y}$, as in (4.102), one has

$$(4.123) \quad \mathbb{E}[\mathbf{Y}] = \sum_{x \in B_{v_1}(c_{10}n)} \mathbb{P}(\hat{\mathbf{C}}[(x, v_2, w_1, w_2), c_7]) \asymp n^d \cdot \mathbb{P}(\mathbf{C}[\psi]).$$

For the second moment, by (4.104) and Lemma 3.10, we have

$$\begin{aligned} & \mathbb{E}[\mathbf{Y}^2] \leq \sum_{x_1, x_2 \in B_{v_1}(c_{10}n)} \mathbb{P}(\{x_1 \leftrightarrow w_1, x_2 \leftrightarrow w_1\} \mid v_2 \leftrightarrow w_2) \\ & \stackrel{(4.104)}{\lesssim} N^{d-2} \cdot \mathbb{P}(\mathbf{C}[\psi]) \sum_{x_1, x_2 \in B_{v_1}(c_{10}n)} \mathbb{P}(x_1 \leftrightarrow w_1, x_2 \leftrightarrow w_1) \\ & \stackrel{(\text{Lemma 3.10})}{\lesssim} n^{d+4} \cdot \mathbb{P}(\mathbf{C}[\psi]). \end{aligned} \tag{4.124}$$

Recall the event $\mathring{\mathbf{C}}[\psi]$ below (4.106). By the Paley-Zygmund inequality, (4.123) and (4.124), we obtain that

$$(4.125) \quad \mathbb{P}(\mathring{\mathbf{C}}[\psi]) = \mathbb{P}(\mathbf{Y} > 0) \gtrsim n^{d-4} \cdot \mathbb{P}(\mathbf{C}[\psi]).$$

As in (4.120), repeating this second moment method yields that

$$(4.126) \quad \mathbb{P}(\widehat{\mathbf{C}}^{\text{in}}[\psi, c_{10}, c_{11}]) \gtrsim n^{d-4} \cdot \mathbb{P}(\mathring{\mathbf{C}}[\psi]).$$

Combining (4.125) and (4.126), we get (4.100), thereby completing the proof. $\square$

5. ESTIMATES ON HETEROCHROMATIC FOUR-POINT FUNCTIONS

The goal of this section is to establish the upper bound in (1.16) and the lower bound in (1.17), which is crucial for proving Theorem 1.1. For the first part, we prove the following estimate. Recall the notation $\mathbf{C}[\cdot]$ in (4.23).

Lemma 5.1. *For any $d \geq 3$ with $d \neq 6$, there exist $C_{11}, C_{12} > 0$ such that for any $v_1 \neq v_2 \in \widetilde{B}(C_{11})$, $N \geq C_{12}$ and $w_1, w_2 \in \widetilde{B}(10N) \setminus \widetilde{B}(N)$ with $|w_1 - w_2| \geq N$,*

$$(5.1) \quad \mathbb{P}(\mathbf{C}[(v_1, v_2, w_1, w_2)]) \lesssim |v_1 - v_2|^{\frac{3}{2}} \cdot N^{-[(\frac{3d}{2}-1)\square(2d-4)]}.$$

Proof. For simplicity, we denote $\epsilon_N := N^{-[(\frac{3d}{2}-1)\square(2d-4)]}$ and $\psi = (v_1, v_2, w_1, w_2)$. We begin by showing that for some constant $C_{\dagger} > 0$,

$$(5.2) \quad \mathbb{P}(\mathbf{C}[\psi]) \lesssim \epsilon_N, \quad \forall \psi \in \Psi(C_{\dagger}, N).$$

When $d \geq 7$, this follows directly from the BKR inequality (see [3, Lemma 3.3]):

$$(5.3) \quad \mathbb{P}(\mathbf{C}[\psi]) \leq \mathbb{P}(v_1 \leftrightarrow w_1) \cdot \mathbb{P}(v_2 \leftrightarrow w_2) \lesssim N^{4-2d}.$$

In the low-dimensional case $3 \leq d \leq 5$, we arbitrarily take $w \in \widetilde{B}(10N) \setminus \widetilde{B}(5N)$ and let $\widetilde{S}^w \sim \widetilde{\mathbb{P}}_w$ be a Brownian motion starting from w . By (1.4), one has

$$(5.4) \quad \begin{aligned} \mathbb{P}(\mathbf{A}) &:= \mathbb{P}(\widetilde{S}^w \text{ hits } \mathcal{C}_0 \cap [\widetilde{B}(2N) \setminus \widetilde{B}(N)]) \\ &\leq \mathbb{P}(\mathcal{C}_0 \cap [\widetilde{B}(2N) \setminus \widetilde{B}(N)] \neq \emptyset) \leq \theta_d(N) \lesssim N^{-\frac{d}{2}+1}. \end{aligned}$$

Meanwhile, for any $x \in \mathbb{Z}^d$, we define the event

$$(5.5) \quad \mathbf{A}_x := \{\widetilde{S}^w \text{ first hits } \mathcal{C}_0 \cap [\widetilde{B}(2N) \setminus \widetilde{B}(N)] \text{ within } \widetilde{B}_x(C_{\dagger})\}.$$

Note that $\cup_{x \in B(2N-2C_{\dagger}) \setminus B(N+C_{\dagger})} \mathbf{A}_x \subset \mathbf{A}$, and that $\mathbf{A}_{x_1}$ and $\mathbf{A}_{x_2}$ are disjoint whenever $B_{x_1}(C_{\dagger}) \cap B_{x_2}(C_{\dagger}) = \emptyset$. Therefore,

$$(5.6) \quad \mathbb{P}(\mathbf{A}) \gtrsim \sum_{x \in B(2N-2C_{\dagger}) \setminus B(N+C_{\dagger})} \mathbb{P}(\mathbf{A}_x).$$

For each $x \in B(2N-2C_{\dagger}) \setminus B(N+C_{\dagger})$, we arbitrarily take $x' \in \partial B_x(C_{\dagger}-1)$. In order for $\mathbf{A}_x$ to occur, it suffices to ensure that $x \in \mathcal{C}_0$, and that $\widetilde{S}^w$ reaches x' before $\mathcal{C}_0$ and then hit $\mathcal{C}_0$ before leaving $\widetilde{B}_x(C_{\dagger})$. As a result,

$$(5.7) \quad \mathbb{P}(\mathbf{A}_x) \geq \mathbb{E}[\mathbb{1}_{x \in \mathcal{C}_0} \cdot \widetilde{\mathbb{P}}_w(\tau_{x'} < \tau_{\mathcal{C}_0}) \cdot \widetilde{\mathbb{P}}_{x'}(\tau_{\mathcal{C}_0} < \tau_{\partial B_x(C_{\dagger})})].$$

When $\mathcal{C}_0$ is disjoint from $\tilde{B}_w(1)$, one has $\tilde{G}_{\mathcal{C}_0}(w, w) \asymp 1$ and hence,

$$(5.8) \quad \tilde{\mathbb{P}}_w(\tau_{x'} < \tau_{\mathcal{C}_0}) \gtrsim \tilde{\mathbb{P}}_w(\tau_{x'} < \tau_{\mathcal{C}_0}) \cdot \sqrt{\frac{\tilde{G}_{\mathcal{C}_0}(x', x')}{\tilde{G}_{\mathcal{C}_0}(w, w)}} \stackrel{(3.28)}{\asymp} \mathbb{P}(x' \xrightarrow{(\mathcal{C}_0)} w).$$

In addition, when $x \in \mathcal{C}_0$, one has

$$(5.9) \quad \tilde{\mathbb{P}}_{x'}(\tau_{\mathcal{C}_0} < \tau_{\partial B_x(C_\dagger)}) \geq \tilde{\mathbb{P}}_{x'}(\tau_x < \tau_{\partial B_x(C_\dagger)}) \gtrsim 1.$$

Combining (5.7), (5.8) and (5.9), we get

$$(5.10) \quad \begin{aligned} \mathbb{P}(\mathbf{A}_x) &\geq \mathbb{E}[\mathbb{1}_{x \in \mathcal{C}_0, \tilde{B}_w(1) \cap \mathcal{C}_0 = \emptyset} \cdot \mathbb{P}(x' \xrightarrow{(\mathcal{C}_0)} w)] \\ &= \mathbb{P}(\{x \leftrightarrow \mathbf{0} \parallel x' \leftrightarrow w\}, \tilde{B}_w(1) \cap \mathcal{C}_0 = \emptyset) \\ &\stackrel{(\text{Lemma 4.5})}{\asymp} \mathbb{P}(\{x \leftrightarrow \mathbf{0} \parallel x' \leftrightarrow w\}) \stackrel{(\text{Lemma 4.8})}{\asymp} \mathbb{P}(\mathbf{C}[\psi]), \end{aligned}$$

for all $\psi \in \Psi(C_\dagger, N)$. Putting this together with (5.4) and (5.6), one obtain

$$(5.11) \quad \mathbb{P}(\mathbf{C}[\psi]) \lesssim \epsilon_N, \quad \forall \psi \in \Psi(C_\dagger, N)$$

By (5.3) and (5.11), we confirm (5.2).

For any $v_1, v_2 \in \tilde{B}(\frac{C_\dagger}{10})$, and any $w_1, w_2 \in \tilde{B}(10N) \setminus \tilde{B}(N)$ with $|w_1 - w_2| \geq N$, on the event $\mathbf{C}[\psi]$, the cluster $\mathcal{C}_{w_1}$ (resp. $\mathcal{C}_{w_2}$) must intersect $\partial B(2C_\dagger)$ (resp. $\partial B(4C_\dagger)$). Combined with (5.2), it implies that

$$(5.12) \quad \mathbb{P}(\mathbf{C}[\psi]) \leq \sum_{v'_1 \in \partial B(2C_\dagger), v'_2 \in \partial B(4C_\dagger)} \mathbb{P}(\mathbf{C}[(v'_1, v'_2, w_1, w_2)]) \lesssim \epsilon_N.$$

Thus, it remain to prove (5.1) in the case when $\chi := |v_1 - v_2|$ is sufficiently small.

Note that one of $\mathbf{B}_{v_1}(\frac{\chi}{3})$ and $\mathbf{B}_{v_2}(\frac{\chi}{3})$ must be contained in some interval I_e . Otherwise, there exist $x_1, x_2 \in \mathbb{Z}^d$ such that $|x_i - v_i| \leq \frac{\chi}{3}$ for $i \in \{1, 2\}$.

- If $x_1 = x_2$, then one has $\chi \leq |x_1 - v_1| + |x_1 - v_2| \leq \frac{2\chi}{3}$, a contradiction.
- If $x_1 \neq x_2$, then one has $\chi \geq |x_1 - x_2| - |x_1 - v_1| - |x_2 - v_2| \geq d - \frac{2\chi}{3} > \chi$, again a contradiction.

Without loss of generality, we assume $\mathbf{B}_{v_1}(\frac{\chi}{3}) \subset I_{\{x, y\}}$. By the restriction property,

$$(5.13) \quad \begin{aligned} \mathbb{P}(\mathbf{C}[\psi]) &= \mathbb{E}[\mathbb{1}_{\{v_1 \leftrightarrow w_1\} \cap \{v_1 \leftrightarrow v_2\}^c} \cdot \mathbb{P}(v_2 \xrightarrow{(\mathcal{C}_{v_1})} w_2 \mid \mathcal{F}_{\mathcal{C}_{v_1}})] \\ &\stackrel{(\text{Lemma 3.12})}{\lesssim} \chi^{\frac{1}{2}} \sum_{z \in \partial \mathcal{B}_{v_1}(C_\dagger)} \mathbb{E}[\mathbb{1}_{\{v_1 \leftrightarrow w_1\} \cap \{v_1 \leftrightarrow v_2\}^c} \cdot \mathbb{P}(z \xrightarrow{(\mathcal{C}_{v_1})} w_2 \mid \mathcal{F}_{\mathcal{C}_{v_1}})] \\ &\leq \chi^{\frac{1}{2}} \sum_{z \in \partial \mathcal{B}_{v_1}(C_\dagger)} \mathbb{P}(v_1 \leftrightarrow w_1 \parallel z \leftrightarrow w_2) \stackrel{(5.2)}{\lesssim} \chi^{\frac{1}{2}} \epsilon_N. \end{aligned}$$

This estimate does not yet yield (5.1). We proceed by iterating it.

For $z \in \partial\mathcal{B}_{v_1}(C_\dagger)$, we denote $\mathcal{C}_z := \{v_1 \leftrightarrow w_1 \parallel z \leftrightarrow w_2\}$. We claim that

$$(5.14) \quad \mathbb{P}(\mathcal{C}[\psi]) \lesssim \chi^{\frac{3}{2}} \max_{z \in \partial\mathcal{B}_{v_1}(C_\dagger)} \mathbb{E}[\mathbb{1}_{\mathcal{C}_z \cap \{\text{dist}(\mathcal{C}_{w_2}, v_1) \geq \frac{\chi}{3}\}} \cdot (\text{dist}(\mathcal{C}_{w_2}, v_1))^{-1}] + \chi^2 \epsilon_N.$$

Before proving (5.14), we use it to derive (5.1). Note that for any $z \in \partial\mathcal{B}_{v_1}(C_\dagger)$, the event $\{z \leftrightarrow w_2\}$ implies that $\text{dist}(\mathcal{C}_{w_2}, v_1) \leq |z - v_1| \leq 10dC_\dagger$. Thus, assuming that $\mathbb{P}(\mathcal{C}[\psi]) \lesssim f(\chi) \cdot \epsilon_N$ holds for the function $f(\cdot)$ (which has been confirmed by (5.13) for $f(a) = a^{\frac{1}{2}}$), we have

$$(5.15) \quad \begin{aligned} & \mathbb{E}[\mathbb{1}_{\mathcal{C}_z \cap \{\text{dist}(\mathcal{C}_{w_2}, v_1) \geq \frac{\chi}{3}\}} \cdot (\text{dist}(\mathcal{C}_{w_2}, v_1))^{-1}] \\ & \leq \mathbb{P}(\mathcal{C}_z) + \int_{(10dC_\dagger)^{-1} \leq t \leq 3\chi^{-1}} \mathbb{P}(\mathcal{C}_z, 0 < \text{dist}(\mathcal{C}_{w_2}, v_1) \leq t^{-1}) dt \\ & \leq \mathbb{P}(\mathcal{C}_z) + \int_{(10dC_\dagger)^{-1} \leq t \leq 3\chi^{-1}} \sum_{v'_1 \in \tilde{\mathbb{Z}}^d: |v_1 - v'_1| = t^{-1}} \mathbb{P}(\mathcal{C}_{v'_1}) dt \\ & \lesssim (1 + \int_{(10dC_\dagger)^{-1} \leq t \leq 3\chi^{-1}} f(t^{-1}) dt) \cdot \epsilon_N. \end{aligned}$$

Plugging (5.15) into (5.14), we obtain that

$$(5.16) \quad \mathbb{P}(\mathcal{C}[\psi]) \lesssim \chi^{\frac{3}{2}} \cdot (1 + \int_{(10dC_\dagger)^{-1} \leq t \leq 3\chi^{-1}} f(t^{-1}) dt) \cdot \epsilon_N.$$

Combined with (5.13), it implies $\mathbb{P}(\mathcal{C}[\psi]) \lesssim \chi \cdot \epsilon_N$. Inserting this stronger estimate into (5.16), we further get $\mathbb{P}(\mathcal{C}[\psi]) \lesssim \chi^{\frac{3}{2}} \ln(\chi^{-1}) \cdot \epsilon_N$. Iterating this argument once more, we finally arrive at (5.1), since $\int_{1 \leq t < \infty} t^{-\frac{3}{2}} \ln(t) dt < \infty$.

We divide the proof of (5.14) into two cases, depending on whether $v_2 \in I_{\{x, y\}}$.

Case (1): $v_2 \in I_{\{x, y\}}$. Recall that the first two lines of (5.13) yield

$$(5.17) \quad \begin{aligned} \mathbb{P}(\mathcal{C}[\psi]) & \lesssim \chi^{\frac{1}{2}} \sum_{z \in \partial\mathcal{B}_{v_1}(C_\dagger)} \mathbb{E}[\mathbb{1}_{\{v_1 \leftrightarrow w_1\} \cap \{v_1 \leftrightarrow v_2\}^c} \cdot \mathbb{P}(z \xrightarrow{(C_{v_1})} w_2 \mid \mathcal{F}_{C_{v_1}})] \\ & = \chi^{\frac{1}{2}} \sum_{z \in \partial\mathcal{B}_{v_1}(C_\dagger)} \mathbb{E}[\mathbb{1}_{z \leftrightarrow w_2} \cdot \mathbb{P}(\{v_1 \xrightarrow{(C_{w_2})} w_1\} \cap \{v_1 \xrightarrow{(C_{w_2})} v_2\}^c \mid \mathcal{F}_{C_{w_2}})]]. \end{aligned}$$

Without loss of generality, assume that x is the endpoint of $I_{\{x, y\}}$ such that $v_2 \in I_{[v_1, x]}$. On $\{v_1 \xrightarrow{(C_{w_2})} w_1\} \cap \{v_1 \xrightarrow{(C_{w_2})} v_2\}^c$, one of the following occurs:

- (a) $\mathcal{C}_{w_2}$ intersects v_2 , and hence $\mathcal{C}[\psi]$ occurs;
- (b) $\mathcal{C}_{w_2}$ intersects y but not v_2 . As a result, $v_1 \xrightarrow{(C_{w_2})} w_1$ and $\{v_1 \xrightarrow{(C_{w_2})} v_2\}^c$ are incompatible.
- (c) $\mathcal{C}_{w_2}$ is disjoint from both y and v_2 , and thus $\text{dist}(\mathcal{C}_{w_2}, v_1) \geq \frac{\chi}{3}$.

In conclusion, the expectation on the right-hand side of (5.17) is at most

$$(5.18) \quad \mathbb{P}(\mathcal{C}[\psi]) + \mathbb{E}[\mathbb{1}_{\{z \leftrightarrow w_2, \text{dist}(\mathcal{C}_{w_2}, v_1) \geq \frac{\chi}{3}\}} \mathbb{P}(v_1 \xrightarrow{(C_{w_2})} w_1, \{v_1 \xrightarrow{(C_{w_2})} v_2\}^c \mid \mathcal{F}_{C_{w_2}})].$$

Moreover, by the isomorphism theorem we have

$$\begin{aligned}
(5.19) \quad & \mathbb{P}(v_1 \xleftrightarrow{(\mathcal{C}_{w_2})} w_1, \{v_1 \xleftrightarrow{(\mathcal{C}_{w_2})} v_2\}^c \mid \mathcal{F}_{\mathcal{C}_{w_2}}) \\
& \asymp \int_{a>0} \frac{\exp\left(-\frac{a^2}{2\tilde{G}_{\mathcal{C}_{w_2}}(v_1, v_1)}\right)}{\sqrt{2\pi\tilde{G}_{\mathcal{C}_{w_2}}(v_1, v_1)}} \cdot \mathbb{P}^{\mathcal{C}_{w_2}}(v_1 \xleftrightarrow{\geq 0} w_1, \{v_1 \xleftrightarrow{\geq 0} v_2\}^c \mid \tilde{\phi}_{v_1} = a) da,
\end{aligned}$$

where the conditional probability, by the FKG inequality, is upper-bounded by

$$\begin{aligned}
(5.20) \quad & \mathbb{P}^{\mathcal{C}_{w_2}}(v_1 \xleftrightarrow{\geq 0} w_1 \mid \tilde{\phi}_{v_1} = a) \cdot \mathbb{P}^{\mathcal{C}_{w_2}}(\{v_1 \xleftrightarrow{\geq 0} v_2\}^c \mid \tilde{\phi}_{v_1} = a) \\
& \stackrel{(\text{Lemmas 3.13 and 3.15})}{\lesssim} \mathbb{P}^{\mathcal{C}_{w_2}}(v_1 \xleftrightarrow{\geq 0} w_1) \cdot \frac{a \cdot e^{-ca^2\chi^{-1}}}{\sqrt{\tilde{G}_{\mathcal{C}_{w_2}}(v_1, v_1)}}.
\end{aligned}$$

By (5.19), (5.20) and $\tilde{G}_{\mathcal{C}_{w_2}}(v_1, v_1) \asymp \text{dist}(\mathcal{C}_{w_2}, v_1) \wedge 1$ (by (3.6)), we obtain that on the event $\{z \leftrightarrow w_2, \text{dist}(\mathcal{C}_{w_2}, v_1) \gtrsim \chi\}$,

$$\begin{aligned}
(5.21) \quad & \mathbb{P}(v_1 \xleftrightarrow{(\mathcal{C}_{w_2})} w_1, \{v_1 \xleftrightarrow{(\mathcal{C}_{w_2})} v_2\}^c \mid \mathcal{F}_{\mathcal{C}_{w_2}}) \\
& \lesssim \left(\int_{a>0} a \cdot e^{-ca^2\chi^{-1}} da \right) \cdot (\text{dist}(\mathcal{C}_{w_2}, v_1) \wedge 1)^{-1} \cdot \mathbb{P}^{\mathcal{C}_{w_2}}(v_1 \xleftrightarrow{\geq 0} w_1) \\
& \lesssim \chi \cdot (\text{dist}(\mathcal{C}_{w_2}, v_1))^{-1} \cdot \mathbb{P}^{\mathcal{C}_{w_2}}(v_1 \xleftrightarrow{\geq 0} w_1),
\end{aligned}$$

where in the last line we used the inclusion $\{z \leftrightarrow w_2\} \subset \{\text{dist}(\mathcal{C}_{w_2}, v_1) \leq 10dC_{\dagger}\}$. Plugging (5.21) into (5.18), we get

$$\begin{aligned}
(5.22) \quad & \mathbb{E}[\mathbb{1}_{z \leftrightarrow w_2} \cdot \mathbb{P}(\{v_1 \xleftrightarrow{(\mathcal{C}_{w_2})} w_1\} \cap \{v_1 \xleftrightarrow{(\mathcal{C}_{w_2})} v_2\}^c \mid \mathcal{F}_{\mathcal{C}_{w_2}})] \\
& \lesssim \mathbb{P}(\mathcal{C}[\psi]) + \mathbb{E}[\mathbb{1}_{\mathcal{C}_z \cap \{\text{dist}(\mathcal{C}_{w_2}, v_1) \geq \frac{\chi}{3}\}} \cdot (\text{dist}(\mathcal{C}_{w_2}, v_1))^{-1}].
\end{aligned}$$

Inserting (5.22) into (5.17), and recalling that χ is small, we obtain (5.14).

