



UNIVERSIDADE DE LISBOA
INSTITUTO SUPERIOR TÉCNICO

THERMODYNAMICS AND
STATISTICAL MECHANICAL
ENSEMBLES OF BLACK HOLES AND
SELF-GRAVITATING MATTER

TIAGO VASQUES FERNANDES

Supervisor: Doctor José Pizarro de Sande e Lemos

Co-Supervisor: Doctor Vítor Manuel dos Santos Cardoso

THESIS APPROVED IN PUBLIC SESSION TO OBTAIN THE PhD DEGREE IN:
PHYSICS

JURY FINAL CLASSIFICATION: **PASS WITH DISTINCTION AND HONOUR**

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À memória da minha avó, Ana...
Dedicado ao meu avô, Albino.

RESUMO

Buracos negros existem por todo o nosso universo, e possuem uma larga variedade de massas. Até ao momento, os buracos negros têm sido usados para testar a relatividade geral em escalas astrofísicas, mas também poderão dar no futuro informação sobre a gravidade em escalas microscópicas. Os buracos negros parecem ter propriedades termodinâmicas como a entropia de Bekenstein e Hawking, que são mais relevantes quando se consideram buracos negros do tamanho de alguns centímetros ou mais pequenos ainda. Como a entropia está relacionada com os micro-estados de um sistema em mecânica estatística, isto levanta certas questões: o que dá origem à entropia de um buraco negro? Poderá esta origem ser explicada por uma descrição quântica da gravidade? Para compreender estas questões, a conexão entre a termodinâmica e a gravidade tem de ser explorada.

Nesta tese de doutoramento, pretendemos compreender esta conexão usando duas descrições que levam à termodinâmica de espaços-tempos curvos. Começamos por impôr a primeira lei da termodinâmica a uma camada fina auto-gravitante com carga elétrica em dimensões arbitrárias. A camada fina com carga pode assumir uma equação de estado fundamental para a pressão, que é obtida apenas pela relatividade geral. Uma equação de estado para a temperatura da camada é escolhida para permitir o estudo do limite de buraco negro e a consequente recuperação da termodinâmica de buracos negros.

Para além disso, usamos a abordagem da gravidade quântica através do integral de caminho Euclideano para construir *ensembles* estatísticos de espaços-tempos com buracos negros e com matéria auto-gravitante, com o objetivo de estudar semiclasicamente as possíveis transições de fase entre matéria quente e buracos negros. Mostramos também a capacidade que o formalismo tem para descrever as propriedades termodinâmicas de espaços-tempos curvos. Especificamente, estudamos os *ensembles* canónicos e grão-canónicos de buracos negros com carga elétrica, dentro de uma cavidade com raio finito ou infinito. Adicionalmente, construímos *ensembles* de camadas finas em anti-de Sitter e em espaços assintoticamente planos, para compreender as características termodinâmicas de camadas finas e as possíveis transições de fase para configurações de buracos negros.

PALAVRAS-CHAVE: TERMODINÂMICA, *ensembles* ESTATÍSTICOS, RELATIVIDADE GERAL, BURACOS NEGROS, MATÉRIA QUENTE .

ABSTRACT

Black holes exist all over our Universe, possessing a very wide range of masses. At the moment, they serve as a probe to test general relativity at astrophysical scales, but in the future they may also give us information about gravity at the microscale. Black holes seem to have thermodynamic properties, such as the Bekenstein-Hawking entropy, which are important when considering black holes with size of a few centimeters or smaller. Since entropy in statistical mechanics is related to the number of possible microstates of a system, several questions arise: what gives rise to the black hole entropy? Can it be explained by a quantum description of gravity? In order to further study these questions, the connection between thermodynamics and gravity must be explored at the microscale.

In this doctoral thesis, we aim to understand this connection using two descriptions that yield the thermodynamics of curved spacetimes. We start by imposing the first law of thermodynamics to a charged self-gravitating matter thin shell in higher dimensions. The fundamental pressure equation of state can be used for the shell, which is given solely by general relativity. An equation of state for temperature of the shell is also chosen, so that it allows the study of the black hole limit and the recovery of black hole thermodynamics.

Furthermore, we use the Euclidean path integral approach to quantum gravity to construct statistical ensembles of black hole spacetimes and self-gravitating matter, in order to study semiclassically the possible phase transitions between hot matter and black holes. We also show the power of the formalism in obtaining the thermodynamic properties of curved spacetimes. Namely, we study the canonical and grand canonical ensemble of charged black holes inside a cavity, which may have a finite or infinite radius. We also construct ensembles of a self-gravitating matter thin shell, both in anti-de Sitter and in asymptotically flat spaces, in order to understand the thermodynamic features of the shell and the possible phase transitions to black hole configurations.

KEYWORDS: THERMODYNAMICS, STATISTICAL ENSEMBLES, GENERAL RELATIVITY, BLACK HOLES, HOT MATTER .

PREFACE

During the four years and five months of the Doctoral Programme, the official research that led to the development of this thesis has been conducted at Centro de Astrofísica e Gravitação (CENTRA), in the Physics Department at Instituto Superior Técnico (IST) - University of Lisbon. The research done during the Doctoral Programme was financially supported by Fundação para a Ciência e Tecnologia - FCT through the project UIDB/00099/2020, with the grant FCT no. RD0970, and through the project UIDB/00099/2025, with the grant FCT no. RD1415. Furthermore, I acknowledge the support from the project 2024.04456.CERN. I declare that this thesis and its content has not been submitted for a degree, diploma or other qualification at any other university and it has been made specifically to obtain the PhD in Physics at IST.

The research developed in Chapters 2, 4, 5 and 6 was done in collaboration with José Sande Lemos. The research developed in Chapter 7 was done in collaboration with Francisco J. Gandum, José Sande Lemos and Oleg Zaslavskii. The research developed in Chapter 8 was done in collaboration with Francisco J. Gandum and José Sande Lemos. Finally, the research developed in Chapter 9 was done in collaboration with José Sande Lemos and Oleg Zaslavskii. Chapters 2, 4 and 6 have been published, while Chapter 5 has been accepted but not yet published and 8 has been submitted to a journal, with the manuscripts being available in the arXiv. The Chapters 7 and 9 are in preparation. In sum, this doctoral thesis is mainly based on the following works:

[1] T. V. Fernandes and J. P. S. Lemos, “Electrically charged spherical matter shells in higher dimensions: Entropy, thermodynamic stability, and the black hole limit,” *Phys. Rev. D* **106**, 104008 (2022), [arXiv:2208.11127 \[gr-qc\]](#) (Chapter 2);

[2] T. V. Fernandes and J. P. S. Lemos, “Grand canonical ensemble of a d-dimensional Reissner-Nordström black hole in a cavity,” *Phys. Rev. D* **108**, 084053 (2023), [arXiv:2309.12388 \[hep-th\]](#) (Chapter 4);

[3] T. V. Fernandes and J. P. S. Lemos, “Gibbons-Hawking action for electrically charged black holes in the canonical ensemble and Davies’ thermodynamic theory of black holes,” (accepted in *Proc. R. Soc. Lond. A*), [arXiv:2410.12902 \[hep-th\]](#) (Chapter 5);

[4] T. V. Fernandes and J. P. S. Lemos, “Canonical ensemble of a d-dimensional Reissner-Nordström black hole spacetime in a cavity,” 2025, [arXiv:2504.15339 \[hep-th\]](#) (Chapter 6);

[5] T. V. Fernandes, F. J. Gandom, J. P. S. Lemos, and O. B. Zaslavskii, “Limits in hot spaces with negative cosmological constant in the canonical ensemble: Thermal anti-de Sitter solution, Schwarzschild-anti de Sitter black hole, Hawking-Page solution, and planar AdS black hole,” to be submitted (Chapter 7);

[6] T. V. Fernandes, F. J. Gandom, and J. P. S. Lemos, “The canonical ensemble of a self-gravitating matter thin shell in AdS,” submitted to Phys. Rev. D, [arXiv:2504.08059 \[hep-th\]](#) (Chapter 8);

[7] T. V. Fernandes, J. P. S. Lemos, and O. B. Zaslavskii, “Thermodynamic ensembles of a black hole and a self-gravitating matter thin shell with a fixed chemical potential: equilibrium, stability and Le Chatelier-Braun principle,” to be submitted (Chapter 9).

There are also the following works which were started recently and came out of the ideas of this PhD thesis:

[8] F. J. Gandom, T. V. Fernandes, and J. P. S. Lemos, “The canonical ensemble of a self-gravitating thin shell in AdS inside a cavity,” to be submitted;

[9] T. V. Fernandes, R. André, and J. P. S. Lemos, “Canonical ensemble of self-gravitating photon gas inside a cavity,” to be submitted.

During the years of my doctoral program, I also coauthored other works which are not discussed in this thesis. Two of these works were done in collaboration with David Hilditch, Vítor Cardoso and José Sande Lemos, published in Phys. Rev. D. Two of the works have been done in collaboration with David Lopes and José Sande Lemos, as part of David Lopes’ master thesis, with one published in Phys. Rev. D and the other to be submitted. The remaining work was done in collaboration with Julian Barragán Amado and David Lopes, and it has been submitted to JHEP, with the manuscript being available in the arXiv. These works are the following:

[10] T. V. Fernandes, D. Hilditch, J. P. S. Lemos, and V. Cardoso, “Quasinormal modes of Proca fields in a Schwarzschild-AdS spacetime,” *Phys. Rev. D* **105**, 044017 (2022), [arXiv:2112.03282 \[gr-qc\]](#).

[11] T. V. Fernandes, D. Hilditch, J. P. S. Lemos, and V. Cardoso, “Normal modes of Proca fields in AdS spacetime,” *Gen. Rel. Grav.* **55**, 5 (2023), [arXiv:2301.10248 \[gr-qc\]](#).

- [12] D. C. Lopes, T. V. Fernandes, and J. P. S. Lemos, “Normal modes of Proca fields in AdS_d spacetime,” *Phys. Rev. D* **109**, 064041 (2024), [arXiv:2401.13030 \[gr-qc\]](#).
- [13] D. C. Lopes, T. V. Fernandes, and J. P. S. Lemos, “Quasinormal modes of a Proca and Maxwell field in Schwarzschild- AdS_d spacetime,” to be submitted.
- [14] J. Barragán Amado, T. V. Fernandes, and D. C. Lopes, “Quasinormal modes of a Proca field in Schwarzschild- AdS_5 spacetime via the isomonodromy method,” (2025), [arXiv:2504.00080 \[gr-qc\]](#).

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CONVENTIONS, NOTATION AND UNITS

In this thesis, the conventions of Refs. [15, 16] are followed. There are four constants throughout the thesis that establish the units, the gravitational constant G in arbitrary dimensions d , the speed of light c , the Planck constant $\hbar$ and the Boltzmann constant. Unless stated otherwise, I use Planck units $G = \hbar = c = 1$ and the Boltzmann constant is set to 1 as well. In some places, the Planck scale or the gravitational constant $l_p = G^{\frac{1}{d-2}}$ is kept. Lorentzian spacetimes have the most positive metric signature, while Euclidean spaces have the typical positive metric signature.

$\alpha, \beta, \gamma, \dots$	Lorentzian spacetime and Riemannian space indices;
$a, b, c, d, \dots, h$	timelike hypersurface indices in Lorentzian spacetime and indices of a hypersurface parametrized by imaginary time in Riemannian space;
$i, j, k, \dots$	spacelike hypersurface indices in Lorentzian spacetime and indices of a hypersurface of constant imaginary time in Riemannian space;
$A, B, C, \dots$	$d - 2$ -surface indices in Lorentzian spacetime and in Euclidean space;
$V_\alpha W^\alpha \equiv \sum_{\alpha=0}^3 V_\alpha W^\alpha$	Einstein's notation;
$T_{(\alpha_1 \dots \alpha_l)} \equiv \frac{1}{l!} \sum_\sigma T_{\alpha_{\sigma(1)} \dots \alpha_{\sigma(l)}}$	symmet. over all permutat. σ ;
$T_{[\alpha_1 \dots \alpha_l]} \equiv \frac{1}{l!} \sum_\sigma \epsilon_\sigma T_{\alpha_{\sigma(1)} \dots \alpha_{\sigma(l)}}$	anti-symmet. over all permutat. σ ;
$g_{\alpha\beta}$	curved Lorentzian/Riemannian metric;
$(\cdot)_{,\alpha} = \partial_\alpha(\cdot) = \frac{\partial}{\partial x^\alpha}(\cdot)$	coord. derivative;
$(\cdot)_{;\alpha} = \nabla_\alpha(\cdot)$	Levi-Civita derivative;
$\square(\cdot) \equiv \nabla_\alpha \nabla^\alpha(\cdot)$	Levi-Civita d'Alembertian.

INTRODUCTION

1.1 CLASSICAL BLACK HOLES IN GENERAL RELATIVITY

The theory of general relativity, with its definite formulation in [17] by Einstein, has withstood for now as the theory that describes gravity at large length scales. Gravity in this theory is described as the link between the curvature of spacetime and the presence of matter, through the Einstein equations. As a consequence, the presence of matter curves spacetime in a nontrivial way, giving origin to a number of effects such as the precession of orbits, the deflection of light near massive objects, the gravitational redshift of light's frequency and the time delay of light as it travels near a massive object. Surprisingly, right at its conception, general relativity was able to explain the perihelion motion of Mercury [18], which could not be explained by Newton's gravity. Moreover, the deflection of light by the Sun was observed by Eddington [19]. Further tests were made, with the measurement of the gravitational redshift [20] and the measurement of the gravitational time delay [21, 22], or Shapiro delay, which agreed with general relativity. Another characteristic of the theory is the existence of gravitational waves, ripples of spacetime originated from the motion of two massive compact objects. These were eventually measured indirectly by Hulse and Taylor [23, 24] and first directly measured by LIGO [25]. The present evidence continues to strengthen the position of general relativity as the theory of gravitation, at least at large length scales.

Although these effects are quite important, one of the most important predictions of general relativity is the existence of black holes. These objects are defined, in a general sense, as regions of spacetime from which light cannot escape. The first solution which describes such an object was given by Schwarzschild [26] and additionally with electric charge in [27, 28], although initially these metrics were only thought to describe the exterior region of spherically symmetric self-gravitating objects. However, it was only with the work by Oppenheimer and Snyder [29], where they studied the gravitational collapse of dust in spherical symmetry, that black holes were put in the spotlight as remnants of gravitational collapse. Penrose [30] showed that gravitational collapse in general settings would occur, giving birth to black holes and to the occurrence of singularities. In general relativity, these black holes can be described by the Kerr-Newman family [31, 32], which extends the Schwarzschild and Reissner-Nordström solutions to include rotation. The fact that black holes can only be described by three parameters has

been a quite enticing feature of general relativity and the possibility of probing regions of strong gravitational field with current technology could give us more insights in the validity of general relativity. Based on recent observations, it is most certain that black holes exist and that they populate our whole universe. The first detection of gravitational waves [25] not only demonstrated the existence of these objects but also initiated an era of probing the strong field regime of general relativity with gravitational waves. Moreover, the observations of the center of the M87 galaxy [33], and the center of the Milky Way [34–36], are in complete agreement with the existence of a supermassive black hole at the center of these galaxies and also with the predictions of general relativity.

At the theoretical level, the analysis of global causality provided by Penrose and Hawking, which were needed for the development of the singularity theorems [30, 37], allowed for a better understanding of the properties of a black hole and its boundary, the event horizon. It was realized by Hawking [38] that under the weak cosmic censorship and under the weak energy condition, the area of the event horizon always increases. Furthermore, using the Kerr solution, Smarr [39] showed that the mass of the black hole is related to the area of the event horizon and its angular momentum. These developments led to the establishment of the four laws of black hole mechanics [40] for black holes in stationary spacetimes. While these revealed crucial properties of classical black holes, the most impact was felt in black hole thermodynamics.

Astrophysical black holes, which we discussed above, can have a mass ranging from a few solar masses up to millions of solar masses. There may be however another class of black holes, with masses much smaller than one solar mass [41] but larger than the Planck mass. These are micro black holes or quantum black holes, where their thermodynamic properties have the most importance. Theoretically, they may be formed by extreme heat, by a collision of two particles or even by overdensities in the early universe. Micro black holes are of great importance to probe the gravitational interaction at quantum scales and, in the particular context of this thesis, they are an important avenue to study the link between gravity and thermodynamics.

1.2 THERMODYNAMIC PROPERTIES OF BLACK HOLES

Influenced by Wheeler’s thought experiments regarding matter entropy and black holes, Bekenstein [42, 43] proposed that black holes should have an entropy proportional to its event horizon area and generalized the second law of thermodynamics to include black hole entropy. The argument towards such proposal was based on the lowest integer power of the horizon radius that allowed for the entropy to always increase, following from the second law of black hole mechanics given by Hawking [38]. Bekenstein’s proposal was seen with skepticism by Hawking, which in [40] points out that classical black holes do not radiate and so the only connection between thermodynamics and the four laws of black hole mechanics was purely an analogy.

It was around this time that quantum field theory in curved spacetime was being explored. The notion of particle states provided by the Fock space of a field in a curved spacetime depends on the choice of a basis of positive frequencies for the field, i.e. it depends on the observer. However, such dependence seems to be akin to the choice of coordinates of a manifold in general relativity. Hawking [44, 45] used the treatment of quantum fields in a collapsing spacetime that would asymptotically tend to a stationary black hole spacetime. Using the geometric optics approximation, he showed that black holes radiate neutral quanta with a spectrum similar to a black body at Hawking temperature $T = \frac{\hbar\kappa}{2\pi k}$, where $\hbar$ is the Planck constant, k is the Boltzmann constant and κ is the surface gravity of the event horizon. Boulware [46], upon the work by Hawking, showed that the vacuum prescribed by choosing positive frequencies via the timelike Killing field led to no radiation for an eternal Schwarzschild black hole and that Hawking radiation was mostly due to the existence of gravitational collapse. The picture of black hole evaporation for an eternal Schwarzschild black hole was then developed by Unruh [47], by choosing a different vacuum at the past horizon, as the one that could mimic gravitational collapse. However, both Boulware and Unruh vacuum states are not well-defined in the maximally extended Schwarzschild spacetime. Hartle and Hawking [48] obtained the unique vacuum state that is well-defined in the whole maximally extended Schwarzschild spacetime through Euclidean path integral techniques. An observer following the orbits of the timelike Killing vector in the exterior region of Schwarzschild will observe the Hartle-Hawking state as a thermal state. By consequence, a Schwarzschild black hole can be in thermal equilibrium [49] with a heat bath described by the Hartle-Hawking state, revealing the thermodynamic nature of black holes. Since these developments, quantum field theory in curved spacetime has established itself as a promising area of research, with the study of vacuum states in black hole spacetimes being extended to several cases, e.g. see [50].

These developments established the first strong link between thermodynamics and black hole mechanics that was missing. Black holes indeed radiate at the Hawking temperature and possess the Bekenstein-Hawking entropy $S = \frac{A_+}{4A_p}$, where A_+ is the event horizon area and A_p is the Planck area. The four laws of black hole mechanics in [40] correspond to the laws of black hole thermodynamics. These laws were used in tandem with the Hawking temperature and the Bekenstein-Hawking temperature to describe the thermodynamic theory of black holes, by Davies [51]. Since then, the first law of thermodynamics has been used to study black hole solutions and their stability [52–63]. However, it must be noted that such prescription is heuristic and so a fundamental formalism is required to describe the thermodynamics in black hole spacetimes.

1.3 THERMODYNAMICS AND STATISTICAL ENSEMBLES IN CURVED SPACETIMES

From the existence of a thermal state describing a heat bath, one can extract thermodynamic properties of a curved spacetime. But the realization of temperature

in curved spacetimes predates the substantial works on black hole thermodynamics. To tackle this issue, Tolman and Ehrenfest [64] considered a system in thermal equilibrium with an external heat reservoir, but with the space between the system and the heat reservoir filled with electromagnetic radiation. The electromagnetic field was then averaged to describe black body radiation in the form of a perfect fluid with a radiative equation of state. From the equilibrium equations, Tolman and Ehrenfest showed that the local temperature is given by the redshifted temperature of the heat reservoir. This means that thermodynamic equilibrium in a gravitating system is not described by a constant local temperature but rather must be described by the redshift factor. In some sense, temperature follows the same behavior as the frequency of photons in a stationary curved spacetime. This evidently distorts the notions of intensive and extensive variables in thermodynamics involving curved spacetimes.

To understand further thermodynamics in curved spacetimes and its properties, one needs to have a more fundamental approach. A formal way of formulating thermodynamics is by using the tools of statistical mechanics. However, for that construction, one needs to have a microscopic description of gravity. For quantum systems without gravity, the microscopic description is given by quantum field theory and one can build the partition function of statistical ensembles using the statistical path integral [65, 66]. The time parameter associated to the evolution of states is extended to the complex plane and one then works with an imaginary time, which has a period equal to the inverse temperature of the ensemble. In this framework, quantum field theory is transformed into thermal quantum field theory. Gibbons and Hawking [67] extended this formalism to curved spacetimes, where the partition function is given by the Euclidean path integral approach to quantum gravity. Using the semiclassical approximation, they obtained the grand canonical ensembles of Kerr-Newman and Reissner-Nordström spaces and also the canonical ensemble of Schwarzschild space, in four dimensions. For the specific case of Schwarzschild, the heat capacity of the black hole is negative, which makes the canonical ensemble unstable and the semiclassical approximation not valid. York [68] understood, motivated by the results of Page and Hawking [69], that introducing a cavity at finite radius makes the canonical ensemble of a black hole stable and the semiclassical approximation valid in a specific range of the parameter space. Specifically, the introduction of the cavity gives rise to an additional black hole solution in thermal equilibrium with the cavity with positive heat capacity. This is the York formalism for the construction of canonical and grand canonical ensembles in curved spacetimes. Even though one is making a semiclassical approximation, the formalism allows the study of phase transitions between stable configurations, namely between matter at finite temperature and black hole configurations, which arise purely from quantum effects. Note that the configurations we are discussing must be microscopic so that semiclassical effects come into play, but they must also be far away from the Planck scale so that additional quantum effects can be ignored. This motivates the exploration of the York formalism to understand the implications of the Euclidean path integral approach to quantum gravity in thermodynamic and microscopic systems.

1.4 HIGHER DIMENSIONAL CURVED SPACETIMES

From our sensorial perspective, we perceive the universe as being four dimensional. As we are observers, our spatial awareness tells us that there are three dimensions of space and we move in worldlines with a dedicated clock, i.e. one dimension of time. Up to now, it seems that all events that we observe are consistent with a four dimensional universe. However, the interest in higher dimensions increased mainly by the prospects of unifying the fundamental forces of the Universe. The original work by Kaluza [70] and Klein [71] tried to unify electromagnetism with gravity by considering a five dimensional spacetime with a cylindrical coordinate, with the metric being a solution of the five dimensional Einstein field equations. The metric did not depend on this cylindrical coordinate, and its effects on the matter fields could be minimized by having a scale much smaller than the Planck scale associated to the length of the extra dimension, i.e. one has a compactified dimension. This approach was expanded in order to include the forces of the standard model by considering supersymmetry [72], which is named as Kaluza-Klein supergravity theory. It was shown that this could be done with an eleven dimensional supergravity theory [73]. However, many difficulties arose, e.g. the difficulty in including chirality for fermion fields and the presence of anomalies when quantizing the theory. The unification attempts were later focused on superstring theories, which had supergravity as their low energy limit, and ultimately M-theory, to tackle these difficulties. Still, there is an apparent absence of clear physical predictions, apart from supersymmetry, that can be extracted from these unified theories. Additionally, from the observation of the gravitational wave signal and electromagnetic counterpart of GW170817 [74, 75], an event classified as a neutron star binary merger, there are constraints on the dimensions of the Universe that are compatible with four non-compactified dimensions.

Apart from unification theories, there has been a large interest in higher dimensional spacetimes due to the AdS/CFT conjecture [76]. This conjecture is a correspondence between a string theory in $(d + 1)$ dimensions and a quantum conformal field theory without gravity in d dimensions. The main advantages of the conjecture is that one has a strong/weak coupling duality, i.e. string theory in the weak coupling regime is dual to a conformal field theory in the strong coupling regime. This motivated the study of strings and branes in higher dimensional AdS spacetimes, which can be done using higher dimensional general relativity, in order to obtain the properties of non-perturbative conformal fluids, with many applications to condensed matter and particle physics.

1.5 OUTLINE OF THE THESIS

In this thesis, we focus on the study of thermodynamic self-gravitating systems with the objective to further understand the interplay between gravity and thermodynamics. We work with microscopic systems much larger than the Planck length, where thermodynamics is important and semiclassical effects are present. This thesis consists on two main Parts. In each Part, we choose a specific formulation to

describe thermodynamics in curved spacetime. Moreover, each Part of the thesis is self-contained.

In the first Part, we formulate the thermodynamics for self-gravitating matter by imposing the first law of thermodynamics. Namely, in Chapter 2, we impose the first law of thermodynamics to a self-gravitating matter thin shell with electric charge, in arbitrary dimensions. The purpose is to understand the possibility and the implications of imposing the Martinez pressure equation of state, which arises naturally from the Einstein equations. Indeed, the Martinez pressure equation of state can in fact be imposed, and we obtain the entropy of the shell in terms solely of its gravitational radius and its Cauchy radius, related to the electric charge. We impose further equations of state that have a black hole-like behaviour, allowing the recovery of black hole thermodynamics from the thin shell in the black hole limit. The intrinsic thermodynamic stability of the thin shell is then analyzed, showing that the case of a thin shell with black hole equations of state in the black hole limit is marginally stable.

In the second Part, we construct statistical ensembles of curved spacetimes including matter in order to obtain their thermodynamic properties. To obtain the partition function of a statistical ensemble, we use the Euclidean path integral approach to quantum gravity, which gives a microscopic description of gravity, in the zeroth order of the saddle point approximation, i.e. the zero loop approximation. The state of the art and the formalism restricted to spherically symmetric metrics are explained in Chapter 3, which is taken as a reference in the rest of the Chapters. Apart from this relationship between Chapter 3 and the rest of the Chapters, the remaining content of the Chapters is self-contained. Throughout the second Part, we apply this formalism to various cases involving a gauge field and matter in order to understand the intricacies of the formalism and the phase diagrams of the configurations considered. In Chapter 4, we consider the grand canonical ensemble of a Reissner-Nordström black hole inside a cavity, in arbitrary dimensions. In Chapter 5, we consider the canonical ensemble of a Reissner-Nordström black hole with cavity at infinity, in arbitrary dimensions, where we establish a link between the Euclidean path integral approach to quantum gravity and the strategy of imposing the first law of thermodynamics. In Chapter 6, we consider the canonical ensemble of a Reissner-Nordström black hole inside a cavity, in arbitrary dimensions. Note that for Chapters 4, 5, and 6, we obtain the phase diagrams between black hole configurations and hot flat space, where the models of hot flat space, i.e. flat space at a certain temperature, are considered for fixed electric potential and for fixed electric charge, which is a novelty. In Chapter 7, we analyze the limits of the canonical ensemble of a black hole in AdS inside a cavity, which unify the black hole solutions existing in the literature. In Chapter 8, we consider the canonical ensemble of a self-gravitating matter thin shell in anti-de Sitter (AdS), showing that for a particular equation of state, it mimics hot thermal AdS, i.e. pure AdS space with a thermal graviton gas, for a wide range of temperatures. In Chapter 9, we build the grand canonical ensemble of a matter thin shell with chemical potential involving a black hole, all surrounded by a cavity, showing the

power of the formalism. We further show certain connections between the validity of the zero loop approximation, mechanical stability and thermodynamic stability.

Finally in Chapter [10](#), we present some conclusion remarks regarding the main results of the thesis, including caveats of the study and possible future work.

Part I

THERMODYNAMICS OF SELF-GRAVITATING MATTER
USING THE FIRST LAW

ELECTRICALLY CHARGED SPHERICAL MATTER SHELLS IN HIGHER DIMENSIONS

2.1 INTRODUCTION

The study of self-gravitating matter is fundamental to understand the effects of general relativity and ultimately to describe the astrophysical objects of the universe. While matter is typically distributed through the three dimensions of space, self-gravitating thin shells in general relativity [77] have proven to be of great significance towards the understanding of the interaction between gravitational and matter fields. Namely, the dynamics of thin shells in Schwarzschild and Reissner-Nordström spacetimes, together with generalizations to higher dimensions [78–83], are able to capture in detail the main features of gravitational collapse and the corresponding black hole formation. Related to gravitational collapse and the stability of self-gravitating matter, the maximum compactness of stars has been studied through neutral and electrically charged thin shells [84]. Moreover, the landscape of spacetimes that can be constructed using thin shells is vast and uncovers the possible exotic configurations provided by general relativity such as wormhole spacetimes [85, 86], bubble universes [87], tension shells and stars [88], and regular black holes [89–91], with the case of an electrically charged shell with two different normal orientations being able to describe even more objects [92].

The interest in self-gravitating matter thin shells also extends to the treatment of thermodynamics within general relativity, as shells can be used to understand the thermodynamics of matter in a gravitational field and even the thermodynamics of the gravitational field itself. While one has the freedom to choose equations of state for the shell, there is a particular pressure equation of state that can be provided by general relativity in the case of a static spherical thin shell with Minkowski in its interior. This can be called as a fundamental pressure equation of state and allows the radius of the shell to be arbitrary. By imposing the first law of thermodynamics to the shell, one can restrict the expression for the temperature equation of state using the integrability conditions, leaving some freedom for its choice that can be motivated by a fundamental description of matter or one can always make a reasonable guess based on the behaviour that one wants to imprint to the shell. This treatment was first performed for a static shell with an exterior Schwarzschild spacetime in [93, 94] and extended to higher dimensions in [95]. The inclusion of electric charge in the case of four dimensions has also been treated in [96, 97], with

the extremal case being analyzed in [98, 99], where one also has a fundamental pressure equation of state and the exterior region is now Reissner-Nordström spacetime. As we shall see in the next chapters, the thermodynamics of these matter thin shells has some relation with the thermodynamics of black holes and their statistical ensembles inside a cavity [68, 100–102]. As the boundary of such cavity is given by a non-massive spherically symmetric shell, the thermodynamic variables of the two systems can be similar.

The thermodynamics of matter thin shells can also yield black hole thermodynamics using the quasiblack hole approach [103–106]. This approach avoids the setting of the specific equations of state, as one keeps the shell's gravitational radius fixed and one changes its proper mass and radius, so that the configuration is maintained near the black hole threshold. By using the integration of the first law over these configurations, one retrieves the black hole properties in a model independent way.

Therefore, the study of thin shells, together with black holes and quasiblack holes, is of great importance in the understanding of thermodynamics of spacetimes. Following these themes, in this chapter, we analyze the entropy and the thermodynamic stability of a static electrically charged spherical thin shell in d dimensions, with the fundamental pressure equation of state, and we also study the black hole limit, extending the analysis of [95] to the electrically charged case and of [96] to higher dimensions. We impose an equation of state for the temperature and the electric potential so that the entropy of the shell is described by a power law in the gravitational radius, and find that intrinsic stability for the possible fluctuations of the shell requires positive heat capacity, positive isothermal compressibility and positive isothermal electric susceptibility. By performing the black hole limit, we find the black hole thermodynamic properties, such as the Smarr formula and thermodynamic stability. The intrinsic stability analysis followed here is provided in [107].

The work presented in this chapter is mainly based on [1]. The chapter is organized as follows. In Sec. 2.2.2, we construct the spacetime solution containing an electrically charged matter thin shell using the thin shell formalism. In Sec. 2.3, we apply the first law of thermodynamics of the shell together with the fundamental pressure equation of state to obtain the entropy of the shell for two specific equations of state and further analyze the black hole limit. In Sec. 2.4.9, we analyze the intrinsic stability of the shell with the specific equations of state, including the case of the black hole limit. In Sec. 2.5.3, we treat the intrinsic stability in terms of physical quantities, namely the heat capacity, the isothermal compressibility and the isothermal electric susceptibility. In Sec. 2.6, we conclude.

2.2 ELECTRICALLY CHARGED MATTER THIN SHELL IN HIGHER DIMENSIONS

2.2.1 Formalism

In order to construct the spacetime with an electrically matter thin shell, we first consider a curved spacetime containing a Maxwell field and additional matter. The metric is described by the Einstein equations

$$G_{\alpha\beta} = 8\pi G T_{\alpha\beta}, \quad (2.1)$$

where $G_{\alpha\beta}$ is the Einstein tensor given in terms of the metric $g_{\alpha\beta}$ and its derivatives, $T_{\alpha\beta}$ is the stress-energy tensor, while the Maxwell field is described by the Maxwell equations

$$\nabla_b F^{ab} = J^a, \quad (2.2)$$

where $F_{\alpha\beta}$ is the Maxwell tensor obeying the internal equations $\nabla_{[\alpha} F_{\beta\gamma]}$, with the Maxwell tensor being described by the vector potential A_α as $F_{\alpha\beta} = \nabla_{[\alpha} A_{\beta]}$, and J^a is the electric current. For the case of an electrovacuum spacetime, the stress-energy tensor $T_{\alpha\beta}$ is

$$T_{\alpha\beta} = \varepsilon \left(F_{\alpha}{}^{\gamma} F_{\beta\gamma} - \frac{1}{4} g_{\alpha\beta} F^{\gamma\nu} F_{\gamma\nu} \right), \quad (2.3)$$

where we define the parameter ε as $\varepsilon = \epsilon \frac{(d-3)}{\Omega}$, the parameter ϵ being the electromagnetic coupling constant, and the area of a $d-2$ unit sphere is $\Omega = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma[\frac{d-1}{2}]}$.

As we have an electrically charged thin shell, the spacetime is divided into two bulk regions, the interior region $\mathcal{V}_1$ and the exterior region $\mathcal{V}_2$, both obeying to Eqs. (2.1) and (2.2), with the stress energy tensor given by Eq. (2.3) and zero electric current. Moreover, there is a boundary timelike hypersurface, Σ , corresponding to the thin shell, between the two regions. In order to match the two regions at the thin shell, one requires the fulfillment of appropriate junction conditions, according to [77] in general relativity.

For the interior region $\mathcal{V}_1$, the coordinates associated to this region are x_1^α and the metric is $g_{1\alpha\beta}$, with the corresponding covariant derivative $\nabla_{1\alpha}$. The covector normal to the thin shell in this region is $n_{1\alpha}$, and so one can build tangent vectors $(e_1)^\alpha_a = \frac{\partial x_1^\alpha}{\partial y^a}$, where y^a are the associated coordinates to the thin shell. The vector potential in the interior region is $A_{1\alpha}$ with the field strength $F_{1\alpha\beta} = \nabla_{1[\alpha} A_{1\beta]}$. In the same way, for the exterior region $\mathcal{V}_2$, we have the same definitions but with the subscript 2, i.e. the coordinates x_2^α , metric $g_{2\alpha\beta}$, covariant derivative $\nabla_{2\alpha}$, the normal covector $n_{2\alpha}$, the tangent vectors on the hypersurface $(e_2)^\alpha_a$, the vector potential $A_{2\alpha}$ and strength field tensor $F_{2\alpha\beta}$.

The boundary timelike hypersurface Σ , with coordinates y^a , is the thin shell and it is shared by the two regions $\mathcal{V}_1$ and $\mathcal{V}_2$. One can perform the pull-back of the tensorial quantities living in the product of cotangent spaces of both regions to define these quantities at the hypersurface. The junction conditions then yield

the relation between the quantities evaluated in both regions. For the interior region, the pull-back of the metric is defined as $g_{1\alpha\beta}(e_1)^\alpha_a(e_1)^\beta_b \equiv h_{iab}$, and the pull-back of $\nabla_{1\alpha}n_{1\beta}$ is $\nabla_{1\alpha}n_{1\beta}(e_1)^\alpha_a(e_1)^\beta_b = \nabla_{1a}n_{1b} \equiv K_{1ab}$, where K_{1ab} is the extrinsic curvature of the hypersurface measured in the interior region. Additionally, in the interior region, the pull-back of the vector potential is $A_{1\alpha}(e_1)^\alpha_a \equiv A_{1a}$, and the strength field tensor can be decomposed in two parts, the complete pull-back $F_{1\alpha\beta}(e_1)^\alpha_a(e_1)^\beta_b \equiv F_{1ab}$ and the pull-back of $F_{1\alpha\beta}n^\beta$ as $F_{1\alpha\beta}n^\beta(e_1)^\alpha_a \equiv F_{1a}$. For the exterior region, the same definitions and pullbacks apply but where subscript 1 is replaced by the subscript 2.

The junction conditions to relate the above quantities in both regions can be obtained by assigning a normal geodesic coordinate common to both regions in the neighbourhood of the hypersurface and the conditions for the metric are obtained by imposing regularity of the Levi-Civita connection and imposing the Einstein equations, considering only the Dirac delta terms. This can also be achieved by variational principle using the Einstein-Hilbert action together with the Gibbons-Hawking-York boundary terms at the hypersurface. From existence of the common normal coordinate, one has that the normal covectors in both regions must be the same at the hypersurface, but it does not apply to its derivatives. From the regularity of Levi-Civita connection, the junction condition reads

$$[h_{ab}] = 0, \quad (2.4)$$

where $[h_{ab}]$ means $[h_{ab}] = h_{2ab} - h_{1ab}$ and the same applies for other quantities. The junction condition in Eq (2.4) means that the induced metric at the hypersurface must be recovered from both sides, i.e. $h_{ab} = h_{1ab} = h_{2ab}$, and this establishes the relation between the coordinates chosen in both regions. From the Einstein equations, the junction condition reads

$$-\left([K_{ab}] - [K]h_{ab}\right) = 8\pi GS_{ab}, \quad (2.5)$$

where K is defined as the trace of the extrinsic curvature K_{ab} , and S_{ab} is defined as the stress-energy tensor for matter in the shell. For the stress-energy tensor, it is assumed that the matter is described by a perfect fluid, i.e.

$$S_{ab} = (\sigma + p)u_a u_b + p h_{ab}, \quad (2.6)$$

where σ is the matter density, p is the matter pressure and u_a is the velocity of the fluid on the boundary.

One also has junction conditions for the vector potential and the strength field tensor, as they can be extracted by imposing the continuity of the vector potential and the Maxwell equations. Indeed, one has

$$[A_a] = 0, \quad (2.7)$$

$$[F_{ab}] = 0, \quad [F_a] = j_a, \quad (2.8)$$

where F_a is the pull-back of $F_{\alpha\beta}n^\beta$ for each region at the hypersurface, and j_a is the electric current given by

$$j_a = \zeta \sigma_e u_a, \quad (2.9)$$

with σ_e being the electric charge density, ζ defined as $\zeta = \frac{\Omega}{\epsilon_q}$, and ϵ_q being the electric permittivity.

The formalism can now be applied to a d -dimensional spacetime with a Minkowski interior in region $\mathcal{V}_1$, an electrically charged shell at the hypersurface and a Reissner-Nordström-Tangherlini exterior in region $\mathcal{V}_2$. The electromagnetic coupling ϵ appearing in Eq. (2.3) and the electric permittivity ϵ_q appearing in Eq. (2.9) are set to unity as a choice of convention, i.e. $\epsilon = 1$ and $\epsilon_q = 1$. However, the next subsection, the metric and vector potential dependence on these parameters are shown. The reason for not setting them at the start is to show the possible conventions in the literature and make the conversions easier.

2.2.2 The solution of a spacetime with an electrically charged thin shell

As the interior region is a vacuum d -dimensional spherically symmetric Minkowski spacetime, the line element for the metric in region $\mathcal{V}_1$ is

$$ds_1^2 = -dt_1^2 + dr^2 + r^2 d\Omega^2, \quad 0 \leq r \leq R_1, \quad (2.10)$$

where spherical coordinates have been adopted, i.e. $x_1^\alpha = (t_1, r_1, \theta_1^A)$ together with $r \equiv r_1$, $d\Omega^2$ is the line element of a $(d-2)$ unit sphere, and R_1 is the radius of the shell at region $\mathcal{V}_1$. Regarding the vector potential $A_{1\alpha}$, we have

$$A_{1t_1} = A_1, \quad (2.11)$$

where A_1 is a constant, with the other remaining components being zero.

The exterior region $\mathcal{V}_2$ is described by the d -dimensional Reissner-Nordström spacetime, also known as Reissner-Nordström-Tangherlini spacetime, with the line element

$$ds_2^2 = -f(r) dt_2^2 + f(r)^{-1} dr^2 + r^2 d\Omega^2, \quad R_2 \leq r \leq \infty, \quad (2.12)$$

where Schwarzschild-like coordinates $x_2^\alpha = (t_2, r_2, \theta_2^A)$ have been adopted, $r \equiv r_2$ has been used, R_2 is the radius of the thin shell at the region $\mathcal{V}_2$, and the function $f(r)$ is

$$f(r) = 1 - \frac{2\mu m}{r^{d-3}} + \frac{qQ^2}{r^{2(d-3)}}, \quad (2.13)$$

with m being the ADM mass, Q being the total electric charge and

$$\mu = \frac{8\pi G}{(d-2)\Omega}, \quad \lambda = \frac{8\pi G}{(d-2)\Omega}. \quad (2.14)$$

It must be noted that the choice of ϵ and ϵ_q has been made to be unity, otherwise the quantity λ would be given by $\lambda = \frac{8\pi G\epsilon}{(d-2)\Omega\epsilon_q^2}$. Also, the definition of electric charge used here is $\frac{1}{2} \int F^{\alpha\beta} dS_{\alpha\beta} = \frac{\Omega Q}{\epsilon_q}$, where $dS_{\alpha\beta}$ is the surface element.

The exterior Reissner-Nordström metric has its gravitational radius, r_+ , and Cauchy radius, r_- , given by the parameters m and Q in the following way

$$r_\pm^{d-3} = \mu m \pm \sqrt{\mu^2 m^2 - \lambda Q^2}, \quad (2.15)$$

Since in this case, only the exterior region is Reissner-Nordström spacetime, the gravitational radius and the Cauchy radius are not horizon radii if the shell radius is larger than the gravitational radius. The extremal case is defined by the relation $r_+ = r_-$, which in terms of the mass and charge is $\mu m = \sqrt{\lambda} Q$. This last relation for the choice of $\epsilon = \epsilon_q = 1$ is $\sqrt{\mu} m = \sqrt{\mu} Q$. The area associated to the gravitational radius, A_+ , is an important quantity that is considered in the analysis, and it is given by

$$A_+ = \Omega r_+^{d-2}. \quad (2.16)$$

It is also helpful to invert Eq. (2.15), to obtain the mass and the electric charge in terms of the horizon and Cauchy radii as

$$m = \frac{1}{2\mu} (r_+^{d-3} + r_-^{d-3}), \quad Q = \frac{(r_+ r_-)^{\frac{d-3}{2}}}{\sqrt{\lambda}}. \quad (2.17)$$

With the choice of $\epsilon = \epsilon_q = 1$, the parameter λ can be replaced by μ . Nevertheless, the parameter λ is kept throughout the chapter whenever the coefficient is associated to the electric charge Q . The function $f(r)$ can then be rewritten in terms of r_+ and r_- as

$$f(r) = \left(1 - \left(\frac{r_+}{r}\right)^{d-3}\right) \left(1 - \left(\frac{r_-}{r}\right)^{d-3}\right). \quad (2.18)$$

For the exterior region as well, the vector potential that solves the vacuum Maxwell equations with the presence of a total electric charge is

$$A_{2t_2} = -\frac{Q}{(d-3)r^{d-3}}, \quad (2.19)$$

where the constant of integration has been set to zero, as one can always make a gauge choice, and the other remaining components vanish. The strength field tensor has a non-zero component $F_2^{t_2 r} = \frac{Q}{r^{d-2}}$, which can be understood as the electric field with respect to a stationary observer. Notice that $\epsilon_q = 1$ here, otherwise one would have $A_{2t_2} = -\frac{Q}{(d-3)\epsilon_q} r^{d-3}$ and $F_2^{t_2 r} = \frac{Q}{\epsilon_q r^{d-2}}$.

For the boundary hypersurface, describing the history of a thin shell, we assume spherical symmetry and so the induced metric h_{ab} or the line element $ds_\Sigma^2 = h_{ab} dy^a dy^b$ is given by

$$ds_\Sigma^2 = -d\tau^2 + R(\tau)^2 d\Omega^2, \quad (2.20)$$

where the coordinate system $y^a = (\tau, \theta^A)$ has been adopted, with the coordinate τ being the proper time of the shell, and $R(\tau)$ being the radius of the shell. The vector potential must be constant along the shell due to spherical symmetry, up to gauge transformations, as

$$A_{\Sigma\tau} = A_\Sigma, \quad (2.21)$$

with A_Σ being a constant and with the remaining components being zero.

With the two bulk regions and the hypersurface described, one now can proceed with the junction conditions. The pull-back of the metric in the interior region $\mathcal{V}_1$ evaluated at the hypersurface Σ is

$$ds_{1,\Sigma}^2 = (-\dot{t}_1^2 + \dot{R}_1^2) d\tau^2 + R_1(\tau)^2 d\Omega^2, \quad (2.22)$$

where the hypersurface Σ is described by $r = R_1(t_1)$, $R_1(t_1)$ being the radius of the shell in function of the time coordinate in $\mathcal{V}_1$, and where $\dot{} = \frac{d}{dt}$. The pull-back of the metric in the exterior region $\mathcal{V}_2$ evaluated at the hypersurface is

$$ds_{2,\Sigma}^2 = \left[-f(R_2(\tau))\dot{t}_2^2 + f(R_2(\tau))^{-1}\dot{R}_2^2 \right] d\tau^2 + R_2(\tau)^2 d\Omega^2, \quad (2.23)$$

here the hypersurface Σ is described by $r = R_2(t_2)$, $R_2(t_2)$ being the radius of the shell in function of the time coordinate in $\mathcal{V}_2$. The first junction condition, Eq. (2.4) now states that the induced metrics in Eqs. (2.22) and (2.23) must correspond to the same physical induced metric and additionally must correspond to Eq. (2.20). Since the three metrics are in the same coordinate system, the first junction condition yields

$$R_2(\tau) = R_1(\tau) = R(\tau), \quad (2.24)$$

$$-\dot{t}_1^2 + \dot{R}^2 = -f(R)\dot{t}_2^2 + f(R)^{-1}\dot{R}^2 = -1. \quad (2.25)$$

It must be noted that condition Eq. (2.24) motivates the usage of the same coordinate r for the interior and exterior region, as it was done apriori. The area of the shell can then be defined as

$$A = \Omega R^{d-2}. \quad (2.26)$$

Moving to the second junction condition, the normal covector must be specified for each region in order to compute the extrinsic curvature K_{ab} . For the interior, the normal covector can be deduced from the hypersurface equation $r = R_1(t_1)$ as

$$n_{1\alpha} dx^\alpha = \left(1 - \left(\frac{dR}{dt_1} \right)^2 \right)^{-\frac{1}{2}} \left(-\frac{dR}{t_1} dt_1 + dr \right). \quad (2.27)$$

It is useful to write the components of the normal covector in terms of the coordinate τ at the hypersurface. Using the first junction condition in Eq. (2.25), one has

$$\left(1 - \left(\frac{dR}{dt_1} \right)^2 \right)^{-\frac{1}{2}} \Big|_\Sigma = \sqrt{1 + \dot{R}^2}, \quad (2.28)$$

$$\frac{dR}{t_1} \Big|_\Sigma = \frac{\dot{R}}{\sqrt{1 + \dot{R}^2}}, \quad (2.29)$$

so that

$$n_{1\alpha}|_\Sigma = (-\dot{R}, \sqrt{1 + \dot{R}^2}, 0, 0). \quad (2.30)$$

For the exterior region, the normal covector is given by

$$n_{2\alpha} dx^\alpha = \sqrt{f(r_2)} \left(f(r_2)^2 - \left(\frac{dR}{dt_2} \right)^2 \right)^{-\frac{1}{2}} \left(-\frac{dR}{dt_2} dt_2 + dr \right). \quad (2.31)$$

Written in terms of the coordinate τ , and using

$$\sqrt{f(r_2)} \left(f(r_2)^2 - \left(\frac{dR}{dt_2} \right)^2 \right)^{-\frac{1}{2}} \Big|_\Sigma = \frac{\sqrt{f(R) + \dot{R}^2}}{f(R)}, \quad (2.32)$$

$$\frac{dR}{dt_2} \Big|_\Sigma = \frac{f(R)\dot{R}}{\sqrt{f(R) + \dot{R}^2}}, \quad (2.33)$$

the normal covector at the hypersurface is

$$n_{2\alpha}|_\Sigma = \left(-\dot{R}, \frac{\sqrt{f(R) + \dot{R}^2}}{f(R)}, 0, 0 \right). \quad (2.34)$$

The extrinsic curvature can now be calculated, giving for the interior region

$$K_1{}^\tau{}_\tau = \frac{\ddot{R}}{\sqrt{1 + \dot{R}^2}}, \quad K_1{}^{\theta^A}{}_{\theta^A} = \frac{\sqrt{1 + \dot{R}^2}}{R}, \quad (2.35)$$

and for the exterior region

$$K_2{}^\tau{}_\tau = \frac{\ddot{R} + \frac{\partial_R f(R)}{2}}{\sqrt{f(R) + \dot{R}^2}}, \quad K_2{}^{\theta^A}{}_{\theta^A} = \frac{\sqrt{f(R) + \dot{R}^2}}{R}, \quad (2.36)$$

where the indices A in this case are not to be summed over, The shell is assumed to be static and in equilibrium, meaning $\dot{R} = 0$ and $\ddot{R} = 0$, together with $u^\alpha = (1, 0, 0)$, the velocity of the shell. From the second junction condition, i.e. Eqs. (2.5) and (2.6), together with the expressions for the extrinsic curvature in Eqs. (2.35)-(2.36), the energy density and the pressure can be obtained in terms of the gravitational and Cauchy radii of the exterior region, as

$$\sigma = \frac{1 - k}{\mu \Omega R}, \quad (2.37)$$

$$p = \frac{1}{2\mu \Omega R^{2d-5}k} \frac{d-3}{d-2} \left[(1 - k)^2 R^{2(d-3)} - \lambda Q^2 \right], \quad (2.38)$$

where k is defined as the redshift function evaluated at the shell radius R , as

$$k = \sqrt{f(R)}. \quad (2.39)$$

As a reminder, the parameter λ could be renamed as μ in Eq. (2.38) due to the choice in Eq. (2.14). Knowing the energy density, the rest mass of the shell can be defined by

$$M = \Omega R^{d-2} \sigma = \frac{R^{d-3}}{\mu} (1 - k) , \quad (2.40)$$

This relation can be inverted to get the ADM mass in terms of the rest mass of the shell and the electric charge as

$$m = M - \frac{\mu M^2}{2R^{d-3}} + \frac{Q^2}{2R^{d-3}} , \quad (2.41)$$

where it was used $k(M, R, Q) = \sqrt{1 - \frac{2\mu m}{R^{d-3}} + \frac{\lambda Q^2}{R^{2(d-3)}}}$. The expression in Eq. (2.41) can be interpreted as the total energy of the self-gravitating shell being given by the rest mass plus the gravitational potential energy and the electric potential energy.

Written with generic ϵ and ϵ_q , Eq. (2.41) is $m = M - \frac{\mu M^2}{2R^{d-3}} + \frac{\epsilon Q^2}{2\epsilon_q^2 R^{d-3}}$.

With respect to the junction conditions of the Maxwell vector potential, the first junction condition in Eq. (2.7), which is $[A_\tau] = 0$, together with the expressions of the vector potential in Eqs. (2.11) and (2.19), determines the constant

$$A_i = -\frac{Q}{(d-3)R^{d-3}k} , \quad (2.42)$$

which can be written as $A_i = -\frac{Q}{(d-3)\epsilon_q R^{d-3}k}$ for generic ϵ and ϵ_q . The relevant junction condition for the strength field tensor in Eq. (2.8) is $[F_\alpha] = j_\alpha$, which upon using Eq. (2.9) becomes $-(F_2)_{tr} k \dot{t}_2 = \zeta \sigma_e$, with $\zeta = \frac{\Omega}{\epsilon_q}$, see Eq. (2.9), $\epsilon_q = 1$ and $\zeta = \Omega$. The junction condition implies, with Eqs. (2.11) and (2.19), that

$$Q = R^{d-2} \sigma_e . \quad (2.43)$$

This relates the total electric charge with its corresponding charge density.

2.3 THERMODYNAMICS OF THE ELECTRICALLY CHARGED THIN SHELL FROM THE FIRST LAW

2.3.1 The entropy of the shell from the first law of thermodynamics

With the electrically charged matter thin shell spacetime constructed, one way to study its thermodynamics is by imposing the first law of thermodynamics. By matching the internal energy of the shell to its rest mass, the first law for the shell can be written in a way to determine the differential of the shell entropy S as

$$dS = \beta dM + \beta p dA - \beta \Phi dQ , \quad (2.44)$$

where

$$\beta = \frac{1}{T} , \quad (2.45)$$

is the inverse of the local temperature, T , of the shell, M is the rest mass of the shell determined in the previous section, p is the pressure of the shell determined in the previous section, A is the area of the shell, Q is the electric charge of the shell and Φ is the thermodynamic electric potential. The main objective is to compute the entropy of the shell by integrating Eq. (2.44). In order to proceed, one must provide an equation of state for the three quantities

$$\beta = \beta(M, A, Q) , \quad (2.46)$$

$$p = p(M, A, Q) , \quad (2.47)$$

$$\Phi = \Phi(M, A, Q) , \quad (2.48)$$

in function of the rest mass M , the area of the shell A and the electric charge Q . We now choose the pressure equation of state as the one in Eq. (2.38), stemming from the junction conditions, with the function k being written in terms of M , A and Q , and additionally R being written in terms of A . This is referred to the fundamental pressure equation of state coming from the Einstein equations, and we shall see its consequences. The choice of the remaining equations of state $\beta(M, A, Q)$ and $\Phi(M, A, Q)$ is not completely free as the functions β and Φ must satisfy integrability conditions. These conditions must be first analyzed before the equations of state are chosen.

The integrability conditions are related to the fact that S is a function or scalar, depending on the thermodynamic parameters (M, A, Q) . So, its differential must be exact by definition, which is translated to the Hessian matrix of S being a symmetric matrix. From the first law in Eq. (2.44), the first derivatives of the entropy are

$$\left(\frac{\partial S}{\partial M} \right)_{A,Q} = \beta , \quad (2.49)$$

$$\left(\frac{\partial S}{\partial A} \right)_{M,Q} = \beta p ,$$

$$\left(\frac{\partial S}{\partial Q} \right)_{M,A} = -\beta \Phi , \quad (2.50)$$

and so the integrability conditions read

$$\begin{aligned} \left(\frac{\partial \beta}{\partial A} \right)_{M,Q} &= \left(\frac{\partial \beta p}{\partial M} \right)_{A,Q} , \\ \left(\frac{\partial \beta}{\partial Q} \right)_{M,A} &= - \left(\frac{\partial \beta \Phi}{\partial M} \right)_{A,Q} , \\ \left(\frac{\partial \beta p}{\partial Q} \right)_{M,A} &= - \left(\frac{\partial \beta \Phi}{\partial A} \right)_{M,Q} . \end{aligned} \quad (2.51)$$

The idea is to use the fundamental pressure equation of state to determine the restrictions to the expression of the inverse temperature and to the thermodynamic electric potential. In order to simplify this task, it is useful to proceed with a parameter transformation from (M, A, Q) or (M, R, Q) to the parameters (r_+, r_-, R) ,

which can be done using Eq. (2.17) and Eq. (2.26). For completeness, the redshift function in these parameters is

$$k(r_+, r_-, R) = \sqrt{\left(1 - \left(\frac{r_+}{R}\right)^{d-3}\right) \left(1 - \left(\frac{r_-}{R}\right)^{d-3}\right)}. \quad (2.52)$$

By using the chain rule, one can transform the differential of the entropy to be dependent on the parameters (r_+, r_-, R) with the derivatives of the entropy being

$$\left(\frac{\partial S}{\partial r_{\pm}}\right)_{r_{\mp}, R} = \beta \left(\frac{\partial M}{\partial r_{\pm}}\right)_{r_{\mp}, R} - \beta \Phi \left(\frac{\partial Q}{\partial r_{\pm}}\right)_{r_{\mp}, R}. \quad (2.53)$$

and

$$\left(\frac{\partial S}{\partial R}\right)_{r_+, r_-} = \beta \left(\frac{\partial M}{\partial R}\right)_{r_+, r_-} + \beta p \left(\frac{\partial A}{\partial R}\right)_{r_+, r_-} = 0, \quad (2.54)$$

where it was used

$$\left(\frac{\partial M}{\partial R}\right)_{r_+, r_-} = -p \left(\frac{\partial A}{\partial R}\right)_{r_+, r_-}, \quad (2.55)$$

from Eqs. (2.38), (2.40) and (2.17). As it can be seen from Eq. (2.54), the consequence of having the fundamental pressure equation of state is that the entropy, supposedly a function $S = S(r_+, r_-, R)$, does not depend on the radius of the shell R , it only depends on r_+ and r_- , i.e.

$$S = S(r_+, r_-). \quad (2.56)$$

The integrability conditions can be used to further restrict the remaining derivatives of the entropy in the parameters (r_+, r_-, R) , by finding the expressions for β and Φ .

For β , one can compute the derivative of β using the chain rule as

$$\left(\frac{\partial \beta}{\partial R}\right)_{r_+, r_-} = \left(\frac{\partial \beta}{\partial M}\right)_{A, Q} \left(\frac{\partial M}{\partial R}\right)_{r_+, r_-} + \left(\frac{\partial \beta}{\partial A}\right)_{M, Q} \left(\frac{\partial A}{\partial R}\right)_{r_+, r_-}. \quad (2.57)$$

Using the first integrability condition in Eq. (2.51), together with

$$\left(\frac{\partial p}{\partial M}\right)_{A, Q} = \frac{1}{(d-2)\Omega k R^{d-3}} \left(\frac{\partial k}{\partial R}\right)_{r_+, r_-}, \quad (2.58)$$

one obtains a differential equation for β

$$\left(\frac{\partial \beta}{\partial R}\right)_{r_+, r_-} = \frac{\beta}{k} \left(\frac{\partial k}{\partial R}\right)_{r_+, r_-}, \quad (2.59)$$

which can be integrated to give

$$\beta(r_+, r_-, R) = b(r_+, r_-)k, \quad (2.60)$$

where $b(r_+, r_-)$ is a reduced equation of state and only depends on the nature of matter in the shell. The meaning of $b(r_+, r_-)$ is made clear when the limit of $R \rightarrow +\infty$ is made in the expression of β , i.e. $\beta(r_+, r_-, +\infty) = b(r_+, r_-)$, where we emphasize that k is given by Eq. (2.52) and becomes unity in the limit $R \rightarrow +\infty$. Therefore, $b(r_+, r_-)$ is the inverse temperature of the shell measured by a stationary observer at infinity, while expression Eq. (2.60) describes the Tolman's formula for the temperature. We must stress that this is indeed a consequence of the choice of the fundamental pressure equation of state.

For Φ , one can use the chain rule once again to obtain the derivative

$$\left(\frac{\partial \Phi}{\partial R}\right)_{r_+, r_-} = \left(\frac{\partial A}{\partial R}\right)_{r_+, r_-} \left(\frac{\partial \Phi}{\partial A}\right)_{Q, M} - p \left(\frac{\partial A}{\partial R}\right)_{r_+, r_-} \left(\frac{\partial \Phi}{\partial M}\right)_{A, Q}, \quad (2.61)$$

using Eq. (2.58). The integrability conditions in Eq. (2.51) can be rearranged to transform the last equation into

$$\left(\frac{\partial \Phi}{\partial R}\right)_{r_+, r_-} = - \left(\frac{\partial A}{\partial R}\right)_{r_+, r_-} \left(\frac{\partial p}{\partial Q}\right)_{M, A} - \Phi \left(\frac{\partial A}{\partial R}\right)_{r_+, r_-} \left(\frac{\partial p}{\partial M}\right)_{A, Q}. \quad (2.62)$$

Using the fundamental pressure equation of state, with its derivatives

$$\left(\frac{\partial p}{\partial M}\right)_{A, Q} = \frac{1}{(d-2)\Omega k R^{(d-3)}} \left(\frac{\partial k}{\partial R}\right)_{r_+, r_-}, \quad (2.63)$$

$$\left(\frac{\partial p}{\partial Q}\right)_{M, A} = - \frac{Q(d-3)}{(d-2)\Omega k R^{2d-5}}, \quad (2.64)$$

the differential equation for the electric potential Φ is obtained as

$$\left(\frac{\partial k \Phi}{\partial R}\right)_{r_+, r_-} = \frac{(d-3)Q}{R^{d-2}}, \quad (2.65)$$

which can be readily integrated into

$$\Phi(r_+, r_-, R) = \frac{Q}{k} \left[c(r_+, r_-) - \frac{1}{R^{d-3}} \right], \quad (2.66)$$

where $c(r_+, r_-)$ is a reduced equation of state like $b(r_+, r_-)$, and it depends on the nature of matter in the shell. Performing the limit $R \rightarrow +\infty$, one can see that $c(r_+, r_-) = \frac{\Phi(r_+, r_-, \infty)}{Q}$ and so $c(r_+, r_-)$ is the electric potential per charge measured by a stationary observer at infinity.

With the expressions for β in Eq. (2.60) and for Φ in Eq. (2.66), one can obtain the derivatives of the entropy in Eq. (2.53) in terms of the reduced equations of state. The differential of the entropy dS in the parameters (r_+, r_-, R) becomes

$$\begin{aligned} dS = & \frac{(d-3)b(r_+, r_-)}{2\mu} \left[\left(1 - r_-^{d-3}c(r_+, r_-)\right) r_+^{d-4} dr_+ \right. \\ & \left. + \left(1 - r_+^{d-3}c(r_+, r_-)\right) r_-^{d-4} dr_- \right]. \end{aligned} \quad (2.67)$$

It must be noted that there are still integrability conditions that must be satisfied between $b(r_+, r_-)$ and $c(r_+, r_-)$ to ensure that the differential is exact, yielding

$$\begin{aligned} & \frac{\partial b}{\partial r_-} (1 - c r_-^{d-3}) r_+^{d-4} - \frac{\partial b}{\partial r_+} (1 - c r_+^{d-3}) r_-^{d-4} \\ &= \frac{\partial c}{\partial r_-} b r_-^{d-3} r_+^{d-4} - \frac{\partial c}{\partial r_+} b r_+^{d-3} r_-^{d-4}. \end{aligned} \quad (2.68)$$

The consequence of having the fundamental pressure equation of state is that the entropy $S(r_+, r_-)$ depends on two reduced equations of state $b(r_+, r_-)$ and $c(r_+, r_-)$, related by Eq. (2.68). These two functions cannot be further specified by general relativity or the first law of thermodynamics, and so their expression must be chosen depending on the class of matter that it is of interest.

It may also be interesting to rewrite the entropy in terms of the ADM mass, i.e. $S(m, Q)$, with its differential being given by $dS = \frac{b}{2\mu} d(r_+^{d-3} + r_-^{d-3}) - \frac{b}{2\mu} c d[(r_+ r_-)^{d-3}]$, or in a cleaner way, $dS = b dm - b \phi dQ$, where $\phi = Q c$. In this case, the equations of state are a function of m and Q as $b = b(m, Q)$ and $c(m, Q)$. Notice that we are using the convention $\lambda = \mu$, and we write Q here as the modulus of the electric charge. This further stresses the meaning of b and c as the inverse temperature and the electric potential per charge at infinity.

2.3.2 The entropy of the shell for a specific choice of equations of state

We are now going to choose the two reduced equations of state for $b(r_+, r_-)$ and $c(r_+, r_-)$. The choice for the reduced equation of state for the inverse temperature of the shell is

$$b(r_+, r_-) = \frac{a\gamma\Omega^{a-1}}{d-3} \frac{r_+^{a(d-2)}}{r_+^{d-3} - r_-^{d-3}}, \quad (2.69)$$

where a is a free exponent and γ is a free parameter. The equation of state is only valid for $r_- \leq r_+$, with r_+ and r_- assume real values from Eq. (2.15). The shell for this choice of equation of state can be either undercharged or extremely charge but not overcharged.

From the integrability condition Eq. (2.68) with the choice of the reduced equation of state in Eq. (2.69), one possible solution that we choose for $c(r_+, r_-)$ is

$$c(r_+, r_-) = \frac{1}{r_+^{d-3}}, \quad (2.70)$$

yielding the typical Reissner-Nordström equation of state for the electric potential.

We should give a comment regarding the constants appearing in the equation of state $b(r_+, r_-)$ in Eq. (2.69). One has two parameters a and γ . The power law exponent a is adimensional and it is the most relevant in the analysis. The constant γ should be determined by the features of matter, including quantum effects, and so it is expected that depends on the Planck constant and the Boltzmann constant, which we set to unity here. Moreover, γ must have the units of length to the power

$(d-2)(1-a)$. Regarding the equation of state for the electric potential, it does not depend on any new free parameter. We also note that both equations of state depend on the dimensions d , and one can treat d as a free parameter, as long as it is a finite positive integer and $d > 3$, which is case of interest here. There may be some interest in performing the limit of infinite d , but that depends on the way the limit is taken. We do not pursue that limit here.

With the reduced equations of state Eqs. (2.69) and (2.70), the differential of the entropy given in Eq. (2.67) can be integrated, yielding

$$S = \frac{\gamma}{16\pi G} A_+^a, \quad (2.71)$$

and so the entropy of the shell, being dimensionless in our convention, is proportional to a power of the gravitational area A_+ . Indeed, the specific choice of equations of state make the entropy S only dependent on the gravitational radius r_+ only as $S = S(r_+)$. It is also convenient to restrict the parameter a to $a > 0$, so that the entropy does not diverge when $r_+ = 0$. An additional note is that this is still the entropy of the shell and r_+ is the gravitational radius of the shell and not the horizon radius of a black hole.

We now explain the motivation for the choice of the reduced equations of state in Eqs. (2.69) and (2.70). First, power laws in thermodynamics and statistical mechanics emerge ubiquitously, therefore it is natural for $b(r_+, r_-)$ and $c(r_+, r_-)$ to be described by power laws in r_+ and r_- , these parameters being dependent on the rest mass M and the charge Q . Second, it is of interest to assign black hole like behaviour to the shell, so that it is possible to perform the black hole limit $R = r_+$ and study its implications. Moreover, if one sets $a = 1$, the inverse temperature has the same dependence as the Hawking temperature of a black hole, while the electric potential is identical as the one from a black hole. Consequently, the entropy of the shell in the case $a = 1$ is $S = \frac{\gamma}{16\pi G} A_+$, which has the same functional dependence as the Bekenstein-Hawking black hole entropy. However, one indeed could choose other power laws for the equation of state and they could possibly have the same black hole features for appropriate choice of the exponents. Another possibility for the equations of state that could be worth exploring and that generalizes the choice in [95] for higher dimensions is choosing power laws in the ADM mass and the charge, i.e. $b(r_+, r_-) = a(r_+^{d-3} + r_-^{d-3})^\alpha$ and $c(r_+, r_-) = \frac{f(r_+, r_-)}{(r_+^{d-3} + r_-^{d-3})^\alpha}$, which obey the integrability conditions. However, we did not explored such choices here.

The expression for the entropy in Eq. (2.71) brings an important point, the entropy is the same if r_+ is fixed for any radius R of the shell. If one imagines the process of increasing the radius R of the shell with fixed r_+ and fixed r_- , this implies that electric charge Q and the entropy are constant, while the area of the shell increases and the internal energy of the shell decreases. In the limit $R \rightarrow +\infty$, the internal energy assumes the value of the ADM mass. And so this is an isentropic process.

To end this subsection, one can also obtain an integrated version of the first law of thermodynamics to the shell, knowing the entropy with Eq. (2.71), the internal energy or rest mass M with Eq. (2.40), the electric charge with Eq. (2.17) and the

area of the shell with Eq. (2.26). The energy of the shell can be written in terms of the entropy, area and the charge of the shell as

$$M(S, A, Q) = \frac{1}{\mu} \left(\frac{A}{\Omega} \right)^{\frac{d-3}{d-2}} \left[1 - \sqrt{\left(1 - \left(\frac{16\pi GS}{\gamma A^a} \right)^{\frac{d-3}{a(d-2)}} \right) \left(1 - \frac{qQ^2 \Omega^{\frac{d-3}{d-2}}}{\left(\frac{16\pi GSA^a}{\gamma} \right)^{\frac{d-3}{a(d-2)}}} \right)} \right]. \quad (2.72)$$

The function $M(S, A, Q)$ has the scaling property

$$M\left(\nu S^{\frac{1}{a}}, \nu A, \nu Q^{\frac{d-2}{d-3}}\right) = \nu^{\frac{d-3}{d-2}} M\left(S^{\frac{1}{a}}, A, Q^{\frac{d-2}{d-3}}\right), \quad (2.73)$$

and since the first law of thermodynamics states $dM = TdS - pdA + \Phi dQ$, one can use the Euler relation theorem for homogeneous functions to get

$$\frac{d-3}{d-2} M = aTS - pA + \frac{d-3}{d-2} \Phi Q. \quad (2.74)$$

This shows that the choice of equations of state alter the homogeneity of the variables compared to what is typical in homogeneous thermodynamic systems.

2.3.3 The case of a shell with black hole features and the black hole limit

In this subsection, we focus on the shell with black hole features. This can be done by setting $a = 1$ in Eqs. (2.69)-(2.71). The resulting temperature of the shell for this case $T_0 = \frac{1}{b|_{a=0}}$ is $T_0 = \frac{d-3}{\gamma} \frac{r_+^{d-3} - r_-^{d-3}}{r_+^{d-2}}$, which translates into the Hawking temperature of the shell if additionally one considers $\gamma = 4\pi$. The reduced equation of state for the electric potential is still described by $c(r_+, r_-) = \frac{1}{r_+^{d-3}}$, which corresponds to the black hole electric potential. Therefore, the shell with black hole features has the reduced equations of state

$$b_+(r_+, r_-) = \frac{4\pi}{d-3} \frac{r_+^{d-2}}{r_+^{d-3} - r_-^{d-3}}, \quad c_+(r_+, r_-) = \frac{1}{r_+^{d-3}}, \quad (2.75)$$

where the subscript $+$ indicates thermodynamic quantities characteristic of black holes. The entropy of the shell becomes

$$S_+ = \frac{1}{4} \frac{A_+}{A_p}, \quad (2.76)$$

with $A_p = l_p^{d-2} = G$ being the Planck area. This means the shell with black hole features, i.e. $a = 1$ and $\gamma = 4\pi$, has precisely the Bekenstein-Hawking entropy. While the spacetime does not describe a black hole spacetime but rather a shell spacetime, we have that the shell mimics thermodynamically the black hole spacetime with horizon radius equal to the gravitational radius of the shell, r_+ . Even more, this is independent of the radius of the shell R , with the condition $R \geq r_+$, since the entropy remains the same as the Bekenstein-Hawking entropy.

In order to get a shell which not only has black hole features but it is almost a black hole, meaning a quasiblack hole, we must take the limit of the shell radius $R \rightarrow r_+$. This can only be done through a sequence of quasistatic thermodynamic equilibrium configurations if the temperature shell is precisely the Hawking temperature, $T_+(r_+, r_-) = \frac{d-3}{4\pi} \frac{r_+^{d-3} - r_-^{d-3}}{r_+^{d-2}}$, with the entropy of the shell being the Bekenstein-Hawking entropy, in Eq. (2.76). This is to avoid divergences from the backreaction of the matter quantum fluctuations. When the shell is placed at its own gravitational radius, the shell spacetime describes a quasiblack hole state, with the gravitational radius being a quasihorizon radius. The pressure of the shell is very large as one approaches the radius of the shell to its gravitational radius, becoming divergent the limit, see Eq. (2.38). In some sense, this means the shell must have all its degrees of freedom excited in this limit to maintain equilibrium and to make the entropy maximal. As such, the limit $R \rightarrow r_+$ must be envisioned as R being very close to r_+ but not at exactly r_+ . While the shell formalism indeed provides the black hole features in the appropriate limit, the quasiblack hole formalism, having some correspondence with the membrane paradigm formalism, can deal with generic matter systems on the verge of becoming a black hole and provides also the thermodynamic properties of black holes [104–106].

The extremal case of the charged shell deserves a complete study but we give some highlights here in connection to the extremal Reissner-Nordström black hole in d dimensions. The extremal Reissner-Nordström spacetime satisfies the relation $r_+ = r_-$. For a reduced equation of state of the form given in Eq. (2.69), we have that the extremal charged shell case has zero temperature, whereas the reduced electric potential is still given by Eq. (2.70), thus both are well-defined in the extremal case. The expression for the entropy of the shell, however, requires some subtleties and depends on the order of the limits performed. The first case is when we start from a nonextremal shell with $R > r_+$ and we take the limit $r_+ = r_-$. The entropy of the shell is then Eq. (2.71), by continuity. The second case is when we consider from the start an extremal shell. Then, the entropy is some function of the gravitational radius $S(A_+)$, and we are free to choose it, see [98]. Proceeding with the black hole limit in the extremal case, the first case gives the Bekenstein-Hawking entropy $S_+ = \frac{1}{4} \frac{A_+}{A_p}$ and the second case gives an entropy determined by an unspecified function of the gravitational radius. There is an additional third case in the black hole limit, which is when we start from an undercharged shell with $R > r_+$ and then we bring simultaneously the shell to the brink of extremality while approaching it to its gravitational radius [99]. But in the third case, the entropy of the shell also becomes the Bekenstein-Hawking entropy. Connecting these cases to the properties of the black hole, as one makes the black hole limit of the shell, the entropy of the extremal black hole depends on its past, or better, on how it was formed, see [105].