Case (2): $v_2 \notin I_{\{x, y\}}$. Without loss of generality, assume that x is the endpoint of $I_{\{x, y\}}$ that is closer to v_2 , and that $m := |v_1 - x| \geq |v_2 - x|$. Since $v_2 \in \mathbf{B}_x(m)$,

$$(5.23) \quad \mathbb{P}(\mathcal{C}[\psi]) \leq \mathbb{I} := \sum_{v'_2 \in \tilde{\partial}\mathbf{B}_x(m): v'_2 \notin I_{\{x, y\}}} \mathbb{P}(\mathcal{C}[(v_1, v'_2, w_1, w_2)]).$$

In addition, it follows from (5.17) that

$$(5.24) \quad \mathbb{I} \lesssim \chi^{\frac{1}{2}} \sum_{z \in \partial\mathcal{B}_{v_1}(C_{\dagger})} \sum_{v'_2 \in \tilde{\partial}\mathbf{B}_x(m): v'_2 \notin I_{\{x, y\}}} \mathbb{J}_{z, v'_2},$$

where $\mathbb{J}_{z, v'_2} := \mathbb{E}[\mathbb{1}_{z \leftrightarrow w_2} \cdot \mathbb{P}(\{v_1 \xleftrightarrow{(\mathcal{C}_{w_2})} w_1\} \cap \{v_1 \xleftrightarrow{(\mathcal{C}_{w_2})} v'_2\}^c \mid \mathcal{F}_{\mathcal{C}_{w_2}})]$. Moreover, on the event $\{v_1 \xleftrightarrow{(\mathcal{C}_{w_2})} w_1\} \cap \{v_1 \xleftrightarrow{(\mathcal{C}_{w_2})} v'_2\}^c$, one of the following occurs:

- (i) $\mathcal{C}_{w_2}$ intersects y . Therefore, since $v_1 \xleftrightarrow{(\mathcal{C}_{w_2})} w_1$, the cluster $\mathcal{C}_{w_1}$ must intersect x , and thus $\mathcal{C}[(x, v'_2, w_1, w_2)]$ occurs.
- (ii) $\mathcal{C}_{w_2}$ intersects $\mathbf{B}_x(m)$ but not y . Consequently, $\mathcal{C}[(v_1, v'_2, w_1, w_2)]$ occurs for some $v'_2 \in \tilde{\partial}\mathbf{B}_x(m)$ with $v'_2 \notin I_{\{x, y\}}$.

(iii) $\mathcal{C}_{w_2}$ is disjoint from both y and $\mathbf{B}_x(m)$. Hence, $\text{dist}(\mathcal{C}_{w_2}, v_1) \geq \frac{\chi}{3}$.

As a result, for any $v'_2 \in \tilde{\partial}\mathbf{B}_x(m)$ with $v'_2 \notin I_{\{x,y\}}$, we have

$$(5.25) \quad \mathbb{J}_{z,v'_2} \lesssim \mathbb{P}(\mathbf{C}[(x, v'_2, z, w_2)]) + \mathbb{I} \\ + \mathbb{E}[\mathbb{1}_{\{z \leftrightarrow w_2, \text{dist}(\mathcal{C}_{w_2}, v_1) \geq \frac{\chi}{3}\}} \mathbb{P}(v_1 \xleftrightarrow{(\mathcal{C}_{w_2})} w_1, \{v_1 \xleftrightarrow{(\mathcal{C}_{w_2})} v'_2\}^c \mid \mathcal{F}_{\mathcal{C}_{w_2}})]].$$

Recall that it has been proved in Case (1) that

$$(5.26) \quad \mathbb{P}(\mathbf{C}[(x, v'_2, z, w_2)]) \lesssim \chi^{\frac{3}{2}} \epsilon_N.$$

Meanwhile, by the same argument as in proving (5.21), we have

$$(5.27) \quad \mathbb{E}[\mathbb{1}_{\{z \leftrightarrow w_2, \text{dist}(\mathcal{C}_{w_2}, v_1) \geq \frac{\chi}{3}\}} \mathbb{P}(v_1 \xleftrightarrow{(\mathcal{C}_{w_2})} w_1, \{v_1 \xleftrightarrow{(\mathcal{C}_{w_2})} v'_2\}^c \mid \mathcal{F}_{\mathcal{C}_{w_2}})] \\ \lesssim \mathbb{E}[\mathbb{1}_{\mathcal{C}_z \cap \{\text{dist}(\mathcal{C}_{w_2}, v_1) \geq \frac{\chi}{3}\}} \cdot (\text{dist}(\mathcal{C}_{w_2}, v_1))^{-1}].$$

Plugging (5.26) and (5.27) into (5.25), we get

$$(5.28) \quad \mathbb{J}_{z,v'_2} \lesssim \mathbb{E}[\mathbb{1}_{\mathcal{C}_z \cap \{\text{dist}(\mathcal{C}_{w_2}, v_1) \geq \frac{\chi}{3}\}} \cdot (\text{dist}(\mathcal{C}_{w_2}, v_1))^{-1}] + \chi^{\frac{3}{2}} \cdot \epsilon_N + \mathbb{I}.$$

Inserting (5.28) into (5.24), and using the fact that χ is small, we obtain

$$(5.29) \quad \mathbb{I} \lesssim \chi^{\frac{3}{2}} \max_{z \in \partial\mathcal{B}_{v_1}(C_{\dagger})} \mathbb{E}[\mathbb{1}_{\mathcal{C}_z \cap \{\text{dist}(\mathcal{C}_{w_2}, v_1) \geq \frac{\chi}{3}\}} \cdot (\text{dist}(\mathcal{C}_{w_2}, v_1))^{-1}] + \chi^2 \epsilon_N.$$

This together with (5.23) implies the claim (5.14) and thus completes the proof. $\square$

For the lower bound in (1.17), we first address the low-dimensional case (i.e., $3 \leq d \leq 5$) by proving the following lemma.

Lemma 5.2. *For any $3 \leq d \leq 5$, there exist $C_{13}, C_{14} > 0$ such that for any $n \geq C_{13}$, $N \geq C_{14}n$ and $\psi \in \Psi(n, N)$,*

$$(5.30) \quad \mathbb{P}(\mathbf{C}[\psi]) \gtrsim n^{3-\frac{d}{2}} N^{-\frac{3d}{2}+1}.$$

To derive this, we need an auxiliary lemma as follows.

Lemma 5.3. *For any $3 \leq d \leq 5$, $N \geq 1$ and $z \in \partial B(4N)$,*

$$(5.31) \quad \tilde{\mathbb{P}}_z \times \mathbb{P}(\tau_{\mathcal{C}_0} = \tau_{\mathcal{C}_0 \cap [\tilde{B}(2N) \setminus \tilde{B}(N)]} < \infty) \gtrsim N^{-\frac{d}{2}+1}.$$

Proof. We claim that for any $x \in B(\frac{5N}{3}) \setminus B(\frac{4N}{3})$,

$$(5.32) \quad \tilde{\mathbb{P}}_z \times \mathbb{P}(\mathbf{0} \leftrightarrow x, \tau_x < \tau_{\mathcal{C}_0 \cap [\tilde{B}(2N) \setminus \tilde{B}(N)]^c}) \gtrsim N^{4-2d}.$$

In fact, assuming this claim, the desired bound (5.31) can be derived via the second moment method. Precisely, we consider the quantity

$$(5.33) \quad \mathbf{X} := \sum_{x \in B(\frac{5N}{3}) \setminus B(\frac{4N}{3})} \mathbb{1}_{\mathbf{0} \leftrightarrow x, \tau_x < \tau_{\mathcal{C}_0 \cap [\tilde{B}(2N) \setminus \tilde{B}(N)]^c}}.$$

It directly follows from (5.32) that $\mathbb{E}[\mathbf{X}] \gtrsim N^{4-d}$. Moreover, using the isomorphism theorem and Lemma 3.9, we have

$$(5.34) \quad \begin{aligned} \mathbb{E}[\mathbf{X}^2] &\lesssim \sum_{x_1, x_2 \in B(\frac{5N}{3}) \setminus B(\frac{4N}{3})} \tilde{\mathbb{P}}_z(\tau_{x_1} < \infty, \tau_{x_2} < \infty) \cdot \mathbb{P}(\mathbf{0} \xrightarrow{\geq 0} x_1, \mathbf{0} \xrightarrow{\geq 0} x_2) \\ &\lesssim N^{-\frac{3d}{2}+7}. \end{aligned}$$

Thus, by the Paley-Zygmund inequality, we obtain (5.31) as follows:

$$(5.35) \quad \begin{aligned} &\tilde{\mathbb{P}}_z \times \mathbb{P}(\tau_{\mathcal{C}_0} = \tau_{\mathcal{C}_0 \cap [\tilde{B}(2N) \setminus \tilde{B}(N)]} < \infty) \\ &\geq \mathbb{P}(\mathbf{X} > 0) \gtrsim \frac{(\mathbb{E}[\mathbf{X}])^2}{\mathbb{E}[\mathbf{X}^2]} \gtrsim N^{-\frac{d}{2}+1}. \end{aligned}$$

It remains to prove the claim (5.32). We take a path $\tilde{\eta}$ consisting of at most $2d$ line segments, connecting z and $\partial B(\frac{17N}{9})$, and satisfying $\text{ran}(\tilde{\eta}) \cap B(\frac{16N}{9}) = \emptyset$ and $\text{vol}(\text{ran}(\tilde{\eta})) \leq CN$. For $\delta > 0$, we denote $D_{\dagger}^{\delta} := \cup_{v \in \text{ran}(\tilde{\eta}_{\dagger})} \mathbf{B}_v(\delta N)$. It follows from (4.28) that for a sufficiently small $\delta > 0$,

$$(5.36) \quad \mathbb{P}(\mathbf{0} \leftrightarrow x, \{\mathbf{0} \leftrightarrow D_{\dagger}^{\delta}\}^c) \asymp \mathbb{P}(x \in \mathcal{C}_0) \asymp N^{2-d}.$$

Meanwhile, note that $D_{\dagger}^{\delta} \cup [\tilde{B}(2N) \setminus \tilde{B}(N)]$ contains both x and z , and the distance between its boundary and $\{x, z\}$ is of order N . Therefore, by Harnack's inequality we have (letting w be an arbitrary point in $\partial B_x(\frac{\delta N}{10})$)

$$(5.37) \quad \tilde{\mathbb{P}}_z(\tau_x < \tau_{(D_{\dagger}^{\delta} \cup [\tilde{B}(2N) \setminus \tilde{B}(N)])^c}) \asymp \tilde{\mathbb{P}}_w(\tau_x < \tau_{(D_{\dagger}^{\delta} \cup [\tilde{B}(2N) \setminus \tilde{B}(N)])^c}) \asymp N^{2-d}.$$

In particular, when the two events on the left-hand sides of (5.36) and (5.37) both occur, the Brownian motion starting from z indeed reaches x before hitting the set $\mathcal{C}_0 \cap [\tilde{B}(2N) \setminus \tilde{B}(N)]^c$ (since it is disjoint from $D_{\dagger}^{\delta} \cup [\tilde{B}(2N) \setminus \tilde{B}(N)]$). Thus, by (5.36) and (5.37), we obtain the claim (5.32) as follows:

$$\begin{aligned} &\tilde{\mathbb{P}}_z \times \mathbb{P}(\mathbf{0} \leftrightarrow x, \tau_x < \tau_{\mathcal{C}_0 \cap [\tilde{B}(2N) \setminus \tilde{B}(N)]^c}) \\ &\geq \mathbb{P}(\mathbf{0} \leftrightarrow x, \{\mathbf{0} \leftrightarrow D_{\dagger}^{\delta}\}^c) \cdot \tilde{\mathbb{P}}_z(\tau_x < \tau_{(D_{\dagger}^{\delta} \cup [\tilde{B}(2N) \setminus \tilde{B}(N)])^c}) \asymp N^{4-2d}. \end{aligned} \quad \square$$

With Lemma 5.3 in hand, we are ready to prove Lemma 5.2.

Proof of Lemma 5.2. We arbitrarily choose $z \in \partial B(4N)$, and denote by $\tilde{S}^z \sim \tilde{\mathbb{P}}_z$ a Brownian motion starting from z , independent of $\tilde{\mathcal{L}}_{1/2}$.

Let $\Omega_{n,N} := \{x \in n \cdot \mathbb{Z}^d : B_x(n) \cap [B(2N) \setminus B(N)] \neq \emptyset\}$. For $x \in \Omega_{n,N}$, we define

$$(5.38) \quad \bar{A}_x := \{\tilde{S}^z \text{ first hits } \mathcal{C}_0 \text{ within } \tilde{B}_x(n)\}.$$

By $\tilde{B}(2N) \setminus \tilde{B}(N) \subset \cup_{x \in \Omega_{n,N}} \tilde{B}_x(n)$ and Lemma 5.3, we have

$$(5.39) \quad \sum_{x \in \Omega_{n,N}} \tilde{\mathbb{P}}_z \times \mathbb{P}(\bar{A}_x) \geq \tilde{\mathbb{P}}_z \times \mathbb{P}(\tau_{\mathcal{C}_0} = \tau_{\mathcal{C}_0 \cap [\tilde{B}(2N) \setminus \tilde{B}(N)]} < \infty) \gtrsim N^{-\frac{d}{2}+1}.$$

As a result, there exists $x_{\dagger} \in \Omega_{n,N}$ such that

$$(5.40) \quad \tilde{\mathbb{P}}_z \times \mathbb{P}(\bar{\mathbf{A}}_{x_{\dagger}}) \gtrsim |\Omega_{n,N}|^{-1} N^{-\frac{d}{2}+1} \asymp n^d N^{-\frac{3d}{2}+1}.$$

Since $\bar{\mathbf{A}}_{x_{\dagger}} \subset \{\mathbf{0} \leftrightarrow \partial B_{x_{\dagger}}(n)\}$, we have

$$(5.41) \quad \tilde{\mathbb{P}}_z \times \mathbb{P}(\bar{\mathbf{A}}_{x_{\dagger}}) \leq \mathbb{E}[\mathbb{1}_{\mathbf{0} \leftrightarrow \partial B_{x_{\dagger}}(n)} \cdot \tilde{\mathbb{P}}_z(\tau_{B_{x_{\dagger}}(n)} < \tau_{\mathcal{C}_0})].$$

By the last-exit decomposition (see e.g. [33, Section 8.2]), one has

$$(5.42) \quad \begin{aligned} & \tilde{\mathbb{P}}_z(\tau_{B_{x_{\dagger}}(n)} < \tau_{\mathcal{C}_0}) \\ & \lesssim \sum_{y \in \partial B_{x_{\dagger}}(4dn), y' \in B_{x_{\dagger}}(4dn-1): \{y, y'\} \in \mathbb{L}^d} \tilde{G}_{\mathcal{C}_0}(z, y) \cdot \tilde{\mathbb{P}}_{y'}(\tau_{B_{x_{\dagger}}(n)} < \tau_{\mathcal{C}_0 \cup \partial B_{x_{\dagger}}(4dn)}). \end{aligned}$$

Moreover, the formula (3.28) implies that

$$(5.43) \quad \tilde{G}_{\mathcal{C}_0}(z, y) \lesssim \mathbb{P}(z \xleftrightarrow{(\mathcal{C}_0)} y).$$

Meanwhile, applying the potential theory, one has

$$(5.44) \quad \tilde{\mathbb{P}}_{y'}(\tau_{B_{x_{\dagger}}(n)} < \tau_{\mathcal{C}_0 \cup \partial B_{x_{\dagger}}(4dn)}) \leq \tilde{\mathbb{P}}_{y'}(\tau_{B_{x_{\dagger}}(n)} < \tau_{\partial B_{x_{\dagger}}(4dn)}) \lesssim n^{-1}.$$

Combining (5.41)-(5.44), we obtain

$$(5.45) \quad \begin{aligned} & \tilde{\mathbb{P}}_z \times \mathbb{P}(\bar{\mathbf{A}}_{x_{\dagger}}) \lesssim n^{-1} \sum_{y \in \partial B_{x_{\dagger}}(4dn)} \mathbb{E}[\mathbb{1}_{\mathbf{0} \leftrightarrow \partial B_{x_{\dagger}}(n)} \cdot \mathbb{P}(z \xleftrightarrow{(\mathcal{C}_0)} y)] \\ & \stackrel{(\text{restriction property})}{=} n^{-1} \sum_{y \in \partial B_{x_{\dagger}}(4dn)} \mathbb{E}[\mathbb{1}_{z \leftrightarrow y} \cdot \mathbb{P}(\mathbf{0} \xleftrightarrow{(\mathcal{C}_z)} \partial B_{x_{\dagger}}(n))] \\ & \stackrel{(\text{Lemma 3.6})}{\lesssim} n^{-\frac{d}{2}-1} \sum_{y \in \partial B_{x_{\dagger}}(4dn), w \in \partial \mathcal{B}_{x_{\dagger}}(dn)} \mathbb{P}(\mathbf{C}[(y, w, z, \mathbf{0})]), \end{aligned}$$

where, according to Lemma 4.8, each $\mathbb{P}(\mathbf{C}[(y, w, z, \mathbf{0})])$ is proportional to $\mathbb{P}(\mathbf{C}[\psi])$ for all $\psi \in \Psi(n, N)$. Consequently,

$$(5.46) \quad \tilde{\mathbb{P}}_z \times \mathbb{P}(\bar{\mathbf{A}}_{x_{\dagger}}) \lesssim n^{\frac{3d}{2}-1} \cdot \mathbb{P}(\mathbf{C}[\psi]), \quad \forall \psi \in \Psi(n, N).$$

Combined with (5.40), it yields the desired bound (5.30). $\square$

For the high-dimensional case $d \geq 7$, we present the following stronger result, which includes additional restrictions on the cluster volume. Recall $\mathcal{V}(\cdot)$ in (4.92).

Lemma 5.4. *For any $d \geq 7$, there exist $C_{15}, C_{16}, c_{13} > 0$ such that for any $M \geq 1$, $n \geq C_{15}$, $N \geq C_{16}n$ and $\psi \in \Psi(n, N)$,*

$$(5.47) \quad \mathbb{P}(\mathcal{V}_{v_1}(M) \wedge \mathcal{V}_{v_2}(M) \geq c_{13}M^4, \mathbf{C}[\psi]) \gtrsim N^{4-2d}.$$

To establish Lemma 5.4, we need the following technical lemma, which shows that in high dimensions, imposing an absorbing boundary on a large cluster decreases the Green's function by only a small proportion.

Lemma 5.5. *For any $d \geq 7$, there exists $C_{17} > 0$ such that for any $n \geq C_{17}$, $N \geq C_{17}n$, $\psi = (v_1, v_2, w_1, w_2) \in \Psi(n, N)$,*

$$(5.48) \quad \mathbb{E} \left[\mathbb{1}_{v_1 \leftrightarrow w_1} \cdot (\tilde{G}(v_2, w_2) - \tilde{G}_{\mathcal{C}_{v_1}}(v_2, w_2)) \right] \lesssim n^{6-d} N^{4-2d}.$$

Proof. By (3.4), the left-hand side of (5.48) can be written as

$$(5.49) \quad \begin{aligned} & \mathbb{E} \left[\mathbb{1}_{v_1 \leftrightarrow w_1} \cdot \sum_{z \in \mathcal{C}_{v_1}} \tilde{\mathbb{P}}_{v_2}(\tau_{\mathcal{C}_{v_1}} = \tau_z < \infty) \cdot \tilde{G}(z, w_2) \right] \\ & \lesssim \mathbb{I}(B_{v_1}(\frac{n}{10})) + \mathbb{I}(B_{w_1}(\frac{N}{10})) + \mathbb{J}, \end{aligned}$$

where $\mathbb{I}(\cdot)$ and $\mathbb{J}$ are defined by

$$(5.50) \quad \mathbb{I}(A) := \sum_{x \in A} \mathbb{P}(v_1 \leftrightarrow w_1, v_1 \leftrightarrow x) \cdot \tilde{\mathbb{P}}_{v_2}(\tau_x < \infty) \cdot \tilde{G}(x, w_2),$$

$$(5.51) \quad \mathbb{J} := \mathbb{E} \left[\mathbb{1}_{v_1 \leftrightarrow w_1} \cdot \sum_{z \in \partial \hat{\mathcal{C}}_{v_1}} \tilde{\mathbb{P}}_{v_2}(\tau_{\hat{\mathcal{C}}_{v_1}} = \tau_z < \infty) \cdot \tilde{G}(z, w_2) \right]$$

with $\hat{\mathcal{C}}_{v_1} := \mathcal{C}_{v_1} \setminus [B_{v_1}(\frac{n}{10}) \cup B_{w_1}(\frac{N}{10})]$.