Another property that we can obtain from the black hole limit of the charged thin shell is the Smarr formula for the charged black hole, starting from the integrated first law formula, Eq. (2.74). Setting $a = 1$ in the reduced equations of state, the integrated first law formula is $\frac{d-3}{d-2} kM = T_+ S_+ - kpA + \frac{d-3}{d-2} k\Phi_+ Q$, where the factor

k was multiplied and Φ_+ is Φ defined in Eq. (2.66) with black hole characteristics, i.e., $\Phi_+(r_+, r_-, R) = Q \frac{r_+^{-(d-3)} - R^{-(d-3)}}{k}$. We now must take the black hole limit with care, as $R = r_+$ means that $k = 0$, and so $kM = 0$ and $k\Phi_+ = 0$. The remaining equation is $T_+S_+ - kpA = 0$. Now, $kpA = \frac{1}{2\mu} \frac{d-3}{d-2} (r_+^{d-3} + r_-^{d-3}) - \frac{1}{\mu} \frac{d-3}{d-2} r_-^{d-3}$ in the black hole limit. This can be simplified using Eq. (2.17), as the first term becomes the ADM mass $\frac{1}{2\mu} (r_+^{d-3} + r_-^{d-3}) = m$, while second term becomes $\frac{d-3}{d-2} \phi_+ Q$, where the black hole potential ϕ_+ is naturally defined as $\phi_+ = Qr_+^{-(d-3)}$. We then recover the Smarr formula in the black hole limit as

$$m = \frac{d-2}{d-3} T_+ S_+ + \phi_+ Q. \quad (2.77)$$

For the extremal case, $r_+ = r_-$, the Smarr formula in Eq. (2.77) can also be applied, with $T_+ = 0$ and $\phi_+ Q = \frac{Q}{\sqrt{\mu}}$, where Eq. (2.17) has been used, and the equality $\mu = \lambda$ in our convention of units has been applied. For the case $d = 4$, the original Smarr formula is recovered, together with the extremal case.

2.4 INTRINSIC THERMODYNAMIC STABILITY FOR THE CHARGED THIN SHELL

2.4.1 Stability conditions for the charged thin shell

With the thermodynamic equilibrium of the shell described, we must analyze the intrinsic thermodynamic stability of the shell. We perform the analysis according to an extended formalism in Callen's book[107].

If we consider a system in thermodynamic equilibrium, the system is always susceptible to fluctuations. Let's consider an isolated system with entropy S , which can always be split into two equal subsystems. For perturbations within the system, there can be exchanges in the thermodynamic variables (M, A, Q) between the two subsystems. These fluctuations can lead the system slightly away from the initial equilibrium and the system's entropy becomes $S + \Delta S$, which can be assumed to be described by the sum of the entropies of the two subsystems. Using the second law of thermodynamics, systems tend always to be at the configuration that maximizes the entropy and so the system returns to the initial equilibrium if $\Delta S < 0$, i.e. the system is stable. Otherwise, the system departs from the initial equilibrium configuration, developing inhomogeneities, and so the system is unstable. Thus, for small fluctuations, the conditions of intrinsic stability are such that $d\bar{S} = 0$ and $d^2\bar{S} < 0$. Note that these conditions are applied to a generalized entropy $\bar{S}$, which covers configurations not in equilibrium. An example of such generalized entropy is precisely the one of [107], where $2\bar{S} = S(M + \delta M, A, Q) + S(M - \delta M, A, Q)$ for small mass fluctuations only, with $S(M, A, Q)$ being precisely the entropy of the configuration in equilibrium, δM being the variable establishing the non-equilibrium configuration. The condition $d\bar{S} = 0$ is satisfied if $\delta M = 0$, indeed $S(M, A, Q)$ is the entropy of the configuration in equilibrium. The condition $d^2\bar{S} < 0$ evaluated at $\delta M = 0$ translates into the condition that the hessian of $S(M, A, Q)$

must be negative definite. Note that for the case of the shell, we must calculate the second derivatives in order to the parameters (M, A, Q) and not to the parameters (r_+, r_-, R) . First, they are not equivalent as hessians of scalars are not tensors. Second, the independent thermodynamic parameters that can be directly exchanged by the subsystems are the quantities (M, A, Q) .

The stability conditions can be written in terms of the hessian components of the entropy, i.e. the second derivatives $S_{h_i h_j} = \frac{\partial^2 S}{\partial h_i \partial h_j}$, where $h_1 = M$, $h_2 = A$ and $h_3 = Q$. The negative definite condition is equivalent to the condition that the eigenvalues of the hessian are negative. Since the hessian is a 3×3 matrix, the expression of the eigenvalues is not trivial but one can get the sufficient conditions through the Sylvester's criterion by looking at the leading principal minors of the matrix, which are related to the pivots when one does Gauss elimination in the hessian to have a matrix in row-echelon form. The full conditions are

$$\begin{aligned}
S_{MM} &\leq 0, \quad S_{AA} \leq 0, \quad S_{QQ} \leq 0, \\
S_{MM}S_{AA} - S_{MA}^2 &\geq 0, \\
S_{MM}S_{QQ} - S_{MQ}^2 &\geq 0, \\
S_{QQ}S_{AA} - S_{QA}^2 &\geq 0, \\
(S_{MM}S_{AQ} - S_{MA}S_{MQ})^2 - (S_{AA}S_{MM} - S_{AM}^2)(S_{QQ}S_{MM} - S_{QM}^2) &\leq 0, \text{ if } S_{MM} \neq 0, \\
-S_{MA}^2S_{QQ} + 2S_{MQ}S_{MA}S_{AQ} - S_{AA}S_{MQ}^2 &\leq 0, \text{ if } S_{MM} = 0,
\end{aligned} \tag{2.78}$$

where the conditions were extended to the marginal case, seminegative definiteness. We must make a correction to [10] here. While the negative definiteness follows sufficiently from only the leading principal minors or the pivots, the condition of semi-negative definiteness does not follow by simply including the equal case, with the failing cases being when one has vanishing leading principal minors. One instead must look at all the principal minors of the matrix. It turns out that for the case of the shell, we verified that this does not make any difference and the analysis of [10] follows.

For convenience, we present the entropy and its first derivatives here for the chosen equations of state. The entropy of the shell is

$$S(M, A, Q) = \frac{\gamma}{16\pi G} A_+^a, \tag{2.79}$$

where $A_+ = \Omega r_+^{d-2}$ and r_+ and r_- are functions of (M, A, Q) from Eqs. (2.15), (2.41), and (2.26). From Eq. (2.60) and the specific choice of the reduced equation of state, the inverse temperature is given by

$$\beta(M, A, Q) = \frac{a\gamma\Omega^{a-1}}{d-3} \frac{r_+^{a(d-2)}}{r_+^{d-3} - r_-^{d-3}} k. \tag{2.80}$$

The pressure is given by the fundamental pressure equation of state as

$$p(M, A, Q) = \frac{1}{2\mu\Omega} \frac{d-3}{d-2} \left[(1-k)^2 R^{2(d-3)} - \lambda Q^2 \right] \frac{1}{R^{2d-5}k}, \tag{2.81}$$

Finally, the electric potential in Eq. (2.66) with the choice of the reduced equation of state in Eq. (2.70) is given by

$$\Phi(M, A, Q) = Q \left(\frac{1}{r_+^{d-3}} - \frac{1}{R^{d-3}} \right) \frac{1}{k}. \quad (2.82)$$

The first derivatives of the entropy follow easily from the first law given in Eq. (2.44) together with Eqs. (2.80)-(2.82), yielding

$$\begin{aligned} S_M &= \frac{a\gamma\Omega^{a-1}}{d-3} \frac{r_+^{a(d-2)}}{r_+^{d-3} - r_-^{d-3}} k, \\ S_A &= \frac{a\gamma\Omega^{a-2} r_+^{a(d-2)} \left[(1-k)^2 R^{2(d-3)} - \lambda Q^2 \right]}{2\mu(d-2) R^{2d-5} (r_+^{d-3} - r_-^{d-3})}, \\ S_Q &= -\frac{a\gamma\Omega^{a-1} Q}{(d-3)} \left(\frac{r_+^{3-d} - R^{3-d}}{r_+^{d-3} - r_-^{d-3}} \right) r_+^{a(d-2)}. \end{aligned} \quad (2.83)$$

To compute the second derivatives of the entropy, it is useful to present the derivatives of $r_{\pm}$ with respect to the thermodynamic variables as

$$\frac{\partial r_{\pm}}{\partial M} = \pm 2\mu \frac{r_{\pm} k}{(d-3)(r_+^{d-3} - r_-^{d-3})}, \quad (2.84)$$

$$\frac{\partial r_{\pm}}{\partial R} = \pm \mu \frac{r_{\pm}}{r_+^{d-3} - r_-^{d-3}} \frac{\mu M^2 - Q^2}{R^{d-2}}, \quad (2.85)$$

$$\frac{\partial r_{\pm}}{\partial Q} = \mp \frac{2\lambda Q r_{\pm} (r_{\pm}^{3-d} - R^{3-d})}{(d-3)(r_+^{d-3} - r_-^{d-3})}. \quad (2.86)$$

The components of the hessian of the entropy are

$$\begin{aligned} S_{MM} &= \frac{a\gamma\Omega^{a-2} 8\pi G r_+^{a(d-2)}}{(d-3)(d-2)(r_+^{d-3} - r_-^{d-3}) R^{d-3}} S_1, \\ S_{AA} &= \frac{a\gamma\Omega^{a-3} r_+^{a(d-2)}}{2\mu(d-2)^2 (r_+^{d-3} - r_-^{d-3}) R^{d-1}} S_2, \\ S_{QQ} &= \frac{a\gamma\Omega^{a-1} r_+^{a(d-2)} (1-x)}{(d-3)(r_+^{d-3} - r_-^{d-3}) r_+^{d-3}} S_3, \\ S_{MA} &= \frac{a\gamma\Omega^{a-2} r_+^{a(d-2)}}{(d-2)(r_+^{d-3} - r_-^{d-3}) R^{d-2}} S_{12}, \\ S_{MQ} &= -\frac{2\mu a\gamma\Omega^{a-1} r_+^{a(d-2)} Q k}{(d-3)^2 (r_+^{d-3} - r_-^{d-3}) r_+^{2d-6}} S_{13}, \\ S_{AQ} &= -\frac{a\gamma\Omega^{a-2} r_+^{a(d-2)} Q}{(d-2)(r_+^{d-3} - r_-^{d-3}) r_+^{d-3} R^{d-2}} S_{23}, \end{aligned} \quad (2.87)$$

with

$$\begin{aligned}
S_1 &= \frac{2k^2\mathcal{G}}{(d-3)x} - 1, \quad S_2 = \mathcal{F} \left[\frac{\mathcal{F}\mathcal{G}}{x} - 2d + 5 \right], \\
S_3 &= -1 + \frac{2y}{d-3} \left[\mathcal{G}(1-x) - \frac{2(d-3)}{1-y} \right], \\
S_{12} &= 1 - k + \frac{k\mathcal{G}}{x(d-3)}\mathcal{F}, \quad S_{13} = \mathcal{G}(1-x) - \frac{(d-3)}{1-y}, \\
S_{23} &= x + \frac{\mathcal{F}}{x(d-3)} \left[\mathcal{G}(1-x) - \frac{(d-3)}{1-y} \right],
\end{aligned} \tag{2.88}$$

The auxiliary functions $\mathcal{G}$, $\mathcal{F}$ and k are given by

$$\begin{aligned}
\mathcal{G} &= \frac{1}{1-y} \left[a(d-2) - (d-3) \frac{1+y}{1-y} \right], \quad \mathcal{F} = 2 - 2k - x(1-y), \\
k &= \sqrt{(1-x)(1-xy)},
\end{aligned} \tag{2.89}$$

and the parameters x and y are defined as

$$x = \frac{r_+^{d-3}}{R^{d-3}}, \quad y = \frac{r_-^{d-3}}{r_+^{d-3}}. \tag{2.90}$$

Note that the definition of k is the same as above, but given in terms of x and y .

The set of inequalities in Eq. (2.78) with the entropy equation given in Eq. (2.79) together with the equations of state given in Eqs. (2.80)-(2.82) can be written as conditions in terms of the functions given in Eq. (2.88). The conditions restrict the parameter space described by the points (d, a, x, y) for stable configurations. We constrain the parameter space to the region

$$d \geq 4, \quad a > 0, \quad 0 < x < 1, \quad 0 < y < 1. \tag{2.91}$$

We constrain the dimension d as $d \geq 4$, since for lower d there is no proper Reissner-Nordström solution. We constrain the parameter a as $a > 0$, because in the no black hole limit, $A_+ = 0$, and the entropy expression cannot diverge, see Eq. (2.79). We constrain also the parameter x as $0 < x < 1$, because the shell has to be in the limits between no shell, $x = 0$, and the black hole state, $x = 1$. Finally, we constrain the parameter y as $0 < y < 1$, due to the validity of the equations of state. Overcharged shell, with $y > 1$, are not treated here since the equations of state, Eqs. (2.80)-(2.82), do not apply to overcharged shells.

In what follows, we present the analysis of the stability conditions for each possible combination of fluctuations, accompanied by plots to further understand these conditions.

2.4.2 Stability of the shell for mass fluctuations only

Considering a shell with only mass fluctuations, the stability condition is given by $S_{MM} \leq 0$, see Eq. (2.78). For the chosen equations of state, and with the help of Eq. (2.87), the condition $S_{MM} \leq 0$ can be written as

$$S_1 \leq 0. \quad (2.92)$$

This inequality can be simplified using Eq. (2.88) to the condition

$$a \leq \frac{x(d-3)(1-y)}{2(d-2)k^2} + \frac{(d-3)}{(d-2)} \frac{(1+y)}{(1-y)}, \quad (2.93)$$

where Eqs. (2.89) and (2.90) were used. We should give some comments about this condition. The right-hand side of Eq. (2.90) tends to infinity at the points $x = 1$ or $y = 1$. Moreover, it has its minimum value at $(x, y) = (0, 0)$, corresponding to $a = \frac{d-3}{d-2}$. We plot the right-hand side for $d = 5$ in Figs. 2.1(a) and 2.1(b), where the region below the curves is stable. The curves increase overall with d . The case $a = 1$ has some interest as it represents a shell with thermodynamic black hole features, we plot this case in Figs. 2.2(a) and 2.2(b). For the uncharged case $y = 0$, there is thermodynamic stability for $\frac{2}{d-1} < x < 1$, in agreement with [104]. Increasing the value of y also increases the range of x for thermodynamic stable configurations, meaning that a shell with more electric charge may have higher radius R and remain stable. Thermodynamic stability is guaranteed in the full range of x if $y \geq \frac{1}{2d-5}$, see also Fig. 2.2(b) for this case. It is also interesting to see the stability with respect to the variables $\frac{\mu M}{R^{d-3}}$ and $\frac{\sqrt{\mu} Q}{R^{d-3}}$, shown in Fig. 2.2(a), which follow from the analysis above by a transformation of variables.

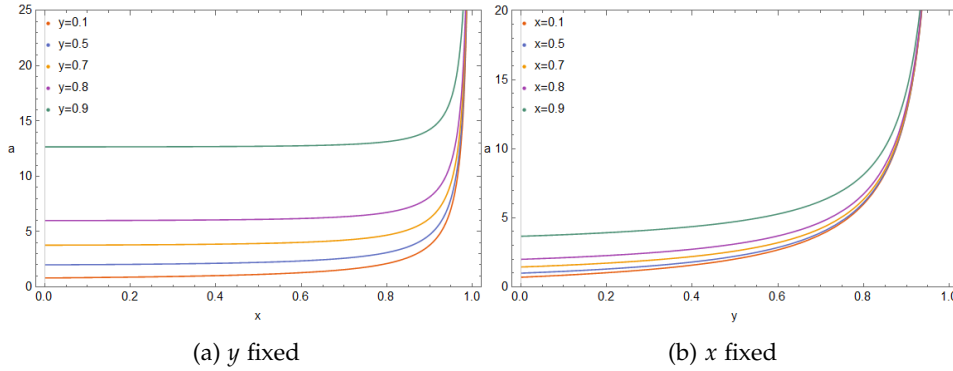


Figure 2.1: Region of thermodynamic stability of the shell for mass fluctuations only for $d = 5$ with the curves of marginal stability $a(x, y)$ plotted in function $x = r_+^2/R^2$, and $y = r_-^2/r_+^2$: (a) certain values of y ; (b) certain values of y . The regions below the curves describe the stable configurations.

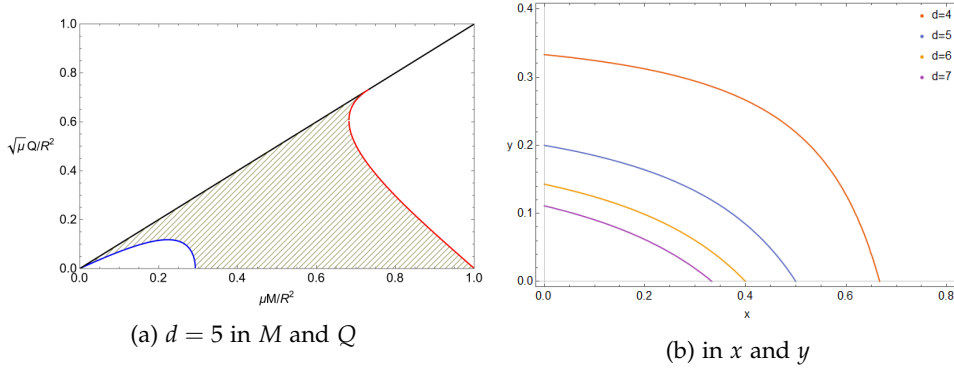


Figure 2.2: Region of thermodynamic stability of the shell for mass fluctuations only and $a = 1$: (a) in stripes in terms of $\mu M/R^2$ and $\sqrt{\mu}Q/R^2$; (b) above the curves for different d in terms of $x = r_+^{d-3}/R^{d-3}$ and $y = r_-^{d-3}/r_+^{d-3}$.

2.4.3 Stability of the shell for area fluctuations only

Considering a shell with only area fluctuations, the stability condition is given by $S_{AA} \leq 0$, see Eq. (2.78). Using Eq. (2.87), we have that $S_{AA} \leq 0$ can be written as

$$S_2 \leq 0. \quad (2.94)$$

Now Eq. (2.88) can also be used to rearrange this inequality into

$$a \leq \frac{(2d-5)x(1-y)}{(d-2)\mathcal{F}} + \frac{(d-3)(1+y)}{(d-2)(1-y)}, \quad (2.95)$$

where Eqs. (2.89) and (2.90) have been used. We also used that the factor $\mathcal{F}$ is always positive for $0 < x < 1$ and $0 < y < 1$, being proportional to $M - m$. Regarding some properties of the condition, the right-hand side of Eq. (2.95) has the minimum at $(x = 1, y = 0)$, with the value $a = 3 - \frac{2}{d-2}$. Moreover, the function increases in the direction of $x \rightarrow 0$ or $y \rightarrow 1$, where it tends to infinity. The right-hand side is plotted for $d = 5$ in Figs. 2.3(a) and 2.3(b). The curves increase with d . The case of the shell with $a = 1$, which has thermodynamic black hole features, is always below the surface of marginal stability, therefore it is stable to area perturbations only.

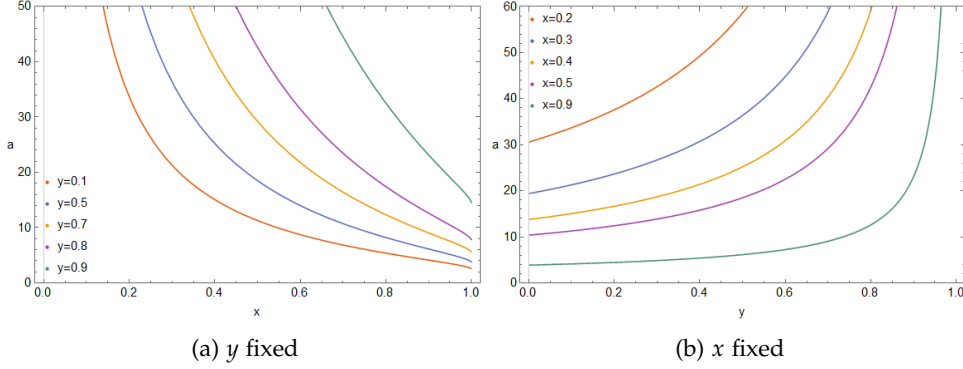


Figure 2.3: Region of thermodynamic stability of the shell for area fluctuations only for $d = 5$ with the curves of marginal stability $a(x, y)$ plotted in function $x = r_+^2/R^2$, and $y = r_-^2/r_+^2$: (a) certain values of y ; (b) certain values of x . The regions below the curves describe the stable configurations.

2.4.4 Stability of the shell for charge fluctuations only

For a shell with only electric charge fluctuations, the stability condition is given by $S_{QQ} \leq 0$, see Eq. (2.78). With the help of Eq. (2.87), the condition $S_{QQ} \leq 0$ can be written as

$$S_3 \leq 0. \quad (2.96)$$

Using Eq. (2.88), this inequality can be simplified into

$$a \leq \frac{(d-3)(1-y)}{2(d-2)y(1-x)} + \frac{2(d-3)}{(d-2)(1-x)} + \frac{(d-3)(1+y)}{(d-2)(1-y)}, \quad (2.97)$$

where Eqs. (2.89) and (2.90) have been used. The right-hand side of Eq. (2.97) depicts a concave surface, faced to $a \rightarrow +\infty$. In the restricted parameter space, the minimum is at $(x = 0, y = \frac{1}{3})$, where its value is $a = 5\frac{d-3}{d-2}$. The right-hand side diverges to infinity at the axes $x = 1, y = 0$ and $y = 1$. The right-hand side is plotted in Figs. 2.4(a) and 2.4(b) for $d = 5$. For increasing d , the curves increase overall. The shell with thermodynamic black hole features, described by $a = 1$, finds itself always below the surface of marginal stability, therefore it is stable to charge perturbations only.

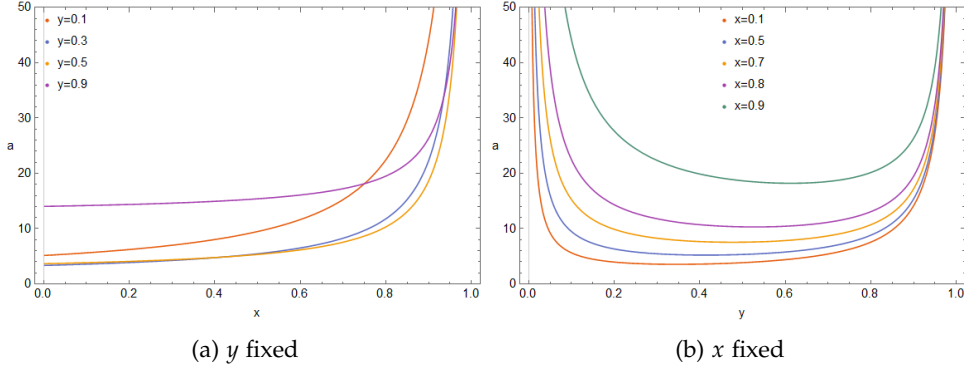


Figure 2.4: Region of thermodynamic stability of the shell for charge fluctuations only for $d = 5$ with the curves of marginal stability $a(x, y)$ plotted in function $x = r_+^2 / R^2$, and $y = r_-^2 / r_+^2$: (a) certain values of y ; (b) certain values of x . The regions below the curves describe the stable configurations.

2.4.5 Stability of the shell for mass and area fluctuations together

Considering now a shell with mass and area fluctuations, the stability conditions including the marginal condition is given by $S_{MM} \leq 0$, $S_{AA} \leq 0$, and $S_{MM}S_{AA} - S_{MA}^2 \geq 0$, see Eq. (2.78). Without the marginal condition, it suffices to consider $S_{MM} < 0$ and $S_{MM}S_{AA} - S_{MA}^2 > 0$. However, the condition $S_{MM}S_{AA} - S_{MA}^2 \geq 0$ for the case of the shell is the strongest even including the marginal case. With Eq. (2.87), we have that $S_{MM}S_{AA} - S_{MA}^2 \geq 0$ can be written as

$$S_4 = -\frac{1}{2(d-3)}S_1S_2 + S_{12}^2 \leq 0. \quad (2.98)$$

From Eq. (2.88), this inequality can be simplified into

$$a \leq \frac{(1-y)x\left((d-\frac{5}{2})\mathcal{F} - (d-3)(1-k)^2\right)}{(d-2)\mathcal{F}\left(\frac{k^2}{d-3} + 2k + \frac{\mathcal{F}}{2}\right)} + \frac{(d-3)(1+y)}{(d-2)(1-y)}, \quad (2.99)$$

where Eqs. (2.89) and (2.90) have been used. The right-hand side of Eq. (2.99) assumes the minimum value at $x = 1$, where $a = 1$. From $x = 1$ towards $x = 0$, the function bends towards $a = \frac{d-3}{d-2} \frac{1+y}{1-y}$. At $y = 1$, the right-hand side tends to infinity. The right-hand side is plotted in Figs. 2.5(a) and 2.5(b) for $d = 5$. For higher d , the curves increase overall. The case $a = 1$ of the shell, having thermodynamic black hole features, has the property that increasing the value of y decreases the range of x for thermodynamic stable configurations, meaning, if the shell has more electric charge than it needs to have lower R for stability, see also Fig. 2.6 for this $a = 1$ case.

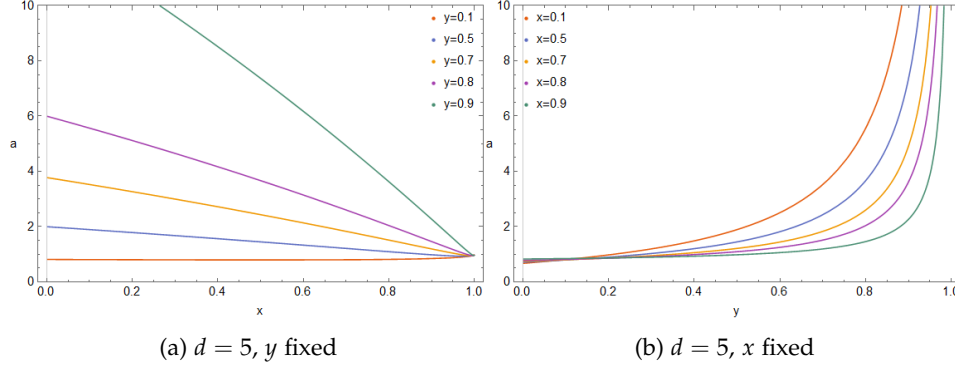


Figure 2.5: Region of thermodynamic stability of the shell for mass and area fluctuations together for $d = 5$ with the curves of marginal stability $a(x, y)$ plotted in function $x = r_+^2/R^2$, and $y = r_-^2/r_+^2$: (a) certain values of y ; (b) certain values of x . The regions below the curves describe the stable configurations.

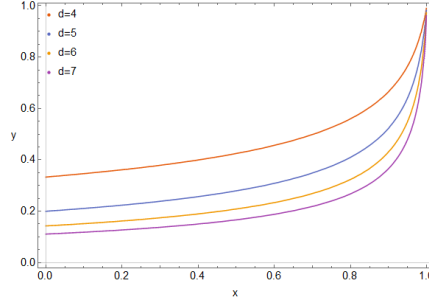


Figure 2.6: Region of thermodynamic stability for mass and area fluctuations and for $a = 1$, for different values of d , in terms of $x = r_+^{d-3}/R^{d-3}$ and $y = r_-^{d-3}/r_+^{d-3}$. Region below curves describes stability.

2.4.6 Stability of the shell for mass and charge fluctuations together

We consider now the shell with mass and charge fluctuations. The stability conditions including the marginal case are given by $S_{MM} \leq 0$, $S_{QQ} \leq 0$, and $S_{MM}S_{QQ} - S_{MQ}^2 \geq 0$, see Eq. (2.78). Without considering the marginal case, the sufficient conditions are $S_{MM} < 0$ and $S_{MM}S_{QQ} - S_{MQ}^2 > 0$. However, for the case of the shell, the condition $S_{MM}S_{QQ} - S_{MQ}^2 \geq 0$ is the strongest. Using Eq. (2.87), we have that $S_{MM}S_{QQ} - S_{MQ}^2 \geq 0$ can be written as

$$S_5 = -x(1-x)S_1S_3 + \frac{4yk^2}{(d-3)^2}S_{13}^2 \leq 0. \quad (2.100)$$

Using Eq. (2.88), the inequality above can be rearranged as

$$a \leq \frac{(d-3)}{2(d-2)} \frac{2-x(1+y)}{1-x}, \quad (2.101)$$

where Eqs. (2.89) and (2.90) have been used. Some properties of right-hand side follow. At $x = 0$ or $y = 1$, the right-hand side takes the value $a = \frac{d-3}{d-2}$. The function diverges to infinity at $x = 1$. The function then bends from a constant value to $a = \frac{d-3}{2(d-2)} \frac{2-x}{1-x}$, going from $y = 1$ to $y = 0$. We present the plot of the right-hand side for $d = 5$ in Figs. 2.7(a) and 2.7(b). The curves further increase with d . For the case with $a = 1$, increasing the value of y decreases the range of x for thermodynamic stable configurations, see also Fig. 2.8 for this $a = 1$ case.

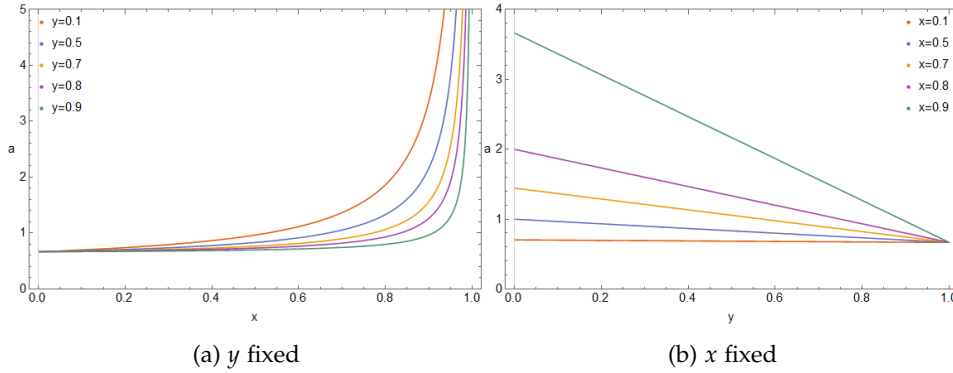


Figure 2.7: Region of thermodynamic stability of the shell for mass and charge fluctuations together for $d = 5$ with the curves of marginal stability $a(x, y)$ plotted in function $x = r_+^2/R^2$, and $y = r_-^2/r_+^2$: (a) certain values of y ; (b) certain values of x . The regions below the curves describe the stable configurations.

2.4.7 Stability of the shell for area and charge fluctuations together

Regarding the case of a shell with area and charge fluctuations, the stability conditions including the marginal case are given by $S_{AA} \leq 0$, $S_{QQ} \leq 0$, and $S_{AA}S_{QQ} - S_{AQ}^2 \geq 0$, see Eq. (2.78). Without the marginal case, the sufficient conditions can be chosen to be $S_{AA} < 0$ and $S_{AA}S_{QQ} - S_{AQ}^2 > 0$. For the specific case of the shell, it turns out that $S_{AA}S_{QQ} - S_{AQ}^2 \geq 0$ is sufficient. Using Eq. (2.87), we have that $S_{AA}S_{QQ} - S_{AQ}^2 \geq 0$ can be written as

$$S_6 = -\frac{(1-x)}{2(d-3)} S_2 S_3 + xy S_{23} \leq 0, \quad (2.102)$$

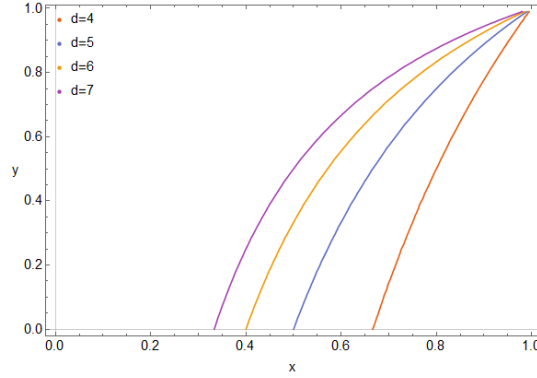


Figure 2.8: Region of thermodynamic stability for mass and area fluctuations and for $a = 1$, for different values of d , in terms of $x = r_+^{d-3}/R^{d-3}$ and $y = r_-^{d-3}/r_+^{d-3}$. Region below curves describes stability.

which can be simplified using Eq. (2.88) into

$$a \leq \frac{\frac{(1-x)\mathcal{F}(2d-5)}{2(d-3)}(1+3y) - x^3y(1-y) + 2\mathcal{F}xy - \frac{y\mathcal{F}^2}{x(1-y)}}{(d-2)(1-x)\left(\frac{\mathcal{F}^2}{2x(d-3)} + \frac{2d-5}{(d-3)^2}y(1-x)\mathcal{F} + \frac{2\mathcal{F}xy}{(d-3)}\right)} + \frac{(d-3)}{(d-2)} \frac{1+y}{1-y}. \quad (2.103)$$

where Eqs. (2.89) and (2.90) have been used. At $y = 0$, the right-hand side function intersects S_2 . It then grows without bound at $(x = 0, y = 0)$ or $y = 1$. In the limit of $x \rightarrow 1$, the right-hand side approaches the value of $a = \frac{8+6y-3d(1+y)}{(d-2)(1+3y)}$. At $x = 0$, the right-hand side function approaches S_3 from below. The right-hand side function is plotted for $d = 5$ in Figs. 2.9(a) and 2.9(b). The curves increase with d . The case with $a = 1$ is always below the surface of marginal stability, therefore it is stable to area and charge perturbations.

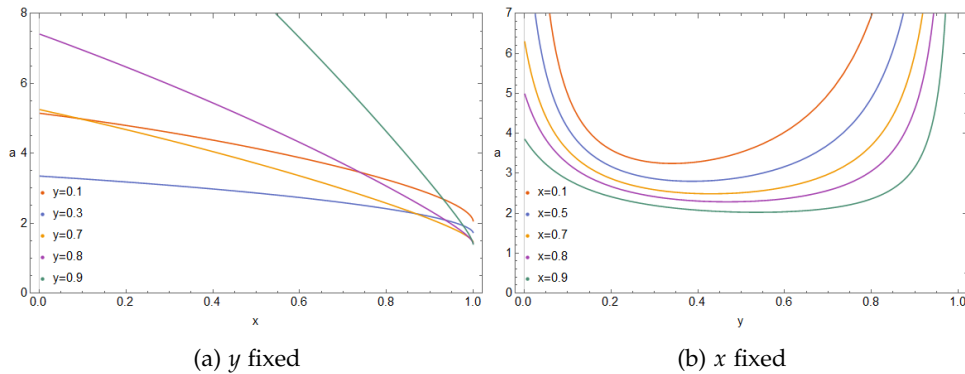


Figure 2.9: Region of thermodynamic stability of the shell for area and charge fluctuations together for $d = 5$ with the curves of marginal stability $a(x, y)$ plotted in function $x = r_+^2/R^2$, and $y = r_-^2/r_+^2$: (a) certain values of y ; (b) certain values of x . The regions below the curves describe the stable configurations.

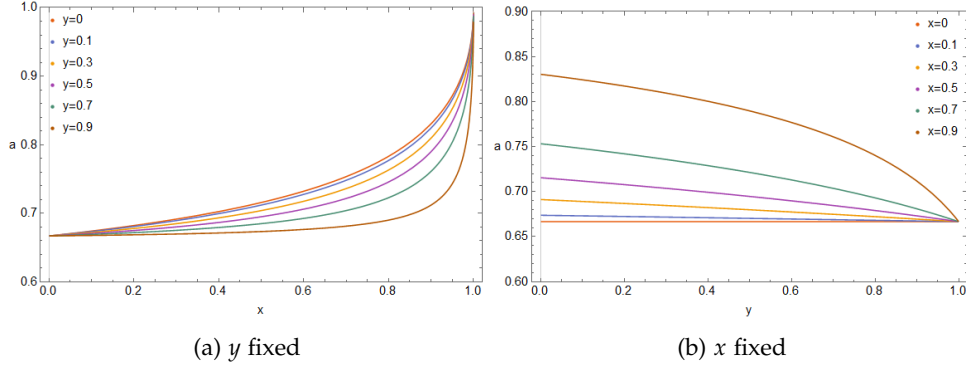


Figure 2.10: Region of thermodynamic stability of the shell for mass, area and charge fluctuations together for $d = 5$ with the curves of marginal stability $a(x, y)$ plotted in function $x = r_+^2/R^2$, and $y = r_-^2/r_+^2$: (a) certain values of y ; (b) certain values of x . The regions below the curves describe the stable configurations.

2.4.8 Stability of the shell for mass, area and charge fluctuations

For the shell with full perturbations, i.e. mass, area, and charge fluctuations, the stability conditions including the marginal case are given by all the inequalities in Eq. (2.78). Without the marginal case, the sufficient conditions are $S_{MM} < 0$, $S_{MM}S_{AA} - S_{MA}^2 > 0$, and $(S_{MM}S_{AQ} - S_{MA}S_{MQ})^2 - (S_{AA}S_{MM} - S_{AM}^2)(S_{QQ}S_{MM} - S_{QM}^2) < 0$. However, for the case of the shell, it is sufficient to consider $(S_{MM}S_{AQ} - S_{MA}S_{MQ})^2 - (S_{AA}S_{MM} - S_{AM}^2)(S_{QQ}S_{MM} - S_{QM}^2) \leq 0$ for $S_{MM} \neq 0$ and $-S_{MA}^2 S_{QQ} + 2S_{MQ}S_{MA}S_{AQ} - S_{AA}S_{MQ}^2 \leq 0$ for $S_{MM} = 0$. For $S_{MM} \neq 0$, the condition simplifies to

$$S_7 = \left(xS_1S_{23} - \frac{2k}{d-3}S_{12}S_{13} \right)^2 y - S_4S_5 \leq 0, \quad (2.104)$$

while for $S_{MM} = 0$, we must divide by S_{MM} and make the limit $S_{MM} = 0$. In both cases, the inequality reduces to

$$a \leq \frac{d-3}{d-2} \left(\frac{4-4k+x^2(d(1-y)^2+C)}{4-4k+x^2d(1-y)^2+xD} \right), \quad (2.105)$$

where $C = 2x(1+y)(k-2) - 2 - 2(y-4)y$, and $D = 4k - 2y - 6 - x(1+y(3y-8))$, and Eqs. (2.89) and (2.90) have been used. The right-hand side in the condition given in Eq. (2.105) has its lowest value of $a = \frac{d-3}{d-2}$ at $x = 0$, for every y . The function then increases towards $x = 1$, with the limit $x = 1$ giving $a = 1$. At the limit of $y = 1$, the right-hand side is given by the lowest value $a = \frac{d-3}{d-2}$ for every x , except for the limit $x = 1$ where it gives $a = 1$. Therefore, the condition for

stability in Eq. (2.105) implies that every configuration with $a \leq \frac{d-3}{d-2}$ is stable. For $\frac{d-3}{d-2} < a < 1$, the stability region decreases with increasing y , being zero in the limit of $y = 1$. And so shells with more electric charge have less configurations of stability. The space of stable configurations in the $a - d$ plane is similar to the analysis made for the uncharged case in [95]. The right-hand side is plotted for $d = 5$ in Figs. 2.10(a) and 2.10(b). The curves increase with increasing d . The case of the shell with thermodynamic black hole features, with $a = 1$, is always above the surface of marginal stability, hence unstable, except for the points with $x = 1$ which lie on the limit of the surface, hence marginally stable. This means that, in the black hole limit $a = 1$ and $x = 1$, the configurations for every value of y are marginally stable.

2.4.9 Behaviour of the intrinsic stability with the parameter a : some comments

We give here some additional comments on the analysis of the stability conditions above, namely on the stability of mass fluctuations only in terms of M and Q , and also on the effect of the electric charge in the stability.

For mass fluctuations only, Sec. 2.4.2, we barely mentioned the stability analysis in terms of $\frac{M}{R^{d-3}}$ and $\frac{Q}{R^{d-3}}$. However, there are still some interesting insights to be made. The condition given in Eq. (2.93) in terms of $\frac{M}{R^{d-3}}$ and $\frac{Q}{R^{d-3}}$ instead of $x = \frac{r_+^{d-3}}{R^{d-3}}$ and $y = \frac{r_-^{d-3}}{r_+^{d-3}}$ becomes

$$a \leq \frac{\mu}{2} \frac{(d-3)}{(d-2)} \frac{\left(\frac{Q^2}{R^{2(d-3)}} + \frac{\mu M^2}{R^{2(d-3)}} - \frac{2M}{R^{d-3}} \right)}{\left(1 - \frac{\mu M}{R^{d-3}} \right)^2} \frac{\left(\frac{2\mu M}{R^{d-3}} + \frac{\mu(Q^2 - M^2 \mu^2)}{R^{2(d-3)}} - 2 \right)}{\sqrt{\frac{\mu(\mu M^2 - Q^2)}{R^{2(d-3)}} \left(\left(2 - \frac{\mu M}{R^{d-3}} \right)^2 - \frac{\mu Q^2}{R^{2(d-3)}} \right)}}. \quad (2.106)$$

The parameter space is restricted to the condition of subextremality, namely $\sqrt{\mu}M > Q$, and to the condition of no trapped surface, $\frac{r_+}{R} < 1$. From the stability condition in Eq. (2.106), we find that the shell with small $\frac{M}{R^{d-3}}$ requires at least a minimum value of electric charge $\frac{Q}{R^{d-3}}$ to be stable. When $\frac{M}{R^{d-3}}$ assumes the value corresponding to $x = \frac{2}{d-1}$, the minimum charge for stability becomes zero, or $y = 0$. For higher mass $\frac{M}{R^{d-3}}$, the region of stable configurations is only constrained by the restrictions of the parameter space, meaning $\frac{\sqrt{\mu}M}{R^{d-3}} > \frac{Q}{R^{d-3}}$ and $\frac{r_+}{R} < 1$. The important point is that the condition in Eq. (2.106) means that thermodynamic stability for small $\frac{M}{R^{d-3}}$ only happens for sufficiently large electric charge. Moreover, it is also interesting to consider the $y = 0$ case. The shell is only stable for $x \geq \frac{2}{d-1}$, with equality being the marginally stable case. Since $x = \frac{2}{d-1}$ corresponds to the photonic orbit, the stable shell must always lay inside the photonic orbit, in agreement with [95]. We must note that this behaviour is similar to black holes in the canonical ensemble [68, 100] and its generalization to higher dimensions [101, 102], indeed there is a stable black hole solution which must be larger than $x = \frac{2}{d-1}$, where x is the ratio between the horizon radius and the cavity radius. For larger values of $\frac{M}{R^{d-3}}$,

it seems that increasing the electric charge does not change the stability of the shell, apart from the subextremality and no trapped surface conditions. This can be interpreted in terms of a thermal length scale, which is proportional to the reduced inverse temperature b . We have that, for small $\frac{M}{R^{d-3}}$ and $Q = 0$, the thermodynamic unstable shells have radii higher than the photonic orbit. Since the thermal length b is proportional to M in the uncharged case, the thermal length is smaller or of the order of the radius of the shell, and so the shell loses energy and mass along these thermal lengths. The effect of loosing mass causes the thermal length b to decrease and so we have a runaway process, the shell is unstable. For the case that charge Q is increased, the thermal length b gets also increased and so it happens that for sufficiently large charge, b becomes greater than the radius of the shell, quenching the loss of energy. And so the shell becomes stable for charges larger than this minimum electric charge. For a shell close to extremality, the thermal length is proportional to $\frac{1}{\sqrt{M-Q}}$, which is divergent and so larger than R . For the exact value of $Q = 0$ and $\frac{M}{R^{d-3}}$ corresponding to $x = \frac{2}{d-1}$, the shell is at the photonic orbit and it is marginally stable. Indeed, the thermal length is barely larger than the radius of the shell so that the shell is in the cusp of losing energy. For larger $\frac{M}{R^{d-3}}$ and $Q = 0$, the shell resides inside the photonic orbit and the thermal length is larger enough to avoid the loss of energy, being thus stable. If we increase the charge, the thermal length increases even more and the shell remains stable. The discussion we presented here for generic dimensions d also applies to the $d = 4$ electric charged case studied in [96] and is exemplified for $d = 5$ in Fig. 2.2(a).

For the case of full fluctuations, i.e. mass, area and charge fluctuations, shells with more electric charge have a lesser amount of stable configurations. This behaviour does differ from the case of mass fluctuations only, where more electric charge contributes to stability. But of course, the thermal length analysis is not enough to explain such stability since there are area and charge fluctuations to take into account. Stability is then more restrictive, meaning that configurations that are stable to mass fluctuations may not be stable to full fluctuations, while stable configurations to full perturbations must be stable to mass fluctuations only.

We must indicate another point regarding the stable values of a in the case of one or two fluctuations together. Indeed, there are stable shell configurations with $a \geq 1$. And in turn, shells with higher a for the same thermodynamic configurations have higher entropy as it goes with the power of a . For example, for area fluctuations only, the value of a for marginal stability is $a = 3 - \frac{2}{d-2}$ for $x = 1$ and $y = 0$, and then increases for other values. Since $a = 3 - \frac{2}{d-2}$ is always larger than one, it may mean that a shell with lower a suffers a transition to a shell with larger a , since the latter has more entropy. For that to happen, the matter of the shell would have to rearrange in order to change its equation of state. Note however that the stability analysis here is for fixed a , and so one would need to have a more fundamental description for the shell to understand if such transition is possible.

With the thermodynamic stability conditions worked out, we present the physical meaning of these conditions below.

2.5 INTRINSIC THERMODYNAMIC STABILITY IN TERMS OF LABORATORY VARIABLES

2.5.1 The case for mass fluctuations only

In thermodynamics, the stability conditions are linked to thermodynamic quantities that are measured in a laboratory. Here, we establish this link for the self-gravitating thin shell. A simple example is the one given by a shell with mass fluctuations only. The stability condition is tied to the heat capacity at constant area and charge, $C_{A,Q}$, defined as $C_{A,Q}^{-1} = \left(\frac{\partial T}{\partial M}\right)$, since $S_{MM} = -\beta^2 C_{A,Q}^{-1}$. For the shell to be stable in terms of mass fluctuations only, one has $S_{MM} \leq 0$, and so the shell must have a positive heat capacity. We extend this analysis below for mass and charge fluctuations together and for full fluctuations, as they are the most interesting cases in the context of the thesis.

2.5.2 The case for mass and charge fluctuations

We discuss here the stability conditions for mass and charge fluctuations in terms of laboratory variables. The interest in this case stems from the fact that, in canonical ensembles, the area is fixed, and this extends even to the case of black holes, where the area of the cavity is fixed. There are two variables that have an important role which are the two heat capacities $C_{A,Q}$ and $C_{A,\Phi}$, i.e. the heat capacity at constant area and charge, and the heat capacity at constant area and electric potential. There is an additional variable, the susceptibility at constant entropy and area $\chi_{S,A}$ which comes into play.

The idea is to write the second derivatives of the entropy in terms of the laboratory variables or thermodynamic coefficients. For that, we must start from the equations of state $T(M, A, Q)$, $p(M, A, Q)$ and $\Phi(M, A, Q)$, given in Eqs. (2.80)-(2.82), and rewrite them in terms of such variables. For mass and charge fluctuations, we only need to consider $T(M, A, Q)$ and $\Phi(M, A, Q)$.

Regarding the equation of state for the temperature, $T(M, A, Q)$, it is convenient to define the laboratory quantities in terms of the derivatives of $S(T, A, Q)$. The heat capacity $C_{A,Q}$ is defined as $\frac{1}{T}C_{A,Q} = \left(\frac{\partial S}{\partial T}\right)_{A,Q}$, which is equivalent to the usual definition $C_{A,Q}^{-1} = \left(\frac{\partial T}{\partial M}\right)_{A,Q}$. The latent heat capacity at constant temperature and charge, $\lambda_{T,Q}$, is defined by the derivative $\lambda_{T,Q} = \left(\frac{\partial S}{\partial A}\right)_{T,Q}$. The latent heat capacity at constant temperature and area, $\lambda_{T,A}$, is defined by the derivative $\lambda_{T,A} = \left(\frac{\partial S}{\partial Q}\right)_{T,A}$. With these definitions, we can write the differential of $S(T, A, Q)$ and then invert the relation to get the differential of $T(S, A, Q)$. Using the first law as $TdS = dM + pdA - \Phi dQ$, the differential of $T(S, A, Q)$ can be transformed into the differential of $T(M, A, Q)$, yielding

$$dT = \frac{1}{C_{A,Q}}dM - \frac{T\lambda_{T,Q} - p}{C_{A,Q}}dA - \frac{T\lambda_{T,A} + \Phi}{C_{A,Q}}dQ. \quad (2.107)$$

Regarding the equation of state for the electric potential, $\Phi(M, A, Q)$, we can define the laboratory variables in terms of the derivatives of $\Phi(S, A, Q)$. The adia-

batic electric susceptibility, $\chi_{S,A}$, is defined as $\frac{1}{\chi_{S,A}} = \left(\frac{\partial\Phi}{\partial Q}\right)_{S,A}$. The electric pressure at constant entropy and charge, $P_{S,A}$, is defined as $P_{S,Q} = \left(\frac{\partial\Phi}{\partial A}\right)_{S,Q}$. The remaining derivative of Φ is given by the Maxwell relation $\left(\frac{\partial\Phi}{\partial S}\right)_{A,Q} = \left(\frac{\partial T}{\partial Q}\right)_{S,A} = -\frac{T\lambda_{T,A}}{C_{A,Q}}$. The differential of $\Phi(S, A, Q)$ can be written directly in terms of laboratory variables, and using the first law $TdS = dM + pdA - \Phi dQ$, we obtain the differential of $\Phi(M, A, Q)$ as

$$d\Phi = -\frac{\lambda_{T,A}}{C_{A,Q}}dM + \left(P_{S,Q} - p\frac{\lambda_{T,A}}{C_{A,Q}}\right)dA + \left(\frac{1}{\chi_{S,A}} + \frac{\Phi\lambda_{T,A}}{C_{A,Q}}\right)dQ. \quad (2.108)$$

Additionally, it is important to define the heat capacity at constant area and electric potential as $C_{A,\Phi} = T\left(\frac{\partial S}{\partial T}\right)_{A,\Phi}$, which can be written as $C_{A,\Phi} = C_{A,Q}\left(1 - \frac{T\lambda_{T,A}^2}{C_{A,Q}}\chi_{S,A}\right)^{-1}$.

Returning to the stability conditions of a shell for mass and charge fluctuations, the relevant stability conditions are $S_{MM} \leq 0$ and $S_{MM}S_{QQ} - S_{MQ}^2 \geq 0$, see Eq. (2.78). The first condition is identical to the one of the mass fluctuations only, as we have $S_{MM} = -\beta^2 \frac{1}{C_{A,Q}}$ from Eq. (2.107). The second condition can be rewritten using Eqs. (2.107) and (2.108), together with the definition of the heat capacity $C_{A,\Phi}$ to obtain $S_{MM}S_{QQ} - S_{MQ}^2 = \beta^2 \frac{1}{C_{A,\Phi}\chi_{S,A}}$. And so the thermodynamic stability for mass and charge fluctuations reduces to the conditions

$$\begin{aligned} C_{A,Q} &\geq 0, \\ C_{A,\Phi} \chi_{S,A} &\geq 0. \end{aligned} \quad (2.109)$$

For the equations of state chosen, i.e. Eqs. (2.80)-(2.82) the adiabatic susceptibility is given by $\chi_{S,A}^{-1} = \Phi^2 \frac{\mu}{R^{d-3}\left(1-\mu\frac{M}{R^{d-3}}\right)} + \frac{\Phi}{Q}$, and so for the physical parameters of (M, A, Q) , $\chi_{S,A} \geq 0$. Therefore, the stability conditions become $C_{A,Q} \geq 0$ and $C_{A,\Phi} \geq 0$. For the case of the shell, the condition $S_{MM}S_{QQ} - S_{MQ}^2 \geq 0$ is sufficient and so the condition for thermodynamic stability for mass and charge fluctuations is

$$C_{A,\Phi} \geq 0. \quad (2.110)$$

Note that Eq. (2.110) is equivalent to Eq. (2.101), but it is important to stress that equality in Eq. (2.101) means that the heat capacity diverges to positive infinity.

2.5.3 The case for mass, area and charge fluctuations

In this subsection, we treat the thermodynamic stability of a thin shell with full fluctuations, i.e. mass, area and charge fluctuations, in terms of the laboratory variables. The analysis of the previous subsection highlighted the importance of the heat capacities $C_{A,Q}$ and $C_{A,\Phi}$ in the description of thermodynamic stability with fixed area. For the case of full fluctuations, there are other thermodynamic coefficients that play an important role, namely the expansion coefficient at constant temperature and electric charge, $\kappa_{T,Q}$, and the electric susceptibility at constant

pressure and temperature $\chi_{p,T}$. Additionally, the heat capacity $C_{A,Q}$ also appears here.

As in the previous subsection, it is helpful to obtain the differential of the equations of state $T(M, A, Q)$, $p(M, A, Q)$ and $\Phi(M, A, Q)$ in terms of the laboratory variables or thermodynamic coefficients. These can then be related to the second order derivatives of the entropy since $dS = \beta dM + \beta p dA - \beta \Phi dQ$. To that effect, we define the laboratory variables considering the derivatives of $S(T, p, Q)$, $A(T, p, Q)$ and $\Phi(T, p, Q)$, as they simplify the considered stability conditions. Note that the three functions $S(T, p, Q)$, $A(T, p, Q)$ and $\Phi(T, p, Q)$ are precisely the derivatives of the Gibbs potential, i.e. $dG = -SdT + Adp + \Phi dQ$. Starting with the equation of state $A(T, p, Q)$, the expansion coefficient $\alpha_{p,Q}$ is defined as $\alpha_{p,Q} = \frac{1}{A}(\frac{\partial A}{\partial T})_{p,Q}$, the isothermal compressibility $\kappa_{T,Q}$ is defined as $\kappa_{T,Q} = -\frac{1}{A}(\frac{\partial A}{\partial p})_{T,Q}$, and the electric compressibility $\kappa_{p,T}$ is defined as $\kappa_{p,T} = -\frac{1}{A}(\frac{\partial A}{\partial Q})_{T,p}$. For the equation of state $S(T, p, Q)$, the derivative $(\frac{\partial S}{\partial T})_{p,Q}$ can be written as $(\frac{\partial S}{\partial T})_{p,Q} = \frac{C_{A,Q}}{T} + A \frac{\alpha_{p,Q}^2}{\kappa_{T,Q}}$, while the derivative $(\frac{\partial S}{\partial p})_{T,Q}$ can be calculated using the Maxwell relation $(\frac{\partial S}{\partial p})_{T,Q} = -(\frac{\partial A}{\partial T})_{p,Q}$ to get $(\frac{\partial S}{\partial T})_{p,Q} = \frac{C_{A,Q}}{T} + A \frac{\alpha_{p,Q}^2}{\kappa_{T,Q}}$, $(\frac{\partial S}{\partial p})_{T,Q}$. For the remaining derivative, the latent heat capacity $\lambda_{p,T}$ is defined as $\lambda_{p,T} = (\frac{\partial S}{\partial Q})_{p,T}$. For the equation of state $\Phi(T, p, Q)$, two of its derivatives are given by the Maxwell relations as $(\frac{\partial \Phi}{\partial T})_{p,Q} = -(\frac{\partial S}{\partial Q})_{p,T} = \lambda_{p,T}$ and $(\frac{\partial \Phi}{\partial p})_{T,Q} = (\frac{\partial A}{\partial Q})_{p,T} = -A\kappa_{p,T}$, while the isothermal electric susceptibility $\frac{1}{\chi_{p,T}}$ is defined as $\frac{1}{\chi_{p,T}} = (\frac{\partial \Phi}{\partial Q})_{p,T}$.

Having these definitions together with the differentials $dA(T, p, Q)$ and $dS(T, p, Q)$ in terms of laboratory variables, we can invert the relations to obtain $dT(S, A, Q)$ and $dp(S, A, Q)$. Using then the first law $TdS = dM + pdA - \Phi dQ$, the differentials $dT(M, A, Q)$ and $dp(M, A, Q)$ are obtained. Inserting these two differentials into the differential of $\Phi(T, p, Q)$, we obtain the differential $d\Phi(M, A, Q)$. The differentials in terms of the laboratory variables and the thermodynamic variables (M, A, Q) are

$$dT = \frac{dM}{C_{A,Q}} + \left(\frac{p}{C_{A,Q}} - T \frac{\alpha_{p,Q}}{C_{A,Q}\kappa_{T,Q}} \right) dA - \left(\frac{\Phi}{C_{A,Q}} + T \frac{\lambda_{p,T}}{C_{A,Q}} + A \frac{\alpha_{p,T}\kappa_{p,T}}{\kappa_{T,Q}C_{A,Q}} \right) dQ, \quad (2.111)$$

$$dp = \frac{\alpha_{p,Q}}{C_{A,Q}\kappa_{T,Q}} dM - \left[\frac{1}{A\kappa_{T,Q}} - \frac{\alpha_{p,Q}}{C_{A,Q}\kappa_{T,Q}} \left(p - T \frac{\alpha_{p,Q}}{\kappa_{T,Q}} \right) \right] dA - \left(\frac{\kappa_{p,T}}{\kappa_{T,Q}} - \frac{\alpha_{p,Q}}{C_{A,Q}\kappa_{T,Q}} \mathcal{C} \right) dQ, \quad (2.112)$$

$$d\Phi = -\mathcal{B}dM + \left[\frac{\kappa_{p,T}}{\kappa_{T,Q}} - \left(p - T \frac{\alpha_{p,Q}}{\kappa_{T,Q}} \right) \mathcal{B} \right] dA + \left(\mathcal{B}\mathcal{C} + \frac{1}{\chi_{p,T}} + A \frac{\kappa_{p,T}^2}{\kappa_{T,Q}} \right) dQ, \quad (2.113)$$

where $\mathcal{B}$ is defined as $\mathcal{B} = A \frac{\kappa_{p,T}\alpha_{p,Q}}{C_{A,Q}\kappa_{T,Q}} + \frac{\lambda_{p,T}}{C_{A,Q}}$, and $\mathcal{C}$ is defined as $\mathcal{C} = TA \frac{\kappa_{p,T}\alpha_{p,Q}}{\kappa_{T,Q}} + T\lambda_{p,T} + \Phi$. With Eqs. (2.111)-(2.113), the second derivatives of the entropy in terms of the laboratory variables can be found through the first law of thermodynamics.

The intrinsic thermodynamic stability of the shell for mass, area and charge fluctuations is by the relevant stability conditions $S_{MM} \leq 0$, $S_{MM}S_{AA} - S_{MA}^2 \geq 0$, and $(S_{MM}S_{AQ} - S_{MA}S_{MQ})^2 - (S_{AA}S_{MM} - S_{AM}^2)(S_{QQ}S_{MM} - S_{QM}^2) \leq 0$, taken from Eq. (2.78). Now, in terms of laboratory variables, the first condition is given by $S_{MM} = -\beta^2 \frac{1}{C_{A,Q}}$, the second condition is given by $S_{MM}S_{AA} - S_{MA}^2 = -\beta^3 \frac{1}{A\kappa_{T,Q}C_{A,Q}}$ and finally the third condition is given by $(S_{MM}S_{AQ} - S_{MA}S_{MQ})^2 - (S_{AA}S_{MM} - S_{AM}^2)(S_{QQ}S_{MM} - S_{QM}^2) - \beta^6 \frac{1}{AC_{A,Q}^2\kappa_{T,Q}\chi_{p,T}}$. It follows that the stability conditions for mass, area and charge fluctuations in terms of the laboratory variables reduce to

$$\begin{aligned} C_{A,Q} &\geq 0, \\ \kappa_{T,Q} &\geq 0, \\ \chi_{p,T} &\geq 0. \end{aligned} \tag{2.114}$$

Hence, all the three laboratory quantities have to be positive, specifically, the heat capacity $C_{A,Q}$ related to the temperature equation of state, the isothermal compressibility $\kappa_{T,Q}$ related to the pressure equation of state, and the isothermal electric susceptibility $\chi_{p,T}$ related to the electric potential equation of state, have to be positive, with the case of marginal stability corresponding to these physical variables going to infinity.

From the conditions in Eqs. (2.114), the sufficient stability condition for the case of the shell with the specific choice of equations of state is the last condition in Eq. (2.114), namely

$$\chi_{p,T} \geq 0. \tag{2.115}$$

In connection with Sec. 2.4.8, the condition in Eq. (2.115) is equivalent to Eq. (2.105), meaning that for $a < \frac{d-3}{d-2}$, the isothermal electric susceptibility is positive, and for $\frac{d-3}{d-2} < a < 1$ it is positive depending on the values of (r_+, r_-, R) . For $a \geq 1$ and $r_+ < R$, the susceptibility is negative. The shell with black hole features has to be treated carefully as it resides in the marginal surface. If the shell with $a = 1$ and $r_+ < R$ approaches its own gravitational radius, it is thermodynamically unstable as the susceptibility tends to $\chi_{p,T} \rightarrow -\infty$. However, there can be a configuration with $R = r_+$ from the start that does not belong to this sequence of quasistatic configurations. The stability of the black hole limit depends on whether the exponent a of the equation of state approaches $a = 1$ from below or from above. If it is possible to have the exponent a of the temperature equation of state to approach $a = 1$ from below, the configuration with $R = r_+$ is marginally stable with $\chi_{p,T} \rightarrow +\infty$. Having such diverging susceptibility means that changes in the electric charge of the configuration don't have any impact on the electric potential. For the case that the exponent a approaches $a = 1$ from above, the configuration with $R = r_+$ is unstable, and $\chi_{p,T} \rightarrow -\infty$.

2.6 CONCLUSIONS

In this chapter, we used the thin shell formalism to determine the mechanics of a static charged spherical thin shell in d dimensions in general relativity. Furthermore,

we studied the thermodynamics of the shell by imposing the first law of thermodynamics. The use of the pressure equation of state as given by general relativity and the relation between the rest mass of the shell and the quasilocal energy give special thermodynamic properties to the shell, indicating a link between thermodynamics and general relativity. One of such remarkable thermodynamic properties is that the entropy of shell depends on r_- and r_+ and not on the radius of the shell. Note that this property has also been found for other thin shell spacetimes.

In order to proceed with the thermodynamic analysis of the shell, we provided two equations of state to the shell, one where the temperature is described by a power law in r_+ with exponent a , and another where the electric potential is described by the typical potential of the Reissner-Nordström spacetime. We were interested in these specific shells due to the possibility of performing the black hole limit, and also for having shells with thermodynamic black hole features.

We studied the thermodynamic intrinsic stability of the shell. A shell is stable if the hessian of the entropy is negative semidefinite, where the marginal case was included. We analyzed the stability for seven types of fluctuations. The most general case constitutes the one with mass, area and charge fluctuations, for which the shell is always stable in the case $0 < a \leq \frac{d-3}{d-2}$. For $\frac{d-3}{d-2} < a < 1$, the stability depends on the mass and electric charge, while for $a \geq 1$ and $r_+ < R$ the shell is unstable. For the shell with $a = 1$ at its own gravitational radius, there is marginal stability.

We have seen the thermodynamic intrinsic stability of the shell from the perspective of laboratory variables. For the generic type of fluctuations, stable shells have positive heat capacity, positive isothermal compressibility and positive isothermal electric susceptibility. We found, for the specific shells considered, that the positivity of the isothermal electric susceptibility is sufficient for the thermodynamic intrinsic stability of the shell. The marginal stability case corresponds to an infinite electric susceptibility, with its positivity depending on the way one approaches the marginal points. If the shell has negative susceptibility, there is a runaway process, making them depart from equilibrium towards a stable equilibrium configuration or even towards a breakdown of the shell.

In this chapter, we have derived some thermodynamic properties for electrically charged spherical matter shells in higher dimensions, complementing a set of works on the thermodynamics of thin shells. There is still more future work that has to be done in regarding the link between thermodynamics and general relativity, hopefully contributing to the understanding of black hole physics and with it to grasp gravitation at the tiniest possible scales.

Part II

STATISTICAL MECHANICAL ENSEMBLES OF BLACK HOLES AND MATTER USING THE EUCLIDEAN PATH INTEGRAL APPROACH

THERMODYNAMICS IN CURVED SPACES THROUGH THE EUCLIDEAN PATH INTEGRAL APPROACH

3.1 THERMODYNAMIC BLACK HOLE ENSEMBLES

3.1.1 *The Gibbons-Hawking statistical path integral and York formalism*

The thermodynamics of stationary configurations involving gravity can be obtained from the construction of statistical ensembles through the Euclidean path integral approach to quantum gravity. The approach is based on extending the statistical path integral to the gravitational sector, where one performs a map from the Lorentzian metric to a Riemannian or pseudo-Riemannian metric [108, 109]. This map is usually a Wick transformation $t \rightarrow -i\tau$, where t is a Lorentzian time coordinate and τ is an imaginary time. The path integral of the Euclidean gravitational action is then performed over the possible metrics with fixed boundary conditions which are extracted from the configuration one wishes to study. In the canonical ensemble, one fixes the inverse temperature given by the total imaginary proper time at the boundary. The boundary of the space then acts as a heat reservoir. The statistical path integral gives the partition function of the ensemble, which is associated to a thermodynamic potential. One can use this link to extract the thermodynamic properties of the system.

The Euclidean path integral approach has several shortcomings. To start, the map between Lorentzian spacetimes and Riemannian or quasi-Riemannian spaces is not well-defined in general. Moreover, the Euclidean gravitational action can be unbounded from below, which makes the path integral ill-defined. It has been suggested that this last issue can be tackled by using conformal classes of metrics and change the contour of the integration [110, 111]. There is also a problem regarding the measure of the gravitational metric. For gauge fields, the measure is well understood, where the overcounting coming from the gauge freedom is removed using ghost contributions. For general metrics, it is not yet clear how to remove the overcounting from diffeomorphisms. Moreover, it is also not clear what is the relative measure between metrics of different topology. Nevertheless, the Euclidean path integral approach yields interesting results when the saddle point approximation is performed. This approximation or the zeroth order version of it, the zero loop approximation, avoids the shortcomings of the Euclidean path integral. It consists on expanding the Euclidean action over the paths that

extremize the action. In the semiclassical limit, these paths contribute the most to the partition function and have a correspondence to physical Lorentzian spacetimes. For the approximation to be valid, one must consider the stationary points that minimize the action. This can be seen from the first order loop corrections, where the integrand can be put in terms of the eigenvalues of an operator. For the stationary points that minimize the action, the operator has positive eigenvalues, yielding a path integral with real values. Otherwise, the stationary point is called an instanton and the first loop corrections yield complex contributions to the partition function with the imaginary part indicating the decay probability of the instanton.

The construction of statistical ensembles through the Euclidean path integral approach to quantum gravity was first applied by Gibbons and Hawking [67]. In the zero loop approximation, the grand canonical and canonical ensembles of Kerr-Newmann black hole spacetimes in four dimensions was considered with boundary at infinity. For the case of a Schwarzschild black hole as a stationary point of the Euclidean gravitational action, it was observed that the heat capacity of the black hole was negative. In the canonical ensemble, this means that the Schwarzschild black hole is thermodynamically unstable and also deems the zero loop approximation invalid. It was further shown in [112] that the Schwarzschild instanton was a saddle point of the Euclidean action and not a maximum, and its existence caused a global instability of flat space. The first loop corrections of the Schwarzschild black hole led to a complex contribution to the partition function due to a negative mode perturbation, which disappeared if the boundary was put at a finite radius [113]. An additional analysis of the negative mode was done in [114]. Moreover, Hawking and Page [69] applied the same formalism to the Schwarzschild-anti de Sitter black hole in four dimensions, with boundary at infinity. In this case, two possible solutions for the black hole mass were found for a fixed temperature, with one having a positive heat capacity and thus being stable. The existence of the stable solution is related to the fact that anti-de Sitter space acts as a finite box. In order to cure the canonical ensemble of a Schwarzschild black hole, taking into consideration the works above, York analyzed the Schwarzschild black hole inside a finite cavity [68]. Two solutions for the Schwarzschild radius were found, in analogy to the Schwarzschild-anti de Sitter case, with one having again a positive heat capacity and thus being stable. By putting the stationary configuration in a finite cavity, the zero loop approximation becomes valid. This is the York formalism in the construction of statistical ensembles of curved spaces. Moreover, York constructed a generalized free energy that allowed the study of phase transitions between hot flat space, i.e. vacuum flat space at a fixed temperature, and the stable Schwarzschild black hole. The motivation for the generalized free energy was given in [115], where the generalized free energy is obtained from the reduced action, which is the Euclidean action with imposed constraint equations that partially extremize the action.

In this part of the thesis, we are interested in exploring the York and Gibbons-Hawking formalism to construct statistical ensembles of various spacetimes. We focus on charged black hole spacetimes and spacetimes involving self-gravitating matter thin shells. The objective is to further understand the phase diagrams

when we include matter and gauge fields, and to uncover possible links between dynamics and thermodynamics.

3.1.2 *Application to different configurations*

The construction of statistical ensembles through the Euclidean path integral approach was further extended to other stationary spacetime configurations and different ensembles. Namely, the formulation of different ensembles with a gravitational action was done in [116], and specifically the microcanonical ensemble was formulated more explicitly in [117]. Regarding other stationary black hole spacetimes without matter, the ensemble and thermodynamics of a two-dimensional black hole in the Teitelboim-Jackiw theory was treated in [118], the formalism was extended to anti-de Sitter black holes [119, 120], and to de Sitter spaces [121–125]. An important study of the canonical ensemble of five-dimensional and d -dimensional Schwarzschild black holes was done in [101, 102], where a link was established between the Buchdahl bound [126, 127] and the radius marking the phase transition from hot flat space to a black hole phase. It is important to note that the Buchdahl bound indicates the maximum bound for the compactness of fluid spheres above which the configuration is singular, when certain energy conditions are obeyed. This bound has been generalized to charged configurations [84], for positive cosmological constant [128] and for higher dimensions [129]. Therefore, the work in [101, 102] suggests the existence of a link between matter dynamics and black hole thermodynamics, which shall be explored in this thesis.

The formalism was extended as well to include the Maxwell vector potential, allowing the treatment of charged black holes. The grand canonical ensemble for Reissner-Nordström black holes in four dimensions was done in [130] and its extension to anti-de Sitter in [131]. The thermodynamics and the construction of the ensembles of Kerr-Newmann black holes through the York formalism was sketched in [132]. Moreover, the canonical ensemble of a Reissner-Nordström black hole in four dimensions was worked out in [133, 134], and the d dimensional Reissner-Nordström-anti-de Sitter was worked out in [135]. The inclusion of matter, namely of a spherical matter thin shell with a black hole inside was done in [136], where it was shown the additivity of the matter and black hole entropies. A more thorough study of this case was done in [137]. The canonical ensemble for arbitrary configurations of a self-gravitating system was studied in [138].

The analysis of ensembles of anti-de Sitter spaces has a deeper motivation related to superstring and supergravity theories, and gauge/gravity duality. In supergravity theories, one usually has a collection of branes living in a world space. Through the gauge/gravity duality, the low energy supergravity in the world space can have a correspondence to a strongly coupled field theory at the boundary. An example of a gauge/gravity duality is the AdS/CFT correspondence [76, 139]. The important feature here of the gauge/gravity duality is that the thermodynamic properties of the branes carry over to the field theory. In that regard, the thermodynamics of black branes through statistical ensembles have been studied [140–144], including electric charge as well. Another motivation for the study of such thermodynamic ensembles

is the Gubser-Mitra conjecture [145]. This conjecture states that black branes are stable to linear perturbations if and only if they are thermodynamically stable. The linear instability of black branes is mainly driven by the Gregory-Laflame instability [146, 147] and it was shown to have a connection with the negative mode arising in the perturbation of the respective instantons for particular cases [142, 148], thus complying with the conjecture.

3.1.3 *Physical scales and the applicability of the zero loop approximation*

It is important to state the applicability of the zero loop approximation to obtain the partition function of self-gravitating systems. The calculation of the one loop corrections can give us a hint, by evaluating when these corrections are negligible. Formally, the loop contributions can be computed by renormalization and regularization techniques, which for the case of statistical path integrals, the zeta regularization procedure [149] and the expansion of the heat kernel through DeWitt-Schwinger proper time [150, 151] are the most utilized. The one loop corrections arise in the form of logarithmic terms that are added to the thermodynamic potential associated to the ensemble, for example see [152–155] where the one loop contributions for different fields have been computed arising from higher order local and non-local curvature terms. There is also a computation of the one loop contributions arising from thermodynamic fluctuations [156], where the heat capacity plays an important role.

The scale controlling the one loop contributions is generally attributed to the Planck scale, $l_p = 1.6 \times 10^{-35}m$. It is expected that one loop contributions are negligible for scales much larger than l_p . Another scale which is fixed in the canonical and grand canonical ensemble is the temperature and the radius of the cavity. It is then useful to understand what are the scales of interest [157, 158] in the semiclassical regime where the zero loop approximation is still valid. We can work with the stable black hole of York [68], for which the zero loop approximation is valid. First, we require that the regime must be far from the Planck length. In the canonical ensemble, this means that the temperature of ensemble must be below Planck temperature $T_p = 10^{32}K$. For this temperature and higher, the stable black hole is close enough to the cavity such that the full quantum regime must be taken into account. Moreover, we want to have a cavity radius far from the Planck length but still small compared to SI units in order to probe the semiclassical regime, e.g. we can choose a cavity radius $R = 10^{20}l_p$. For the parameters chosen, the stable black hole solution exists for temperatures higher than $T = 2 \times 10^{11}K$. However, one loop corrections seem quite relevant near the temperatures at which the large black hole solution starts to exist [156]. Hence, the zero loop approximation is valid for the range of temperatures $2 \times 10^{11}K \ll T \ll 10^{32}K$, with the large black hole radius being of the order $r_+ = 6 \times 10^{19}l_p$. Indeed, these orders of magnitude imply a cavity with $R = 10^{-13}cm$ and a black hole with mass $m = 6 \times 10^{14}g$, which means the system is microscopic, where semiclassical effects enter into play. We could also make the same analysis for the case of an infinite cavity, where we have to consider the Gibbons and Hawking black hole. For such a black hole, the zero

loop approximation of the canonical ensemble is not valid but it is useful for the study of Hawking radiation [159].

The scale analysis above has the purpose of motivating the study of thermodynamic ensembles through the zero loop approximation in order to probe semiclassical effects. Such effects can lead to phase transitions between the stable black hole phase and hot space, which is our main object of study here. Although these phase transitions may occur for temperatures close to the starting point of existence of the stable black hole, where loop corrections may be relevant, we extrapolate the analysis of the zero loop approximation to this regime with the expectation that the qualitative behaviour remains the same.

3.1.4 Outline

The role of this chapter is to introduce the Euclidean path integral approach to quantum gravity and its application to the construction of the statistical path integral for curved spaces. Moreover, this chapter serves as a preparation for the remaining chapters of the second part of the thesis.

In Sec. 3.3.1, we discuss the extension of the Euclidean path integral to obtain the partition function of curved spaces. In Sec. 3.4.1, we work out the restriction of the path integral to spherically symmetric metrics. In Sec. 3.4, we explain the regularity conditions for the spherically symmetric metrics that enter in the path integral. In Sec. 3.5, we present the boundary conditions that establish the data that is fixed in the path integral. In Sec. 3.6.2.1, we calculate the gravitational action that enters in the path integral for spherically symmetric metrics, which is relevant for the upcoming chapters. In Sec. 3.7, we connect the statistical path integral to the relevant thermodynamic potential of an ensemble, from which we can derive the thermodynamic quantities of the ensemble. Finally, in Sec. 3.8, we summarize the chapter.

3.2 THE EUCLIDEAN PATH INTEGRAL APPROACH

In order to construct the generating function of the statistical ensemble, also called the partition function, we employ the Euclidean path integral approach. This approach is mainly used to obtain the partition function of systems with quantum fields, giving origin to the study of thermal field theory. Suppose that one has a quantum system being described by a quantum field $\hat{\psi}$, with its classical counterpart ψ , and with the associated Hamiltonian operator H . The ensemble of the system with fixed temperature and volume, i.e. the canonical ensemble, has the partition function given by $Z = \text{Tr} [e^{-\beta H}]$, where Tr is the trace of the operator over a basis of the Hilbert space, where the quantum field theory is modelled. This trace can be rewritten in terms of a path integral. In order to see this, one can use the formula of the Feynman path integral. Let the system be at the quantum state $|\psi_1\rangle$ at a time t_1 , the amplitude for the system to be at a quantum state $|\psi_2\rangle$ at time t_2 is $\langle\psi_2| e^{-i(t_2-t_1)H} |\psi_1\rangle$, but in turn this can also be given by the

Feynman path integral $\int D\psi e^{iI_L}$, where I_L is the Lorentzian action of the field ψ , and where the path integral has boundary conditions $\psi(t_1) = \psi_1$ and $\psi(t_2) = \psi_2$. One can make now the transformation $i(t - t_1) = \tau'$, where τ' is an imaginary time with period $\beta = i(t_2 - t_1)$, and also set $\psi_1 = \psi_2$. Therefore, the partition function of the ensemble is given by $Z = \int D\psi e^{-I}$, where I is the Euclidean action and the integration is done with periodic boundary conditions with period β for bosonic fields, and anti-periodic boundary conditions for fermionic fields, due to the properties of the commutator and anticommutator between the fields.

The idea of the Euclidean path integral approach to quantum gravity is to apply the aforementioned logic to the gravitational field. The first ingredient is the map between a d dimensional Lorentzian spacetime M_L and a d dimensional Riemannian space M , through a Wick rotation $t \rightarrow -i\tau'$, where t is a Lorentzian time coordinate and τ' is the imaginary time. Of course, such map is not covariant and may be ill-defined. Usually, this issue can be overlooked for static spacetimes, while for stationary spacetimes one must consider a map to a quasi-Riemannian space instead, which satisfies allowable conditions, see [160, 161]. The time coordinate chosen for the map is usually associated to a Killing vector, which is timelike in some region. For black hole spacetimes, the time coordinate chosen is associated to the Killing vector that is timelike near the horizon and becomes null at the horizon. At the boundary of the spacetime, ∂M_L , one has the heat reservoir represented by a timelike hypersurface in the Lorentzian spacetime. This hypersurface can be brought to the Riemannian or quasi-Riemannian space through the map, obtaining a hypersurface ∂M . Now, there must be an identification of points such that the imaginary time τ' is periodic with some constant period. It is better to perform a coordinate transformation $\tau'(\tau)$ and work with an imaginary time τ with period 2π . This map allows the correspondence of a Riemannian or quasi-Riemannian space with the physical spacetime, and more importantly the physical boundary data of the heat reservoir can be established, namely its geometry and its quasilocal quantities, such as the energy and angular momentum. This data at the boundary of space is what we need to consider in the construction of an ensemble, while the specific geometry of the Riemannian or quasi-Riemannian space is not needed in principle but it plays a huge role as we shall see.

The partition function of an ensemble for a curved spacetime including matter fields, through the Euclidean path integral approach, is formally defined by

$$Z = \int Dg_{\alpha\beta} D\psi e^{-I[g_{\alpha\beta}, \psi]}, \quad (3.1)$$

where I is the Euclidean action, $g_{\alpha\beta}$ is the Euclidean metric of the Riemannian space (not to be confused with the Euclidean flat metric), ψ represents any kind of matter or gauge field, $Dg_{\alpha\beta}$ is the integration measure over the paths of $g_{\alpha\beta}$ and $D\psi$ is the integration measure over the path of ψ . The path integral is done over periodic $g_{\alpha\beta}$ and ψ , if bosonic. In general, all paths of $g_{\alpha\beta}$ and ψ may not have locally a physical correspondence, but one does need to give fixed data at the boundary of the Riemannian space, ∂M , where the heat reservoir sits, corresponding to the same data of the stationary Lorentzian spacetime one wants to study, through the Wick rotation mentioned above. Namely for static configurations, one can use the

Dirichlet boundary conditions for the induced metric at the boundary, with fixed inverse temperature defined by $\beta = \int_0^{2\pi} b|_{\partial M} d\tau$, where $b = 1/\sqrt{g^{\tau\tau}}$, and with the remaining components describing the spatial geometry of the boundary in the Lorentzian static configuration. Moreover, one must also give data for the field ψ at the boundary, depending on the type of ensemble one wants to consider. We explain the matter boundary conditions in the following chapters, according to the ensemble under study. With such boundary conditions, the partition function can then be determined. Note that the reason for associating the boundary conditions of the Riemannian space to the data of a Lorentzian configuration allows the Riemannian space obtained from the map of such configuration to be included in the sum of paths, with this Riemannian space extremizing the Euclidean action. This is how a physical meaning is given to the Euclidean path integral, since there is a correspondence to a physical spacetime.

We assume that the Euclidean action $I[g_{\alpha\beta}, \psi]$ is given by the sum $I[g_{\alpha\beta}, \psi] = I_g[g_{\alpha\beta}] + I_m[g_{\alpha\beta}, \psi]$, where $I_g[g_{\alpha\beta}]$ is the gravitational Euclidean action given by the Euclidean Einstein-Hilbert action with the Gibbons-Hawking-York boundary term, i.e.

$$I_g = -\frac{1}{16\pi l_p^{d-2}} \int_M (R - 2\Lambda) \sqrt{g} d^d x - \frac{1}{8\pi l_p^{d-2}} \int_{\partial M} K \sqrt{\gamma} d^{d-1} x - I_{\text{ref}} , \quad (3.2)$$

where R is the Ricci scalar, Λ is the cosmological constant, g is the metric determinant, $K = n^\alpha{}_{;\alpha}$ is trace of the extrinsic curvature of ∂M , n^α is the unit normal vector to ∂M , γ is the determinant of the induced metric γ_{ab} of ∂M and I_{ref} is the action of a reference metric to make I_g finite. The action $I_m[g_{\alpha\beta}, \psi]$ is the matter action which is specified in the following chapters depending on the case of study. One can obtain the Euclidean action I_g from the Lorentzian action I_L by performing the map referred above from a Lorentzian spacetime to a Riemannian space. Neglecting the boundary term on the spacelike hypersurfaces, the effect of the map can be seen by changing the volume elements as $\sqrt{-g_L} d^d x \rightarrow -i\sqrt{g} d^d x$ and $\sqrt{-\gamma_L} d^{d-1} x \rightarrow -i\sqrt{\gamma} d^{d-1} x$, as the integrands are left invariant, where g_L is the determinant of the Lorentzian metric and γ_L is the determinant of the induced Lorentzian metric. The Euclidean action is then defined with an overall minus sign so that $I_L \rightarrow iI_g$, which explains the minus sign in the Ricci term in Eq. (3.2), as the Lorentzian action is defined with a positive sign in the Ricci term. The analysis of the gravitational Euclidean action is going to be split into two cases for spherically symmetric metrics: the zero cosmological constant case and the negative cosmological case. For the negative cosmological case, the anti-de Sitter or AdS length is defined by $l^2 = \frac{(d-1)(d-2)}{-2\Lambda}$.

3.3 THE CLASS OF SPHERICALLY SYMMETRIC METRICS

3.3.1 Smooth metrics

In this thesis, we focus on statistical ensembles of configurations with spherical symmetry. In order to avoid the repetition in the upcoming chapters, we analyze here in detail the Euclidean gravitational action in spherical symmetry.

To avoid the problems coming from a sum over topologies, we can restrict the paths in the path integral to metrics with spherical symmetry. In cases where the system is described by a finite cavity, this can be motivated by the fact that spherically symmetric metrics are expected to contribute the most to the path integral. The Euclidean metric for the Riemannian space M can then be written as

$$ds^2 = b(u)^2 d\tau^2 + a(u)^2 du^2 + r(u)^2 d\Omega_{d-2}^2, \quad (3.3)$$

where $b(u)$, $a(u)$ and $r(u)$ are arbitrary functions of u , the coordinate τ is spanned by $\tau \in]0, 2\pi[$, the coordinate u is spanned by $u \in]0, 1[$ and $d\Omega_{d-2}^2$ is the $(d-2)$ -sphere metric in spherical coordinates θ^A with total area $\Omega_{d-2} = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}$, where Γ is the gamma function.

The boundary of the space is described by the hypersurface $u = 1$, which may be singular for the reservoir at infinity or smooth for the reservoir at a finite radius. It is then useful to analyze hypersurfaces of constant u , which have an induced metric

$$ds^2|_u = b(u)^2 d\tau^2 + r(u)^2 d\Omega_{d-2}^2. \quad (3.4)$$

The dependence on u is now going to be dropped for convenience, except for occasions where clarity demands it. The extrinsic curvature K_{ab} of the constant u hypersurfaces can be calculated using the unit normal $n_a dx^a = a du$ as

$$K_{ab} dx^a dx^b = \frac{b'b}{a} d\tau^2 + \frac{r'r}{a} d\Omega_{d-2}^2, \quad (3.5)$$

where a prime means the derivative over u , i.e. $b' = \frac{db}{du}$. The trace of the extrinsic curvature is given by

$$K = \frac{b'}{ab} + (d-2) \frac{r'}{ar}. \quad (3.6)$$

One can use the Cartan structure equations to determine the Ricci tensor, $R_{\alpha\beta}$, of the metric in Eq. (3.3), together with a differential relation between the components of $d\Omega_{d-2}^2$. The components of the Ricci tensor have the following expression

$$\begin{aligned} R^\tau_\tau &= -\frac{1}{abr^{d-2}} \left(\frac{b'r^{d-2}}{a} \right)', \\ R^y_y &= R^\tau_\tau - \frac{d-2}{ra} \left(\frac{r'}{a} \right)' + (d-2) \frac{b'r'}{a^2 br}, \\ R^{\theta^A}_{\theta^A} &= -\frac{b'r'}{ba^2 r} + \frac{1}{ra} \left(\frac{r'}{a} \right)' - \frac{1}{r'r^{d-2}} \left[r^{d-3} \left(\left(\frac{r'}{a} \right)^2 - 1 \right) \right]', \end{aligned} \quad (3.7)$$

where the indices A are not being summed. The Ricci scalar, R , is then given by

$$R = -\frac{2}{abr^{d-2}} \left(\frac{b'r^{d-2}}{a} \right)' - 2G^\tau_\tau, \quad (3.8)$$

where G^τ_τ is the $\tau\tau$ component of the Einstein tensor, given by

$$G^\tau_\tau = \frac{(d-2)}{2r'r^{d-2}} \left[r^{d-3} \left(\left(\frac{r'}{a} \right)^2 - 1 \right) \right]'. \quad (3.9)$$

3.3.2 C^0 metrics

For the purpose of the thesis, we assume metrics with the form of Eq. (3.3) to be smooth except when there is the presence of a spherical matter thin shell, described by the hypersurface $\mathcal{C}$. In such case, the hypersurface $\mathcal{C}$ separates the space M into the inner region M_1 and the outer region M_2 with metrics

$$ds_1^2 = b_1(u)^2 \frac{b_2(u_m)^2}{b_1(u_m)^2} d\tau^2 + a_1(u)^2 du^2 + r(u)^2 d\Omega_{d-2}^2, \quad (3.10)$$

$$ds_2^2 = b_2(u)^2 d\tau^2 + a_2(u)^2 du^2 + r(u)^2 d\Omega_{d-2}^2, \quad (3.11)$$

respectively, where u_m is the position label of the matter thin shell with the hypersurface $\mathcal{C}$ being described by $u = u_m$, and, $b_1(u)$, $b_2(u)$, $a_1(u)$ and $a_2(u)$ are smooth arbitrary functions. The coordinate u in M_1 has the range $u \in]0, u_m[$ and in M_2 it has the range $u \in]u_m, u[$. The metrics in Eqs. (3.10) and (3.11) can be described by the metric in Eq. (3.3), with metric components

$$b(u) = \begin{cases} \frac{b_1(u)b_2(u_m)}{b_1(u_m)} & 0 < u < u_m \\ b_2(u) & u_m \leq u < 1 \end{cases},$$

$$a(u) = \begin{cases} a_1(u) & 0 < u < u_m \\ a_2(u) & u_m \leq u < 1 \end{cases}, \quad (3.12)$$

While in this case $b(u)$ is continuous, $a(u)$ is not necessarily. Using further a coordinate transformation to a geodesic coordinate $\rho = \int_{u_m}^u a(s)ds$, one turns the discontinuity in a into the property that $r(\rho)$ is C^0 in function of the geodesic coordinate. The metric with these properties is then C^0 with the form

$$ds^2 = b(\rho)^2 d\tau^2 + d\rho^2 + r(\rho)^2 d\Omega_{d-2}^2, \quad (3.13)$$

where

$$b(\rho) = b_1(u(\rho)) \frac{b_2(u_m)}{b_1(u_m)} \theta[-\rho] + b_2(u(\rho)) \theta[\rho], \quad (3.14)$$

where $\theta[\rho]$ is the Heaviside function and the inverse of $\rho(u)$, i.e. $u(\rho)$ was used.

The matter shell and the boundary of the space are described by the hypersurfaces of constant u , i.e. $u = u_m$ and $u = 1$ respectively. As the previous case of smooth metrics, it is useful to analyze hypersurfaces of constant u , which have an induced metric as Eq. (3.4), or explicitly at the shell one has

$$ds^2|_{u=u_m} = b_2(u_m)^2 d\tau^2 + r(u_m)^2 d\Omega_{d-2}^2, \quad (3.15)$$

and at the boundary one has

$$ds^2|_{u \rightarrow 1} = b_2(u \rightarrow 1)^2 d\tau^2 + r(u \rightarrow 1)^2 d\Omega_{d-2}^2, \quad (3.16)$$

The extrinsic curvature K_{ab} of the constant u hypersurfaces is given by Eq. (3.5), taking into account the metrics in Eqs. (3.10) and (3.11). Explicitly at the shell, the extrinsic curvature suffers a jump in general, i.e. the extrinsic curvature computed in one side of the shell is not the same as the one computed at the other side. The jump is defined by square brackets on a tensor living in the hypersurface, e.g. $[K_{ab}] = K_{2ab} - K_{1ab}$, where K_{2ab} is the extrinsic curvature evaluated at the side towards $u > u_m$ and K_{1ab} is the extrinsic curvature evaluated at the side towards $u < u_m$. Namely, the extrinsic curvature at the shell from the side of M_1 is

$$K_{1ab} dx^a dx^b = \frac{b'_1 b_2^2}{a_1 b_1} d\tau^2 + \frac{r' r}{a_1} d\Omega_{d-2}^2, \quad (3.17)$$

and from the side of M_2 is

$$K_{2ab} dx^a dx^b = \frac{b'_2 b_2}{a_2} d\tau^2 + \frac{r' r}{a_2} d\Omega_{d-2}^2, \quad (3.18)$$

where the components are written here in terms of the coordinate u and the prime means the derivative in u . Moreover, the trace of the extrinsic curvature at each side is given by

$$\begin{aligned} K_1 &= \frac{b'_1}{a_1 b_1} + (d-2) \frac{r'}{a_1 r}, \\ K_2 &= \frac{b'_2}{a_2 b_2} + (d-2) \frac{r'}{a_2 r}. \end{aligned} \quad (3.19)$$

At the boundary of space $u = 1$, one has the extrinsic curvature K_{2ab} and its trace K_2 with the same form of Eqs. (3.18) and (3.19) evaluated at $u = 1$.

In the presence of a matter thin shell, the C^0 metric induces Dirac delta terms in the Ricci tensor. We are interested here on the Ricci scalar and the Einstein tensor in particular, which in this case have the expression

$$\begin{aligned} R &= R_1 \theta[-\rho] + R_2 \theta[\rho] - 2[K] \delta[\rho], \\ G^\tau_\tau &= G_1^\tau \theta[-\rho] + G_2^\tau \theta[\rho] + ([K] - [K^\tau_\tau]) \delta[\rho], \end{aligned} \quad (3.20)$$

where R_1 is the Ricci scalar evaluated at $\rho < 0$ or $u < u_m$, R_2 is the Ricci scalar evaluated at $\rho > 0$ or $u > u_m$, G_1^τ is the Einstein tensor component evaluated at

$\rho < 0$ or $u < u_m$, and $G_2^\tau{}_\tau$ is the Einstein tensor component evaluated at $\rho > 0$ or $u > u_m$, with these quantities being given by

$$R_1 = -\frac{2}{a_1 b_1 r^{d-2}} \left(\frac{b_1' r^{d-2}}{a_1} \right)' \frac{b_2(u_m)^2}{b_1(u_m)^2} - 2G_1^\tau{}_\tau, \quad (3.21)$$

$$R_2 = -\frac{2}{a_2 b_2 r^{d-2}} \left(\frac{b_2' r^{d-2}}{a_2} \right)' - 2G_2^\tau{}_\tau, \quad (3.22)$$

$$G_1^\tau{}_\tau = \frac{(d-2)}{2r' r^{d-2}} \left[r^{d-3} \left(\left(\frac{r'}{a_1} \right)^2 - 1 \right) \right]', \quad (3.23)$$

$$G_2^\tau{}_\tau = \frac{(d-2)}{2r' r^{d-2}} \left[r^{d-3} \left(\left(\frac{r'}{a_2} \right)^2 - 1 \right) \right]', \quad (3.24)$$

written in terms of the coordinate u . Notice that the expansion in Eq. (3.20) can be computed by writing the Ricci scalar and the Einstein tensor in terms of the first and second derivatives of b and r , use the chain rule $\frac{d}{d\rho} = \frac{1}{a(u)} \frac{d}{du}$, and then use the expansion in Heaviside functions, with the identity $\frac{d\theta[\rho]}{d\rho} = \delta(\rho)$. This is indeed the same procedure as the thin shell formalism [77], where the continuity of the metric is imposed as the first junction condition. The expression for the Dirac delta term in the Einstein tensor is given by

$$[K] - [K^\tau{}_\tau] = \frac{(d-2)}{r} \left(\frac{r'}{a_2} - \frac{r'}{a_1} \right) \Big|_{u=u_m}, \quad (3.25)$$

written in terms of the coordinate u . The expression in Eq. (3.25) is useful further on and constitutes one component of the gravitational part of the second junction condition.

We have to insert the metric, in Eq. (3.3) for smooth metrics or in Eqs. (3.10) and (3.11) for C^0 metrics, in the path integral in Eq. (3.1), together with the boundary conditions at the hypersurface $u = 1$ describing the boundary of space ∂M and also the heat reservoir. These boundary conditions are fixed while performing the path integral. One then should sum over all the possible metrics on the path integral. In principle, the sum over the metrics can be decomposed in terms of their topology class. In the case here treated, the topology class depends on a set of regularity conditions for the spherically symmetric metric at $u = 0$. Below, we present the regularity and boundary conditions used in the thesis for spherically symmetric metrics.

3.4 METRIC REGULARITY CONDITIONS

3.4.1 Black hole-like conditions

For spherically symmetric metrics, we need to impose regularity conditions at the center of the space or at its minimal surface, which for the metric in Eq. (3.3) is situated at $u = 0$. The black hole-like conditions correspond to the choice

$$b(0) = 0, \quad r(0) = r_+, \quad (3.26)$$

for the components of the metric, where r_+ is the horizon radius.

This choice alone can induce possible divergences in the Ricci scalar and topological defects, which here they must be avoided by imposing conditions to the derivatives of the components of the metric. These conditions can be found by expanding the metric near $u = 0$ as

$$\begin{aligned} ds^2 = & \left[\left(\frac{b'}{a} \right)^2 \Big|_{u=0} \varepsilon^2 + \left(\frac{b'}{a^2} \left(\frac{b'}{a} \right)' \right) \Big|_{u=0} \varepsilon^3 + \mathcal{O}(\varepsilon^4) \right] d\tau^2 \\ & + d\varepsilon^2 + \left[r_+ + (r'a^{-1}) \Big|_{u=0} \varepsilon + \mathcal{O}(\varepsilon^2) \right]^2 d\Omega_{d-2}^2, \end{aligned} \quad (3.27)$$

where $\varepsilon = \int_0^\delta a du$ for small δ , assuming that $\int_0^\delta a du$ is finite. Otherwise, $u = 0$ should be understood as a boundary of the space. The condition $b(0) = 0$ means that a hypersurface with constant u , having topology $\mathbb{S}^1 \times \mathbb{S}^{d-2}$, becomes topologically $\{u = 0\} \times \mathbb{S}^{d-2}$ in the limit of $u = 0$, i.e. a point times a $(d-2)$ -sphere and the hypersurface volume becomes zero. This behaviour is precisely described by the metric in Eq. (3.27), namely, the (τ, ε) sector describes approximately the metric of a cone in general, with a possible conical singularity which introduces a topological defect in the Riemannian space. In order to avoid the existence of such singularity, we impose the regularity condition

$$\frac{b'}{a} \Big|_{u=0} = 1, \quad (3.28)$$

and so the (τ, ε) sector of metric describes Euclidean flat space near $u = 0$. The remaining conditions are found from avoiding the divergence of the Ricci scalar. The Ricci scalar near $u = 0$ is given by

$$R = -\frac{2(d-2)}{\varepsilon r_+} \left(\frac{r'}{a} \right) \Big|_{u=0} - \frac{2}{\varepsilon} \left(\frac{1}{a} \left(\frac{b'}{a} \right)' \right) \Big|_{u=0} + \mathcal{O}(1), \quad (3.29)$$

and so the regularity conditions are

$$\left(\frac{r'}{a} \right) \Big|_{u=0} = 0, \quad \left(\frac{1}{a} \left(\frac{b'}{a} \right)' \right) \Big|_{u=0} = 0. \quad (3.30)$$

It is interesting to note that the first equation of Eq. (3.30) is equivalent, in even dimensions, to the condition that the Riemannian space must have an Euler characteristic $\chi = 2$.

The conditions in Eqs. (3.26)–(3.30) are precisely the conditions of the metric that one would obtain if the Wick transformation of a stationary black hole spacetime metric was performed. The $(d-2)$ -surface at $u=0$ coincides with the bifurcate $(d-2)$ -sphere of the horizon of the stationary black hole. The topology of the Riemannian space is $\mathbb{R}^2 \times \mathbb{S}^{d-2}$ with these conditions.

3.4.2 Flat conditions

Other possible regularity conditions are the flat conditions, which are achieved by choosing

$$b(0) \text{ finite and non zero}, \quad r(0) = 0, \quad (3.31)$$

for the components of the metric. By expanding the metric near $u=0$ with the conditions in Eq. (3.31), one has

$$\begin{aligned} ds^2 = & \left(b(0) + \left(\frac{b'}{a} \right) \Big|_{u=0} \varepsilon + \mathcal{O}(\varepsilon^2) \right) d\tau^2 + d\varepsilon^2 \\ & + \left[\left(\frac{r'}{a} \right) \Big|_{u=0} \varepsilon + \frac{1}{a} \left(\frac{r'}{a} \right)' \varepsilon^2 + \mathcal{O}(\varepsilon) \right]^2 d\Omega_{d-2}^2, \end{aligned} \quad (3.32)$$

where again $\varepsilon = \int_0^\delta adu$ for small δ and it is assumed that $\int_0^\delta adu$ is finite. The remaining regularity conditions must be extracted from the condition that the Ricci scalar is well-behaved at $u=0$. The Ricci scalar is

$$\begin{aligned} R = & -\frac{2(d-2)}{b\varepsilon} \left(\frac{b'}{a} \right) \Big|_{u=0} - \frac{2(d-2)}{\varepsilon} \left(\frac{1}{r'} \left(\frac{r'}{a} \right)' \right) \Big|_{u=0} \\ & - \frac{(d-2)(d-3)}{\varepsilon^2} \left[1 - \left(\frac{a}{r'} \right)^2 \right] \Big|_{u=0}, \end{aligned} \quad (3.33)$$

near $u=0$. Therefore, in order to avoid the divergence of the Ricci scalar, the following regularity conditions

$$\left(\frac{b'}{a} \right) \Big|_{u=0} = 0, \quad \left(\frac{r'}{a} \right) \Big|_{u=0} = 1, \quad \left(\frac{1}{a} \left(\frac{r'}{a} \right)' \right) \Big|_{u=0} = 0, \quad (3.34)$$

are necessary.

The regularity conditions in Eqs. (3.31) and (3.34) are the conditions of the Riemannian metric if the Wick transformation was performed to a flat Lorentzian metric. The topology of the Riemannian space with these regularity conditions is $\mathbb{S}^1 \times \mathbb{R}^{d-1}$.

3.5 METRIC BOUNDARY CONDITIONS

3.5.1 Finite cavity

The boundary conditions that we impose at the boundary of the Riemannian space for the metric are given here by the Dirichlet boundary conditions. In the case of the

spherically symmetric metric in Eq. (3.3), the boundary of the space is positioned at $u = 1$ with induced metric

$$ds^2|_{u=1} = b(1)^2 d\tau^2 + r(1)^2 d\Omega_{d-2}^2 . \quad (3.35)$$

According to the Dirichlet boundary conditions, for a finite boundary, we must fix

$$\beta = 2\pi b(1) , \quad R = r(1) , \quad (3.36)$$

that is, we must fix the inverse temperature β of the spherical shell at $u = 1$, representing the heat reservoir, that is given as the total imaginary time length, and moreover we fix the radius R of the shell. We therefore have a Riemannian space which represents a finite cavity, assuming the regularity conditions in the previous section.

3.5.2 Infinite cavity: zero cosmological constant

For the case where the boundary of the space is infinite, i.e. when $r(u)|_{u \rightarrow 1}$ is infinite, the boundary conditions of the Riemannian space are given according to the asymptotic behaviour of the metric when $u \rightarrow 1$. The cases for zero and negative cosmological constant must be analyzed separately as the metric has different asymptotic behaviour.

When the cosmological constant is zero, the boundary conditions imposed are the same as the asymptotically flat spacetime conditions but translated to Riemannian space. In this sense, the behaviour of the metric components must be that

$$b(u)|_{u \rightarrow 1} = \frac{\beta}{2\pi} , \quad \frac{r'}{a}|_{u \rightarrow 1} = 1 , \quad (3.37)$$

where $b(u)|_{u \rightarrow 1}$ must be a fixed constant and it is given by β , the inverse temperature measured at infinity.

3.5.3 Infinite cavity: negative cosmological constant

When the cosmological constant is negative, for an infinite hypersurface ∂M , the boundary conditions imposed are the ones of asymptotically anti-de Sitter or AdS, but translated to Riemannian space. This amounts to the metric satisfying asymptotically the Euclidean Einstein equations with a negative cosmological constant, i.e. $R_{\alpha\beta} = -\frac{(d-1)}{l^2} g_{\alpha\beta}$ and fixing the remaining freedom in the metric. In order to put these conditions in terms of the components of the metric and their asymptotic behaviours, we must perform a conformal transformation in the metric $g_{\alpha\beta} = w^2 \bar{g}_{\alpha\beta}$, where $\bar{g}_{\alpha\beta}$ is the conformal metric and w is the conformal factor. In order to have a nonsingular conformal metric, the conformal transformation must have a behaviour $w = \frac{c}{r(u)}$, where c is a constant. We choose $c = 1$. We further make the coordinate transformation $w = w(u)$ so that the conformal metric assumes the form in the neighbourhood $\mathcal{N}(\partial M)$ of the hypersurface ∂M as

$$d\bar{s}^2|_{\mathcal{N}(\partial M)} = \frac{b(u)^2}{r(u)^2} d\tau^2 + \left(\frac{a(u)r(u)}{r'(u)} \right)^2 dw(u)^2 + d\Omega_{d-2}^2 . \quad (3.38)$$

The hypersurface ∂M is then defined by $w = 0$. The asymptotic behaviour of the metric $g_{\alpha\beta}$ translates into conditions for the metric $\bar{g}_{\alpha\beta}$, see [162, 163]. In fact, the condition $R_{\alpha\beta} = -\frac{(d-1)}{l^2}g_{\alpha\beta}$ can be split into two conditions for the conformal metric, namely that the boundary $w = 0$ is described by the metric $d\bar{s}^2|_{w=0} = d\bar{\tau}^2 + d\Omega_{d-2}^2$ and that $\bar{g}^{\alpha\beta}\nabla_\alpha w \nabla_\beta w = \frac{1}{l^2}$, where $\bar{\tau}$ is proportional to τ by some constant. And so the boundary conditions chosen for the metric elements are

$$\frac{b(u)}{r(u)} \Big|_{u \rightarrow 1} = \frac{\bar{\beta}}{2\pi l}, \quad \frac{a(u)r(u)}{r'(u)} \Big|_{u \rightarrow 1} = l, \quad (3.39)$$

where $\bar{\beta}$ is defined as the inverse temperature measured at the conformal boundary of the asymptotically AdS space.

3.6 THE GRAVITATIONAL PATH INTEGRAL IN SPHERICAL SYMMETRY

3.6.1 Smooth metrics

3.6.1.1 General considerations

We now proceed to reduce the path integral in Eq. (3.1) to the case of spherically symmetric Riemannian spaces. For the case of zero cosmological constant, one has the partition function $Z = \int Dg_{\alpha\beta} D\psi e^{-I_g[g_{\mu\nu}] - I_m[g_{\mu\nu}, \psi]}$ with the gravitational action being given by

$$I_g = -\frac{1}{16\pi l_p^{d-2}} \int_M (R - 2\Lambda) \sqrt{g} d^d x - \frac{1}{8\pi l_p^{d-2}} \int_{\partial M} K \sqrt{\gamma} d^{d-1} x - I_{\text{ref}}. \quad (3.40)$$

For the case of a smooth metric, one can use the expression of the Ricci scalar in Eq. (3.8) and the expression of the trace of the extrinsic curvature in Eq. (3.6) to obtain

$$\begin{aligned} & -\frac{1}{16\pi l_p^{d-2}} \int_M R \sqrt{g} d^d x = -\frac{\Omega_{d-2}}{4l_p^{d-2}} \left(\frac{b'r^{d-2}}{a} \right) \Big|_{u=0} + \frac{\Omega_{d-2}}{4l_p^{d-2}} \left(\frac{b'r^{d-2}}{a} \right) \Big|_{u \rightarrow 1} \\ & + \frac{1}{8\pi l_p^{d-2}} \int_M a b r^{d-2} G^\tau_\tau d^d x, \\ & -\frac{1}{8\pi l_p^{d-2}} \int_{\partial M} K \sqrt{\gamma} d^{d-1} x = -\frac{2\pi}{\mu} \left(\frac{b'r^{d-3}}{a} \right) \Big|_{u \rightarrow 1} - \frac{\Omega_{d-2}}{4l_p^2} \left(\frac{b'r^{d-2}}{a} \right) \Big|_{u \rightarrow 1}, \end{aligned} \quad (3.41)$$

where $\sqrt{g} = a b r^{d-2}$, $\sqrt{\gamma} = b r^{d-2}$ and $\mu = \frac{8\pi l_p^{d-2}}{(d-2)\Omega_{d-2}}$. Putting together the expressions in Eqs. (3.41) into Eq. (3.40), one finally has

$$\begin{aligned} I_g = & -\left(\frac{2\pi b r^{d-3}}{\mu} \left(\frac{r'}{a} \right) \right) \Big|_{u \rightarrow 1} - \frac{\Omega_{d-2}}{4l_p^{d-2}} \left(\frac{b'r^{d-2}}{a} \right) \Big|_{u=0} \\ & + \frac{1}{8\pi l_p^{d-2}} \int_M a b r^{d-2} (G^\tau_\tau + \Lambda) d^d x - I_{\text{ref}}. \end{aligned} \quad (3.42)$$

The form of the gravitational action in Eq. (3.42) assumes the form of the typical decomposition of the space in a foliation of hypersurfaces, with G^τ_τ term and the boundary term at $u \rightarrow 1$ being part of the Hamiltonian of the space. The remaining terms are determined by the regularity and boundary conditions, and also by the choice of the reference space. It is interesting to note that the term at $u = 0$ seems to be topological, due to its link with the regularity conditions.

3.6.1.2 Zero cosmological constant

For the case of zero cosmological constant, the reference space is the Riemannian space obtained by performing the map to flat Lorentzian spacetime at the same temperature, which we call hot flat space, giving

$$ds_{\text{flat}}^2 = b(1)^2 d\tau^2 + dr^2 + r^2 d\Omega_{d-2}^2, \quad (3.43)$$

where the coordinate transformation $r = r(y)$ was performed and $r \in]0, r(1)[$. It is important to distinguish this space from the flat Riemannian space since the former has topology $S^1 \times \mathbb{R}^{d-1}$ while the latter has topology $\mathbb{R}^d$. The action for hot flat space can be written as

$$I_{\text{flat}} = -\frac{1}{8\pi l_p^{d-2}} \int_{\partial M} K_{\text{flat}} d^{d-1}x, \quad (3.44)$$

or alternatively can be obtained from Eq. (3.42) by setting $\left(\frac{r'}{a}\right)_{\text{flat}} = 1$, $\left(\frac{b'}{a}\right)_{\text{flat}} = 0$, $G^\tau_\tau = 0$, with $\Lambda = 0$ and flat regular conditions, yielding

$$I_{\text{flat}} = -\frac{2\pi}{\mu} \left(b r^{d-3} \right) \Big|_{u \rightarrow 1}. \quad (3.45)$$

The gravitational action for a spherically smooth metric with zero cosmological constant is then

$$\begin{aligned} I_{\text{gf}} = & \left(\frac{2\pi b r^{d-3}}{\mu} \left(1 - \frac{r'}{a} \right) \right) \Big|_{u \rightarrow 1} - \frac{\Omega_{d-2}}{4l_p^{d-2}} \left(\frac{b' r^{d-2}}{a} \right) \Big|_{u=0} \\ & + \frac{1}{8\pi l_p^{d-2}} \int_M a b r^{d-2} G^\tau_\tau d^d x. \end{aligned} \quad (3.46)$$

3.6.1.3 Negative cosmological constant

For negative cosmological constant, the reference space chosen is the Riemannian space obtained from performing the map to AdS spacetime, which we call hot AdS space, at the same temperature. The metric describing hot AdS space is

$$ds_{\text{AdS}}^2 = b_{\text{AdS}}(y(r))^2 d\tau^2 + \left(\frac{a}{r'}(r) \right)_{\text{AdS}}^2 dr^2 + r^2 d\Omega_{d-2}^2, \quad (3.47)$$

where $b_{\text{AdS}}(1) = b(1)$, $\left(\frac{b'}{a}\right)_{\text{AdS}}\big|_{u=0} = 0$ and $\left(\frac{r'}{a}\right)_{\text{AdS}}^2 = 1 + \frac{r^2}{l^2}$. The action for the hot AdS space is more contrived compared to hot flat space, since the Ricci scalar is $R = -\frac{d(d-1)}{l^2}$, and it is given by

$$I_{\text{AdS}} = \frac{(d-1)}{8\pi l_p^{d-2}} \int_M \sqrt{g} d^d x - \frac{1}{8\pi l_p^{d-2}} \int_{\partial M} K_{\text{AdS}} \sqrt{\gamma} d^{d-1} x. \quad (3.48)$$

Alternatively, one can use also Eq. (3.42) to evaluate the action of hot AdS space by using the expression of the components of the metric at the boundary, plus that $G^\tau_\tau = -\Lambda$ and the flat regularity conditions, to obtain

$$I_{\text{AdS}} = - \left(\frac{2\pi b r^{d-3}}{\mu} \left(\frac{r'}{a} \right)_{\text{AdS}} \right)_{u \rightarrow 1}. \quad (3.49)$$

The gravitational action for a smooth spherically symmetric metric with a negative cosmological constant is then

$$\begin{aligned} I_{gl} = & \left(\frac{2\pi b r^{d-3}}{\mu} \left(\left(\frac{r'}{a} \right)_{\text{AdS}} - \frac{r'}{a} \right) \right) \bigg|_{u \rightarrow 1} - \frac{\Omega_{d-2}}{4l_p^{d-2}} \left(\frac{b' r^{d-2}}{a} \right) \bigg|_{u=0} \\ & + \frac{1}{8\pi l_p^{d-2}} \int_M a b r^{d-2} \left(G^\tau_\tau - \frac{(d-1)(d-2)}{2l^2} \right) d^d x. \end{aligned} \quad (3.50)$$

3.6.2 C^0 metrics

3.6.2.1 General considerations

For the case where the metric is C^0 with non-differentiability at the hypersurface $\mathcal{C}$, described by $u = u_m$, the gravitational action can be given by

$$\begin{aligned} I_g = & -\frac{1}{16\pi l_p^{d-2}} \int_{M_1} (R_1 - 2\Lambda) \sqrt{g} d^d x - \frac{1}{16\pi l_p^{d-2}} \int_{M_2} (R_2 - 2\Lambda) \sqrt{g} d^d x \\ & + \frac{1}{8\pi l_p^{d-2}} \int_{\mathcal{C}} [K] \sqrt{\gamma} d^{d-1} x - \frac{1}{8\pi l_p^{d-2}} \int_{\partial M} K \sqrt{\gamma} d^{d-1} x - I_{\text{ref}}, \end{aligned} \quad (3.51)$$

where Eq. (3.20) was used to decompose the Ricci scalar in terms of Heaviside functions and the Dirac delta. Another way of getting Eq. (3.51) is by summing the Einstein-Hilbert action with the Gibbons-Hawking-York boundary term in each region M_1 and M_2 , noting that the difference on the trace of the extrinsic curvature

is not zero. The integrals can be decomposed in terms of the spherically symmetric metric components as

$$\begin{aligned}
& -\frac{1}{16\pi l_p^{d-2}} \int_{M_1} R \sqrt{g} d^d x = -\frac{\Omega_{d-2}}{4l_p^{d-2}} \left(\frac{b'_1 b_2(u_m) r^{d-2}}{a_1 b_1(u_m)} \right) \Big|_{u=0} \\
& + \frac{\Omega_{d-2}}{4l_p^{d-2}} \left(\frac{b'_1 b_2(u_m) r^{d-2}}{a_1 b_1(u_m)} \right) \Big|_{u=u_m} + \frac{1}{8\pi l_p^{d-2}} \int_{M_1} a_1 b_1 \frac{b_2(u_m)}{b_1(u_m)} r^{d-2} G_1^\tau d^d x, \\
& -\frac{1}{16\pi l_p^{d-2}} \int_{M_2} R_2 \sqrt{g} d^d x = -\frac{\Omega_{d-2}}{4l_p^{d-2}} \left(\frac{b'_2 r^{d-2}}{a_2} \right) \Big|_{u=u_m} + \frac{\Omega_{d-2}}{4l_p^{d-2}} \left(\frac{b'_2 r^{d-2}}{a_2} \right) \Big|_{u \rightarrow 1} \\
& + \frac{1}{8\pi l_p^{d-2}} \int_{M_2} a_2 b_2 r^{d-2} G_2^\tau d^d x, \\
& \frac{1}{8\pi l_p^{d-2}} \int_C [K] \sqrt{\gamma} d^{d-1} x = -\frac{1}{8\pi l_p^{d-2}} \int_C ([K^\tau] - [K]) \sqrt{\gamma} d^{d-1} x \\
& + \frac{\Omega_{d-2}}{4l_p^2} \left(\frac{b'_2 r^{d-2}}{a_2} \right) \Big|_{u \rightarrow u_m} - \frac{\Omega_{d-2}}{4l_p^2} \left(\frac{b'_1 b_2(u_m) r^{d-2}}{a_1 b_1(u_m)} \right) \Big|_{u \rightarrow u_m}, \\
& -\frac{1}{8\pi l_p^{d-2}} \int_{\partial M} K \sqrt{\gamma} d^{d-1} x = -\frac{2\pi}{\mu} \left(\frac{b_2 r' r^{d-3}}{a_2} \right) \Big|_{u \rightarrow 1} - \frac{\Omega_{d-2}}{4l_p^2} \left(\frac{b'_2 r^{d-2}}{a_2} \right) \Big|_{u \rightarrow 1}. \quad (3.52)
\end{aligned}$$

Putting together the terms in Eq. (3.52), the action for the spherically symmetric C^0 metric is

$$\begin{aligned}
I_g = & -\frac{2\pi}{\mu} \left(\frac{b_2 r' r^{d-3}}{a_2} \right) \Big|_{u \rightarrow 1} - \frac{\Omega_{d-2}}{4l_p^{d-2}} \left(\frac{b'_1 b_2(u_m) r^{d-2}}{a_1 b_1(u_m)} \right) \Big|_{u=0} \\
& + \frac{1}{8\pi l_p^{d-2}} \int_{M_1} a_1 b_1 \frac{b_2(u_m)}{b_1(u_m)} r^{d-2} (G_1^\tau + \Lambda) d^d x + \frac{1}{8\pi l_p^{d-2}} \int_{M_2} a_2 b_2 r^{d-2} (G_2^\tau + \Lambda) d^d x \\
& - \frac{1}{8\pi l_p^{d-2}} \int_C ([K^\tau] - [K]) \sqrt{\gamma} d^{d-1} x - I_{\text{ref}}, \quad (3.53)
\end{aligned}$$

which is basically Eq. (3.42) but with $b(u)$, $a(u)$ and G^τ expanded in Heaviside functions and Dirac delta. The non-smoothness of the metric leads to the additional boundary term of the action at the hypersurface $\mathcal{C}$ compared to the action of smooth metrics. Indeed, this is expected as the term $[K^\tau] - [K]$ represents the junction condition for the shell that comes from the Dirac delta term of G^τ .

3.6.2.2 Zero cosmological constant

The action for spherically symmetric C^0 metric with zero cosmological constant follows from the analysis with the smooth metric. The action for the reference space is I_{flat} from Eq. (3.44), which in terms of the metric components is

$$I_{\text{flat}} = -\frac{2\pi}{\mu} \left(b_2 r^{d-3} \right) \Big|_{u \rightarrow 1}. \quad (3.54)$$

And so the action of a C^0 metric with a zero cosmological constant in Eq. (3.53) becomes

$$\begin{aligned}
I_{gf} = & \left(\frac{2\pi b_2 r^{d-3}}{\mu} \left(1 - \frac{r'}{a_2} \right) \right) \Big|_{u \rightarrow 1} - \frac{\Omega_{d-2}}{4l_p^{d-2}} \left(\frac{b'_1 b_2(u_m) r^{d-2}}{a_1 b_1(u_m)} \right) \Big|_{u=0} \\
& + \frac{1}{8\pi l_p^{d-2}} \int_{M_1} a_1 b_1 \frac{b_2(u_m)}{b_1(u_m)} r^{d-2} G_1^\tau d^d x + \frac{1}{8\pi l_p^{d-2}} \int_{M_2} a_2 b_2 r^{d-2} G_2^\tau d^d x \\
& - \frac{1}{8\pi l_p^{d-2}} \int_{\mathcal{C}} ([K^\tau_\tau] - [K]) \sqrt{\gamma} d^{d-1} x .
\end{aligned} \tag{3.55}$$

3.6.2.3 Negative cosmological constant

For the case of negative cosmological constant, the reference action is I_{AdS} from Eq. (3.48), which in terms of the metric components is

$$I_{\text{AdS}} = - \left(\frac{2\pi b_2 r^{d-3}}{\mu} \left(\frac{r'}{a_2} \right)_{\text{AdS}} \right) \Big|_{u \rightarrow 1} . \tag{3.56}$$

Therefore, the action of a spherically symmetric C^0 metric with negative cosmological constant, in Eq. (3.53), becomes

$$\begin{aligned}
I_{gl} = & \left(\frac{2\pi b_2 r^{d-3}}{\mu} \left(\left(\frac{r'}{a} \right)_{\text{AdS}} - \frac{r'}{a_2} \right) \right) \Big|_{u \rightarrow 1} - \frac{\Omega_{d-2}}{4l_p^{d-2}} \left(\frac{b'_1 b_2(u_m) r^{d-2}}{a_1 b_1(u_m)} \right) \Big|_{u=0} \\
& + \frac{1}{8\pi l_p^{d-2}} \int_{M_1} a_1 b_1 \frac{b_2(u_m)}{b_1(u_m)} r^{d-2} \left(G_1^\tau - \frac{(d-1)(d-2)}{2l^2} \right) d^d x \\
& + \frac{1}{8\pi l_p^{d-2}} \int_{M_2} a_2 b_2 r^{d-2} \left(G_2^\tau - \frac{(d-1)(d-2)}{2l^2} \right) d^d x \\
& - \frac{1}{8\pi l_p^{d-2}} \int_{\mathcal{C}} ([K^\tau_\tau] - [K]) \sqrt{\gamma} d^{d-1} x .
\end{aligned} \tag{3.57}$$

3.7 THE STATISTICAL PATH INTEGRAL AND ITS CONNECTION TO THERMODYNAMICS

With the shape of the action decomposed into the spherically symmetric metric components, the Euclidean path integral is composed by the sum over the possible paths of the metric components and matter fields as

$$Z = \int DbDaDrD\psi e^{-I_g - I_m} , \tag{3.58}$$

where I_g can be given by I_{gf} in the case of zero cosmological constant or I_{gl} for the case of negative cosmological constant, with the sum being made over components obeying the boundary conditions and over the possible regularity conditions.

As discussed above, even in this form, an expression for the path integral seems quite elusive. Typically, one performs the saddle point approximation to find the paths of the metric components b , a , r and ψ , that minimize the full action. We

are going to apply this approximation in the upcoming chapters for the particular cases of interest, namely for black hole spaces with a static electromagnetic field or with matter, and also for the case of a self-gravitating matter thin shell. In the saddle point approximation, one can consider only the zeroth order contribution which translates into a partition function $Z = e^{-I_0}$, where I_0 is the action evaluated at a minimum path. This is the zero loop approximation.

Depending on the ensemble, the partition function is tied to a thermodynamic potential. This can be seen from the definition of the partition function and the possible mean thermodynamic values that one can obtain. For example, in the canonical ensemble, with the inverse temperature and area fixed, one can obtain the mean energy through

$$E = -\frac{\partial \log(Z)}{\partial \beta} , \quad (3.59)$$

since $-\frac{\partial \log(Z)}{\partial \beta}$ can be formally written as $\sum_i E_i e^{-\beta E_i} / Z$, where the i subscript means with respect to each microstate. Moreover, the entropy is defined as the Gibbs entropy with the formula $S = -\sum_i p_i \log(p_i)$, with $p_i = e^{-\beta E_i} / Z$, which can be written in terms of $\log(Z)$ as

$$S = -\beta \frac{\partial \log(Z)}{\partial \beta} + \log(Z) . \quad (3.60)$$

Using the formula for the energy and the entropy, the partition function can be related to the free energy F in the canonical ensemble as

$$\beta F = -\log(Z) , \quad (3.61)$$

where the free energy is defined by the Legendre transform of the mean energy, $F = E - TS$. We can now establish the connection between the zero loop approximation and thermodynamics. Since $\log(Z) = -I_0$, then we have

$$F = T I_0 , \quad (3.62)$$

which means that the action evaluated at the minimizing paths translates into the free energy of the ensemble. Having the action in the zero loop approximation, we can obtain straightforwardly the free energy and the remaining thermodynamic quantities, i.e. the mean energy, the entropy and the pressure, by calculating the derivatives of the free energy.

A similar analysis holds for the case of the grand canonical ensemble, where the thermodynamic potential that has the connection with the partition function is the grand potential $W = -\log(Z)$, where we define $W = E - TS - \mu N$, with μ being a chemical potential and N being a mean number. Hence, the grand potential is given by the action at the minimum path as $W = T I_0$, and we can obtain the thermodynamic quantities by calculating the derivatives of W or I_0 .

3.8 SUMMARY

In this chapter, we gave an introduction to the construction of statistical ensembles of curved spacetimes, here with focus on the metric. In order to study the statistical ensemble of a configuration described by a stationary Lorentzian spacetime, with a timelike hypersurface as the boundary and describing the heat reservoir, we must perform a map of the spacetime to a Riemannian or pseudo-Riemannian metric. The shape of the Riemannian or pseudo-Riemannian metric has to be relaxed except for the fixed data at the boundary, which must be the same data of the configuration that we want to study. For spherically symmetric spaces, we impose Dirichlet boundary conditions that compose the fixed data at the boundary space, which in this case is described by a spherical shell and the inverse temperature of the ensemble is fixed to be the total imaginary time length at the boundary. The partition function is then given by the Euclidean path integral over the Riemannian or pseudo-Riemannian metrics with fixed boundary data. For spherically symmetric metrics, we also need to sum over the discrete set of regularity conditions which are tied to the topology of the Riemannian space.

Here, we restricted the shape of the metric to spherically symmetric metrics as the boundary data is given for a spherical shell. We explained the possible regularity and boundary conditions, which are going to be used in the next chapters. Moreover, we decomposed the gravitational action in terms of the spherically symmetric metric components, ready to be used for the analysis of specific configurations including matter. We performed the calculations in this chapter to avoid repetition in the following chapters.

Finally, we established the connection of the partition function through the Euclidean path integral with the thermodynamics of the ensemble. In order to obtain the partition function, we are going to employ the zero loop approximation in the upcoming chapters. The action evaluated at the minimizing paths gives the relevant thermodynamic potential of the ensemble and through its derivatives, we can obtain the thermodynamic properties of the system. This analysis is expanded further in the upcoming chapters.

GRAND CANONICAL ENSEMBLE OF A d -DIMENSIONAL REISSNER-NORDSTRÖM BLACK HOLE IN A CAVITY

4.1 INTRODUCTION

As previously discussed in Chapter 3, the Euclidean path integral approach [67] allows the construction of statistical ensembles in curved spaces. Moreover, the use of the zero loop approximation allows the computation of the partition function in terms of the classical paths of the action. The approach was extended to the York formalism [68, 115], where a finite cavity is introduced, allowing for stable equilibrium configurations. The formalism was used to construct the grand canonical ensemble of a charged black hole inside a cavity in four dimensions [130, 131] and for black branes [143]. Also, York's analysis was extended to higher dimensions [101, 102], where there was an emphasis on the connection between the statistical ensemble and matter dynamic stability in curved spacetime. Namely, the two black hole solutions of the ensemble bifurcate when the cavity is at the light ring radius, and also the stable black hole starts to be more favorable than hot flat space when its radius corresponds to the Buchdahl bound.

Motivated by these developments, in this chapter, we construct the grand canonical ensemble of a Reissner-Nordström black hole inside a cavity in higher dimensions using the Euclidean path integral approach to quantum gravity, with fixed temperature and fixed electric potential. We perform the zero loop approximation in steps. First, the Hamiltonian and Gauss constraints are imposed to find a reduced action and then the stationary points of the reduced action are found, corresponding to black hole solutions of the ensemble. It is found that there are up to two solutions of the ensemble, with its qualitative behaviour being presented. The two solutions bifurcate at a certain ratio between horizon radius and cavity radius, which does not correspond to the light ring ratio. The main objective was to study the phase transitions and read off possible connections to matter dynamic stability. We therefore analyzed the phase transitions between the black hole solutions and hot flat space, finding a first order phase transition. We found that the black hole solution is more favorable when its radius is slightly lower than the Buchdahl-Andréasson-Wright bound [129], which is related to the maximum compactness of a charged configuration obeying certain energy conditions. We present in detail the five dimensional case, $d = 5$, where an analytic expression for the black hole solutions was found.

This chapter is organized as follows. In Sec. 4.2, we consider the partition of the grand canonical ensemble for spherically symmetric metrics, obeying regularity and boundary conditions, namely the fixed inverse temperature is established as the total imaginary proper time at the boundary of the cavity and the radius of the cavity is fixed. In Sec. 4.3.2, we perform the zero loop approximation, where we first impose the constraints to find the reduced action and we find the stationary points of the reduced action. In Sec. 4.4, we obtain the thermodynamics of the system from the partition function in the zero loop approximation and we further analyze the possible phase transitions. In Sec. 4.6.1, we present in detail the five dimensional case, $d = 5$. In Sec. 4.6, we compare the bifurcation radius with the light ring radius, and also the thermodynamic radius with the Buchdahl-Andre sson-Wright bound. In Sec. 4.8, we conclude the chapter. We note that the work in this chapter is based on [2].

4.2 THE GRAND CANONICAL ENSEMBLE OF A CHARGED BLACK HOLE IN THE EUCLIDEAN PATH INTEGRAL APPROACH

4.2.1 The partition function

Through the Euclidean path integral approach, we can construct the grand canonical ensemble of a charged black hole inside a finite cavity, in d dimensions, by considering the partition function

$$Z = \int Dg_{\alpha\beta} DA_\gamma e^{-I[g_{\mu\nu}, A_\sigma]}, \quad (4.1)$$

with the Euclidean action

$$I = - \int_{\mathcal{M}} \left(\frac{R}{16\pi l_p^{d-2}} - \frac{F_{ab}F^{ab}}{4} \right) \sqrt{g} d^d x - \frac{1}{8\pi l_p^{d-2}} \int_{\partial M} (K - K_0) \sqrt{\gamma} d^{d-1} x, \quad (4.2)$$

where R is the Ricci scalar, g is the determinant of the Euclidean metric $g_{\alpha\beta}$, $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ is the strength field tensor of the Maxwell vector potential A_α , γ is the determinant of the induced metric γ_{ab} of the hypersurface describing the boundary ∂M , $K = n^\alpha_{;\alpha}$ is the trace of the extrinsic curvature of the hypersurface with n^α being the outward unit normal to it, and K_0 is the extrinsic curvature of the boundary embedded in flat space, giving the action of hot flat space. The action in Eq. (4.2) can be split into $I = I_{\text{gf}} + I_A$, where

$$I_{\text{gf}} = - \frac{1}{16\pi l_p^{d-2}} \int_{\mathcal{M}} R \sqrt{g} d^d x - \frac{1}{8\pi l_p^{d-2}} \int_{\partial M} (K - K_0) \sqrt{\gamma} d^{d-1} x, \quad (4.3)$$

$$I_A = \int_M \frac{F_{ab}F^{ab}}{4} \sqrt{g} d^d x. \quad (4.4)$$

The path integral in the partition function in Eq. (4.1) is performed along Riemannian metrics with fixed boundary conditions, which are periodic in the imaginary time. We make a reminder that among the possible paths are Riemannian metrics

that correspond to physical static Lorentzian spacetimes by a Wick transformation in the imaginary time. For further details on the construction of the path integral, see Chapter 3.

4.2.2 Geometry and boundary conditions

In this case, we consider the boundary of space to be described by a spherical shell with fixed temperature and electric potential. This means we are dealing with the grand canonical ensemble. Due to the spherical symmetry of the boundary, the metrics with spherical symmetry should contribute the most to the path integral. And so, we restrict the path integral to spherically symmetric metrics of the form

$$ds^2 = b(u)^2 d\tau^2 + a(u)^2 du^2 + r(u)^2 d\Omega_{d-2}^2, \quad (4.5)$$

where $b(u)$, $a(u)$ and $r(u)$ are arbitrary smooth functions of u , the coordinates have the range $\tau \in]0, 2\pi[$ and $u \in]0, 1[$, and $d\Omega_{d-2}^2$ is the $(d-2)$ -sphere line element.

In principle, the path integral should include a sum over topologies of the Riemannian space with a metric of the form of Eq. (4.5). The sum over topologies is related to the sum over metrics with different regularity conditions. Here, we choose the black hole-like regularity conditions, described by

$$\begin{aligned} b(0) &= 0, \\ r(0) &= r_+, \\ (b'\alpha^{-1}) \Big|_{u=0} &= 1, \\ \alpha^{-1}(b'\alpha^{-1})' \Big|_{u=0} &= 0, \\ \left(\frac{r'}{\alpha} \right) \Big|_{u=0} &= 0, \end{aligned} \quad (4.6)$$

where r_+ is the horizon radius and a prime denotes the derivative of a function in u , e.g. $b' = \frac{db}{du}$. The boundary conditions, as already stated, are such that the boundary is described by a spherical shell, in this case with the boundary at $u = 1$, with induced metric

$$ds_{\partial M}^2 = b(1)^2 d\tau^2 + R^2 d\Omega_{d-2}^2, \quad (4.7)$$

where R is the radius of the shell. As part of the boundary conditions, we fix the radius of the shell R or equivalently its area defined by

$$A = \Omega_{d-2} R^{d-2}, \quad (4.8)$$

where $\Omega_{d-2} = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}$ is the area of the unit $(d-2)$ -sphere, with Γ being the gamma function. For $d = 4$, we have $\Omega_4 = 4\pi$ and, for $d = 5$, we have $\Omega_5 = 2\pi^2$. We also fix the inverse temperature β at the boundary of space, which corresponds to the component $b(1)$ as

$$\beta = 2\pi b(1), \quad (4.9)$$

where $\beta = 1/T$, with T being the temperature of the heat reservoir.

We also need to provide regularity and boundary conditions for the Maxwell field A_α according to the regularity conditions of the metric and to the ensemble in question. In spherical symmetry, and without considering magnetic monopoles, the strength field tensor is zero except for the component $F_{y\tau} = -F_{\tau y} = A'_\tau$, where we choose a gauge in which only the Maxwell component A_τ is non-zero. At $u = 0$, we enforce the regularity condition

$$A(0) = 0 , \quad (4.10)$$

while at the boundary, $u = 1$, we fix the electric potential given by

$$\beta\phi = 2\pi i A_\tau(1) . \quad (4.11)$$

We note that the correspondence between the electric potential and the Maxwell field can be deduced by defining ϕ as the electric potential measured by a stationary observer in the physical Lorentzian spacetime and then use the Euclidean Maxwell field instead of the physical one.

4.2.3 Action in spherical symmetry

Having the expression of the metric with the regularity and the boundary conditions, we can now simplify the action in Eq. (4.2) by using the explicit form of the spherically symmetric metrics in Eq. (4.5). By using the results of Chapter 3, the gravitational action can be written as

$$I_{\text{gf}} = \left(\frac{2\pi b r^{d-3}}{\mu} \left(1 - \frac{r'}{a} \right) \right) \Big|_{u=1} - \frac{\Omega_{d-2}}{4l_p^{d-2}} \left(\frac{b' r^{d-2}}{a} \right) \Big|_{u=0} + \frac{1}{8\pi l_p^{d-2}} \int_M ab r^{d-2} G^\tau_\tau d^d x , \quad (4.12)$$

where

$$\mu = \frac{8\pi l_p^{d-2}}{(d-2)\Omega_{d-2}} , \quad (4.13)$$

and the Einstein tensor component G^τ_τ is given by

$$G^\tau_\tau = \frac{(d-2)}{2r' r^{d-2}} \left(r^{d-3} \left(\frac{r'^2}{a^2} - 1 \right) \right)' . \quad (4.14)$$

Together with the regularity and boundary conditions in Eqs. (4.6)–(4.9), the action in Eq. (4.12) becomes

$$I_{\text{gf}} = \left(\frac{\beta R^{d-3}}{\mu} \left(1 - \frac{r'}{a} \right) \right) \Big|_{u=1} - \frac{\Omega_{d-2} r_+^{d-2}}{4l_p^{d-2}} + \frac{1}{8\pi l_p^{d-2}} \int_M ab r^{d-2} G^\tau_\tau d^d x . \quad (4.15)$$

The action for the Maxwell field can also be simplified using that $F^{\alpha\beta}F_{\alpha\beta} = 2F_{u\tau}F^{u\tau} = 2\frac{A_\tau'^2}{b^2a^2}$. It is more convenient to work with the integrand written as

$$\frac{\sqrt{g}}{4}F_{\alpha\beta}F^{\alpha\beta} = -\frac{1}{2}\frac{r^{d-2}A_\tau'^2}{ba} + \left(\left(\frac{r^{d-2}A_\tau'}{ba}\right)A_\tau\right)' - \left(\frac{r^{d-2}A_\tau'}{ba}\right)'A_\tau, \quad (4.16)$$

and so the Maxwell action is given by

$$I_A = -\frac{1}{2}\int_M \frac{r^{d-2}A_\tau'^2}{ba}d^dx - i\beta\phi\Omega_{d-2}\left(\frac{r^{d-2}A_\tau'}{ba}\right)\Big|_{u=1} - \int_M \left(\frac{r^{d-2}A_\tau'}{ba}\right)'A_\tau d^dx, \quad (4.17)$$

where the regularity and boundary conditions in Eqs. (4.10) and (4.11) have been used. Finally, putting together both actions, we have that the full action I is

$$I = \left(\frac{\beta R^{d-3}}{\mu}\left(1 - \frac{r'}{a}\right)\right)\Big|_{u=1} - \frac{\Omega_{d-2}r_+^{d-2}}{4l_p^{d-2}} - i\beta\phi\Omega_{d-2}\left(\frac{r^{d-2}A_\tau'}{ba}\right)\Big|_{u=1} + \frac{1}{8\pi l_p^{d-2}}\int_M ab r^{d-2}\left(G^\tau_\tau - 4\pi l_p^{d-2}\frac{A_\tau'^2}{b^2a^2}\right)d^dx - \int_M \left(\frac{r^{d-2}A_\tau'}{ba}\right)'A_\tau d^dx, \quad (4.18)$$

which must be inserted in the Euclidean path integral

$$Z = \int DbDaDrDA_\tau e^{-I}. \quad (4.19)$$

4.3 ZERO LOOP APPROXIMATION AND THE BLACK HOLE SOLUTIONS

4.3.1 The constrained path integral and reduced action

As a step towards the zero loop approximation, we constrain the path integral along metrics that obey the constraint equations that partially minimize the action. The constraint equations are composed by the Hamiltonian constraint, momentum constraint and the Gauss constraint. The momentum constraint is satisfied apriori since the metric is static. The Hamiltonian constraint is given by $G^\tau_\tau = 8\pi l_p^{d-2}T^\tau_\tau$, where $T_{\alpha\beta} = F_{\alpha\mu}F_{\beta\nu}g^{\mu\nu} - \frac{1}{4}g_{\alpha\beta}F_{\nu\mu}F^{\nu\mu}$ is the stress energy tensor of the Maxwell field. This Einstein equation is precisely obtained by calculating the first order variation of the Euclidean action in the metric component $b(u)$. In terms of the components of the metric, the Hamiltonian constraint is given by

$$\frac{(d-2)}{2r'r^{d-2}}\left(r^{d-3}\left(\frac{r'^2}{a^2} - 1\right)\right)' = 4\pi l_p^{d-2}\frac{A_\tau'^2}{a^2b^2}. \quad (4.20)$$

The Gauss constraint is given by the Maxwell equation $\nabla_u F^{\tau u} = 0$, which in terms of the metric and Maxwell field components yields

$$\left(\frac{r^{d-2}A_\tau'}{ab}\right)' = 0, \quad (4.21)$$

which can be obtained by performing the first order variation of the Euclidean action in the Maxwell component A_τ . The Gauss constraint can be first integrated to give

$$\frac{r^{d-2}}{ab} A'_\tau = -i \frac{q}{\Omega_{d-2}}, \quad (4.22)$$

where q is the electric charge, having dimensions of length $L^{\frac{d}{2}-2}$, where L is some unit of length. Plugging this into the Hamiltonian constraint, one can integrate Eq. (4.20) to obtain

$$\left(\frac{r'}{a}\right)^2 = f(r; r_+, q) = 1 - \frac{r_+^{d-3}}{r^{d-3}} - \frac{\lambda q^2}{r^{2d-6}} + \frac{\lambda q^2}{r^{2d-6}}, \quad (4.23)$$

where we used the regularity condition $r(0) = r_+$, we defined $f(r; r_+, q)$, and λ is given by

$$\lambda = \frac{8\pi l_p^{d-2}}{\Omega_{d-2}^2 (d-2)(d-3)}. \quad (4.24)$$

The action in Eq. (4.18) with the Hamiltonian and Gauss constraints imposed is called the reduced action and assumes a simple expression, since the bulk terms disappear, yielding

$$I_*[\beta, \phi, R; r_+, q] = \frac{R^{d-3}\beta}{\mu} \left(1 - \sqrt{f[R; r_+, q]}\right) - q\beta\phi - \frac{\Omega_{d-2} r_+^{d-2}}{4l_p^{d-2}}. \quad (4.25)$$

The reduced action I_* becomes a functional of the parameters r_+ and q , and a function of the fixed parameters β , ϕ and R . With the constraints applied to the path integral in Eq. (4.1), the integral over b can be neglected as the reduced action does not depend on b . Also, the integral over A_τ can also be neglected for the same reason. The remaining integrals over a , r and A_τ transform into integrals over r_+ and q . In order to see this, first one can perform an arbitrary coordinate transformation $r = r(y)$, which gives a metric only as functional of $b(y)$, r_+ and q . And so, the sum over configurations obeying the constraints must be done only in r_+ and q . Then, the constrained path integral can be written as

$$Z[\beta, \phi, R] = \int D[r_+] D[q] e^{-I_*[\beta, \phi, R; r_+, q]}. \quad (4.26)$$

To proceed with the zero loop approximation, one must impose the remaining Einstein-Maxwell equations. These equations are equivalent to the conditions for the stationary points of the reduced action in the plane $r_+ \times q$. For the zero loop approximation to remain valid, the stationary points must be local minima of the action. The motivation of imposing the constraint equations to the path integral is that we are able from Eq. (4.26) to verify the stability of the solutions given by the stationary point, at least along the hypersurface of metrics obeying the constraints, see [115].

4.3.2 Stationary points of the reduced action

The partition function in Eq. (4.26) describes the grand canonical ensemble of a charged black hole inside a cavity constrained to the hypersurface where the Hamiltonian and Gauss constraints are satisfied. Here, we are interested in performing the full zero loop approximation of the path integral to obtain the equilibrium solutions for the black hole. The solutions are described by the stationary points of the reduced action in Eq. (4.25), which satisfy the conditions

$$\frac{\partial I_*}{\partial r_+} = 0, \quad (4.27)$$

$$\frac{\partial I_*}{\partial q} = 0. \quad (4.28)$$

These two conditions can be written in terms of the fixed variables of the ensemble, β and ϕ , and the variables evaluated at the stationary points, r_+ and q , as

$$\beta = \frac{4\pi}{(d-3)} \frac{r_+^{2d-5}}{r_+^{2d-6} - \lambda q^2} \sqrt{f[R, r_+, q]}, \quad (4.29)$$

$$\phi = \frac{q}{(d-3)\Omega_{d-2}\sqrt{f[R, r_+, q]}} \left(\frac{1}{r_+^{d-3}} - \frac{1}{R^{d-3}} \right), \quad (4.30)$$

respectively. In order to find the solutions of the ensemble, one must solve the inverse problem of the system in Eqs. (4.29) and (4.30) to have the functions $r_+ = r_+(\beta, \phi, R)$ and $q = q(\beta, \phi, R)$. The reduced action evaluated at the stationary points $r_+ = r_+(\beta, \phi, R)$ and $q = q(\beta, \phi, R)$ is defined as

$$I_0[\beta, \phi, R] = I_*[\beta, \phi, R; r_+[\beta, \phi, R], q[\beta, \phi, R]]. \quad (4.31)$$

Using the expression of the reduced action in Eq. (4.25), the action I_0 can be further written as

$$\begin{aligned} I_0[\beta, \phi, R] = & \frac{R^{d-3}\beta}{\mu} \left(1 - \sqrt{f[R; r_+[\beta, \phi, R], q[\beta, \phi, R]]} \right) \\ & - q[\beta, \phi, R]\beta\phi - \frac{\Omega_{d-2}r_+^{d-2}[\beta, \phi, R]}{4l_p^{d-2}}. \end{aligned} \quad (4.32)$$

In the zero loop approximation, the partition function is given solely by the contribution of the stationary point, i.e.

$$Z[\beta, \phi, R] = e^{-I_0[\beta, \phi, R]}, \quad (4.33)$$

where $I_0[\beta, \phi, R]$ is taken from Eq. (4.32).

We analyze now the solutions of the stationary conditions. It is useful to make the following definitions

$$\gamma = \frac{16\pi^2 R^2}{(d-3)^2} \frac{\Phi^2}{\beta^2(1-\Phi^2)^2}, \quad (4.34)$$

$$\Phi = (d-3)\Omega_{d-2}\sqrt{\lambda}\phi, \quad (4.35)$$

$$x = \frac{r_+}{R}, \quad (4.36)$$

$$y = \frac{\lambda q^2}{R^{2d-6}}. \quad (4.37)$$

The parameter γ takes the role of the temperature $T = \frac{1}{\beta}$, while Φ takes the role of ϕ . The radius of the reservoir R is taken to be a scale, that can be absorbed in the variables r_+ and q , having thus the horizon radius in units of R , yielding x , and the charge squared in units of R multiplied by the ratio between the Planck length and R , yielding y . The system of equations in Eqs. (4.29) and (4.30) can be inverted to give equations for x and y . Inverting Eq. (4.30), i.e. $y = \frac{x^{2d-6}\Phi^2}{1-(1-\Phi^2)x^{d-3}}$, and substituting into Eq. (4.29), one arrives to the equation

$$(1-\Phi^2)x^{d-1} - x^2 + \frac{\Phi^2}{\gamma} = 0. \quad (4.38)$$

Now using Eq. (4.38) into $y = \frac{x^{2d-6}\Phi^2}{1-(1-\Phi^2)x^{d-3}}$, one gets the second equation as

$$y = \gamma x^{2(d-2)}. \quad (4.39)$$

Hence, the solutions to the stationary conditions are obtained by solving Eq. (4.38) for $x(\gamma, \Phi)$ and the value of $y(\gamma, \Phi)$ can then be read off from Eq. (4.39).

The solutions for the horizon radius satisfying the polynomial equation Eq. (4.38) can only be found analytically for specific values of d . For example, analytical solutions can be found for $d = 4$ as we have a third order polynomial equation for x [130], and also for $d = 5$ as we have a second order polynomial equation for x^2 . We analyze this last case below, separately. For generic d , however, it is not possible to find an analytic expression for x .

Notwithstanding, we can study the qualitative behaviour of Eq. (4.38) in terms of the dimension d and the parameters γ , which represents the fixed temperature of the ensemble, and Φ , which represents the fixed electric potential of the ensemble. We impose that the solutions for x and y must be physical. This is so since the Riemannian space that minimizes the action must be correspondent to a Lorentzian space through the inverse Wick rotation. Here, we assume that the black hole must lie inside the cavity and that it must be subextremal, i.e.

$$0 \leq x < 1, \quad (4.40)$$

$$0 \leq \frac{y}{x^{2(d-3)}} < 1, \quad (4.41)$$

respectively. One can use Eq. (4.39) to rewrite the last condition Eq. (4.41) into

$$\gamma x^2 < 1. \quad (4.42)$$

Moreover, using Eq. (4.38) in Eq. (4.42), one obtains a condition for Φ as

$$0 \leq \Phi^2 < 1. \quad (4.43)$$

Therefore, the conditions for physical solutions reduce to the restrictions of the parameters γ and Φ through Eqs. (4.42) and (4.43), respectively.

In order to study further Eq. (4.38), it is useful to define

$$h(x) = (1 - \Phi^2)x^{d-1} - x^2 + \frac{\Phi^2}{\gamma}. \quad (4.44)$$

The values of the function $h(x)$ at the boundaries established by the condition Eq. (4.40) are $h(0) = \frac{\Phi^2}{\gamma} > 0$ and $h(1) = h(0)(1 - \gamma)$. The parameter γ , although being restricted by Eq. (4.42), can assume values higher than unity. Since γ is proportional to the temperature, γ can attain large values for large values of the temperature, for fixed ϕ or Φ . We thus need to separate the analysis into three regions, $\gamma < 1$, $\gamma = 1$, and $\gamma > 1$.