For $\mathbb{I}(B_{v_1}(\frac{n}{10}))$, since $\text{dist}(v_2, B_{v_1}(\frac{n}{10})) \gtrsim n$ and $\text{dist}(v_2, B_{w_1}(\frac{N}{10})) \gtrsim N$, we have

$$(5.52) \quad \begin{aligned} \mathbb{I}(B_{v_1}(\frac{n}{10})) & \lesssim n^{2-d} N^{2-d} \sum_{x \in B_{v_1}(\frac{n}{10})} \mathbb{P}(v_1 \leftrightarrow w_1, v_1 \leftrightarrow x) \\ & \stackrel{(\text{Lemma 3.10})}{\lesssim} n^{6-d} N^{4-2d}. \end{aligned}$$

Similarly, $\mathbb{I}(B_{w_1}(\frac{N}{10}))$ can be estimated as follows:

$$(5.53) \quad \mathbb{I}(B_{w_1}(\frac{N}{10})) \lesssim N^{4-2d} \sum_{x \in B_{w_1}(\frac{N}{10})} \mathbb{P}(v_1 \leftrightarrow w_1, v_1 \leftrightarrow x) \stackrel{(\text{Lemma 3.10})}{\lesssim} N^{10-3d}.$$

It remains to bound the quantity $\mathbb{J}$. By (3.4) and Lemma 4.2, one has

$$(5.54) \quad \begin{aligned} \mathbb{J}_1 & := \mathbb{E} \left[\mathbb{1}_{v_1 \leftrightarrow w_1, \hat{\mathcal{L}}_{1/2}^{w_1} \geq [\ln(n)]^2} \cdot \sum_{z \in \partial \hat{\mathcal{C}}_{v_1}} \tilde{\mathbb{P}}_{v_2}(\tau_{\hat{\mathcal{C}}_{v_1}} = \tau_z < \infty) \cdot \tilde{G}(z, w_2) \right] \\ & = \mathbb{E} \left[\mathbb{1}_{v_1 \leftrightarrow w_1, \hat{\mathcal{L}}_{1/2}^{w_1} \geq [\ln(n)]^2} \cdot \max_{z \in \partial \hat{\mathcal{C}}_{v_1}} \tilde{G}(z, w_2) \right] \\ & \lesssim N^{2-d} \cdot \mathbb{P}(v_1 \leftrightarrow w_1, \hat{\mathcal{L}}_{1/2}^{w_1} \geq [\ln(n)]^2) \stackrel{(\text{Lemma 4.2})}{\lesssim} e^{-c[\ln(N)]^2} N^{4-2d}. \end{aligned}$$

Let $\mathbb{J}_2 := \mathbb{J} - \mathbb{J}_1$. By the switching identity (see Lemma 3.18) one has

$$(5.55) \quad \begin{aligned} \mathbb{J}_2 & \lesssim N^{2-d} \int_{0 \leq a, b \leq [\ln(n)]^2} \check{\mathbb{E}}_{v_1 \leftrightarrow w_1, a, b} \left[\sum_{z \in \partial \hat{\mathcal{C}}_{v_1}} \tilde{\mathbb{P}}_{v_2}(\tau_{\hat{\mathcal{C}}_{v_1}} = \tau_z < \infty) \tilde{G}(z, w_2) \right] \\ & \quad \cdot \mathbf{p}_{v_1 \leftrightarrow w_1, a, b} da db. \end{aligned}$$

Moreover, under the measure $\check{\mathbb{E}}_{v_1 \leftrightarrow w_1, a, b}$, the number of excursions in $\mathcal{P}^{(2)}$ that intersect $\partial B_{v_1}(\frac{n}{100})$ is a Poisson random variable with parameter

$$(5.56) \quad a \cdot \mathbf{e}_{v_1, v_1}^{\{w_1\}}(\{\tilde{\eta} : \text{ran}(\tilde{\eta}) \cap \partial B_{v_1}(\frac{n}{100}) \neq \emptyset\}) \stackrel{(\text{Lemma 3.17})}{\lesssim} a \cdot n^{2-d}.$$

Therefore, for any $0 \leq a \leq [\ln(n)]^2$, we have

$$\begin{aligned}
(5.57) \quad & \check{\mathbb{E}}_{v_1 \leftrightarrow w_1, a, b} \left[\mathbb{1}_{(\cup \mathcal{P}^{(2)}) \cap \partial B_{v_1}(\frac{n}{100}) \neq \emptyset} \cdot \sum_{z \in \tilde{\partial} \hat{\mathcal{C}}_{v_1}} \tilde{\mathbb{P}}_{v_2}(\tau_{\hat{\mathcal{C}}_{v_1}} = \tau_z < \infty) \cdot \tilde{G}(z, w_2) \right] \\
& \leq \check{\mathbb{E}}_{v_1 \leftrightarrow w_1, a, b} \left[\mathbb{1}_{(\cup \mathcal{P}^{(2)}) \cap \partial B_{v_1}(\frac{n}{100}) \neq \emptyset} \cdot \max_{z \in \tilde{\partial} \hat{\mathcal{C}}_{v_1}} \tilde{G}(z, w_2) \right] \\
& \lesssim N^{2-d} \cdot \check{\mathbb{P}}_{v_1 \leftrightarrow w_1, a, b}((\cup \mathcal{P}^{(2)}) \cap \partial B_{v_1}(\frac{n}{100}) \neq \emptyset) \stackrel{(5.56)}{\lesssim} [\ln(n)]^2 n^{2-d} N^{2-d}.
\end{aligned}$$

Similarly, we also have: for any $0 \leq b \leq [\ln(n)]^2$,

$$\begin{aligned}
(5.58) \quad & \check{\mathbb{E}}_{v_1 \leftrightarrow w_1, a, b} \left[\mathbb{1}_{(\cup \mathcal{P}^{(3)}) \cap \partial B_{w_1}(\frac{N}{100}) \neq \emptyset} \cdot \sum_{z \in \tilde{\partial} \hat{\mathcal{C}}_{v_1}} \tilde{\mathbb{P}}_{v_2}(\tau_{\hat{\mathcal{C}}_{v_1}} = \tau_z < \infty) \cdot \tilde{G}(z, w_2) \right] \\
& \lesssim N^{2-d} \cdot \check{\mathbb{P}}_{v_1 \leftrightarrow w_1, a, b}((\cup \mathcal{P}^{(3)}) \cap \partial B_{w_1}(\frac{N}{100}) \neq \emptyset) \lesssim [\ln(n)]^2 N^{4-2d}.
\end{aligned}$$

Let $\mathbf{F} := \{\cup \mathcal{P}^{(2)} \subset \tilde{B}_{v_1}(\frac{n}{100}), \cup \mathcal{P}^{(3)} \subset \tilde{B}_{w_1}(\frac{N}{100})\}$. By (5.55), (5.57) and (5.58),

$$\begin{aligned}
(5.59) \quad & \mathbb{J}_2 \lesssim N^{2-d} \int_{0 \leq a, b \leq [\ln(n)]^2} \check{\mathbb{E}}_{v_1 \leftrightarrow w_1, a, b} \left[\mathbb{1}_{\mathbf{F}} \sum_{z \in \tilde{\partial} \hat{\mathcal{C}}_{v_1}} \tilde{\mathbb{P}}_{v_2}(\tau_{\hat{\mathcal{C}}_{v_1}} = \tau_z < \infty) \tilde{G}(z, w_2) \right] \\
& \quad \cdot \mathbf{p}_{v_1 \leftrightarrow w_1, a, b} da db + [\ln(n)]^2 n^{2-d} N^{4-2d}.
\end{aligned}$$

On the event $\mathbf{F}$, when $\tilde{S} \sim \tilde{\mathbb{P}}_{v_2}$ first hits $\hat{\mathcal{C}}_{v_1}$ at z , one of the following occurs:

- (i) z is connected to $[\cup(\mathcal{P}^{(2)} + \mathcal{P}^{(4)})] \cap B_{v_1}(\frac{n}{100})$ by $\cup \mathcal{P}^{(1)}$;
- (ii) z is connected to $[\cup(\mathcal{P}^{(3)} + \mathcal{P}^{(4)})] \cap B_{w_1}(\frac{N}{100})$ by $\cup \mathcal{P}^{(1)}$;
- (iii) z is connected to $(\cup \mathcal{P}^{(4)}) \setminus [B_{v_1}(\frac{n}{100}) \cup B_{w_1}(\frac{N}{100})]$ by $\cup \mathcal{P}^{(1)}$.

For $\diamond \in \{\text{i, ii, iii}\}$, we define $\mathbf{F}_\diamond$ as the sub-event of $\mathbf{F}$ restricted to Case $(\diamond)$, and let

$$(5.60) \quad \mathbb{U}_\diamond := \check{\mathbb{E}}_{v_1 \leftrightarrow w_1, a, b} \left[\mathbb{1}_{\mathbf{F}_\diamond} \cdot \sum_{z \in \tilde{\partial} \hat{\mathcal{C}}_{v_1}} \tilde{\mathbb{P}}_{v_2}(\tau_{\hat{\mathcal{C}}_{v_1}} = \tau_z < \infty) \tilde{G}(z, w_2) \right].$$

For $\mathbb{U}_i$, by the union bound, $\mathbb{U}_i$ is at most proportional to

$$\begin{aligned}
(5.61) \quad & \left(\check{\mathbb{E}}_{v_1 \leftrightarrow w_1, a, b} [\text{vol}(\cup \mathcal{P}^{(2)} \cap B_{v_1}(\frac{n}{100}))] + \check{\mathbb{E}}_{v_1 \leftrightarrow w_1, a, b} [\text{vol}(\cup \mathcal{P}^{(4)} \cap B_{v_1}(\frac{n}{100}))] \right) \\
& \cdot \max_{y \in B_{v_1}(\frac{n}{100})} \sum_{z \in [B_{v_1}(\frac{n}{10}) \cup B_{w_1}(\frac{N}{10})]^c} \mathbb{P}(y \leftrightarrow \tilde{B}_z(1)) \tilde{\mathbb{P}}_{v_2}(\tau_z < \infty) \tilde{G}(z, w_2).
\end{aligned}$$

Since $\mathbf{e}_{v_1, v_1}(\{\tilde{\eta} : y \in \text{ran}(\tilde{\eta})\}) \lesssim |v_1 - y|^{4-2d}$ for all $y \in [\mathbf{B}_{v_1}(1)]^c$, one has

$$(5.62) \quad \check{\mathbb{E}}_{v_1 \leftrightarrow w_1, a, b} [\text{vol}(\cup \mathcal{P}^{(2)} \cap B_{v_1}(\frac{n}{100}))] \lesssim \sum_{y \in B_{v_1}(\frac{n}{100})} (|v_1 - y| + 1)^{4-2d} \stackrel{(4.78)}{\lesssim} 1.$$

Meanwhile, under the measure $\mathbf{e}_{v_1, w_1}$, the probability of an excursion intersecting $y \in B_{v_1}(\frac{n}{100})$ is at most $(|v_1 - y| + 1)^{2-d}$. Therefore, similar to (4.61), we have

$$(5.63) \quad \check{\mathbb{E}}_{v_1 \leftrightarrow w_1, a, b} [\text{vol}(\cup \mathcal{P}^{(4)} \cap B_{v_1}(\frac{n}{100}))] \lesssim \sum_{y \in B_{v_1}(\frac{n}{100})} (|v_1 - y| + 1)^{2-d} \stackrel{(4.78)}{\lesssim} n^2.$$

For any $y \in B_{v_1}(\frac{n}{100})$ and $z \in [B_{v_1}(\frac{n}{10}) \cup B_{w_1}(\frac{N}{10})]^c$, since $|y - z| \asymp |z - v_1|$, one has

$$(5.64) \quad \mathbb{P}(y \leftrightarrow \tilde{B}_z(1)) \asymp (|z - v_1| + 1)^{2-d}.$$

Thus, plugging (5.62), (5.63) and (5.64) into (5.61), we derive that

$$(5.65) \quad \mathbb{U}_i \lesssim n^2 \sum_{z \in \mathbb{Z}^d} (|z - v_1| + 1)^{2-d} (|v_2 - z| + 1)^{2-d} (|z - w_2| + 1)^{2-d}.$$

Since $|v_2 - w_2| \gtrsim N$, one has $|v_2 - z| \vee |z - w_2| \gtrsim N$. Restricted to $|v_2 - z| \gtrsim N$, the sum in (5.65) is bounded from above by

$$(5.66) \quad CN^{2-d} \sum_{z \in \mathbb{Z}^d} (|z - v_1| + 1)^{2-d} (|z - w_2| + 1)^{2-d} \stackrel{(4.66)}{\lesssim} N^{6-2d}.$$

Similarly, when restricted to $|z - w_2| \gtrsim N$, this sum is upper-bounded by

$$(5.67) \quad CN^{2-d} \sum_{z \in \mathbb{Z}^d} (|z - v_1| + 1)^{2-d} (|v_2 - z| + 1)^{2-d} \lesssim n^{4-d} N^{2-d}.$$

Combining (5.65), (5.66) and (5.67), we get

$$(5.68) \quad \mathbb{U}_i \lesssim n^2 (N^{6-2d} + n^{4-d} N^{2-d}) \lesssim n^{6-d} N^{2-d}.$$

For $\mathbb{U}_{ii}$, by the same argument as in the proof of (5.65), one has

$$(5.69) \quad \begin{aligned} \mathbb{U}_{ii} &\lesssim N^2 \sum_{z \in \mathbb{Z}^d} (|z - w_1| + 1)^{2-d} (|v_2 - z| + 1)^{2-d} (|z - w_2| + 1)^{2-d} \\ &\stackrel{(|z - w_1| \gtrsim N)}{\lesssim} N^{4-d} \sum_{z \in \mathbb{Z}^d} (|v_2 - z| + 1)^{2-d} (|z - w_2| + 1)^{2-d} \stackrel{(4.66)}{\lesssim} N^{8-2d}. \end{aligned}$$

We now estimate $\mathbb{U}_{iii}$. Recalling the construction of Case (iii), we have

$$(5.70) \quad \begin{aligned} \mathbb{U}_{iii} &\lesssim \sum_{y \in [B_{v_1}(\frac{n}{100}) \cup B_{w_1}(\frac{N}{100})]^c, z \in [B_{v_1}(\frac{n}{10}) \cup B_{w_1}(\frac{N}{10})]^c} \check{\mathbb{P}}_{v_1 \leftrightarrow w_1, a, b}(y \in \cup \mathcal{P}^{(4)}) \\ &\quad \cdot \mathbb{P}(y \leftrightarrow \tilde{B}_z(1)) \cdot \tilde{\mathbb{P}}_{v_2}(\tau_z < \infty) \cdot \tilde{G}(z, w_2). \end{aligned}$$

In addition, parallel to (4.61), the probability $\check{\mathbb{P}}_{v_1 \leftrightarrow w_1, a, b}(y \in \cup \mathcal{P}^{(4)})$ is at most

$$(5.71) \quad \begin{aligned} &\max_{y' \in \partial B_{v_1}(\frac{n}{200})} \tilde{\mathbb{P}}_{y'}(\tau_y < \tau_{\{v_1, w_1\}} \mid \tau_{\{v_1, w_1\}} = \tau_{w_1} < \infty) \\ &\lesssim N^{d-2} (|v_1 - y| + 1)^{2-d} (|y - w_1| + 1)^{2-d} \stackrel{(|y - w_1| \gtrsim N)}{\lesssim} (|v_1 - y| + 1)^{2-d}. \end{aligned}$$

By (5.70), (5.71) and $\mathbb{P}(y \leftrightarrow \tilde{B}_z(1)) \asymp (|y - z| + 1)^{2-d}$, we have

$$(5.72) \quad \begin{aligned} \mathbb{U}_{iii} &\lesssim \sum_{y, z \in \mathbb{Z}^d} (|v_1 - y| + 1)^{2-d} (|y - z| + 1)^{2-d} \\ &\quad \cdot (|v_2 - z| + 1)^{2-d} (|z - w_2| + 1)^{2-d} \\ &\stackrel{(4.66)}{\lesssim} \sum_{z \in \mathbb{Z}^d} (|v_1 - z| + 1)^{4-d} (|v_2 - z| + 1)^{2-d} (|z - w_2| + 1)^{2-d}. \end{aligned}$$

As in (5.66), restricted to $|v_2 - z| \gtrsim N$, this sum is bounded from above by

$$(5.73) \quad CN^{2-d} \sum_{z \in \mathbb{Z}^d} (|v_1 - z| + 1)^{4-d} (|z - w_2| + 1)^{2-d} \lesssim N^{8-2d}.$$

Here we used the elementary estimate that (see e.g. [3, (4.10)])

$$(5.74) \quad \sum_{x \in \mathbb{Z}^d} (|v - x| + 1)^{4-d} (|x - w| + 1)^{2-d} \lesssim (|v - w| + 1)^{6-d}.$$

This estimate also implies that restricted to $|z - w_2| \gtrsim N$, the sum is at most

$$(5.75) \quad CN^{2-d} \sum_{z \in \mathbb{Z}^d} (|v_1 - z| + 1)^{4-d} (|v_2 - z| + 1)^{2-d} \lesssim n^{6-d} N^{2-d}.$$

Combining (5.72), (5.73) and (5.75), we have

$$(5.76) \quad \mathbb{U}_{\text{iii}} \lesssim N^{8-2d} + n^{6-d} N^{2-d} \lesssim n^{6-d} N^{2-d}.$$

Plugging (5.68), (5.69), (5.76) into (5.59), we obtain

$$(5.77) \quad \mathbb{J}_2 \lesssim N^{2-d} \cdot (n^{6-d} N^{2-d} + N^{8-2d}) + [\ln(n)]^2 n^{2-d} N^{4-2d} \lesssim n^{6-d} N^{4-2d}.$$

By substituting (5.52), (5.53), (5.54), (5.77) into (5.49), we derive (5.48). $\square$

Now we are ready to prove Lemma 5.4.

Proof of Lemma 5.4. According to Lemma 3.11, there exists $c_{\dagger} > 0$ such that for any $\psi = (v_1, v_2, w_1, w_2) \in \Psi(n, N)$,

$$(5.78) \quad \mathbb{P}(\mathcal{V}_{v_i}(M) \geq c_{\dagger} M^4, v_i \leftrightarrow w_i) \geq \frac{3}{4} \cdot \mathbb{P}(v_i \leftrightarrow w_i), \quad \forall i \in \{1, 2\}.$$

For any $i \in \{1, 2\}$ and $\epsilon > 0$, we denote the event

$$(5.79) \quad \mathbf{F}_{i,\epsilon} := \{\mathcal{V}_{v_i}(M) \geq c_{\dagger} M^4, v_i \leftrightarrow w_i, \mathcal{C}_{v_i} \cap [B_{v_{3-i}}(\epsilon n) \cup B_{w_{3-i}}(\epsilon N)] = \emptyset\}.$$

By the restriction property, one has

$$(5.80) \quad \begin{aligned} \mathbb{P}(\mathcal{V}_{v_i}(M) \geq c_{\dagger} M^4, \mathbf{C}[\psi]) &\geq \mathbb{E} \left[\mathbb{1}_{\mathbf{F}_{i,\epsilon}} \cdot \mathbb{P}(v_{3-i} \xrightarrow{(\mathcal{C}_{v_i})} w_{3-i}) \right] \\ &\stackrel{(\text{Lemma 3.3})}{\geq} \mathbb{I}_1 - C \cdot (\mathbb{I}_2 + \mathbb{I}_3), \end{aligned}$$

where $\mathbb{I}_1$, $\mathbb{I}_2$ and $\mathbb{I}_3$ are defined by

$$(5.81) \quad \mathbb{I}_1 := \mathbb{P}(\mathbf{F}_{i,\epsilon}) \cdot \mathbb{P}(v_{3-i} \leftrightarrow w_{3-i}),$$

$$(5.82) \quad \mathbb{I}_2 := \mathbb{E} \left[\mathbb{1}_{v_i \leftrightarrow w_i} \cdot (\tilde{G}(v_{3-i}, w_{3-i}) - \tilde{G}_{\mathcal{C}_{v_i}}(v_{3-i}, w_{3-i})) \right],$$

$$(5.83) \quad \mathbb{I}_3 := \mathbb{P}(v_1 \leftrightarrow w_1) \cdot \mathbb{P}(v_2 \leftrightarrow w_2) \cdot (\epsilon n)^{2-d}.$$

For $\mathbb{I}_1$, by (4.28) and (5.78), we have

$$(5.84) \quad \mathbb{I}_1 \geq \left[\frac{3}{4} - o_{\epsilon}(1) \right] \cdot \mathbb{P}(v_1 \leftrightarrow w_1) \cdot \mathbb{P}(v_2 \leftrightarrow w_2).$$

For $\mathbb{I}_2$, it directly follows from Lemma 5.5 that

$$(5.85) \quad \mathbb{I}_2 \lesssim n^{6-d} N^{4-2d} \asymp n^{6-d} \cdot \mathbb{P}(v_1 \leftrightarrow w_1) \cdot \mathbb{P}(v_2 \leftrightarrow w_2).$$