Starting with $\gamma < 1$, one has that $h(x)$ is positive at the boundaries $h(0) > 0$ and $h(1) > 0$. To further gain knowledge on the amount of zeros, one must compute the zeros of the first derivative $h'(x)$ and the sign of the second derivative $h''(x)$. The derivatives are given by

$$h'(x) = x(d-1)(1 - \Phi^2) \left(x^{d-3} - \frac{2}{(d-1)(1 - \Phi^2)} \right), \quad (4.45)$$

$$h''(x) = (d-1)(d-2)(1 - \Phi^2)x^{d-3} - 2. \quad (4.46)$$

The derivative of $h(x)$ vanishes at a bifurcation point

$$x_{\text{bif}} = \left(\frac{2}{(d-1)(1 - \Phi^2)} \right)^{\frac{1}{d-3}}, \quad (4.47)$$

i.e. $h'(x_{\text{bif}}) = 0$, and the second derivative is positive there, i.e. $h''(x_{\text{bif}}) > 0$. So, the bifurcation point x_{bif} marks the location of the only minimum of $h(x)$. In the case of $\gamma < 1$, if the location of the minimum lies out of bounds, $x_{\text{bif}} > 1$, then it is certain that there are no zeros of $h(x)$ in the interval $0 \leq x < 1$. If the minimum lies in the interval $0 < x_{\text{bif}} < 1$, then $h(x)$ may have one or two zeros in $0 \leq x < 1$. This happens for the range of $0 \leq \Phi^2 < \frac{d-3}{d-1}$. However, in order for the zeros to exist, one must require that $h(x_{\text{bif}}) < 0$, which only happens when

$$\gamma \geq \gamma_{\text{bif}}(\Phi, d) \equiv \frac{(d-1)^{\frac{d-1}{d-3}}}{4^{\frac{1}{d-3}}(d-3)} \Phi^2 (1 - \Phi^2)^{\frac{2}{d-3}}. \quad (4.48)$$

Summarizing the results in the interval $\gamma < 1$ and $0 \leq \Phi^2 < \frac{d-3}{d-1}$, for

$$\gamma < \gamma_{\text{bif}}, \quad (4.49)$$

there are no solutions. For

$$\gamma_{\text{bif}} \leq \gamma < 1, \quad (4.50)$$

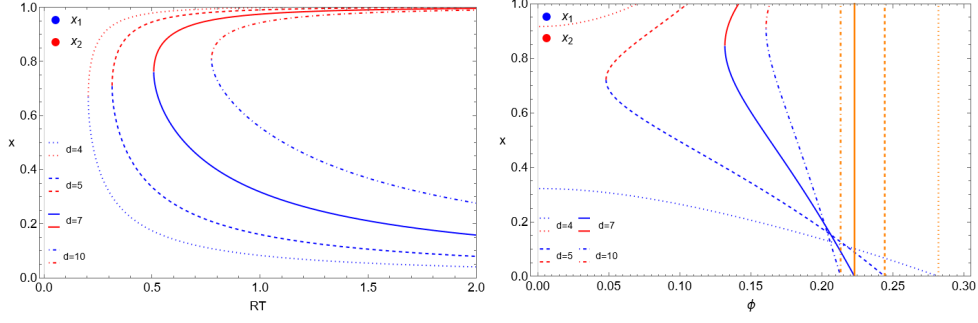


Figure 4.1: Left plot: Stationary points of the action are plotted, with x_1 in blue and x_2 in red, in function of RT , for a constant $\phi = 0.02$ and for four values of d : $d = 4$ in dotted lines, $d = 5$ in dashed lines, $d = 7$ in solid lines, and $d = 10$ in dot dashed lines. Right plot: Stationary points of the action are plotted, with x_1 in blue and x_2 in red, in function of ϕ , for a constant $RT = 0.3$, and the maximum value of ϕ (in orange) corresponding to $\Phi = 1$, for four values of d : $d = 4$ in dotted lines, $d = 5$ in dashed lines, $d = 7$ in solid lines, and $d = 10$ in dot dashed lines. Here, l_p is set to one.

there are two solutions. The solutions can be denominated by x_1 and x_2 , with $x_1 \leq x_2$. Moreover, one must have

$$x_1 \leq x_{\text{bif}} \leq x_2. \quad (4.51)$$

When the equality is reached in Eq. (4.50), i.e. $\gamma = \gamma_{\text{bif}}$, the two solutions merge into $x_1 = x_2 = x_{\text{bif}}$, hence the nomenclature of x_{bif} as the bifurcation point. For $\gamma < 1$ and $\frac{d-3}{d-1} \leq \Phi^2 < 1$, there are no solutions.

For $\gamma = 1$, the function $h(x)$ has always one zero at $x = 1$. We note that this point is a critical point since the derivatives of the action are not well-defined there. If $0 \leq \Phi^2 < \frac{d-3}{d-1}$, the other zero lies in the interval $0 < x < 1$ and so it corresponds to x_1 while $x_2 = 1$. For the case of equality $\Phi^2 = \frac{d-3}{d-1}$, one has $x_1 = x_2 = 1$. For $\frac{d-3}{d-1} < \Phi^2 < 1$, $x_1 = 1$ while x_2 lies out of bounds as $x_2 > 1$ and it is unphysical.

For $\gamma > 1$, the function $h(x)$ has always one zero, x_1 , in the interval $0 \leq x < 1$, while the remaining zero x_2 lies in the interval $x > 1$, all of this for $0 \leq \Phi^2 < 1$. The zero x_1 is physical while the zero x_2 is unphysical.

We must highlight some important comments on the behaviour of the solutions. From the definition of γ in Eq. (4.34), one has that $\gamma \propto \frac{R^2}{\beta^2} \propto (RT)^2$. Considering the solution x_2 in function of the temperature RT , one has that x_2 exists with an image in the interval $0 < x_2 < 1$ for a finite range of $T = \frac{1}{\beta}$ and for $\Phi^2 < \frac{d-3}{d-1}$. Specifically, for $RT = \frac{d-3}{4\pi|\Phi|}(1 - \Phi^2)$, i.e. $\gamma = 1$, one has $x_2 = 1$ and so for higher values of RT , the solution x_2 goes out of bounds and becomes unphysical. This behaviour is not present in the neutrally charged case, analyzed in [102]. The plot of the solutions x_1 and x_2 as functions of RT for a constant $\phi = 0.02$ and for four values of d is shown in the left part of Fig. 4.1, with $l_p = 1$. Also, the plot of the solutions x_1 and x_2 as functions of ϕ for a constant $RT = 0.3$ and for four values of d is shown in the right part of Fig. 4.1, with $l_p = 1$. In these plots, we chose the quantity ϕ instead of Φ

since we want to showcase the full dependence in the dimension d . Indeed, ϕ is the fixed independent parameter at the cavity while Φ is a quantity proportional to d .

4.3.3 Beyond zero loop approximation and stability of the stationary points

With the stationary points analyzed, we proceed with the analysis of their stability. In order to probe the stability, we can go beyond the zero loop approximation of the reduced action. The reduced action is to be expanded around the stationary points up to second order terms, i.e. the reduced action can be written as

$$I_*[\beta, \phi, R; r_+, q] = I_0[\beta, \phi, R] + \sum_{ij} I_{*0ij} \delta i \delta j, \quad (4.52)$$

where $I_0[\beta, \phi, R]$ is defined generically in Eq. (4.31) and specifically for the grand canonical ensemble of a charged black hole in Eq. (4.32), and I_{*0ij} are the second derivatives of the reduced action $I_{*ij} = \frac{\partial^2 I_*}{\partial i \partial j}$ evaluated at an extremum of the action, with i and j being either r_+ or q . The partition function with this expansion around a particular stationary point is given by

$$Z[\beta, \phi, R] = e^{-I_0[\beta, \phi, R]} \int D[\delta q] D[\delta r_+] e^{-\sum_{ij} I_{*0ij} \delta i \delta j}. \quad (4.53)$$

In order to have a well-defined path integral in this approximation, which takes into account the static one loop corrections obeying the Hamiltonian and Gauss constraints, the exponent must be always negative, i.e. the stationary point must be a minimum of the reduced action.

To obtain the condition of stability in terms of the solutions of the ensemble, one must analyze the hessian of the reduced action, which has in this case the components

$$I_{*0r_+r_+} = \frac{(d-2)\Omega_{d-2}R^{d-3}\beta}{16\pi\sqrt{f}r_+^2l_p^{d-2}}\mathcal{I}_{r_+r_+}, \quad (4.54)$$

$$I_{*0r_+q} = \frac{(d-2)\Omega_{d-2}R^{d-3}\beta}{16\pi\sqrt{f}r_+ql_p^{d-2}}\mathcal{I}_{r_+q}, \quad (4.55)$$

$$I_{*0qq} = \frac{(d-2)\Omega_{d-2}R^{d-3}\beta}{16\pi\sqrt{f}q^2l_p^2}\mathcal{I}_{qq}, \quad (4.56)$$

with

$$\mathcal{I}_{r_+r_+} = \frac{d-3}{fx^{2d-6}} \left[\frac{d-3}{2} (x^{2d-6} - y)^2 - (x^{2d-6} - (2d-5)y) (1 - x^{d-3}) (x^{d-3} - y) \right], \quad (4.57)$$

$$\mathcal{I}_{r_+q} = -\frac{(d-3)}{x^{d-3}} \frac{(2x^{d-3} - x^{2d-6} - y)}{x^{d-3} - y} y, \quad (4.58)$$

$$\mathcal{I}_{qq} = 2 \frac{1 - x^{d-3}}{x^{d-3} - y} y. \quad (4.59)$$

For the stationary point to be a minimum, the matrix I_{*0ij} must be positive definite. In turn, this condition can be translated into the principal minors being positive through the Sylvester criterion, i.e.

$$\mathcal{I}_{r_+r_+} > 0, \quad (4.60)$$

$$\mathcal{I}_{r_+r_+}\mathcal{I}_{qq} - \mathcal{I}_{r_+q}^2 > 0. \quad (4.61)$$

In this case, $\mathcal{I}_{qq}$ is always positive and so the last condition, Eq. (4.61), is sufficient to ensure positive definiteness. Using Eqs. (4.57)-(4.59) and (4.39), the condition in Eq. (4.61) reduces to

$$-(d-3)\gamma x^{d-1} + (d-1)x^{d-3} - 2 > 0. \quad (4.62)$$

This is the sufficient condition for a stationary point, given by Eqs. (4.38) and (4.39), to be a minimum of the action.

Since the stability condition in Eq. (4.62) is to be evaluated at the stationary points, one can use Eq. (4.38) to further simplify Eq. (4.62). By rewriting Eq. (4.38) as $\gamma = \frac{\Phi^2}{x^2 - (1-\Phi^2)x^{d-1}}$ and by substituting γ in Eq. (4.62), the stability condition simplifies into a factorized polynomial

$$\frac{((d-1)(1-\Phi^2)x^{d-3} - 2)(1 - x^{d-3})}{1 - (1-\Phi^2)x^{d-3}} > 0. \quad (4.63)$$

Now, the physical range of solutions is $0 \leq x^{d-3} < 1$ which means the denominator is always greater than zero. Therefore, the condition can be reduced to $(d-1)(1-\Phi^2)x^{d-3} - 2 > 0$. The stationary point is thus stable if

$$x > x_{\text{bif}}, \quad (4.64)$$

where $x_{\text{bif}} = \left(\frac{2}{(d-1)(1-\Phi^2)} \right)^{\frac{1}{d-3}}$. Since x_{bif} marks the point of bifurcation of the two solutions x_1 and x_2 , one has $x_1 < x_{\text{bif}} < x_2$. The bifurcation radius thus marks marginal stability. It must be noted that in the uncharged case, the bifurcation radius also marks marginal stability and it coincides with the photon sphere radius. It turns out that in the case of the grand canonical ensemble, this is not the case, see Sec. 4.6.

Coming back to the analysis of the stability, for $\gamma_{\text{bif}} < \gamma < 1$ and $\Phi^2 < \frac{d-3}{d-1}$, one has $x_1 < x_{\text{bif}} < x_2$ and so x_1 is unstable, corresponding to a saddle point of the reduced action, while x_2 is stable, corresponding to a minimum of the reduced action. For the case $\frac{d-3}{d-1} \leq \Phi^2 < 1$, there are no solutions in the physical interval $0 < x < 1$. In case of the equality $\gamma = \gamma_{\text{bif}}$, the solutions $x_1 = x_2 = x_{\text{bif}}$ coincide. While this may signal marginality on the stability, one should be more careful and analyze higher derivatives at this point. By inspection of the plots of the action $I(x, y)$, one finds that x_{bif} is a saddle point.

For the case of $\gamma = 1$, and with $\Phi^2 < \frac{d-3}{d-1}$, solution x_1 is unstable, while the solution x_2 resides at the boundary of the cavity, $x_2 = 1$. At $x_2 = 1$, the derivatives of the action are not well-defined and so the stability cannot be specified. For

$\Phi^2 = \frac{d-3}{d-1}$, the two solutions x_1 and x_2 coincide and reside at the boundary of the cavity, sharing the properties of $x_2 = 1$. For $\frac{d-3}{d-1} < \Phi^2 < 1$, the solution $x_1 = 1$ resides at the cavity, so the stability cannot be specified, while x_2 lies outside the bounds of the cavity and it is unphysical.

Finally for $\gamma > 1$ and for $0 < \Phi^2 < 1$, the solution x_1 is the only physical solution and it is unstable.

4.3.4 Most probable configurations of the ensemble

With the stationary points obtained and their stability characterized, we are interested to see the configurations that are most favorable or most probable. From Eq. (4.26), it can be seen that the paths with the lowest I_* , or from Eq. (4.33) the paths with the lowest I_0 , are the ones that contribute the most to the partition function, and so they correspond to the most probable states. Here, we make a comparison between the critical points of the reduced action obtained above.

In the electrically uncharged case done in [68] for $d = 4$ and in [101, 102] for generic d , the comparison between the stable black hole solution and hot flat space was made regarding what was the most favorable state. The stable black hole is a stationary point of the reduced action, and the hot flat space solution is an extra stationary point existing in another topological sector. Hot flat space here is defined by the solution of the vacuum Einstein equations with topology $S^1 \times \mathbb{R}^{d-1}$, where the total imaginary time is the inverse temperature. As already stated, the most probable state is the one with the lowest value of the action. In the case of no electric charge, the value of the action I_0 depends on β , while in the case of hot flat space one has $I_{\text{hot flat space}} = 0$. In [101, 102], it was shown for any dimension $d \geq 4$ that the black hole is more favorable than hot flat space, $I_0 < I_{\text{hot flat space}}$, if β is such that $\frac{r_+}{R} > \frac{r_+}{r_{\text{Buch}}}$, where r_{Buch} is the Buchdahl radius. Additionally, in [101, 102], a comparison between the stable black hole solution and quantum hot flat space was also done.

In the electrically charged case, we can also make a comparison of the stable black hole with an equivalent of the uncharged hot flat space. The electrically charged case is more rich than the uncharged one since the dimensionality of the reduced action increases. In the charged case, besides the stationary point related to the stable black hole, there are two additional critical points that are possible stable solutions of the ensemble. One critical point is $r_+ = 0$ and $q = 0$, corresponding to a cavity without a black hole and without charge. Note however that this critical point must be seen as a limit, since the regularity conditions must be changed to cover this point. The configuration with $r_+ = 0$ and $q = 0$ seems unphysical for a fixed nonzero value of ϕ . For this configuration, there is a difference of electric potential, which in turn implies the existence of an electric field and thus of an electric charge. For this reason, $q = 0$ seems rather unphysical. Nevertheless, one must consider that the path integral approach in the semiclassical approximation deals intrinsically with quantum systems. And so, when one writes $q = 0$, one should mean q of the order of the Planck charge, and a particle, say, carrying such a charge should be envisaged as having the dimensions of the order of or slightly

larger than a Planck length. Therefore, one has to seek a corresponding action for such a particle in a reservoir of fixed R and β . The other critical point is described by $r_+ = R$ and $\sqrt{\lambda}q = R^{d-3}$, so that $r_+ = (\sqrt{\lambda}q)^{\frac{1}{d-3}} = R$. This critical point actually corresponds to an extremal black hole with the horizon localized at the radius of the cavity, and for this case the volume of the Riemannian space is zero, which may require a different procedure to be analyzed. Again, this critical point should be understood as a quantum system treated semiclassically, and so one should think of a black hole almost at its extremal state, failing to be extremal by a Planck charge and not touching the reservoir at R by a Planck length. The question of which configuration is the ground state is a pertinent one.

Starting with the first critical point $r_+ = 0$ and $q = 0$, as stated before, the derivative of the action in order to q is not well-defined. Nonetheless, we argue that this critical point can be considered as a local minimum of the action in the physical domain but not in the typical sense, see Sec. 4.7. In order to find an equivalent of hot flat space for the charged case, we consider a hot sphere, made of a perfect conductor material, with a certain radius r_{hs} , inside the reservoir at constant β and ϕ , and with its center situated at the center of the reservoir. For simplicity, the gravitational interaction is neglected, i.e., the constant of gravitation is put to zero. The action is then composed solely by the Maxwell term in Eq. (4.2). The hot sphere conductor depends on a fixed radius r_{hs} , in the boundary conditions. Using the Gauss constraint, the charge of the conducting sphere can be related to the value of ϕ , as $\phi = \frac{q}{(d-3)\Omega_{d-2}} \left(\frac{1}{r_{\text{hs}}^{d-3}} - \frac{1}{R^{d-3}} \right)$, see also Eq. (4.30). The action for this cavity becomes $I = -\frac{1}{2}q\beta\phi$, and one can use the relation between the electric charge and the electric potential to obtain the action of a hot spherical sphere in function of β , ϕ , r_{hs} and R , yielding

$$I_{\text{hot sphere}} = -\frac{1}{2} \frac{(d-3)\Omega_{d-2}}{\frac{1}{r_{\text{hs}}^{d-3}} - \frac{1}{R^{d-3}}} \beta \phi^2. \quad (4.65)$$

We can then make a comparison between the action of the conducting hot sphere with radius r_{hs} given in Eq. (4.65), with the action of the stable configuration of the charged black hole, which is Eq. (4.25) with the r_{+2} solution of Eq. (4.38). Analyzing Eq. (4.65), if r_{hs} is high, of the order of R , then $I_{\text{hot sphere}}$ is large and negative and so the hot flat sphere is the most probable solution when compared to the stable black hole r_{+2} . On the other hand, if r_{hs} is small, as we expect to be when dealing with a case analogous to hot flat space, then $I_{\text{hot sphere}} = 0$ or close to zero. In the limit of $r_{\text{hs}} = 0$, we can say that $I_{\text{hot sphere}}$ is indeed $I_{\text{hot flat space}}$ which is a configuration with zero action. The configuration of the hot conducting sphere shows clearly that the critical point $r_+ = q = 0$ is rather a limit of a very small electric charge at the center with a very small radius but such that ϕ is kept finite. The stable black hole has a positive action for low temperatures T , specifically, near the minimum temperature where the stable black hole exists. Therefore, the very small charged sphere that emulates hot flat space is more probable for a short interval of low temperatures when compared with the stable black hole. For a long interval of temperatures, the

black hole is eventually more probable. More specifically, when the solution of the stable black hole has a horizon radius

$$\begin{aligned} \frac{r_{+2}^{d-3}}{R^{d-3}} &\geq \frac{\mu m_0}{R^{d-3}} + \sqrt{\frac{\mu^2 m_0^2}{R^{2d-6}} - \frac{\lambda q^2}{R^{2d-6}}} , \\ \frac{\mu m_0}{R^{d-3}} &= -\frac{4(d-2)^2}{(d-1)^2(d-3)^2} + \frac{2(d-2)((d-2)^2+1)}{(d-1)^2(d-3)^2} \sqrt{1 + \frac{(d-1)^2(d-3)^2}{4(d-2)^2} \frac{\lambda q^2}{R^{2d-6}}} . \end{aligned} \quad (4.66)$$

When Eq. (4.66) is satisfied, then the action for the black hole r_{+2} is negative and the black hole is more probable. When Eq. (4.66) is not satisfied, the very small charged sphere is more probable. This radius R does not correspond to the Buchdahl-Andre sson-Wright radius [129], a radius that generalizes the Buchdahl bound for d -dimensional self-gravitating electric charged spheres. In fact, the horizon radius with zero action is equal or lower than the Buchdahl-Andre sson-Wright radius in the case of $d = 4$, with a difference up to 0.004 in $\frac{\mu m}{R}$, and being equal in the uncharged case and the extreme case $\sqrt{\lambda}q = R$. So, the equality in the uncharged situation of the minimum most probable radius of a black hole in the canonical ensemble and the Buchdahl radius does not seem to hold when other fields are added. It is a very restricted equality holding only in the pure gravitational situation.

Regarding the second critical point $r_+ = R$ and $\sqrt{\lambda}q = R^{d-3}$, it describes an extremal black hole with the horizon localized at the radius of the cavity, bearing in mind that the precise extremality and the precise location can fluctuate by Planck order quantities. This is a critical point in the sense that the gradient of the action is not defined, even as a limit. To analyze the limit, one can calculate the gradient of the reduced action in Eq. (4.25) and perform the limit to $r_+ = (\sqrt{\lambda}q)^{\frac{1}{d-3}} = R$ along the curve $\frac{r_+}{R} = (1 - \epsilon)^{\frac{1}{d-3}}$ and $\frac{\sqrt{\lambda}q}{R^{d-3}} = \sqrt{(1 - \eta\epsilon)}$, where η is a positive constant and ϵ parametrizes the curve. The constant η is restricted here to the physical domain of the action, with the condition $\eta > 2$. After substituting the variables by the parameterization of the curve in the expression of the gradient and performing the limit $\epsilon \rightarrow 0^+$, one obtains an expression that depends on the constant η . Since the limit is different for different values of η , then the gradient cannot be extended as a limit to that point, but one can still analyze the directional derivatives along the considered paths. The directional derivatives along decreasing ϵ , that go from lower r_+ and q towards $r_+ = (\sqrt{\lambda}q)^{\frac{1}{d-3}} = R$, may be either positive, zero, or negative, depending on the value of η . So, the critical point does not resemble a local minimum restricted to a corner. Particularly, there is a set of temperatures and electric potential given by the condition $\gamma = 1$, where the stable black hole solution tends to this extremal black hole. It can be seen that for such values of temperature and electric potential, there is a value of η in which the limit of the gradient vanishes, but the fact still remains that the gradient is undefined here, see Sec. 4.7 for a detailed analysis of the gradient at this critical point. It may be that this critical point smooths up by taking in consideration higher loops in the path integral or a different theory of gravity. The action for this critical point can be evaluated from

Eq. (4.25), i.e., $I_{\text{extreme black hole}} = \frac{R^{d-3}\beta}{\mu} \left(1 - \sqrt{f(R, r_+, q)}\right) - q\beta\phi - \frac{\Omega_{d-2}r_+^{d-2}}{4l_p^{d-2}}$, where r_+ and q have extremal values, so that $R = r_+$ and $f(R, r_+, q) = 0$. Then,

$$I_{\text{extreme black hole}} = \frac{R^{d-3}\beta}{\mu} - \frac{R^{d-3}}{\sqrt{\lambda}}\beta\phi - \frac{\Omega_{d-2}R^{d-2}}{4l_p^{d-2}}. \quad (4.67)$$

Comparing the action of the critical point, $I_{\text{extreme black hole}}$, it seems that the stable black hole is always a more probable configuration than the extreme black hole with horizon at the cavity.

4.4 THERMODYNAMICS OF A CHARGED BLACK HOLE IN HIGHER DIMENSIONS INSIDE A CAVITY

4.4.1 Thermodynamic properties from the grand canonical ensemble

Having the grand canonical ensemble constructed, with the partition function Z obtained in the zero loop approximation, we can obtain the thermodynamic properties of the system by relating the partition function Z to the thermodynamic grand potential W . The relation is established by

$$Z(\beta, \phi, R) = e^{-\beta W[\beta, \phi, R]}, \quad (4.68)$$

or $\beta W = -\ln Z$.

From the semiclassical zero loop approximation, the partition function is $Z = e^{-I_0}$, hence one has the correspondence $\beta W[\beta, \phi, R] = I_0[\beta, \phi, R]$. Considering $\beta = \frac{1}{T}$, the grand potential is

$$W[T, \phi, A(R)] = T I_0[T, \phi, R], \quad (4.69)$$

where $A(R) = \Omega_{d-2}R^{d-2}$, or written explicitly

$$W = \frac{R^{d-3}}{\mu} \left(1 - \sqrt{f(R, T, \phi)}\right) - T \frac{\Omega_{d-2}r_+^{d-2}(R, T, \phi)}{4l_p^{d-2}} - q(R, T, \phi)\phi, \quad (4.70)$$

where $f(R, T, \phi)$ is given by Eq. (4.23) and the solutions satisfy Eqs. (4.29) and (4.30).

We must comment about the connection of the reduced action to the grand potential in the zero loop approximation. The reduced action can be seen as a generalized grand potential where the relation between the first derivatives of the mean energy and the quantities fixed at the ensemble is relaxed. In fact, the minimization of the reduced action leads to the identification of the first derivatives of the mean energy to be the temperature T fixed at the cavity and the electric potential ϕ fixed at the cavity.

The grand canonical potential W , which is the potential directly related to the partition function of the grand canonical ensemble, is defined by the Legendre transformation of the mean energy E as

$$W = E - TS - Q\phi, \quad (4.71)$$

where $E(S, Q, A)$. The thermodynamic quantities which are the entropy, the mean charge, and the thermodynamic pressure can be obtained by evaluating the derivatives of the grand potential, and, from Eq. (4.71), one can also extract the mean energy. The differential of the grand potential $W = W[T, \phi, A(R)]$ can be written as

$$dW = -SdT - pdA - Qd\phi, \quad (4.72)$$

i.e. the first derivatives of the grand potential are $S = -\left(\frac{\partial W}{\partial T}\right)_{A,\phi}$, $p \equiv -\left(\frac{\partial W}{\partial A}\right)_{S,Q} = -\left(\frac{\partial W}{\partial A}\right)_{T,\phi}$, and $Q = -\left(\frac{\partial W}{\partial \phi}\right)_{A,T}$. The subscript in this section means that the derivative is taken with the variables in subscript kept constant.

From Eq. (4.70), the entropy can be computed through $S = -\left(\frac{\partial W}{\partial T}\right)_{A,\phi}$. It is useful to consider the grand potential with the dependence of the reduced action evaluated at the stationary points, meaning $W = W(T, \phi, A, r_+(T, \phi, R), q(T, \phi, R))$. Using the chain rule, one has

$$S = -\left(\frac{\partial W}{\partial T}\right)_{A,\phi} = -\left(\frac{\partial W}{\partial T}\right)_{r_+,q,A,\phi} - \left(\frac{\partial W}{\partial r_+}\right)_{q,T,A,\phi} \left(\frac{\partial r_+}{\partial T}\right) - \left(\frac{\partial W}{\partial q}\right)_{r_+,T,A,\phi} \left(\frac{\partial q}{\partial T}\right). \quad (4.73)$$

Now, using the fact that the derivatives must be evaluated at the solutions $r_+(T, \phi, R)$ and $q(T, \phi, R)$, one has that the partial derivatives with respect to r_+ and q vanish. Hence, one only has to evaluate the partial derivative $\left(\frac{\partial W}{\partial T}\right)_{r_+,q,A,\phi}$ to give

$$S = \frac{A_+}{4l_p^{d-2}}, \quad (4.74)$$

with A_+ is the area of the horizon given by $A_+ = \Omega_{d-2}r_+^{d-2}$. The entropy of the system is then the Bekenstein-Hawking entropy of the electrically charged black hole.

In the same way, one can calculate the electric charge Q to give the expression $Q = -\left(\frac{\partial W}{\partial \phi}\right)_{T,A} = -\left(\frac{\partial W}{\partial \phi}\right)_{r_+,q,T,A}$, yielding

$$Q = q, \quad (4.75)$$

so the thermodynamic value of the electric charge Q is equal to the typical electric charge q of an electrically charged black hole.

The thermodynamic pressure is given by $p = -\left(\frac{\partial W}{\partial A}\right)_{T,\phi}$, and so we obtain

$$p = \frac{d-3}{16\pi l_p^{d-2} R \sqrt{f}} \left((1 - \sqrt{f})^2 - \frac{\lambda q^2}{R^{2d-6}} \right), \quad (4.76)$$

which is the gravitational tangential pressure at the heat reservoir of radius R .

The remaining quantity to be calculated is the mean energy, which can be achieved by putting Eqs. (4.74)-(4.76) into Eq. (4.71). The mean energy is

$$E = \frac{(d-2)\Omega_{d-2}R^{d-3}}{8\pi l_p^{d-2}} (1 - \sqrt{f}), \quad (4.77)$$

which is the quasilocal energy evaluated at radius R .

One can verify from the previous equations that the first law of thermodynamics is satisfied, i.e.

$$TdS = dE + pdA - \phi dQ. \quad (4.78)$$

In this spirit, one can rewrite the expression of the energy in function of S , A and Q as

$$E = \frac{(d-2)A^{\frac{d-3}{d-2}}\Omega_{d-2}^{\frac{1}{d-2}}}{8\pi l_p^{d-2}} \left(1 - \sqrt{\left(1 - \left(\frac{4S}{A} \right)^{\frac{d-3}{d-2}} \right) \left(1 - \frac{\lambda Q^2 \Omega_{d-2}^{\frac{2}{d-2}}}{(4SA)^{\frac{d-3}{d-2}} l_p^{d-3}} \right)} \right). \quad (4.79)$$

Using the Euler's homogeneous function theorem and the rescaling property $E(\nu S, \nu A, \nu Q^{\frac{d-2}{d-3}}) = \nu^{\frac{d-3}{d-2}} E(S, A, Q^{\frac{d-2}{d-3}})$ with ν being a constant, one can find an integrated version of the first law given by

$$E = \frac{d-2}{d-3} (TS - pA) + \phi Q, \quad (4.80)$$

which is also called the Euler equation, in this case for system of a d -dimensional electrically charged black holes in a heat reservoir. By differentiating Eq. (4.80) and considering that $dE = TdS - pdA + \phi dQ$, one obtains a modified version of the Gibbs-Duhem relation

$$TdS - pdA + (d-2)(SdT - Adp) + (d-3)Qd\phi = 0. \quad (4.81)$$

Note that this relation depends on the differential of T and S , as well as p and A simultaneously. This is an indication of the lack of homogeneity of degree one of the system.

It is also interesting to consider the limit of infinite radius of the cavity. In fact, the integrated first law in Eq. (4.80) becomes the Smarr formula for the charged black hole. To see this, one must consider the limit of infinite radius for the thermodynamic quantities. The temperature in Eq. (4.29) reduces for the infinite cavity to the Hawking temperature $T = T_H = \frac{d-3}{4\pi} \left(\frac{1}{r_+} - \frac{\lambda q^2}{r_+^{2d-5}} \right)$, while the electric potential in Eq. (4.30) reduces to the electric potential of the Reissner-Nordström black hole $\phi = \phi_H = \frac{q}{(d-3)\Omega_{d-2} r_+^{d-3}}$. The quantity pA with p in Eq. (4.76) being proportional to $\frac{1}{R^{d-3}}$ vanishes in the limit of infinite cavity. The mean energy reduces to the ADM mass $E = m$ determined by $m = \frac{1}{2\mu} \left(r_+^{d-3} + \frac{\lambda q^2}{r_+^{d-2}} \right)$. With these ingredients, the integrated first law becomes

$$m = \frac{d-2}{d-3} T_H S + \phi_H Q, \quad (4.82)$$

which is the Smarr formula. However, the Smarr formula is only valid for the solution that exists in the limit of infinite cavity, i.e. r_{+1} , where the zero loop approximation is not valid.

4.4.2 Thermodynamic stability

We now analyze the thermodynamic stability of the system in the zero loop approximation. We must note that thermodynamic stability in general has different connections to the stability of the stationary points of the reduced action. In this case, we show that they coincide.

To understand the thermodynamic stability of an ensemble, we must consider the following. In a thermodynamic system with fixed size, fixed temperature, and fixed electric potential at a heat reservoir, there can be an exchange of energy, entropy, and electric charge between the heat reservoir and the system. In any thermodynamic process within the system, the grand canonical potential W tends to decrease down to its minimum or stay at its minimum. In particular, a spontaneous process in the grand canonical ensemble can never increase the grand canonical potential W , otherwise it violates the second law of thermodynamics.

To see this, we must resort to the second law of thermodynamics applied to the total structure. A variation dS in entropy of the system plus a variation $dS_{\text{reservoir}}$ in entropy of the reservoir add up to a variation dS_{total} of the total entropy of the system plus reservoir, as $dS_{\text{total}} = dS + dS_{\text{reservoir}}$. Now consider a perturbation in which the thermodynamic system absorbs energy dE and charge dQ from the reservoir. By conservation of energy and charge, the reservoir has to absorb energy $-dE$ and charge $-dQ$. The first law of thermodynamics states that the change in entropy of the reservoir is $TdS_{\text{reservoir}} = -dE + \phi dQ$, where the reservoir's temperature and electric potential are kept constant due to the quality of the reservoir. The total change in entropy can be written then as $TdS_{\text{total}} = TdS - dE + \phi dQ = -d(E - TS - \phi Q) = -d\bar{W}$, where T and ϕ are constant since they are the reservoir values. The potential $\bar{W}$ has been defined as

$$\bar{W}[\bar{T}, A, \bar{\phi}] \equiv E(\bar{T}, A, \bar{\phi}) - TS(\bar{T}, A, \bar{\phi}) - \phi Q(\bar{T}, A, \bar{\phi}), \quad (4.83)$$

as the grand canonical potential related to the nonequilibrium situation. Due to the variation towards a nonequilibrium situation, the thermodynamic system attains in general a new temperature $\bar{T}$ and a new potential $\bar{\phi}$ different from T and ϕ of the reservoir. The energy E , the entropy S and the charge Q that arise in the variation of the nonequilibrium situation have the same functional form of $\bar{T}$, A , and $\bar{\phi}$, as they had of T , A , and ϕ before the nonequilibrium process set in, but $\bar{W}[\bar{T}, A, \bar{\phi}]$ has a different functional dependence than the typical grand potential, since T and ϕ that appear in Eq. (4.83) are quantities of the heat reservoir fixed by assumption. The area A has been kept fixed in the system and reservoir. In brief, one has $TdS_{\text{total}} = -d\bar{W}$. Since one must have $dS_{\text{total}} \geq 0$ by the second law of thermodynamics, one deduces that $d\bar{W} \leq 0$. Any spontaneous process decreases the grand canonical potential. For a review of this discussion, see Sec. 8.2 and the following sections of [164].

The equilibrium situation is reached when

$$\bar{T} = T, \quad \bar{\phi} = \phi, \quad (4.84)$$

in which case $\bar{W}$ must reach a minimum. Therefore, to be stable, the hessian of the potential $\bar{W}$ must be positive definite, which can be summarized into the conditions

$$\left(\frac{\partial^2 \bar{W}}{\partial \bar{T}^2}\right)_{\bar{\phi}, A} > 0, \quad (4.85)$$

$$\left(\frac{\partial^2 \bar{W}}{\partial \bar{T}^2}\right)_{\bar{\phi}, A} \left(\frac{\partial^2 \bar{W}}{\partial \bar{\phi}^2}\right)_{\bar{T}, A} - \left(\frac{\partial^2 \bar{W}}{\partial \bar{T} \partial \bar{\phi}}\right)_A^2 > 0, \quad (4.86)$$

$$\left(\frac{\partial^2 \bar{W}}{\partial \bar{\phi}^2}\right)_{\bar{T}, A} > 0, \quad (4.87)$$

where all the derivatives are to be calculated at the solutions of the ensemble. Only two conditions from Eqs. (4.85)-(4.87) are sufficient, and so we choose Eqs. (4.85) and (4.86). From the expression of $\bar{W}$, the second derivative in order to T is $\left(\frac{\partial^2 \bar{W}}{\partial \bar{T}^2}\right)_{\bar{\phi}, A} = \left(\frac{\partial S}{\partial \bar{T}}\right)_{A, \bar{\phi}}$, where the bars have been dropped on the right-hand side of the equality because S has the same functional form of $\bar{T}$, A , and $\bar{\phi}$, as it has of T , A , and ϕ , and at equilibrium $\bar{T} = T$. In the same way, one has $\left(\frac{\partial^2 \bar{W}}{\partial \bar{\phi}^2}\right)_{\bar{T}, A} = \left(\frac{\partial Q}{\partial \phi}\right)_{T, A}$, and $\left(\frac{\partial^2 \bar{W}}{\partial \bar{T} \partial \bar{\phi}}\right)_{\bar{T}, \bar{\phi}, A} = \left(\frac{\partial Q}{\partial \bar{T}}\right)_{A, \bar{\phi}} = \left(\frac{\partial S}{\partial \phi}\right)_{T, A}$. The two sufficient conditions, Eqs. (4.85) and (4.86), can be written in terms of first derivatives of the entropy and charge as

$$\left(\frac{\partial S}{\partial T}\right)_{A, \phi} > 0, \quad (4.88)$$

$$\left(\frac{\partial Q}{\partial \phi}\right)_{T, A} \left(\frac{\partial S}{\partial T}\right)_{A, \phi} - \left(\frac{\partial S}{\partial \phi}\right)_{T, A}^2 > 0 \quad (4.89)$$

respectively.

We can now link the thermodynamic stability to thermodynamic coefficients. First, we can define the isochoric heat capacity at constant electric potential as

$$C_{A, \phi} = T \left(\frac{\partial S}{\partial T}\right)_{A, \phi}. \quad (4.90)$$

Second, we can define the adiabatic electric susceptibility as

$$\chi_{S, A} = \left(\frac{\partial Q}{\partial \phi}\right)_{S, A}. \quad (4.91)$$

From a change of variables $Q(T, A, \phi)$ to $Q(T(S, A, \phi), A, \phi)$, where $T(S, A, \phi)$ is the inverse function of $S(T, A, \phi)$, one gets

$$\chi_{S, A} = \frac{\left(\frac{\partial Q}{\partial \phi}\right)_{T, A} \left(\frac{\partial S}{\partial T}\right)_{A, \phi} - \left(\frac{\partial S}{\partial \phi}\right)_{T, A}^2}{\left(\frac{\partial S}{\partial T}\right)_{A, \phi}}, \quad (4.92)$$

Hence, the two stability conditions, Eqs. (4.88) and (4.89), are now

$$C_{A,\phi} > 0, \quad (4.93)$$

$$\chi_{S,A} C_{A,\phi} > 0, \quad (4.94)$$

respectively. The above analysis to obtain the stability conditions is equivalent to the requirement that the matrix of variances in the grand canonical ensemble is positive definite. The matrix of variances contains the variances ΔE^2 , ΔQ^2 and the correlation $\Delta E \Delta Q$, where E and Q are the quantities that are exchanged with the heat reservoir. By working out the conditions of positive definiteness through the Sylvester's criterion, one recovers also the conditions Eqs. (4.93) and (4.94).

For the specific case of the electrically charged black hole in a cavity, the susceptibility is

$$\chi_{S,A} = \frac{(d-3)\Omega_{d-2}r_+^{d-3}\left(1 - \frac{r_+^{d-3}}{R^{d-3}}\right)}{(1 - (1 - \Phi^2)\left(\frac{r_+}{R}\right)^{d-3})^{\frac{3}{2}}}. \quad (4.95)$$

The adiabatic susceptibility is then positive for all physical configurations of the charged black hole. Therefore, the two conditions for stability can be reduced to a single one given in Eq. (4.93), $C_{A,\phi} > 0$, which in terms of the ensemble quantities is

$$C_{A,\phi} = \frac{A(d-3)^2(d-2)x^{d-4}(1-\Phi^2)^2}{32l_p^{d-2}(\pi RT)^2((d-1)(1-\Phi^2)x^{d-3}-2)} > 0, \quad (4.96)$$

with the dependence on the variable $x = \frac{r_+}{R}$ being maintained for convenience. With Eq. (4.96), one recovers Eq. (4.63) for thermodynamic stability. This means that the stability of the stationary points, that point towards the validity of the zero loop approximation, coincides with thermodynamic stability, in this case. For the case of $\Phi^2 = 0$, $C_{A,\phi}$ becomes the heat capacity at constant area C_A with the expression given in [102]. Moreover, the bifurcation radius, indicating marginal stability, and the photon sphere radius are the same for $\Phi^2 = 0$. We make a comparison between the bifurcation radius and the photon sphere radius in Sec. 4.6, showing that these radii do not coincide. The connection displayed in the uncharged case does not seem to be generic, it seems to hold only in the pure gravitational situation.

It is worth making a comparison of the thermodynamic analysis that we have done here with the case of a self-gravitating static electrically charged thin shell in d -dimensions presented in Chapter 2, or in [1]. It is quite remarkable that the thermodynamic pressure given in Eq. (4.76) and the thermodynamic energy given in Eq. (4.77) in the grand canonical ensemble have the same expression as the matter pressure and the matter rest energy of the corresponding self-gravitating charged spherical shell in equilibrium. Additionally, by choosing for the matter of the thin shell the equations of state corresponding to the temperature and electric potential of the black hole, the shell also has the Bekenstein-Hawking entropy and its stability at constant area is given by the same condition, i.e., positive heat capacity at constant electric potential.

4.4.3 Thermodynamic phases and phase transitions

In a thermodynamic system characterized by the grand canonical ensemble, spontaneous processes always occur in order to decrease W to its lowest value. As shown above, this is a consequence of the second law of thermodynamics. The configuration we are studying here is a black hole inside a reservoir characterized by a fixed area A , a fixed temperature T , and a fixed electric potential ϕ . So thermodynamically, W is the most suited thermodynamic potential to be analyzed. It is relevant to know whether the stable black hole is the thermodynamic state with less energy W , or if there is another state to which the black hole can make a phase transition. Indeed, the stable black hole is a local minimum but may not be the global minimum of the potential. Here, we can use the thermodynamic language now, and so we can analyze phase transitions instead of quantum transitions as done previously in the analyses of probable configurations. But the results are the same, as here we use W instead of I_0 , with the connection $TI_0 = W$. We summarize the results using the grand canonical potential W .

In the uncharged case, one has $W_{\text{hot flat space}} = 0$ and so the favorability of the black hole depends on whether the black hole with horizon radius r_{+2} has a W lower or greater than zero. We found that the radius where the stable black hole with r_{+2} yields $W = 0$ when the cavity is at the Buchdahl radius, r_{Buch} , where the first order phase transition from hot flat space to black hole phase occurs. For the ratio $\frac{r_{+2}}{R}$ higher than $\frac{r_{+2}}{r_{\text{Buch}}}$, the black hole phase is favored. In the electrically charged case, the grand potential for hot flat space is $W_{\text{hot flat space}} = 0$, and corresponds to a cavity without a black hole and without charge. We emulated hot flat space by a very small electric hot sphere in flat space. Its grand potential is $W_{\text{hot sphere}} = TI_{\text{hot sphere}}$, which tends to zero as the radius of the sphere tends to zero. Essentially, in this setting, the black hole phase is favored when its grand potential W is less than zero. We established that the ratio $\frac{r_{+2}}{R}$ which yields $W = 0$ is not related to the Buchdahl-Andréasson-Wright ratio, a generalization for the Buchdahl ratio to any higher dimension d that includes electric charge, see Sec. 4.6. There is also a critical phase, the extreme black hole solution localized at the radius of the cavity. We found that the stable black hole r_{+2} has always lower or equal W than $W_{\text{extreme black hole}}$, and hence the stable black hole is always more favorable than the extremal black hole with horizon at the cavity.

4.5 ZERO LOOP APPROXIMATION AND THERMODYNAMICS FOR $d = 5$

4.5.1 Zero loop approximation

4.5.1.1 Reduced action, stationary solutions and stability conditions

Here, we apply the whole formalism to the specific five dimensional case, $d = 5$, since we want to stress some analytical results pertaining this case. The $d = 4$ case

recovers the analysis of [130], see [2]. In $d = 5$, the reduced action in Eq. (4.25) can be rewritten as

$$I_* = \frac{3\pi}{4l_p^3} \beta R^2 \left(1 - \sqrt{f}\right) - q\beta\phi - \frac{\pi^2 r_+^3}{2l_p^3}, \quad (4.97)$$

where

$$f = \left(1 - \frac{r_+^2}{R^2}\right) \left(1 - \frac{1}{3\pi^3} \frac{q^2}{r_+^2 R^2}\right). \quad (4.98)$$

with $I_* = I_*[\beta, \phi, R; r_+, q]$ and $f = f[R; r_+, q]$. For $d = 5$, one has $\Omega = 2\pi^2$, $\mu = \frac{4l_p^3}{3\pi}$ and $\lambda = \frac{l_p^3}{3\pi^3}$.

The stationary solutions for r_+ that minimize the reduced action in Eq. (4.97) satisfy the relation in Eq. (4.38), which for $d = 5$ is

$$(1 - \Phi^2) \left(\frac{r_+}{R}\right)^4 - \left(\frac{r_+}{R}\right)^2 + (1 - \Phi^2)^2 \frac{1}{(2\pi RT)^2} = 0, \quad (4.99)$$

where $\gamma = (2\pi RT)^2 \frac{\Phi^2}{(1 - \Phi^2)^2}$. The electric charge is given in terms of the horizon radius r_+ by Eq. (4.39), becoming

$$\frac{l_p^{\frac{3}{2}} |q|}{R^2} = 2\sqrt{3} \pi^{\frac{5}{2}} \frac{RT |\Phi|}{1 - \Phi^2} \left(\frac{r_+}{R}\right)^3. \quad (4.100)$$

The stationary condition in Eq. (4.99) must be solved to obtain the stationary point $r_+ = r_+(T, \Phi, R)$, which in turn can be plugged in Eq. (4.100) to get $q = q(T, \Phi, R)$ of the stationary point. Since Eq. (4.99) is a quadratic equation for r_+^2 , analytic expressions can be obtained for the two solutions r_{+1} and r_{+2} . Namely, the solution r_{+1} is given by

$$\frac{r_{+1}}{R} = \frac{1}{\sqrt{2(1 - \Phi^2)}} \left[1 - \sqrt{1 - \frac{(1 - \Phi^2)^3}{(\pi RT)^2}} \right]^{\frac{1}{2}}, \quad (4.101)$$

$$\frac{l_p^{\frac{3}{2}} |q_1|}{R^2} = \sqrt{\frac{3}{2}} \frac{\pi^{\frac{5}{2}} RT \Phi}{(1 - \Phi^2)^{\frac{5}{2}}} \left[1 - \sqrt{1 - \frac{(1 - \Phi^2)^3}{(\pi RT)^2}} \right]^{\frac{3}{2}}. \quad (4.102)$$

The solution r_{+1} for the horizon radius of the charged black hole was designated x_1 in the analysis above for generic d . The second solution r_{+2} , designated as x_2 above, is given by

$$\frac{r_{+2}}{R} = \frac{1}{\sqrt{2(1 - \Phi^2)}} \left[1 + \sqrt{1 - \frac{(1 - \Phi^2)^3}{(\pi RT)^2}} \right]^{\frac{1}{2}}, \quad (4.103)$$

$$\frac{l_p^{\frac{3}{2}} |q_2|}{R^2} = \sqrt{\frac{3}{2}} \frac{\pi^{\frac{5}{2}} RT \Phi}{(1 - \Phi^2)^{\frac{5}{2}}} \left[1 + \sqrt{1 - \frac{(1 - \Phi^2)^3}{(\pi RT)^2}} \right]^{\frac{3}{2}}. \quad (4.104)$$

The analysis of the behaviour of the solutions can be easily understood compared to the d generic case. For the two solutions to exist, one requires that Eq. (4.50), which here reduces to

$$0 \leq (1 - \Phi^2)^3 \leq (\pi RT)^2 < \infty, \quad (4.105)$$

in $d = 5$ dimensions. For the uncharged case, i.e. $\Phi = 0$, the condition of existence of both solutions Eq. (4.105) reduces to the interval $1 \leq (\pi RT)^2 < \infty$, being precisely the interval of existence for the $d = 5$ Schwarzschild-Tangherlini black hole solutions, see [101].

Before proceeding to a careful analysis of the stationary points, it is useful to make an analysis of the limits. First, for very large πRT , or $(\pi RT)^2 \rightarrow \infty$, and constant Φ , the solution r_{+1} behaves as $\frac{r_{+1}}{R} \rightarrow \frac{(1-\Phi^2)}{2\pi RT}$, and, since $|\Phi| < 1$, the solution always exists. For very large πRT , or $(\pi RT)^2 \rightarrow \infty$, and constant Φ , the solution r_{+2} behaves as $\frac{r_{+2}}{R} \rightarrow \frac{1}{\sqrt{1-\Phi^2}}$, which for values of $\Phi^2 < 1$, one has $r_{+2} > R$, so the solution is unphysical. This situation is different from the uncharged case, where the solution with larger mass, r_{+2} , only meets the cavity at infinite temperature, while in the charged case, the solution r_{+2} meets the cavity at finite temperature, as already seen above in qualitative terms. Second, for $\Phi^2 \rightarrow 1$, and constant $(\pi RT)^2$, the solution r_{+1} tends to $r_{+1} \rightarrow 0$. For $\Phi^2 \rightarrow 1$, and constant $(\pi RT)^2$, the solution r_{+2} tends to $r_{+2} \rightarrow \infty$, which is unphysical.

We now give a careful analysis of the stationary points described by the solution r_{+1} of Eqs. (4.101)-(4.102) and by the solution r_{+2} of Eqs. (4.103)-(4.104). We show the plots of the solutions in Figs. 4.2–4.4, which complement the behaviour expressed in Eqs. (4.101)–(4.105). In $d = 5$, one has that the value $\Phi^2 = \frac{1}{2}$ plays an important role in the analysis. Thus, the analysis is divided into two parts, namely, $0 \leq \Phi^2 \leq \frac{1}{2}$ and $\frac{1}{2} < \Phi^2 < 1$.

(i) For $0 \leq \Phi^2 \leq \frac{1}{2}$, there are three different branches.

(a) for $0 \leq (\pi RT)^2 < (1 - \Phi^2)^3$, there are no stationary points or solutions for the charged black hole.

(b) For $(1 - \Phi^2)^3 \leq (\pi RT)^2 \leq \frac{(1-\Phi^2)^2}{4\Phi^2}$, there are two black hole solutions and they lie inside the cavity, i.e., $r_{+1} \leq R$ and $r_{+2} \leq R$. In the case of the equality $(\pi RT)^2 = \frac{(1-\Phi^2)^2}{4\Phi^2}$, the solution r_{+1} obeys $r_{+1} < R$, and the solution r_{+2} satisfies $r_{+2} = R$ with the charge q_2 obeying $l_p^{\frac{3}{2}} |q_2| = \sqrt{3\pi^3} r_+^2$, which means that the r_{+2} solution is an extremal electrically charged black hole. The particular case $\Phi^2 = \frac{1}{2}$ and $(\pi RT)^2 = \frac{(1-\Phi^2)^2}{4\Phi^2}$ yields that $(1 - \Phi^2)^3 = \frac{(1-\Phi^2)^2}{4\Phi^2} = (\pi RT)^2 = \frac{1}{8}$, and the r_{+1} and r_{+2} solutions merge into one, an extremal electrically charged black hole that obeys $r_{+1} = r_{+2} = R$.

(c) For $\frac{(1-\Phi^2)^2}{4\Phi^2} < (\pi RT)^2 < \infty$, the solution r_{+1} has always $r_+ < R$ and so it is physical. For Φ near zero, r_{+1} is small and as the value of $(\pi RT)^2$ increases, r_{+1} tends to zero. For Φ near $\frac{1}{2}$ approaching from below, r_{+1} approaches R from below and as $(\pi RT)^2$ increases, r_{+1} tends to zero. Regarding the other solution r_{+2} , it obeys $r_{+2} > R$, so it is unphysical.

(ii) For $\frac{1}{2} < \Phi^2 < 1$, there are three different branches.

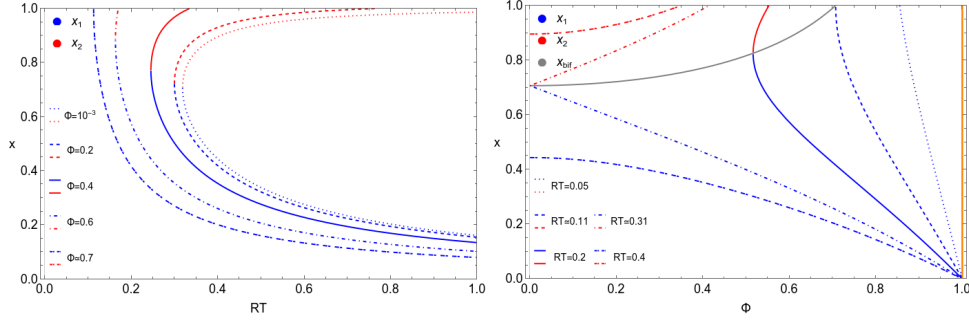


Figure 4.2: Left plot: Stationary points $\frac{r_{+1}}{R} = x_1$ (in blue) and $\frac{r_{+2}}{R} = x_2$ (in red) of the reduced action I_* as a function of RT , for $d = 5$ dimensions, and for five values of Φ , namely, $\Phi = 0.001$ in dotted lines, $\Phi = 0.2$ in dashed lines, $\Phi = 0.4$ in solid lines, $\Phi = 0.6$ in dot dashed lines and $\Phi = \frac{1}{\sqrt{2}} = 0.7$, the last equality is approximate, in dot double dashed lines. Right plot: Stationary points $\frac{r_{+1}}{R} = x_1$ (in blue) and $\frac{r_{+2}}{R} = x_2$ (in red) of the reduced action I_* as a function of Φ , for $d = 5$ dimensions, and for five values of RT , namely, $RT = 0.05$ in dotted lines, $RT = \frac{1}{2\sqrt{2}\pi} = 0.112$, the last equality is approximate, in dashed lines, $RT = 0.2$ in solid lines, $RT = \frac{1}{\pi} = 0.318$, the last equality is approximate, in dot dashed lines, and $RT = 0.4$ in dot double dashed lines. The gray line corresponds to the bifurcation points where the solutions x_1 and x_2 coincide. The orange line corresponds to $\Phi = 1$, which is the maximum possible electric potential of the ensemble.

- (a) For $0 \leq (\pi RT)^2 < (1 - \Phi^2)^3$, there are no black hole solutions.
- (b) For $(1 - \Phi^2)^3 \leq (\pi RT)^2 \leq \frac{(1 - \Phi^2)^2}{4\Phi^2}$, both solutions r_{+1} and r_{+2} exist but lie outside the cavity and so they are unphysical. This means that within this range there are no physical black hole solutions.
- (c) For $\frac{(1 - \Phi^2)^2}{4\Phi^2} < (\pi RT)^2 < \infty$, the solution r_{+1} starts at $r_{+1} = R$ in the case of $(\pi RT)^2 = \frac{(1 - \Phi^2)^2}{4\Phi^2}$ and then decreases toward zero as the temperature increases. On the other hand, the solution r_{+2} remains outside the cavity, being thus unphysical.

In Fig. 4.3, we present a contour plot of the reduced action I_* , for $RT = 0.5$ and $\Phi = 0.2$, as a function of $\frac{r_{+}}{R} = x$ and $\frac{l_p^{\frac{3}{2}}|q|}{\sqrt{3\pi^3}R^2} = \sqrt{y}$. The reduced action is given by Eq. (4.25) with $d = 5$. The two stationary points $\frac{r_{+1}}{R} = x_1$ and $\frac{r_{+2}}{R} = x_2$ are displayed as a blue dot and a red dot, respectively. The contour plot allows for a visual identification of the nature of the stationary points, with r_{+1} being a saddle point and r_{+2} being a minimum.

It is interesting to see the effects of changing T and Φ to the contour plot, namely to see the trajectories of the solutions x_1 and x_2 . In the left plot of Fig. 4.4, the migration path of the two stationary points $\frac{r_{+1}}{R} = x_1$ and $\frac{r_{+2}}{R} = x_2$ from a point in the central region where they coincide (a bifurcation point) to the two points at the corners is shown as a function of RT for four different values of Φ . The gray line corresponds to the condition of extremal black holes inside a cavity, namely, $\sqrt{y} = x^2$, i.e., $\frac{l_p^{\frac{3}{2}}|q|}{\sqrt{3\pi^3}} = r_+^2$. The black line corresponds to the points x and $\sqrt{y}$ where the solutions x_1 and x_2 coincide, i.e. the class of bifurcation points. For

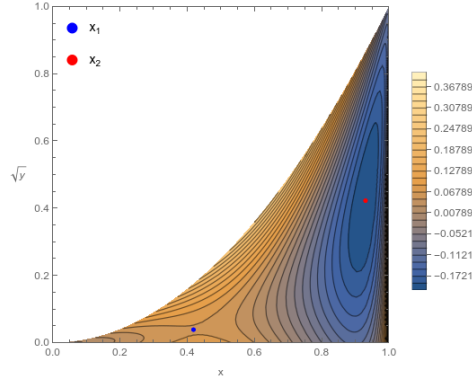


Figure 4.3: Contour plot of the reduced action $\frac{4l_p^3 I_*}{3\pi R^2}$ in $d = 5$ dimensions, in function of

$\frac{r_+}{R} = x$ and $\frac{l_p^2 |q|}{\sqrt{3\pi^3 R^2}} = \sqrt{y}$, for $\Phi = 0.2$ and $RT = 0.4$. The blue dot corresponds to $\frac{r_{+1}}{R} = x_1$ and it is a saddle point, while the red dot corresponds to $\frac{r_{+2}}{R} = x_2$ and it is a minimum.

the minimum possible temperature in each case, the solutions start at the black line, and as one increases the temperature, x_1 decreases toward the origin $x = \sqrt{y} = 0$, where $RT \rightarrow +\infty$, and x_2 increases toward $x = \sqrt{y} = 1$, where $RT \rightarrow \frac{(1-\Phi^2)^2}{4\Phi^2}$. In the right plot of Fig. 4.4, the migration path of the two stationary points $\frac{r_{+1}}{R} = x_1$ and $\frac{r_{+2}}{R} = x_2$ from a point in the central region where they coincide to the two points at the corners is shown as a function of Φ for four different values of RT . In these plots, the quantity Φ was chosen instead of ϕ so that the comparison between the analytical study and the plots is straightforward, and also to avoid setting Planck units. Since $\Phi = \sqrt{\frac{16\pi}{3}} \frac{l_p^2}{R} \phi$, one has that Φ is fixed as a consequence of fixing ϕ . The gray line corresponds to the condition of extremal black holes inside a cavity, namely $\sqrt{y} = x^2$, i.e., $\frac{l_p^2 |q|}{\sqrt{3\pi^3 R^2}} = r_+^2$. The black line corresponds to the points x and $\sqrt{y}$ where solutions x_1 and x_2 coincide, i.e. the bifurcation points. For minimum potential, the solutions either start from the black line where the solutions coincide at the bifurcation points or start separated in the $\sqrt{y} = 0$ line. As one increases further the potential, x_1 tends to the origin $x = \sqrt{y} = 0$, where $\Phi \rightarrow 1$, and x_2 tends to $x = \sqrt{y} = 1$, where $\Phi \rightarrow \sqrt{(\pi RT)^2 + 1} - \pi RT$.

Regarding stability, using Eq. (4.63) with $x \equiv \frac{r_{\pm}}{R}$, one finds that the solutions are stable if they obey

$$\frac{\left(4(1-\Phi^2)\left(\frac{r_{\pm}}{R}\right)^2 - 2\right)\left(1 - \left(\frac{r_{\pm}}{R}\right)^2\right)}{(1 - (1-\Phi^2)\left(\frac{r_{\pm}}{R}\right)^2)} > 0, \quad (4.106)$$

for $d = 5$, where the physical range is $\frac{r_{\pm}}{R} < 1$. Hence, the solutions are stable if $r_+ > r_{\text{bif}}$, where $r_{\text{bif}} = \frac{R}{\sqrt{2(1-\Phi^2)}}$ is the bifurcation radius from which the solutions r_{+2} and r_{+1} bifurcate at $(\pi RT)^2 = (1-\Phi^2)^3$. For r_{+1} , this condition means that for $(1-\Phi^2)^3 \leq (\pi RT)^2 \leq \frac{(1-\Phi^2)^2}{4\Phi^2}$, in the case $0 \leq \Phi^2 \leq \frac{1}{2}$, the solution does not obey the stability condition, and so it is thermodynamically unstable, and

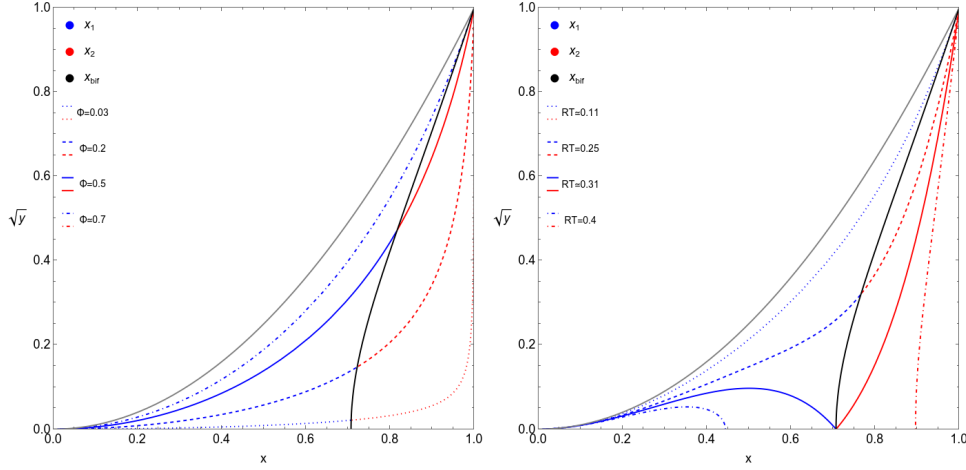


Figure 4.4: Left plot: Curves describing the path of the solutions $\frac{r_{+1}}{R} = x_1$ (in blue) and $\frac{r_{+2}}{R} = x_2$ (in red) in the $x \times \sqrt{y}$ plane, for $d = 5$, with x and $\sqrt{y}$ parametrized by RT , for $\Phi = 0.03$ in dotted lines, $\Phi = 0.2$ in dashed lines, $\Phi = 0.5$ in solid lines and $\Phi = \frac{1}{\sqrt{2}} = 0.707$, the last equality being approximate, in dot dashed lines. The gray line corresponds to the class of extremal black holes inside the cavity, i.e., $\sqrt{y} = x^2$. The black line corresponds to the bifurcation points x and $\sqrt{y}$, where x_1 and x_2 coincide. Right plot: Curves describing the path of solutions x_1 (in blue) and x_2 (in red) in the $x \times \sqrt{y}$ plane, for $d = 5$, with x and $\sqrt{y}$ parametrized by Φ , for $RT = \frac{1}{2\sqrt{2}\pi} = 0.11$, the last equality being approximate, in dotted line, $RT = 0.25$ in dashed lines, $RT = \frac{1}{\pi} = 0.31$, the last equality being approximate, in solid lines, and $RT = 0.4$ in dot dashed lines. The gray line corresponds to the condition of extremal black holes inside the cavity, i.e., $\sqrt{y} = x^2$. The black line corresponds to the bifurcation points x and $\sqrt{y}$ where x_1 and x_2 coincide.

in the case $\frac{1}{2} < \Phi^2 < 1$ the solution r_{+1} does not physically exist as it lies outside the cavity. For $\frac{(1-\Phi^2)^2}{4\Phi^2} < (\pi RT)^2 < \infty$ and $0 \leq \Phi^2 < 1$, the solution r_{+1} does not obey the stability condition, and so it is thermodynamically unstable. Moreover, r_{+1} corresponds to a saddle point of the action as seen from Fig. 4.3. For r_{+2} , this condition means that for $(1 - \Phi^2)^3 \leq (\pi RT)^2 \leq \frac{(1-\Phi^2)^2}{4\Phi^2}$, in the case $0 \leq \Phi^2 \leq \frac{1}{2}$, the solution obeys the stability condition, therefore for this range of parameters the solution is thermodynamically stable, and it is also a minimum of the action, as seen in Fig. 4.3. In the case $\frac{1}{2} < \Phi^2 < 1$, the solution r_{+2} lies outside the cavity and it is not physical. For $\frac{(1-\Phi^2)^2}{4\Phi^2} < (\pi RT)^2 < \infty$ and $0 < \Phi^2 < 1$ the solution r_{+2} does not physically exist also, lying outside the cavity.

4.5.1.2 Most favorable or probable configurations

We study here the most probable configurations in the case $d = 5$. The analysis follows from the generic d case, and additionally, we present the phase diagram for this case. Essentially, we perform the comparison between the stable black hole and the charged equivalent hot flat space. The reduced action has two stable stationary points, in particular, the stationary point r_{+2} related to the stable black hole, and

the stationary point $r_+ = 0$ and $q = 0$, which corresponds to a cavity without a black hole and without charge. The action also has a critical point corresponding to an extremal black hole with the radius of the cavity, $r_+ = R$ and $\frac{q}{\sqrt{3\pi^3}} = R$.

In order to model the stationary point $r_+ = 0$ and $q = 0$, we have put forward a model described by a nongravitating perfect conductor hot sphere with radius r_{hs} , inside the reservoir at constant β and ϕ . The electric potential for the case of the perfect conductor is $\phi = \frac{q}{4\pi^2} \left(\frac{1}{r_{\text{hs}}^2} - \frac{1}{R^2} \right)$, see also Eq. (4.30). The action for a hot sphere, as a model of hot flat space, in five dimensions is then

$$I_{\text{hot sphere}} = -\frac{1}{2} \frac{4\pi^2}{\frac{1}{r_{\text{hs}}^2} - \frac{1}{R^2}} \beta \phi^2. \quad (4.107)$$

We now compare the action of the conducting hot sphere given in Eq. (4.107) with the action of the stable configuration of the charged black hole given in Eq. (4.97) together with Eqs. (4.103) and (4.104). From Eq. (4.107), one can see that for small r_{hs} , which is the case analogous to hot flat space, the action is approximately zero, $I_{\text{hot sphere}} = 0$, and so one can assign essentially $I_{\text{hot sphere}} = I_{\text{hot flat space}}$ in this case.

Regarding the stable black hole solution, it assumes a positive action only in a small range of low temperatures, namely, for temperatures near the minimum temperature for which the stable black hole exists. For higher temperatures, the action for the stable black hole solution is negative. Hence, one finds that the small charged sphere that emulates hot flat space is more probable or favorable than the stable black hole solution for a small interval of temperatures. In fact, when the solution of the stable black hole obeys the condition for its horizon radius

$$\frac{r_{+2}^2}{R^2} \geq \frac{4l_p^3 m}{3\pi} + \sqrt{\frac{16l_p^6 m^2}{9\pi^2} - \frac{l_p^3 q^2}{3\pi^3}}, \quad (4.108)$$

with $\frac{4l_p^3 m}{3\pi} = -\frac{9}{16} + \frac{15}{16} \sqrt{1 + \frac{16l_p^3 q^2}{27\pi^3 R^4}}$, the corresponding action is negative and the black hole is more probable than the very small charged sphere. The radii ratio in Eq. (4.108) does not have a connection to the Buchdahl-Andréasson-Wright bound, in contrast to the uncharged case, see also Sec. 4.6.

The comparison between the hot flat sphere and the stable black hole is shown in Fig. 4.5 for $d = 5$ dimensions. In the two plots of the figure, the gray region represents the points (RT, Φ) in which the stable black hole solution r_{+2} is more favorable or probable. The regions in purple in the left plot of the figure, and in blue in the right plot, represent the points in which the charged conducting sphere with radius r_{hs} is more probable. The regions in white represent points where there is no stable black hole solution, so presumably the most favorable state is hot flat space. The upper white region is quite different from the uncharged case, see [102], because in the uncharged case the stable black hole solution exists for temperatures up to infinite ones, whereas in the electrically charged case, the stable black hole solution only exists within a range of finite temperatures. In the left plot of Fig. 4.5, one can see that for small values of r_{hs} , the larger the region gets where the stable black hole solution is more favorable over the conducting sphere,

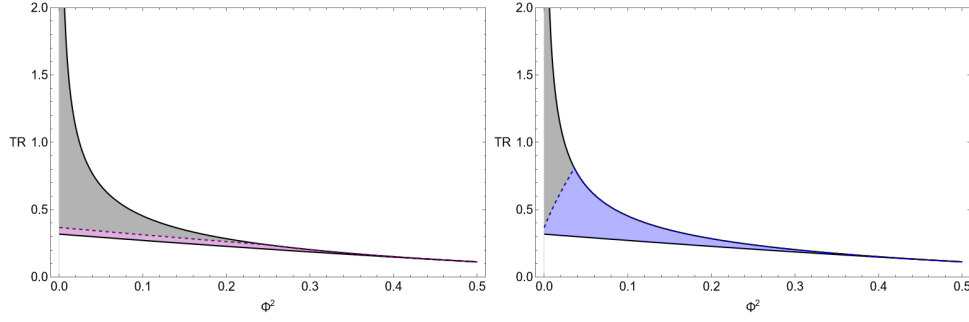


Figure 4.5: Regions of favorability in five dimensions, $d = 5$, between the stable black hole solution and the charged conducting sphere, in function of RT and Φ . Left plot: $\frac{r_{\text{hs}}}{R} \rightarrow 0$. The region in gray represents the points where the black hole solution is more favorable. The region in purple represents the points where the infinitesimal charged conducting sphere, emulating electrically charged hot flat space, is more favorable. The regions in white do not have a stable black hole solution, so presumably the most favorable state is hot flat space. Right plot: $\frac{r_{\text{hs}}}{R} = 0.99$. The region in gray represents the points where the black hole solution is more favorable. The region in blue represents the points where the charged conducting sphere is more favorable, with $\frac{r_{\text{hs}}}{R} = 0.99$. The regions in white do not have a stable black hole solution, so presumably the most favorable state is hot flat space.

until the point where one has a microscopic sphere. This case of a microscopic electrically charged sphere is precisely the case that emulates hot flat space. In the right plot of Fig. 4.5, one can see that for large values of r_{hs} , the smaller the region gets where the stable black hole solution is more favorable, but this case is contrived, as it does not emulate hot flat space. Moreover, it must be stated that for relatively small values of r_{hs} , the region of favorability for the electrically charged shell does not change much, as even with $r_{\text{hs}} = 0.7$, the difference to $r_{\text{hs}} = 0$ is considerably small. Only variations of r_{hs} close to R induce substantial changes to the regions of favorability. We must give some comments regarding the uncharged case, in $d = 5$, and the comparison between the stable black hole solution and hot flat space in [101]. It was shown that the black hole is favorable, $I_0 < I_{\text{hot flat space}}$, if β is such that $\frac{r_+}{R} > \frac{r_+}{r_{\text{Buch}}}$, where r_{Buch} is the Buchdahl radius. In the pure gravitational case, the radii ratio that establishes favorability agrees with the Buchdahl bound. However, here we have shown that this agreement does not seem to hold when other fields are present, since the radii ratio for the electrically charged black hole to be the dominant phase does not coincide with the Buchdahl-Andréasson-Wright bound.

Finally, in $d = 5$, one can make also a comparison of the stable black hole r_{+2} with the critical point given by $r_+ = R$ and $\frac{l_p^3 q}{\sqrt{3\pi^3}} = R$, which is an extremal black hole with the horizon localized at the radius of the cavity, bearing in mind that the precise extremality and the precise location can fluctuate by Planck order quantities. The gradient of the action is not defined at this critical point but it may be smoothed up by taking in consideration higher loops in the path integral or a different theory

of gravity, see Sec. 4.7 for more details on the critical point. The action for the extremal black hole at the cavity is Eq. (4.67) in the $d = 5$ case, i.e.,

$$I_{\text{extreme black hole}} = \frac{3\pi R^2 \beta}{4} - \sqrt{3\pi^3} R^2 \beta \phi - \frac{\pi^2 R^3}{2}. \quad (4.109)$$

We found that, for every instance, the stable black hole is a more favorable configuration than the extreme black hole with horizon at the cavity.

4.5.2 Thermodynamics

Here, we analyze the thermodynamics for the particular case of $d = 5$ dimensions. The grand potential W has the dependence $W = W[T, \phi, A]$, where A is the surface area of the 3-sphere at the boundary ∂M . The correspondence between thermodynamics and the action of the system is given by Eq. (4.69). For $d = 5$, one has

$$W = \frac{3\pi}{4l_p^3} R^2 \left(1 - \sqrt{\left(1 - \frac{r_+^2}{R^2}\right) \left(1 - \frac{1}{3\pi^3} \frac{q^2}{r_+^2 R^2}\right)} \right) - T \frac{\pi^2 r_+^3}{2l_p^3} - q\phi, \quad (4.110)$$

The grand potential is defined by the expression $W = E - ST - Q\phi$, with $dW = -SdT - Qd\phi - pdA$ and with the first law of thermodynamics $TdS = dE - \phi dQ + pdA$ holding, see Eq. (4.69).

The physical quantities of the system such as the entropy, electric charge, surface pressure, thermodynamic energy, and area can be given in this case, through the derivatives of the grand potential. The entropy can be directly obtained from Eq. (4.74) in $d = 5$ as

$$S = \frac{A_+}{4l_p^3}, \quad (4.111)$$

which is the Bekenstein-Hawking entropy of a black hole, with $A_+ = 2\pi^2 r_+^3$. The electric charge can be computed from Eq. (4.75), which in $d = 5$ it has the same appearance as in general d , i.e. $Q = q$. The gravitational thermodynamic surface pressure at R can be calculated from Eq. (4.76) to yield

$$p = \frac{1}{8\pi R l_p^3 \sqrt{f}} \left(\left(1 - \sqrt{\left(1 - \frac{r_+^2}{R^2}\right) \left(1 - \frac{q^2}{3\pi^3 r_+^2 R^2}\right)} \right)^2 - \frac{q^2}{3\pi^3 R^4} \right), \quad (4.112)$$

where f is given in Eq. (4.98). The tangential surface pressure p acts along an area A that in $d = 5$ is $A = 2\pi^2 R^3$. Finally, the mean thermodynamic energy can be taken from Eq. (4.77) to the $d = 5$ case and is given by

$$E = \frac{3\pi R^2}{4l_p^3} \left(1 - \sqrt{\left(1 - \frac{r_+^2}{R^2}\right) \left(1 - \frac{q^2}{3\pi^3 r_+^2 R^2}\right)} \right). \quad (4.113)$$

This is again the same expression as the quasilocal energy evaluated at a spherical shell of radius R in $d = 5$.

From Eq. (4.113), one can write the mean energy in terms of the entropy S of Eq. (4.111), electric charge Q , and surface area of the cavity A , as

$$E = \frac{3(2\pi^2)^{\frac{1}{3}} A^{\frac{2}{3}}}{8\pi l_p^3} \left(1 - \sqrt{\left(1 - \left(\frac{4S}{A} \right)^{\frac{2}{3}} \right) \left(1 - \frac{Q^2 (2\pi^2)^{\frac{4}{3}}}{3\pi^3 (4SA)^{\frac{2}{3}}} \right)} \right). \quad (4.114)$$

Hence, one can use the Euler's homogeneous function theorem considering that under rescaling of its arguments, the energy has the property $E(\nu S, \nu A, \nu Q^{\frac{3}{2}}) = \nu^{\frac{2}{3}} E(S, A, Q^{\frac{3}{2}})$. An integrated version of the first law of thermodynamics can be obtained from the theorem, Eq. (4.80), which in $d = 5$ is

$$\frac{2}{3}E = TS - pA + \frac{2}{3}\phi Q. \quad (4.115)$$

This is the Euler equation for the system of a $d = 5$ electrically charged black hole in a heat reservoir. By differentiating Eq. (4.115) and considering that $dE = TdS - pdA + \phi dQ$, the Gibbs-Duhem relation

$$TdS - pdA + 3(SdT - Adp) + 2Qd\phi = 0. \quad (4.116)$$

is obtained for the $d = 5$ electrically charged black hole in a heat reservoir. By putting the reservoir at infinity, the integrated first law yields the Smarr formula in $d = 5$

$$m = \frac{3}{2}T_{\text{H}}S + \phi_{\text{H}}Q, \quad (4.117)$$

see Eq. (4.82). We must note that the Smarr formula is valid for the small black hole solution only.

Regarding thermodynamic stability, the heat capacity $C_{A,\phi} = T \left(\frac{\partial S}{\partial T} \right)_{A,\phi}$ controls the stability of the ensemble and it is given by Eq. (4.96). By setting $d = 5$, the heat capacity is given by

$$C_{A,\phi} = \frac{3 \left(\frac{r_+}{R} \right) (1 - \Phi^2)^2}{8\pi l_p^3 T^2 \left(2(1 - \Phi^2) \left(\frac{r_+}{R} \right)^2 - 1 \right)}, \quad (4.118)$$

where we have that A is the area of the reservoir, and $x = \frac{r_+}{R}$. So $C_{A,\phi} > 0$ is the same condition as the validity of the zero loop approximation Eq. (4.106).

The most favorable thermodynamic configuration is found from the state with the lowest value of the grand potential W , as we have done previously for generic d . Since $W = TI_0$, the analysis is practically the same if done in I_0 or in W . The only difference is the perspective. In I , one talks about the most probable state and about quantum transitions, and when using W one talks about the most favorable state and thermodynamic phase transitions. See the analysis in the subsection above, Sec. 4.5.1.2.

4.6 THERMODYNAMIC RADII AND SPACETIME RADII COMPARISON

4.6.1 Thermodynamic bifurcation radius and the photon sphere radius comparison

In the case of the grand canonical ensemble of a d -dimensional Reissner-Nordström black hole in a cavity, we have seen in Eq. (4.47) that the two thermodynamic black hole solutions, represented by r_{+1} and r_{+2} , bifurcate from a horizon radius obeying $\frac{r_{+bif}}{R} = \frac{2^{\frac{1}{d-3}}}{((d-1)(1-\Phi^2))^{\frac{1}{d-3}}}$, or in terms of R ,

$$R = \left(\frac{(d-1)}{2} (1-\Phi^2) \right)^{\frac{1}{d-3}} r_+. \quad (4.119)$$

We have shown that a black hole for which the horizon radius r_+ satisfies Eq. (4.119) is marginally stable to thermodynamic perturbations, and that black holes with larger radius r_+ are thermodynamically stable. Hence, the bifurcation radius is also the marginal thermodynamic stable radius.

The photon sphere radius R of a d -dimensional Reissner-Nordström black hole is given by

$$R = \left(\frac{(d-1)}{2} \left(1 + \frac{d-3}{d-2} \Phi^2 \right) \right)^{\frac{1}{d-3}} r_+. \quad (4.120)$$

At this radius, null geodesics and photons can have circular trajectories.

From direct comparison between Eqs. (4.119) and (4.120), we observe that the two radii are distinct in any dimension d , therefore in the grand canonical ensemble of the Reissner-Nordström black hole there is no connection between them. Of course, when there is no charge or electric potential, the two radii coincide as Eqs. (4.119) and (4.120) both yield $R = \left(\frac{d-1}{2} \right)^{\frac{1}{d-3}} r_+$, and so the radius of the cavity at which a stable black hole appears corresponds to the photon sphere radius, as seen in [102].

4.6.2 The marginal favorability radius and the Buchdahl-Andréasson-Wright sphere radius comparison

In the case of grand canonical ensemble of a d -dimensional Reissner-Nordström black hole in a cavity, the stable black hole solution has a negative action I_0 , see Eq. (4.32), or equivalently, a negative grand potential W , see Eq. (4.70), if

$$\begin{aligned} \frac{\mu m}{R^{d-3}} &\leq -\frac{4(d-2)^2}{(d-1)^2(d-3)^2} \\ &+ \frac{2(d-2)((d-2)^2+1)}{(d-1)^2(d-3)^2} \sqrt{1 + \frac{(d-1)^2(d-3)^2}{4(d-2)^2} \frac{\lambda q^2}{R^{2d-6}}}. \end{aligned} \quad (4.121)$$

The condition in Eq. (4.121) for $d = 4$ is given by $\frac{l_p^2 m}{R} \leq -\frac{16}{9} + \frac{20}{9} \sqrt{1 + \frac{9}{16} \frac{q^2}{4\pi R^2}}$.

The Buchdahl-Andréasson-Wright bound is the minimum radius, below which, an electrically charged matter distribution obeying certain conditions, in general

relativity coupled to Maxwell electromagnetism in d dimensions, the spacetime is singular. The Buchdahl-Andréasson-Wright radius was obtained in [129] and can also be found from [1] by imposing that the trace of the stress-energy tensor of the matter in the thin shell is zero. The bound is given by

$$\frac{\mu m}{R^{d-3}} = \frac{d-2}{(d-1)^2} + \frac{1}{d-1} \frac{\lambda q^2}{R^{2d-6}} + \frac{d-2}{(d-1)^2} \sqrt{1 + (d-1)(d-3) \frac{\lambda q^2}{R^{2d-6}}}. \quad (4.122)$$

For $d = 4$, this is $\frac{l_p^2 m}{R} \leq \left(\frac{1}{3} + \sqrt{\frac{1}{9} + \frac{1}{3} \frac{\lambda q^2}{R^2}} \right)^2$, i.e., $\frac{m}{R} \leq \frac{2}{9} + \frac{1}{3} \frac{l_p^2 q^2}{4\pi R^2} + \frac{2}{3} \sqrt{\frac{1}{9} + \frac{1}{3} \frac{l_p^2 q^2}{4\pi R^2}}$.

Comparing Eqs. (4.121) and (4.122), it can be seen that the marginal favorability radius and the Buchdahl-Andréasson-Wright radius are distinct for any dimension d , and so there is no connection between them. When there is no charge, $q = 0$, and no electric potential, $\Phi = 0$, both radii are equal to Buchdahl radius, i.e. $\frac{r_+}{R} \geq \left(\frac{4(d-2)}{(d-1)^2} \right)^{\frac{1}{d-3}}$. In this case, the stable solution has a negative free energy if the radius of the black hole is larger than the Buchdahl radius, see [102].

4.7 GRADIENT OF THE ACTION FOR THE TWO ILL-BEHAVED CRITICAL POINTS

We analyze the gradient of the action near the critical point $r_+ = q = 0$, and also $r_+^{2d-6} = \lambda q^2 = R^{2d-6}$, to understand their metastability. The gradient of the action in Eq. (4.25) can be written as

$$\frac{\mu}{R^{d-2}} \frac{\partial I_*}{\partial x} = \frac{(d-3)\beta}{2Rx\sqrt{f}} \left[x^{d-3} - \frac{y}{x^{d-3}} \right] - 2\pi x^{d-3}, \quad (4.123)$$

$$\frac{\mu}{R^{d-2}} \frac{\partial I_*}{\partial \sqrt{y}} = \frac{\beta\sqrt{y}}{Rx^{d-3}\sqrt{f}} (1 - x^{d-3}) - \frac{\beta\Phi}{R}, \quad (4.124)$$

where $x = \frac{r_+}{R}$, $y = \frac{\lambda q^2}{R^{2d-6}}$ and $\Phi = (d-3)\Omega\sqrt{\lambda}\phi$.

First, we proceed with the analysis for hot flat space $r_+ = q = 0$. For that, the limit of the gradient for $x = y = 0$ is done along a family of curves $y = (\eta)^2 x^{2d-6}$, where η is a positive constant of the curve. One must consider $\eta < 1$ so that the curve is inside the physical domain of the action and it covers all the possible directions inside it. The gradient near $r_+ = q = 0$ is given by

$$\frac{\mu}{R^{d-2}} \frac{\partial I_*}{\partial x} = \frac{(d-3)\beta x^{d-4}}{2R} (1 - \eta^2), \quad (4.125)$$

$$\frac{\mu}{R^{d-2}} \frac{\partial I_*}{\partial \sqrt{y}} = \frac{\beta}{R} (\eta - \Phi). \quad (4.126)$$

The dependence in x^{d-4} was left since it gives different limits for the case of $d = 4$ and $d > 4$. Since the gradient depends on η , then the limit of the gradient is not defined. Nevertheless, one can calculate the directional derivative along the vector $v = \frac{1}{\sqrt{1+(d-3)^2\eta^2x^{2d-8}}} (1, (d-3)\eta x^{d-4})^T$, which is given by

$$D_v I_* = \frac{(d-3)\beta x^{d-4}}{2R} (1 + \eta^2 - 2\eta\Phi), \quad (4.127)$$

so for $d > 4$, the directional derivative vanishes. Additionally, the directional $(d-3)$ th derivative is positive since $1 + \eta^2 - 2\eta\Phi > 0$ for $\eta < 1$ and $\Phi \leq 1$, and so this can be considered as a minimum, although formally the partial derivative in $\sqrt{y}$ is undefined. For $d = 4$ case, the directional derivative does not vanish, however, since $1 + \eta^2 - 2\eta\Phi > 0$ for $\eta < 1$ and $\Phi \leq 1$, one can observe that the directional derivative is positive in the physical domain. Hence, the action resembles a conical potential well at the origin and so hot flat space can be considered as a solution.

Now, we analyze the extremal black hole located at the cavity. In order to study the gradient in the critical point $x = 1$ and $y = 1$, one can calculate the gradient of the action in this limit along the curve $x^{d-3} = 1 - \epsilon$ and $y = 1 - \eta\epsilon$, where η is a constant of the curve and ϵ parametrizes the curve. The limit of $\epsilon \rightarrow 0^+$ is then performed, giving the gradient

$$\frac{\mu}{R^{d-2}} \frac{\partial I_*}{\partial x} = \frac{(d-3)\beta}{2R\sqrt{\eta-1}}(\eta-2) - 2\pi, \quad (4.128)$$

$$\frac{\mu}{R^{d-2}} \frac{\partial I_*}{\partial \sqrt{y}} = \frac{\beta}{R\sqrt{\eta-1}} - \frac{\beta\Phi}{R}, \quad (4.129)$$

where it is required that $\eta > 2$ so that the curve is done along configurations of subextremal black holes, coming from the condition $y < x^{2d-6}$. Since there is a dependence on the curve one chooses to perform the limit, the limit of the gradient at the extremal point is not defined.

Interestingly, for $\gamma = 1$, i.e., $\beta = \frac{4\pi}{d-3} \frac{|\Phi|}{1-\Phi^2} R$, the gradient vanishes in the limit along a curve with $\frac{1}{\eta} = 1 + \frac{1}{\Phi^2}$. In fact, this set of temperatures corresponds to the stable black hole solution hitting the extremal point $x = y = 1$. But this only happens in one particular curve, the limit of the gradient is still undefined.

Finally, one should consider the directional derivative along these curves, in the direction of smaller ϵ . Indeed, the direction can be described by the vector $v = \frac{1}{\sqrt{1+(d-3)^2\eta^2/4}}(1, \frac{\eta(d-3)}{2})^T$, and so the directional derivative gives

$$\frac{\mu}{R^{d-2}} D_v I_* = \frac{\frac{\beta(d-3)}{2R} (2\sqrt{\eta-1} - \eta\Phi) - 2\pi}{\sqrt{1+(d-3)^2\eta^2/4}}. \quad (4.130)$$

The directional derivative depends also on η and it can be either positive or negative. Particularly, for values of η and Φ where $\gamma_{\text{bif}}(\Phi, d) < \frac{4(\eta-1)\Phi^2}{(1-\Phi^2)^2} \left(1 - \frac{\eta}{2\sqrt{\eta-1}}\Phi\right)^2$, the directional derivative in Eq. (4.130) can be positive in a region $\gamma_{\text{bif}}(\Phi, d) < \gamma < \frac{4(\eta-1)\Phi^2}{(1-\Phi^2)^2} \left(1 - \frac{\eta}{2\sqrt{\eta-1}}\Phi\right)^2$, with γ_{bif} given in Eq. (4.48). Hence, the action near this critical point does not resemble a potential well.

4.8 CONCLUSIONS

In this Chapter, we built the grand canonical ensemble of a d -dimensional Reissner-Nordström space in a cavity, using the path integral approach. We obtained the

partition function of the space in a cavity, by performing the zero loop approximation to the path integral relative to the Euclidean action, where only the term which minimizes the action contributes to the path integral. There are two stationary points of the action that correspond to a black hole in equilibrium with a heat reservoir with the temperature and the electric potential fixed at the boundary of the cavity. We have shown that the stationary point with lower horizon radius is unstable, while the stationary point with higher horizon radius is stable. We could not find analytically the corresponding values of the event horizon radius depending on the temperature and electric potential of the two stationary points for arbitrary dimensions. However, we were able to find analytical expressions for the event horizon radius in $d = 5$, where the equation reduces to a quadratic polynomial.

From our analysis, there are some features of the stationary points in the electrically charged case that differ from the electrically uncharged case. First, the event horizon radius corresponding to the lowest temperature allowed does not correspond to the photon sphere, unlike the uncharged case. This indicates that the correspondence in the uncharged case is a coincidence. Second, the larger horizon radius solution reaches the radius of the cavity at finite temperature, unlike the uncharged case, where the horizon radius only reaches the cavity radius at infinite temperature.

We have obtained the thermodynamics of the system, by connecting the partition function given by the path integral in the zero loop approximation with the partition function of the grand canonical ensemble. The grand potential of the system can be obtained in terms of the action in the zero loop approximation. We thus recover the thermodynamics of the black hole corresponding to the stable stationary point. We have shown that the system's entropy corresponds to the Bekenstein-Hawking entropy, the pressure corresponds to the pressure of a self-gravitating static electrically charged spherical thin shell in equilibrium, and the thermodynamic energy has the same expression as the expression for the quasilocal energy. The first law of thermodynamics with constant area is obeyed at the stationary points of the action, as we would expect. The stability of the stationary points is described by the heat capacity at constant area and electric potential. If this heat capacity is positive, then the stationary point is stable. This fits well with the relationship between thermodynamic stability and the heat capacity.

Additionally, we made the comparison between the stable black hole solution and an electrically charged conducting hot sphere in flat space, in order to see the most favorable phase of the system. In this case, a configuration is more favorable than the other when its grand potential W is lower. This in turn depends on the value of the temperature, of the electric potential of the reservoir, and of the radius of the conducting sphere. Moreover, the smaller the radius of the conducting sphere, the larger the region where the stable black hole is favored. We also made the comparison of the Buchdahl-Andréasson-Wright bound radius in d -dimensional Reissner-Nordström spacetimes with the minimum radius for which the stable black hole phase is thermodynamically favored. We have shown that both radius do not coincide, thus showing that the connection displayed in the Schwarzschild

case is not generic, rather it is a very restricted equality holding only in the pure gravitational situation.

GIBBONS-HAWKING ACTION FOR ELECTRICALLY CHARGED BLACK HOLES IN THE CANONICAL ENSEMBLE AND DAVIES' THERMODYNAMIC THEORY OF BLACK HOLES

5.1 INTRODUCTION

In the previous chapter, we have analyzed the grand canonical ensemble of a charged black hole inside a cavity, using the Euclidean path integral approach. Apart from this statistical treatment, there is another thermodynamic approach based on the use of Bekenstein-Hawking entropy [42, 43, 45] and the first law of black hole mechanics [40] to obtain the thermodynamic properties of black holes, namely the first law of thermodynamics. This law and its consequences were summarized in Davies' thermodynamic theory of black holes [51, 165]. An important feature described by Davies was the case of an infinite discontinuity in the heat capacity with constant electric charge for electric charged black holes, which was described as being similar to a second order phase transition. The thermodynamic analysis of charged black holes was also performed at the same time in [166]. The characterization of this discontinuity as a second order phase transition was further scrutinized in [52, 53, 167, 168], where it was established that the discontinuity described a turning point or a condition of stability rather than a phase transition. Further works using the first law of thermodynamics for black hole spacetimes were done afterwards [54–63].

Even though the use of the first law of thermodynamics to describe the thermodynamics of black holes is well-motivated, there is a lack of analysis establishing that the construction of statistical ensembles using the Euclidean path integral approach yields the same results as just simply imposing the first law of thermodynamics as described in Davies' thermodynamic theory. Hence, in this chapter, we construct the canonical ensemble of a charged black hole with the cavity at infinity in higher dimensions through the Euclidean path integral approach with fixed temperature and electric charge. The objective is to compare the results from the Euclidean path integral approach, as in Gibbons and Hawking, with the results from Davies' thermodynamic theory. We perform the zero loop approximation to the partition function, as in Gibbons and Hawking, and find two possible black hole solutions for the ensemble, with the larger black hole being unstable and the smaller black hole being stable. We find the thermodynamic properties of the black hole, and show that the Davies' thermodynamic theory for the four dimensional case, $d = 4$,

is in agreement with our results. We also briefly analyze the five dimensional case, $d = 5$. We observe that the heat capacity of the black hole has precisely the discontinuity found by Davies [51], and we show that it is in fact a turning point. Finally, we construct a model for charged hot flat space, which is described by hot flat space with electric charge at infinity. This allows us to study the phase transitions between this configuration and the stable black hole, which is lacking in the literature. We find that the charged hot flat space is always favorable compared to the stable black hole solution.

This chapter is presented as follows. In Sec. 5.2, we construct the canonical ensemble using the partition function and perform the zero loop approximation. In Sec. 5.3.3, we obtain the thermodynamics of the system from the partition function and we perform the analysis of phase transitions. In Sec. 5.4, we present the case $d = 4$, showing that the results agree with Davies' thermodynamic theory. In Sec. 5.5, we present briefly the case $d = 5$. In Sec. 5.6, we conclude the chapter. The work in this chapter is based on [3].

5.2 THE CANONICAL ENSEMBLE OF A CHARGED BLACK HOLE IN ASYMPTOTICALLY FLAT SPACE THROUGH THE EUCLIDEAN PATH INTEGRAL APPROACH

5.2.1 The Euclidean path integral and Euclidean action for the canonical ensemble

The canonical ensemble of a charged black hole in asymptotically flat space can be constructed through the Euclidean path integral approach, in d dimensions, with the partition function given formally by

$$Z = \int Dg_{\alpha\beta} DA_\gamma e^{-I[g_{\mu\nu}, A_\sigma]}, \quad (5.1)$$

with the Euclidean action

$$\begin{aligned} I = & -\frac{1}{16\pi l_p^{d-2}} \int_M R \sqrt{g} d^d x - \frac{1}{8\pi l_p^{d-2}} \int_{\partial M} (K - K_0) \sqrt{\gamma} d^{d-1} x \\ & + \frac{(d-3)}{4\Omega_{d-2}} \int_M F_{ab} F^{ab} \sqrt{g} d^d x \\ & + \frac{(d-3)}{\Omega_{d-2}} \int_{\partial M} F^{ab} A_a n_b \sqrt{\gamma} d^{d-1} x, \end{aligned} \quad (5.2)$$

where R is the Ricci scalar given by first and second order derivatives of the Euclidean metric $g_{\alpha\beta}$, g is the determinant of $g_{\alpha\beta}$, K is the trace of the extrinsic curvature K_{ab} of the space boundary, K_0 is the trace of the extrinsic curvature of the space boundary embedded in flat Euclidean space, γ_{ab} is the induced metric on the space boundary, γ is the determinant of γ_{ab} , $\Omega_{d-2} = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}$ is the surface area of the unit $(d-2)$ -sphere, $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ is the Maxwell tensor given by derivatives of the electromagnetic vector potential A_α , and n_β is the outward unit normal vector to the space boundary. The Gibbons-Hawking-York boundary

term involving the extrinsic curvature must be present in the action in order to have a well-defined variational principle with Dirichlet boundary conditions. The boundary term depending on the Maxwell tensor must be present so that the canonical ensemble may be prescribed, see [130]. This can be seen from the variation of the action, as one must get a boundary term with the variation of flux density and not the variation of the Maxwell field. This term gives the correct identification of the action with fixed electric flux given by the integral of the Maxwell tensor on a $(d-2)$ -surface, which has the meaning of electric charge. In the other situation, the potential vector A_a must be fixed at the boundary in order to have a well-described system, which means the grand canonical ensemble should be prescribed as was done in [67], see also [130].

5.2.2 Zero loop approximation

5.2.2.1 Euclidean Reissner-Nordström line element and Maxwell field

Differently from the other chapters, here we apply the zero loop approximation directly in the sense of Gibbons-Hawking [67], meaning that the action in Eq. (5.2) is evaluated for a space that is a solution to the Euclidean Einstein-Maxwell equations. This solution for arbitrary d dimensions with $d \geq 4$, is described by the d -dimensional Reissner-Nordström line element

$$ds^2 = \left(\frac{1}{2\pi t_H} \right)^2 f(r) d\tau^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{d-2}^2, \quad (5.3)$$

also called Tangherlini line element, where the function t_H , the Hawking function or Hawking temperature function, is given by

$$t_H = \frac{(d-3) \left(r_+^{d-3} - \frac{\mu q^2}{r_+^{d-3}} \right)}{4\pi r_+^{d-2}}, \quad (5.4)$$

with r_+ being the horizon radius of the black hole, q its electric charge, the function $f(r)$ defined by

$$f(r) = \left(1 - \frac{r_+^{d-3}}{r^{d-3}} \right) \left(1 - \frac{\mu q^2}{r_+^{d-3} r^{d-3}} \right), \quad (5.5)$$

with

$$\mu = \frac{8\pi l_p^{d-2}}{(d-2)\Omega_{d-2}}, \quad (5.6)$$

and $d\Omega_{d-2}^2$ being the line element of the $(d-2)$ -sphere with surface area $\Omega_{d-2} = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}$. The coordinate range for the Euclidean time is $\tau \in]0, 2\pi[$, the range for the radius coordinate is $r \in]r_+, \infty[$, and the ranges of the angular coordinates are the usual ones. The Maxwell electromagnetic potential field is described by

$$A_\tau(y) = -\frac{iq}{2\pi(d-3)t_H} \left(\frac{1}{r_+^{d-3}} - \frac{1}{r^{d-3}} \right). \quad (5.7)$$

We now discuss the considerations used to obtain the precise forms of the line element and the Maxwell field given in Eqs. (5.4)-(5.7). First for the line element, we choose a smooth metric, i.e., the metric cannot have conical singularities or curvature singularities. In order to avoid a conical singularity at the horizon and since we have chosen 2π periodicity in the Euclidean imaginary time, we added the factor $\frac{1}{(2\pi t_H)^2} = \frac{1}{(\sqrt{f}\partial_r\sqrt{f})_{y=0}^2}$ to the usual $\tau\tau$ component of the Reissner-Nordström line element. Secondly, for the Maxwell field, we have chosen a gauge for A_μ such that only A_τ is non-zero and we have assumed the nonexistence of magnetic monopoles. Also, the gauge was chosen such that $A_\mu(r_+) = 0$. In this gauge, the Maxwell field in the Riemannian metric is tied to the physical electric potential given by $(d-3)\frac{2\pi i t_H}{\sqrt{f}}A_\tau$, which should be bounded at the horizon.

The Reissner-Nordström line element characterized by Eqs. (5.3)-(5.6) has several features. The main features are the two parameters, namely, the horizon radius r_+ , and the electric charge q . There is an instance where the line element is characterized by one parameter alone, instead of two, which is the extremal case

$$r_{+e} = (\mu q^2)^{\frac{1}{2d-6}}. \quad (5.8)$$

From Eq. (5.8), one sees that for a given electric charge q the extremal horizon radius r_{+e} has a precise value. One can invert Eq. (5.8) so that, for a given horizon radius r_+ , there is an extremal electric charge q_e given by $q_e = \sqrt{\frac{r_+^{2d-6}}{\mu}}$. When it is convenient, we shall trade the horizon radius r_+ for the space mass m and the electric charge q as

$$r_+ = \left(\mu m + \sqrt{\mu^2 m^2 - \mu q^2} \right)^{\frac{1}{d-3}}. \quad (5.9)$$

This equation can be inverted to give $m = \frac{r_+^{d-3}}{2\mu} + \frac{q^2}{2r_+^{d-3}}$. In terms of the mass, the extremal black hole of Eq. (5.8) obeys the relation $\sqrt{\mu} m = q$, where here q means the absolute value of the electric charge.

5.2.2.2 The ensemble and its solutions

We are considering here the canonical ensemble of a charged black with the boundary at infinity. This boundary characterizes the heat reservoir with a fixed temperature T and fixed electric charge of the whole space Q . The inverse temperature at infinity, $\beta = \frac{1}{T}$, is determined by the Euclidean proper time at the boundary of the space, i.e., $\beta = 2\pi \left(\frac{\sqrt{f}}{2\pi t_H} \right) \Big|_{r \rightarrow \infty}$. Using that $f(r \rightarrow \infty) = 1$, one has that β must be equal to the inverse of the Hawking function t_H . Now, from the path integral formalism, β is the fixed inverse temperature of the ensemble. Therefore, the ensemble temperature T and the Hawking temperature function $t_H(r_+, q)$ of Eq. (5.4) satisfy the relation $T = t_H(r_+, q)$. Notice that, since the period of the Euclidean time τ is 2π , the factor $(2\pi t_H)^{-2}$ was introduced on the time-time

component of the metric in order to have regularity, therefore one links the temperature function t_h to the avoidance of a conical singularity at the horizon if the Einstein equations are solved. In addition, in this canonical ensemble, the electric flux $\int F^{ab} dS_{ab} = -i \frac{\Omega_{d-2}}{2} Q$ or, equivalently, the total electric charge, with the reservoir at spatial infinity, is fixed to be Q so that the electric charge of the black hole q obeys $q = Q$. In brief, the considered canonical ensemble with fixed temperature T and fixed electric charge Q at infinity imposes the following constraints to the possible black hole solutions,

$$T = t_H(r_+, Q), \quad (5.10)$$

$$Q = q. \quad (5.11)$$

The latter equation means that black holes that are solutions of this ensemble must have their electric charge q equal to the ensemble electric charge Q .

Inverting Eqs. (5.10) and (5.11), we can see that the black hole solutions have the generic form

$$r_+ = r_+(T, Q), \quad (5.12)$$

$$q = q(T, Q), \quad (5.13)$$

with this later equality having a direct solution $q = Q$. Specifically, by rearranging Eq. (5.10) and taking into account Eq. (5.11), the black hole solutions r_+ , which are formally represented in Eq. (5.12), obey the condition

$$\left(\frac{d-3}{4\pi T} \right) (r_+^{2d-6} - \mu Q^2) - r_+^{2d-5} = 0. \quad (5.14)$$

This equation, Eq. (5.14), is not solvable analytically for generic d . However, we can perform an analysis of its solutions. The function $t_H(r_+, Q)$ in Eq. (5.4), see also Eq. (5.10), has a maximum at

$$r_{+s} = \left(\sqrt{(2d-5)\mu} Q \right)^{\frac{1}{d-3}}, \quad (5.15)$$

which is a saddle point of the action for the black hole. From now onwards, Q stands for the absolute value of the electric charge Q itself for convenience. The saddle point r_{+s} of the action of the black hole has temperature

$$T_s = \frac{(d-3)^2}{2\pi(2d-5)(\sqrt{(2d-5)\mu} Q)^{\frac{1}{d-3}}}. \quad (5.16)$$

Eliminating Q in Eqs. (5.15) and (5.16), one finds r_{+s} in terms of a given temperature T , $r_{+s} = \frac{(d-3)^2}{2\pi(2d-5)T}$, or inverting, for a given r_+ , one finds $T_s = \frac{(d-3)^2}{2\pi(2d-5)r_+}$. In $d = 4$, the temperature T_s in Eq. (5.16) reduces to the Davies temperature, and so, one can see Eq. (5.16) as the generalization of the Davies temperature to d dimensions.

By inspection of Eq. (5.14), for temperatures in the interval $0 < T \leq T_s$, there are two solutions, the solution $r_{+1}(T, Q)$ and the solution $r_{+2}(T, Q)$, while for $T > T_s$ there are no black hole solutions. The solution $r_{+1}(T, Q)$ exists in the interval $r_{+e} < r_{+1}(T, Q) \leq r_{+s}$, so we can summarize for the solution 1

$$\begin{aligned} r_{+1} &= r_{+1}(T, Q), & 0 < T \leq T_s, \\ q_1 &= Q, & r_{+e} < r_{+1}(T, Q) \leq r_{+s}, \end{aligned} \quad (5.17)$$

where r_{+e} is the radius of the extremal black hole given by $r_{+e} = r_{+1}(T \rightarrow 0, Q) = (\mu Q^2)^{\frac{1}{2d-6}}$, see Eq. (5.8), and $r_{+s} = r_{+1}(T_s, Q)$ is given in Eq. (5.15). This solution, $r_{+1}(T, Q)$, is an increasing monotonic function of T . The solution $r_{+2}(T, Q)$ exists in the interval $r_{+s} < r_{+2}(T, Q) < \infty$, so we can summarize for the solution 2

$$\begin{aligned} r_{+2} &= r_{+2}(T, Q), & 0 < T \leq T_s, \\ q_2 &= Q, & r_{+s} < r_{+2}(T, Q) < \infty, \end{aligned} \quad (5.18)$$

where $r_{+s} = r_{+2}(T_s, Q)$, i.e., the solution 2 is bounded from below, and is unbounded from above, since at $T \rightarrow 0$, the solution tends to infinity. This solution, $r_{+2}(T, Q)$, is a decreasing monotonic function of T . When the ensemble is only characterized by the temperature T , with vanishing Q , only the black hole solution r_{+2} survives which corresponds to the Gibbons-Hawking black hole solution.

There is however another solution which exists for all temperatures. This solution can be described by a limit of solutions in the charged matter sector. In order to keep a vanishing mass of space and to keep a fixed electric charge, one must have charged matter at infinity, at the boundary of space. We refer to this configuration as the charged hot flat space, i.e. hot flat space with electric charged Q dispersed at infinity. For $T > T_s$, there are no black hole solutions and one is left with hot flat space with electric charged Q dispersed at infinity, and so the solution of the ensemble at this temperature range can be summarized as

$$\begin{aligned} &\text{charged hot flat space,} & T_s < T < \infty, \\ &Q \text{ dispersed at } r = \infty, & 0 \leq r < \infty. \end{aligned} \quad (5.19)$$

Thus, the three solutions of the ensemble are displayed in Eqs. (5.17)-(5.19).

5.2.2.3 Action of the Reissner-Nordström black hole space and partition function

We now evaluate the action given in Eq. (5.2) for the metric in Eq. (5.3) and for the Maxwell field in Eq. (5.7), with the black hole solutions of the ensemble obeying Eq. (5.14), i.e., those formally shown in Eq. (5.17) and Eq. (5.18).

It is useful to split the action I into the gravitational action plus the Maxwell action, i.e. $I = I_{\text{gf}} + I_q$, where

$$I_{\text{gf}} = -\frac{1}{16\pi l_p^{d-2}} \int_{\mathcal{M}} R \sqrt{g} d^d x - \frac{1}{8\pi l_p^{d-2}} \int_{\partial \mathcal{M}} (K - K_0) \sqrt{\gamma} d^{d-1} x, \quad (5.20)$$

$$I_q = \frac{(d-3)}{4\Omega_{d-2}} \int_{\mathcal{M}} F_{\alpha\beta} F^{\alpha\beta} \sqrt{g} d^d x + \frac{(d-3)}{\Omega_{d-2}} \int_{\partial \mathcal{M}} F^{\alpha\beta} A_\alpha n_\beta \sqrt{\gamma} d^{d-1} x. \quad (5.21)$$

Starting with the gravitational action, one can obtain it generally for a spherically symmetric metric, see Chapter 3. Together with the metric form in Eq. (5.3), one has that

$$I_{\text{gf}} = \left(\frac{\sqrt{f} r^{d-3}}{\mu t_H} (1 - \sqrt{f}) \right) \Big|_{r \rightarrow +\infty} - \frac{\Omega_{d-2}}{4l_p^{d-2}} \left(\frac{\partial_r f r^{d-2}}{4\pi t_H} \right) \Big|_{r \rightarrow r_H} - \frac{1}{2t_H} \frac{q^2}{r_+^{d-3}}. \quad (5.22)$$

Regarding the action for the Maxwell field, one can simplify the Maxwell term as $F^{\alpha\beta} F_{\alpha\beta} = 2F_{u\tau} F^{u\tau} = -2 \frac{q^2}{r^{2d-4}}$ and the boundary term as $F^{\alpha\beta} A_\alpha n_\beta = \frac{2\pi i t_H q}{r^{d-2} \sqrt{f}} A_\tau$ to obtain

$$I_q = -\frac{q^2}{2t_H r_+^{d-3}} + \left(\frac{q^2}{t_H \sqrt{f}} \left(\frac{1}{r_+^{d-3}} - \frac{1}{r^{d-3}} \right) \right) \Big|_{r \rightarrow +\infty}, \quad (5.23)$$

With the action written explicitly in terms of the important quantities of the ensemble solution, one can now further perform the limits using the properties of the function f and the Maxwell field A_τ , in Eq. (5.5) and (5.7) respectively. The gravitational action has two limits that must be performed. The first limit yields $(\sqrt{f} r^{d-3} (1 - \sqrt{f})) \Big|_{r \rightarrow +\infty} = \frac{r_+^{d-3}}{2} + \frac{\mu q^2}{2r_+^{d-3}}$ while the second limit yields $(\partial_r f r^{d-2}) \Big|_{r \rightarrow r_H} = 4\pi t_H r_+^{d-2}$. The action for the Maxwell field has one limit which yields $\left(\frac{q^2}{t_H \sqrt{f}} \left(\frac{1}{r_+^{d-3}} - \frac{1}{r^{d-3}} \right) \right) \Big|_{r \rightarrow +\infty} = \frac{q^2}{t_H r_+^{d-3}}$. Therefore, the full action is given by

$$I_0[T, Q] = \frac{1}{\mu T} \left(\frac{r_+^{d-3}}{2} + \frac{\mu Q^2}{2r_+^{d-3}} \right) - \frac{\Omega_{d-2}}{4l_p^{d-2}} r_+^{d-2}, \quad (5.24)$$

where, $T = t_H(r_+, Q)$ was used, having two black hole solutions for $T \geq T_s$, $r_{+1}(T, Q)$ and $r_{+2}(T, Q)$, each of which gives an expression in terms of T and Q that replace r_+ in Eq. (5.24). Explicitly, the actions for each solution are of the form $I_0(T, Q, r_{+1}(T, Q))$ and $I_0(T, Q, r_{+2}(T, Q))$. There is a third solution that must be considered, corresponding to the case of having no black hole solutions. This case is described by hot flat space with fixed temperature of the reservoir at infinity and with fixed electric charge residing near the reservoir at infinity, in order to satisfy the Gauss constraint of the electromagnetic field without contributing to the energy content of the space. This hot flat space in this zero loop approximation is simply classical flat space at some temperature T with no matter fields present. The zero loop action for classical hot flat space with electric charge at infinity is then zero, i.e., $I_0[T, Q] = 0$. The partition function Z in the zero loop approximation for the canonical ensemble is then

$$Z = e^{-I_0[T, Q]}, \quad (5.25)$$

with $I_0[T, Q]$ given in Eq. (5.24).

The partition function given in Eq. (5.25), with the action described in Eq. (5.24), is valid for d dimensions. In four dimensions, $d = 4$, the partition function will give origin to Davies results [51], see also [166]. This means that Davies' thermodynamic

theory of black holes, in the case of electrically charged black holes, can be seen within the canonical ensemble formalism. Here, the results are generalized to arbitrary d dimensions, $d = 4$ being a particular case.

5.3 THERMODYNAMICS

5.3.1 Thermodynamic quantities and properties

We have used the Gibbons-Hawking Euclidean path integral approach to construct the canonical ensemble of an asymptotically flat spherically symmetric electrically charged black hole space in arbitrary d dimensions. With the system being in equilibrium with a heat reservoir at infinity with temperature T and electric charge Q , the thermodynamics of the system can now be obtained by considering that the partition function of the canonical ensemble is related to the Helmholtz free energy F through $Z = e^{-\frac{F}{T}}$, i.e., $F = -T \ln Z$. From the zero loop approximation, Eq. (5.25), this means $F = TI_0$. With I_0 given in Eq. (5.24) one finds that the free energy is $F = \frac{1}{\mu} \left(\frac{r_+^{d-3}}{2} + \frac{\mu Q^2}{2r_+^{d-3}} \right) - \frac{\Omega_{d-2} r_+^{d-2}}{4} T$. Substituting T for t_H , see Eqs. (5.4) and (5.10), one obtains for the free energy the expression

$$F(T, Q) = \frac{1}{\mu(d-2)} \left(\frac{r_+^{d-3}}{2} + (2d-5) \frac{\mu Q^2}{2r_+^{d-3}} \right), \quad (5.26)$$

where r_+ should be envisaged as $r_+ = r_+(T, Q)$, since it is one of the solutions $r_{+1}(T, Q)$ or $r_{+2}(T, Q)$, given in Eq. (5.17) or Eq. (5.18), respectively. Thus, the Helmholtz free energy F for each solution is a function only of T and Q , namely, $F(T, Q, r_{+1}(T, Q))$ and $F(T, Q, r_{+2}(T, Q))$.

With the free energy F given by Eq. (5.26), one can obtain the thermodynamic quantities through its differential, $dF = -SdT + \phi dQ$. The first component of the differential yields the entropy

$$S = \frac{A_+}{4l_p^{d-2}}, \quad (5.27)$$

where $A_+ = \Omega_{d-2} r_+^{d-2}$ is the area of the horizon, and so S is the Bekenstein-Hawking entropy, valid for the two solutions $r_{+1}(T, Q)$ or $r_{+2}(T, Q)$. The second component of the differential yields the thermodynamic electric potential

$$\phi = \frac{Q}{r_+^{d-3}}, \quad (5.28)$$

i.e., the Coulombic electric potential, with r_+ being $r_{+1}(T, Q)$ or $r_{+2}(T, Q)$. The thermodynamic energy, given by $E = F + TS$, has the form $E = \frac{r_+^{d-3}}{2\mu} + \frac{Q^2}{2r_+^{d-3}}$, and since r_+ is $r_{+1}(T, Q)$ or $r_{+2}(T, Q)$, there are two solutions for E . This can be connected to the space mass m given by $m = \frac{r_+^{d-3}}{2\mu} + \frac{Q^2}{2r_+^{d-3}}$, see Eq. (5.9), so that here

the thermodynamic energy and the black hole mass are equal, i.e., they obey the relation

$$E = m. \quad (5.29)$$

Thus, one can write the free energy given in Eq. (5.26) as

$$F = m - TS. \quad (5.30)$$

The energy in the form $E = \frac{r_+^{d-3}}{2\mu} + \frac{Q^2}{2r_+^{d-3}}$ can also be rewritten in terms of the entropy and electric charge as $E = \frac{1}{2\mu} \left(\frac{4l_p^{d-2}S}{\Omega_{d-2}} \right)^{\frac{d-3}{d-2}} + \frac{Q^2}{2} \left(\frac{4l_p^{d-2}S}{\Omega_{d-2}} \right)^{\frac{3-d}{d-2}}$. This energy function has the scaling property $\nu^{\frac{d-3}{d-2}} E = E(\nu S, \nu^{\frac{d-3}{2(d-2)}} Q)$, for some ν , and so through the Euler relation theorem, one has $E = \frac{d-3}{d-2} TS + \phi Q$. Together with Eq. (5.29), i.e. $E = m$, one obtains

$$m = \frac{d-3}{d-2} TS + \phi Q, \quad (5.31)$$

which is the Smarr formula for an electrically charged black hole in d dimensions. The Smarr formula is valid for the two solutions, $r_{+1}(T, Q)$ or $r_{+2}(T, Q)$.

One can also verify that the first law of thermodynamics,

$$dm = TdS + \phi dQ, \quad (5.32)$$

holds. It holds for the two solutions, $r_{+1}(T, Q)$ or $r_{+2}(T, Q)$. But, Eq. (5.32) is also the first law of black hole mechanics since it involves pure black hole quantities. This shows that the thermodynamics that follow from the electrically charged canonical ensemble statistical mechanics is equivalent to the thermodynamics that follows from the first law of black hole mechanics. The first law of black hole mechanics was the starting point of Davies' analysis, while here it is a result of the statistical mechanics formalism.

5.3.2 Heat capacity and thermodynamic stability

The thermodynamic stability of the system is given by the condition that the heat capacity at constant electric charge must be positive, ensuring that the respective solution is stable. The heat capacity at constant electric charge is defined by $C_Q = \left(\frac{\partial E}{\partial T} \right)_Q$. Since $E = m = \frac{r_+^{d-3}}{2\mu} + \frac{Q^2}{2r_+^{d-3}}$ and $r_+ = r_+(T, Q)$, one has

$$\begin{aligned} C_Q &= \frac{1}{l_p^{d-2}} \frac{(d-2)\Omega_{d-2}r_+^{d-2}(r_+^{2d-6} - \mu Q^2)}{4 \left((2d-5)\mu Q^2 - r_+^{2d-6} \right)} \\ &= \frac{mS^3 T}{\frac{(d-3)\Omega_{d-2}^3}{4^5 l_p^{3d-6} \pi^2} \left[\frac{(3d-8)\mu^2 Q^4}{\left(\frac{4l_p^{d-2}S}{\Omega_{d-2}} \right)^{\frac{d-4}{d-2}}} + (d-4) \left(\frac{4l_p^{d-2}S}{\Omega_{d-2}} \right)^{\frac{3d-8}{d-2}} \right] - T^2 S^3}, \end{aligned} \quad (5.33)$$

where in the second equality, we have written the heat capacity in terms of the thermodynamic variables m , S , T , and Q . Note however that the heat capacity must be understood as a function of T and Q , as these are the quantities controlled in the ensemble. This means that r_+ must be understood as either $r_{+1}(T, Q)$ or $r_{+2}(T, Q)$, as well as m and S must be understood as $m = m(T, Q)$ and $S = S(T, Q)$. The thermodynamic stability is satisfied if the heat capacity is positive. According to Eq. (5.33), the ensemble is stable in the range $r_{+e} \leq r_+ < r_{+s}$, where $r_{+e} = (\mu Q^2)^{\frac{1}{2d-6}}$ and $r_{+s} = \left(\sqrt{(2d-5)\mu} Q\right)^{\frac{1}{d-3}}$. This is precisely the range of the solution r_{+1} . Therefore, one has

$$\text{stability if } C_Q \geq 0, \quad \text{i.e., } r_+ = r_{+1}. \quad (5.34)$$

In opposition, the ensemble is unstable in the range $r_{+s} < r_+ < \infty$, which is the range of the solution r_{+2} . Hence, one has

$$\text{instability if } C_Q < 0, \quad \text{i.e., } r_+ = r_{+2}. \quad (5.35)$$

So, from Eqs. (5.34) and (5.35) one has that the solution r_{+1} is stable whereas the solution r_{+2} is unstable, see Eqs. (5.17) and (5.18). We note also that r_{+1} is an increasing monotonic function of T , so that the energy of the system increases when the temperature increases, as it is expected from a stable system. The opposite happens to the solution r_{+2} , since it is a decreasing monotonic function of T and so the energy of the black hole decreases when the temperature increases. From Eq. (5.34), one also finds that the radius r_{+s} acts as the generalization of the Davies point for higher dimensions. Indeed, for r_{+s} fixed, for steady addition of electric charge Q , one finds that the solution passes from an r_{+2} solution to an r_{+1} solution, and eventually at the transition, a negative C_Q turns into a positive C_Q . In thermodynamics, this could signal a phase transition of second order, since the free energy F and its first derivatives are continuous, but second derivatives are discontinuous. However, this is not the case here, we are instead in the presence of a turning point which determines the relative scale of r_+ and Q at which a black hole can be in stable or metastable equilibrium when in thermal contact with a heat reservoir that holds T and Q fixed at infinity. Indeed, in the canonical ensemble, the parameters that we can control are T and Q . Maintaining Q fixed, and for a given sufficiently low T , there are two solutions, the stable solution $r_{+1}(T, Q)$ and the unstable solution $r_{+2}(T, Q)$. We could try to start with the stable solution at low T and devise a change of parameters T and Q such that r_+ was kept fixed. Eventually, we are able to reach the turning point and beyond it, the character of the solutions changes, i.e., the unstable solution r_{+2} would have a fixed r_+ , while the stable solution r_{+1} still exists and would suffer a change in r_+ . But any thermal perturbations would make the unstable solution r_{+2} to run away from equilibrium, thus the unstable solution r_{+2} is impossible to be maintained. And so, even for this specific change of parameters, with temperature up to T_s , we are always in the presence of the stable solution $r_{+2}(T, Q)$, this being the existing solution of the ensemble at T_s , and so we should not classify this point as a phase transition.

Bear in mind that the thermodynamic quantities, the first law of thermodynamics, and the Smarr formula as an integrated first law of thermodynamics, are only valid

strictly for the stable black hole solution r_{+1} , since the solution r_{+2} is unstable and does not allow a proper thermodynamic treatment. Note also, that in the limit of zero electric charge, $Q = 0$, there is only the r_{+2} solution corresponding to the Gibbons-Hawking black hole solution which is unstable. Indeed, the heat capacity in the zero electric charge case is $C = -\frac{(d-2)\Omega_{d-2}r_+^{d-2}}{4J_p^{d-2}}$, thus negative for all $r_+(T, Q)$.

5.3.3 Favorable phases

In a thermodynamic system, if different thermodynamic phases can take place, we are interested to know which are the favored phases for a given set of parameters. For temperature T and electric charge Q fixed by the reservoir, a thermodynamic system tends to be in a state in which its Helmholtz free energy F has the lowest value. If a system is in a stable state but with a higher free energy F than another stable state, it is probable that the system undergoes a transition to the state with the lowest free energy. Returning to the path integral calculation and the corresponding partition function, one sees that if there are two stable configurations, i.e., two states that minimize the action, then the largest contribution to the partition function is given by the configuration with the lowest action or, in thermodynamic language, with the lowest free energy. In order to analyze these phase transitions, one must obtain the critical regions where the free energy is the same for both configurations. Generally, at these transition points, the free energy's derivatives are different, signaling first order phase transitions.

In the case of a cavity within a heat reservoir at infinity kept at T and Q constants, we have seen that within the context of this chapter there are three solutions. One is the stable black hole r_{+1} , Eq. (5.17), which counts as a thermodynamic phase and exists for $T \leq T_s$. The other is the unstable black hole r_{+2} , Eq. (5.18), which also exists for $T \leq T_s$, but does not count as a thermodynamic phase since it is unstable. The other is hot flat space with electric charge at infinity that exists for $T > T_s$, Eq. (5.19). We have considered this phase, where there are no black holes, to be hot flat space with electric charge dispersed at infinity, because it seems the most natural solution, as electric charge of the same sign repels, and eventually disperse to infinity but it can also be motivated by certain limits of charged matter configurations.

Thus, there are two possible phases, namely, the black hole r_{+1} phase and hot flat space with electric charge at infinity. For $T > T_s$, only hot flat space with electric charge at infinity exists, as seen above. But for $T < T_s$, both r_{+1} and hot flat space with electric charge at infinity can exist. The one that is going to dominate for $T \leq T_s$ is the one that has the lowest free energy. Now, the free energy of hot flat space with electric charge at infinity is zero,

$$F_{\text{hfs}} = 0. \quad (5.36)$$

The free energy of the r_{+1} black hole is always positive, $F(T, Q, r_{+1}(T, Q)) > 0$. This can be seen from the on-shell expression Eq. (5.26) which for the r_{+1} solution reads

$$F(T, Q, r_{+1}(T, Q)) = \frac{1}{\mu(d-2)} \left(\frac{r_{+1}^{d-3}}{2} + (2d-5) \frac{\mu Q^2}{2r_{+1}^{d-3}} \right). \quad (5.37)$$

One finds that Eq. (5.37) has strictly positive terms. Thus, since

$$F_{\text{hfs}} < F(T, Q, r_{+1}(T, Q)), \quad (5.38)$$

hot flat space with electric charge at infinity is the favored phase for $T \leq T_s$. If the system finds itself in the black hole phase, it will make a transition to hot flat space with electric charge at infinity since it has lower free energy. We note however that the free energy of these two phases never intersects and so we cannot call this a first order phase transition. An analog to this transition is the one between supercooled water and ice. Moreover, hot flat space is the only phase for $T > T_s$.

5.3.4 Interpretation

We have deduced the thermodynamic results above starting from the path integral approach. The action that has entered into the path integral is the classical action, corresponding thus to a zero loop approximation. Although in this order of approximation there is no mention of matter fields, which would enter in a first loop approximation, we can try to interpret some of the results found in zero order, in terms of wavelengths of packets of thermal energy inside the cavity of a heat reservoir at infinity. This is because there is a given temperature T within the system, and at a quantum level, for a given T , there is an associated thermal wavelength λ , which is $\lambda = \frac{(d-3)^2}{2\pi(2d-5)T}$. The interpretation of the results in terms of matter fields is useful as we shall see now, even if it is beyond the formalism used here.

We can start by interpreting the existence and nonexistence of the two black hole solutions r_{+1} and r_{+2} . For small enough temperature T , and so large thermal wavelength λ , there are two solutions for r_+ . The r_+ of the small solution is sufficiently small so that it is smaller than λ , and so energy packets with typical wavelength λ are trapped in the black hole geometry and do not escape, making the black hole a possible solution and a stable one. The r_+ of the large solution is sufficiently large so that it is of the order of λ , with r_+ being a bit larger, and so energy packets with typical wavelength λ can escape, and backreact to turn the black hole unstable. Indeed, this case, with r_+ of the order $\frac{1}{T}$ and so of the order of λ , corresponds to the black hole with the Gibbons-Hawking black hole solution properties. Now, for larger reservoir temperature T , the thermal wavelength λ gets smaller. The r_+ of the small solution increases, now r_+ being barely smaller than λ . The r_+ of the large solution decreases, with r_+ being barely larger than λ . This latter solution is still the one with the Gibbons-Hawking black hole solution properties. At a saddle or critical temperature T_s , the two solutions meet. For even

higher reservoir temperature T , and so lower thermal wavelength λ , there is no way to make a black hole. The wavelength λ is low enough that it disperses without being able to aggregate the energy and the electric charge in a black hole state. In this case the electric charge disperses to infinity, yielding hot flat space for the whole space with the electric charge at infinity, and so vanishing electric charge density.

We could try to interpret the favorable phases in terms of wavelengths of packets of thermal energy inside the cavity, but we have not found a direct way to see how the behaviour of these packets lead to hot flat space with electric charge at infinity having always, for all parameters, a free energy lower than the small black hole free energy. However, it is clear what happens when one looks at the free energy expressions. Looking at the original expression for the free energy of the stable black hole, i.e., $F = \frac{1}{\mu} \left(\frac{r_+^{d-3}}{2} + \frac{\mu Q^2}{2r_+^{d-3}} \right) - \frac{\Omega_{d-2} r_+^{d-2}}{4l_p^{d-2}} T$, one sees that the entropy term which is negative has a small contribution because r_+ is small, and there is the electric charge term which goes as $\frac{Q^2}{2r_+^{d-3}}$ which gives a large positive contribution, since r_+ is small, all contributing for F never being zero for any set of parameters T and Q .

To better understand all the issues that we have worked out so far and to make further progress, we have to pick up definite dimensions. We specify the generic d -dimensional results above to the case of $d = 4$ and $d = 5$ dimensions. We perform a thorough analysis for the dimension $d = 4$, while we briefly analyze the case of $d = 5$ dimensions.

5.4 THE CASE $d = 4$: DAVIES' THERMODYNAMIC THEORY OF BLACK HOLES AND DAVIES POINT FROM THE CANONICAL ENSEMBLE

5.4.1 Solutions and action in $d = 4$

The dimension $d = 4$ is specially interesting since it gives the results of Davies' thermodynamic theory of black holes [51], see also [166].

We must start from the canonical ensemble characterized by a heat reservoir at infinity with temperature T and electric charge Q in $d = 4$. The black hole solutions r_+ of the ensemble are taken from solving Eq. (5.10) together with Eq. (5.4), which in $d = 4$ they yield

$$T = t_H(r_+, Q), \quad t_H(r_+, Q) = \frac{r_+ - \frac{l_p^2 Q^2}{r_+}}{4\pi r_+^2}, \quad (5.39)$$

where again T is the temperature kept fixed at the reservoir at infinity and $t_H(r_+, Q)$ is the original Hawking function in $d = 4$. When the electric charge of the reservoir at infinity is zero, $Q = 0$, then $t_H(r_+, 0) = \frac{1}{4\pi r_+}$, which is the Hawking temperature of a Schwarzschild black hole. The electric charge Q is the electric charge kept fixed at the reservoir at infinity, and the black hole electric charge q must match it to have a consistent solution, $q = Q$, see Eq. (5.11).

To find the solutions of the canonical ensemble, one inverts Eq. (5.39) to yield $(\frac{1}{4\pi T})(r_+^2 - l_p^2 Q^2) - r_+^3 = 0$, which is Eq. (5.14) for $d = 4$. This equation can be solved analytically as it is a cubic equation. However, we do not present the expression here. Alternatively, the solutions can be analyzed qualitatively or solved numerically. One can find that the function $t_H(r_+, Q)$ in Eq. (5.39) has a maximum at $r_{+s} = \sqrt{3}Q$, which is a saddle or critical point of the action of the black hole and which it is defined as

$$r_{+D} = \sqrt{3}l_p Q, \quad (5.40)$$

since in $d = 4$ it gives the Davies black hole horizon radius. This saddle point of the action is at the temperature given by

$$T_D = \frac{1}{6\sqrt{3}\pi l_p Q}, \quad (5.41)$$

see Eq. (5.16), when $d = 4$. From Eqs. (5.40) and (5.41), one can eliminate Q to give for a given T , $r_{+D} = \frac{1}{6\pi T}$, or inverting, for a given r_+ , $T_D = \frac{1}{6\pi r_+}$. The temperature given in Eq. (5.41) is the Davies temperature, and it is a result that can be extracted from [51, 166]. One finds that for temperatures $T \leq T_D$ there are two black holes, the solution $r_{+1}(T, Q)$ and the solution $r_{+2}(T, Q)$, while for $T > T_D$ there are no black hole solutions. The solution $r_{+1}(T, Q)$ is bounded in the interval $r_{+e} < r_{+1}(T, Q) \leq r_{+D}$, where $r_{+e} = r_{+1}(T \rightarrow 0, Q) = Q$ is the radius of the extremal black hole and $r_{+D} = r_{+1}(T_D, Q) = \sqrt{3}l_p Q$, so we can summarize for the solution 1

$$\begin{aligned} r_{+1} &= r_{+1}(T, Q), & 0 < T \leq T_D, \\ q_1 &= Q, & r_{+e} < r_{+1}(T, Q) \leq r_{+D}. \end{aligned} \quad (5.42)$$

This solution, $r_{+1}(T, Q)$, is an increasing monotonic function of T . The solution $r_{+2}(T, Q)$ is in the interval $r_{+D} < r_{+2}(T, Q) < \infty$, so we can summarize for the solution 2

$$\begin{aligned} r_{+2} &= r_{+2}(T, Q), & 0 < T \leq T_D, \\ q_2 &= Q, & r_{+D} < r_{+2}(T, Q) < \infty, \end{aligned} \quad (5.43)$$

where, at $T \rightarrow 0$, the solution tends to infinity there. This solution, $r_{+2}(T, Q)$, is a decreasing monotonic function of T . When the ensemble is only characterized by the temperature T , with Q vanishing, $Q = 0$, only the black hole r_{+2} survives which is the Gibbons-Hawking black hole solution. For $T > T_D$, there are no black hole solutions and one is left with hot flat space with electric charge Q dispersed at infinity, i.e.,

$$\begin{aligned} &\text{charged hot flat space,} & T_D < T < \infty, \\ &Q \text{ dispersed at } r = \infty, & 0 \leq r < \infty. \end{aligned} \quad (5.44)$$

We plot the two solutions $r_{+1}(T, Q)$ and $r_{+2}(T, Q)$ as functions of the temperature in Fig. 5.1 for two different values of the electric charge, which displays the features of the solutions just mentioned. For $T > T_D$, there are no solutions, only hot flat space with electric charge Q at infinity.

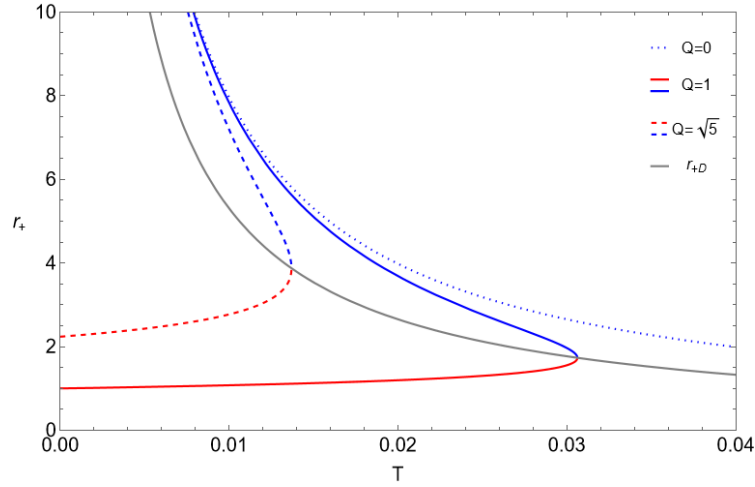


Figure 5.1: Plot of the two solutions $r_{+1}(T, Q)$, in red, and $r_{+2}(T, Q)$, in blue, of the charged black hole in the canonical ensemble for infinite cavity radius, as a function of T , for three values of the charge, $Q = 0$ in a dotted line, $Q = 1$ in filled lines, and $Q = \sqrt{5}$ in dashed lines, in $d = 4$. The case $Q = 0$ is the Gibbons-Hawking black hole, there is only the $r_{+1}(T, Q)$ solution, which is clearly unstable. It is also plotted, in a gray line, the critical Davies radius as a function of T , $r_{+D} = \frac{1}{6\pi T}$.

The zero loop action of the canonical ensemble characterized by the temperature T and the electric charge Q for $d = 4$ can be found using directly Eq. (5.24), i.e.,

$$I_0[T, Q] = \frac{1}{2T} \left(r_+ + \frac{l_p^2 Q^2}{r_+} \right) - \pi \frac{r_+^2}{l_p^2}, \quad (5.45)$$

where $\mu = l_p^2$ and $\Omega_2 = 4\pi$ were used. The black hole horizon radii r_+ that enter into this action are the r_{+1} given in Eq. (5.42) or the r_{+2} given in Eq. (5.43).

5.4.2 Thermodynamics in $d = 4$

With the solutions and the action of the canonical ensemble found, we can obtain the thermodynamics through the correspondence $F = TI_0$, where F again is the Helmholtz free energy of the system. From Eq. (5.45), F in $d = 4$ is $F = \frac{1}{2l_p^2} \left(r_+ + \frac{l_p^2 Q^2}{r_+} \right) - T \pi \frac{r_+^2}{l_p^2}$, which upon using Eq. (5.39) gives

$$F(T, Q) = \frac{1}{4l_p^2} \left(r_+ + \frac{3l_p^2 Q^2}{r_+} \right), \quad (5.46)$$

where r_+ should be envisaged as $r_+ = r_+(T, Q)$, since it is one of the solutions $r_{+1}(T, Q)$ or $r_{+2}(T, Q)$, given in Eq. (5.42) or Eq. (5.43), respectively. From the derivatives of the free energy, one can obtain the entropy as $S = \pi \frac{r_+^2}{l_p^2}$, i.e., $S = \frac{A_+}{4l_p^2}$, the electric potential, $\phi = \frac{Q}{r_+}$, and the thermodynamic energy, $E = \frac{1}{2l_p^2} \left(r_+ + \frac{l_p^2 Q^2}{r_+} \right)$,

where $E = F + TS$ was used. These expressions are valid for both solutions, r_{+1} and r_{+2} . The expression for the energy is precisely the expression for the space mass m , so $E = m$. The free energy of Eq. (5.46) is then $F = m - TS$.

Here, the Smarr formula in $d = 4$ is clearly

$$m = \frac{1}{2} TS + \phi Q, \quad (5.47)$$

see Eq. (5.31) for $d = 4$. Also, one has that the law $dm = TdS + \phi dQ$ holds, which ties the first law of black hole mechanics with the first law of thermodynamics. The first law of black hole mechanics is the expression from which Davies [51] started his thermodynamic analysis, see also [166]. Our analysis here started from the canonical ensemble theory and the Euclidean path integral approach with the action of Eq. (5.45), which yields naturally the first law of thermodynamics.

The heat capacity C_Q of Eq. (5.33), for $d = 4$, is given by

$$C_Q = \frac{1}{l_p^2} \frac{2\pi r_+^2 \left(1 - \frac{l_p^2 Q^2}{r_+^2}\right)}{3 \frac{l_p^2 Q^2}{r_+^2} - 1} = \frac{m S^3 T}{\frac{\pi Q^4}{4l_p^{d-2}} - T^2 S^3}, \quad (5.48)$$

where in the second equality the heat capacity was written in terms of the thermodynamic variables m , S , T , and Q . Note that C_Q is a function of T and Q , which are the parameters that are controlled. Thermodynamic stability is governed by the positivity of the heat capacity, $C_Q \geq 0$. From Eq. (5.48), one finds that the range of stability is $r_{+e} \leq r_+ < r_{+D}$, where r_{+e} is the radius of the extremal black hole given by $r_{+e} = l_p Q$ and r_{+D} is the Davies horizon radius given in Eq. (5.40). This range for r_+ corresponds to the solution r_{+1} , and so one has

$$\text{stability if } C_Q \geq 0, \quad \text{i.e., } r_+ = r_{+1}. \quad (5.49)$$

Since $r_{+D} = \sqrt{3} l_p Q$, Eq. (5.49) is equivalent to $l_p Q \geq \frac{1}{\sqrt{3}} r_+$, i.e., one has $\frac{1}{\sqrt{3}} r_+ \leq l_p Q \leq r_+$, the latter term being the extremal case. Now, the relation between the horizon radius, the mass, and the electric charge of the black hole is $r_+ = l_p^2 m + \sqrt{l_p^4 m^2 - l_p^2 Q^2}$, so $l_p Q \geq \frac{1}{\sqrt{3}} r_+$ is the same as $Q \leq l_p m \leq \frac{2}{\sqrt{3}} Q$, which is another manner of writing the condition for stability, and is the expression that can be found in [51], see also [166]. The heat capacity goes to zero at the extremal case $\frac{l_p Q}{r_+} = 1$. Moreover, from Eq. (5.48), one finds that the range of instability is $r_{+D} < r_+ < \infty$. This range for r_+ corresponds to the solution r_{+2} , hence there is

$$\text{instability if } C_Q < 0, \quad \text{i.e., } r_+ = r_{+2}. \quad (5.50)$$

The inequality on the horizon radius for the case of instability can be rewritten as $0 \leq l_p Q < \frac{1}{\sqrt{3}} r_+$. Note that when the electric charge is zero, the heat capacity is negative for all r_+ , indeed for $Q = 0$ the heat capacity is $\frac{l_p^2 C}{r_+^2} = -2\pi$. Note that C_Q given in the second part of Eq. (5.48) is the same formula found in [51] by performing in Eq. (5.48) the redefinitions $S \rightarrow 8\pi S$, $T \rightarrow \frac{1}{8\pi} T$ and $\frac{C_Q}{8\pi} \rightarrow C_Q$, and

additionally by using Planck units. In [166], the conventions are yet different from the ones we use here and from [51].

The heat capacity C_Q in units of Q^2 , i.e., $\frac{C_Q}{Q^2}$, as a function of the temperature parameter, i.e., $Tl_p Q$ is plotted in Fig. 5.2. For each $Tl_p Q$, the heat capacity is double-valued, being positive for r_{+1} in the red curve and negative for r_{+2} in the blue curve. Therefore, the solution r_{+1} is stable as it is expected from the increasing monotonic behavior of r_{+1} with increasing temperature, while the solution r_{+2} is unstable, having the opposite monotonic behavior. When $Q = 0$, there is only the r_{+2} solution corresponding to the unstable Gibbons-Hawking black hole solution. At the Davies point, corresponding to $T_D l_p Q = \frac{1}{6\pi\sqrt{3}}$, the heat capacity goes to plus infinity for the solution r_{+1} , and to minus infinity for the solution r_{+2} . If, for some T , the configuration of the ensemble happens to be in the unstable r_{+1} solution, then it will transition to the stable r_{+2} , since any thermal perturbations make the solution r_{+2} run away from equilibrium. This happens for all temperatures up to T_D , where the two solutions coincide, and for higher T , there are no more black hole solutions. Thus, the point with temperature T_D characterizes a turning point. It was stated by Davies that such point might be classified as a second order phase transition. However, this cannot be the case for the canonical ensemble, as we discussed above, because only the stable solutions must be considered and the temperature T_D signals the upper limit of existence of the stable solution. Another way of looking at the Davies point, through the ranges of the horizon radius, is that it provides the relative scale between r_+ and Q at which one has black hole stability or metastability in the canonical ensemble with a heat reservoir at infinity. In [51], a plot $C_Q \times Q$ was presented in some units of C_Q and of Q at constant mass m , whereas, here, we present the plot $\frac{C_Q}{Q^2} \times Tl_p Q$, where $Tl_p Q$ is a temperature parameter, as Q is kept constant in the calculation of C_Q .

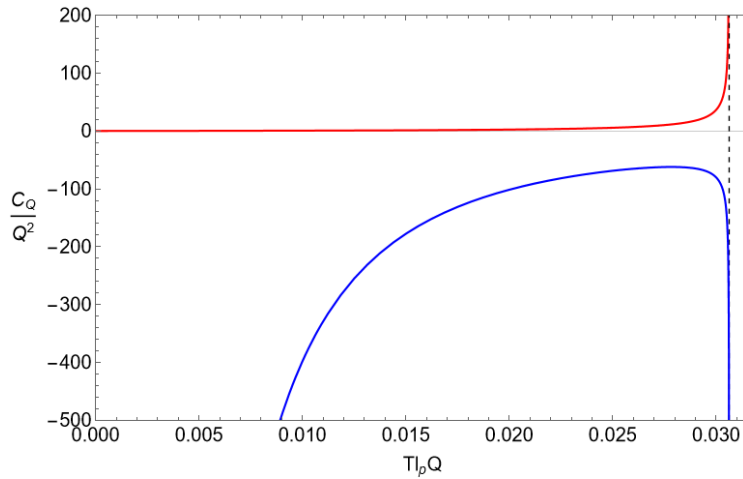


Figure 5.2: The heat capacity C_Q in Q^2 units, $\frac{C_Q}{Q^2}$, is given as a function of the temperature parameter $Tl_p Q$ in $d = 4$, for the stable solution r_{+1} in red and unstable solution r_{+2} in blue. The heat capacity diverges for both solutions at the turning point $T_D l_p Q = \frac{1}{6\pi\sqrt{3}} \approx 0.03$, the latter equality being approximate.

The analysis of the favorable thermodynamic states for $d = 4$ does not differ from the analysis for generic d given above. We summarize the analysis here for completeness. There are two possible phases, the stable black hole r_{+1} phase and the phase with hot flat space with electric charge at infinity. For $T \leq T_D$, both r_{+1} and hot flat space with electric charge at infinity can exist. Since the free energy for hot flat space is zero and the free energy for the black hole r_{+1} is positive for all T and Q , hot flat space with electric charge at infinity is the favored phase for $T \leq T_D$. In this range of temperatures, if the system is in the black hole phase, it will settle upon perturbation in a hot flat space with charge at infinity phase which has lower free energy, in the same way that supercooled water phase changes into ice. For $T > T_D$, there are no black holes, hot flat space with electric charge at infinity is the only phase.

The same interpretation in terms of wavelengths λ of packets of thermal energy inside the cavity, that we gave above, can be applied to the specific case $d = 4$. Note that this interpretation goes beyond the formal results found here, since we carried out the zero loop approximation and we do not treat quantum matter fields. Nevertheless, it is beneficial to give an interpretation. The essential idea is that at a given T and so at a given λ , the small black hole is smaller than λ and the radiation is trapped outside, while the large black hole is larger than λ and the radiation can escape the black hole. For sufficient high T , there is too much agitation in packets of energy with small wavelength λ , and these packets wonder undisturbed by gravity in hot flat space with the electric charge being deposited uniformly at infinity. In Fig. 5.1, the curve $r_{+D} = \frac{1}{6\pi T}$ is drawn in gray, but this is the definition of $\lambda = \frac{1}{6\pi T}$ for $d = 4$. And so, Fig. 5.1 describes precisely the interpretation in terms of wavepackets given above. Indeed, from small T up to T_D , the gray curve is larger than the horizon radius of the smaller black hole, while it is smaller, although of the same order, than the horizon radius of the larger black hole. At T_D , the gray curve and both solutions meet. For larger temperatures than T_D , there are no black hole solutions.

We must comment on the comparison between the approach we followed and the approach followed by Davies. The first law of black hole mechanics is the expression from which Davies [51] started his analysis, see also [166]. Our analysis here started from the statistical mechanics canonical ensemble theory using the Euclidean path integral approach and the action of Eq. (5.45) rather than starting from the first law of black hole thermodynamics. In the Reissner-Nordström black hole case in the canonical ensemble, as opposed to the Schwarzschild case, there is true thermodynamics, since there are instances where the system is thermodynamically stable. This thermodynamic stability of black holes for a heat reservoir at constant T and Q contrasts with the thermodynamic instability of all electrically charged black holes in a heat reservoir at constant T and constant electric potential ϕ , i.e., Reissner-Nordström black holes in the grand canonical ensemble. This latter case was the case analyzed in [67] using the Euclidean path integral approach for the grand canonical ensemble, where this instability was noticed but there was no attempt to cure the problem. The appropriate setting that gives a meaningful path

integral and a corresponding thermodynamics is within the electrically charged canonical ensemble rather than the grand canonical one.

5.5 THE CASE $d = 5$: A TYPICAL HIGHER-DIMENSIONAL CASE

5.5.1 Solutions and action in $d = 5$

Here, we present the case with dimension $d = 5$, as it is a typical higher dimension, and it is the first possible extension of the results provided by Davies.

We must start from the canonical ensemble characterized by the temperature T and the electric charge Q at infinity in $d = 5$. The black hole solutions r_+ of the ensemble are taken from Eq. (5.10) together with Eq. (5.4) which in $d = 5$ give

$$T = t_H(r_+, Q), \quad t_H(r_+, Q) = \frac{r_+^2 - \frac{4l_p^3 Q^2}{3\pi r_+^2}}{2\pi r_+^3}, \quad (5.51)$$

where T is the temperature kept by the reservoir at infinity and $t_H(r_+, Q)$ is the Hawking function in $d = 5$. When the electric charge of the reservoir at infinity is zero, $Q = 0$, then $t_H(r_+, Q) = \frac{1}{2\pi r_+}$, which is the Hawking temperature of a Schwarzschild black hole in $d = 5$. The electric charge Q is the electric charge kept by the reservoir at infinity, and the black hole electric charge q must match it to have a consistent solution, $q = Q$.

To find the solutions of the canonical ensemble, one inverts Eq. (5.51) to yield $(\frac{1}{2\pi T})(r_+^4 - \frac{4l_p^3}{3\pi}Q^2) - r_+^5 = 0$, which is Eq. (5.14) for $d = 5$, a quintic equation not easily solvable analytically. However, it can be analyzed qualitatively or solved numerically. One finds that the function $t_H(r_+, Q)$ in Eq. (5.39) has a maximum at

$$r_{+s} = \left(\sqrt{\frac{20}{3\pi}} l_p^{\frac{3}{2}} Q \right)^{\frac{1}{2}}. \quad (5.52)$$

which is a saddle point of the action of the black hole, with a corresponding temperature at the reservoir given by

$$T_s = \frac{2}{5\pi \left(\sqrt{\frac{20}{3\pi}} l_p^{\frac{3}{2}} Q \right)^{\frac{1}{2}}}. \quad (5.53)$$

From Eqs. (5.52) and (5.53), one can eliminate Q to give for a given T , $r_{+s} = \frac{2}{5\pi T}$, or inverting, for a given r_+ , $T_s = \frac{2}{5\pi r_+}$. One finds that for temperatures $T \leq T_s$, there are two black hole solutions, the solution $r_{+1}(T, Q)$ and the solution $r_{+2}(T, Q)$, while for $T > T_s$ there are no black hole solutions. The solution $r_{+1}(T, Q)$ is bounded in the interval $r_{+e} < r_{+1}(T, Q) \leq r_{+s}$, where $r_{+e} = r_{+1}(T \rightarrow 0, Q) = \left(\frac{2}{\sqrt{3\pi}} l_p^{\frac{3}{2}} Q \right)^{\frac{1}{2}}$ is the radius of the extremal black hole and $r_{+s} = r_{+1}(T_s, Q) = \left(\sqrt{\frac{20}{3\pi}} l_p^{\frac{3}{2}} Q \right)^{\frac{1}{2}}$, so one can summarize solution 1 in the form $r_{+1} = r_{+1}(T, Q)$,

$q_1 = Q$, with $0 < T \leq T_s$ and $r_{+e} < r_{+1}(T, Q) \leq r_{+s}$. This solution, $r_{+1}(T, Q)$, is an increasing monotonic function of T . The solution $r_{+2}(T, Q)$ is in the interval $r_{+s} < r_{+2}(T, Q) < \infty$, so one can summarize solution 2 in the form $r_{+2} = r_{+2}(T, Q)$, $q_2 = Q$, with $0 < T \leq T_s$ and $r_{+s} < r_{+2}(T, Q) < \infty$, where the solution tends to infinity at $T \rightarrow 0$. This solution, $r_{+2}(T, Q)$, is a decreasing monotonic function of T . When the ensemble is only characterized by the temperature T , with Q vanishing, $Q = 0$, only the black hole r_{+2} survives which has the Gibbons-Hawking black hole solution properties. For $T > T_s$, there are no black hole solutions and one is left with hot flat space with electric charge Q dispersed at infinity, i.e., one has charged hot flat space for $T_s < T < \infty$ with Q dispersed at $r = \infty$.