Plugging (5.83), (5.84) and (5.85) into (5.80), we have

$$\begin{aligned}
(5.86) \quad & \mathbb{P}(\mathcal{V}_{v_i}(M) \geq c_{\dagger} M^4, \mathbb{C}[\psi]) \\
& \geq \left[\frac{3}{4} - o_{\epsilon}(1) - C(\epsilon n)^{2-d} - Cn^{6-d} \right] \cdot \mathbb{P}(v_1 \leftrightarrow w_1) \cdot \mathbb{P}(v_2 \leftrightarrow w_2) \\
& \geq \frac{2}{3} \cdot \mathbb{P}(v_1 \leftrightarrow w_1) \cdot \mathbb{P}(v_2 \leftrightarrow w_2),
\end{aligned}$$

where in the last inequality, we choose a sufficiently small $\epsilon > 0$, and then take n large enough to ensure that $\frac{3}{4} - o_{\epsilon}(1) - C(\epsilon n)^{2-d} - Cn^{6-d} \geq \frac{2}{3}$. Moreover, applying the BRK inequality, one has

$$(5.87) \quad \mathbb{P}(\mathbb{C}[\psi]) \leq \mathbb{P}(v_1 \leftrightarrow w_1) \cdot \mathbb{P}(v_2 \leftrightarrow w_2).$$

Thus, applying the inclusion-exclusion principle, we obtain (5.47) as follows:

$$\begin{aligned}
& \mathbb{P}(\mathcal{V}_{v_1}(M) \wedge \mathcal{V}_{v_2}(M) \geq c_{\dagger} M^4, \mathbb{C}[\psi]) \\
& \geq \sum_{i \in \{1,2\}} \mathbb{P}(\mathcal{V}_{v_i}(M) \geq c_{\dagger} M^4, \mathbb{C}[\psi]) - \mathbb{P}(\mathbb{C}[\psi]) \\
& \stackrel{(5.86), (5.87)}{\geq} \frac{1}{3} \cdot \mathbb{P}(v_1 \leftrightarrow w_1) \cdot \mathbb{P}(v_2 \leftrightarrow w_2) \gtrsim N^{4-2d}. \quad \square
\end{aligned}$$

By Lemmas 5.1, 5.2 and 5.4, we have verified the upper bound in (1.16) and the lower bound in (1.17). $\square$

6. ESTIMATES ON HETEROCHROMATIC TWO-ARM PROBABILITIES

This section is mainly devoted to proving Theorem 1.1, which provides the exact order of the probability $\mathbb{P}(\mathbf{H}_{v'}^v(N))$. Unless specified otherwise, we assume that N is sufficiently large and that $v, v' \in \tilde{B}(cN)$. By the isomorphism theorem, it suffices to prove the analogue of Theorem 1.1 for $\bar{\mathbf{H}}_{v'}^v(N) := \{v \leftrightarrow \partial B(N) \parallel v' \leftrightarrow \partial B(N)\}$:

$$(6.1) \quad \text{when } \chi =: |v - v'| \leq C, \quad \mathbb{P}(\bar{\mathbf{H}}_{v'}^v(N)) \asymp \chi^{\frac{3}{2}} N^{-[(\frac{d}{2}+1)\Box 4]},$$

$$(6.2) \quad \text{when } \chi \geq C, \quad \mathbb{P}(\bar{\mathbf{H}}_{v'}^v(N)) \asymp \chi^{(3-\frac{d}{2})\Box 0} N^{-[(\frac{d}{2}+1)\Box 4]}.$$

We will establish (6.1) and (6.2) in Sections 6.1 and 6.2 respectively. In Section 6.3, we prove the remaining estimates in Theorem 1.4.

6.1. Proof of (6.1). Without loss of generality, we assume that $v \neq v' \in \tilde{B}(C_{11})$ (recall the constant C_{11} from Lemma 5.1).

For the upper bound in (6.1), let Ω_1 (resp. Ω_2) denote a collection of $O(1)$ -many boxes of the form $B_x(c_{10}N)$ whose union contains $\partial B(\frac{N}{4})$ (resp. $\partial B(\frac{N}{2})$). By the union bound, one has (recalling $\mathbb{C}^{\text{out}}[\cdot]$ from (4.90)),

$$(6.3) \quad \mathbb{P}(\bar{\mathbf{H}}_{v'}^v(N)) \leq \sum_{\psi=(v,v',w_1,w_2): w_1 \in \Omega_1, w_2 \in \Omega_2} \mathbb{P}(\mathbb{C}^{\text{out}}[\psi, c_{10}]).$$

Moreover, by the same argument as in the proof of (4.97), we have

$$(6.4) \quad \mathbb{P}(\mathbb{C}^{\text{out}}[\psi, c_{10}]) \lesssim N^{(d-2)\Box(2d-8)} \cdot \mathbb{P}(\mathbb{C}[\psi])$$

for all $\psi = (v, v', w_1, w_2)$ with $w_1 \in \Omega_1$ and $w_2 \in \Omega_2$. Plugging (6.4) into (6.3), and then using Lemma 5.1, we obtain the upper bound in (6.1).

For the lower bound in (6.1), we prove the following stronger result, which will be useful for the companion paper [7].

Lemma 6.1. *For any $d \geq 3$ with $d \neq 6$, there exist $C_{18}, C_{19}, C_{20}, c_{14} > 0$ such that for any $v \neq v' \in \tilde{B}(C_{18})$ and any sufficiently large $N \geq 1$,*

$$(6.5) \quad \mathbb{P}(\hat{H}_{v'}^v(N)) \gtrsim |v - v'|^{\frac{3}{2}} N^{-[(\frac{d}{2}+1)\square 4]},$$

where $\hat{H}_{v'}^v(N)$ is defined as $H_{v'}^v(N) \cap \{v \xleftrightarrow{\geq 0} \partial B(C_{19}N)\}^c \cap \{v' \xleftrightarrow{\leq 0} \partial B(C_{19}N)\}^c \cap \{|\tilde{\phi}_v|, |\tilde{\phi}_{v'}| \in [c_{14}, C_{20}]\}$.

Proof. Recall $C^\diamond[\cdot]$ in (4.90), and recall $S^\diamond[\cdot]$ below (4.91). By Lemmas 4.9, 5.2 and 5.4, there exist $C_\dagger, c_\star, c_\Delta > 0$ such that for any $\psi \in \Psi(C_\dagger, N)$,

$$(6.6) \quad \begin{aligned} \mathbb{P}(C^{\text{out}}[\psi, c_\star]) &\stackrel{(\text{Lemma 4.9})}{\asymp} \mathbb{P}(C^{\text{out}}[\psi, c_\star] \cap (S^{\text{out}}[\psi, c_\Delta])^c) \\ &\stackrel{(\text{Lemma 4.9})}{\asymp} N^{(d-2)\square(2d-8)} \mathbb{P}(C[\psi]) \stackrel{(\text{Lemmas 5.2 and 5.4})}{\gtrsim} N^{-[(\frac{d}{2}+1)\square 4]}. \end{aligned}$$

We arbitrarily take $v_1 \in B(2C_\dagger)$, $v_2 \in \partial B(4C_\dagger)$, $w_1 \in \partial B(2N)$ and $w_2 \in \partial B(4N)$. Note that (6.6) holds for $\psi_\spadesuit = (v_1, v_2, w_1, w_2)$. On $C^{\text{out}}[\psi_\spadesuit, c_\star] \cap (S^{\text{out}}[\psi_\spadesuit, c_\Delta])^c$, there exist two disjoint paths $\tilde{\eta}_1$ and $\tilde{\eta}_2$ such that for $j \in \{1, 2\}$, $\tilde{\eta}_j$ is contained in $\mathcal{C}_{v_j}$, starts from w_j and ends at v_j . Let x_j be the point where $\tilde{\eta}_j$ first hits $\tilde{B}(8C_\dagger)$. It follows from the construction that the events

$$(6.7) \quad A_{x_1, x_2} := \{x_1 \xleftrightarrow{\mathcal{C}_{x_1} \cap [\tilde{B}(8C_\dagger)]^c} B_{w_1}(c_\star N) \parallel x_2 \xleftrightarrow{\mathcal{C}_{x_2} \cap [\tilde{B}(8C_\dagger)]^c} B_{w_2}(c_\star N)\}$$

and $(S^{\text{out}}[(x_1, x_2, w_1, w_2), c_\Delta])^c$ both occur. Combined with (6.6), it yields that for some $x_1^* \neq x_2^* \in \tilde{\partial}\tilde{B}(8C_\dagger)$ (letting $\psi_* := (x_1^*, x_2^*, w_1, w_2)$),

$$(6.8) \quad \mathbb{P}(\hat{A}_{x_1^*, x_2^*}) := \mathbb{P}(A_{x_1^*, x_2^*} \cap (S^{\text{out}}[\psi_*, c_\Delta])^c) \gtrsim N^{-[(\frac{d}{2}+1)\square 4]}.$$

For any measurable set $F \subset (0, \infty)$, any $x_1 \neq x_2 \in \tilde{\partial}\tilde{B}(8C_\dagger)$ and $j \in \{1, 2\}$, by the restriction property, we have (letting $\psi := (x_1, x_2, w_1, w_2)$)

$$(6.9) \quad \begin{aligned} &\mathbb{P}(A_{x_1, x_2} \cap (S^{\text{out}}[\psi, c_\Delta])^c, \hat{\mathcal{L}}_{1/2}^{x_j} \in F) \\ &\leq \mathbb{E} \left[\mathbb{1}_{x_{3-j} \leftrightarrow B_{w_{3-j}}(c_\star N), \{x_{3-j} \leftrightarrow B_{x_j}(c_\Delta C_\dagger)\}^c} \right. \\ &\quad \cdot \mathbb{P}(x_j \xleftrightarrow{\mathcal{C}_{x_{3-j}}} B_{w_j}(c_\star N), \hat{\mathcal{L}}_{1/2}^{\mathcal{C}_{x_{3-j}}, x_j} \in F \mid \mathcal{F}_{\mathcal{C}_{x_{3-j}}}) \Big]. \end{aligned}$$

On the event $\{x_{3-j} \leftrightarrow B_{x_j}(c_\Delta C_\dagger)\}^c$, one has $\tilde{G}_{\mathcal{C}_{x_{3-j}}}(x_j, x_j) \asymp 1$ and hence,

$$(6.10) \quad \mathbb{P}(\hat{\mathcal{L}}_{1/2}^{\mathcal{C}_{x_{3-j}}, x_j} \leq t \mid \mathcal{F}_{\mathcal{C}_{x_{3-j}}}) = \mathbb{P}^{\mathcal{C}_{x_{3-j}}}(|\tilde{\phi}_{x_j}| \leq \sqrt{2t}) \lesssim t^{\frac{1}{2}}, \quad \forall t > 0.$$

Therefore, applying the FKG inequality, we get

$$(6.11) \quad \begin{aligned} & \mathbb{P}(x_j \xrightarrow{(C_{x_{3-j}})} B_{w_j}(c_\star N), \hat{\mathcal{L}}_{1/2}^{C_{x_{3-j}}, x_j} \leq t \mid \mathcal{F}_{C_{x_{3-j}}}) \\ & \lesssim t^{\frac{1}{2}} \cdot \mathbb{P}(x_j \xrightarrow{(C_{x_{3-j}})} B_{w_j}(c_\star N) \mid \mathcal{F}_{C_{x_{3-j}}}). \end{aligned}$$

Combined with (6.9), it yields that

$$(6.12) \quad \begin{aligned} & \mathbb{P}(A_{x_1, x_2} \cap (S^{\text{out}}[\psi, c_\Delta])^c, \hat{\mathcal{L}}_{1/2}^{x_j} \leq t) \\ & \lesssim t^{\frac{1}{2}} \cdot \mathbb{P}(C^{\text{out}}[\psi, c_\star]) \leq t^{\frac{1}{2}} \cdot \mathbb{P}(\bar{H}_{x_2}^{x_1}(N)) \lesssim t^{\frac{1}{2}} N^{-[(\frac{d}{2}+1)\Box 4]}. \end{aligned}$$

Here in the last inequality we used the upper bound in (6.1), which has been confirmed below (6.4). Moreover, for any $T > 0$, by Markov's inequality, one has

$$(6.13) \quad \begin{aligned} & \mathbb{P}(x_j \xrightarrow{(C_{x_{3-j}})} B_{w_j}(c_\star N), \hat{\mathcal{L}}_{1/2}^{C_{x_{3-j}}, x_j} \geq T \mid \mathcal{F}_{C_{x_{3-j}}}) \\ & \leq T^{-1} \cdot \mathbb{E} \left[\mathbb{1}_{x_j \xrightarrow{(C_{x_{3-j}})} B_{w_j}(c_\star N)} \cdot \hat{\mathcal{L}}_{1/2}^{C_{x_{3-j}}, x_j} \mid \mathcal{F}_{C_{x_{3-j}}} \right] \\ & \lesssim T^{-1} \cdot \mathbb{P}(x_j \xrightarrow{(C_{x_{3-j}})} B_{w_j}(c_\star N) \mid \mathcal{F}_{C_{x_{3-j}}}), \end{aligned}$$

where the last inequality follows from [5, (2.67)]. Combined with (6.9), it implies

$$(6.14) \quad \begin{aligned} & \mathbb{P}(A_{x_1, x_2} \cap (S^{\text{out}}[\psi, c_\Delta])^c, \hat{\mathcal{L}}_{1/2}^{x_j} \geq T) \\ & \lesssim T^{-1} \cdot \mathbb{P}(C^{\text{out}}[\psi, c_\star]) \lesssim T^{-1} N^{-[(\frac{d}{2}+1)\Box 4]}. \end{aligned}$$

Putting (6.8), (6.12) and (6.14) together, we obtain that for some $C_\clubsuit > 0$,

$$(6.15) \quad \mathbb{P}(\hat{A}_{x_1^*, x_2^*}, \bigcap_{y \in (C_{x_1^*} \cup C_{x_2^*}) \cap \partial \tilde{B}(8C_\dagger)} \{\hat{\mathcal{L}}_{1/2}^y \in [C_\clubsuit^{-1}, C_\clubsuit]\}) \gtrsim N^{-[(\frac{d}{2}+1)\Box 4]}.$$

We define the clusters

$$(6.16) \quad C_1^* := \{v \in \tilde{\mathbb{Z}}^d : x_1^* \xrightarrow{\tilde{E} \geq 0 \cap [\tilde{B}(8C_\dagger)]^c} v\}, \quad C_2^* := \{v \in \tilde{\mathbb{Z}}^d : x_2^* \xrightarrow{\tilde{E} \leq 0 \cap [\tilde{B}(8C_\dagger)]^c} v\}.$$

The estimate (6.15) together with the isomorphism theorem implies that

$$(6.17) \quad \mathbb{P}(A_*^\pm) \gtrsim N^{-[(\frac{d}{2}+1)\Box 4]}.$$

Here the event $A_*^\pm$ is defined by

$$A_*^\pm := \bigcap_{j \in \{1, 2\}} \{C_j^* \cap B_{w_j}(c_\star N) \neq \emptyset\} \cap \bigcap_{y \in (C_1^* \cup C_2^*) \cap \partial \tilde{B}(8C_\dagger)} \{\sqrt{2C_\clubsuit^{-1}} \leq |\tilde{\phi}_y| \leq \sqrt{2C_\clubsuit}\},$$

which is measurable with respect to $\mathcal{F}_{C_1^* \cup C_2^*}$.

We enumerate the points in $[(C_1^* \cup C_2^*) \cap \partial \tilde{B}(8C_\dagger)] \setminus \{x_1^*, x_2^*\}$ as $y_1, \dots, y_K$. For each $1 \leq j \leq K$, we denote by I_j^* the interval incident to y_j such that $I_j^* \subset C_1^* \cup C_2^*$. When the distance between y_j and the boundary with GFF values fixed is at least

d and the absolute values of all boundary conditions lie in $[\sqrt{2C_{\clubsuit}^{-1}}, \sqrt{2C_{\clubsuit}}]$, the conditional probability of $D_j := \{y_j \xrightarrow{\tilde{E} \geq 0 \setminus I_j^*} \tilde{\partial} \mathbf{B}_{y_j}(1)\}^c \cup \{y_j \xrightarrow{\tilde{E} \leq 0 \setminus I_j^*} \tilde{\partial} \mathbf{B}_{y_j}(1)\}^c$ is of order 1. As a result, on the event $\mathbf{A}_*^\pm$,

$$(6.18) \quad \mathbb{P}(\cap_{1 \leq j \leq K} D_j \mid \mathcal{F}_{\mathcal{C}_1^* \cup \mathcal{C}_2^*}) = \prod_{1 \leq j \leq K} \mathbb{P}(D_j \mid \mathcal{F}_{\mathcal{C}_1^* \cup \mathcal{C}_2^*}, \cap_{1 \leq l \leq j-1} D_l) \stackrel{K=O(1)}{\gtrsim} 1,$$

which together with (6.17) implies that

$$(6.19) \quad \mathbb{P}(\hat{\mathbf{A}}_*^\pm) := \mathbb{P}(\cap_{1 \leq j \leq K} D_j \mid \mathbf{A}_*^\pm) \gtrsim N^{-[(\frac{d}{2}+1)\Box 4]}.$$

For each $i \in \{1, 2\}$ and $y \in [\mathcal{C}_i^* \cap \tilde{\partial} \tilde{B}(8C_+)] \setminus \{x_i^*\}$, we define the cluster

$$(6.20) \quad \mathcal{C}_{i,y} := \{z \in \mathbf{B}_y(1) : y \xrightarrow{\tilde{E} \geq 0 \setminus I_j^*} z \text{ or } y_j \xrightarrow{\tilde{E} \leq 0 \setminus I_j^*} z\}.$$

We then define $\hat{\mathcal{C}}_i^*$ as the union of all these clusters and $\mathcal{C}_i^*$. Note that the event $\hat{\mathbf{A}}_*^\pm$ is measurable with respect to $\mathcal{F}_{\hat{\mathcal{C}}_1^* \cup \hat{\mathcal{C}}_2^*}$.

We first present the remainder of the proof for the case $\chi := |v - v'| \leq 1$. The case $\chi > 1$ follows from a similar argument, which is considerably simpler. Let y_1^* and y_2^* be two lattice points that are contained in the same interval as v and v' respectively, and satisfy $\text{dist}(I_{[y_1^*, v]}, I_{[y_2^*, v']}) = \chi$. Note that when $\chi \leq 1$, at least one of the intervals $I_{[y_1^*, v]}$ and $I_{[y_2^*, v']}$ has length at least 1 (recall that the length of a single interval is $d \geq 3$). Without loss of generality, we assume that $I_{[y_1^*, v]}$ does. We then take two disjoint paths η_1^* and η_2^* on $\mathbb{Z}^d$ (with lengths l_1^* and l_2^*) such that η_i^* starts from x_i^* and ends at y_i^* . The k -th point of η_i^* is denoted by $\eta_i^*(k)$. For each $0 \leq k \leq l_1^* - 1$, we define the event

$$(6.21) \quad \begin{aligned} \mathbf{F}_k^+ := & \{ \sqrt{2C_{\clubsuit}^{-1}} \leq \tilde{\phi}_z \leq \sqrt{2C_{\clubsuit}}, \forall z \in I_{\{\eta_1^*(k), \eta_1^*(k+1)\}} \} \\ & \cap \{ \eta_1^*(k) \xrightarrow{\tilde{E} \geq 0 \setminus I_{\{\eta_1^*(k), \eta_1^*(k+1)\}}} \tilde{\partial} \mathbf{B}_{\eta_1^*(k)}(1) \}^c. \end{aligned}$$

For the same reason as in (6.18), we have: on the event $\hat{\mathbf{A}}_*^\pm$,

$$(6.22) \quad \mathbb{P}(\cap_{0 \leq k \leq l_1^*-1} \mathbf{F}_k^+ \mid \mathcal{F}_{\hat{\mathcal{C}}_1^* \cup \hat{\mathcal{C}}_2^*}) = \prod_{0 \leq k \leq l_1^*-1} \mathbb{P}(\mathbf{F}_k^+ \mid \mathcal{F}_{\mathcal{C}_1^* \cup \mathcal{C}_2^*}, \cap_{0 \leq l \leq k-1} \mathbf{F}_l^+) \gtrsim 1.$$

Moreover, we consider the event

$$(6.23) \quad \mathbf{G}^+ := \{ \tilde{\phi}_v \in [\chi^{\frac{1}{2}}, 2\chi^{\frac{1}{2}}] \} \cap \{ y_1^* \xrightarrow{\tilde{E} \geq 0 \cap I_{[y_1^*, v]}} v \} \cap \{ v \xrightarrow{\tilde{E} \geq 0 \setminus I_{[y_1^*, v]}} \tilde{\partial} \mathbf{B}_v(\frac{\chi}{10}) \}^c.$$

Conditioned on $\cap_{0 \leq k \leq l_1^*-1} \mathbf{F}_k^+ \cap \hat{\mathbf{A}}_*^\pm$, since the distance between v and the boundary with GFF values fixed is at least 1 (here we used the assumption that the length of $I_{[y_1^*, v]}$ is at least 1) and the absolute values of all boundary conditions are contained in $[\sqrt{2C_{\clubsuit}^{-1}}, \sqrt{2C_{\clubsuit}}]$, the probability of $\tilde{\phi}_v \in [\chi^{\frac{1}{2}}, 2\chi^{\frac{1}{2}}]$ is of order $\chi^{\frac{1}{2}}$. Moreover, conditioned on $\{ \tilde{\phi}_v \in [\chi^{\frac{1}{2}}, 2\chi^{\frac{1}{2}}] \}$, the formula (3.29) implies that the probability of

$\{y_1^* \xleftrightarrow{\tilde{E} \geq 0 \cap I_{[y_1^*, v]}} v\}$ is at least of order $\chi^{\frac{1}{2}}$. Furthermore, after sampling $\tilde{\phi}$ on $I_{[y_1^*, v]}$ with $\tilde{\phi}_v \in [\chi^{\frac{1}{2}}, 2\chi^{\frac{1}{2}}]$, the conditional probability of the event $\{v \xleftrightarrow{\geq 0} \tilde{\partial} \mathbf{B}_v(\frac{\chi}{10})\}^c$ is of order 1. Combined with (6.22), it yields that on the event $\hat{\mathbf{A}}_*^\pm$,

$$(6.24) \quad \mathbb{P}(\cap_{0 \leq k \leq l_1^* - 1} \mathbf{F}_k^+ \cap \mathbf{G}^+ \mid \mathcal{F}_{\hat{\mathcal{C}}_1^* \cup \hat{\mathcal{C}}_2^*}) \gtrsim \chi.$$

Note that $\hat{\mathbf{A}} := \cap_{0 \leq k \leq l_1^* - 1} \mathbf{F}_k^+ \cap \mathbf{G}^+ \cap \hat{\mathbf{A}}_*^\pm$ is measurable with respect to $\mathcal{F}_{\mathcal{C}_{x_1^*}^+ \cup \hat{\mathcal{C}}_2^*}$.