The zero loop action of the canonical ensemble, which is characterized by the temperature T and the electric charge Q at infinity, for $d = 5$ can be found using directly Eq. (5.24), i.e.,

$$I_0[T, Q] = \frac{1}{2Tl_p^3} \left(\frac{3\pi r_+^2}{4} + \frac{l_p^3 Q^2}{r_+^2} \right) - \frac{\pi^2 r_+^3}{2l_p^3}, \quad (5.54)$$

where $\mu = \frac{4l_p^3}{3\pi}$ and $\Omega_3 = 2\pi^2$ were used. The black hole horizon radii r_+ that enter into this action are r_{+1} or r_{+2} .

5.5.2 Thermodynamics in $d = 5$

With the solutions and the action of the canonical ensemble found, we can obtain the thermodynamics through the correspondence $F = TI_0$, that comes from the zero loop approximation of the path integral, where F again is the Helmholtz free energy of the system. From Eq. (5.54), F in $d = 5$ is $F = \frac{1}{2l_p^3} \left(\frac{3\pi r_+^2}{4} + \frac{l_p^3 Q^2}{r_+^2} \right) - T \frac{\pi^2 r_+^3}{2l_p^3}$. Substituting T for t_H , see Eq. (5.51), one obtains for the free energy the expression

$$F(T, Q) = \frac{\pi}{8l_p^3} \left(r_+^2 + \frac{20l_p^3 Q^2}{3\pi r_+^2} \right), \quad (5.55)$$

where r_+ should be envisaged as $r_+ = r_+(T, Q)$, since it is one of the solutions $r_{+1}(T, Q)$ or $r_{+2}(T, Q)$. Thus, the Helmholtz free energy F for each solution is a function only of T and Q , namely, $F(T, Q, r_{+1}(T, Q))$ and $F(T, Q, r_{+2}(T, Q))$. By computing the derivatives of the free energy, one can obtain the entropy as $S = \frac{A_+}{4l_p^3}$, $A_+ = 2\pi^2 r_+^3$, the thermodynamic electric potential, which is $\phi = \frac{Q}{r_+^2}$, and the energy, which is $E = \frac{3\pi r_+^2}{8l_p^3} + \frac{l_p^3 Q^2}{2r_+^2}$, where it was used $E = F - TS$. These expressions are valid for both solutions, r_{+1} and r_{+2} . The energy has precisely the expression for the space mass m , so $E = m$. The free energy of Eq. (5.55) is then $F = m - TS$.

Here, in $d = 5$, the Smarr formula takes the form

$$m = \frac{2}{3} TS + \phi Q. \quad (5.56)$$

Also, one has that the law $dm = TdS + \phi dQ$ holds. This is the first law of black hole mechanics, which is also the first law of black hole thermodynamics. And in fact, the first law of black hole thermodynamics is valid in the electrically charged case for the instances where the system is thermodynamically stable.

The heat capacity of Eq. (5.33) is in $d = 5$ given by

$$C_Q = \frac{1}{l_p^3} \frac{3\pi^2 r_+^3 \left(1 - \frac{4}{3\pi} \frac{l_p^3 Q^2}{r_+^4}\right)}{2 \left(\frac{20}{3\pi} \frac{l_p^3 Q^2}{r_+^4} - 1\right)} = \frac{mS^3 T}{\frac{7\pi^2}{36l_p^3} Q^4 \left(\frac{2l_p^3 S}{\pi^2}\right)^{-\frac{1}{3}} + \frac{\pi^4}{4^3} \left(\frac{2l_p^3 S}{\pi^2}\right)^{\frac{7}{3}} - T^2 S^3}, \quad (5.57)$$

where in the second equality, the heat capacity was written in terms of the thermodynamic variables m , S , T , and Q . The heat capacity must be seen as a function of T and Q , with r_+ being given by either the solutions $r_{+1}(T, Q)$ and $r_{+2}(T, Q)$, or as well $m = m(T, Q)$ and $S = S(T, Q)$. In order to have thermodynamic stability, the heat capacity must be positive, i.e., $C_Q \geq 0$, which is accomplished by the range $r_{+e} \leq r_+ \leq r_{+s}$ or in terms of electric charge $\left(\frac{3\pi}{20}\right)^{\frac{1}{2}} r_+^2 \leq l_p^{\frac{3}{2}} Q \leq \left(\frac{3\pi}{4}\right)^{\frac{1}{2}} r_+^2$, with r_{+s} given in Eq. (5.52). This range is precisely the one of the solution r_{+1} , and so the solution r_{+1} is stable. For the remaining range, satisfied by the solution r_{+2} , the heat capacity is negative, thus the solution r_{+2} is unstable. The heat capacity, when the electric charge is zero, is negative for all r_+ , given by $\frac{l_p^3 C}{r_+^3} = -\frac{3\pi^2}{2}$. The heat capacity has the feature that diverges for each solution at T_s , which is a turning point of the two solutions. The heat capacity goes to zero at the extremal case $\frac{l_p^{\frac{3}{2}} Q}{r_+^2} = \left(\frac{3\pi}{4}\right)^{\frac{1}{2}}$. One can also infer that the solution is stable if the radius r_+ increases as the temperature increases, yielding the same analysis above.

The analysis of the favorable thermodynamic states for $d = 5$ follows the same reasoning as for generic d . There is the small stable black hole phase and the hot flat space with electric charge at infinity phase. Depending on the temperature, either the latter is favored or it is the only phase.

An interpretation of the results in $d = 5$ in terms of wavelengths λ of packets of thermal energy inside the cavity of a heat reservoir at infinity also follows the analysis for generic d given above.

5.6 CONCLUSIONS

In this chapter, we have shown that the Gibbons-Hawking Euclidean path integral approach for electrically charged black holes in the canonical ensemble has in its core the Davies' thermodynamic theory of black holes. Since statistical mechanics and its ensembles provide a deeper description of the physics world, the results of this chapter place Davies' thermodynamic theory on a firm basis.

To determine this connection, we computed the canonical partition function in the Gibbons-Hawking Euclidean path integral approach for a Reissner-Nordström

black hole in d dimensions. The Euclidean action that enters into the path integral consists of the Einstein-Hilbert-Maxwell action with the Gibbons-Hawking-York boundary term and an additional Maxwell boundary term so that the canonical ensemble is well-defined. We have assumed that the heat reservoir resides at the boundary of space, at infinite radius, where the temperature T is fixed as the inverse of the Euclidean proper time length at the boundary, and also the electric charge is fixed by fixing the electric flux at the boundary. We then performed the zero loop approximation by giving the expressions for the metric and the Maxwell tensor of the Einstein-Maxwell system, obtaining the black hole solutions of the ensemble, $r_+(T, Q)$. We have shown that there are two solutions for temperatures below a critical value. The smaller black hole solution is stable, while the larger one is unstable. The two solutions meet at a saddle or critical point, given formally by $r_{+s} = r_+(T_s, Q_s)$. Above the saddle value for the temperature, there are no black hole solutions, only hot flat space with electric charge dispersed at infinity. The thermodynamics of the system follows, since the canonical partition function connects directly to the Helmholtz free energy. The entropy obtained from the free energy is the Bekenstein-Hawking entropy, the electric potential is the usual Coulombic potential, and the thermodynamic energy is the mass of the black hole. The thermodynamic stability is controlled by the heat capacity at constant electric charge, which must be positive for stable solutions and negative for unstable solutions. There is a turning point precisely at the saddle values T_s and Q_s . The solution with smaller radius is thermodynamically stable while the solution with larger radius is thermodynamically unstable. The Smarr formula relating mass, temperature, entropy, electric potential, and electric charge follows naturally. In addition, the first law of thermodynamics reduces to the first law of black hole mechanics, which, strictly speaking, is valid only for the case of the stable solution. We have studied the favorable phases, comparing the free energies of the stable black hole and the hot flat space with electric charge at infinity. We have obtained that hot flat space with electric charge at infinity is favorable throughout the configuration space. If, for some reason, the system finds itself in the black hole phase, it will make a transition to hot flat space with electric charge at infinity. This fact is due to the black hole phase not being a global minimum of the free energy F , the global minimum of F being hot flat space with charge at infinity. Since the free energy of these two phases never intersects, one cannot call this a first order phase transition. However, if one includes the matter sector, it may be possible that a first order phase transition exists between black hole and matter. We also gave an interpretation for the solutions and their stability in terms of wavelengths of energy packets. By considering the dimension $d = 4$, we have shown that Davies' thermodynamic theory of black holes follows directly from the whole formalism presented. Davies' starting point for the theory was the first law of black hole thermodynamics, our starting point here was the path integral approach with its action, and from it, we have deduced the first law of black hole thermodynamics and the critical points found by Davies. The theory to the case $d = 5$ was also applied. The analysis of this chapter generically points towards the equivalence

between the black hole mechanics and black hole thermodynamics through the canonical ensemble with an appropriate heat reservoir at infinity.

6

CANONICAL ENSEMBLE OF A D-DIMENSIONAL
REISSNER-NORDSTRÖM BLACK HOLE SPACETIME IN A
CAVITY

6.1 INTRODUCTION

The York formalism [68, 115], which uses the Euclidean path integral approach to quantum gravity [67], was applied to charged black holes to construct its grand canonical ensemble in [130], in the case of four dimensions. One can also construct the canonical ensemble of charged black holes by adding a boundary term to the Einstein-Maxwell action, as explained in [130], with the analysis of the solutions being done in [134] and with the phase transitions between the black hole solutions having been done in [133], both in four dimensions. Also, in [144], the canonical ensemble of charged black branes was analyzed.

As a continuity of the work presented in Chapters 4 and 5, we construct in this chapter the canonical ensemble of a charged black hole inside a cavity for higher dimensions, with fixed temperature and electric charge. Again, we use the Euclidean path integral approach in the zero loop approximation to obtain the solutions of the ensemble and analyze the validity of the approximation. We generalize the results about the solutions of the ensemble for d dimensions, giving analytical results for the bifurcation and meeting points. In sum, there are three solutions for electric charge lower than a critical charge, and there is only one solution for electric charge larger than a critical charge. Another novelty of the work centers on the analysis of the phase transitions between the stable black holes and a charged sphere with no gravity, that can model in a certain limit charged hot flat space, described by hot flat space with electric charge near the boundary of the cavity. From this analysis, there is a horizon radius at which the black hole phase starts to be more favorable. The comparison between this horizon radius and the Buchdahl-Andre sson-Wright bound [129] is done, together also with the relevant horizon radius of the grand canonical ensemble. It is shown that both horizon radii do not correspond to the Buchdahl-Andre sson-Wright bound. This puts in question the link between matter dynamics and black hole thermodynamics for more general configurations than in [102]. Further interpretation is given, and the particular analysis for the cases $d = 4$ and $d = 5$ is made.

This chapter is organized as follows. In Sec. 6.2, we construct the partition function for spherically symmetric metrics with a Maxwell field. In Sec. 6.3.5.2, we

perform the zero loop approximation, and, we present the analysis of the solutions and their stability. In Sec. 6.4, we obtain the thermodynamic quantities of the system from the canonical ensemble partition function. In Sec. 6.5, we make a comparison between the solutions regarding their favorability and we show the presence of phase transitions. In Sec. 6.6, we analyze the limit of infinite cavity, recovering the results of Chapter 5. In Sec. 6.7, we compare the thermodynamic radii obtained here and in Chapter 4 with the Buchdahl-Andre sson-Wright bound. Finally, in Sec. 6.8, we present the conclusions. The work in this chapter is based on [4].

6.2 THE CANONICAL ENSEMBLE OF A CHARGED BLACK HOLE IN THE EUCLIDEAN PATH INTEGRAL APPROACH

6.2.1 The partition function

Here, we build the canonical ensemble of a charged black hole inside cavity, in d dimensions, using the Euclidean path integral approach to quantum gravity. The partition function of the system is given by

$$Z = \int Dg_{\alpha\beta} DA_\gamma e^{-I[g_{\mu\nu}, A_\sigma]}, \quad (6.1)$$

where the integral of paths must be done over periodic $g_{\mu\nu}$ and A_σ in imaginary time, see for more details Chapter 3. The Euclidean action is written in this case as

$$\begin{aligned} I[g_{\mu\nu}, A_\sigma] = & - \int_{\mathcal{M}} \left(\frac{R}{16\pi l_p^{d-2}} - \frac{(d-3)}{4\Omega_{d-2}} F_{\alpha\beta} F^{\alpha\beta} \right) \sqrt{g} d^d x \\ & - \frac{1}{8\pi l_p^{d-2}} \int_{\partial M} (K - K_0) \sqrt{\gamma} d^{d-1} x + \frac{(d-3)}{\Omega_{d-2}} \int_{\partial M} F^{\alpha\beta} A_\alpha n_\beta \sqrt{\gamma} d^{d-1} x, \end{aligned} \quad (6.2)$$

where R is the Ricci scalar given by derivatives and second derivatives of the Riemannian metric $g_{\alpha\beta}$, g is the determinant of $g_{\alpha\beta}$, K is the trace of the extrinsic curvature of the boundary of the cavity defined as $K_{\alpha\beta}$, K_0 is the trace of the extrinsic curvature of the boundary of the cavity embedded in flat Euclidean space, γ is the determinant of the induced metric γ_{ab} on the boundary of the cavity, Ω_{d-2} is the surface area of the unit $(d-2)$ -sphere, $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ is the Maxwell tensor given by derivatives of the vector potential A_α , n_α is the outward unit normal vector to the boundary of the cavity.

The action in Eq. (6.2) can be written in terms of two separate actions $I = I_{\text{gf}} + I_q$, whereas

$$I_{\text{gf}} = - \frac{1}{16\pi l_p^{d-2}} \int_{\mathcal{M}} R \sqrt{g} d^d x - \frac{1}{8\pi l_p^{d-2}} \int_{\partial M} (K - K_0) \sqrt{\gamma} d^{d-1} x, \quad (6.3)$$

$$I_q = \frac{(d-3)}{4\Omega_{d-2}} \int_{\mathcal{M}} F_{\alpha\beta} F^{\alpha\beta} \sqrt{g} d^d x + \frac{(d-3)}{\Omega_{d-2}} \int_{\partial M} F^{\alpha\beta} A_\alpha n_\beta \sqrt{\gamma} d^{d-1} x. \quad (6.4)$$

The action I_{gf} is the gravitational action with a zero cosmological constant, while I_q is the Maxwell action with an additional boundary term. The boundary term

depending on the Maxwell tensor must be included so that the canonical ensemble may be prescribed, see [130]. This term allows us to fix the electric flux given by the integral of the Maxwell tensor on a $(d - 2)$ -surface, which has the meaning of an electric charge. This can be seen by performing the functional variation of the action and fixing data such that the variations at the boundary vanish.

6.2.2 Geometry and boundary conditions

The path integral is formally performed by summing over the Riemannian metrics with fixed boundary data. We choose the boundary data to be compatible with the data of a spherical shell with finite radius embedded in the Reissner-Nordström black hole spacetime. Namely, the boundary of the Riemannian space describes a spherically symmetric heat reservoir with fixed inverse temperature, defined by the total imaginary proper time of the boundary, and with fixed electric flux, meaning a fixed electric charge. Due to the spherical symmetry of the boundary, it is expected that the paths having spherical symmetry contribute the most to the path integral. In order to simplify the analysis and towards the zero loop approximation, the path integral is restricted to spherical symmetric metrics of the form

$$ds^2 = b(u)^2 d\tau^2 + a(u)^2 du^2 + r(u)^2 d\Omega_{d-2}^2, \quad (6.5)$$

where $b(u)$, $a(u)$ and $r(u)$ are arbitrary smooth functions of u , the coordinates have the range $\tau \in]0, 2\pi[$ and $u \in]0, 1[$, and $d\Omega_{d-2}^2$ is the $(d - 2)$ -sphere line element.

Moreover, the path integral also includes a sum over the possible topologies of the Riemannian space. Each topology, in the case of a spherically symmetric metric, is related to a set of regularity conditions. In the line of the zero loop approximation, we select the black hole sector, which resumes into the following regularity conditions at $u = 0$

$$\begin{aligned} b(0) &= 0, \\ r(0) &= r_+, \\ (b'a^{-1}) \Big|_{u=0} &= 1, \\ a^{-1}(b'a^{-1})' \Big|_{u=0} &= 0, \\ \left(\frac{r'}{a} \right) \Big|_{u=0} &= 0, \end{aligned} \quad (6.6)$$

where r_+ is the horizon radius, and also where a prime denotes the derivative of a function in u , e.g. $b' = \frac{db}{du}$. The boundary conditions are set at the boundary of space ∂M , which is assumed to be a spherical shell located at $u = 1$ with induced metric

$$ds_{\partial M}^2 = b(1)^2 d\tau^2 + R^2 d\Omega_{d-2}^2, \quad (6.7)$$

having components fixed by the inverse temperature and the radius of the reservoir as

$$\begin{aligned} b(1) &= \frac{\beta}{2\pi}, \\ r(1) &= R. \end{aligned} \quad (6.8)$$

For the electromagnetic Maxwell field, due to spherical symmetry, the only nonvanishing components of the Maxwell tensor $F_{\alpha\beta}$ are $F_{u\tau} = -F_{\tau u}$. Note that we are assuming the non-existence of magnetic monopoles. Moreover, we choose the gauge such that the only nonvanishing component of the vector potential is $A_\tau(u)$. Therefore, the Maxwell tensor $F_{\alpha\beta}$ is described only by the term

$$F_{u\tau}(u) = \frac{dA_\tau(u)}{du}. \quad (6.9)$$

At $u = 0$, we impose the regularity condition

$$A_\tau(0) = 0, \quad (6.10)$$

which fixes completely the gauge of the Maxwell field. The boundary condition at $u = 1$ for the Maxwell field consists on a fixed electric charge. The electric charge can be written in terms of the electric flux $\int_{u=1}^{\tau=c} F^{\alpha\beta} dS_{\alpha\beta} = 2i\Omega_{d-2}Q$, where c is a constant, Q is the electric charge in the cavity, $dS_{\alpha\beta} = 2u_{[\alpha}n_{\beta]}dS$ is the surface element of the $y = 1$ and $\tau = c$ surface, $u_\alpha dx^\alpha = bd\tau$, $n_\alpha dx^\alpha = ady$, and dS is the surface volume. For this case, the boundary condition reduces to

$$(bar^{d-2} F^{u\tau})(1) = -iQ. \quad (6.11)$$

6.2.3 Action in spherical symmetry

With the restriction to spherical symmetric metrics, the regularity conditions and the boundary conditions, we can simplify the action in the path integral. We can start with the gravitational action, which can be simplified into

$$\begin{aligned} I_{gf} &= \left(\frac{2\pi b r^{d-3}}{\mu} \left(1 - \frac{r'}{a} \right) \right) \Big|_{u=1} - \frac{\Omega_{d-2}}{4l_p^{d-2}} \left(\frac{b' r^{d-2}}{a} \right) \Big|_{u=0} \\ &+ \frac{1}{8\pi l_p^{d-2}} \int_M ab r^{d-2} G^\tau_\tau d^d x, \end{aligned} \quad (6.12)$$

where

$$\mu = \frac{8\pi l_p^{d-2}}{(d-2)\Omega_{d-2}}, \quad (6.13)$$

and the Einstein tensor component G^τ_τ is given by

$$G^\tau_\tau = \frac{(d-2)}{2r' r^{d-2}} \left(r^{d-3} \left(\frac{r'^2}{a^2} - 1 \right) \right)'. \quad (6.14)$$

Using the regularity conditions in Eq. (6.6) and the boundary conditions in Eq. (6.8), the gravitational action can be written as

$$I_{\text{gf}} = \left(\frac{\beta R^{d-3}}{\mu} \left(1 - \frac{r'}{a} \right) \right) \Big|_{u=1} - \frac{\Omega_{d-2} r_+^{d-2}}{4l_p^{d-2}} + \frac{1}{8\pi l_p^{d-2}} \int_M ab r^{d-2} G^\tau_\tau d^d x . \quad (6.15)$$

Regarding the action for the Maxwell field, one can use that $F^{\alpha\beta} F_{\alpha\beta} = 2F_{u\tau} F^{u\tau} = 2\frac{A_\tau'^2}{b^2 a^2}$ and also that $F^{\alpha\beta} A_\alpha n_\beta = -\frac{A'_\tau}{b^2 a} A_\tau$ to obtain

$$I_q = -\frac{(d-3)}{2\Omega_{d-2}} \int_M \left(r^{d-2} \frac{A_\tau'^2}{ab} + 2 \left(\frac{A'_\tau r^{d-2}}{ab} \right)' A_\tau \right) d^d x , \quad (6.16)$$

where the regularity condition $A_\tau(0) = 0$ was used and the boundary term was transformed into a bulk integration term. The full action for the spherically symmetric metric with a Maxwell field in the canonical ensemble is then

$$\begin{aligned} I = & \left(\frac{\beta R^{d-3}}{\mu} \left(1 - \frac{r'}{a} \right) \right) \Big|_{u=1} - \frac{\Omega_{d-2} r_+^{d-2}}{4l_p^{d-2}} - \frac{(d-3)}{\Omega_{d-2}} \int_M \left(\frac{r^{d-2} A'_\tau}{ba} \right)' A_\tau d^d x \\ & + \frac{1}{8\pi l_p^{d-2}} \int_M ab r^{d-2} \left(G^\tau_\tau - 4\pi l_p^{d-2} \frac{(d-3)}{\Omega_{d-2}} \frac{A_\tau'^2}{b^2 a^2} \right) d^d x . \end{aligned} \quad (6.17)$$

The statistical path integral that yields the partition function can then be written as

$$Z = \int DbDaDrDA_\tau e^{-I} , \quad (6.18)$$

with the action in Eq. (6.17). For more details about the statistical ensemble through the Euclidean path integral approach, the gravitational action in spherical symmetry, the regularity and boundary conditions, one can find them in Chapter 3.

6.3 THE ZERO LOOP APPROXIMATION

6.3.1 The constrained path integral and reduced action in the canonical ensemble

Given the action and the path integral for a spherically symmetric metric with a Maxwell field, we can proceed with the zero loop approximation through incremental steps. First, we impose the Hamiltonian and momentum constraints to the metric and the Gauss constraint to the Maxwell field. This results in a constrained path integral with a reduced action. We then can use the reduced action to study the validity of the zero loop approximation under static perturbations, which have a connection to the thermodynamic stability as we will show.

Starting with the constraints for the metric, the Hamiltonian constraint is $G^\tau_\tau = 8\pi l_p^{d-2} T^\tau_\tau$, with G^τ_τ given by Eq. (6.14), and

$$T^\tau_\tau = \frac{(d-3)}{\Omega_{d-2}} \frac{A_\tau'^2}{2a^2 b^2} , \quad (6.19)$$

where T^τ_τ is the time-time component of the stress-energy tensor T^α_β . Thus, the Hamiltonian constraint is

$$\frac{d-2}{2r'r^{d-2}} \left[r^{d-3} \left(\frac{r'^2}{a^2} - 1 \right) \right]' = \frac{4\pi l_p^{d-2} (d-3) A_\tau'^2}{\Omega_{d-2} a^2 b^2}. \quad (6.20)$$

The momentum constraint is trivially satisfied since the metric Eq. (6.5) is diagonal and does not depend on the imaginary time τ . The Gauss constraint is $\nabla_u F^{\tau u} = 0$, which explicitly is

$$\left(\frac{r^{d-2} A'_\tau}{ba} \right)' = 0, \quad (6.21)$$

The two constraint equations, Eqs. (6.20) and (6.21), are coupled, but they can be integrated in the following way. It is better to start first by integrating Eq. (6.21). Its integration yields

$$A'_\tau = -i \frac{q}{r^{d-2}} ba, \quad (6.22)$$

where q is an integration constant. If one evaluates Eq. (6.22) at $u = 1$ and uses the boundary condition Eq. (6.11), then one obtains that

$$q = Q, \quad (6.23)$$

and so the integration constant q of the Gauss constraint is precisely the fixed electric charge Q of the ensemble. From this point onward, Q is used to describe the fixed electric charge. By using Eq. (6.22) and Eq. (6.23), the Hamiltonian constraint becomes

$$\frac{d-2}{2r'r^{d-2}} \left[r^{d-3} \left(\frac{r'^2}{a^2} - 1 \right) \right]' = - \frac{4\pi (d-3) l_p^{d-2} Q^2}{\Omega_{d-2} r^{2d-4}}, \quad (6.24)$$

which can be integrated to obtain

$$\frac{r'^2}{a^2} \equiv f(r, Q, r_+), \quad (6.25)$$

where

$$f(r, Q, r_+) \equiv 1 - \frac{r_+^{d-3} + \frac{\mu Q^2}{r_+^{d-3}}}{r^{d-3}} + \frac{\mu Q^2}{r^{2d-6}}, \quad (6.26)$$

with μ given in Eq. (6.13). We define the function f in Eq. (6.26) for convenience, and the regularity conditions in Eq. (6.6) were used to determine the integration constant r_+ . Although the second to last condition in Eq. (6.6) is not used anywhere, notice for bookkeeping that, if $u = r$ is chosen, $r' = 1$ and a diverges at $r = r_+$, therefore the condition should be satisfied if $\left(\frac{b'}{a} \right)'_{u=0}$ is finite. The function A'_τ in Eq. (6.22) is related to the Coulomb electric field in Lorentzian curved spacetime as $n_\alpha E^\alpha = \frac{i A'_\tau}{ba} = \frac{Q}{r^{d-2}}$, where E^α is the electric field measured by a static observer. It is

important to write explicitly the extremal case, i.e., when $r_+^{2d-6} = \mu Q^2$. The horizon radius for the extremal case, r_{+e} , can be defined as

$$r_{+e} = (\mu Q^2)^{\frac{1}{2d-6}}, \quad (6.27)$$

and function $f(r, Q, r_+)$ in Eq. (6.26) in the extremal case is $f(r, Q, r_{+e}) = \left(1 - \frac{\sqrt{\mu}Q}{r^{d-3}}\right)^2$.

The Hamiltonian, momentum, and Gauss constraints simplify the action in Eq. (6.17) considerably. One can see that the last term in Eq. (6.17) has an integrand proportional to $G^\tau_\tau - 8\pi T^\tau_\tau$ and so, applying the Hamiltonian constraint given in Eq. (6.20), this term vanishes. Moreover, the third term in Eq. (6.17) is proportional to $\left(\frac{r^{d-2}A'_\tau}{b\alpha}\right)'$ which vanishes also if the Gauss constraint given in Eq. (6.21) is applied. Therefore, the action Eq. (6.17) becomes the reduced action I_* written as

$$I_*[\beta, Q, R; r_+] = \frac{\beta R^{d-3}}{\mu} (1 - \sqrt{f(R, Q, r_+)}) - \frac{\Omega_{d-2} r_+^{d-2}}{4l_p^{d-2}}, \quad (6.28)$$

which is the Euclidean action evaluated on the paths that obey the Hamiltonian and Gauss constraints, where $(r'\alpha^{-1})_{y=1}$ was substituted by the solution to the Hamiltonian constraint given in Eq. (6.25). From Eq. (6.26), one has that $f(r, Q, r_+)$ appearing in Eq. (6.28) evaluated at the cavity radius R is given by

$$f(R, Q, r_+) \equiv 1 - \frac{r_+^{d-3} + \frac{\mu Q^2}{r_+^{d-3}}}{R^{d-3}} + \frac{\mu Q^2}{R^{2d-6}}. \quad (6.29)$$

The function $f(R, Q, r_+)$ for the extremal case characterized by Eq. (6.27) is given by $f(R, Q, r_{+e}) = \left(1 - \frac{\sqrt{\mu}Q}{R^{d-3}}\right)^2$.

The Hamiltonian, momentum, and Gauss constraints, together with the boundary conditions and the requirement of spherical symmetry, restrict the path integral considerably. The Riemannian space is determined by the functional r_+ , so the path integral is the sum of spaces with all possible r_+ . Indeed, the partition function is given by the path integral

$$Z = \int D r_+ e^{-I_*[\beta, Q, R; r_+]}, \quad (6.30)$$

where $I_*[\beta, Q, R; r_+]$ is the reduced action described in Eq. (6.28). There is formally another functional, the Maxwell field A_τ , but the action does not depend explicitly on A_τ , it only depends on the electric charge which is fixed at the cavity. This means the integration over paths of A_τ can be absorbed by a normalization and thus yielding no additional contributions to the constrained path integral.

6.3.2 Stationary points of the reduced action

Having the constrained path integral in Eq. (6.30), we can perform the zero loop approximation, which takes into consideration only the paths that minimize the action. The partition function in the zero loop approximation is given by

$$Z[\beta, R, Q] = e^{-I_0[\beta, R, Q]}, \quad (6.31)$$

where

$$I_0[\beta, R, Q] = I_*[\beta, R, Q; r_+[\beta, R, Q]], \quad (6.32)$$

is the action in Eq. (6.28) evaluated at the path that minimizes the action with respect to r_+ . The function $r_+[\beta, R, Q]$ corresponds to a black hole solution that is in thermal equilibrium with the cavity and it is determined by a stationary point of the action, i.e., $\left(\frac{\partial I_*}{\partial r_+}\right)_{r_+=r_+[\beta, R, Q]} = 0$. Using Eq. (6.28), the stationary condition reduces to the equation

$$\beta = \iota(r_+), \quad \iota(r_+) \equiv \frac{4\pi}{(d-3)} \frac{r_+^{d-2}}{r_+^{d-3} - \frac{\mu Q^2}{r_+^{d-3}}} \sqrt{f(R, Q, r_+)}, \quad (6.33)$$

where $\iota(r_+)$ is the inverse temperature function, defined here for convenience. The function ι for fixed R and Q , which are the fixed quantities of the ensemble, only depends on r_+ alone. The solutions $r_+[\beta, R, Q]$ of Eq. (6.33) are the stationary points or the paths that minimize the action in Eq. (6.28), and they are obtained from inverting Eq. (6.33). For convenience, we can define a horizon radius parameter x and an electric charge parameter y as

$$x = \frac{r_+}{R}, \quad y = \frac{\mu Q^2}{R^{2d-6}}. \quad (6.34)$$

Rearranging Eq. (6.33), one obtains

$$(x^{2d-6} - y)^2 \left(\frac{(d-3)\beta}{4\pi R} \right)^2 - x^{3d-7} (1 - x^{d-3}) (x^{d-3} - y) = 0. \quad (6.35)$$

The equation above for the horizon radius, Eq. (6.35), can be reduced at most to sixth polynomial order for $d = 5$, while for other dimensions the polynomial order is higher. We have not found an analytical solution for any specific value of d . We note that the non-extremal condition for the black hole can be put in the form

$$x_e \leq x \leq 1, \quad (6.36)$$

where x_e is the extremal x related to the extremal y , denoted as y_e , by

$$y_e = x_e^{2d-6}, \quad (6.37)$$

see Eq. (6.27).

Even though we may not find the exact solutions for x , it is possible to obtain analytically the limiting values for the solutions. These limiting values are determined by the saddle points of the action I_* described as $\left(\frac{\partial^2 I_*}{\partial r_+^2}\right)_0 = 0$, where the subscript 0 means that the quantity inside parenthesis is evaluated at the stationary point. Now, $\left(\frac{\partial^2 I_*}{\partial r_+^2}\right)_0 = -\frac{\Omega_{d-2}(d-2)r_+^{d-3}}{4} \beta^{-1} \frac{\partial \iota}{\partial r_+}$, so the saddle points of the action are

given by $\frac{\partial t}{\partial r_+} = 0$ together with Eq. (6.33). This condition in terms of x and y yields the equation

$$\begin{aligned} & \frac{d-1}{2}x^{4d-12} - (1+y)x^{3d-9} - 3(d-3)yx^{2d-6} \\ & + (2d-5)y(1+y)x^{d-3} - \frac{3d-7}{2}y^2 = 0, \end{aligned} \quad (6.38)$$

which is a polynomial equation of order four in x^{d-3} and it can be solved analytically. Its solutions are relevant in the qualitative behaviour of the horizon radius x of the black hole in thermal equilibrium. For the electrically uncharged case $y = 0$, the limiting values were discussed in [68] for $d = 4$, [101] for $d = 5$, and [102] for generic d . For $0 < y < y_s$, there are four real roots of Eq. (6.38), from which only two obey the non-extremal condition $0 < y < x^{2d-6}$, and where y_s is a saddle or critical electric charge parameter to be given below. The two saddle points of the action, being the solutions of interest of Eq. (6.38), are designated by $x_{s1} = x_{s1}(y)$ and $x_{s2} = x_{s2}(y)$, where $x_{s1} \leq x_{s2}$. Explicitly, they are given by the expressions

$$x_{s1}^{d-3} = \frac{1+y}{2(d-1)} + \xi - \frac{1}{2}\sqrt{2\eta + \frac{\zeta}{\xi} - 4\xi^2}, \quad (6.39)$$

$$x_{s2}^{d-3} = \frac{1+y}{2(d-1)} + \xi + \frac{1}{2}\sqrt{2\eta + \frac{\zeta}{\xi} - 4\xi^2}, \quad (6.40)$$

where

$$\begin{aligned} \eta &= \frac{3(1+y)^2 + 12(d-1)(d-3)y}{2(d-1)^2}, \\ \zeta &= \frac{(1+y)}{(d-1)^3} (y^2 - (4d^3 - 24d^2 + 48d - 30)y + 1), \\ \xi &= \frac{1}{2}\sqrt{\frac{2}{3}\eta + \frac{2}{3(d-1)}\frac{\sigma^2 + \sigma_0}{\sigma}}, \\ \sigma &= \left(\frac{\sigma_1 + \sqrt{\sigma_1^2 - 4\sigma_0^3}}{2} \right)^{\frac{1}{3}}, \\ \sigma_0 &= 3(2d-5)y(1-y)^2, \\ \sigma_1 &= 54(d-3)(d-2)^2(1-y)^2y^2. \end{aligned} \quad (6.41)$$

For the critical charge $y = y_s$, both saddle points merge into a single one. The saddle point of the action at $y = y_s$ is designated by $x_s \equiv x_{s1} = x_{s2}$, which is a saddle point with the feature that the third derivative of the action also vanishes. The saddle point $x_s \equiv x_{s1} = x_{s2}$ is given by

$$\begin{aligned} x_s^{d-3} &= \frac{1}{2(d-1)(2d-5)} \\ &\times \left[(d-1)(3d-7)(3d^2 - 16d + 22) \right. \\ &\quad \left. - 3\sqrt{3}(d-2)^2(d-3)\sqrt{(d-1)(3d-7)} \right], \end{aligned} \quad (6.42)$$

which occurs at $y = y_s$ given by

$$y_s = \frac{1}{4(d-1)(2d-5)^3(3d-7)} \times \left[(d-1)(3d-7)(3d^2-16d+22) - 3\sqrt{3}(d-3)(d-2)^2\sqrt{(d-1)(3d-7)} \right]^2. \quad (6.43)$$

We must be note that to x_s corresponds an r_{+s} through $r_{+s} = x_s R$, and to y_s corresponds a Q_s through $Q_s = \frac{y_s R^{2d-6}}{\mu}$, where the subscript s was not put in R in these formulas because, for finite R , one can always assume R fixed. Putting the values given in Eqs. (6.42) and (6.43) into Eq. (6.35), one finds the temperature of the saddle point RT_s ,

$$RT_s = RT_s(x_s, y_s) \quad (6.44)$$

the temperature parameter at which x_s is a solution of the black hole for $y = y_s$. The values of x_s , y_s , and RT_s are displayed for different values of d in Fig. 6.1. We

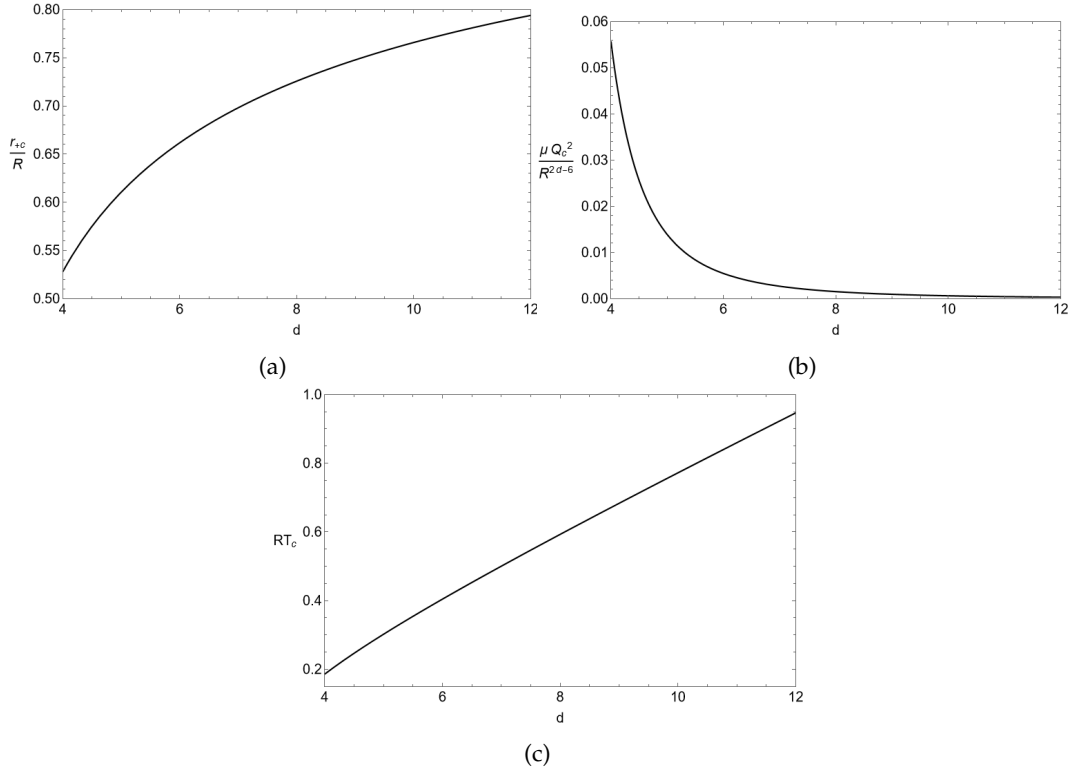


Figure 6.1: Plots of the saddle point (x_s, y_s, T_s) of the action as functions of the number of dimensions d . (a) Plot of $x_s = \frac{r_{+s}}{R}$ as a function of d ; (b) plot of $y_s = \frac{\mu Q_s^2}{R^{2d-6}}$ as a function of d ; (c) plot of RT_s as a function of d .

can see that both x_s and RT_s increase as d increases, and y_s decreases as d increases.

For $y_s < y < 1$, there are no roots of Eq. (6.38) that obey the non-extremal condition $0 < y < x^{2d-6}$ and so there are no saddle points of the action.

Having the limiting values, we can now perform a qualitative analysis of the solutions for the horizon radius of the black hole in thermal equilibrium with the reservoir. For the uncharged case $y = 0$, the analysis has been discussed in [68] for $d = 4$, [101] for $d = 5$, and [102] for generic d . For $0 < y < y_s$, one can find that there are three solutions $x(\beta, y)$, or if one prefers $x(T, y)$, of Eq. (6.35). These three solutions are designated by x_1 , x_2 , and x_3 . The solution x_1 exists in the interval of temperatures $0 < T < T_1$ and it is bounded by $x_e < x_1(T, y) < x_{s1}(y)$, where the values of the solution at the bounds are $x_1(0, y) = x_e$, with x_e defined in Eq. (6.37), and $x_1(T_1, y) = x_{s1}(y)$, with T_1 being defined by the latter relation. The solution x_2 exists in the interval of temperatures $T_1 > T > T_2$ and it is bounded by $x_{s1}(y) < x_2(T, y) < x_{s2}(y)$, where the values of the solution at the bounds are $x_2(T_1, y) = x_{s1}(y)$ and $x_2(T_2, y) = x_{s2}(y)$, with T_2 being defined by the former relation. The solution x_3 exists in the interval of temperatures $T_2 < T < \infty$, and it is bounded by $x_{s2}(y) < x_3(T, y) < 1$, where the values of the solution at the bounds are $x_3(T_2, y) = x_{s2}(y)$ and $x_3(T \rightarrow \infty, y) = 1$. As y_s decreases with the increase of d , the region of existence of these solutions is squeezed towards lower values of the electric charge with an increase of d . For $y = y_s$, there are still three solutions x_1 , x_2 , and x_3 , with the solution x_2 being reduced to a point, more precisely to the saddle point of $\iota(r_+)$ given as $x_2(T_s, y_s) = x_s$, with T_s being defined by the latter relation. The bounds of x_1 and x_3 are the same as the case $0 < y < y_s$, except that $x_{s1}(y_s) = x_{s2}(y_s) = x_s$ and $T_s = T_1 = T_2$. For $y_s < y < 1$, there is only one solution x_4 that exists for all T and it is bounded by $x_e < x_4(T, y) < 1$, where $x_4(0, y) = x_e$ and $x_4(T \rightarrow \infty, y) = 1$.

6.3.3 Stability conditions

To determine if the solutions are minima of the action and thus stable, we must go beyond the zero loop approximation. This means we must expand the action and the path integral around the stationary point. The action can be expanded as $I_* = I_0 + \left(\frac{\partial I_*}{\partial r_+}\right)_0 \delta r_+ + \left(\frac{\partial^2 I_*}{\partial r_+^2}\right)_0 \delta r_+^2$, where the subscript 0 means that the quantity inside parenthesis is evaluated at the stationary point, $I_0 = I_*(\beta, Q, R; (r_+)_0)$, and $\delta r_+ = r_+ - (r_+)_0$. Then, the partition function can be expanded as

$$Z = e^{-I_0} \int D\delta r_+ e^{-\left(\frac{\partial^2 I_*}{\partial r_+^2}\right)_0 \delta r_+^2}. \quad (6.45)$$

The partition function in Eq. (6.45) contains one loop contributions that obey the spherical symmetry of the geometry, the boundary conditions, and the Hamiltonian and Gauss constraints. For the path integral to be well-defined, one must have

$$\left(\frac{\partial^2 I_*}{\partial r_+^2}\right)_0 > 0, \quad (6.46)$$

so that the stationary point is a minimum and stable, otherwise the integral may blow up or be continued to a complex function, indicating that the stationary

point is not a minimum and it is therefore an instanton. The second derivative of the action Eq. (6.28) can be simplified into $\left(\frac{\partial^2 I_*}{\partial r_+^2}\right)_0 = -\frac{\Omega_{d-2}(d-2)r_+^{d-3}}{4\beta} \frac{\partial I}{\partial r_+}$. Thus, the stability condition reduces to $\frac{\partial I}{\partial r_+} < 0$, meaning that the solution is stable when $\frac{r_+}{R}$ increases with a decrease in the inverse temperature, and so with an increase in the temperature. In terms of the variables x and y , see Eq. (6.34), the stability condition is

$$\begin{aligned} & \frac{d-1}{2}x^{4d-12} - (1+y)x^{3d-9} - 3(d-3)yx^{2d-6} \\ & + (2d-5)y(1+y)x^{d-3} - \frac{3d-7}{2}y^2 > 0. \end{aligned} \quad (6.47)$$

The range of x is $x_e < x < 1$, where x_e is a function of y_e , see Eq. (6.37). In the case of $0 \leq y < y_s$, the condition of stability reduces to two intervals in x , one is $0 < x < x_{s1}(y)$ and the other is $x_{s2}(y) < x < 1$. Therefore, the solutions x_1 and x_3 are stable, while the solution x_2 is unstable. Moreover, the points $x = x_{s1}$ and $x = x_{s2}$ are saddle points of the action as previously stated, and so they are neutrally stable. In the case of $y = y_s$, the same applies as the previous case. In the case of $y_s < y < 1$, the stability condition is satisfied in the interval $x_e < x < 1$ and so the solution x_4 is stable.

It is of interest to us to pick specific dimensions d . Due to its real importance, we review the case $d = 4$, and as a typical case of higher dimension, we analyze the case $d = 5$ carefully.

6.3.4 The case of $d = 4$: stationary points and stability conditions

We analyze briefly the particular case of four dimensions, $d = 4$. The original results were presented in [133, 134], here we show that the results above are in agreement with the original results, and we display also new and interesting features for this case.

First, we should look at the qualitative behaviour of the solutions $x \equiv \frac{r_+}{R}$ as a function of the temperature parameter RT , i.e., $x(RT)$, for the several distinct electric charge parameter y regions. Recall that the value of y_s is important since it separates the behavior of the solutions. From Eq. (6.43), in $d = 4$ it is $y_s = (\sqrt{5} - 2)^2 = 0.056$, the latter equality being approximate. The solutions can then be divided using the electric charge parameter y in the solution for the no charge case $y = 0$, solutions for the charge parameter in the region $0 < y < (\sqrt{5} - 2)^2$, the solution for $y = y_s = (\sqrt{5} - 2)^2$, and solutions for the charge parameter in the region $(\sqrt{5} - 2)^2 < y < 1$. For $y = 0$, the function $x(RT)$ describes the uncharged case and the solution is known, it is the original York solution [68], and consists of two solutions, here represented as x_2 and x_3 . The solution x_{s2} happens when x_2 and x_3 meet at temperature $RT = \frac{3\sqrt{3}}{8\pi} = 0.207$, the latter equality being approximate. For the electric charge in the range $0 < y < (\sqrt{5} - 2)^2$, there are three solutions x_1 , x_2 and x_3 , where x_1 is stable, x_2 is unstable, and x_3 is stable. For very small charges, the temperature T_1 , which is the temperature at which x_{s1} is a solution for the black hole at the given charge, is very high, tending to infinite when the charge tends to

zero. For very small charges, the temperature T_2 , which is the temperature at which x_{s2} is a solution for the black hole at the given charge, is very near the minimum temperature of the solutions of the canonical ensemble of the Schwarzschild black hole in four dimensions, i.e., $RT = \frac{3\sqrt{3}}{8\pi}$, mentioned above. Increasing the electric charge from small values, one has that the saddle points x_{s1} and x_{s2} approach each other. For the electric charge parameter given by $y = (\sqrt{5} - 2)^2 = y_s$, the saddle points x_{s1} and x_{s2} meet, and at this electric charge, the solution x_1 is described by a curve, the solution x_2 is now reduced to a point that coincides with $x_s = x_{s1} = x_{s2}$, and the solution x_3 is described by another curve. All solutions are stable, more precisely, x_1 is stable, x_2 is neutrally stable, and x_3 is stable. For electric charge in the range $(\sqrt{5} - 2)^2 < y < 1$, there is only one solution x_4 which represents the union of x_1 and x_3 , with x_2 having disappeared. Also, the solution x_4 is stable.

Second, we should look at the qualitative behaviour of the solutions $x \equiv \frac{r_+}{R}$ as a function of the electric charge parameter $y \equiv \frac{\mu Q^2}{R^2}$, with $\mu = l_p^2$ here, i.e., $x(y)$, for the several distinct temperature parameter RT regions. Recall that the value of RT_s and the value of minimum temperature in the uncharged case $RT = \frac{3\sqrt{3}}{8\pi}$ are important since they separate the behavior of the solutions. In $d = 4$, the value of the temperature corresponding to y_s and x_s is $RT_s = 0.185$, this equality being approximate. Thus, the temperature parameter regions are $0 < RT < 0.185$, $RT_s = 0.185$, $0.185 < RT \leq \frac{3\sqrt{3}}{8\pi} = 0.207$, and $\frac{3\sqrt{3}}{8\pi} < RT < \infty$. For $0 < RT < 0.185$, there are only two solutions, which are x_1 in the interval $0 < y < y_s$ and x_4 in the interval $y_s \leq y < 1$, with $y_s = (\sqrt{5} - 2)^2$. For $RT_s = 0.185$ corresponding to y_s and x_s , with this equality being approximate, there are four solutions, but two of them are degenerate. Indeed, there is the x_1 solution, there are the x_2 and x_3 solutions that degenerate into a point $x_2 = x_3$ at $y = y_s$, and there is the x_4 solution. For $0.185 < RT \leq \frac{3\sqrt{3}}{8\pi} = 0.207$, the latter equality being approximate, there are the four solutions x_1 , x_2 , x_3 and x_4 . The solutions x_1 , x_2 , and x_3 lie in the range $0 < y < y_s$, and the solution x_4 exists only for $y_s < y < 1$. The solution x_4 can be seen as a continuation in y , i.e., in Q , of the solutions x_1 and x_3 , and so in a sense x_4 is the union of x_1 and x_3 . For $\frac{3\sqrt{3}}{8\pi} < RT < \infty$, there are also the four solutions but x_2 and x_3 are discontinuous.

6.3.5 The case of $d = 5$: stationary points and stability conditions

6.3.5.1 Behaviour of solutions and stability

We present here the case for $d = 5$ in detail, namely we explain the behaviour of the solutions with the aid of plots.

First, we can analyze $x \equiv \frac{r_+}{R}$ as a function of the temperature parameter RT , for the several regions of the electric charge parameter y . Once more, the value of y_s is important for the analysis since it separates the regions of different behavior for the solutions. From Eq. (6.43), in $d = 5$ it is $y_s = \frac{(68-27\sqrt{6})^2}{250} = 0.014$, the latter equality being approximate. We can divide the analysis into the following regions of the electric charge parameter y : the no charge case $y = 0$, the electric charge parameter

in the region $0 < y < \frac{(68-27\sqrt{6})^2}{250}$, the specific case of the critical charge $y = y_s = \frac{(68-27\sqrt{6})^2}{250}$, and the electric charge parameter in the region $\frac{(68-27\sqrt{6})^2}{250} < y < 1$. We now describe the solutions $x(RT)$ for each region of y , according to Fig. 6.2, where the plots of the solutions $x \equiv \frac{r_+}{R}$ as a function of RT of the canonical ensemble in five dimensions, $d = 5$, are displayed. An important line in such plots is the gray

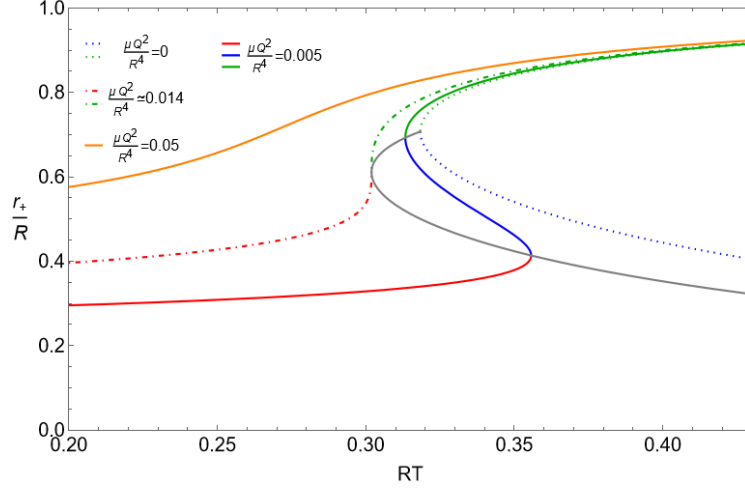


Figure 6.2: Plots of the solutions $x \equiv \frac{r_+}{R}$ as a function of RT of the canonical ensemble in five dimensions, $d = 5$, for four values of the electric charge parameter $y \equiv \frac{\mu Q^2}{R^4}$, with $\mu = \frac{4l_p^3}{3\pi}$ here. The four values of the electric charge parameter y are $y = 0$ in dotted lines, $y = 0.005$ in full lines, $y = \frac{(68-27\sqrt{6})^2}{250} = 0.014$ in dot dashed lines, the latter equality being approximate, and $y = 0.05$ in an orange full line. The solution $x_1 = \frac{r_{+1}}{R}$ is represented in red, $x_2 = \frac{r_{+2}}{R}$ is represented in blue, $x_3 = \frac{r_{+3}}{R}$ is represented in green, and $x_4 = \frac{r_{+4}}{R}$ is represented in orange. The gray curve describes the trajectory of the saddle points of the action $x_{s1} = \frac{r_{+s1}}{R}$ and $x_{s2} = \frac{r_{+s2}}{R}$ by changing the electric charge parameter, and it separates the regions of existence of the solutions $x_1 = \frac{r_{+1}}{R}$, $x_2 = \frac{r_{+2}}{R}$, and $x_3 = \frac{r_{+3}}{R}$.

line, that represents the trajectory of the saddle points x_{s1} and x_{s2} of the action by varying the electric charge. This gray line separates the regions where the solutions x_1 , x_2 , and x_3 can be found. More precisely, the two saddle points x_{s1} and x_{s2} are the bounds of the solution x_2 . For $y = 0$, one has the uncharged case, which has been analyzed in [101], and consists of two solutions, here represented as x_2 and x_3 . At the saddle point x_{s2} , the solutions x_2 and x_3 meet at temperature $RT = \frac{1}{\pi}$. For the electric charge parameter y in the region $0 < y < \frac{(68-27\sqrt{6})^2}{250}$, which can be visualized by the $y = 0.005$ case in the plot, there are three solutions x_1 , x_2 , and x_3 , where again x_1 is stable, x_2 is unstable, and x_3 is stable, see below for the discussion of thermodynamic stability. This case is representative of small electric charges. For very small charges, the temperature T_1 corresponding to the saddle point x_{s1} assumes very large values and tends to infinity when the charge tends to zero. Moreover, the temperature T_2 , corresponding to the saddle point x_{s2} is close to the minimum temperature of the solutions of the canonical ensemble of the Schwarzschild black hole in five dimensions $RT = \frac{1}{\pi}$. Note that the figure with the

plots for small electric charge parameter yields a unification of York and Davies, as the two solutions are here represented. More precisely, the blue and green lines correspond to the unstable and stable black holes of York [68], respectively, and the red and blue lines correspond to the stable and unstable black holes of Davies [51], respectively, see below for these latter black holes. Increasing the electric charge from small values, one sees that the saddle points x_{s1} and x_{s2} approach each other along the gray curve. For the saddle electric charge $y = y_s = \frac{(68-27\sqrt{6})^2}{250} = 0.014$, with the latter equality being approximate, the saddle points x_{s1} and x_{s2} are equal as $x_{s1} = x_{s2} = x_s$. While x_1 and x_3 are described by a curve, the solution x_2 reduces to a point $x_2 = x_s$ that connects both solutions x_1 and x_3 . Regarding stability, x_1 is stable, x_2 is neutrally stable, and x_3 is stable. For the electric charge parameter y in the region $\frac{(68-27\sqrt{6})^2}{250} < y < 1$, which is represented in the plot by the case $y = 0.05$, there is only one solution x_4 , that is in a sense the continuation of x_1 and x_3 , with x_2 having disappeared. It must be noted that x_4 is a stable solution.

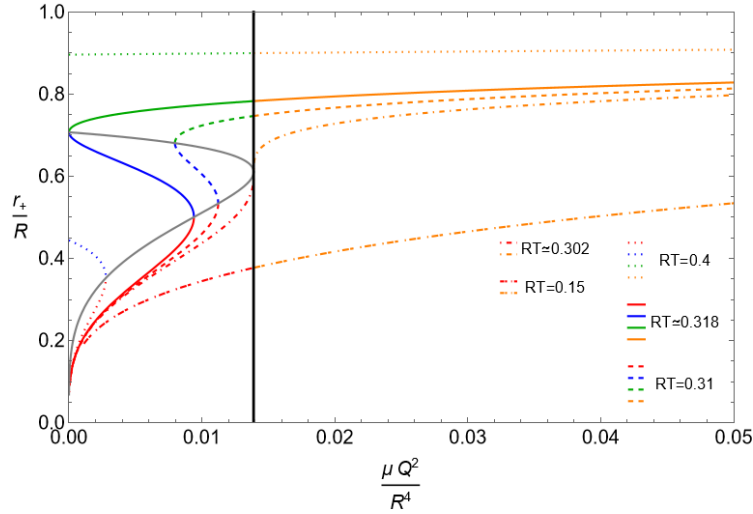


Figure 6.3: Plots of the solutions $x \equiv \frac{r_+}{R}$ as a function of $y \equiv \frac{\mu Q^2}{R^4}$ of the canonical ensemble in five dimensions, $d = 5$, for five values of the temperature parameter RT , with $\mu = \frac{4}{3\pi}$. The five values of RT are $RT = 0.15$ in double dashed lines, $RT = RT_s = 0.302$ in dot dashed lines, $RT = 0.31$ in dashed lines, $RT = \frac{1}{\pi} = 0.318$, in full lines, the latter equality being approximate, and $RT = 0.4$ in dotted lines. The solution $x_1 = \frac{r_{+1}}{R}$ is represented in red, $x_2 = \frac{r_{+2}}{R}$ is represented in blue, $x_3 = \frac{r_{+3}}{R}$ is represented in green, and $x_4 = \frac{r_{+4}}{R}$ is represented in orange. The black line, corresponding to $y = y_s = \frac{(68-27\sqrt{6})^2}{250}$, separates the solution $x_4 = \frac{r_{+4}}{R}$ from the remaining solutions. The gray line corresponds to the trajectory of the saddle points of the action $x_{s1} = \frac{r_{+s1}}{R}$ and $x_{s2} = \frac{r_{+s2}}{R}$, which bounds the region where $x_2 = \frac{r_{+2}}{R}$ exists.

Second, we can describe $x \equiv \frac{r_+}{R}$ as a function of the electric charge parameter $y \equiv \frac{\mu Q^2}{R^4}$, with $\mu = \frac{4l_p^3}{3\pi}$ here, for the several regions of the temperature parameter RT . Here, the value of RT_s and the value of the minimum temperature of the uncharged case $RT = \frac{1}{\pi}$ are important since they separate the regions of different

behavior for the solutions. In $d = 5$, the temperature corresponding to $x_s(y_s)$ is $RT_s = 0.302$, with this equality being approximate. The temperature parameter regions $0 < RT < 0.302$, $RT_s = 0.302$, $0.302 < RT \leq \frac{1}{\pi} = 0.318$, the latter equality being approximate, and $\frac{1}{\pi} < RT < \infty$ are then considered. We describe the solution $x(y)$ here within each RT region, in agreement with Fig. 6.3, where the plots of the solutions $x \equiv \frac{r_+}{R}$ as a function of $y \equiv \frac{\mu Q^2}{R^4}$, $\mu = \frac{4}{3\pi}$, of the canonical ensemble in five dimensions, $d = 5$, are displayed. For the temperature parameter RT in the range $0 < RT < 0.302$, of which $RT = 0.15$ is represented in the figure, there are only two solutions to display, which are x_1 in the interval $0 < y < y_s$, and x_4 in the interval $y_s \leq y < 1$, with $y_s = \frac{(68-27\sqrt{6})^2}{250}$. For the temperature parameter RT given by $RT = RT_s = 0.302$, this equality being approximate, one has the curves of the x_1 solution and the x_4 solution, while the x_2 and x_3 solutions degenerate into a point $x_2 = x_3$ at $y = y_s$. For the temperatures $0.302 < RT \leq \frac{1}{\pi} = 0.318$, of which $RT = 0.31$ and $RT = \frac{1}{\pi}$ are represented in the figure, one has the solutions x_1 , x_2 and x_3 lying in the range $0 < y < y_s$, while the solution x_4 lies in the range $y_s < y < 1$. The figure shows explicitly that the solution x_4 is a continuation in the electric charge parameter y of the solutions x_1 and x_3 . Note also that the gray curve in the figure bounds the solution x_2 . For $\frac{1}{\pi} < RT < \infty$, which is represented by $RT = 0.4$ in the figure, one has also the four solutions but the segments of x_2 and x_3 are discontinuous.

6.3.5.2 Interpretation through the thermal length

The behaviour of the solutions merits some underlying understanding of the physics at play, which we now give in terms of the thermal wavelength λ , which is proportional to the inverse of the temperature, $\lambda = \frac{1}{T}$. We present the reasoning here for the plots of the solutions $x \equiv \frac{r_+}{R}$ as a function of RT of the canonical ensemble shown in Fig. 6.2. The solutions are analyzed from small electric charge to large electric charge, and from low to high temperature T with R fixed. We must note that small RT corresponds to low T here.

We analyze the case for a given small electric charge first. For small T , the associated thermal wavelength λ is large and is stuck to the cavity walls, which means that if there were no electric charge, there would be no black hole. But since there is a fixed electric charge, there is a small black hole with radius r_+ of the order of the length scale set by the charge itself. This black hole does not form by collapse, its presence comes from topological constraints. The black hole is stable, small perturbations cannot evaporate it. For the smallest possible T , $T = 0$, the black hole is an extremal black hole. For small temperature, there is only one black hole solution which is this one. For an intermediate T , as the temperature increases, one has that the associated thermal wavelength λ decreases. The black hole with small r_+ is still there, but there is now the possibility of forming black holes via collapse, indeed the thermal wavelengths are no more stuck to the cavity walls and the existent thermal energy can collapse. One black hole that can form in this way has radius r_+ of the order of λ and is thermodynamically unstable since clearly it can evaporate. The other black hole that can form in this way has radius r_+ large

such that $R - r_+$ is of the order of λ , and is thermodynamically stable, the reservoir and the black hole exchange quanta of λ in a stabilizing way. For intermediate temperatures, there are thus three black hole solutions for each temperature. For high T , as the temperature increases and the associated wavelength λ gets even smaller. The smallest black hole r_+ ceases to exist because, due to the turbulence created by the high temperature, there is no way to maintain the electric charge coherently at the center of the cavity. The intermediate black hole r_+ ceases to exist because the electric charge repulsion is sufficient to halt gravitational collapse of this black hole with intermediate r_+ . The large black hole r_+ still exists, as it has sufficient mass to overcome the electric repulsion and still collapses. For high T , therefore only the large black hole exists. This is for a typical reasonably low electric charge Q , and there is an interplay between the two quantities that characterize the ensemble, namely, the temperature T and the electric charge Q .

Second, we analyze the case of high electric charge. Again here, for small T , the associated thermal wavelength λ is large and is stuck and cannot collapse. But since there is a fixed electric charge, there is a small black hole with radius r_+ of the order of the length set by the charge itself, its presence comes from topological constraints, is stable, i.e., small perturbations cannot evaporate it. $T = 0$ yields an extremal black hole. At intermediate T , there is turbulence to disperse the black hole with topological features but it is possible to have sufficient mass to collapse the existent thermal energy into the large black hole, with $R - r_+$ starting to be comparable to λ . Note that the intermediate black hole does not exist because the electric charge is large enough to counter its collapse. For high T , as the temperature increases and the associated wavelength λ gets smaller, the large black hole r_+ has sufficient mass to overcome the electric repulsion and the thermal energy collapses, being stable. For all temperatures, there is thus one black hole solution only for each temperature. It is in a sense the union of the topological black hole with the large collapsed black hole as the temperature T increases, the intermediate one having disappeared. Following this reasoning, one could also extend this interpretation to the plots of the solutions $x \equiv \frac{r_+}{R}$ as a function of $\frac{\mu Q^2}{R^4}$ in Fig. 6.3.

6.4 THERMODYNAMICS OF A CHARGED BLACK HOLE INSIDE A CAVITY IN d DIMENSIONS THROUGH THE CANONICAL ENSEMBLE

6.4.1 Thermodynamic properties and stability for d dimensions

6.4.1.1 Thermodynamic properties

We have constructed above the canonical ensemble for a charged black hole in d dimensions through the Euclidean path integral approach subjected to the zero loop approximation. With the partition function calculated, we can now proceed to obtain the thermodynamic properties of the system. Namely, the partition function is given by $Z = e^{-I_0[\beta, R, Q]}$, where I_0 is the action evaluated at the stationary points. Thermodynamically, the partition function in the canonical ensemble is also related to the Helmholtz free energy F as $Z = e^{-\beta F}$. Therefore, one has that the free energy is given by

$$F = T I_0[\beta, R, Q] \text{ or} \quad (6.48)$$

or explicitly,

$$F = \frac{R^{d-3}}{\mu} \left(1 - \sqrt{f(R, Q, r_+)} \right) - T \frac{\Omega_{d-2} r_+^{d-2}}{4l_p^{d-2}}, \quad (6.49)$$

with $f(R, Q, r_+) \equiv 1 - \frac{r_+^{d-3} + \frac{\mu Q^2}{r_+^{d-3}}}{R^{d-3}} + \frac{\mu Q^2}{R^{2d-6}}$, see Eq. (6.29).

The Helmholtz free energy by definition is given in terms of the internal energy E , the temperature T , and the entropy S by the relation

$$F = E - TS, \quad (6.50)$$

and it has the differential

$$dF = -SdT - pdA + \phi dQ, \quad (6.51)$$

where, in addition to the entropy S , the area A , and the electric charge Q , there is the thermodynamic pressure p , and the thermodynamic electric potential ϕ . The thermodynamic quantities can then be obtained from the derivatives of the free energy F , more precisely, the entropy is $S = -\left(\frac{\partial F}{\partial T}\right)_{A, Q}$, the pressure is $p = -\left(\frac{\partial F}{\partial A}\right)_{T, Q}$, and the electric potential is $\phi = \left(\frac{\partial F}{\partial Q}\right)_{T, A}$, where here the subscript indicates the quantities that are fixed while performing the derivative. In Eq. (6.51), a part of the dependence on T , A , and Q is implicit on the solution for the horizon radius $r_+ = r_+(T, A, Q)$, as it is evaluated at the minima of the action. To simplify the calculation of the derivatives, one can perform the chain rule and the fact that, since $r_+ = r_+(T, A, Q)$, the derivative of the reduced action obeys $\left(\frac{\partial I_*}{\partial r_+}\right)_{T, R, Q} = \left(\frac{\partial F}{\partial r_+}\right)_{T, R, Q} = 0$, to get for example $S = -\left(\frac{\partial F}{\partial T}\right)_{A, Q} = -\left(\frac{\partial F}{\partial T}\right)_{R, Q, r_+} - \left(\frac{\partial F}{\partial r_+}\right)_{T, R, Q} \frac{\partial r_+}{\partial T} = -\left(\frac{\partial F}{\partial T}\right)_{R, Q, r_+}$, and this also holds similarly for the computation of

the pressure and the electric potential. Therefore, the thermodynamic quantities can be computed as

$$S = - \left(\frac{\partial F}{\partial T} \right)_{R,Q,r_+}, \quad p = - \frac{1}{(d-2)\Omega_{d-2}R^{d-3}} \left(\frac{\partial F}{\partial R} \right)_{T,Q,r_+}, \quad \phi = \left(\frac{\partial F}{\partial Q} \right)_{T,R,r_+}, \quad (6.52)$$

and $E = F - TS$. The entropy is then given as

$$S = \frac{A_+}{4l_p^{d-2}}, \quad (6.53)$$

where $A_+ \equiv \Omega_{d-2}r_+^{d-2}$ is the area of the event horizon, and so this is the usual Bekenstein-Hawking expression for the entropy of a black hole. The thermodynamic pressure is

$$p = \frac{d-3}{16\pi R\sqrt{f}l_p^{d-2}} \left((1-\sqrt{f})^2 - \frac{\mu Q^2}{R^{2d-6}} \right), \quad (6.54)$$

the thermodynamic electric potential is

$$\phi = \frac{Q}{\sqrt{f}} \left(\frac{1}{r_+^{d-3}} - \frac{1}{R^{d-3}} \right), \quad (6.55)$$

and finally, from Eq. (6.50), the thermodynamic energy is given by

$$E = \frac{R^{d-3}}{\mu} \left[1 - \sqrt{\left(1 - \frac{r_+^{d-3}}{R^{d-3}} \right) \left(1 - \frac{\mu Q^2}{r_+^{d-3}R^{d-3}} \right)} \right], \quad (6.56)$$

Collecting Eqs. (6.53)-(6.56), one finds that the first law of thermodynamics in the form

$$dE = TdS - pdA + \phi dQ, \quad (6.57)$$

holds. It is interesting to note, and surely not a coincidence, that these thermodynamic quantities are identical to the ones calculated for a self-gravitating charged shell, where the first law of thermodynamics is imposed, and, the charged shell assumes the temperature equation of state of a black hole and the thermodynamic pressure equation of state of the cavity, see [1].

With the thermodynamic quantities obtained in Eqs. (6.53)-(6.56), we can get an integrated first law of thermodynamics known as the Euler equation. For that, one rewrites the energy in Eq. (6.56) in terms of the entropy in Eq. (6.53), the area $A = \Omega_{d-2}R^{d-2}$, and the electric charge Q as

$$E = \frac{(d-2)A^{\frac{d-3}{d-2}}\Omega_{d-2}^{\frac{1}{d-2}}}{8\pi l_p^{d-2}} \times \left(1 - \sqrt{\left(1 - \left(\frac{4Sl_p^{d-2}}{A} \right)^{\frac{d-3}{d-2}} \right) \left(1 - \frac{\mu Q^2 \Omega_{d-2}^{\frac{2d-3}{d-2}}}{(4Sl_p^{d-2}A)^{\frac{d-3}{d-2}}} \right)} \right). \quad (6.58)$$

If a scaling is performed on the thermodynamic quantities $S \rightarrow \nu S$, $A \rightarrow \nu A$ and $Q \rightarrow \nu^{\frac{d-3}{d-2}} Q$, then it can be verified that $E(\nu S, \nu A, \nu^{\frac{d-3}{d-2}} Q) = \nu^{\frac{d-3}{d-2}} E(S, A, Q)$. According to the Euler relation theorem, and considering that the differential of the energy is given by the first law of thermodynamics Eq. (6.57), the Euler equation is given by

$$E = \frac{d-2}{d-3}(TS - pA) + \phi Q. \quad (6.59)$$

One can furthermore differentiate Eq. (6.59) and use the first law of thermodynamics to obtain

$$\frac{1}{d-3}(TdS - pdA) + \frac{d-2}{d-3}(SdT - Adp) + Qd\phi = 0, \quad (6.60)$$

which is the Gibbs-Duhem relation.

6.4.1.2 Thermodynamic stability and the heat capacity

A system to be thermodynamically stable must have positive heat capacity at constant area and constant electric charge $C_{A,Q}$, i.e.,

$$C_{A,Q} \geq 0, \quad (6.61)$$

where $C_{A,Q} \equiv T \left(\frac{\partial S}{\partial T} \right)$. We have shown in Sec. 6.3.3 that the stability condition in the ensemble formalism was reduced to the condition $\frac{\partial \iota}{\partial r_+} < 0$. The derivative above can be put in terms of thermodynamic variables, and then in terms of the heat capacity. The inverse temperature function $\iota(r_+)$ is a function of r_+ , R and Q . The variables Q and R are already thermodynamic variables. The quantity r_+ is also in some sense a thermodynamic variable since one has that $S = \frac{\Omega_{d-2} r_+^{d-2}}{4}$. Therefore, using $\beta = \iota(r_+)$, one has $\frac{\partial \iota}{\partial r_+} = -\frac{1}{T} \left(\frac{\partial S}{\partial r_+} \right) \frac{1}{C_{A,Q}}$, where the definition of the heat capacity at constant area and constant electric charge was used.

The heat capacity is then written as

$$C_{A,Q} = \frac{(d-2)R^{d-2}f\left(\frac{r_+^{d-3}}{R^{d-3}} - \frac{\mu Q^2}{R^{d-3}r_+^{d-3}}\right)\frac{\Omega_{d-2}r_+^{d-2}}{4R^{d-2}}}{l_p^{d-2}\frac{d-3}{2}\left(\frac{r_+^{d-3}}{R^{d-3}} - \frac{\mu Q^2}{r_+^{d-3}R^{d-3}}\right)^2 - f\left(\frac{r_+^{d-3}}{R^{d-3}} - (2d-5)\frac{\mu Q^2}{r_+^{d-3}R^{d-3}}\right)}. \quad (6.62)$$

Since one has that $C_{A,Q} \geq 0$ for the system to be thermodynamically stable, thermodynamic stability reduces to Eq. (6.47) after rearrangements and definitions. Thus, the physical interpretation is that the stability of the solutions is controlled by the heat capacity at constant area and charge, as it should be in the canonical ensemble. This quantity is tied to the derivative of the inverse temperature given by Eq. (6.33) and so the condition reduces to the intervals given by the stationary points of $\iota(r_+, R, Q)$, or the saddle points of the action. Moreover, solutions where r_+ increases as T increases are stable and solutions where r_+ decreases as T increases are unstable.

It is interesting to see what happens when one fixes $\frac{r_+}{R}$ and changes the electric charge parameter $\frac{\mu Q^2}{R^{2d-6}}$. For $\frac{r_+}{R} > \left(\frac{2}{d-1}\right)^{\frac{1}{d-3}}$, the heat capacity is always positive for any electric charge, with the limit of the bound matching the one for the uncharged black hole. For $0 \leq \frac{r_+}{R} \leq \left(\frac{2}{d-1}\right)^{\frac{1}{d-3}}$, the sign of the heat capacity $C_{A,Q}$ changes according to the electric charge. $C_{A,Q}$ is positive for sufficiently high electric charge parameter $\frac{\mu Q^2}{R^{2d-6}}$, and is negative for sufficiently low electric charge parameter $\frac{\mu Q^2}{R^{2d-6}}$, the change in sign happening at the definite value of the charge satisfying Eq. (6.38) with fixed $\frac{r_+}{R}$. It is important to note that this does not indicate a phase transition since $\frac{r_+}{R}$ is not a thermodynamic variable controlled in the ensemble. At that definite value of the charge parameter, there is rather a turning point describing the ratio of scales at which there is stability.