As in (6.21), for each $0 \leq k \leq l_2^* - 1$, we define the event

$$(6.25) \quad \begin{aligned} \mathbf{F}_k^- := & \{ -\sqrt{2C_\clubsuit} \leq \tilde{\phi}_v \leq -\sqrt{2C_\clubsuit^{-1}}, \forall v \in I_{\{\eta_2^*(k), \eta_2^*(k+1)\}} \} \\ & \cap \{ \eta_2^*(k) \xleftrightarrow{\tilde{E} \leq 0 \setminus I_{\{\eta_2^*(k), \eta_2^*(k+1)\}}} \tilde{\partial} \mathbf{B}_{\eta_2^*(k)}(1) \}^c. \end{aligned}$$

Similar to (6.18), we have: on the event $\hat{\mathbf{A}}$,

$$(6.26) \quad \mathbb{P}(\cap_{0 \leq k \leq l_2^* - 1} \mathbf{F}_k^- \mid \mathcal{F}_{\mathcal{C}_{x_1^*}^+ \cup \hat{\mathcal{C}}_2^*}) = \prod_{0 \leq k \leq l_2^* - 1} \mathbb{P}(\mathbf{F}_k^- \mid \mathcal{F}_{\mathcal{C}_{x_1^*}^+ \cup \hat{\mathcal{C}}_2^*}, \cap_{0 \leq l \leq k-1} \mathbf{F}_l^-) \gtrsim 1.$$

In fact, conditioned on $\cap_{0 \leq k \leq l_2^* - 1} \mathbf{F}_k^- \cap \hat{\mathbf{A}}$, since the distance between v' and the boundary with GFF values fixed is of order χ , the mean and variance of $\tilde{\phi}_{v'}$ are both $O(\chi)$. As a result, the event $\{\tilde{\phi}_{v'} \in [\chi^{\frac{1}{2}}, 2\chi^{\frac{1}{2}}]\}$ occurs with uniform positive probability. Moreover, conditioned on this event, the probability of $\mathbf{G}^- := \{y_2^* \xleftrightarrow{\leq 0} v'\} \cap \{v' \xleftrightarrow{\tilde{E} \leq 0 \setminus I_{[y_2^*, v']}} \tilde{\partial} \mathbf{B}_{v'}(1)\}^c$ is proportional to $\chi^{\frac{1}{2}}$ (using (3.29)). Combined with (6.24) and (6.26), it yields that on the event $\hat{\mathbf{A}}_*^\pm$,

$$(6.27) \quad \mathbb{P}(\cap_{0 \leq k \leq l_2^* - 1} \mathbf{F}_k^- \cap \mathbf{G}^- \cap (\cap_{0 \leq k \leq l_1^* - 1} \mathbf{F}_k^+) \cap \mathbf{G}^+ \mid \mathcal{F}_{\hat{\mathcal{C}}_1^* \cup \hat{\mathcal{C}}_2^*}) \gtrsim \chi^{\frac{3}{2}}.$$

This together with (6.19) implies that

$$(6.28) \quad \mathbb{P}(\cap_{0 \leq k \leq l_2^* - 1} \mathbf{F}_k^- \cap \mathbf{G}^- \cap \hat{\mathbf{A}}) \gtrsim \chi^{\frac{3}{2}} \cdot N^{-[(\frac{d}{2}+1)\Box 4]}.$$

Noting that the event on the left-hand side implies $\hat{\mathbf{H}}_{v'}^v(N)$ with $C_{19} = c_\star^{-1}$, $c_{14} = \sqrt{2C_\clubsuit^{-1}}$ and $C_{20} = \sqrt{2C_\clubsuit}$, we obtain the desired bound (6.5) for $\chi \leq 1$.

For the case $\chi > 1$, we select two paths $\tilde{\eta}_1$ and $\tilde{\eta}_2$ on $\tilde{\mathbb{Z}}^d$ such that $\tilde{\eta}_1$ connects x_1^* and v , $\tilde{\eta}_2$ connects x_2^* and v' , and $\text{dist}(\text{ran}(\tilde{\eta}_1), \text{ran}(\tilde{\eta}_2)) \geq 1$. We then define $\mathbf{H}_1$ (resp. $\mathbf{H}_2$) as the event that x_1^* and v (resp. x_2^* and v') are connected by a positive (resp. negative) cluster within $\cup_{z \in \text{ran}(\tilde{\eta}_1)} \mathbf{B}_z(\frac{1}{10})$ (resp. $\cup_{z \in \text{ran}(\tilde{\eta}_2)} \mathbf{B}_z(\frac{1}{10})$). Note that $\hat{\mathbf{A}}_*^\pm \cap \mathbf{H}_1 \cap \mathbf{H}_2 \subset \hat{\mathbf{H}}_{v'}^v(N)$. Moreover, following an argument similar to that for the case $\chi \leq 1$, we can also derive that given $\hat{\mathbf{A}}_*^\pm$ occurs, the conditional probability of $\mathbf{H}_1 \cap \mathbf{H}_2$ is of order 1. To sum up, we obtain (6.5) for $\chi > 1$. $\square$

Since $\hat{\mathbf{H}}_{v'}^v(N) \subset \mathbf{H}_{v'}^v(N)$, the proof of (6.1) is now complete. $\square$

Lemmas 5.2 and 5.4 imply that for any sufficiently large $N \geq 1$ and any $\psi = (v, v', w, w') \in \Psi(C, N)$,

$$(6.29) \quad \mathbb{P}(\mathbf{H}_{v', w'}^{v, w}) \gtrsim N^{-[(\frac{3d}{2}-1)\Box(2d-4)]}.$$

By employing the arguments in the proof of Lemma 6.1 (where we only need to replace the bound (6.6) with (6.29)), we derive the following strengthened version of the lower bound in (1.16).

Lemma 6.2. *For any $d \geq 3$ and $d \neq 6$, there exist $C_{21}, C_{22}, C_{23}, c_{15} > 0$ such that for any $v \neq v' \in \tilde{B}(C_{21})$, any sufficiently large $N \geq 1$, and $w, w' \in \tilde{B}(10N) \setminus \tilde{B}(N)$ with $|w - w'| \geq N$,*

$$(6.30) \quad \mathbb{P}(\hat{\mathbf{H}}_{v', w'}^{v, w}) \gtrsim |v - v'|^{\frac{3}{2}} N^{-[(\frac{3d}{2}-1)\Box(2d-4)]},$$

where $\hat{\mathbf{H}}_{v', w'}^{v, w}$ is defined as $\mathbf{H}_{v', w'}^{v, w} \cap \{v \xleftrightarrow{\geq 0} \partial B(C_{22}N)\}^c \cap \{v' \xleftrightarrow{\leq 0} \partial B(C_{22}N)\}^c \cap \{|\tilde{\phi}_v|, |\tilde{\phi}_{v'}| \in [c_{15}, C_{23}]\}$.

6.2. Proof of (6.2). We begin by verifying the lower bound in (6.2). By Lemmas 4.9, 5.2 and 5.4, we have: for any $\chi \geq C$, $N \geq C'\chi$ and $\psi \in \Psi(\chi, 2N)$,

$$(6.31) \quad \mathbb{P}(\mathbf{C}^{\text{out}}[\psi, c_{10}]) \gtrsim \chi^{(3-\frac{d}{2})\Box 0} N^{-[(\frac{d}{2}+1)\Box 4]}.$$

For any $\psi = (v_1, v_2, w_1, w_2) \in \Psi(\chi, 2N)$, since w_1, w_2 are contained in $[B(2N)]^c$, one has $B_{w_i}(c_{10}N) \subset [B(N)]^c$ for $i \in \{1, 2\}$. Hence, we have $\mathbf{C}^{\text{out}}[\psi, c_{10}] \subset \bar{\mathbf{H}}_{v'}^v(N)$. Combined with (6.31), it implies the lower bound in (6.2).

For the upper bound in (6.2), when $d \geq 7$, by the BKR inequality we have

$$(6.32) \quad \mathbb{P}(\bar{\mathbf{H}}_{v'}^v(N)) \lesssim [\theta_d(cN)]^2 \lesssim N^{-4}.$$

It remains to prove the upper bound for $3 \leq d \leq 5$. Using the decomposition in (6.3), it suffices to prove that for any $\psi = (v_1, v_2, w_1, w_2) \in \Psi(\chi, N)$,

$$(6.33) \quad \mathbb{P}(\mathbf{C}^{\text{out}}[\psi, c_{10}]) \lesssim \chi^{3-\frac{d}{2}} N^{-\frac{d}{2}-1}.$$

To achieve this, we arbitrarily take $\psi' := (x_1, x_2, y_1, y_2) \in \Psi(C_{10}, c_{\dagger}\chi)$, where $c_{\dagger}$ will be determined later. We take two paths $\tilde{\eta}_1$ and $\tilde{\eta}_2$ satisfying the following:

- $\tilde{\eta}_i$ starts from v_j and ends at y_j ;
- $\tilde{\eta}_i$ consists of at most $2d$ line segments, and is contained in $\tilde{B}(10\chi) \setminus \tilde{B}(\frac{c_{\dagger}\chi}{10})$;
- $\text{dist}(\text{ran}(\tilde{\eta}_1), \text{ran}(\tilde{\eta}_2)) \geq c_{\star}\chi$ for some constant $c_{\star} > 0$.

We denote $D_j^{\delta} := \cup_{v \in \text{ran}(\tilde{\eta}_j)} \tilde{B}_v(\delta\chi)$. It follows from Lemma 4.6 that for $j \in \{1, 2\}$, the conditional probability $\mathbb{P}(w_j \leftrightarrow D_{3-j}^{\delta} \mid v_j \leftrightarrow w_j)$ uniformly converges to zero as $\delta \rightarrow 0$. This together with Lemma 4.1 implies that

$$(6.34) \quad \mathbb{P}(\mathbf{D}_{\delta} \mid \mathbf{C}[\psi]) := \mathbb{P}(\cup_{j \in \{1, 2\}} \{w_j \leftrightarrow D_{3-j}^{\delta}\} \mid \mathbf{C}[\psi])$$

also uniformly converges to zero as $\delta \rightarrow 0$. Combined with Lemma 4.5, it yields that (recall the notation $\widehat{\mathbf{C}}[\cdot]$ before Lemma 4.5)

$$(6.35) \quad \mathbb{P}(\widehat{\mathbf{C}}[\psi, c_7] \cap (\mathbf{D}_{\delta_{\dagger}})^c) \asymp \mathbb{P}(\mathbf{C}[\psi])$$

for a sufficiently small constant $\delta_{\dagger} > 0$. Based on this estimate, we may employ the second moment method as in the proof of Lemma 4.9 to derive that for some $c_{\dagger}^{(1)}, c_{\dagger}^{(2)}, c_{\dagger}^{(3)} > 0$ depending on d and $\delta_{\dagger}$ (recall $\widehat{\mathbf{C}}^{\circ}[\cdot]$ before Lemma 4.9),

$$(6.36) \quad \mathbb{P}(\mathbf{A}) := \mathbb{P}(\widehat{\mathbf{C}}^{\text{in}}[\psi, c_{\dagger}^{(1)}, c_{\dagger}^{(2)}, c_{\dagger}^{(3)}] \cap (\mathbf{D}_{\delta_{\dagger}})^c) \asymp \mathbb{P}(\mathbf{C}[\psi]) \cdot \chi^{d-2}.$$

Specifically, similar to (4.101), we take a point $z \in \partial B_{v_1}(4c_{\dagger}^{(1)}\chi)$, define $\widetilde{S}^z \sim \widetilde{\mathbb{P}}_z$, and then consider the quantity

$$(6.37) \quad \mathbf{X} := \sum_{x \in B_{v_1}(c_{\dagger}^{(1)}\chi)} \mathbb{1}_{\{\widetilde{S}^z \text{ hits } x\} \cap \widehat{\mathbf{C}}^{\text{in}}[(x, v_2, w_1, w_2), c_{\dagger}^{(1)}, c_{\dagger}^{(2)}, c_{\dagger}^{(3)}] \cap (\mathbf{D}_{\delta_{\dagger}})^c}.$$

By (6.36), the first moment of $\mathbf{X}$ satisfies $\mathbb{E}[\mathbf{X}] \asymp \chi^2 \mathbb{P}(\mathbf{C}[\psi])$. Moreover, by (4.103), the second moment of $\mathbf{X}$ is at most $C\chi^{5-\frac{d}{2}} \cdot \mathbb{P}(\mathbf{C}[\psi])$. Thus, as in (4.106), applying the Paley-Zygmund inequality, one has $\mathbb{P}(\mathbf{X} > 0) \gtrsim \chi^{\frac{d}{2}-1} \mathbb{P}(\mathbf{C}[\psi])$. Next, following the same arguments as in (4.107)-(4.117), this yields

$$(6.38) \quad \mathbb{P}(\{\overline{\mathbf{G}}\|v_2 \leftrightarrow w_2\} \cap (\mathbf{D}_{\delta_{\dagger}})^c) \gtrsim \chi^{\frac{d}{2}-1} \cdot \mathbb{P}(\mathbf{C}[\psi]),$$

where $\overline{\mathbf{G}}$ is defined by $\{\mathcal{V}_{w_1, v_1}(c_{\dagger}^{(2)}\chi) \geq c_{\dagger}^{(3)}\chi^{\frac{d}{2}+1}\} \cap \{\mathcal{T}_{w_1, v_1}(c_{\dagger}^{(2)}\chi) \geq c_{\dagger}^{(3)}\chi^{d-2}\}$. We then repeat this second moment method for the quantity (here $\widetilde{S}^{z'} \sim \widetilde{\mathbb{P}}_{z'}$ is an independent Brownian motion with $z' \in \partial B_{v_2}(4c_{\dagger}^{(1)}\chi)$)

$$(6.39) \quad \mathbf{X}' := \sum_{x \in B_{v_2}(c_{\dagger}^{(1)}\chi)} \mathbb{1}_{\{\widetilde{S}^{z'} \text{ hits } x\} \cap \{\overline{\mathbf{G}}\|v_2 \leftrightarrow w_2\} \cap (\mathbf{D}_{\delta_{\dagger}})^c},$$

thereby deriving the claimed bound (6.36).

Similarly to (6.36), we also have (letting $\mathbf{D}'_{\delta} := \cup_{j \in \{1,2\}} \{x_j \leftrightarrow D_{3-j}^{\delta}\}$)

$$(6.40) \quad \mathbb{P}(\mathbf{A}') := \mathbb{P}(\widehat{\mathbf{C}}^{\text{out}}[\psi, c_{\dagger}^{(1)}, c_{\dagger}^{(2)}, c_{\dagger}^{(3)}] \cap (\mathbf{D}'_{\delta_{\dagger}})^c) \asymp \mathbb{P}(\mathbf{C}[\psi']) \cdot \chi^{d-2}.$$

We require that $c_{\dagger} < \frac{1}{10}(c_{\dagger}^{(2)})^2$. Since the event $\mathbf{A}$ depends only on the loops outside $\widetilde{B}(c_{\dagger}^{(2)}\chi)$, while $\mathbf{A}'$ depends only on the loops within $\widetilde{B}((c_{\dagger}^{(2)})^{-1}c_{\dagger}\chi)$, one has

$$(6.41) \quad \mathbb{P}(\mathbf{A} \cap \mathbf{A}') = \mathbb{P}(\mathbf{A}) \cdot \mathbb{P}(\mathbf{A}') \gtrsim \chi^{\frac{d}{2}-3} \cdot \mathbb{P}(\mathbf{C}[\psi]),$$

where in the inequality we used (6.36), (6.40) and Lemma 5.2.

Conditioned on $\{\mathcal{C}_{w_1} = A_1, \mathcal{C}_{w_2} = A_2, \mathcal{C}_{x_1} = A'_1, \mathcal{C}_{x_2} = A'_2\}$, where A_1, A_2, A'_1 and A'_2 are proper realizations of the corresponding clusters such that $\mathbf{A} \cap \mathbf{A}'$ occurs, the distribution of the remaining loops is given by $\widetilde{\mathcal{L}}_{1/2}^{\overline{A}}$ with $\overline{A} := A_1 \cup A_2 \cup A'_1 \cup A'_2$. For any $U \subset \widetilde{\mathbb{Z}}^d$ and $\epsilon > 0$, we define the event

$$(6.42) \quad \mathbf{F}_{\epsilon}^U := \cup_{j \in \{1,2\}} \{D_j^{\epsilon, \delta_{\dagger}} \xleftrightarrow{(U)} \widetilde{\partial} D_j^{\delta_{\dagger}}\}.$$

In addition, we define F_ϵ as F_ϵ^A when $\{\mathcal{C}_{w_1} = A_1, \mathcal{C}_{w_2} = A_2, \mathcal{C}_{x_1} = A'_1, \mathcal{C}_{x_2} = A'_2\}$ occurs. Since $\tilde{\mathcal{L}}_{1/2}^A \subset \tilde{\mathcal{L}}_{1/2}$, it follows from (1.7) and the FKG inequality that for a small constant $\epsilon_\dagger > 0$, the conditional probability of $(F_{\epsilon_\dagger})^c$ given $\mathcal{F}_{\mathcal{C}_{\{w_1, w_2, x_1, x_2\}}}$ is uniformly bounded away from zero. Thus, we have

$$(6.43) \quad \mathbb{P}(\mathbf{A} \cap \mathbf{A}' \cap (F_{\epsilon_\dagger})^c) \asymp \mathbb{P}(\mathbf{A} \cap \mathbf{A}').$$

Moreover, on the event $\mathbf{A} \cap \mathbf{A}' \cap (F_{\epsilon_\dagger})^c$, if one adds two loops $\tilde{\ell}_1$ and $\tilde{\ell}_2$ to $\tilde{\mathcal{L}}_{1/2}$ such that $\tilde{\ell}_j$ is contained in $D_j^{\epsilon_\dagger \cdot \delta_\dagger}$ and connects $\mathcal{C}_{w_j}$ and $\mathcal{C}_{x_j}$, then the following occur (see Figure 1 for an illustration):

- (i) For $j \in \{1, 2\}$, the new cluster of $\tilde{\mathcal{L}}_{1/2}$ containing x_j (denoted by $\bar{\mathcal{C}}_j$) consists of $\mathcal{C}_{w_j}$, $\mathcal{C}_{x_j}$, $\tilde{\ell}_j$ and loops of $\tilde{\mathcal{L}}_{1/2}$ contained in $D_j^{\delta_\dagger}$.
- (ii) The clusters $\bar{\mathcal{C}}_1$ and $\bar{\mathcal{C}}_2$ are disjoint.
- (iii) The cluster $\bar{\mathcal{C}}_j$ connects x_j and w_j , and has volume at least $c_\dagger^{(3)} n^{d-2}$ within $\tilde{B}_{x_j}(20\chi)$.

We denote the intersection of these events by $\mathbf{G}$. In particular, one has $\mathbf{G} \subset \mathbf{C}[(x_1, x_2, w_1, w_2)]$. Note that the total mass of loops $\tilde{\ell}_1$ and $\tilde{\ell}_2$ is of order 1 since $\text{cap}(\mathcal{C}_{w_j} \cap \tilde{B}_{w_j}(c_\dagger^{(1)} n)) \geq c_\dagger^{(3)} \chi^{d-2}$ and $\text{cap}(\mathcal{C}_{x_j} \cap \tilde{B}_{x_j}(c_\dagger^{(1)} \chi)) \geq c_\dagger^{(3)} (c_\dagger \chi)^{d-2}$. Thus,

$$(6.44) \quad \mathbb{P}(\mathbf{G}) \asymp \mathbb{P}(\mathbf{A} \cap \mathbf{A}' \cap (F_{\epsilon_\dagger})^c) \stackrel{(6.41), (6.43)}{\gtrsim} \chi^{\frac{d}{2}-3} \cdot \mathbb{P}(\mathbf{C}[\psi]).$$

Combined with $\mathbf{G} \subset \mathbf{C}[(x_1, x_2, w_1, w_2)]$ and Lemma 5.1, this implies (6.33), which in turn yields the upper bound in (6.2). In conclusion, we complete the proof of Theorem 1.1. $\square$

Remark 6.3 (extension with zero boundary conditions). Throughout the proof of Theorem 1.1, all relevant quantities—including the hitting probabilities of Brownian motion, Green's functions, two-point functions, one-arm probabilities, and crossing probabilities—will change by at most a constant factor when zero boundary conditions are imposed on either a distant region or a small neighborhood (see e.g., [5, Proposition 1.8, Lemmas 2.1 and 2.2]). Thus, applying the same argument yields analogous bounds in Theorem 1.1 for the heterochromatic two-arm probability under appropriate zero boundary conditions. Specifically, for any $d \geq 3$ with $d \neq 6$, there exist $C, C', c, c' > 0$ such that for any $N \geq C$ and $v, v' \in \tilde{B}(cN)$, the following hold:

- (1) When $|v - v'| \leq 1$, for any $D \subset [\tilde{B}(C'N)]^c$,
- $$(6.45) \quad \mathbb{P}^D(\mathbf{H}_{v'}^v(N)) \asymp |v - v'|^{\frac{3}{2}} N^{-[(\frac{d}{2}+1)\Box 4]},$$
- (2) When $|v - v'| \geq 1$, if there exists $n \in [1, c'N]$ such that $v, v' \in \tilde{B}(10n) \setminus \tilde{B}(n)$, then for any $D \subset \tilde{B}(\frac{n}{C'}) \cup [\tilde{B}(C'N)]^c$,
- $$(6.46) \quad \mathbb{P}^D(\mathbf{H}_{v'}^v(N)) \asymp |v - v'|^{(3-\frac{d}{2})\Box 0} N^{-[(\frac{d}{2}+1)\Box 4]}.$$

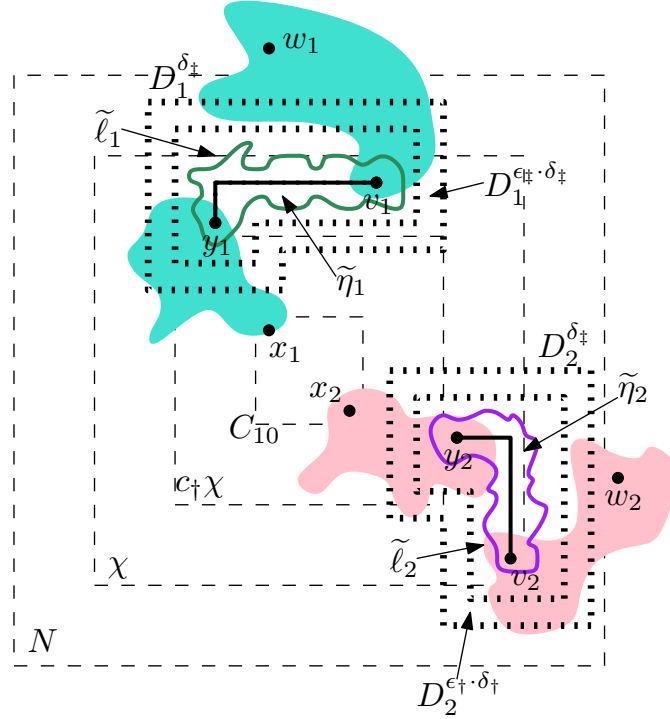


FIGURE 1. In this illustration, the cyan (resp. pink) regions represent the disjoint clusters $\mathcal{C}_{w_1}$ and $\mathcal{C}_{x_1}$ (resp. $\mathcal{C}_{w_2}$ and $\mathcal{C}_{x_2}$). In order to connect these two pairs of clusters separately, the loops $\tilde{\ell}_1$ and $\tilde{\ell}_2$ (i.e., the green and purple curves) are added to $\tilde{\mathcal{L}}_{1/2}$. For $j \in \{1, 2\}$, since $\tilde{\ell}_j$ is contained in $D_j^{\epsilon_{\dagger} \cdot \delta_{\dagger}}$, the absence of the event $\mathbf{F}_{\epsilon_{\dagger}}$ ensures that after adding $\tilde{\ell}_j$, the enlarged part of the new cluster $\bar{\mathcal{C}}_j$ containing x_j (i.e., the union of clusters intersecting $\tilde{\ell}_j$) is contained in $D_j^{\delta_{\dagger}}$ and hence, is disjoint from $\bar{\mathcal{C}}_{3-j}$. Thus, after this modification, the event $\mathbf{C}[(x_1, x_2, w_1, w_2)]$ occurs.

6.3. Proof of Theorem 1.4. Recall that (1.16) has been verified by Lemmas 5.1 and 6.2, and that the lower bound in (1.17) has been proved in Lemmas 5.2 and 5.4. Hence, it remains to consider the upper bound in (1.17). By the isomorphism theorem and Lemma 4.9, we have: for any $\psi = (v, v', w, w') \in \Psi(\chi, N)$,

$$\begin{aligned}
 \mathbb{P}(\mathbf{H}_{v', w'}^{v, w}) &\asymp \mathbb{P}(\mathbf{C}[\psi]) \asymp N^{-[(d-2)\Box(2d-8)]} \cdot \mathbb{P}(\mathbf{C}^{\text{out}}[\psi, c_{10}]) \\
 (6.47) \quad &\asymp N^{-[(d-2)\Box(2d-8)]} \cdot \mathbb{P}(\mathbf{H}_{v', B_{w_2}(c_{10}N)}^{v, B_{w_1}(c_{10}N)}).
 \end{aligned}$$

Meanwhile, by $\mathbf{H}_{v', B_{w_2}(c_{10}N)}^{v, B_{w_1}(c_{10}N)} \subset \mathbf{H}_{v'}^v(\frac{N}{10})$ and Theorem 1.1, one has

$$(6.48) \quad \mathbb{P}(\mathbf{H}_{v', B_{w_2}(c_{10}N)}^{v, B_{w_1}(c_{10}N)}) \leq \mathbb{P}(\mathbf{H}_{v'}^v(\frac{N}{10})) \lesssim \chi^{(3-\frac{d}{2})\Box 0} \cdot N^{-[(\frac{d}{2}+1)\Box 4]}.$$

Plugging (6.48) into (6.47), we get the upper bound in (1.17), and thus conclude the proof of Theorem 1.4. $\square$

As noted in Remark 6.3, imposing suitable zero boundary conditions also preserves the estimates in Theorem 1.4, which leads to the following result:

Lemma 6.4. *For any $d \geq 3$ with $d \neq 6$, there exist $C, C', c, c' > 0$ such that for any $N \geq C$, $v, v' \in \tilde{B}(cN)$ and $w, w' \in \tilde{B}(10N) \setminus \tilde{B}(N)$ with $|w - w'| \geq N$,*

(1) *When $|v - v'| \leq 1$, for any $D \subset [\tilde{B}(C'N)]^c$,*

$$(6.49) \quad \mathbb{P}^D(\mathbf{H}_{v',w'}^{v,w}) \asymp |v - v'|^{\frac{3}{2}} N^{-[(\frac{3d}{2}-1)\square(2d-4)]},$$

(2) *When $|v - v'| \geq 1$, if there exists $n \in [1, c'N]$ such that $v, v' \in \tilde{B}(10n) \setminus \tilde{B}(n)$, then for any $D \subset \tilde{B}(\frac{n}{C'}) \cup [\tilde{B}(C'N)]^c$,*

$$(6.50) \quad \mathbb{P}^D(\mathbf{H}_{v',w'}^{v,w}) \asymp |v - v'|^{(3-\frac{d}{2})\square 0} N^{-[(\frac{3d}{2}-1)\square(2d-4)]}.$$

7. CLUSTER VOLUMES UNDER HETEROCHROMATIC TWO-ARM EVENTS

In this section, we aim to establish Theorem 1.5, which states that conditioned on the two-arm events, the volume growth of each involved cluster is of the same order as that of an unconditioned cluster (as shown in (1.12)). The proof is divided into the following two parts.

Section 7.1: We prove through constructions that given the occurrence of two distinct large clusters, both clusters have sufficiently large volume with a uniform positive probability.

Section 7.2: We first establish an upper bound on the expected volume of the clusters involved in two-arm events. This bound, combined with Markov's inequality, yields the desired upper bound on their typical volumes.

7.1. Lower bound on the growth rate. Recall $\mathcal{V}_{z,z'}$ and $\mathcal{T}_{z,z'}$ in (4.92). For brevity, we denote $\mathcal{V}_z := \mathcal{V}_{z,z}$ and $\mathcal{T}_z := \mathcal{T}_{z,z}$. We aim to prove that for some $c > 0$,

$$(7.1) \quad \mathbb{P}(\mathcal{V}_v(M) \wedge \mathcal{V}_{v'}(M) \geq cM^{(\frac{d}{2}+1)\square 4} \mid \overline{\mathbf{H}}_{v'}^v(N)) \gtrsim 1.$$

In what follows, we first prove this inequality separately in the low- and high-dimensional cases, under the restriction $|v - v'| \geq C$. We then extend it to the case when $|v - v'| \leq C$.

When $3 \leq d \leq 5$. We employ χ , N , $\psi = (v_1, v_2, w_1, w_2)$, $\psi' := (x_1, x_2, y_1, y_2)$ and $\mathbf{G}$ as in Section 6.2, and we let $M = 20\chi$, $x_1 = v$ and $x_2 = v'$. Recall that the event $\mathbf{G}$ before (6.44), and note that $\mathbf{G}$ implies both $\{\mathcal{V}_v(M) \wedge \mathcal{V}_{v'}(M) \geq cM^{(\frac{d}{2}+1)\square 4}\}$ and $\overline{\mathbf{H}}_{v'}^v(N)$. As a result,

$$(7.2) \quad \begin{aligned} & \mathbb{P}(\mathcal{V}_v(M) \wedge \mathcal{V}_{v'}(M) \geq cM^{(\frac{d}{2}+1)\square 4}, \overline{\mathbf{H}}_{v'}^v(N)) \\ & \geq \mathbb{P}(\mathbf{G}) \stackrel{(6.44)}{\gtrsim} \chi^{\frac{d}{2}-3} \cdot \mathbb{P}(\mathbf{C}[\psi]) \stackrel{(\text{Lemma 5.2})}{\gtrsim} N^{-\frac{d}{2}-1}, \end{aligned}$$

which together with Theorem 1.1 implies (7.1) for $|v - v'| \geq C$.

When $d \geq 7$. Without loss of generality, we assume $v, v' \in \tilde{B}(\chi)$ with $|v - v'| = \chi$ for some $\chi \geq C$. We arbitrarily take $y_1, y_2 \in \partial B(2N)$ such that $|y_1 - y_2| \geq dN$. Note that $\text{dist}(\tilde{B}_{y_1}(d^{-1}N), \tilde{B}_{y_2}(d^{-1}N)) \geq N$. We consider the quantity

$$(7.3) \quad \mathbf{X} := \sum_{x_1 \in B_{y_1}(d^{-1}N), x_2 \in B_{y_2}(d^{-1}N)} \mathbb{1}_{\{v \leftrightarrow x_1 \leftrightarrow v' \leftrightarrow x_2\} \cap \{\mathcal{V}_v(M) \wedge \mathcal{V}_{v'}(M) \geq c_1 M^4\}}.$$

It follow from Lemma 5.4 that

$$(7.4) \quad \mathbb{E}[\mathbf{X}] \gtrsim |B(d^{-1}N)|^2 \cdot N^{4-2d} \asymp N^4.$$

For the second moment of $\mathbf{X}$, by the BKR inequality we have

$$\begin{aligned} \mathbb{E}[\mathbf{X}^2] &= \sum_{x_1, x'_1 \in B_{y_1}(d^{-1}N), x_2, x'_2 \in B_{y_2}(d^{-1}N)} \mathbb{P}(v \leftrightarrow x_1 \leftrightarrow x'_1 \parallel v' \leftrightarrow x_2 \leftrightarrow x'_2) \\ &\leq \left(\sum_{x_1, x'_1 \in B_{y_1}(d^{-1}N)} \mathbb{P}(v \leftrightarrow x_1 \leftrightarrow x'_1) \right) \cdot \left(\sum_{x_2, x'_2 \in B_{y_2}(d^{-1}N)} \mathbb{P}(v' \leftrightarrow x_2 \leftrightarrow x'_2) \right), \end{aligned}$$

which is bounded from above by CN^{12} according to Lemma 3.10. Therefore, by the Paley-Zygmund inequality, we obtain (7.1) for $|v - v'| \geq C$:

$$(7.5) \quad \mathbb{P}(\mathcal{V}_v(M) \wedge \mathcal{V}_{v'}(M) \geq cM^{(\frac{d}{2}+1)\Box 4}, \bar{\mathbf{H}}_v^v(N)) \geq \mathbb{P}(\mathbf{X} > 0) \gtrsim \frac{(\mathbb{E}[\mathbf{X}])^2}{\mathbb{E}[\mathbf{X}^2]} \gtrsim N^{-4}.$$

As for the case $|v - v'| \leq C$, recall the argument in Section 6.1, which was used to extend the lower bound in (6.1) from $|v - v'| \geq C$ to $|v - v'| \leq C$. Specifically, we apply a union bound to identify two distinct points $x_1^*, x_2^* \in \partial B(C')$ such that the parts of their sign clusters outside $\tilde{B}(C')$ reach $\partial B(N)$ with significant probability, and then explore $\tilde{\phi}$ inside $\tilde{B}(C')$ to connect these two partial clusters to v and v' respectively. By repeating the same argument, we can similarly extend (7.1) from $|v - v'| \geq C$ to $|v - v'| \leq C$ with an additional factor of $|v - v'|^{\frac{3}{2}}$, i.e.,

$$(7.6) \quad \mathbb{P}(\mathcal{V}_v(M) \wedge \mathcal{V}_{v'}(M) \geq cM^{(\frac{d}{2}+1)\Box 4}, \bar{\mathbf{H}}_{x_2}^{x_1}(N)) \gtrsim |v - v'|^{\frac{3}{2}} N^{-\frac{d}{2}-1}.$$

Combined with Theorem 1.1, it yields (7.1) for $|v - v'| \leq C$. $\square$

7.2. Upper bound on the growth rate. The goal of this subsection is to prove that for any $\lambda > 1$,

$$(7.7) \quad \mathbb{P}(\mathcal{V}_v(M) \vee \mathcal{V}_{v'}(M) \geq \lambda M^{(\frac{d}{2}+1)\Box 4} \mid \bar{\mathbf{H}}_{v'}^v(N)) \lesssim \lambda^{-1}.$$

To achieve this, it suffices to bound the conditional expectations of the volumes $\mathcal{V}_v(M)$ and $\mathcal{V}_{v'}(M)$ given $\bar{\mathbf{H}}_{v'}^v(N)$. In what follows, we carry out the proof separately for the low- and high-dimensional cases.

When $3 \leq d \leq 5$. We aim to prove that for any $x \in B_v(M)$,

$$(7.8) \quad \mathbb{P}(v \leftrightarrow x \mid \bar{\mathbf{H}}_{v'}^v(N)) \lesssim (|v - x| + 1)^{-\frac{d}{2}+1}.$$

It is sufficient to consider the case when $r := |v - x|$ is large. By the restriction property, we have

$$(7.9) \quad \begin{aligned} \mathbb{P}(v \leftrightarrow x, \bar{H}_{v'}^v(N)) &\leq \mathbb{E}[\mathbb{1}_{v \leftrightarrow x, v \leftrightarrow \partial B(N)} \cdot \mathbb{P}^{\mathcal{C}_v}(v' \leftrightarrow \partial B(\frac{N}{2}))] \\ &\stackrel{(\text{Lemma 3.6})}{\lesssim} N^{-\frac{d}{2}} \sum_{z_1 \in \partial \mathcal{B}(\frac{N}{2d})} \mathbb{P}(v \leftrightarrow x, v \leftrightarrow \partial B(N) \| v' \leftrightarrow z_1). \end{aligned}$$

Next, we bound the probability on the right-hand side in two cases, depending on whether r is comparable to $\chi := |v - v'|$.

When $r \geq c\chi$. It follows from (4.28) that $\mathbb{P}(v' \leftrightarrow B_x(\delta r) \mid v' \leftrightarrow z_1)$ uniformly converges to 0 as $\delta \rightarrow 0$. Combined with Lemma 4.5, it yields that for some $c_\dagger > 0$,

$$(7.10) \quad \begin{aligned} &\mathbb{P}(v \leftrightarrow x, v \leftrightarrow \partial B(N) \| v' \leftrightarrow z_1) \\ &\asymp \mathbb{P}(v \leftrightarrow x, v \leftrightarrow \partial B(N) \| v' \leftrightarrow z_1, \{v' \leftrightarrow B_x(c_\dagger r)\}^c). \end{aligned}$$

Moreover, by the restriction property, the right-hand side equals

$$(7.11) \quad \begin{aligned} &\mathbb{E}[\mathbb{1}_{v' \leftrightarrow z_1, \{v' \leftrightarrow B_x(c_\dagger r)\}^c} \cdot \mathbb{P}(v \xrightarrow{(C_{v'})} x, v \xrightarrow{(C_{v'})} \partial B(N))] \\ &\stackrel{(\text{Lemma 3.8})}{\lesssim} r^{-\frac{d}{2}+1} N^{-\frac{d}{2}} \sum_{z_2 \in \partial \mathcal{B}(d^{-1}N)} \mathbb{E}[\mathbb{1}_{v' \leftrightarrow z_1, \{v' \leftrightarrow B_x(c_\dagger r)\}^c} \cdot \mathbb{P}(v \xrightarrow{(C_{v'})} z_2)] \\ &\lesssim r^{-\frac{d}{2}+1} N^{-\frac{d}{2}} \sum_{z_2 \in \partial \mathcal{B}(d^{-1}N)} \mathbb{P}(v' \leftrightarrow z_1 \| v \leftrightarrow z_2) \\ &\stackrel{(\text{Lemma 5.1})}{\lesssim} f(\chi) \cdot r^{-\frac{d}{2}+1} N^{-d}, \end{aligned}$$

where $f(\chi)$ is defined by $f(\chi) = \chi^{3/2}$ for $\chi \leq 1$, and $f(\chi) = \chi^{3-\frac{d}{2}}$ for $\chi \geq 1$. Combined with (7.9) and (7.10), it implies that

$$(7.12) \quad \mathbb{P}(v \leftrightarrow x, \bar{H}_{v'}^v(N)) \lesssim r^{-\frac{d}{2}+1} \cdot f(\chi) N^{-\frac{d}{2}-1},$$

which together with Theorem 1.1 yields (7.8).

When $r \leq c\chi$. This case is specific to $\chi \gtrsim 1$. Similar to (7.10), we have

$$(7.13) \quad \begin{aligned} &\mathbb{P}(v \leftrightarrow x, v \leftrightarrow \partial B(N) \| v' \leftrightarrow z_1) \\ &\asymp \mathbb{P}(v \leftrightarrow x, v \leftrightarrow \partial B(N) \| v' \leftrightarrow z_1, \{v' \leftrightarrow B_v(c_\dagger \chi)\}^c). \end{aligned}$$

As in (7.11), the right-hand side can be written as

$$(7.14) \quad \begin{aligned} &\mathbb{E}[\mathbb{1}_{v' \leftrightarrow z_1, \{v' \leftrightarrow B_v(c_\dagger \chi)\}^c} \cdot \mathbb{P}(v \xrightarrow{(C_{v'})} x, v \xrightarrow{(C_{v'})} \partial B(N))] \\ &\stackrel{(\text{Lemma 3.7})}{\lesssim} r^{-\frac{d}{2}+1} \cdot \mathbb{E}[\mathbb{1}_{v' \leftrightarrow z_1, \{v' \leftrightarrow B_v(c_\dagger \chi)\}^c} \cdot \mathbb{P}(v \xrightarrow{(C_{v'})} \partial B(N))] \\ &\stackrel{(\text{Lemma 3.6})}{\lesssim} r^{-\frac{d}{2}+1} N^{-\frac{d}{2}} \sum_{z_2 \in \partial \mathcal{B}(d^{-1}N)} \mathbb{P}(v' \leftrightarrow z_1 \| v \leftrightarrow z_2) \stackrel{(\text{Lemma 5.1})}{\lesssim} r^{-\frac{d}{2}+1} N^{-d}. \end{aligned}$$

Similarly, this together with (7.9), (7.13) and Theorem 1.1 implies (7.8).