The thermodynamic variables are the temperature and the electric charge, and therefore the heat capacity must be analyzed in terms of these quantities, instead of $\frac{r_+}{R}$ and the electric charge. For the range of electric charges $0 < \frac{\mu Q^2}{R^{2d-6}} < \frac{\mu Q_s^2}{R^{2d-6}}$, one has three curves for the heat capacity as a function of the temperature, one for each solution. The heat capacity is positive for the solutions r_{+1} and r_{+3} , while it is negative for r_{+2} . The heat capacity diverges when the solutions reach the temperatures of the saddle points of the action, which are the turning points. For the critical charge parameter $\frac{\mu Q_s^2}{R^{2d-6}}$, one has two curves for the heat capacity as a function of the temperature. In this particular case, the two curves are described by the solutions r_{+1} and r_{+3} and it is positive for both. Moreover, there is a discontinuity between the two curves at RT_s , where the heat capacity diverges. This point indeed does mark a second order phase transition between r_{+1} and r_{+3} , as both solutions are stable and it can be seen that the free energy is continuous at RT_s for $\frac{\mu Q_s^2}{R^{2d-6}}$. For the range $\frac{\mu Q^2}{R^{2d-6}} > \frac{\mu Q_s^2}{R^{2d-6}}$, there is only one curve for the heat capacity as a function of the temperature, corresponding to the solution r_{+4} and it is always positive.

We now specify the results in this subsection for the cases $d = 4$ and $d = 5$ dimensions, supported by further comments and a figure.

6.4.2 Thermodynamic properties and stability for $d = 4$ dimensions

For the case $d = 4$, we can write straightforwardly the results from the subsection above. The entropy is given as

$$S = \pi \frac{r_+^2}{l_p^2}, \quad (6.63)$$

which is the usual Hawking-Bekenstein formula $S = \frac{A_+}{4l_p^2}$, with $A_+ = 4\pi r_+^2$ being the area of the event horizon. The pressure is

$$p = \frac{1}{16\pi R l_p^2 \sqrt{f}} \left((1 - \sqrt{f})^2 - \frac{l_p^2 Q^2}{R^2} \right), \quad (6.64)$$

where it was used $\mu = l_p^2$ and $f = 1 - \frac{r_+ + \frac{l_p^2 Q^2}{r_+}}{R} + \frac{l_p^2 Q^2}{R^2}$. The electric potential is

$$\phi = \frac{Q}{\sqrt{f}} \left(\frac{1}{r_+} - \frac{1}{R} \right). \quad (6.65)$$

Finally, the mean energy is given by

$$E = \frac{R}{l_p^2} \left[1 - \sqrt{\left(1 - \frac{r_+}{R} \right) \left(1 - \frac{l_p^2 Q^2}{r_+ R} \right)} \right]. \quad (6.66)$$

One can then write the energy in terms of S , $A = 4\pi R^2$, and Q , i.e., $E = E(S, A, Q)$ to obtain the Euler relation $E = 2(TS - pA) + \phi Q$. The Gibbs-Duhem relation is $TdS - pdA + 2(SdT - Adp) + Qd\phi = 0$.

The heat capacity, the quantity that controls thermodynamic stability, is

$$C_{A,Q} = \frac{1}{l_p^2} \frac{2R^2 f \left(\frac{r_+}{R} - \frac{l_p^2 Q^2}{R^2} \frac{R}{r_+} \right) \frac{\pi r_+^2}{R^2}}{\frac{1}{2} \left(\frac{r_+}{R} - \frac{Q^2}{R^2} \frac{R}{r_+} \right)^2 - f \left(\frac{r_+}{R} - \frac{3Q^2}{R^2} \frac{R}{r_+} \right)}. \quad (6.67)$$

One could fix $\frac{r_+}{R}$ and change the electric charge parameter $\frac{l_p^2 Q^2}{R^2}$ in Eq. (6.67). As seen in the general d case, one finds that for $\frac{r_+}{R} > \frac{2}{3}$, the heat capacity is always positive, and for $0 \leq \frac{r_+}{R} \leq \frac{2}{3}$, the sign of the heat capacity $C_{A,Q}$ changes depending on the electric charge, being positive for a region of high electric charge parameter $\frac{l_p^2 Q^2}{R^2}$, and being negative for a region of low electric charge parameter $\frac{l_p^2 Q^2}{R^2}$. This does not indicate a phase transition but rather a turning point. To see this fact and verify the true phase transitions, one must analyze the heat capacity in terms of the fixed quantities of the ensemble, i.e., the temperature and the electric charge. For the range of charge parameters $0 < \frac{l_p^2 Q^2}{R^2} < (\sqrt{5} - 2)^2$, where in $d = 4$ one has $\frac{l_p^2 Q_s^2}{R^2} = (\sqrt{5} - 2)^2$, the heat capacity has a curve for each solution r_{+1} , r_{+2} , and r_{+3} , being positive for r_{+1} and r_{+3} , and being negative for r_{+2} . When the solutions reach the temperatures of the saddle points of the action, i.e., the turning points, the heat capacity diverges but this only indicates conditions for stability of the ensemble, there are no phase transitions at these points. For the critical charge $\frac{l_p^2 Q_s^2}{R^2} = (\sqrt{5} - 2)^2$, the heat capacity has two curves as a function of the temperature, r_{+1} and r_{+3} , being positive for both solutions. For this case, there is a discontinuity between the two curves at $RT_s = 0.185$, where the heat capacity diverges. This point indeed signals a second order phase transition between r_{+1} and r_{+3} , as both solutions are stable and it can be seen that the free energy is continuous at $RT_s = 0.185$ for $\frac{l_p^2 Q^2}{R^2} = (\sqrt{5} - 2)^2$. For the range of charge parameters $\frac{l_p^2 Q^2}{R^2} > (\sqrt{5} - 2)^2$, one has that the heat capacity of r_{+4} as a function of the temperature is always positive. In [133, 134] some of these results for $d = 4$ are presented.

6.4.3 Thermodynamic properties and stability for $d = 5$ dimensions

Here, we make the results for the case $d = 5$ explicit. The entropy is given as

$$S = \frac{\pi^2 r_+^3}{2l_p^3}, \quad (6.68)$$

matching the usual Bekenstein-Hawking formula $S = \frac{A_+}{4l_p^3}$, with $A_+ = 2\pi^2 r_+^3$ being the area of the event horizon. The pressure yields

$$p = \frac{2}{16\pi l_p^2 R \sqrt{f}} \left((1 - \sqrt{f})^2 - \frac{4l_p^3 Q^2}{3\pi R^4} \right), \quad (6.69)$$

where it was used $\mu = \frac{4l_p^3}{3\pi}$ and $f = 1 - \frac{r_+^2 + \frac{4l_p^3 Q^2}{3\pi r_+^2}}{R^2} + \frac{4l_p^3 Q^2}{3\pi R^4}$. The electric potential yields

$$\phi = \frac{Q}{\sqrt{f}} \left(\frac{1}{r_+^2} - \frac{1}{R^2} \right). \quad (6.70)$$

And the energy has the expression

$$E = \frac{3\pi R^2}{4l_p^3} \left[1 - \sqrt{\left(1 - \frac{r_+^2}{R^2} \right) \left(1 - \frac{4l_p^3 Q^2}{3\pi r_+^2 R^2} \right)} \right]. \quad (6.71)$$

These thermodynamic quantities are identical to the ones calculated for a self-gravitating charged shell, where the first law of thermodynamics is imposed, and the charged shell assumes the equation of state of the black hole, see [1] or Chapter 2. The energy can be written in terms of S , $A = 2\pi^2 R^3$, and the electric charge Q , as $E = E(S, A, Q)$ to obtain the Euler relation $E = \frac{3}{2}(TS - pA) + \phi Q$. The Gibbs-Duhem relation yields $\frac{1}{2}(TdS - pdA) + \frac{3}{2}(SdT - Adp) + Qd\phi = 0$.

The heat capacity is

$$C_{A,Q} = \frac{1}{l_p^3} \frac{3R^3 f \left(\frac{r_+^2}{R^2} - \frac{4l_p^3 Q^2}{3\pi R^2 r_+^2} \right) \frac{\pi^2 r_+^3}{2R^3}}{\left(\frac{r_+^2}{R^2} - \frac{4l_p^3 Q^2}{3\pi R^4} \frac{R^2}{r_+^2} \right)^2 - f \left(\frac{r_+^2}{R^2} - \frac{20l_p^3 Q^2}{3\pi R^4} \frac{R^2}{r_+^2} \right)}. \quad (6.72)$$

Regarding the behavior of the heat capacity with fixed $\frac{r_+}{R}$ as a function of the electric charge parameter $\frac{l_p^3 Q^2}{R^4}$, one has that the heat capacity is always positive for $\frac{r_+}{R} > \frac{\sqrt{2}}{2}$, and the heat capacity changes signs for $0 \leq \frac{r_+}{R} \leq \frac{\sqrt{2}}{2}$, being positive for high electric charge parameter $\frac{l_p^3 Q^2}{R^4}$, and being negative for low electric charge parameter $\frac{l_p^3 Q^2}{R^4}$. As we already noted, to understand the turning points and the possible phase transitions of the solutions, one must analyze the behavior of the heat capacity through its dependence in the temperature and the electric charge, see Fig. 6.4. For a fixed electric charge parameter in the range $0 < \frac{\mu Q^2}{R^4} < \frac{(68-27\sqrt{6})^2}{250}$,

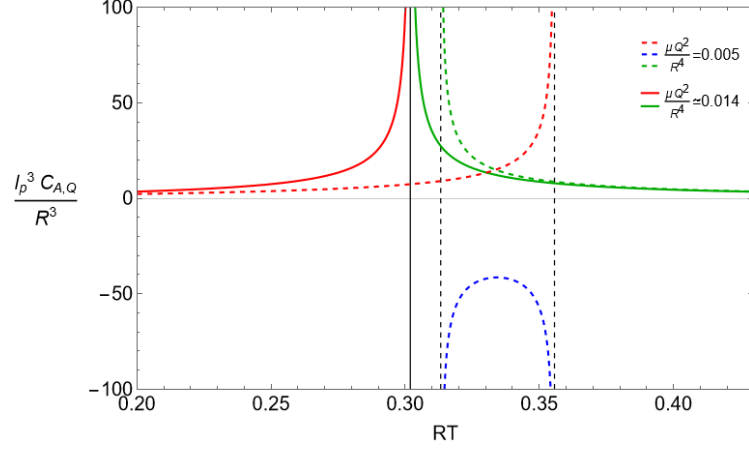


Figure 6.4: The heat capacity $C_{A,Q}$, namely $\frac{C_{A,Q} I_p^3}{R^3}$, as a function of the temperature for two values of the electric charge, $\frac{\mu Q^2}{R^4} = 0.005$ and $\frac{\mu Q^2}{R^4} = \frac{\mu Q_s^2}{R^4} = 0.014$ approximately, for solutions r_{+1} in red, r_{+2} in blue, and r_{+3} in green. The dashed black lines mark the turning points of the solutions and the solid black line marks the second order phase transition between the stable solutions r_{+1} and r_{+3} .

where in $d = 5$ one has $\frac{\mu Q_s^2}{R^4} = \frac{(68-27\sqrt{6})^2}{250}$, the heat capacity is described by three curves, one for each solution r_{+1} , r_{+2} , and r_{+3} , being positive for r_{+1} and r_{+3} , and being negative for r_{+2} , see Fig. 6.4 for the case $\frac{\mu Q^2}{R^4} = 0.005$. The heat capacity in this range of charges diverges at the turning points of the solutions, as seen by the dashed black lines, indicating the conditions for stability of the solutions and not signalling any phase transition. For the electric charge $\frac{\mu Q_s^2}{R^4} = \frac{(68-27\sqrt{6})^2}{250}$, the heat capacity is positive, as it is described by the curves of the solution r_{+1} and r_{+3} . The heat capacity diverges at $RT_s = 0.302$, the solid black line, and here one in fact has a second order transition, from r_{+1} to r_{+3} , as these are both stable solutions, and the free energy is continuous there. For $\frac{\mu Q^2}{R^4} > \frac{(68-27\sqrt{6})^2}{250}$, the heat capacity is always positive, as it is described only by the solution r_{+4} .

6.5 FAVORABLE PHASES IN THE CANONICAL ENSEMBLE OF A d DIMENSIONAL ELECTRICALLY CHARGED BLACK HOLE IN A CAVITY AND PHASE TRANSITIONS

6.5.1 The black hole sector of the canonical ensemble and favorable phases in d dimensions

We plan now to study the favorable phases of the situation at hand. Consider first the black hole sector of the canonical ensemble and the corresponding free energy. Since the free energy F and action I_0 are related by $F = \frac{I}{\beta} = T I_0$, the black hole free energy F_{bh} can be taken directly from Eq. (6.49) to be rewritten as

$$F_{bh} = \frac{R^{d-3}}{\mu} \left(1 - \sqrt{f(R, Q, r_+)} \right) - T \frac{A_+}{4l_p^{d-2}}, \quad (6.73)$$

where, in this section, the bh subscript in F denotes the black hole free energy to distinguish from other possible free energies. Since $A_+ \equiv \Omega_{d-2} r_+^{d-2}$ and $r_+ = r_+(T, R, Q)$, the black hole solutions have their free energies of the form $F_{\text{bh}}(T, R, Q)$. For a system characterized by the free energy, the one that has the lower free energy F_{bh} , for given R , T , and Q , is the one that is thermodynamically favored. Thus, one can find the black hole that is favored.

We have shown above that in the zero loop approximation, there are different black hole solutions depending on the electric charge and temperature of the reservoir, see Sec. 6.3.2. For sufficiently low electric charge parameter, i.e., for $0 \leq \frac{\mu Q^2}{R^{2d-6}} < \frac{\mu Q_s^2}{R^{2d-6}}$, where Q_s is the saddle electric charge value, corresponding to the saddle electric charge parameter $y_s = \frac{\mu Q_s^2}{R^{2d-6}}$, it was seen that there can be up to three solutions $\frac{r_{+1}}{R}$, $\frac{r_{+2}}{R}$, and $\frac{r_{+3}}{R}$. The free energies F_{bh} of these three solutions are now commented. The solution $\frac{r_{+1}}{R}$ has positive free energy for all the temperatures in which the solution exists. The solution $\frac{r_{+2}}{R}$ has also positive free energy always, but it is unstable, so this solution has no interest here. The solution $\frac{r_{+3}}{R}$, has a temperature for each electric charge parameter $\frac{\mu Q^2}{R^{2d-6}}$ at which the free energy becomes zero, which it is defined as $T_{F_{\text{bh}}=0}(Q)$ or $T_{F_{\text{bh}}=0}(\frac{\mu Q^2}{R^{2d-6}})$, thus $\frac{r_{+3}}{R}$ can have positive or negative free energy. For the saddle charge parameter $\frac{\mu Q^2}{R^{2d-6}} = \frac{\mu Q_s^2}{R^{2d-6}}$, the solution $\frac{r_{+1}}{R}$ has positive free energy, there is a solution where $\frac{r_{+1}}{R} = \frac{r_{+2}}{R} = \frac{r_{+3}}{R}$ which has positive free energy, and the solution $\frac{r_{+3}}{R}$ has again a temperature $T_{F_{\text{bh}}=0}(Q_s)$ or $T_{F_{\text{bh}}=0}(\frac{\mu Q_s^2}{R^{2d-6}})$, at which the free energy becomes zero, thus $\frac{r_{+3}}{R}$ can have positive or negative free energy. For higher values of the electric charge parameter, i.e., for $\frac{\mu Q_s^2}{R^{2d-6}} < \frac{\mu Q^2}{R^{2d-6}} < 1$, the solution $\frac{r_{+4}}{R}$ has also a temperature $T_{F_{\text{bh}}=0}(Q)$, or $T_{F_{\text{bh}}=0}(\frac{\mu Q^2}{R^{2d-6}})$, at which the free energy becomes zero, thus $\frac{r_{+4}}{R}$ can have positive or negative free energy. The temperature $T_{F_{\text{bh}}=0}(Q)$ can be calculated by solving $F_{\text{bh}} = 0$, with F_{bh} given in Eq. (6.73) for either the solution $\frac{r_{+3}}{R}$ or $\frac{r_{+4}}{R}$. One can instead put the free energy in terms of the mass m and electric charge Q through Eq. (6.33) and through the relation $2\mu m = r_+^{d-3} + \frac{\mu Q^2}{r_+^{d-3}}$, so that $F_{\text{bh}} = 0$ reduces to a quartic equation for the mass m as a function of the electric charge, see Sec 6.7. After solving it, one can then recover the value of r_+ and consequently the value $T_{F_{\text{bh}}=0}(\frac{\mu Q^2}{R^{2d-6}})$. For temperatures lower than $T_{F_{\text{bh}}=0}(\frac{\mu Q^2}{R^{2d-6}})$, the solutions have positive free energy and for temperatures higher than $T_{F_{\text{bh}}=0}(\frac{\mu Q^2}{R^{2d-6}})$, the solutions have negative free energy.

There is another important temperature, T_f , which depends on the electric charge Q , i.e., on the electric charge parameter $\frac{\mu Q^2}{R^{2d-6}}$, and at which the favorability of one phase over the other changes. For the electric charge parameter within the region $0 \leq \frac{\mu Q^2}{R^{2d-6}} < \frac{\mu Q_s^2}{R^{2d-6}}$, there is a phase favorability temperature T_f at which the solutions $\frac{r_{+1}}{R}$ and $\frac{r_{+3}}{R}$ have the same free energy. In other words, the solutions $\frac{r_{+1}}{R}$ and $\frac{r_{+3}}{R}$ are stable, and thus within the black hole sector they compete between themselves to be the most favored phase. Specifically, for temperatures lower than T_f , the solution $\frac{r_{+1}}{R}$ is either more favorable than $\frac{r_{+3}}{R}$, or is the only existing solution if the temperature is low enough. For a temperature equal to T_f , the solutions $\frac{r_{+1}}{R}$

and $\frac{r_{+3}}{R}$ are equally favorable, i.e., they coexist equally. For temperatures higher than T_f , either the solution $\frac{r_{+3}}{R}$ is more favorable than $\frac{r_{+1}}{R}$, or is the only existing solution if the temperature is high enough. For the electric charge parameter given by $\frac{\mu Q^2}{R^{2d-6}} = \frac{\mu Q_s^2}{R^{2d-6}}$, the temperature T_f is the temperature at which $\frac{r_{+1}}{R} = \frac{r_{+2}}{R} = \frac{r_{+3}}{R}$ and all have the same free energy, i.e., $\frac{r_{+1}}{R}$ and $\frac{r_{+3}}{R}$ coexist. For temperatures lower than T_f , the solution $\frac{r_{+1}}{R}$ is the only existing solution. For temperatures higher than T_f , the solution $\frac{r_{+3}}{R}$ is the only existing solution. For the electric charge parameter within the region $\frac{\mu Q_s^2}{R^{2d-6}} < \frac{\mu Q^2}{R^{2d-6}} < 1$, there is only one black hole solution, it is $\frac{r_{+4}}{R}$. Within the black hole sector it is surely the most favored state since it is stable and there is no other solution. It can have positive or negative free energy.

6.5.2 The hot flat space sector of the electrically charged canonical ensemble in d dimensions

Consider now a possible electrically charged hot flat space sector, i.e., a cavity with nothing in it with its boundaries defined by R , T , and Q , the settings of the canonical ensemble.

To have such a solution one can think in trying to decrease r_+ up to zero, to a point where there is no more a black hole and thus obtain flat space. However, this is not possible, since there is a minimum limit for r_+ given by $r_+ = r_{+e}$ corresponding to the extremal black hole. At r_{+e} , the free energy tends to $F_{bh} = \frac{Q}{\sqrt{\mu}}$, and it is then impossible to decrease r_+ further. Regarding extremal black holes, the only temperature that such solutions exist is at $T = 0$ and they are considered here as it is only one point of the ensemble, although it is a very interesting one. It seems there is no other immediate solution of the action that can be a candidate for a stationary point of the reduced action. Thus, to emulate electrically charged hot flat space one has to go beyond the black hole sector. We consider, for example, a shell with radius r_{shell} , coated with the required electric charge Q , and with gravity turned off, i.e., the constant of gravitation is set to zero. The action of the system, if we consider terms depending only on the Maxwell field, can be calculated to give the free energy as

$$F_{\text{shell}} = \frac{Q^2}{2r_{\text{shell}}^{d-3}} \left(1 - \frac{r_{\text{shell}}^{d-3}}{R^{d-3}} \right). \quad (6.74)$$

Thus, for a given r_{shell} , one has that F_{shell} has a given constant fixed value. There are two limits that we can mention. One limit is when r_{shell} is very small. One could see this limit as an electrically charged central point surrounded by hot flat space, where quantum fluctuations of the hot flat space generate electric charge. But this seems to lead to a divergent free energy. Note that the behavior mentioned for very small r_{shell} contrasts with the grand canonical ensemble case [2], where $r_{\text{shell}} = 0$ corresponds to a zero grand potential. The other limit is when $r_{\text{shell}} = R$ and so the free energy is zero. This means that all the charge is infinitesimally near the boundary of the cavity, i.e., it is at the boundary of the cavity itself and there is hot flat space inside the cavity. Thus, the more interesting limit is the latter

one, when $r_{\text{shell}} = R$, and the charge is gathered near the boundary of the cavity giving $F_{\text{shell}} = 0$. Since, in this case, the shell emulates hot flat space with electric charge at the boundary, one has $F_{\text{shell}} = F_{\text{hfs}} = 0$. Nevertheless, it is interesting to compare the toy model of a shell with free energy F_{shell} given in Eq. (6.74) for several $\frac{r_{\text{shell}}}{R}$, and in particular for $\frac{r_{\text{shell}}}{R} = 1$, with the black hole free energy F_{bh} given in Eq. (6.73).

One could further think in building an equivalent system with the constant of gravitation turned on, such as an electrically charged self-gravitating shell close to the boundary of the cavity. Still, it is unclear if there is a possible conversion of this system to a charged black hole, and vice versa, since the two systems correspond to different topologies and also to a different action, as here we do not consider the matter sector.

6.5.3 First and second order phase transitions

We are interested in studying the favorable states of the ensemble, i.e., of an ensemble of a cavity with fixed radius R , fixed temperature T , and fixed electric charge Q , all values of these quantities being set by the reservoir.

A thermodynamic system tends to be in a state in which its thermodynamic potential, associated to the ensemble considered, has the lowest value. In this case, the thermodynamic potential is the Helmholtz free energy F , and so a state is favored relatively to another if it has lower F for given R , T , and Q . If a system is in a stable state but with a higher free energy F than another stable state, it is probable that the system undergoes a conversion, i.e., a phase transition, to the stable state with the lowest free energy. Indeed, in the calculation of the partition function by the path integral approach, if there are two stable configurations, i.e., two states that minimize the action, then the largest contribution to the partition function is given by the configuration with the lowest action or, in thermodynamic language, with the lowest free energy. This type of phase transitions are first order since the free energy is continuous, but the first derivatives are discontinuous.

In the case of the canonical ensemble of an electrically charged black hole inside a cavity in d dimensions, we must compare the free energy between all the stable black hole solutions of the ensemble, i.e., we have to compute F_{bh} given in Eq. (6.73), for the possible solution $r_+(R, T, Q)$. For any d in this ensemble one can have three solutions for the same temperature, two of them are stable. The stable black hole with lowest F_{bh} is the one that is favored. This means that considering only the two stable black hole solutions, one would then have a first order phase transition from r_{+1} to r_{+3} , for the electric charge parameter in the range $0 < \frac{\mu Q^2}{R^{2d-6}} < \frac{\mu Q_s^2}{R^{2d-6}}$, and in the limit of the charge parameter with value $\frac{\mu Q^2}{R^{2d-6}} = \frac{\mu Q_s^2}{R^{2d-6}}$, this first order phase transition becomes a second order phase transition. It is also interesting to compare the black hole solutions with the non-gravitating electrically charged shell case for the same boundary data, which has free energy given in Eq. (6.74). As we argued above, this shell is useful in mimicking charged hot flat space inside the cavity. Depending on the value of the radius of the shell $\frac{r_{\text{shell}}}{R}$, this free energy can

go from infinity, when $\frac{r_{\text{shell}}}{R} = 0$, to zero, when $\frac{r_{\text{shell}}}{R} = 1$. In the case of $\frac{r_{\text{shell}}}{R} = 0$, the shell is never favored, while for $\frac{r_{\text{shell}}}{R} = 1$, i.e., the case of hot flat space with the electric charge at the boundary, there is a region in which it is favored. In order to proceed, it is essentially assumed a shell with $\frac{r_{\text{shell}}}{R} = 1$, so that $F_{\text{shell}} = F_{\text{hfs}} = 0$.

Another issue that should be raised in the connection to favorable states, although it does not come directly from the ensemble formalism and its thermodynamics, is that there is a black hole radius r_+ , more precisely, there is a ratio $\frac{r_+}{R}$, for which the thermodynamic energy contained within R is higher than the Buchdahl bound or, in this context, the generalized Buchdahl bound [129]. When this happens, that energy content should collapse into a black hole. In this situation there is no more favorable phase considerations, the unique phase is a black hole. Indeed, the generalized Buchdahl bound yields the maximum mass, or maximum energy, that can be enclosed in a d -dimensional cavity with electric charge Q , before the system shows up some kind of singularity. At the bound or above, the system most likely tends to gravitational collapse. Since the mass of a system is related to the gravitational radius, it also sets a bound on the ratio $\frac{r_+}{R}$. In this context, one should consider this bound as yielding, for a fixed R , the mass m , or the gravitational radius r_+ , above which the energy within the system is sufficiently large that the system cannot support itself gravitationally and collapses. We can now apply this concept to the case of interest here.

In the Schwarzschild black hole case in d dimensions it was found in [102], that the canonical ensemble yields $F_{\text{bh}} = 0$ when $\frac{r_+}{R}$ has the Buchdahl bound value, $(\frac{r_+}{R})_{\text{Buch}}$. Since R is fixed, one can write $(\frac{r_+}{R})_{\text{Buch}} \equiv \frac{r_{+\text{Buch}}}{R}$ to simplify the notation.

In a d -dimensional Schwarzschild spacetime one has $\frac{r_{+\text{Buch}}}{R} = \left(\frac{4(d-2)}{(d-1)^2}\right)^{\frac{1}{d-3}}$. One can infer that black hole solutions with higher $\frac{r_+}{R}$, i.e., higher temperatures RT , yield gravitational collapse. Since zero free energy in this electrically uncharged case, is also the free energy of hot flat space, $F_{\text{hfs}} = 0$, one sees that in the uncharged case one passes directly from a situation where a hot flat space phase is favored relatively to a black hole phase, to a situation where the phase is a phase where surely there is a black hole, not merely a phase in which the black hole is favored.

Now, in the canonical ensemble for a black hole with electric charge, one finds that for $F_{\text{bh}} = 0$ only the bigger black hole exists, and it gives a value for $\frac{r_+}{R}$ that is higher than the Buchdahl bound value. Thus, there is a definite F_{bh} value greater than zero where the Buchdahl value $\frac{r_{+\text{Buch}}}{R}$ is met. We found this by numerical means up to $d = 16$, but we have not found an analytical proof for all d . For this definite value of F_{bh} or lower values of it, the system has high enough temperature and high enough self thermodynamic energy to undergo gravitational collapse. When this happens there is no more coexistence of phases, there is only the black hole phase. Below the saddle, or critical, charge, i.e., below the electric charge parameter given by $\frac{\mu Q_s^2}{R^{2d-6}}$, it is the black hole solution $\frac{r_{+3}}{R}$ that achieves $\frac{r_{+\text{Buch}}}{R}$. Above the saddle charge, i.e., above $\frac{\mu Q_s^2}{R^{2d-6}}$, it is the black hole solution $\frac{r_{+4}}{R}$ that achieves $\frac{r_{+\text{Buch}}}{R}$. In contrast, if we considered the grand canonical ensemble with electric charge in Chapter 4 or in [2], rather than the canonical ensemble studied here, the point of interest would be $W_{\text{bh}} = 0$, where W_{bh} is the grand potential free

energy related to the grand canonical ensemble, and it corresponds to a $\frac{r_+}{R}$ which is lower than the Buchdahl bound value. In the grand canonical ensemble, there is only one stable black hole. So, this means that for $W_{\text{bh}} = 0$, the two phases black hole and hot flat space coexist equally. For $W_{\text{bh}} < 0$ up to some definite negative value, then the two phases, black hole and hot flat space, coexist but the black hole dominates. For the definite negative value of W_{bh} , the radius $\frac{r_+}{R}$ is the Buchdahl bound value $\frac{r_{\text{Buch}}}{R}$. For even lower W_{bh} , i.e., for higher temperature parameter RT , one has $\frac{r_+}{R}$ larger than $\frac{r_{\text{Buch}}}{R}$ and the system collapses, or is collapsed, there is thus no coexistence, only the black hole phase remains. Although numerically all three radii $\frac{r_+}{R}$, namely, the canonical zero free energy, the Buchdahl, and the grand canonical zero grand potential, are very close, see Sec. 6.7, it seems that a connection between the ensemble stability and the mechanical stability of matter is elusive here. A comment is in order. The Buchdahl bound applies to a self-gravitating mechanical system consisting of a ball of matter of radius R . The system here is a thermodynamic system, with boundary data, namely R , T , and Q , and contains no matter. One can argue that in higher orders of approximation, the system contains packets of energy and one can plausibly deduce that the system must collapse once the Buchdahl bound is surpassed. Be as it may, the inference made here comes from dynamics, not thermodynamics, and therefore is strictly outside the followed approach.

To better understand the issues and make progress, we have to pick up definite dimensions. We now specify the generic d -dimensional results to the dimensions $d = 4$ and $d = 5$, with a more thorough analysis for $d = 5$.

6.5.4 Full analysis in $d = 4$

For $d = 4$, as for any d , this ensemble can have either one or three black hole solutions for a given temperature. When there are three, two of them are stable and are of interest in the consideration of the most favorable phase, while the remaining solution is unstable and is of no interest in the consideration of the most favorable phase. The two that are stable have to be compared against one another to see which is the most favorable phase.

We can start by comparing the free energy of the several black hole solutions that exist in this ensemble between themselves. From Eq. (6.73), in $d = 4$, the black hole free energy is

$$F_{\text{bh}} = \frac{R}{l_p^2} \left(1 - \sqrt{f(R, Q, r_+)} \right) - T \frac{A_+}{4l_p^2}, \quad (6.75)$$

where here one has $\frac{A_+}{4} = \pi r_+^2$, $f(R, Q, r_+) \equiv 1 - \frac{r_+ + \frac{l_p^2 Q^2}{r_+}}{R} + \frac{l_p^2 Q^2}{R^2}$, where it was used $\mu = l_p^2$, and $r_+ = r_+(T, R, Q)$. In $d = 4$, the saddle electric charge parameter value $\frac{l_p^2 Q_s^2}{R^2} = (\sqrt{5} - 2)^2 = 0.056$, the last equality being approximate, separates the region with only one solution from the region with three solutions.

There is a first set of general and specific comments that we must make, namely about the positivity of the free energy for each solution. For $0 \leq \frac{l_p^2 Q^2}{R^2} < \frac{Q_s^2}{R^2}$, the

stable black hole solution $\frac{r_{+1}}{R}$ has positive F_{bh} for all the temperatures in which the solution exists. The same happens for the solution $\frac{r_{+2}}{R}$, but this solution is of not interest here since it is unstable. The other stable black hole solution $\frac{r_{+3}}{R}$ has a temperature $T_{F_{\text{bh}}=0}$ depending on the electric charge, at which the free energy becomes zero, and so the black hole solution $\frac{r_{+3}}{R}$ can have F_{bh} positive or negative. For the critical charge $\frac{l_p^2 Q^2}{R^2} = \frac{l_p^2 Q_s^2}{R^2}$, with $\frac{l_p^2 Q_s^2}{R^2} = 0.056$ approximately, the stable black hole solution $\frac{r_{+1}}{R}$ has positive free energy, the point $\frac{r_{+1}}{R} = \frac{r_{+2}}{R} = \frac{r_{+3}}{R}$ has positive free energy, and the stable black hole solution $\frac{r_{+3}}{R}$ has a temperature $T_{F_{\text{bh}}=0}$ at which the free energy becomes zero. For $\frac{l_p^2 Q^2}{R^2} < \frac{l_p^2 Q_s^2}{R^2} < 1$, the only black hole solution is $\frac{r_{+4}}{R}$, which is stable, and it has a temperature $T_{F_{\text{bh}}=0}$ depending on the electric charge, at which the free energy becomes zero. So, the free energy of $\frac{r_{+4}}{R}$ can be positive or negative. Quite generally one can calculate $T_{F_{\text{bh}}=0}$ by solving $F_{\text{bh}} = 0$, with F_{bh} given in Eq. (6.75), for either the solution $\frac{r_{+3}}{R}$ or $\frac{r_{+4}}{R}$. The free energy can be written in terms of m and Q through Eq. (6.33) in $d = 4$ and through $2l_p^2 m = r_+ + \frac{l_p^2 Q^2}{r_+}$, allowing us to reduce $F_{\text{bh}} = 0$ into a quartic equation for the mass, see Sec. 6.7. The solutions have positive free energy for temperatures lower than $T_{F_{\text{bh}}=0}$, and the solutions have negative free energy for temperatures higher than $T_{F_{\text{bh}}=0}$.

There is a second set of general and specific comments that we must make, namely about the favorability between black hole solutions. For $0 \leq \frac{l_p^2 Q^2}{R^2} < \frac{l_p^2 Q_s^2}{R^2}$, there is a favorability temperature T_f which depends on the electric charge, and at which the solutions $\frac{r_{+1}}{R}$ and $\frac{r_{+3}}{R}$ have the same free energy. For temperatures lower than T_f , the solution r_{+1} is more favorable than $\frac{r_{+3}}{R}$, or it is the only existing solution. For temperatures higher than T_f , the solution $\frac{r_{+3}}{R}$ is more favorable than $\frac{r_{+1}}{R}$, or it is the only existing solution. For the critical charge $\frac{l_p^2 Q^2}{R^2} = \frac{l_p^2 Q_s^2}{R^2}$, the temperature T_f is the temperature at which $\frac{r_{+1}}{R} = \frac{r_{+2}}{R} = \frac{r_{+3}}{R}$ and all have the same free energy, i.e., the stable solutions $\frac{r_{+1}}{R}$ and $\frac{r_{+3}}{R}$ coexist. For $\frac{l_p^2 Q_s^2}{R^2} < \frac{l_p^2 Q^2}{R^2} < 1$, there is only one black hole solution, it is $\frac{r_{+4}}{R}$, and, since it is stable, it is favored. One can now consider phase transitions between the two stable black hole solutions. There is a first order phase transition from r_{+1} to r_{+3} , for the electric charge parameter in the range $0 < \frac{l_p^2 Q^2}{R^2} < \frac{l_p^2 Q_s^2}{R^2}$ and, additionally, in the limit of the electric charge parameter with value $\frac{l_p^2 Q^2}{R^2} = \frac{l_p^2 Q_s^2}{R^2}$, this first order phase transition turns into a second order phase transition.

We now compare, in $d = 4$, the black hole phases just discussed above with hot flat space phase, which we emulate here by a nonself-gravitating shell. In $d = 4$, the free energy of the shell is

$$F_{\text{shell}} = \frac{Q^2}{2r_{\text{shell}}} \left(1 - \frac{r_{\text{shell}}}{R} \right), \quad (6.76)$$

where r_{shell} is the radius of the shell, see Eq. (6.74). So F_{shell} depends on the electric charge Q , on r_{shell} , and on R , but is a constant as a function of the temperature T . The case of a very small shell will lead to a very high free energy due to the dependence on $\frac{Q}{r_{\text{shell}}}$, and therefore, for this case the region of favorability for the shell lies in very small values of the charge. There are also the cases of intermediate

shell radius which would have to be analyzed specifically. The other limiting case is when the charge is near the boundary of the cavity, with the free energy of this case tending to zero. Ultimately, the black hole is favored when $F_{\text{bh}} < F_{\text{shell}}$, both coexist equally when $F_{\text{bh}} = F_{\text{shell}}$, and the black hole is not favored when $F_{\text{bh}} > F_{\text{shell}}$. When the radius of the shell is at the cavity radius, $\frac{r_{\text{shell}}}{R} = 1$, then the shell has zero free energy and emulates hot flat space with electric charge at the boundary. Then, the free energy of hot flat space is $F_{\text{shell}} = F_{\text{hfs}} = 0$. The black hole is not favored when $F_{\text{bh}} > 0$, both the black hole and hot flat space coexist equally when $F_{\text{bh}} = 0$, and the black hole is favored when $F_{\text{bh}} < 0$. When the system finds itself in a phase that is not favored, it will make a first order phase transition to the favored phase.

The problem of the thermodynamic phases is even more complicated as we have mentioned already. When there is no electric charge, i.e., for the Schwarzschild space in $d = 4$, it was found in [102] that, in the canonical ensemble, the condition $F_{\text{bh}} = 0$ yields a value for $\frac{r_{\pm}}{R}$ that is equal to the generalized Buchdahl bound [129], i.e., the limiting value $(\frac{r_{\pm}}{R})_{\text{Buch}}$ for gravitational collapse of a self-gravitating system of energy E and radius R . Since R is fixed in the ensemble, one can write $(\frac{r_{\pm}}{R})_{\text{Buch}} \equiv \frac{r_{\pm\text{Buch}}}{R}$ to simplify the notation, and in $d = 4$ one has $\frac{r_{\pm\text{Buch}}}{R} = \frac{8}{9} = 0.89$, the latter equality being approximate. This result means that, in the uncharged case, as soon as the black hole phase is favored, there is no further coexistence with hot flat space, and the system collapses. For nonzero electric charge there is no more coincidence. Here, to discuss this issue of favorability between black hole and hot flat space, we consider the case for which the free energy of the shell is zero, $F_{\text{shell}} = 0$, i.e., the case of hot flat space with electric charge at the boundary, $\frac{r_{\text{shell}}}{R} = 1$. In this case, the shell is situated at the cavity, and so F_{shell} is the free energy of hot flat space, F_{hfs} , which is zero. For nonzero electric charge Q , i.e., nonzero charge parameter $\frac{l_p^2 Q^2}{R^2}$, we find that in the canonical ensemble, the condition $F_{\text{bh}} = 0$ yields a $\frac{r_{\pm}}{R}$ value, both for $\frac{r_{+3}}{R}$ and $\frac{r_{+4}}{R}$, that is higher than the generalized Buchdahl bound. Notice that the generalized Buchdahl bound here is the limiting value of $\frac{r_{\pm}}{R}$ for gravitational collapse of a self-gravitating system of energy E , electric charge Q , and radius R . For an electric charge parameter lower or equal than the saddle value $\frac{l_p^2 Q_s^2}{R^2}$, only the solution $\frac{r_{+3}}{R}$ can take the value of the Buchdahl bound, corresponding to a positive free energy and some temperature value RT . For a system with this RT or higher, then the system collapses gravitationally into a black hole with the corresponding $\frac{r_{+3}}{R}$. For an electric charge higher or equal than the saddle value $\frac{l_p^2 Q_s^2}{R^2}$, the solution $\frac{r_{+4}}{R}$ can take the value of the Buchdahl bound, having a definite positive value of F_{bh} , at some temperature parameter RT . For a system with this RT or higher, the system again collapses gravitationally into a black hole with the corresponding $\frac{r_{+4}}{R}$. Interesting to note that in the grand canonical ensemble, where there is only one stable black hole solution, the equation $W_{\text{bh}} = 0$, W_{bh} denoting the grand potential, yields a $\frac{r_{\pm}}{R}$ value that is lower than the Buchdahl bound. Thus, in this case, when $W_{\text{bh}} = 0$ for the system, the two phases coexist, black hole and hot flat space. For $W_{\text{bh}} < 0$, the black hole phase dominates in relation to hot flat space. And for a certain definite negative value of W_{bh} , the value of $\frac{r_{\pm}}{R}$ of the system is the same as the value of the Buchdahl bound. From then on the system collapses, the only

phase being the black hole phase, and there is no coexistence of phases, see also Sec. 6.7.

6.5.5 Full analysis in $d = 5$

For $d = 5$, as for any d , this ensemble has between one and three black hole solutions for a given temperature. When there are three solutions, two of them are stable and are going to be considered here, while the remaining is unstable and is of no interest in this analysis. The two that are stable have to be compared against one another to see which is the most favorable phase.

We can start by comparing the free energy of the several black hole solutions that exist in this ensemble between themselves. In $d = 5$, the black hole free energy is

$$F_{\text{bh}} = \frac{R^2}{\mu} \left(1 - \sqrt{f(R, Q, r_+)} \right) - T \frac{A_+}{4l_p^3}, \quad (6.77)$$

where here $\frac{A_+}{4} = \frac{\pi^2 r_+^3}{2}$, $f(R, Q, r_+) \equiv 1 - \frac{r_+^2 + \frac{\mu Q^2}{r_+^2}}{R^2} + \frac{\mu Q^2}{R^4}$, $\mu = \frac{4l_p^3}{3\pi}$, and $r_+ = r_+(T, R, Q)$. To help in the analysis, the free action F_{bh} is plotted in Fig. 6.5 as a function of the temperature parameter RT , for fixed electric charge parameter $\frac{\mu Q^2}{R^4}$ in $d = 5$. Recall that in $d = 5$, one has the saddle electric charge parameter value $\frac{\mu Q_s^2}{R^4} = \frac{(68-27\sqrt{6})^2}{250} = 0.014$, the last equality being approximate.

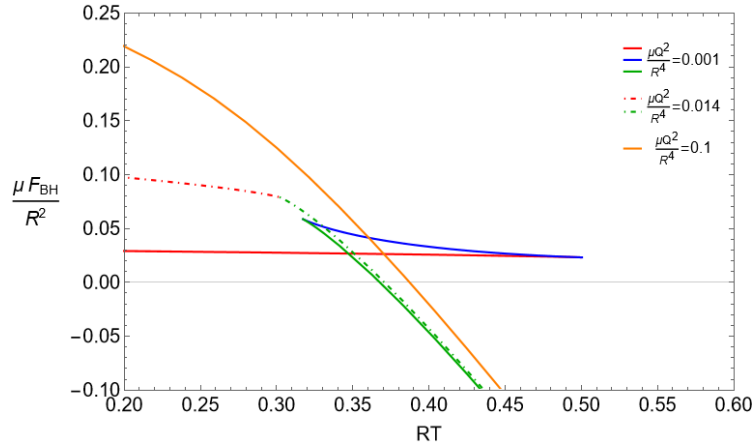


Figure 6.5: Free energy F_{bh} of the charged black hole solutions of the canonical ensemble in $d = 5$, given as a quantity with no units $\frac{\mu F_{\text{bh}}}{R^2}$, as a function of the temperature parameter RT for several electric charge parameters $\frac{\mu Q^2}{R^4}$, where $\mu = \frac{4l_p^3}{3\pi}$. For $\frac{\mu Q^2}{R^4} = 0.001$, the solution r_{+1} is in red, the solution r_{+2} is in blue, and the solution r_{+3} is in green, all of them in solid lines. For $\frac{\mu Q^2}{R^4} = \frac{(68-27\sqrt{6})^2}{250} = 0.014$, the latter equality being approximate, the solution r_{+1} is in red and the solution r_{+3} is in green, all of them in dashed lines. For $\frac{\mu Q^2}{R^4} = 0.1$, the solution r_{+4} is in orange, in solid line.

We can make a first set of general and specific comments directly from Fig. 6.5, regarding the positivity of the free energy for each solution. For relatively low electric charge parameter $0 \leq \frac{\mu Q^2}{R^4} < \frac{\mu Q_s^2}{R^4}$, where $\mu = \frac{4l_p^3}{3\pi}$ in $d = 5$, the solution $\frac{r_{+1}}{R}$ has positive F_{bh} for all the temperatures in which the solution exists. The same happens for the solution $\frac{r_{+2}}{R}$, but this solution is of no interest here since it is unstable. The solution $\frac{r_{+3}}{R}$ has a temperature $T_{F_{bh}=0}$ depending on the electric charge at which the free energy becomes zero, and so $\frac{r_{+3}}{R}$ can have F_{bh} positive or negative. In the figure, this range of the electric charge parameter is represented by the case $\frac{\mu Q^2}{R^4} = 0.001$. For $\frac{\mu Q^2}{R^4} = 0.001$, one has for the $\frac{r_{+3}}{R}$ solution that $T_{F_{bh}=0} = 0.367$ approximately. For the saddle charge $\frac{\mu Q^2}{R^4} = \frac{\mu Q_s^2}{R^4}$, with $\frac{\mu Q_s^2}{R^4} = 0.014$ approximately, the solution $\frac{r_{+1}}{R}$ is positive, the point $\frac{r_{+1}}{R} = \frac{r_{+2}}{R} = \frac{r_{+3}}{R}$ is positive, and the solution $\frac{r_{+3}}{R}$ has a temperature $T_{F_{bh}=0} = 0.37$ at which the free energy becomes zero. For relatively high electric charge parameter $\frac{\mu Q_s^2}{R^4} < \frac{\mu Q^2}{R^4} < 1$, the only solution is $\frac{r_{+4}}{R}$ and it has a temperature $T_{F_{bh}=0}$ depending on the electric charge. So F_{bh} of the black hole $\frac{r_{+4}}{R}$ can be positive or negative. In the figure, this range of $\frac{\mu Q^2}{R^4}$ is represented by the case $\frac{\mu Q^2}{R^4} = 0.1$. For $\frac{\mu Q^2}{R^4} = 0.1$, one has that the solution $\frac{r_{+4}}{R}$ has $T_{F_{bh}=0} = 0.387$ approximately. Quite generally, one can calculate $T_{F_{bh}=0}$ by solving $F_{bh} = 0$, with F_{bh} given in Eq. (6.77) for either the solution $\frac{r_{+3}}{R}$ or $\frac{r_{+4}}{R}$. One obtains a quartic equation for the mass $2\mu m = r_+^2 + \frac{\mu Q^2}{r_+^2}$, with here $\mu = \frac{3}{4\pi}$, as a function of the electric charge, see Sec. 6.7. For temperatures lower than $T_{F_{bh}=0}$, the solutions have positive free energy and for temperatures higher than $T_{F_{bh}=0}$, the solutions have negative free energy.

We can make a second set of general and specific comments directly from Fig. 6.5, regarding the favorability between black hole solutions. For a range of low electric charge parameter $0 \leq \frac{\mu Q^2}{R^4} < \frac{\mu Q_s^2}{R^4}$, the solutions $\frac{r_{+1}}{R}$ and $\frac{r_{+3}}{R}$ have the same free energy at a specific temperature T_f , i.e., the phase favorability temperature which depends on $\frac{\mu Q^2}{R^4}$. For temperatures lower than T_f , the solution $\frac{r_{+1}}{R}$ either has lower free energy than $\frac{r_{+3}}{R}$ or it is the only existing solution, and so $\frac{r_{+1}}{R}$ is more favorable. For a temperature equal to T_f , the solutions $\frac{r_{+1}}{R}$ and $\frac{r_{+3}}{R}$ have the same free energy and they are equally favorable, meaning they coexist equally. For temperatures higher than T_f , the solution $\frac{r_{+3}}{R}$ either has lower free energy than $\frac{r_{+1}}{R}$ or it is the only existing solution, and so $\frac{r_{+3}}{R}$ is more favorable. This is represented for $\frac{\mu Q^2}{R^4} = 0.001$ in the figure. One can see that in this case, the favorability temperature is $RT_f = 0.347$ approximately. Also, for $RT < 0.32$, there is only the $\frac{r_{+1}}{R}$ solution, whereas for $RT > 0.50$ there is only the $\frac{r_{+3}}{R}$ solution. The solution $\frac{r_{+2}}{R}$ is unstable and does not enter in this analysis, however it is plotted in the figure to show a continuity of the free energy on the three solutions. For saddle charge $\frac{\mu Q^2}{R^4} = \frac{\mu Q_s^2}{R^4} = 0.014$, the latter equality being approximate, which is shown in the figure, the temperature $T_f = 0.30$, approximately, is the temperature at which $\frac{r_{+1}}{R} = \frac{r_{+2}}{R} = \frac{r_{+3}}{R}$, and all have the same free energy, i.e., $\frac{r_{+1}}{R}$ and $\frac{r_{+3}}{R}$ coexist. For temperatures lower than T_f , the solution $\frac{r_{+1}}{R}$ is the only existing solution. For temperatures higher than T_f , the solution $\frac{r_{+3}}{R}$ is the only existing solution. For the higher values of the electric

charge parameter, i.e., for $\frac{\mu Q_s^2}{R^4} < \frac{\mu Q^2}{R^4} < 1$, there is only one black hole solution $\frac{r_{+4}}{R}$ that is stable, and so it is favorable. This is represented in the figure by the case $\frac{\mu Q^2}{R^4} = 0.1$. We can now consider phase transitions between the two stable black hole solutions. One has a first order phase transition from r_{+1} to r_{+3} , for the electric charge parameter in the range $0 < \frac{\mu Q^2}{R^4} < \frac{\mu Q_s^2}{R^4}$. Moreover, in the limit of the electric charge parameter given by the value $\frac{\mu Q^2}{R^4} = \frac{\mu Q_s^2}{R^4}$, this first order phase transition becomes a second order phase transition. This can be seen from Fig. 6.5, since the intersection point represents a first order phase transition, and at the limit of the critical charge, this point represents a second order phase transition.

The black hole phases discussed just above with hot flat space phase, which it is emulated by a nonself-gravitating shell, are now compared for $d = 5$, see Fig. 6.6. The favorable states for each electric charge and temperature, and for various values

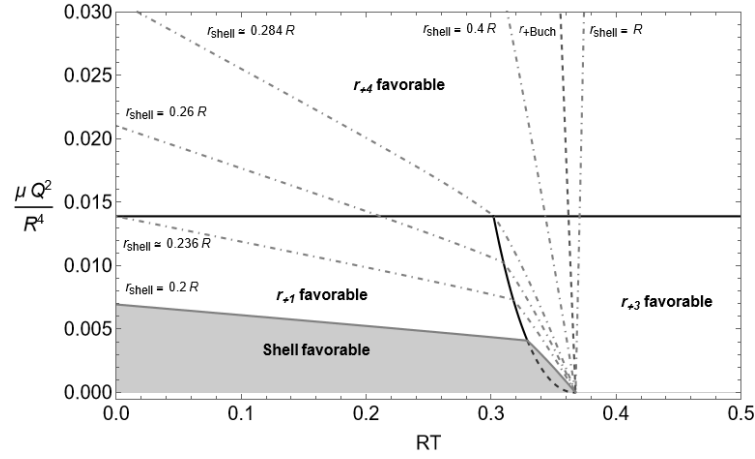


Figure 6.6: Favorable states of the canonical ensemble of an electrically charged black hole inside a cavity in $d = 5$ in an electric charge Q times temperature T , more precisely, $\frac{\mu Q^2}{R^4} \times RT$ plot. It is displayed the region where the black hole r_{+1} is a favorable phase, the region where the black hole r_{+3} is a favorable phase, and the region where the black hole r_{+4} is a favorable phase. The delimiters of the favorable regions of the black hole solutions are the black lines, including the dashed line. It is also incorporated the solution for a nongravitating electrically charged shell as a simulator for hot flat space. The electrically charged shell with $\frac{r_{\text{shell}}}{R} = 0$ is never favored. The electrically charged shell with $\frac{r_{\text{shell}}}{R} = 0.2$ is favored in the region in gray, this case is given as an example. The upper delimiter of the region of favorability of electrically charged shells with $\frac{r_{\text{shell}}}{R} = 0.236$ approximately, $\frac{r_{\text{shell}}}{R} = 0.26$, $\frac{r_{\text{shell}}}{R} = 0.284$ approximately, $\frac{r_{\text{shell}}}{R} = 0.4$ and $\frac{r_{\text{shell}}}{R} = 1$, which better simulates hot flat space, are given by the dot-dashed lines. The Buchdahl condition line, i.e., $r_{+\text{Buch}}$, above which there is presumably collapse is given by a thick black dash line.

of the shell radius can be seen in the figure. The free energy of the shell for the case $d = 5$ is

$$F_{\text{shell}} = \frac{Q^2}{2r_{\text{shell}}^2} \left(1 - \frac{r_{\text{shell}}^2}{R^2} \right), \quad (6.78)$$

where r_{shell} is the radius of the shell, see Eq. (6.74). So the shell free energy F_{shell} has a dependence on electric charge Q , on r_{shell} , and on R , but as a function of the temperature T , the free energy is a constant. Due to the term $\frac{Q^2}{r_{\text{shell}}^2}$, the free energy becomes divergent for a very small shell and fixed electric charge. Therefore, the region of favorability for the very small shell lies in very small values of the electric charge Q . There are the cases of intermediate shell radius that are represented in the figure, namely the cases $\frac{r_{\text{shell}}}{R} = 0.2, 0.236, 0.26, 0.284, 0.4$, with 0.236 and 0.284 being approximate values. The more interesting limiting case is when the electric charge is near or at the boundary of the cavity, $\frac{r_{\text{shell}}}{R} = 1$. The free energy of the shell in this limit is zero. The black hole solution is favored compared to the shell when $F_{\text{bh}} < F_{\text{shell}}$, while both the black hole and the shell coexist equally when $F_{\text{bh}} = F_{\text{shell}}$, and the black hole is not favored compared to the shell when $F_{\text{bh}} > F_{\text{shell}}$. The gray dashed curves in the figure represent the condition $F_{\text{bh}} = F_{\text{shell}}$ for each shell radius, delimiting the regions where the black hole is favorable, for higher temperature, and where the shell is favorable, for lower temperature. When the radius of the shell is at the cavity radius, $\frac{r_{\text{shell}}}{R} = 1$, the free energy of the shell becomes zero, emulating hot flat space with free energy $F_{\text{shell}} = F_{\text{hfs}} = 0$. This is the case of hot flat space with electric charge at the boundary. Again, the black hole is not favored compared to hot flat space when $F_{\text{bh}} > 0$, while both the black hole and hot flat space coexist equally when $F_{\text{bh}} = 0$, and the black hole is favored compared to hot flat space when $F_{\text{bh}} < 0$. The gray dashed curve $r_{\text{shell}} = R$ in the figure corresponds to the boundary of the regions of favorability $F_{\text{bh}} = 0$, and for higher temperature, the black hole is favorable, while for lower temperature, hot flat space is favorable. If for some reason the system is in a not favored phase, then a first order phase transition occurs to a favored phase.

The problem of the thermodynamic phases is more involved as mentioned already above. When there is no electric charge, one has Schwarzschild space in $d = 5$. It was found in [101, 102] that, in the canonical ensemble of Schwarzschild space in $d = 5$, the condition $F_{\text{bh}} = 0$ corresponds to a value for $\frac{r_{\pm}}{R}$ that is equal to the generalized Buchdahl bound radius [129], which is the value $(\frac{r_{\pm}}{R})_{\text{Buch}}$ for gravitational collapse of a self-gravitating system of energy E and radius R . Since R is maintained fixed, it is defined $(\frac{r_{\pm}}{R})_{\text{Buch}} \equiv \frac{r_{\pm\text{Buch}}}{R}$, and in $d = 5$, one has $\frac{r_{\pm\text{Buch}}}{R} = \frac{\sqrt{3}}{2} = 0.86$, the latter equality being approximate. Since for $Q = 0$, the free energy of hot flat space is zero, $F_{\text{hfs}} = 0$, meaning that there is no further coexistence with hot flat space as soon as the black hole phase is favored, because the system tends to collapse. For nonzero electric charge parameter $\frac{\mu Q^2}{R^4}$ there is no coincidence. To compare the free energies, one considers the case in which the shell has radius equal to the cavity radius, $\frac{r_{\text{shell}}}{R} = 1$, and so $F_{\text{shell}} = 0$, meaning that the shell is a surrogate for hot flat space, i.e., $F_{\text{shell}} = F_{\text{hfs}} = 0$, indeed it is hot flat space with electric charge at the boundary. For nonzero $\frac{\mu Q^2}{R^4}$, one finds that in the canonical ensemble $F_{\text{bh}} = 0$ results in a $\frac{r_{\pm}}{R}$ value, both for $\frac{r_{+3}}{R}$ and $\frac{r_{+4}}{R}$, that is higher than the generalized Buchdahl bound, which is the value of $\frac{r_{\pm}}{R}$ for gravitational collapse of a self-gravitating system of energy E , electric charge Q , and radius R , see Fig. 6.7. For an electric charge parameter lower or equal than the saddle value

$\frac{\mu Q_s^2}{R^4}$, there is a temperature RT at which the solution $\frac{r_{+3}}{R}$ can assume the value of the Buchdahl bound, corresponding to a positive free energy lower than the free energy of $\frac{r_{+1}}{R}$. For a system with this RT or higher, the system must suffer gravitational collapse into a black hole with the corresponding $\frac{r_{+3}}{R}$. For an electric charge higher than the saddle value y_s , there is again a temperature RT at which $\frac{r_{+4}}{R}$ assumes the Buchdahl bound, with positive value of F_{bh} . For a system with this RT or higher, then the system must collapse gravitationally into a black hole with the corresponding $\frac{r_{+4}}{R}$. We note that the picture in the grand canonical ensemble is

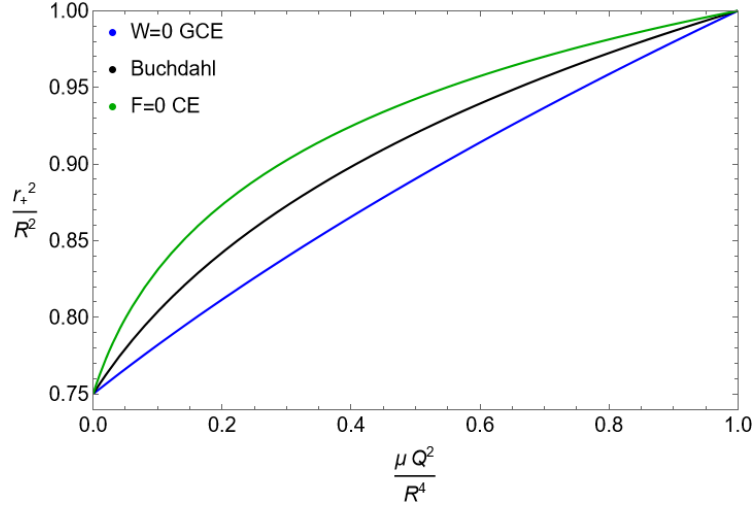


Figure 6.7: Ratio $\frac{r_+^2}{R^2}$ in terms of the electric charge parameter $\frac{\mu Q^2}{R^4}$, $\mu = \frac{4}{3\pi}$, for $d = 5$ for three different cases: given by the condition $F_{\text{bh}} = 0$ in the canonical ensemble in green, representing the stable solution $\frac{r_{+3}}{R}$; given by the condition $W_{\text{bh}} = 0$ in the grand canonical ensemble in blue, representing the only stable solution; and given by generalized Buchdahl condition in black.

different, as the equation $W_{\text{bh}} = 0$, with W_{bh} denoting the grand potential, results in a $\frac{r_+}{R}$ value for the single stable black hole, that is lower than the generalized Buchdahl bound. One has thermodynamically that when the system has $W_{\text{bh}} = 0$ the black hole phase and hot flat space phase coexist, for $W_{\text{bh}} < 0$ the black hole phase dominates, and for a certain definite negative value of W_{bh} the value of $\frac{r_+}{R}$ of the system is the same as the value of the Buchdahl bound. For larger temperatures, therefore the system must collapse gravitationally. The only phase of the system is the black hole phase and so there is no more coexistence, see Fig. 6.7 and Sec. 6.7.

6.6 THE CANONICAL ENSEMBLE IN THE LIMIT OF INFINITE CAVITY RADIUS: THE DAVIES LIMIT AND THE RINDLER LIMIT

6.6.1 Ensemble solutions in the $R \rightarrow +\infty$ limit: the Davies solutions and the Rindler solution

Here, we analyze the infinite cavity radius limit, and we discuss each solution that arises from this limit. The importance of this limit is that it allows us to connect the

York formalism with ensembles treated in [67] and to the Davies thermodynamic theory [51]. By performing $R \rightarrow +\infty$ limit while keeping T fixed and Q fixed, three different solutions are found. One observes from Sec. 6.3.2, that there are three solutions for $r_+(R, T, Q)$ if $\frac{\mu Q^2}{R^{2d-6}} < \frac{\mu Q_s^2}{R^{2d-6}}$. By performing the $R \rightarrow +\infty$ limit, the term $\frac{\mu Q^2}{R^{2d-6}}$ approaches zero, and so the solutions of the ensemble in this limit should correspond to these three solutions under the $R \rightarrow +\infty$ limit. For the smallest and intermediate solutions, the limit $R \rightarrow +\infty$ must be performed by fixing T and Q , while doing $\frac{r_+}{R} \rightarrow 0$. For the largest solution, the limit $R \rightarrow +\infty$ must be performed by fixing T and Q , while doing $\frac{r_+}{R} \rightarrow 1$. The smallest and intermediate solutions correspond to Davies thermodynamic solutions, while the largest solution limit corresponds to the Rindler solution. These solution limits occur for any d . In Fig. 6.8, the behavior of the three solutions in $d = 5$ can be seen for a charge $\mu Q^2 = 0.005$, $\mu = \frac{4l_p^3}{3\pi}$, for two different R , $R = 5$ and $R = 100$, where the latter R gives an idea of the $R \rightarrow \infty$ limit. In this limit the scale R is lost, the scales set by the electric charge Q and temperature T at infinity are now the only two scales of the canonical ensemble. We now briefly describe each solution.

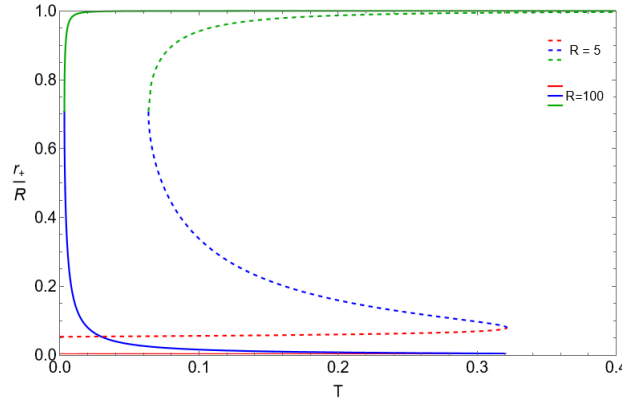


Figure 6.8: Plot of the solutions r_{+1} in red, r_{+2} in blue and r_{+3} in green of the canonical ensemble in $d = 5$ as $\frac{r_+}{R}$ as a function of T in Planck units, for $\mu Q^2 = 0.005$, $\mu = \frac{4l_p^3}{3\pi}$, and for two values of R , $R = 5$ in dashed lines, and $R = 100$ in filled lines. One can see the emergence of the r_{+1} and r_{+2} solution limits corresponding to the Davies limit as they get closer to the $\frac{r_+}{R} = 0$ axis, and the r_{+3} solution limit corresponding to the Rindler limit as it gets closer to the $\frac{r_+}{R} = 1$ axis.

The Davies solution corresponds to the smallest and intermediate solution limits of the canonical ensemble when taking $R \rightarrow +\infty$, with fixed T and Q . Thus, these are the solutions of the electrically charged black hole in the canonical ensemble with reservoir at infinity. This can be seen directly from the expression of the temperature in Eq. (6.33). Since for these solutions the behavior is $\frac{r_+}{R} \rightarrow 0$, one can maintain r_+ finite during the limit $R \rightarrow +\infty$, thus obtaining the temperature formula $T = \frac{d-3}{4\pi} \frac{r_+^{d-3} - \frac{\mu Q^2}{r_+^{d-3}}}{r_+^{d-2}}$, which is obeyed by the smallest and intermediate solutions. This is precisely the Hawking temperature for the electrically charged

black hole. From Fig. 6.8 one sees that the two solutions tend to the axis $\frac{r_+}{R} = 0$ and seem to get overlapped, which is due to the vertical axis being $\frac{r_+}{R}$. If one regularizes the solutions through multiplying by R , one obtains the two solutions in d dimensions, which for $d = 4$ are the Davies thermodynamic solutions. Moreover, one can see that the solutions do not exist for all temperatures. This is because the two solutions only exist up to a critical temperature, the generalized Davies temperature, after which there are no solutions. In the case represented in Fig. 6.8 which is $d = 5$, the generalized Davies temperature, i.e., the temperature when $R \rightarrow \infty$, has the expression $T_s = \frac{4}{10\pi(\sqrt{5\mu Q^2})^{\frac{1}{2}}}$, and so for $\mu Q^2 = 0.005$ as in the figure it yields $T_s = 0.320$, with the last equality being approximate.

The Rindler solution is the largest solution limit that can be obtained from the ensemble by keeping T and Q fixed, while doing $R \rightarrow +\infty$ and $r_+ \rightarrow R$ in Eq. (6.33). In Fig. 6.8, this solution is the one that tends to $\frac{r_+}{R} = 1$. The temperature dependence on the charge goes with $\frac{\mu Q^2}{R^{d-3}r_+^{d-3}}$, therefore such dependence in the limit $r_+ \rightarrow R$ and $R \rightarrow +\infty$ disappears. This happens because the horizon radius of the black hole tends to infinity and any contributions given by the charge become negligible. The expression for the temperature is now the temperature of an electrically uncharged black hole $T = \frac{(d-3)}{4\pi r_+ \sqrt{1 - \frac{r_+^{d-3}}{R^{d-3}}}}$. Imposing that T is fixed and

finite leads to the condition that $r_+ \sqrt{1 - \frac{r_+^{d-3}}{R^{d-3}}}$ must tend to some constant when $R \rightarrow +\infty$ and $r_+ \rightarrow R$. One can show that in this limit the event horizon of the black hole reduces to the Rindler horizon and the cavity boundary is accelerated to yield the Unruh temperature T set by the reservoir.

We now perform the analysis in full detail for the smallest and intermediate solution limits arising from $R \rightarrow +\infty$, i.e., the Davies solution. These are relevant since the formalism in this limit yields the Davies' thermodynamic theory of black holes for $d = 4$. We also analyze the largest solution limit arising from $R \rightarrow +\infty$, i.e., the Rindler solution.

6.6.2 *Infinite cavity radius and Davies' thermodynamic theory of black holes: Canonical ensemble, thermodynamics, and stability of electrically charged black hole solutions in the $R \rightarrow +\infty$ limit*

6.6.2.1 *Action, solutions and stability for the infinite radius limit*

With the limit of infinite cavity radius for the small and intermediate solutions, the canonical ensemble becomes essentially defined by the temperature T and the electric charge Q at infinity. It is this $R \rightarrow +\infty$ limit that in four dimensions gives Davies results [51]. This means that Davies' thermodynamic theory of black holes, in this case of electrically charged black holes, can be seen within the canonical ensemble formalism. Here, we present the results for d dimensions in the $R \rightarrow +\infty$ limit, $d = 4$ being a particular case.

In the limit of infinite radius, we must do the analysis above with care, since the quantities above depend on the scale given by the cavity radius R . To proceed with this limit, we must start from the reduced action in Eq. (6.28) and perform the $R \rightarrow +\infty$ limit to obtain

$$I_* = \frac{\beta}{\mu} \left(\frac{r_+^{d-3}}{2} + \frac{\mu Q^2}{2r_+^{d-3}} \right) - \frac{\Omega_{d-2} r_+^{d-2}}{4l_p^{d-2}}. \quad (6.79)$$

The extrema of the action occurs when

$$\beta = \iota(r_+), \quad \iota(r_+) \equiv \frac{4\pi}{(d-3)} \frac{r_+^{d-2}}{r_+^{d-3} - \frac{\mu Q^2}{r_+^{d-3}}}. \quad (6.80)$$

This is the inverse Hawking temperature of the Reissner-Nordström black hole measured at infinity, i.e., performing the limit of infinite radius into Eq. (6.33).

To find the solutions of this canonical ensemble, one must invert Eq. (6.80) to get $r_+(\beta, Q)$, i.e., $r_+(T, Q)$. This can be done by solving the following equation

$$\left(\frac{(d-3)}{4\pi T} \right) (r_+^{2d-6} - \mu Q^2) - r_+^{2d-5} = 0, \quad (6.81)$$

which generally is not solvable analytically for generic d . However, we can perform some qualitative analysis of the solutions. The function $\iota(r_+)$ in Eq. (6.80) has a minimum at

$$r_{+s1} = \left(\sqrt{(2d-5)\mu Q} \right)^{\frac{1}{d-3}}, \quad (6.82)$$

which is a saddle point of the action for the black hole. This saddle point of the action of the black hole has the temperature

$$T_{s1} = \frac{(d-3)^2}{2\pi(2d-5)(\sqrt{(2d-5)\mu Q})^{\frac{1}{d-3}}}. \quad (6.83)$$

In $d = 4$, this T_{s1} is the Davies temperature, and so Eq. (6.83) is the generalization of Davies temperature for higher dimensions.

By inspection, one finds that for temperatures $T \leq T_{s1}$ there are two black holes, and for $T > T_{s1}$ there are no black hole solutions. Indeed, for temperatures in the interval $0 < T \leq T_{s1}$, there are two solutions, the solution $r_{+1}(T, Q)$ and the solution $r_{+2}(T, Q)$. The solution $r_{+1}(T, Q)$ is bounded in the interval $(\mu Q^2)^{\frac{1}{2d-6}} < r_{+1}(T, Q) \leq r_{+s1}$, where $r_{+1}(T \rightarrow 0, Q) = (\mu Q^2)^{\frac{1}{2d-6}} = r_{+e}$, r_{+e} being the radius of the extremal black hole, and $r_{+1}(T_{s1}, Q) = r_{+s1}$. Moreover, $r_{+1}(T, Q)$ is an increasing monotonic function in T . The solution $r_{+2}(T, Q)$ is bounded from below, i.e., $r_{+2}(T, Q) > r_{+s1}$, where $r_{+2}(T_{s1}, Q) = r_{+s1}$, and is unbounded from above, since at $T \rightarrow 0$, the solution r_{+2} tends to infinity. Moreover, $r_{+2}(T, Q)$ is a decreasing monotonic function in T . The action given in Eq. (6.79) with r_+ holding for $r_{+1}(T, Q)$ or $r_{+2}(T, Q)$ is the action in zero loop approximation that has been found in [3]

directly from the Gibbons-Hawking approach, rather than from York's approach for a given R with subsequently taking the $R \rightarrow \infty$ limit, as it is being done here.

Regarding stability, a solution is stable if $\frac{\partial I(r_+)}{\partial r_+} < 0$, as it was seen in the case of finite cavity. This gives

$$r_+ \leq r_{+s1}, \quad (6.84)$$

with r_{+s1} given in Eq. (6.82). This means that the solution is stable if the radius r_+ increases as the temperature increases. Therefore, the solution r_{+1} is stable since it has this monotonic behavior, while the solution r_{+2} is unstable since it has an opposite monotonic behavior.

6.6.2.2 Thermodynamics in the $R \rightarrow +\infty$ limit

With the solutions of the canonical ensemble found in the limit of infinite radius of the cavity, $R \rightarrow +\infty$, we can obtain the thermodynamics from I_0 , i.e., the action in the zero loop approximation given in Eq. (6.79) evaluated at the extrema of Eq. (6.80). The thermodynamics for the system follows through the correspondence $F = TI_0$, where F again is the Helmholtz free energy of the system and thus it can be written for this case as

$$F = \frac{1}{\mu} \left(\frac{r_+^{d-3}}{2} + \frac{\mu Q^2}{2r_+^{d-3}} \right) - \frac{T\Omega_{d-2}r_+^{d-2}}{4l_p^{d-2}}, \quad (6.85)$$

where r_+ can be $r_{+1}(T, Q)$ or $r_{+2}(T, Q)$. Using the same calculation method from Sec. 6.4.1, we have that the entropy is

$$S = \frac{A_+}{4l_p^{d-2}}. \quad (6.86)$$

Additionally, the thermodynamic pressure p is zero,

$$p = 0. \quad (6.87)$$

And also, the thermodynamic electric potential is

$$\phi = \frac{Q}{r_+^{d-3}}, \quad (6.88)$$

which is equal to the pure electric potential. The energy, given by $E = F + TS$, can be written as $E = \frac{r_+^{d-3}}{2\mu} + \frac{Q^2}{2r_+^{d-3}}$. But the spacetime mass m is given by $m = \frac{r_+^{d-3}}{2\mu} + \frac{Q^2}{2r_+^{d-3}}$, see also Sec. 6.7, so that the thermodynamic energy and the spacetime mass are the same in the $R \rightarrow +\infty$ limit, i.e.,

$$E = m. \quad (6.89)$$

Thus, one can write the free energy given in Eq. (6.85) as

$$F = m - TS. \quad (6.90)$$

We must note that the expressions for the entropy, the pressure, the thermodynamic electric potential, and the energy are consistent with the limit of infinite radius to the respective expressions in Sec. 6.4.1. Moreover, in this limit, the pressure p vanishes, which is consistent with the absence of the variable R in the action.

The energy in Eq. (6.89) can be rewritten in terms of the entropy and the charge as

$$E = \frac{1}{2\mu} \left(\frac{4Sl_p^{d-2}}{\Omega_{d-2}} \right)^{\frac{d-3}{d-2}} + \frac{\mu Q^2}{2} \left(\frac{4Sl_p^{d-2}}{\Omega_{d-2}} \right)^{\frac{3-d}{d-2}}. \quad (6.91)$$

The energy function possesses the scaling property $\nu^{\frac{d-3}{d-2}} E = E(\nu S, \nu^{\frac{d-3}{d-2}} Q)$, which allows the use of the Euler relation theorem to have $E = \frac{d-3}{d-2} TS + \phi Q$, which is the formula obtained in Sec. 6.4.1 without the term pA . Indeed, the term pA in the limit of infinite reservoir radius has leading order $R^{-(d-3)}$, and so it vanishes. Since from Eq. (6.89) $E = m$, one obtains

$$m = \frac{d-3}{d-2} TS + \phi Q, \quad (6.92)$$

which is the Smarr formula.