When $d \geq 7$. We claim that for any $x \in B_v(M)$,

$$(7.15) \quad \mathbb{P}(v \leftrightarrow x \mid \bar{\mathbf{H}}_{v'}^v(N)) \lesssim (|v - x| + 1)^{4-d}.$$

As in the low-dimensional case, it suffices to consider the case when $r := |v - x|$ is large. By the restriction property, $\mathbb{P}(v \leftrightarrow x, \bar{\mathbf{H}}_{v'}^v(N))$ equals

$$(7.16) \quad \begin{aligned} & \mathbb{E}[\mathbb{1}_{v \leftrightarrow x, v \leftrightarrow \partial B(N)} \cdot \mathbb{P}(v' \xleftrightarrow{\mathcal{C}_v} \partial B(N))] \\ & \stackrel{(\text{Lemma 3.6})}{\lesssim} N^{-3} \sum_{z \in \partial B(d^{-1}N)} \mathbb{E}[\mathbb{1}_{v \leftrightarrow x, v \leftrightarrow \partial B(N), \{v \leftrightarrow v'\}^c} \cdot \mathbb{P}(v' \xleftrightarrow{\mathcal{C}_v} z)]. \end{aligned}$$

Let $\hat{\chi} := \chi \wedge 1$. Since $v \in \mathcal{C}_v$, it follows from (3.6) and (3.28) that

$$(7.17) \quad \mathbb{P}(v' \xleftrightarrow{\mathcal{C}_v} z) \leq \mathbb{P}(v' \xleftrightarrow{(v)} z) \stackrel{(3.28)}{\lesssim} \frac{\tilde{G}_v(v', z)}{\sqrt{\tilde{G}_v(v', v') \tilde{G}_v(z, z)}} \stackrel{(3.6)}{\lesssim} \hat{\chi}^{-\frac{1}{2}} \cdot \tilde{\mathbb{P}}_{v'}(\tau_z < \tau_v).$$

Moreover, applying the formula (3.10), one has

$$(7.18) \quad \tilde{\mathbb{P}}_{v'}(\tau_z < \tau_v) \lesssim \hat{\chi} \cdot N^{2-d}.$$

Combining (7.16), (7.17) and (7.18), we get

$$(7.19) \quad \mathbb{P}(v \leftrightarrow x, \bar{\mathbf{H}}_{v'}^v(N)) \lesssim \hat{\chi}^{\frac{1}{2}} N^{-2} \cdot \mathbb{P}(v \leftrightarrow x, v \leftrightarrow \partial B(N), \{v \leftrightarrow v'\}^c).$$

Next, we estimate the probability of $\{v \leftrightarrow x, v \leftrightarrow \partial B(N)\} \cap \{v \leftrightarrow v'\}^c$ using the switching identity. Precisely, recalling $\hat{\mathbb{P}}(\cdot)$ from (3.83), we have

$$(7.20) \quad \begin{aligned} & \mathbb{P}(v \leftrightarrow x, v \leftrightarrow \partial B(N), \{v \leftrightarrow v'\}^c) \\ & = \int_{a, b > 0} \hat{\mathbb{P}}_{v \leftrightarrow x, a, b}(v \leftrightarrow \partial B(N), \{v \leftrightarrow v'\}^c) \mathbf{p}_{v \leftrightarrow w, a, b} da db \\ & \stackrel{(\text{FKG})}{\leq} \int_{a, b > 0} \hat{\mathbb{P}}_{v \leftrightarrow x, a, b}(v \leftrightarrow \partial B(N)) \hat{\mathbb{P}}_{v \leftrightarrow x, a, b}(\{v \leftrightarrow v'\}^c) \mathbf{p}_{v \leftrightarrow w, a, b} da db. \end{aligned}$$

Note that Lemma 3.15 implies

$$(7.21) \quad \hat{\mathbb{P}}_{v \leftrightarrow x, a, b}(\{v \leftrightarrow v'\}^c) \lesssim e^{-ca \cdot \hat{\chi}^{-1}}.$$

Recalling (3.84) and using Lemma 3.11, one has

$$(7.22) \quad \mathbf{p}_{v \leftrightarrow w, a, b} \stackrel{(3.84)}{\lesssim} e^{-c(a+b)} \cdot \mathbb{K}_{\{v, w\}}(v, w) \stackrel{(\text{Lemma 3.11})}{\lesssim} e^{-c(a+b)} \cdot r^{2-d}.$$

We now estimate $\hat{\mathbb{P}}_{v \leftrightarrow x, a, b}(v \leftrightarrow \partial B(N))$. Applying Lemma 3.18, we have

$$(7.23) \quad \begin{aligned} \hat{\mathbb{P}}_{v \leftrightarrow x, a, b}(v \leftrightarrow \partial B(N)) & = \check{\mathbb{P}}_{v \leftrightarrow x, a, b}(\check{\mathcal{C}} \cap \partial B(N) \neq \emptyset) \\ & \lesssim \sum_{i \in \{2, 3, 4\}} \check{\mathbb{P}}_{v \leftrightarrow x, a, b}(\cup \mathcal{P}^{(i)} \xleftrightarrow{\cup \mathcal{P}^{(1)}} \partial B(N)). \end{aligned}$$

For $i = 2$, it follows from Lemma 3.17 that

$$(7.24) \quad \check{\mathbb{P}}_{v \leftrightarrow x, a, b}((\cup \mathcal{P}^{(2)}) \cap \partial B(\frac{N}{2}) \neq \emptyset) \lesssim a N^{2-d}.$$

Meanwhile, using the union bound, we have

$$\begin{aligned}
& \check{\mathbb{P}}_{v \leftrightarrow x, a, b} \left((\cup \mathcal{P}^{(2)}) \cap \tilde{B}(\frac{N}{2}) \xleftrightarrow{\cup \mathcal{P}^{(1)}} \partial B(N) \right) \\
& \leq \sum_{y \in B(N/2)} \check{\mathbb{P}}_{v \leftrightarrow x, a, b} (y \in (\cup \mathcal{P}^{(2)})) \cdot \check{\mathbb{P}}_{v \leftrightarrow x, a, b} (\tilde{B}_y(1) \xleftrightarrow{\cup \mathcal{P}^{(1)}} \partial B(N)) \\
& \stackrel{\text{Lemma 3.16, (1.6)}}{\lesssim} aN^{-2} \sum_{y \in B(N/2)} (|x - y| + 1)^{4-2d} \stackrel{(4.78)}{\lesssim} aN^{-2}.
\end{aligned}$$

Combined with (7.24), it yields that

$$(7.25) \quad \check{\mathbb{P}}_{v \leftrightarrow x, a, b} (\cup \mathcal{P}^{(2)} \xleftrightarrow{\cup \mathcal{P}^{(1)}} \partial B(N)) \lesssim aN^{-2}.$$

By the same reasoning, we also have

$$(7.26) \quad \check{\mathbb{P}}_{v \leftrightarrow x, a, b} (\cup \mathcal{P}^{(3)} \xleftrightarrow{\cup \mathcal{P}^{(1)}} \partial B(N)) \lesssim bN^{-2}.$$

For $i = 4$, recall that the total mass of the excursions in $\mathcal{P}^{(4)}$ is of order $\sqrt{ab}r^{2-d}$. On $\{\cup \mathcal{P}^{(4)} \xleftrightarrow{\cup \mathcal{P}^{(1)}} \partial B(N)\}$, there exists an excursion $\tilde{\eta} \in \mathcal{P}^{(4)}$ such that $\text{ran}(\tilde{\eta}) \xleftrightarrow{\cup \mathcal{P}^{(1)}} \partial B(N)$ occurs. On the one hand, by the strong Markov property of Brownian excursion measure, the probability of $\{\text{ran}(\tilde{\eta}) \cap \partial B(N/2) \neq \emptyset\}$ is at most of order

$$(7.27) \quad \frac{\max_{w \in \partial B(N/2)} \tilde{\mathbb{P}}_w(\tau_x < \infty)}{\mathbb{K}_{\{v, x\}}(v, x)} \asymp \left(\frac{N}{r}\right)^{2-d}.$$

On the other hand, on $\{\text{ran}(\tilde{\eta}) \cap \tilde{B}(\frac{N}{2}) \xleftrightarrow{\cup \mathcal{P}^{(1)}} \partial B(N)\}$, there exists $y \in B(\frac{N}{2})$ such that $y \in \text{ran}(\tilde{\eta})$ and $\tilde{B}_y(1) \xleftrightarrow{\cup \mathcal{P}^{(1)}} \partial B(N)$ both occur, whose probabilities are upper-bounded by $Cr^{d-2}(|v - y| + 1)^{2-d}(|y - x| + 1)^{2-d}$ and CN^{-2} respectively (by Lemma 3.16 and (1.6)). To sum up, from the bound in (4.61), we have

$$\begin{aligned}
& \check{\mathbb{P}}_{v \leftrightarrow x, a, b} (\cup \mathcal{P}^{(4)} \xleftrightarrow{\cup \mathcal{P}^{(1)}} \partial B(N) \neq \emptyset) \\
& \lesssim [(\sqrt{ab} \cdot r^{2-d}) \vee 1] \cdot \left[\left(\frac{N}{r}\right)^{2-d} + r^{d-2}N^{-2} \right] \\
& \quad \cdot \sum_{y \in B(N/2)} (|v - y| + 1)^{2-d}(|y - x| + 1)^{2-d} \\
& \stackrel{(4.66)}{\lesssim} (\sqrt{ab} + r^{d-2}) \cdot (N^{2-d} + r^{4-d}N^{-2}) \lesssim (\sqrt{ab} + r^{d-2}) \cdot r^{4-d}N^{-2}.
\end{aligned}$$

Plugging (7.25), (7.26) and (7.28) into (7.23), we obtain

$$(7.29) \quad \hat{\mathbb{P}}_{v \leftrightarrow x, a, b} (v \leftrightarrow \partial B(N)) \lesssim N^{-2} \cdot (a + b + \sqrt{ab} \cdot r^{4-d} + r^2).$$

Substituting (7.21), (7.22) and (7.29) into (7.20) yields that

$$\begin{aligned}
(7.30) \quad & \mathbb{P}(v \leftrightarrow x, v \leftrightarrow \partial B(N), \{v \leftrightarrow v'\}^c) \\
& \lesssim r^{2-d} N^{-2} \int_{a,b>0} e^{-c(a\hat{\chi}^{-1}+b)} \cdot (a+b+\sqrt{ab} \cdot r^{4-d}+r^2) da db \\
& \lesssim r^{2-d} N^{-2} (\hat{\chi}^2 + \hat{\chi} + \hat{\chi}^{\frac{3}{2}} \cdot r^{4-d} + \hat{\chi} \cdot r^2) \lesssim \hat{\chi} r^{4-d} N^{-2}.
\end{aligned}$$

This together with (7.19) and Theorem 1.1 implies (7.15):

$$(7.31) \quad \mathbb{P}(v \leftrightarrow x \mid \bar{\mathbf{H}}_{v'}^v(N)) = \frac{\mathbb{P}(v \leftrightarrow x, \bar{\mathbf{H}}_{v'}^v(N))}{\mathbb{P}(\bar{\mathbf{H}}_{v'}^v(N))} \lesssim \frac{\hat{\chi}^{\frac{1}{2}} N^{-2} \cdot \hat{\chi} r^{4-d} N^{-2}}{\hat{\chi}^{\frac{3}{2}} N^4} = r^{4-d}.$$

Using (7.8) and (7.15), we have the following bound on the expected volume:

$$\begin{aligned}
(7.32) \quad \mathbb{E}[\mathcal{V}_v(M) \mid \bar{\mathbf{H}}_{v'}^v(N)] &= \sum_{x \in B_v(M)} \mathbb{P}(v \leftrightarrow x \mid \bar{\mathbf{H}}_{v'}^v(N)) \\
&\stackrel{(7.8)}{\lesssim} \sum_{x \in B_v(M)} (|v-x|+1)^{-[(\frac{d}{2}-1)\Box(d-4)]} \stackrel{(4.78)}{\lesssim} M^{(\frac{d}{2}+1)\Box 4}.
\end{aligned}$$

Applying Markov's inequality, and using (7.32), we have

$$(7.33) \quad \mathbb{P}(\mathcal{V}_v(M) \geq \lambda M^{(\frac{d}{2}+1)\Box 4} \mid \bar{\mathbf{H}}_{v'}^v(N)) \lesssim \lambda^{-1}.$$

Similarly, its analogue for $\mathcal{V}_{v'}(M)$ also holds. In conclusion, we obtain (7.7). $\square$

With (7.1) and (7.7) in hand, the proof of Theorem 1.5 is straightforward.

Proof of Theorem 1.5. By (7.1) and (7.7), there exists $C > 0$ such that

$$(7.34) \quad \mathbb{P}(\mathcal{V}_v(M), \mathcal{V}_{v'}(M) \in [C^{-1} M^{(\frac{d}{2}+1)\Box 4}, C M^{(\frac{d}{2}+1)\Box 4}] \mid \bar{\mathbf{H}}_{v'}^v(N)) \gtrsim 1.$$

This together with the isomorphism theorem implies the desired bound (1.18). $\square$

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APPENDIX A. PROOF OF LEMMA 3.9

We prove (3.51) separately for the low- and high-dimensional cases. For convenience, we set $0^{-a} := 1$ for all $a > 0$. Recall the basic computations presented in (4.66), (4.78) and (5.74), as we will refer to them frequently. We will also use the following estimate multiple times:

$$(A.1) \quad \sum_{x \in \mathbb{Z}^d} |v-x|^{6-2d} |x-w|^{2-d} \lesssim |v-w|^{2-d}.$$

When $3 \leq d \leq 5$. By the strong Markov probability of $\tilde{S}^z$ and (3.26), one has

$$\begin{aligned}
\tilde{\mathbb{P}}_z(\tau_{x_1} < \infty, \tau_{x_2} < \infty) &= \tilde{\mathbb{P}}_z(\tau_{x_1} \leq \tau_{x_2} < \infty) + \tilde{\mathbb{P}}_z(\tau_{x_2} \leq \tau_{x_1} < \infty) \\
(A.2) \quad &\lesssim |z - x_1|^{2-d} |x_1 - x_2|^{2-d} + |z - x_2|^{2-d} |x_2 - x_1|^{2-d} \\
&\lesssim M^{2-d} |x_1 - x_2|^{2-d}.
\end{aligned}$$

Meanwhile, it follows from Lemma 3.7 that

$$(A.3) \quad \mathbb{P}(\mathbf{0} \xleftrightarrow{\geq 0} x_1, \mathbf{0} \xleftrightarrow{\geq 0} x_2) \lesssim N^{2-d} |x_1 - x_2|^{-\frac{d}{2}+1}.$$

Combining (A.2) and (A.3), we obtain (3.51):

$$\begin{aligned}
(A.4) \quad &\sum_{x_1, x_2 \in B_y(cN)} \tilde{\mathbb{P}}_z(\tau_{x_1} < \infty, \tau_{x_2} < \infty) \cdot \mathbb{P}(\mathbf{0} \xleftrightarrow{\geq 0} x_1, \mathbf{0} \xleftrightarrow{\geq 0} x_2) \\
&\lesssim M^{2-d} N^{2-d} \sum_{x_1, x_2 \in B(2N)} |x_1 - x_2|^{-\frac{3d}{2}+3} \stackrel{(4.78)}{\lesssim} M^{5-\frac{d}{2}} N^{2-d}.
\end{aligned}$$

When $d \geq 7$. By the isomorphism theorem, (3.51) is equivalent to

$$(A.5) \quad \sum_{x_1, x_2 \in B_y(cN)} \tilde{\mathbb{P}}_z(\tau_{x_1} < \infty, \tau_{x_2} < \infty) \cdot \mathbb{P}(\mathbf{0} \leftrightarrow x_1, \mathbf{0} \leftrightarrow x_2) \lesssim N^{2-d} M^2.$$

For two events $\mathbf{A}_1$ and $\mathbf{A}_2$ measurable with respect to $\tilde{\mathcal{L}}_{1/2}$, we say that they happen disjointly (denoted by $\mathbf{A}_1 \circ \mathbf{A}_2$) if there exist two disjoint collections of loops in $\tilde{\mathcal{L}}_{1/2}$ that certify $\mathbf{A}_1$ and $\mathbf{A}_2$ respectively. Readers may refer to [3, Section 3.3] for more details. For each $i \in \{1, 2\}$, applying the BKR inequality we have

$$\begin{aligned}
(A.6) \quad &\mathbb{P}(\mathbf{G}_i) := \mathbb{P}(\{\mathbf{0} \leftrightarrow x_i\} \circ \{x_{3-i} \leftrightarrow \partial B_{x_{3-i}}(\frac{M}{2})\}) \\
&\leq \mathbb{P}(\mathbf{0} \leftrightarrow x_i) \cdot \mathbb{P}(x_{3-i} \leftrightarrow \partial B_{x_{3-i}}(\frac{M}{2})) \lesssim N^{2-d} M^{-2}.
\end{aligned}$$

Combined with (A.2), it implies that

$$\begin{aligned}
(A.7) \quad &\sum_{x_1, x_2 \in B_y(M)} \tilde{\mathbb{P}}_z(\tau_{x_1} < \infty, \tau_{x_2} < \infty) \cdot \mathbb{P}(\mathbf{G}_1 \cup \mathbf{G}_2) \\
&\lesssim N^{2-d} M^{-d} \sum_{x_1, x_2 \in B_y(M)} |x_1 - x_2|^{2-d} \stackrel{(4.78)}{\lesssim} N^{2-d} M^2.
\end{aligned}$$

Let $\mathbf{A}_{x_1, x_2} := \{\mathbf{0} \leftrightarrow x_1, \mathbf{0} \leftrightarrow x_2\} \cap \mathbf{G}_1^c \cap \mathbf{G}_2^c$. Therefore, it suffices to show that

$$(A.8) \quad \sum_{x_1, x_2 \in B_y(M)} \tilde{\mathbb{P}}_z(\tau_{x_1} < \infty, \tau_{x_2} < \infty) \cdot \mathbb{P}(\mathbf{A}_{x_1, x_2}) \lesssim N^{2-d} M^2.$$

The tree expansion argument (see [3, Section 3.4]) implies that on the event $\{\mathbf{0} \leftrightarrow x_1, \mathbf{0} \leftrightarrow x_2\}$, there exists a loop $\tilde{\ell}_* \in \tilde{\mathcal{L}}_{1/2}$ and three clusters $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{C}_3$, each consisting of disjoint collections of loops in $\tilde{\mathcal{L}}_{1/2} - \mathbb{1}_{\tilde{\ell}_*}$, such that the cluster $\mathcal{C}_1, \mathcal{C}_2$ and $\mathcal{C}_3$ connect $\text{ran}(\tilde{\ell}_*)$ to x_1, x_2 and $\mathbf{0}$ respectively. Moreover, if we further require that $\mathbf{G}_i$ does not occur, then $\mathcal{C}_{3-i} \subset \tilde{B}_{x_{3-i}}(M)$; otherwise, the sets $\text{ran}(\tilde{\ell}_*) \cup \mathcal{C}_i \cup \mathcal{C}_3$ and $\mathcal{C}_{3-i}$ certify $\mathbf{0} \leftrightarrow x_i$ and $x_{3-i} \leftrightarrow \partial B_{x_{3-i}}(\frac{M}{2})$ respectively, and thus $\mathbf{G}_i$ occurs. As a result, on $\mathbf{A}_{x_1, x_2}$, there exist $w_1 \in B_{x_1}(M)$, $w_2 \in B_{x_2}(M)$ and $w_3 \in \mathbb{Z}^d$ such

that $\tilde{\ell}_*$ intersects $\tilde{B}_{w_j}(1)$ for $j \in \{1, 2, 3\}$, and that the events $w_1 \leftrightarrow x_1$, $w_2 \leftrightarrow x_2$ and $w_3 \leftrightarrow \mathbf{0}$ happen disjointly and without using $\tilde{\ell}_*$. Recall that the total mass of loops intersecting $\tilde{B}_{w_j}(1)$ for $j \in \{1, 2, 3\}$ is at most $C|w_1 - w_2|^{2-d}|w_2 - w_3|^{2-d}|w_3 - w_1|^{2-d}$ (see e.g., [3, Lemma 2.3]). In conclusion, we have

$$(A.9) \quad \mathbb{P}(\mathbf{A}_{x_1, x_2}) \lesssim \mathbb{I}_{x_1, x_2} := \sum_{w_1 \in B_{x_1}(M), w_2 \in B_{x_2}(M), w_3 \in \mathbb{Z}^d} |w_1 - w_2|^{2-d} |w_2 - w_3|^{2-d} \cdot |w_3 - w_1|^{2-d} |w_1 - x_1|^{2-d} |w_2 - x_2|^{2-d} |w_3|^{2-d}.$$

We also denote by $\mathbb{I}_{x_1, x_2}^{(1)}$ the partial sum of $\mathbb{I}_{x_1, x_2}$ with w_3 restricted to $B(\frac{N}{10})$. Let $\mathbb{I}_{x_1, x_2}^{(2)} := \mathbb{I}_{x_1, x_2} - \mathbb{I}_{x_1, x_2}^{(1)}$. By (A.2) and (A.9), the left-hand side of (A.8) is at most

$$(A.10) \quad CM^{2-d} \sum_{x_1, x_2 \in B_y(M)} |x_1 - x_2|^{2-d} (\mathbb{I}_{x_1, x_2}^{(1)} + \mathbb{I}_{x_1, x_2}^{(2)}).$$

For each $i \in \{1, 2\}$, when $w_3 \in B(\frac{N}{10})$ and $w_i \in B_{x_i}(M)$, we have $|w_i - w_3| \geq |x_i - w_3| - |w_i - x_i| \gtrsim N$. Therefore, $\sum_{x_1, x_2 \in B_y(M)} |x_1 - x_2|^{2-d} \cdot \mathbb{I}_{x_1, x_2}^{(1)}$ is at most

$$(A.11) \quad \begin{aligned} & CN^{4-2d} \sum_{x_1, x_2 \in B_y(M)} \sum_{w_1 \in B_{x_1}(M), w_2 \in B_{x_2}(M), w_3 \in B(\frac{N}{10})} |x_1 - x_2|^{2-d} \\ & \quad \cdot |w_1 - w_2|^{2-d} |w_1 - x_1|^{2-d} |w_2 - x_2|^{2-d} |w_3|^{2-d} \\ & \stackrel{(4.78)}{\lesssim} N^{6-2d} \sum_{x_1, x_2 \in B_y(M)} \sum_{w_1 \in B_{x_1}(M), w_2 \in B_{x_2}(M)} |x_1 - x_2|^{2-d} |w_1 - w_2|^{2-d} \\ & \quad \cdot |w_1 - x_1|^{2-d} |w_2 - x_2|^{2-d} \\ & \stackrel{(4.66), (5.74)}{\lesssim} N^{6-2d} \sum_{x_1, x_2 \in B_y(M)} |x_1 - x_2|^{8-2d} \\ & \stackrel{(|x_1 - x_2| \lesssim M)}{\lesssim} N^{6-2d} M^2 \sum_{x_1, x_2 \in B_y(M)} |x_1 - x_2|^{6-2d} \stackrel{(4.78)}{\lesssim} N^{6-2d} M^4, \end{aligned}$$

where in the second to last line we first summed over w_1 , then over w_2 .

When $w_3 \in [B(\frac{N}{10})]^c$, $\sum_{x_1, x_2 \in B_y(M)} |x_1 - x_2|^{2-d} \cdot \mathbb{I}_{x_1, x_2}^{(2)}$ is bounded from above by

$$\begin{aligned}
& CN^{2-d} \sum_{x_1, x_2 \in B_y(M)} \sum_{w_1 \in B_{x_1}(M), w_2 \in B_{x_2}(M), w_3 \in \mathbb{Z}^d} |x_1 - x_2|^{2-d} \\
& \cdot |w_1 - w_2|^{2-d} |w_2 - w_3|^{2-d} |w_3 - w_1|^{2-d} |w_1 - x_1|^{2-d} |w_2 - x_2|^{2-d} \\
(A.12) \quad & \stackrel{(4.66), (A.1)}{\lesssim} N^{2-d} \sum_{x_1, x_2 \in B_y(M)} \sum_{w_2 \in B_{x_2}(M)} |x_1 - x_2|^{2-d} |x_1 - w_2|^{2-d} |w_2 - x_2|^{2-d} \\
& \stackrel{(4.66)}{\lesssim} N^{2-d} \sum_{x_1, x_2 \in B_y(M)} |x_1 - x_2|^{6-2d} \stackrel{(4.78)}{\lesssim} N^{2-d} M^d.
\end{aligned}$$

Here in the second to last line we first summed over w_3 , then over w_1 .

By (A.10), (A.11) and (A.12), we obtain (A.8) as follows:

$$\begin{aligned}
(A.13) \quad & \sum_{x_1, x_2 \in B_y(M)} \tilde{\mathbb{P}}_z(\tau_{x_1} < \infty, \tau_{x_2} < \infty) \cdot \mathbb{P}(\mathbf{A}_{x_1, x_2}) \\
& \lesssim M^{2-d} \cdot (N^{6-2d} M^4 + N^{2-d} M^d) \lesssim N^{2-d} M^2,
\end{aligned}$$

which then completes the proof of the lemma. $\square$

APPENDIX B. PROOF OF LEMMA 3.10

Applying the tree expansion argument as in the proof of (A.9), we have

$$\begin{aligned}
& \sum_{x_2 \in B_y(M)} \mathbb{P}(\mathbf{0} \leftrightarrow x_1, \mathbf{0} \leftrightarrow x_2) \\
(B.1) \quad & \lesssim \mathbb{J} := \sum_{x_2 \in B_y(M)} \sum_{w_1, w_2, w_3 \in \mathbb{Z}^d} |w_1 - w_2|^{2-d} |w_2 - w_3|^{2-d} \\
& \cdot |w_3 - w_1|^{2-d} |w_1 - x_1|^{2-d} |w_2 - x_2|^{2-d} |w_3|^{2-d}.
\end{aligned}$$

For $i \in \{1, 2\}$, we denote by $\mathbb{J}_i$ the partial sum of $\mathbb{J}$ restricted to $w_i \in [B_{x_i}(\frac{N}{10})]^c$. In addition, let $\mathbb{J}_3$ denote the partial sum of $\mathbb{J}$ restricted to $w_3 \in [B(\frac{N}{10})]^c$, and let $\mathbb{J}_4 := \mathbb{J} - \sum_{i \in \{1, 2, 3\}} \mathbb{J}_i$.

For $i \in \{1, 2\}$, since $|w_i - x_i| \gtrsim N$, we have

$$\begin{aligned}
& \mathbb{J}_i \lesssim N^{2-d} \sum_{x_2 \in B_y(M)} \sum_{w_1, w_2, w_3 \in \mathbb{Z}^d} |w_1 - w_2|^{2-d} |w_2 - w_3|^{2-d} \\
& \cdot |w_3 - w_1|^{2-d} |w_{3-i} - x_{3-i}|^{2-d} |w_3|^{2-d} \\
& \stackrel{(4.66), (A.1)}{\lesssim} N^{2-d} \sum_{x_2 \in B_y(M)} |x_{3-i}|^{4-d} \stackrel{(|x_{3-i}| \gtrsim N)}{\lesssim} N^{6-2d} M^d,
\end{aligned}$$

where in the second inequality we first summed over w_i , then over w_{3-i} , and finally over w_3 .

For $\mathbb{J}_3$, since $|w_3| \gtrsim N$, one has

$$\begin{aligned} \mathbb{J}_3 &\lesssim N^{2-d} \sum_{x_2 \in B_y(M)} \sum_{w_1, w_2, w_3 \in \mathbb{Z}^d} |w_1 - w_2|^{2-d} |w_2 - w_3|^{2-d} \\ &\quad \cdot |w_3 - w_1|^{2-d} |w_1 - x_1|^{2-d} |w_2 - x_2|^{2-d} \\ &\stackrel{(4.66), (A.1)}{\lesssim} N^{2-d} \sum_{x_2 \in B_y(M)} |x_1 - x_2|^{4-d} \stackrel{(4.78)}{\lesssim} N^{2-d} M^4. \end{aligned}$$

Here in the second inequality we first summed over w_3 , and then over w_1 and w_2 .

Now we estimate $\mathbb{J}_4$. For each $i \in \{1, 2\}$, note that $w_i \in B_{x_i}(\frac{N}{10})$ and $w_3 \in B(\frac{N}{10})$ imply $|w_i - w_3| \gtrsim N$. Therefore, we have

$$\begin{aligned} \mathbb{J}_4 &\lesssim N^{4-2d} \sum_{x_2 \in B_y(M)} \sum_{w_1, w_2, w_3 \in \mathbb{Z}^d} |w_1 - w_2|^{2-d} |w_1 - x_1|^{2-d} |w_2 - x_2|^{2-d} |w_3|^{2-d} \\ &\stackrel{(4.78)}{\lesssim} N^{6-2d} \sum_{x_2 \in B_y(M)} \sum_{w_1, w_2 \in \mathbb{Z}^d} |w_1 - w_2|^{2-d} |w_1 - x_1|^{2-d} |w_2 - x_2|^{2-d} \\ &\stackrel{(4.66), (5.74)}{\lesssim} N^{6-2d} \sum_{x_2 \in B_y(M)} |x_1 - x_2|^{6-d} \stackrel{(4.78)}{\lesssim} N^{6-2d} M^6. \end{aligned}$$

Plugging these estimates for $\{\mathbb{J}_i\}_{1 \leq i \leq 4}$ into (B.1), we obtain the lemma:

$$\begin{aligned} &\sum_{x_2 \in B_y(M)} \mathbb{P}(\mathbf{0} \leftrightarrow x_1, \mathbf{0} \leftrightarrow x_2) \\ &\lesssim N^{6-2d} M^d + N^{2-d} M^4 + N^{6-2d} M^6 \lesssim N^{2-d} M^4. \end{aligned} \quad \square$$

APPENDIX C. PROOF OF LEMMA 3.11

We follow the proof strategy of [2, Theorem 1.2] for $d \geq 7$ (see [2, Section 5.1]). Recall that $y \in \partial B(N)$. For any $m \in [1, M]$, we consider the partial cluster

$$(C.1) \quad \hat{\mathcal{C}}_m^+ := \{v \in \tilde{B}_y(m) : v \xleftrightarrow{\tilde{E}^{\geq 0} \cap \tilde{B}_y(m)} y\},$$

and we define the harmonic average $\hat{\mathcal{H}}_{m,v}^+ := \mathcal{H}_v(\hat{\mathcal{C}}_m^+)$ for $v \in \tilde{\mathbb{Z}}^d$ (recall $\mathcal{H}_v(\cdot)$ below (3.34)). According to [2, Lemma 2.1], one has: for any $w \in [B_y(2dm)]^c$,

$$(C.2) \quad \begin{aligned} \mathbb{P}(z \xleftrightarrow{\geq 0} y \mid \mathcal{F}_{\hat{\mathcal{C}}_m^+}) &\asymp \left(\frac{m}{|w-y|}\right)^{2-d} \cdot |\partial \mathcal{B}_y(dm)|^{-1} \sum_{z \in \partial \mathcal{B}_y(dm)} \hat{\mathcal{H}}_{m,z}^+ \\ &\stackrel{(3.26)}{\asymp} \left(\frac{N}{|w-y|}\right)^{d-2} \cdot \hat{\mathcal{H}}_{m,\mathbf{0}}^+. \end{aligned}$$

The following lemma, analogous to [2, Lemma 5.3], shows that conditioning on $\{\mathbf{0} \xleftrightarrow{\geq 0} y\}$, the aforementioned harmonic average is typically of order $M^2 N^{2-d}$.

Lemma C.1. *For any $d \geq 7$ and $\lambda > 2$, there exist $C_{24}(d, \lambda), C_{25}(d) > 0$ such that for any $m \geq C_{24}$, $N \geq 10dm$ and $y \in \partial B(N)$,*

$$(C.3) \quad \mathbb{P}(\hat{\mathcal{H}}_{m,0}^+ \in [\lambda^{-1}m^2N^{2-d}, \lambda m^2N^{2-d}] \mid \mathbf{0} \xleftrightarrow{\geq 0} y) \leq C_{25}\lambda^{-1}.$$

Proof. On the one hand, it follows from (C.2) that

$$(C.4) \quad \begin{aligned} & \mathbb{P}(0 < \hat{\mathcal{H}}_{m,0}^+ \leq \lambda^{-1}m^2N^{2-d}, \mathbf{0} \xleftrightarrow{\geq 0} y) \\ &= \mathbb{E}[\mathbb{1}_{0 < \hat{\mathcal{H}}_{m,0}^+ \leq \lambda^{-1}m^2N^{2-d}} \cdot \mathbb{P}(\mathbf{0} \xleftrightarrow{\geq 0} y \mid \mathcal{F}_{\hat{C}_m^+})] \\ &\lesssim \lambda^{-1}m^2N^{2-d} \cdot \mathbb{P}(0 < \hat{\mathcal{H}}_{m,0}^+ \leq \lambda^{-1}m^2N^{2-d}) \lesssim \lambda^{-1}N^{2-d}, \end{aligned}$$

Here in the last inequality we used the fact that $\{\hat{\mathcal{H}}_{m,0}^+ > 0\} = \{y \xleftrightarrow{\geq 0} \partial B_y(m)\}$, whose probability is of order m^{-2} (by (1.6)).

On the other hand, by Lemma 3.10 one has

$$(C.5) \quad \mathbb{I} := \sum_{x \in B_y(40dm) \setminus B_y(10dm)} \mathbb{P}(\mathbf{0} \xleftrightarrow{\geq 0} x, \mathbf{0} \xleftrightarrow{\geq 0} y) \lesssim N^{2-d}m^4.$$

Meanwhile, using the FKG inequality and (C.2), we have

$$(C.6) \quad \begin{aligned} \mathbb{I} &\geq \sum_{x \in B_y(40dm) \setminus B_y(10dm)} \mathbb{E}[\mathbb{1}_{\hat{\mathcal{H}}_{m,0}^+ \geq \lambda m^2N^{2-d}} \mathbb{P}(\mathbf{0} \xleftrightarrow{\geq 0} x, \mathbf{0} \xleftrightarrow{\geq 0} y \mid \mathcal{F}_{\hat{C}_m^+})] \\ &\geq \sum_{x \in B_y(40dm) \setminus B_y(10dm)} \mathbb{E}[\mathbb{1}_{\hat{\mathcal{H}}_{m,0}^+ \geq \lambda m^2N^{2-d}} \mathbb{P}(\mathbf{0} \xleftrightarrow{\geq 0} x \mid \mathcal{F}_{\hat{C}_m^+}) \mathbb{P}(\mathbf{0} \xleftrightarrow{\geq 0} y \mid \mathcal{F}_{\hat{C}_m^+})] \\ &\gtrsim \sum_{x \in B_y(40dm) \setminus B_y(10dm)} \lambda m^{4-d} \cdot \mathbb{P}(\hat{\mathcal{H}}_{m,0}^+ \geq \lambda m^2N^{2-d}, \mathbf{0} \xleftrightarrow{\geq 0} y) \\ &\asymp \lambda m^4 \cdot \mathbb{P}(\hat{\mathcal{H}}_{m,0}^+ \geq \lambda m^2N^{2-d}, \mathbf{0} \xleftrightarrow{\geq 0} y). \end{aligned}$$

Combined with (C.5), it implies that

$$(C.7) \quad \mathbb{P}(\hat{\mathcal{H}}_{m,0}^+ \geq \lambda m^2N^{2-d}, \mathbf{0} \xleftrightarrow{\geq 0} y) \lesssim \lambda^{-1}N^{2-d}.$$

By (C.4) and (C.7), we obtain the desired bound (C.3). $\square$

For any $\epsilon > 0$, we denote $C_\star(d, \epsilon) := 4C_{25}\epsilon^{-1}$. For any $M \geq 1$, let $\hat{M} := \frac{M}{C_\dagger C_\star}$, where $C_\dagger > 0$ is a large constant to be determined later. For any $m \geq 1$, we define the event $\mathbf{G}_m^\star := \{C_\star^{-1}m^2N^{2-d} \leq \hat{\mathcal{H}}_{m,0}^+ \leq C_\star m^2N^{2-d}\}$. Lemma C.1 implies that

$$(C.8) \quad \mathbb{P}((\mathbf{G}_M^\star)^c \cup (\mathbf{G}_{\hat{M}}^\star)^c \mid \mathbf{0} \xleftrightarrow{\geq 0} y) \leq \frac{\epsilon}{2}.$$

Thus, to verify Lemma 3.11, it remains to show that for some $c_\clubsuit(d, \epsilon) > 0$,

$$(C.9) \quad \mathbb{P}(\mathcal{V}_y^+(M) \leq c_\clubsuit M^4, \mathbf{G}_M^\star, \mathbf{G}_{\hat{M}}^\star \mid \mathbf{0} \xleftrightarrow{\geq 0} y) \leq \frac{\epsilon}{2}.$$

To achieve (C.9), we employ the exploration martingale argument (see [30, 8]). For any $t > 0$, we define $\mathfrak{B}_t := \{v \in \tilde{B}_y(M) : \text{dist}(v, \tilde{B}_y(\hat{M})) \leq t\}$ and

$$(C.10) \quad \tilde{\mathcal{C}}_t := \{v \in \mathfrak{B}_t : y \xleftrightarrow{\tilde{E}^{\geq 0} \cap \mathfrak{B}_t} v\}.$$

Note that $\tilde{\mathcal{C}}_0 = \hat{\mathcal{C}}_M^+$ and $\tilde{\mathcal{C}}_\infty = \hat{\mathcal{C}}_M^+$. Since the σ -field $\mathcal{F}_{\tilde{\mathcal{C}}_t}$ is increasing in t , the process $\mathcal{M}_t := \mathbb{E}[\phi_{\mathbf{0}} \mid \mathcal{F}_{\tilde{\mathcal{C}}_t}]$ for $t \geq 0$ is a martingale. In particular, $\mathcal{M}_0 = \hat{\mathcal{H}}_{M,\mathbf{0}}^+$ and $\mathcal{M}_\infty = \hat{\mathcal{H}}_{M,\mathbf{0}}^+$. Let $\langle M \rangle_t$ denote the quadratic variation of $\mathcal{M}_t$. According to [8, Corollary 10], $\langle M \rangle_t$ can be equivalently written as

$$(C.11) \quad \begin{aligned} \tilde{G}_{\tilde{\mathcal{C}}_0}(\mathbf{0}, \mathbf{0}) - \tilde{G}_{\tilde{\mathcal{C}}_t}(\mathbf{0}, \mathbf{0}) &\stackrel{(3.4)}{=} \sum_{v \in \partial \tilde{\mathcal{C}}_t} \tilde{\mathbb{P}}_{\mathbf{0}}(\tau_{\tilde{\mathcal{C}}_t} = \tau_v < \tau_{\tilde{\mathcal{C}}_0}) \cdot \tilde{G}_{\tilde{\mathcal{C}}_0}(v, \mathbf{0}) \\ &\stackrel{(\tilde{\mathcal{C}}_t \subset \hat{\mathcal{C}}_M^+ \subset \tilde{B}_y(M))}{\lesssim} N^{2-d} \cdot \tilde{\mathbb{P}}_{\mathbf{0}}(\tau_{\hat{\mathcal{C}}_M^+} < \infty) \\ &\stackrel{(3.26)}{\asymp} N^{4-2d} \cdot \text{cap}(\hat{\mathcal{C}}_M^+) \lesssim N^{4-2d} \cdot \mathcal{V}_y^+(M), \end{aligned}$$

where the last inequality follows from the sub-additivity of the capacity. Therefore, there exists $C_{\dagger} > 0$ such that

$$(C.12) \quad \{\mathcal{V}_y^+(M) \leq c_{\clubsuit} M^4\} \subset \{\langle M \rangle_\infty \leq C_{\dagger} c_{\clubsuit} M^4 N^{4-2d}\}.$$

By the martingale representation theorem (see e.g. [8, Theorem 11]), under certain time change, $\mathcal{M}_t$ behaves as a standard Brownian motion with duration $\langle M \rangle_\infty$. Meanwhile, on the $\mathbf{G}_M^* \cap \mathbf{G}_{\hat{M}}^*$, one has

$$(C.13) \quad \begin{aligned} \mathcal{M}_\infty - \mathcal{M}_0 &= \hat{\mathcal{H}}_{M,\mathbf{0}}^+ - \hat{\mathcal{H}}_{\hat{M},\mathbf{0}}^+ \\ &\geq C_{\star}^{-1} M^2 N^{2-d} - C_{\star} \hat{M}^2 N^{2-d} \geq \frac{1}{2} C_{\star}^{-1} M^2 N^{2-d}, \end{aligned}$$

implying that the aforementioned Brownian motion must reach $\frac{1}{2} C_{\star}^{-1} M^2 N^{2-d}$ before time $\langle M \rangle_\infty$. As shown in [2, (5.66)], this observation yields that for a sufficiently small $c_{\clubsuit} > 0$, on the event $\mathbf{G}_{\hat{M}}^*$,

$$(C.14) \quad \mathbb{P}(\langle M \rangle_\infty \leq C_{\dagger} c_{\clubsuit} M^4 N^{4-2d}, \mathbf{G}_M^* \mid \mathcal{F}_{\hat{\mathcal{C}}_M^+}) \leq e^{-\epsilon^{-1}}.$$

By taking the integral of both sides, and using $\mathbf{G}_{\hat{M}}^* \subset \{y \leftrightarrow \partial B_y(\hat{M})\}$, we get

$$(C.15) \quad \mathbb{P}(\langle M \rangle_\infty \leq C_{\dagger} c_{\clubsuit} M^4 N^{4-2d}, \mathbf{G}_M^*, \mathbf{G}_{\hat{M}}^*) \leq e^{-\epsilon^{-1}} \cdot \mathbb{P}(\mathbf{G}_{\hat{M}}^*) \lesssim e^{-\epsilon^{-1}} \hat{M}^{-2}.$$

Moreover, since the events $\{\langle M \rangle_\infty \leq C_\dagger c_\clubsuit M^4 N^{4-2d}\}$, $\mathbf{G}_M^*$ and $\mathbf{G}_{\hat{M}}^*$ are all measurable with respect to $\mathcal{F}_{\hat{C}_M^+}$, it follows from (C.2) and (C.15) that

$$\begin{aligned}
& \mathbb{P}(\langle M \rangle_\infty \leq C_\dagger c_\clubsuit M^4 N^{4-2d}, \mathbf{G}_M^*, \mathbf{G}_{\hat{M}}^*, \mathbf{0} \xleftrightarrow{\geq 0} y) \\
&= \mathbb{E}[\mathbb{1}_{\langle M \rangle_\infty \leq C_\dagger c_\clubsuit M^4 N^{4-2d}, \mathbf{G}_M^*, \mathbf{G}_{\hat{M}}^*} \cdot \mathbb{P}(\mathbf{0} \xleftrightarrow{\geq 0} y \mid \mathcal{F}_{\hat{C}_M^+})] \\
&\stackrel{(C.2)}{\lesssim} \mathbb{P}(\langle M \rangle_\infty \leq C_\dagger c_\clubsuit M^4 N^{4-2d}, \mathbf{G}_M^*, \mathbf{G}_{\hat{M}}^*) \cdot C_\star M^2 N^{2-d} \\
&\stackrel{(C.15)}{\lesssim} e^{-\epsilon^{-1}} \hat{M}^{-2} \cdot C_\star M^2 N^{2-d}.
\end{aligned}
\tag{C.16}$$

Combined with (C.12), it yields that

$$\mathbb{P}(\mathcal{V}_y^+(M) \leq c_\clubsuit M^4, \mathbf{G}_M^*, \mathbf{G}_{\hat{M}}^* \mid \mathbf{0} \xleftrightarrow{\geq 0} y) \lesssim \epsilon^{-3} e^{-\epsilon^{-1}}.
\tag{C.17}$$

This immediately implies that (C.9) holds for all sufficiently small $\epsilon > 0$, thereby completing the proof of Lemma 3.11. $\square$

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