In this case the law

$$dm = TdS + \phi dQ, \quad (6.93)$$

holds. This is exactly the first law of black hole mechanics. This can be obtained from Eq. (6.57) in the $R \rightarrow \infty$ limit. For R finite, there is a first law of thermodynamics of the cavity and does not correspond to the law of black hole mechanics. For $R \rightarrow \infty$, the first law of black hole thermodynamics and the first law of black hole mechanics coincide into one same law, which is quite remarkable. Moreover, in the electrically charged case, as opposed to the Schwarzschild case, the thermodynamics of the canonical ensemble is valid, since there is a region of the electric charge where the system is thermodynamically stable. It is from Eq. (6.93) that Davies has started his thermodynamic theory of black holes for $d = 4$. Here, we deduced Eq. (6.93) from the action in Eq. (6.79).

The thermodynamic stability can be seen directly from applying the limit of infinite radius of the cavity in Eq. (6.62), which is the condition for the positivity of the heat capacity. This condition ensures that a solution in the limit of infinite cavity is stable. The heat capacity in this limit is

$$\begin{aligned} C_Q &= \frac{(d-2)\Omega_{d-2} r_+^{d-2} (r_+^{2d-6} - \mu Q^2)}{4l_p^{d-2} \left((2d-5)\mu Q^2 - r_+^{2d-6} \right)} \\ &= \frac{S^3 E T}{\frac{(d-3)\Omega_{d-2}^3}{4^5 \pi^2 l_p^{3d-6}} \left[\frac{(3d-8)\mu^2 Q^4}{\left(\frac{4Sl_p^{d-2}}{\Omega_{d-2}} \right)^{\frac{d-4}{d-2}}} + (d-4) \left(\frac{4Sl_p^{d-2}}{\Omega_{d-2}} \right)^{\frac{3d-8}{d-2}} \right] - T^2 S^3}, \end{aligned} \quad (6.94)$$

where the subscript A in $C_{A,Q}$ has been dropped since the evaluation is at infinity, and in the second equality the heat capacity was written in terms of the thermodynamic variables S , E , and T . So there is stability if $C_Q \geq 0$, i.e., $r_+ \leq [(2d-5)\mu Q^2]^{\frac{1}{2d-6}}$, which is Eq. (6.84) together with Eq. (6.82). This means that the solution r_{+1} is thermodynamically stable whereas the solution r_{+2} is unstable. It must be noted also that r_{+1} is an increasing monotonic function in T , which means the energy of the black hole increases of the temperature increases, as it is expected from a stable system. The opposite happens to the solution r_{+2} , since it is a decreasing monotonic function in T and so the energy of the black hole decreases as temperature increases.

6.6.2.3 Favorable phases

There are two stable phases. The small black hole r_{+1} and hot flat space with electric charge at infinity. Since the black hole r_{+1} has positive free energy and hot flat space with electric charge at infinity has zero free energy, and systems with lower free energy are preferred, whenever the system finds itself in the black hole r_{+1} solution it tends to transition to the hot flat space with electric charge at infinity phase.

6.6.2.4 $d = 4$: Analysis leading to Davies' thermodynamic theory of black holes and Davies point

The dimension $d = 4$ is specially interesting since in the $R \rightarrow \infty$ gives the results of Davies' thermodynamic theory of black holes [51]. In this setting, the reservoir of temperature T and electric charge Q is at infinity.

The reduced action in Eq. (6.79) in $d = 4$ gives

$$I_* = \frac{\beta}{2l_p^2} \left(r_+ + \frac{Q^2}{r_+} \right) - \pi \frac{r_+^2}{l_p^2}, \quad (6.95)$$

where $\mu = l_p^2$ and $\Omega_2 = 4\pi$. The stationary points in $d = 4$ occur when

$$\beta = \iota(r_+), \quad \iota(r_+) \equiv \frac{4\pi r_+^2}{r_+ - \frac{Q^2}{r_+}}, \quad (6.96)$$

corresponding to the inverse Hawking temperature of a charged black hole in $d = 4$.

Equation (6.96) must be inverted to get the solutions $r_+(T, Q)$. The solutions satisfy

$$\left(\frac{1}{4\pi T} \right) (r_+^2 - Q^2) - r_+^3 = 0. \quad (6.97)$$

It is possible to write analytically the solutions as the roots of a cubic polynomial, however we do not present them analytically here. The minimum of function $\iota(r_+)$

in Eq. (6.96) occurs at $r_{+s1} = \sqrt{3}Q$, being a saddle point of the action of the black hole. The horizon radius of the saddle point is written here as

$$r_{+D} = \sqrt{3}Q, \quad (6.98)$$

as in $d = 4$ it gives the Davies horizon radius. Since $r_+ = l_p^2 m + \sqrt{l_p^4 m^2 - l_p^2 Q^2}$, this means $l_p m = \frac{2}{\sqrt{3}}Q$ at the saddle point, a result that can be found in [51]. The temperature corresponding to the saddle point is Eq. (6.83) in $d = 4$, or explicitly

$$T_D = \frac{1}{6\sqrt{3}\pi l_p Q}, \quad (6.99)$$

which is the Davies temperature, and it is a result that can be extracted from [51].

We present a summary of the behavior of the solutions for $d = 4$. For $0 < T \leq T_D$, there are two solutions, the solution $r_{+1}(T, Q)$ and the solution $r_{+2}(T, Q)$. The solution $r_{+1}(T, Q)$ increases monotonically with T and lies in the interval $r_{+e} < r_{+1}(T, Q) \leq r_{+D}$, where $r_{+1}(T \rightarrow 0, Q) = r_{+e} = l_p Q$ and $r_{+1}(T_D, Q) = r_{+D} = \sqrt{3}l_p Q$. The solution $r_{+2}(T, Q)$ decreases monotonically with T and lies in the interval $r_{+D} < r_{+2}(T, Q) < \infty$, where $r_{+2}(T_D, Q) = r_{+D} = \sqrt{3}l_p Q$. For $T_D < T$, there are no black hole solutions. Regarding stability, a solution is stable if $\frac{\partial \mathcal{I}(r_+)}{\partial r_+} \leq 0$, i.e.

$$r_+ \leq r_{+D}. \quad (6.100)$$

With r_{+D} given in Eq. (6.98), Eq. (6.100) can be turned in to the region in the electric charge $\frac{1}{\sqrt{3}}r_+ \leq l_p Q \leq r_+$, the latter term being simply the restriction to nonextremal case. From Eq. (6.100), one has that the solution r_{+1} is stable while the solution r_{+2} is unstable.

We now summarize the results for thermodynamics in $d = 4$. The free energy of the system is $F = T I_0$, coming from the zero loop approximation of the path integral. From Eq. (6.95), the free energy is

$$F = \frac{1}{2l_p^2} \left(r_+ + \frac{l_p^2 Q^2}{r_+} \right) - T \pi \frac{r_+^2}{l_p^2}. \quad (6.101)$$

From the derivatives of the free energy, one obtains the entropy $S = \pi \frac{r_+^2}{l_p^2}$, i.e., $S = \frac{1}{4l_p^2} A_+$, the thermodynamic pressure $p = 0$ since there is no area dependence, the electric potential $\phi = \frac{Q}{r_+}$, and the energy $E = \frac{1}{2l_p^2} \left(r_+ + \frac{l_p^2 Q^2}{r_+} \right)$, from $E = F + TS$. Considering that this is the expression for the spacetime mass m , one has $E = m$. The free energy of Eq. (6.101) is then $F = m - TS$.

The Smarr formula for $d = 4$ is

$$m = \frac{1}{2} TS + \phi Q. \quad (6.102)$$

Indeed, the first law of black hole mechanics $dm = TdS + \phi dQ$ coincides with the first law of thermodynamics, see above. The first law of black hole mechanics is

the expression from which Davies [51] started his analysis. Here, we started the analysis from the action Eq. (6.95) and we derived the first law from first principles. Moreover, the system is stable thermodynamically in a range of values of the electric charge. On the other hand, the electrically charged case in the grand canonical ensemble with the reservoir at infinity is unstable. Gibbons and Hawking through the action and the path integral approach [67] noticed this instability problem but did not venture into the electric canonical ensemble to cure it.

The heat capacity described in Eq. (6.94), with $d = 4$, is given by

$$C_Q = \frac{2\pi r_+^2 \left(1 - \frac{Q^2}{r_+^2}\right)}{l_p^2 \left(3\frac{Q^2}{r_+^2} - 1\right)} = \frac{S^3 E T}{\frac{\pi Q^2}{4l_p^6} - T^2 S^3}, \quad (6.103)$$

where in the second equality the heat capacity was written in terms of the thermodynamic variables S , E , and T . The system is thermodynamically stable if $l_p Q \geq \frac{1}{\sqrt{3}} r_+$, i.e., $\frac{1}{\sqrt{3}} r_+ \leq l_p Q \leq r_+$, the latter term being the condition for nonextremal case. The system is thermodynamically unstable if $0 \leq l_p Q < \frac{1}{\sqrt{3}} r_+$. This is the same result as given in Eq. (6.100) together with Eq. (6.98). The heat capacity C_Q is infinitely positive at the point $l_p Q = \frac{1}{\sqrt{3}} r_+$ if one approaches it from higher Q , the heat capacity C_Q is infinitely negative if one approaches the point $l_p Q = \frac{1}{\sqrt{3}} r_+$ from lower Q . The heat capacity goes to zero at the extremal case $l_p Q = r_+$. Precisely at the point $l_p Q = \frac{1}{\sqrt{3}} r_+$, this behavior of the heat capacity was found in [51], and it was classified as being similar to a second order phase transition. However, this point is a turning point rather than a second order phase transition. This turning point indicates the ratio of scales at which one has stability. Indeed, when analyzing the heat capacity in terms of the temperature and electric charge, one has two distinctive curves, one for each solution, diverging at this point. But the unstable solution cannot be considered as a phase, due to its instability. The system will always remain in the stable configuration. Note that the formula for C_Q in the second line of Eq. (6.103) is the same formula found in [51] by performing in Eq. (6.103) the redefinitions $S \rightarrow 8\pi S$, $T \rightarrow \frac{1}{8\pi} T$ and $\frac{C_Q}{8\pi} \rightarrow C_Q$, and using Planck units.

6.6.2.5 $d = 5$: Analysis

The dimension $d = 5$ is a typical higher dimension that we have been analyzing. We continue this trend and we present a summary for this specific case in the $R \rightarrow +\infty$ limit.

The reduced action in Eq. (6.79) in $d = 5$ can be written simply as

$$I_* = \frac{\beta}{2l_p^3} \left(\frac{3\pi r_+^2}{4} + \frac{l_p^3 Q^2}{r_+^2} \right) - \frac{\pi^2 r_+^3}{2l_p^3}, \quad (6.104)$$

where it was used $\mu = \frac{4l_p^3}{3\pi}$ and $\Omega_3 = 2\pi^2$. The stationary points are described by

$$\beta = \iota(r_+), \quad \iota(r_+) \equiv \frac{2\pi r_+^3}{r_+^2 - \frac{4l_p^3 Q^2}{3\pi r_+^2}}. \quad (6.105)$$

again corresponding to the inverse Hawking temperature of a charged black hole in $d = 5$.

The solutions are found by inverting Eq. (6.105) to get $r_+(\beta, Q)$, i.e., $r_+(T, Q)$. This is the same as solving

$$\left(\frac{1}{2\pi T}\right) \left(r_+^4 - \frac{4}{3\pi} l_p^3 Q^2\right) - r_+^5 = 0, \quad (6.106)$$

which cannot be done analytically. However, it can be analyzed qualitatively or solved numerically, see Fig. 6.9 for this case of five dimensions. The function $\iota(r_+)$ in Eq. (6.105) possesses a minimum at

$$r_{+s1} = \left(\sqrt{\frac{20}{3\pi}} l_p^{\frac{3}{2}} Q\right)^{\frac{1}{2}}, \quad (6.107)$$

which corresponds to a saddle point of the action of the black hole. This generalizes the Davies radius to $d = 5$. The temperature at this saddle point is

$$T_{s1} = \frac{4}{10\pi \left(\sqrt{\frac{20}{3\pi}} l_p^{\frac{3}{2}} Q\right)^{\frac{1}{2}}}. \quad (6.108)$$

This generalizes the Davies temperature for $d = 5$.

We now summarize the behavior of the solutions in $d = 5$. For temperatures $0 < T \leq T_s$ there are two solutions, the solution $r_{+1}(T, Q)$ and the solution $r_{+2}(T, Q)$. The solution $r_{+1}(T, Q)$ increases monotonically with the temperature and is bounded by $r_{+e} < r_{+1}(T, Q) \leq r_{+s}$, where $r_{+1}(T \rightarrow 0, Q) = r_{+e} = \left(\sqrt{\frac{4}{3\pi}} l_p^{\frac{3}{2}} Q\right)^{\frac{1}{2}}$ is the extremal black hole, and $r_{+1}(T_{s1}, Q) = r_{+s} = \left(\sqrt{\frac{20}{3\pi}} l_p^{\frac{3}{2}} Q\right)^{\frac{1}{2}}$. The solution $r_{+2}(T, Q)$ decreases monotonically with the temperature and assumes values in the interval $r_{+s1} < r_{+2}(T, Q) < \infty$, where $r_{+2}(T_s, Q) = r_{+s}$. See Fig. 6.9 for the plots of r_{+1} and r_{+2} . Regarding stability, a stable solution obeys $\frac{\partial \iota(r_+)}{\partial r_+} \leq 0$. This condition becomes

$$r_+ \leq r_{+s1}. \quad (6.109)$$

With r_{+s1} given in Eq. (6.107), Eq. (6.109) can be transformed to $\left(\frac{3\pi}{20}\right)^{\frac{1}{2}} r_+^2 \leq l_p^{\frac{3}{2}} Q \leq \left(\frac{3\pi}{4}\right)^{\frac{1}{2}} r_+^2$, the latter term being the restriction to the nonextremal case. From Eq. (6.109), one obtains that r_{+1} is stable and that r_{+2} is unstable. The plots in Fig. 6.9 show the discussion above, namely the stable branch r_{+1} and the unstable

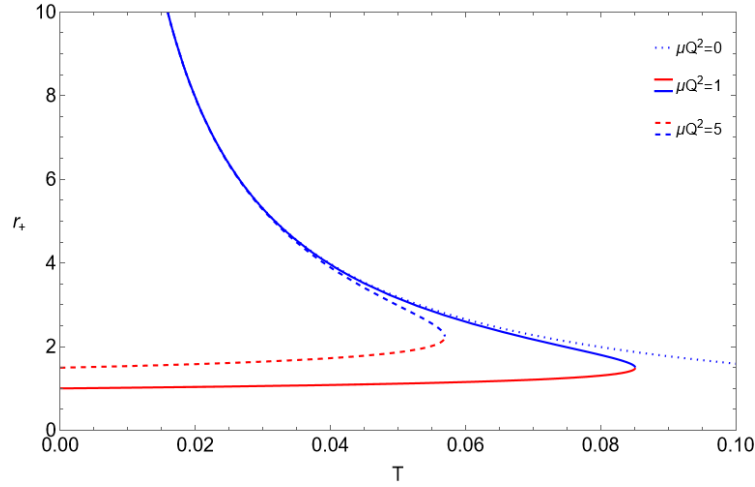


Figure 6.9: Plot of the two solutions $r_{+1}(T, Q)$, in red, and $r_{+2}(T, Q)$, in blue, of the charged black hole in the canonical ensemble for infinite cavity radius, for two values of the charge, $\mu Q^2 = 1$ in filled lines, and $\mu Q^2 = 5$ in dashed lines, $\mu = \frac{4l_p^3}{3\pi}$, in $d = 5$.

branch r_{+2} . It is also seen clearly that the plot of Fig. 6.9 is the limiting $R \rightarrow \infty$ case of Fig. 6.2. From Fig. 6.2 one finds that when $R \rightarrow \infty$, the solution r_{+3} disappears, leaving r_{+1} and r_{+2} , with r_{+1} and r_{+2} meeting at a maximum temperature. Also, from Fig. 6.2 one sees that the r_{+2} and r_{+3} branches meet at a minimum temperature, and these branches are the ones that appears in the zero charge case of York, here slightly modified due to the existence of an electric charge Q . More specifically, comparing Fig. 6.9 with Fig. 6.2, one notes that the red and blue lines of Fig. 6.9 are the stable and unstable black holes of Davies, here in $d = 5$, and the red and blue lines of Fig. 6.2 are precisely these branches of black holes for finite reservoir radius R . The blue and green branches in Fig. 6.2 correspond to York black holes. Thus, Fig. 6.2 is a unified plot of York and Davies black holes. Note further from Fig. 6.9, that for the electric charge going to zero, the branch that survives in Fig. 6.9 is the blue branch, which corresponds to the unstable black hole r_{+2} , and the solution goes up to the point characterized by $T = \infty$ and $r_{+} = 0$. This branch corresponds to the original unstable Hawking black hole, the black hole also found in the Gibbons-Hawking path integral approach.

We present the summary of the results for the thermodynamics in $d = 5$. The free energy can be obtained from the zero loop approximation of the path integral as $F = TI_0$. From Eq. (6.104), the free energy takes the form

$$F = \frac{1}{2l_p^3} \left(\frac{3\pi r_+^2}{4} + \frac{l_p^3 Q^2}{r_+^2} \right) - T \frac{\pi^2 r_+^3}{2l_p^3}. \quad (6.110)$$

From its derivatives, one obtains the entropy as $S = \frac{A_+}{4l_p^3}$, $A_+ = 2\pi^2 r_+^3$, the thermodynamic pressure as $p = 0$, the thermodynamic electric potential as $\phi = \frac{Q}{r_+^2}$, and the energy, given by $E = F - TS$, as $E = \frac{3\pi r_+^2}{8l_p^3} + \frac{Q^2}{2r_+^2}$. Note that this is exactly the

expression for the spacetime mass m , so the mean energy is $E = m$. The free energy of Eq. (6.110) becomes $F = m - TS$.

The Smarr formula in $d = 5$ takes the form

$$m = \frac{2}{3} TS + \phi Q. \quad (6.111)$$

Also, one has that the law $dm = TdS + \phi dQ$ holds. And so the first law of black hole mechanics coincides with the first law of thermodynamics. Also, the system is stable thermodynamically in a small region of the charge, so this correspondence is valid.

The heat capacity of Eq. (6.94) is now in $d = 5$ given by

$$\begin{aligned} C_Q &= \frac{3\pi^2 r_+^3 \left(1 - \frac{4}{3\pi} \frac{Q^2}{r_+^4}\right)}{2l_p^3 \left(\frac{20}{3\pi} \frac{Q^2}{r_+^4} - 1\right)} \\ &= \frac{S^3 E T}{\frac{7\pi^2}{36l_p^3} Q^4 \left(\frac{2Sl_p^3}{\pi^2}\right)^{-\frac{1}{3}} + \frac{\pi^4}{4^3} \left(\frac{2Sl_p^3}{\pi^2}\right)^{\frac{7}{3}} - T^2 S^3}, \end{aligned} \quad (6.112)$$

where in the second equality is in terms of the thermodynamic variables S , E , and T . One has instability if $0 \leq l_p^{\frac{3}{2}} Q < \left(\frac{3\pi}{20}\right)^{\frac{1}{2}} r_+^2$, with Q meaning its absolute modulus. One has thermodynamic stability if $\left(\frac{3\pi}{20}\right)^{\frac{1}{2}} r_+^2 \leq l_p^{\frac{3}{2}} Q \leq \left(\frac{3\pi}{4}\right)^{\frac{1}{2}} r_+^2$, the latter term being the condition for the nonextremal case, and this can also be derived from Eq. (6.109) together with Eq. (6.107). The heat capacity C_Q is infinitely positive at the point $l_p^{\frac{3}{2}} Q = \left(\frac{3\pi}{20}\right)^{\frac{1}{2}} r_+^2$, if this point is approached from higher Q , the heat capacity C_Q is infinitely negative, if the point is approached from lower Q . This is a

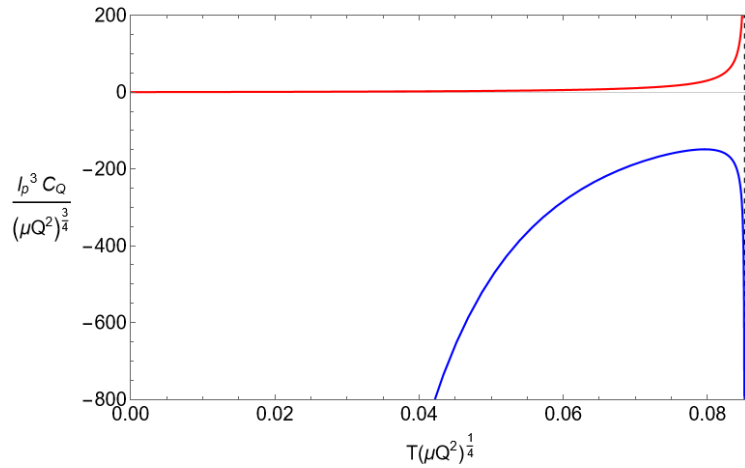


Figure 6.10: The heat capacity $l_p^3 C_Q$ in $(\mu Q^2)^{\frac{3}{4}}$ units, $\frac{l_p^3 C_Q}{(\mu Q^2)^{\frac{3}{4}}}$, is given as a function of the temperature $T(\mu Q^2)^{\frac{1}{4}}$ in $d = 5$. In red, the heat capacity of r_{+1} is represented, while in blue, the heat capacity of r_{+2} is shown. There is a turning point at $T(\mu Q^2)^{\frac{1}{4}} = \frac{4}{10\pi 5^{\frac{1}{4}}}$.

turning point of the solutions, indicating the condition for stability. This is properly seen when analyzing the heat capacity with fixed temperature and electric charge, see Fig. 6.10. Indeed, the heat capacity is described by two curves, one for each solution r_{+1} and r_{+2} , being positive for r_{+1} and negative for r_{+2} . The system cannot be sustained in the solution r_{+2} since it is unstable and so it can only be at the stable solution r_{+1} .

6.6.3 Infinite cavity radius and the Rindler limit: Cavity boundary at the Unruh temperature

The case of the smallest and intermediate solutions was discussed above. We now turn to the largest solution in the limit of infinite cavity. The largest solution in this limit can be obtained by keeping T and Q fixed, while doing $R \rightarrow +\infty$ and $r_+ \rightarrow R$ in Eq. (6.33). The temperature dependence on the charge goes with $\frac{\mu Q^2}{R^{2d-6}}$, therefore such dependence in the limit $r_+ \rightarrow R$ and $R \rightarrow +\infty$ disappears. Intuitively, the black hole becomes very large such that any contributions of the charge become negligible. Then, the expression for the temperature reduces to the non-charged case, $T = (d-3)(4\pi r_+)^{-1}(1 - \frac{r_+^{d-3}}{R^{d-3}})^{-\frac{1}{2}}$, but the limit still needs to be applied. The requirement that T is fixed and so finite leads to the condition that $r_+ \sqrt{1 - \frac{r_+^{d-3}}{R^{d-3}}}$ must tend to some constant under the limit of $R \rightarrow +\infty$ and $r_+ \rightarrow R$. Still, it seems unclear a priori what the system in this limit describes.

In order to understand the limit, we can first consider the Euclidean Schwarzschild metric $ds^2 = R^2 \frac{4r_+^2}{R^2(d-3)^2} (1 - \frac{r_+^{d-3}}{r^{d-3}}) d\tau^2 + (1 - \frac{r_+^{d-3}}{r^{d-3}})^{-1} R^2 d(\frac{r}{R})^2 + R^2 (\frac{r^2}{R^2}) d\Omega_{d-2}^2$, where the normalization by R in the line element was introduced, with $0 \leq \tau < 2\pi$ and $r_+ < r \leq R$. First, we need to consider $r_+ \rightarrow R$ in the limit of infinite cavity and only then perform $R \rightarrow \infty$. Therefore, we must consider the near horizon expansion of the metric. The normalized proper radial length is given by $\epsilon(r) = \frac{1}{R} \int_{r_+}^r (1 - \frac{r_+^{d-3}}{\rho^{d-3}})^{-\frac{1}{2}} d\rho = \frac{2}{d-3} (\frac{R}{r_+})^{\frac{d-5}{2}} \sqrt{(\frac{r}{R})^{d-3} - (\frac{r_+}{R})^{d-3}}$, valid at the near horizon, spanning the interval $0 < \epsilon < \epsilon(R)$. We can therefore rewrite the Schwarzschild metric in this limit as $ds^2 = (R^2 \epsilon^2 + \mathcal{O}(\epsilon^4)) d\tau^2 + R^2 d\epsilon^2 + (R^2 + \mathcal{O}(\epsilon^2)) d\Omega^2$. Notice however that as $r_+ \rightarrow R$, the total normalized radial proper length $\epsilon(R)$ tends to zero. It is now that the limit $R \rightarrow +\infty$ is performed but such that $R\epsilon(R)$ tends to a constant, which is defined as $\bar{R}$, $\bar{R} \equiv R\epsilon(R)$. Thus, one has a new proper length $\bar{r}$, defined as

$$\bar{r} \equiv R\epsilon(r), \quad 0 < \bar{r} < \bar{R}. \quad (6.113)$$

The metric becomes in this limit

$$ds^2 = \bar{r}^2 d\tau^2 + d\bar{r}^2 + R^2 d\Omega^2, \quad (6.114)$$

i.e., it becomes the two-dimensional Euclideanized Rindler metric times a $(d-2)$ -sphere with infinite radius. The metric on the $(d-2)$ -sphere can be normalized by choosing a specific point on the sphere and performing the expansion around such point, obtaining $R^2 d\Omega^2 = \sum_{i=1}^{d-2} (dx^i)^2$, where x^i are the new coordinates. The

metric then reduces to the d dimensional Euclideanized Rindler space. The system can now be interpreted as follows. The event horizon of the black hole reduces to the Rindler horizon at $\bar{r} = 0$, while the cavity boundary is located at $\bar{R}$ and it is being accelerated. The proper acceleration of the cavity is precisely $\frac{1}{\bar{R}}$ and the temperature measured at the boundary of the cavity is $T = \frac{1}{2\pi\bar{R}}$.

We must analyze what happens to the thermodynamic quantities in this Rindler solution limit. First, the temperature in Eq. (6.33) is finite and equals to $T = \frac{1}{2\pi\bar{R}}$. Since T is fixed by the ensemble this gives the solution for the cavity boundary, namely

$$\bar{R} = \frac{1}{2\pi T}. \quad (6.115)$$

To be in equilibrium with the temperature T of the reservoir, the boundary itself $\bar{R}$ has to have a Rindler acceleration that matches its Unruh temperature. The free energy in Eq. (6.49) diverges negatively, $F \rightarrow -\infty$. It diverges as $F = \frac{R^{d-3}}{\mu} - \frac{\Omega_{d-2}}{8\pi\bar{R}l_p^{d-2}}R^{d-2}$, and is negative since the power R^{d-2} is always larger than R^{d-3} for $R \rightarrow +\infty$. This divergence is due to the fact that the area is divergent. Thus, it is better to work with a specific free energy, $\bar{F}$, a free energy per unit area, defined as $\bar{F} \equiv \frac{F}{\Omega_{d-2}R^{d-2}}$. Then,

$$\bar{F} = -\frac{1}{8\pi\bar{R}l_p^{d-2}}, \quad (6.116)$$

so it is negative. From Eq. (6.53), the entropy also diverges, $S \rightarrow \infty$, it diverges as $S = \frac{\Omega_{d-2}R^{d-2}}{4l_p^{d-2}}$. Defining a specific entropy $\bar{S} \equiv \frac{S}{\Omega_{d-2}R^{d-2}}$

$$\bar{S} = \frac{1}{4l_p^{d-2}}, \quad (6.117)$$

so it is a constant. The thermodynamic pressure in Eq. (6.54) is finite, which is written as

$$\bar{p} = \frac{1}{8\pi\bar{R}l_p^{d-2}}, \quad (6.118)$$

so $\bar{p} = \frac{T}{4l_p^{d-2}}$. The electric potential in Eq. (6.55) is zero,

$$\bar{\phi} = 0. \quad (6.119)$$

The thermodynamic energy from Eq. (6.56) obeys $E \rightarrow \infty$, it diverges as $E = \frac{R^{d-3}}{\mu}$ positively. Defining a specific energy, $\bar{E}$, as $\bar{E} \equiv \frac{E}{\Omega_{d-2}R^{d-2}}$, one obtains

$$\bar{E} = 0. \quad (6.120)$$

The heat capacity in Eq. (6.62) goes positively as $C_A = \frac{(d-2)(d-3)\Omega_{d-2}}{2l_p^{d-2}}R^{d-4}\bar{R}^2$. So $C_A = 4\pi\frac{R^2}{l_p^2}$ for $d = 4$ and $C_A \rightarrow \infty$ for $d > 4$, i.e., for $d = 4$ is finite and depends

on the temperature as $C_A = \frac{1}{\pi T^2 l_p^2}$, and for $d > 4$ diverges. Since C_A is positive, this solution can then be considered stable. Defining a specific heat capacity, $\bar{C}_A$, as $\bar{C}_A \equiv \frac{C_A}{\Omega_{d-2} R^{d-2}}$ gives

$$\bar{C}_A = 0. \quad (6.121)$$

Although this solution has divergent quantities, we can resort instead to mean densities or specific quantities, such as the specific heat, thus finding finite thermodynamic quantities.

For the ensemble with infinite radius, we could try to analyze what is the most preferred phase thermodynamically. However, it seems that the two limiting solutions have different character. In the Davies solution there is still a net electric charge Q at infinity. In the Rindler solution the electric charge has disappeared from the context, so it is in fact a zero electric charge solution. Although the starting ensembles are the same, the final ensembles in the infinite radius limit are different. From the free energies, given that the stable black hole in Davies solution has positive free energy and the Rindler one has infinite negative free energy, one would conclude that the Rindler solution is the most preferred phase. But in fact the two solutions belong to different ensembles and cannot be compared. As already mentioned, the Davies stable solution tends to disperse to hot flat space with electric charge at infinity.

6.7 THERMODYNAMIC RADII AND THE GENERALIZED BUCHDAHL RADIUS IN d DIMENSIONS

6.7.1 *The uncharged case*

We analyze the thermodynamic energy or mass to radius ratio for the d -dimensional canonical ensemble, namely, the energy or mass for which the black hole free energy is zero, $F = 0$. We make the comparison between this mass and the Buchdahl bound mass in d dimensions.

In the canonical ensemble of an uncharged spherically symmetric black hole in d dimensions [102], which is described by the Euclidean Schwarzschild-Tangherlini black hole space, the canonical ensemble is realized with a fixed temperature at the boundary of the cavity. There are two black hole solutions, where the one with the largest mass is stable and the one with the least mass is unstable. Here one is interested in the large stable black hole. The free energy of the ensemble also has a critical point at zero horizon radius, which is a minimum, the hot flat space case. Therefore, we can analyze which are the favorable states in comparing the free energies of the zero horizon radius, i.e., hot flat space, and the stable black hole solution. The free energy of hot flat space is zero. The black hole solution also has zero free energy for a given horizon radius, which is thus an important thermodynamic radius. The larger the temperature of the ensemble, the larger this radius, and the lower the corresponding free energy. Thus, we can argue that a stable black hole is favored to hot flat space when the free energy of the black hole

is lower than the zero, which is the free energy of hot flat space. The radius of the black hole horizon that yields zero free energy, i.e., $F = 0$, is $\left(\frac{r_+}{R}\right)_{F=0} = \left(\frac{4(d-2)}{(d-1)^2}\right)^{\frac{1}{d-3}}$. In terms of the spacetime mass m this is

$$\left(\frac{\mu m}{R^{d-3}}\right)_{F=0} = \left(\frac{2(d-2)}{(d-1)^2}\right). \quad (6.122)$$

The Buchdahl bound radius marks the maximum compactness of a spherically symmetric star before spacetime turns singular. The Buchdahl bound for a star or matter configuration of gravitational radius r_+ and radius R is [129] $\left(\frac{r_+}{R}\right)_{\text{Buch}} = \left(\frac{4(d-2)}{(d-1)^2}\right)^{\frac{1}{d-3}}$, which in terms of the spacetime mass m and radius R is

$$\left(\frac{\mu m}{R^{d-3}}\right)_{\text{Buch}} = \left(\frac{2(d-2)}{(d-1)^2}\right). \quad (6.123)$$

It is a structural bound coming from mechanics. Self-gravitating matter for which the mass, or the energy, content within a radius R is above the bound, in principle collapses to a black hole.

It can be seen that both masses, or radii, although conceptually different, have the same expression, indeed, $\left(\frac{\mu m}{R^{d-3}}\right)_{F=0} = \left(\frac{\mu m}{R^{d-3}}\right)_{\text{Buch}}$. Therefore, we can argue that as soon as the black hole phase is thermodynamically favorable over the hot flat space, it is actually the only phase that exists, the energy within the reservoir collapses to form a black hole. This could indicate that there is a link between black hole thermodynamics and matter mechanics.

6.7.2 The charged case

We now analyze the thermodynamic energy or mass to radius ratio for two ensembles, where one is the d -dimensional canonical ensemble with electric charge that we are treating here, and the other is the grand canonical ensemble that was treated in Chapter 4, for which the black hole free energies are zero, i.e., $F = 0$, and $W = 0$, respectively. We make the comparison between these two energy or mass to radius ratio and the generalized Buchdahl bound, i.e. the Buchdahl bound in the electric charged case in d dimensions, also called the Buchdahl-Andréasson-Wright bound, see [129].

In the canonical ensemble of a charged black hole inside a cavity in d dimensions, the construction has been described throughout this chapter. The canonical ensemble in this case is realized with a fixed temperature and fixed electric charge at the boundary of the cavity. One has in this case two stable black hole solutions for a charge below a saddle, or critical, charge Q_s , and one stable black hole solution for a charge larger than Q_s . In this case, it can be shown that the stable solution with the largest mass for every charge can have a negative free energy, if the black hole has a larger mass than the one that solves this equation

$$a \left(\frac{\mu m}{R^{d-3}}\right)^4 + b \left(\frac{\mu m}{R^{d-3}}\right)^3 + c \left(\frac{\mu m}{R^{d-3}}\right)^2 + d \left(\frac{\mu m}{R^{d-3}}\right) + e = 0, \quad (6.124)$$

where

$$\begin{aligned}
a &= \left(\left(\frac{d-3}{d-2} \right)^2 - 4 \right)^2, \\
b &= -4 \left(4 + 8y - \left(\frac{d-3}{d-2} \right)^2 (3+2y) \right), \\
c &= -2 \left(\frac{d-3}{d-2} \right)^4 y - 2 \left(\frac{d-3}{d-2} \right)^2 (y^2 - y + 2) + 4 + 24(y + 6y^2) \\
d &= -4y \left((1+y)(1+2y) + \left(\frac{d-3}{d-2} \right)^2 (3+2y) \right) \\
e &= \left(\frac{d-3}{d-2} \right)^4 y^2 + y^2(1+y)^2 + 2 \left(\frac{d-3}{d-2} \right)^2 y(1+y)(2+y), \quad (6.125)
\end{aligned}$$

with y being the electric charge parameter given by $y \equiv \frac{\mu Q^2}{R^{2d-6}}$, as before. This is a quartic equation in $\frac{\mu m}{R^{d-3}}$ and its solution can be written formally as

$$\left(\frac{\mu m}{R^{d-3}} \right)_{F=0} = g \left(d, \frac{\mu Q^2}{R^{2d-6}} \right), \quad (6.126)$$

for some calculable function $g \left(d, \frac{\mu Q^2}{R^{2d-6}} \right)$. In the case $Q = 0$, one gets $\left(\frac{\mu m}{R^{d-3}} \right)_{F=0} = \left(\frac{2(d-2)}{(d-1)^2} \right)^{\frac{1}{d-3}}$ as required, see Eq. (6.122). The largest stable black hole with this mass has a zero Helmholtz free energy, $F = 0$. Contrasting to the canonical ensemble of the electrically uncharged black hole discussed above, the free energy in the electrically charged case does not include the zero horizon radius case. The minimum possible horizon radius is the extremal black hole point $r_{+e} = (\mu Q^2)^{\frac{1}{2d-6}}$, yielding a free energy $F_{r_{+e}} = \frac{Q}{\sqrt{\mu}}$. To emulate hot flat space, an electrically charged nonself-gravitating shell was used. The comparison was then made between the black hole configuration and the electrically charged shell with no self-gravity at the boundary of the cavity, having then hot flat space inside the cavity with the electric charge near the boundary. This configuration would require us to look into the matter sector which we have not done here. It is unclear if a transition can occur between hot flat space with electric charges near the cavity and the stable black holes. Nevertheless, the thermodynamic radius of zero free energy in the canonical ensemble is still regarded as an important quantity.

In the grand canonical ensemble of a charged Reissner-Nordström black hole inside a cavity for d dimensions, the construction and its thermodynamics were described in [2], and Chapter 4. The grand canonical ensemble is realized with a fixed temperature and fixed electric potential at the boundary of the cavity. In this ensemble, the partition function in the zero loop approximation is given in terms of the grand potential, or Gibbs free energy, $W = E - TS - Q\phi$, where E is the mean energy, T is the temperature, S is the entropy, Q is the mean charge and ϕ is the electric potential. The grand potential yields $W[r_+, Q] = \frac{R^{d-3}}{\mu} (1 - \sqrt{f}) - Q\phi - T \frac{\Omega_{d-2} r_+^{d-2}}{4}$,

with $f = \left(1 - \frac{r_+^{d-3}}{R^{d-3}}\right) \left(1 - \frac{\mu Q^2}{(r_+ R)^{d-3}}\right)$, and the equilibrium equations that yield the black hole solutions are $\frac{1}{T} = \frac{4\pi}{(d-3)} \frac{r_+^{d-2}}{r_+^{2d-6} - \mu Q^2} \sqrt{f}$ and $\phi = \frac{Q}{\sqrt{f}} \left(\frac{1}{R^{d-3}} - \frac{1}{r_+^{d-3}}\right)$, where the convention for the electromagnetic coupling and electric charge was chosen so that $Q \rightarrow \sqrt{(d-3)\Omega_{d-2}}Q$ and $\phi \rightarrow (\sqrt{(d-3)\Omega_{d-2}})^{-1}\phi$ in the expressions in [2] and Chapter 4. One has in this case up to two solutions, depending on the fixed quantities T and ϕ , with only one being stable. The grand canonical free energy of the ensemble also has a critical point at zero horizon radius, which is a minimum, it is the hot flat space case. The stable black hole solution also has zero free energy for a given horizon radius, which is thus an important thermodynamic radius. The larger the temperature of the ensemble, the larger this radius, and the lower the corresponding free energy. Thus, we can argue that a stable black hole is favored to hot flat space when the free energy of the black hole is lower than the zero, which is the free energy of hot flat space. The radius of the black hole horizon that yields zero grand potential energy, i.e., $W = 0$ is complicated to find, but the corresponding mass has a simple expression given by

$$\begin{aligned} \left(\frac{\mu m}{R^{d-3}}\right)_{W=0} &= \frac{-4(d-2)^2}{(d-1)^2(d-3)^2} + \frac{2(d-2)((d-2)^2 + 1)}{(d-1)^2(d-3)^2} \\ &\times \sqrt{1 + \frac{(d-1)^2(d-3)^2}{4(d-2)^2} \frac{\mu Q^2}{R^{2d-6}}}. \end{aligned} \quad (6.127)$$

Since hot flat space is described here by the grand potential $W[r_+, Q]$, a possible transition can occur from the charged hot flat space to the stable black hole for temperatures corresponding to stable black holes with higher mass than Eq. (6.127). In the case $Q = 0$, one has that $W = F$, so one gets $\left(\frac{\mu m}{R^{d-3}}\right)_{W=0} = \left(\frac{\mu m}{R^{d-3}}\right)_{F=0} = \left(\frac{2(d-2)}{(d-1)^2}\right)^{\frac{1}{d-3}}$ as required, see Eq. (6.122).

The Buchdahl bound was originally given for the electrically uncharged case and in $d = 4$. For electrically charged matter in d dimensions one has the generalized Buchdahl bound that is given by [129]

$$\begin{aligned} \left(\frac{\mu m}{R^{d-3}}\right)_{\text{Buch}} &= \frac{d-2}{(d-1)^2} + \frac{1}{d-1} \frac{\mu Q^2}{R^{2d-6}} \\ &+ \frac{d-2}{(d-1)^2} \sqrt{1 + (d-1)(d-3) \frac{\mu Q^2}{R^{2d-6}}}. \end{aligned} \quad (6.128)$$

In the no charge case, $Q = 0$, one gets $\frac{\mu m}{R^{d-3}} = \left(\frac{2(d-2)}{(d-1)^2}\right)^{\frac{1}{d-3}}$, as required.

We see that the three mass to radius ratios, are conceptually different, and now in the electrically charged case, have generically different expressions, indeed, $\left(\frac{\mu m}{R^{d-3}}\right)_{F=0}$, $\left(\frac{\mu m}{R^{d-3}}\right)_{W=0}$, and $\left(\frac{\mu m}{R^{d-3}}\right)_{\text{Buch}}$ are not equal. One has $\left(\frac{\mu m}{R^{d-3}}\right)_{F=0} \geq \left(\frac{\mu m}{R^{d-3}}\right)_{\text{Buch}} \geq \left(\frac{\mu m}{R^{d-3}}\right)_{W=0}$. This is an interesting result. In the canonical ensemble, the thermodynamic energy content within the cavity when the black hole phase

starts to be favorable, i.e., when $F = 0$, is higher than the Buchdahl bound, and so even before the black hole is thermodynamically favored, collapse should occur, i.e., as soon as a black hole forms there is no possibility of a thermodynamic phase transition to hot flat space, indeed the black hole has been formed dynamically. In the grand canonical ensemble, the energy content within the cavity when the black hole phase starts to be favorable, i.e., when $W = 0$, is less than the Buchdahl bound, and so there should be no collapse at this stage, indeed, collapse should only occur when the energy content is increased above the bound. In the grand canonical ensemble this occurs only for some negative W . Both thermodynamic mass to radius ratios are equal to the generalized Buchdahl bound when the electric charge is put to zero, and all the three are also equal at the extremal point. The plots given in Fig. 6.7 for $d = 5$ help in the understanding of this behavior. These results present a counter example to the possible link between the black hole thermodynamics and stability of spherically symmetric matter. The uncharged case seems to be a coincidence.

6.8 CONCLUSIONS

We have analyzed in this chapter the canonical ensemble of a Reissner-Nordström black hole in a cavity for arbitrary dimensions. We have built the canonical ensemble through the Euclidean path integral approach, which specifies the partition function in terms of a path integral involving the Euclidean action. The Euclidean action is the usual Einstein-Hilbert-Maxwell action with the Gibbons-Hawking-York boundary term and an additional Maxwell boundary term so that the canonical ensemble is well-defined, all terms having been Euclideanized. We assumed that the heat reservoir has a spherical boundary at finite radius R , where the temperature is fixed as the inverse of the Euclidean proper time length at the boundary, and also the electric charge is fixed by fixing the electric flux at the boundary. We restricted the summed spaces in the path integral to spherically symmetric spaces and we assumed regularity conditions that avoid the presence of conical and curvature singularities.

We then performed the zero loop approximation by first imposing the Hamiltonian and the Gauss constraints, obtaining a reduced action that depends on the fixed inverse temperature β , electric charge Q , and the radius of the boundary R , and also depends on the radius of the event horizon r_+ as a variable that is integrated through the path integral. We found the equation for the stationary points of the reduced action, which are the solutions $r_+[\beta, Q, R]$, and we found also the condition of stability of the solutions.

We analyzed the existence of the solutions of the ensemble for arbitrary dimensions. For charges smaller than a saddle, or critical, electric charge, there are always three possible solutions where the one with the smallest radius and the one with the largest radius are stable, and the other with intermediate radius is unstable. The value of the saddle charge and the value of the radii that bound these solutions, which are saddle points of the reduced action, were found analytically. For the saddle charge, the unstable solution reduces to a point, having formally only two

solutions which are stable. For charges larger than the saddle charge, there is only one solution, and this solution is stable. This analysis was then applied to the four and five dimensional cases. Regarding stability, the solutions are stable if the radius of the event horizon increases as the temperature increases. For this case, the condition is given in terms of the saddle points of the reduced action.

We obtained the thermodynamics of the electrically charged black hole using that the partition function is related to the Helmholtz free energy of the system in the canonical ensemble. Through the zero loop approximation, we obtained the free energy. We retrieved the entropy, the thermodynamic electric potential, the thermodynamic pressure, and the thermodynamic energy through the derivatives of the free energy. More precisely, the entropy is the Bekenstein-Hawking entropy, the pressure has the same expression of the pressure of a self-gravitating charged shell with radius R , and the thermodynamic electric potential is given by the usual expression. We calculated the mean thermodynamic energy, which can be identified with a quasilocal energy, through the definition of free energy. Regarding thermodynamic stability, the configurations are stable if the heat capacity with constant charge and area is positive. We also found the integrated first law, i.e., the Smarr formula, and the Gibbs-Duhem relation.

We analyzed the favorable states in the canonical ensemble. A favorable state is a stable state of the ensemble that has the lowest value of the free energy. In some sense, transitions can occur between phases. Here, for an electric charge lower than the critical charge, there are two stable black hole solutions that are in competition, with an existing first order phase transition between them. For the critical charge, this first order phase transition becomes a second order phase transition. For a charge larger than the critical charge, there is only one stable black hole solution. In the uncharged case, there is a stable solution and hot flat space. Pure hot flat space does not seem to be a solution of the canonical ensemble since the charge is fixed. Instead, we compare the stable solutions with a nonself-gravitating charged sphere. This covers two limits, the case where one has flat space with a charge at the center, which is not a solution and is never favorable, and another case where the charge resides near the cavity or at the cavity. In this last case, it would act as a hot flat space with electric charge at the boundary and the corresponding free energy vanishes. Considering this latter case, we found a first order phase transition between the largest black hole and hot flat space with electric charge at the boundary.

In this chapter, regarding the canonical ensemble of a Reissner-Nordström black hole in a cavity for four and higher dimensions, there are several main achievements which can be stated:

First, we have made the construction of the canonical ensemble and the thermodynamic analysis of all generic d dimensions in a unified way. Moreover, we presented significant cases in all the detail, namely, the dimension $d = 4$ as the most important case, and the dimension $d = 5$ as a typical higher dimensional case.

Second, in the analysis of the specific heat $C_{A,Q}$ in terms of the temperature and the electric charge, we found the existence of a second order phase transition between the two stable solutions for a critical electric charge parameter $\frac{\mu Q_s}{R^{2d-6}}$ in

arbitrary dimensions. For lower electric charge $\frac{\mu Q}{R^{2d-6}}$, we found two turning points, which indicate the stability of the solutions, where the heat capacity diverges and is double valued. For higher charge $\frac{\mu Q}{R^{2d-6}}$, we found that the heat capacity is always positive.

Third, since in the canonical ensemble one can have two stable black hole solutions, an analysis of the free energy has enabled us to pick the black hole solution that is most favored according to the temperature and electric charge of the ensemble and fine the possible first order phase transitions. Moreover, a comparison with the free energy of hot flat space, emulated by an electric shell at the boundary, has revealed the thermodynamic phase that is favored. We also argued that the Buchdahl bound is important in this context, and the free energies for which the bound is superseded were found, for higher free energies gravitational collapse sets in.

Fourth, we have shown that the Davies thermodynamic theory of black holes follows from the electric charged canonical ensemble in the infinite large reservoir limit when $d = 4$. The two ensemble solutions of lower radii maintain, in this limit, their black hole character. One, with the smallest radius, is the stable one, and the other with intermediate radius is unstable. These two solutions meet at a saddle point. We found the thermodynamic quantities and in particular, we found the heat capacity at constant area and charge. In $d = 4$, the expression of the heat capacity reduces to the expression found by Davies. Here, we started from the action and the path integral approach for a reservoir at infinity and showed that the formalism gives the first law of black hole mechanics which, of course, is also the first law of thermodynamics for black holes. Davies, in the $d = 4$ formulation of the theory, started directly from the first law of black hole mechanics. These results, reached through different means, point towards the equivalence between black hole mechanics and black hole thermodynamics through the canonical ensemble.

Fifth, the limit of infinite radius of the boundary of the cavity has revealed a surprise solution. Indeed, the largest black hole solution of the ensemble, changes character in this limit. The black hole solution turns into a Rindler solution with the ensemble fixed temperature being the Unruh temperature of the now accelerated boundary.

Sixth and last, we have followed the York path integral procedure, which was originally applied to Schwarzschild black holes, throughout this work for Reissner-Nordström black holes. We have shown that the black hole solutions found represent the unification of York electrically uncharged black holes and Davies electric charged black holes, in a remarkable way. Indeed, the two York type solutions, one larger and stable, one smaller and unstable, do appear, and the two Davies type solutions, the smaller and unstable, and the even smaller and stable also do appear, in a remarkable way. York and Davies results follow from two different limits of our analysis in this chapter. York results follow from taking the zero electric charge limit. Davies results follow from taking the infinite cavity radius limit, i.e., by putting the heat reservoir at infinity. This latter case can also be seen to stem from York's generic reduced action approach with the boundary at infinity, which in turn yields the Gibbons-Hawking path integral formulation to black hole thermodynamics.

The Gibbons-Hawking approach was originally applied to electrically uncharged black holes and it was found that there was an unstable black hole solution, the Hawking black hole, and thus no consistent thermodynamics. It was also applied to an electrically charged black hole in the grand canonical ensemble, and it was found a solution that was unstable. Had it been applied directly to electrically charged black holes in the canonical ensemble, one would have found that thermodynamic stable solutions exist to vindicate the approach. We have filled this gap here.

What does remain to be understood? Here, we were interested in the thermodynamic interaction of a black hole in a cavity with a boundary of finite size and fixed temperature, as well as in the interaction of the gravitational field with the electromagnetic field in such a system. The formalism by its very distinctive features, i.e., its Euclidean character, applies only to the outside of a black hole event horizon. The black hole interior and its singularity are not considered in the analysis. Thus, the question about the nature of the singularity remains. It is expected that the singularity is described by a Planck scale object, however intricate the description might be. A canonical formalism for micro black holes, say of the order of ten Planck radii, seems valid, after all Hawking radiation, a tamed radiation at most of the scales, if left by itself, slowly peels the singularity away. If that radiation interacts harmoniously with the boundary of a cavity, a thermodynamic procedure might be valid and show how the black hole horizon and the singularity fuse into one single describable object.

LIMITS IN HOT SPACES WITH NEGATIVE COSMOLOGICAL
CONSTANT IN THE CANONICAL ENSEMBLE: HOT ANTI-DE
SITTER SOLUTION, SCHWARZSCHILD-ANTI DE SITTER
BLACK HOLE, HAWKING-PAGE SOLUTION, AND PLANAR
ADS BLACK HOLE

7.1 INTRODUCTION

In the previous chapter, we studied a charged black hole in the canonical ensemble. We touched briefly on the limits of infinite cavity which connected the solutions that characterize Davies' thermodynamic theory and the solutions inside a finite cavity. We detour then to the analysis of a specific case in which the limits connect various solutions existing in the literature. These are the following. The asymptotically flat black hole solutions in thermal equilibrium are called the Gibbons-Hawking solutions [67]. By putting a cavity at finite radius, York found two solutions for the Schwarzschild black hole, which are called the York black hole solutions. Moreover, Schwarzschild-AdS black holes in thermal equilibrium are described by the Hawking-Page solutions [69]. It is further known that very large black holes in AdS tend to the planar black hole solutions in AdS [169].

In this chapter, we consider the canonical ensemble of a Schwarzschild black hole in AdS inside a cavity, where the emphasis is to unify the aforementioned solutions through limits in the cosmological constant and the radius of the cavity. These limits yield different results for each solution of the Schwarzschild AdS black hole, obtaining thus all the solutions mentioned above.

This chapter is organized as follows. In Sec. 7.2, we construct the ensemble in the zero loop approximation. In Sec. 7.3, we obtain the thermodynamics of the system by using the partition function. In Sec. 7.4, we analyze the solutions of the ensemble and their stability, with the limit of zero cosmological constant being trivial. In Sec. 7.5, we consider the limit of infinite cavity and the limit in the cosmological constant. In Sec. 7.6, we present the conclusions. This chapter is based on the ongoing work [5].

7.2 THERMODYNAMICS OF THE SCHWARZSCHILD-ANTI DE SITTER SPACE IN THE CANONICAL ENSEMBLE: GENERAL RESULTS FOR THE BLACK HOLE HORIZON REGION INSIDE A HEAT RESERVOIR

The setup that we consider here is a space M describing the case of a black hole in a negative cosmological constant background inside a heat reservoir, which is described by the boundary ∂M of a cavity with radius R . We then construct the canonical ensemble of this setup, where at the boundary we specify the data that determine the ensemble, see Fig. 7.1. Namely, we fix the inverse temperature $\beta = \frac{1}{T}$

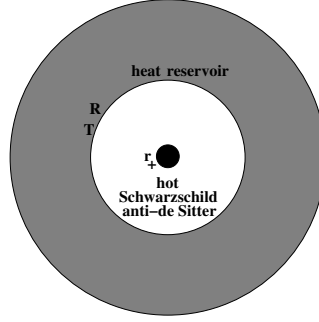


Figure 7.1: A drawing of a black hole in a cavity within a heat reservoir at temperature T and radius R in a space with positive cosmological constant. Outside the black hole radius r_+ the geometry is a Schwarzschild-anti-de Sitter geometry. The Euclideanized space and its boundary have $R^2 \times S^2$ and $S^1 \times S^2$ topologies, respectively, where the S^1 subspace with proper length $\beta = \frac{1}{T}$ is not displayed.

at the boundary with radius R , which we also fix. The inverse temperature β is given by the imaginary proper time at the boundary. Hence, we can construct the canonical ensemble through York formalism [68], see Chapter 3 for more details, by the partition function as

$$Z = Dg_{\alpha\beta} e^{-I} , \quad (7.1)$$

where I is the gravitational action given by

$$I = -\frac{1}{16\pi l_p^2} \int_M (R - 2\Lambda) \sqrt{g} d^4x - \frac{1}{8\pi l_p^2} \int_{\partial M} K \sqrt{\gamma} d^3x - I_{\text{AdS}} , \quad (7.2)$$

where R is the Ricci scalar, Λ is the cosmological constant, g is the metric determinant, $K = n^\alpha{}_{;\alpha}$ is trace of the extrinsic curvature of ∂M , n^α is the unit normal vector to ∂M , γ is the determinant of the induced metric γ_{ab} of ∂M and I_{AdS} is the action of a reference metric to make I finite, which is here the action of pure anti-de Sitter. It is useful to define the anti-de Sitter or AdS length by $l^2 = \frac{3}{-\Lambda}$.

We then proceed with the full zero loop approximation. The spherically symmetric black hole space with topology $\mathbb{R}^2 \times S^2$ that obeys the Euclidean Einstein equations with negative cosmological constant is

$$ds^2 = \frac{1}{(2\pi T_+^H)^2} V(r) d\tau^2 + \frac{dr^2}{V(R)} + r^2 d\Omega_2^2 , \quad (7.3)$$

where $\tau \in]0, 2\pi[$, $r \in]r_+, R[$ with r_+ being the event horizon radius, $d\Omega_2^2$ being the line element of a unit 2-sphere, the function $V(r)$ is

$$V(r) = \left(1 - \frac{r_+}{r}\right) \left(1 + \frac{r^2}{l^2} \left(1 + \frac{r_+}{r} + \frac{r_+^2}{r^2}\right)\right), \quad (7.4)$$

and

$$T_+^H = \frac{1 + 3\frac{r_+^2}{l^2}}{4\pi r_+}, \quad (7.5)$$

is the constant that must be added in Eq. (7.3) so that there is no conical singularity. The line element in Eq. (7.3) can be found either by solving the Einstein equations or by performing the Wick transformation to the Schwarzschild-AdS line element, by compactifying the imaginary time and by avoiding the conical singularity.

The black hole space is then in thermodynamic equilibrium only if the total imaginary proper time at the boundary of the cavity is $\beta = \frac{1}{T}$, hence from Eq. (7.3) one has

$$T = \frac{T_+^H}{\sqrt{V(R)}}. \quad (7.6)$$

Now, the radius of the heat reservoir R sets a scale for our problem. It is then meaningful to gauge all the length scales involved in the problem to R . Thus, the heat reservoir temperature T , the cosmological constant $|\Lambda| = -\Lambda$ or the cosmological length l^2 , and the black hole horizon radius r_+ , are gauged to quantities without units as RT , $\frac{R^2}{l^2}$, and $\frac{r_+}{R}$. The ranges of these quantities are important. They are: $0 \leq RT < \infty$, $0 \leq \frac{R^2}{l^2} < \infty$, and $0 \leq \frac{r_+}{R} \leq 1$.

7.3 ACTION, FREE ENERGY, ENTROPY, MEAN ENERGY, AND HEAT CAPACITY

The Euclidean action in Eq. (7.2) for the space in Eq. (7.3) is precisely the gravitational action with negative cosmological constant I_{gl} in four dimensions, i.e. $I = I_{gl}$. We can easily compute it by substituting the line element into the action written in terms of the components of the spherically symmetric metric

$$\begin{aligned} I = I_{gl} = & \left(\frac{\beta r}{l_p^2} \left(\sqrt{V_{\text{AdS}}(r)} - \sqrt{V(r)} \right) \right) \Big|_{r \rightarrow R} - \frac{\pi}{l_p^2} \left(\frac{V' r^2}{4\pi T_+^H} \right) \Big|_{r=r_+} \\ & + \frac{1}{8\pi l_p^2} \int_M \frac{r^2}{2\pi T_+^H} r^2 \left(G^\tau_\tau - \frac{3}{l^2} \right) d^4x, \end{aligned} \quad (7.7)$$

where $V_{\text{AdS}}(r) = 1 + \frac{r^2}{l^2}$ is the pure AdS redshift factor obtained from $V(r)$ by setting $r_+ = 0$, see Chapter 3 for more details on the action. Now, the line element obeys the Einstein equation $G^\tau_\tau - \frac{3}{l^2} = 0$ and so we obtain the action evaluated at the zero loop approximation, whose designation is kept as I , as

$$I = \beta \frac{R}{l_p^2} \left(\sqrt{V_{\text{AdS}}(R)} - \sqrt{V(R)} \right) - \frac{\pi r_+^2}{l_p^2} \quad (7.8)$$

where $\beta = \frac{1}{T}$ is the inverse temperature of the ensemble, i.e., at the boundary of the heat reservoir, and where $V(R)$ is given by Eq. (7.4) at $r = R$ as

$$V(R) = \left(1 - \frac{r_+}{R}\right) \left(1 + \frac{R^2}{l^2} \left(1 + \frac{r_+}{R} + \left(\frac{r_+}{R}\right)^2\right)\right), \quad (7.9)$$

and also

$$V_{\text{AdS}}(R) = 1 + \frac{R^2}{l^2}. \quad (7.10)$$

Note that l in Eq. (7.8) is $l = l(\beta, R, r_+(R, T, l), l)$. The statistical mechanics ensemble is characterized by l which is fixed for each space, by $T = \frac{1}{\beta}$ and R which are fixed for each ensemble, with $r_+ = r_+(R, T, l)$ solutions of Eq. (7.6). These $r_+ = r_+(R, T, l)$ solutions yield the thermodynamic solutions of the problem. Note that l is also fixed.

In the zero loop approximation, the partition function in Eq. (7.1) becomes $Z = e^{-I}$. Since we are considering the canonical ensemble, the partition function is linked to the free energy F of the system by $Z = e^{-\beta F}$, defined by the Legendre transform of the mean energy as $F = E - TS$. Therefore, the action in the zero loop approximation is connected to the free energy as $I = \beta F$, and so the free energy is given by

$$F = \frac{R}{l_p^2} \left(\sqrt{V_{\text{AdS}}(R)} - \sqrt{V(R)} \right) - T\pi \frac{r_+^2}{l_p^2}. \quad (7.11)$$

Now from the derivatives of F , we are able to obtain the thermodynamic properties of the canonical ensemble of a Schwarzschild-AdS black hole inside a cavity. Namely, the entropy can be given by $S = - \left(\frac{\partial F}{\partial T} \right)_R$, where the subscript means the quantity that is kept fixed while performing the derivative, obtaining thus

$$S = \pi \frac{r_+^2}{l_p^2}. \quad (7.12)$$

The thermodynamic pressure can also be obtained, through the derivative $8\pi R p = - \left(\frac{\partial F}{\partial R} \right)_T$, as

$$p = \frac{1}{8\pi R l_p^2} \left(\frac{1 + 2\frac{R^2}{l^2} - \frac{r_+}{2R}(1 + \frac{r_+^2}{l^2})}{\sqrt{V(R)}} - \frac{1 + 2\frac{R^2}{l^2}}{\sqrt{V_{\text{AdS}}(R)}} \right). \quad (7.13)$$

Finally, the mean energy can be obtained from the Legendre transformation $E = F + TS$ as

$$E = \frac{R}{l_p^2} \left(\sqrt{V_{\text{AdS}}(R)} - \sqrt{V(R)} \right). \quad (7.14)$$

Regarding thermodynamic stability, the quantity that gives information about stability is the heat capacity at constant area $A = 4\pi R^2$, C_A . The heat capacity is

given by $C_A = \left(\frac{\partial E}{\partial T} \right)_A$, which by using Eq. (7.14) together with the solutions of the ensemble $r_+(T, R)$ given by solving Eq. (7.6), we obtain

$$C_A = \frac{4\pi r_+^2 (1 + 3\frac{r_+^2}{l_p^2}) V(R)}{l_p^2 \left(\frac{r_+}{R} \left(1 + 3\frac{r_+^2}{l_p^2} \right)^2 + 2V(R) \left(3\frac{r_+^2}{l_p^2} - 1 \right) \right)}, \quad (7.15)$$

which can also be obtained from the second derivative of the free energy F . Now, one has a thermodynamically stable system if

$$C_A \geq 0, \quad (7.16)$$

which is verified if the denominator in Eq. (7.15) is positive. Another alternative to understand the condition Eq. (7.16) is by relating it to the derivative of the solution, as one has $C_A = -\frac{R}{2l_p^2 \sqrt{V(R)}} \frac{\partial V(R)}{\partial r_+} \frac{\partial r_+}{\partial T}$. Since the derivative $\frac{\partial V(R)}{\partial r_+}$ is negative, the condition for stability in Eq. (7.16) is satisfied if $\frac{\partial r_+}{\partial T}$ is positive, i.e. the solution that obeys $\frac{\partial r_+}{\partial T} > 0$ is stable. We must comment on the type of stability considered here. The stability condition in Eq. (7.16) is the stability condition of the canonical ensemble with fixed area and fixed temperature only. This is different from intrinsic thermodynamic stability, which requires further conditions on the concavity of the free energy. Here, we are only interested on the thermodynamic stability of the ensemble.

7.4 THERMODYNAMIC SOLUTIONS OF SCHWARZSCHILD-ANTI-DE SITTER BLACK HOLES IN THE CANONICAL ENSEMBLE

7.4.1 Temperature equation

The solutions of the event horizon radius can be found by the condition of temperature equilibrium at the boundary of the cavity. This condition is written in Eq. (7.6), which is translated by setting the local temperature according to the Tolman formula at the boundary to be the fixed temperature T . We can indeed write Eq. (7.6) explicitly as

$$4\pi T = \frac{1}{r_+} \frac{1 + 3\frac{R^2}{l_p^2} \frac{r_+^2}{R^2}}{\sqrt{1 - \frac{r_+}{R}} \sqrt{1 + \frac{R^2}{l_p^2} \left(1 + \frac{r_+}{R} + \frac{r_+^2}{R^2} \right)}}. \quad (7.17)$$

The strategy is then to invert Eq. (7.17) in order to find the solutions r_+ in function of the fixed parameters of the ensemble T and R , and also in function of the AdS length l . Depending on the parameters (T, R, l) , there can be no solution, one solution, or two solutions for r_+ . When there are two solutions, these are denoted by

$$r_{+1} = r_{+1}(R, l, T), \quad (7.18)$$

$$r_{+2} = r_{+2}(R, l, T), \quad (7.19)$$

with $r_{+1} \leq r_{+2}$. For a given T and R , the existence of two roots r_{+1} and r_{+2} is similar to the case of the Schwarzschild space [68], with here having additionally the parameter l .

We now proceed with the analysis of the solutions for the cases, $0 \leq \frac{R^2}{l^2} < \infty$ and $0 < RT < \infty$. In general there are no analytical solutions. However, special attention in this paper is given to two limiting cases, for small cosmological constant $0 \leq \frac{R^2}{l^2} \ll 1$, and in the very high temperature $RT \rightarrow \infty$, where analytical solutions can be found.

7.4.2 Solutions in two limiting cases

7.4.2.1 Solutions for small cosmological constant, $\frac{R^2}{l^2} \ll 1$

For very small $\frac{R^2}{l^2}$, $\frac{R^2}{l^2} \ll 1$, we find from Eq. (7.17) that there are no black hole solutions for

$$RT < \frac{\sqrt{27}}{8\pi} \left(1 + \frac{5}{18} \frac{R^2}{l^2} \right), \quad \frac{R^2}{l^2} \ll 1, \quad (7.20)$$

and there are two black hole solutions for

$$RT \geq \frac{\sqrt{27}}{8\pi} \left(1 + \frac{5}{18} \frac{R^2}{l^2} \right), \quad \frac{R^2}{l^2} \ll 1. \quad (7.21)$$

One of the two solutions is the small black hole $r_{+1}(R, l, T)$, and the other solution is the large black hole $r_{+2}(R, l, T)$. The two solutions merge into one sole solution when the equality sign in Eq. (7.21) holds. In this case, the coincident double solution has horizon radius given by

$$\frac{r_{+1}}{R} = \frac{r_{+2}}{R} = \frac{2}{3} \left(1 - \frac{17}{27} \frac{R^2}{l^2} \right), \quad \frac{R^2}{l^2} \ll 1. \quad (7.22)$$

For zero cosmological constant, $|\Lambda|R^2 = 0$, i.e., $\frac{R^2}{l^2} = 0$, we have a pure Schwarzschild black hole and we recover York's result of $RT \geq \frac{\sqrt{27}}{8\pi}$ to have black hole solutions, the solutions merge with $\frac{r_{+1}}{R} = \frac{r_{+2}}{R} = \frac{2}{3}$.

One could work out in the regime $\frac{R^2}{l^2} \ll 1$ the action I , the thermodynamic energy E , the entropy S , and the heat capacity C_A , given through Eqs. (7.8) to (7.15). Apart from the entropy expression $S = 4\pi r_+^2$, valid for each of the two black hole solutions, the calculation of the other quantities is not practical and they are not particularly illuminating. However, an instance where all quantities can be worked out, in particular the heat capacity C_A with a simple expression is the high temperature limit, which we turn now.

7.4.2.2 Solution in the high temperature limit, RT high

For the range of values of the cosmological constant considered in this section, $0 \leq \frac{R^2}{l^2} < \infty$, we can find solutions in the limit of RT goes to infinity, see (7.17).

Since R is the quantity that we consider as the gauge, RT going to infinity is the same in this context as T going to infinity.

For a given T , there are two black hole solutions, the small black hole solution r_{+1} and the large black hole solution r_{+2} . We set the heat reservoir temperature T fixed but very high, in the sense that $T \rightarrow \infty$. From Eq. (7.6) there are two possibilities. Either $T_+^H \rightarrow \infty$ which corresponds to the small black hole solution having a very small r_{+1} , or $V(R) \rightarrow 0$ which corresponds to the large black hole solution r_{+2} approaching the reservoir radius.

The first solution for a very high heat reservoir temperature, $T \rightarrow \infty$, corresponds to the limit $T_+^H \rightarrow \infty$, which from Eq. (7.5) means that $r_+ = r_{+1} \rightarrow 0$. In this limit, we have

$$T_{+1}^H = \frac{1}{4\pi r_{+1}}, \quad (7.23)$$

where the equality sign is valid within the approximation taken. From Eq. (7.17), one can find the leading order behaviour of the small black hole solution r_{+1} as

$$\frac{r_{+1}}{R} = \frac{1}{4\pi RT \sqrt{1 + \frac{R^2}{l_p^2}}}, \quad (7.24)$$

where the equality sign is valid within the approximation taken. The expression inside the square root of Eqs. (7.24) is clearly positive. As a by-product, one can also find the black hole mass $2l_p^2 m = r_+ + \frac{r_+^3}{l_p^2}$ that in this limit one has $m_1 l_p^2 = \frac{r_{+1}}{2}$. One could work out in this order, i.e., $T \rightarrow \infty$, the action I , the energy E , the entropy S , and the heat capacity C_A , given through Eqs. (7.8) to (7.15). The most interesting quantity is the heat capacity C_A , which yields the criterion for thermodynamic stability, indeed when $C_A < 0$ the solution is thermodynamically unstable, when $C_A \geq 0$ the solution is thermodynamically stable. From $C_A = \left(\frac{\partial E}{\partial T}\right)_A$, one finds from Eq. (7.14) that $C_A = \frac{1}{2l_p^2 \sqrt{V(R)}} \left(\frac{\partial r_{+1}}{\partial T}\right)_R$, which upon using Eq. (7.24) yields

$$C_{A+1} = -\frac{1}{8\pi l_p^2 T^2 \left(1 + \frac{R^2}{l_p^2}\right)^{\frac{3}{2}}} < 0, \quad (7.25)$$

so that C_A for the small black hole r_{+1} is negative. The heat capacity C_{A+1} can also be computed through Eq. (7.15) with this limit applied. The small black hole r_{+1} solution is thus unstable. Note that actually, the black hole should be surrounded by quantum fields, with their backreaction on the metric being neglected here. However, if $T^H \rightarrow \infty$, the energy density and other components of the renormalized stress-energy tensor should diverge. To avoid this, we restrict r_{+1} in the sense that it has to be larger than the Planck length scale l_p , i.e., $r_{+1} > l_p$.

The second solution for a very high heat reservoir temperature, $T \rightarrow \infty$ has $V(R) \rightarrow 0$. It is clear from Eqs. (7.6) and (7.9) that the condition $V(R) \rightarrow 0$, implies, for the whole range $0 \leq \frac{R^2}{l_p^2} < \infty$, that r_{+2} should be near the cavity radius, i.e. $r_{+2} = R$ minus corrections. Now, from Eq. (7.5), one has in this limit

$$T_{+2}^H = \frac{1 + \frac{R^2}{l_p^2}}{4\pi R}, \quad (7.26)$$

where the equality sign is valid within the approximation taken. In first order, one can perform a Taylor expansion, and write $V(R) = \left(\frac{dV}{dr}\right)\Big|_{r=r_{+2}} (R - r_{+2})$ plus higher order terms. Since $\left(\frac{dV}{dr}\right)\Big|_{r=r_{+2}} = 4\pi T_{+2}^H$, one can write $V(R) = 4\pi T_{+2}^H (R - r_{+2})$. Using Eq. (7.6), or Eq. (7.17), one has

$$\frac{r_{+2}}{R} = 1 - \frac{1 + 3\frac{R^2}{l_p^2}}{(4\pi RT)^2}, \quad (7.27)$$

where the equality is valid within the approximation taken. As a by-product, the ADM mass can be found through $2ml_p^2 = r_+ + \frac{r_+^3}{l_p^2}$, which in this limit becomes $m_2 l_p^2 = \frac{R}{2} \left[1 + \frac{R^2}{l_p^2} - \frac{(1 + 3\frac{R^2}{l_p^2})^2}{(4\pi RT)^2} \right]$. One could work out in this order, i.e., $T \rightarrow \infty$, the action I , the energy E , the entropy S , and the heat capacity C_A , given through Eqs. (7.8) to (7.15). Again, the most interesting quantity is the heat capacity C_A . For the heat capacity C_A , given by $C_A = \left(\frac{\partial E}{\partial T}\right)_A$, one finds from Eq. (7.14) that $C_A = \frac{1}{l_p^2 \sqrt{V(R)}} \left(\frac{\partial m_2}{\partial T}\right)_R$, where it was used the expression $V(R) = 1 - \frac{2m_2}{R} + \frac{R^2}{l_p^2}$. Thus, using the expression for m_2 just found above, one has $C_A = \frac{1}{l_p^2 \sqrt{V(R)}} \frac{1}{2} \frac{1 + 3\frac{R^2}{l_p^2}}{16\pi^2 T^3 R}$ and since $\sqrt{V(R)} = \frac{1 + 3\frac{R^2}{l_p^2}}{4\pi RT}$ it gives

$$C_{A+2} = \frac{1 + 3\frac{R^2}{l_p^2}}{4\pi l_p^2 T^2} > 0, \quad (7.28)$$

so that C_{A+2} is small and positive. The large black hole r_{+2} solution is thus stable.

7.4.3 Full spectrum of the Schwarzschild-anti de Sitter thermodynamic black hole solutions and diagrams

7.4.3.1 Preliminaries

We now display the solutions in figures accompanied by a qualitative analysis. The figures are important to understand the thermodynamic solutions of the Schwarzschild-anti-de Sitter horizons in a cavity. There are two different figures. The first figure contains the curves $\frac{r_{\pm}}{R}$, that are solution of the thermodynamic equilibrium, for a fixed value of $4\pi RT$, as a function of $\frac{R}{l_p}$, see Fig. 7.2. The second and third figures contain the curves $\frac{r_{\pm}}{R}$, for a fixed value of $\frac{R^2}{l_p^2}$, as a function of $4\pi RT$, see Fig. 7.3, and as a function of $4\pi l_p T$, see Fig. 7.4. We discuss the physical interpretation and present the mathematical analysis of the solutions afterwards.

We can use the variable T , RT , or $l_p T$ to perform the analysis of the solutions. The difference between them relates to the different limiting cases one wishes to analyze. York maintains R fixed and T fixed independently, so RT fixed may be good in certain circumstances, but when one fixes a parameter independently of the other, it may be better to use $l_p T$. Then, T fixed is the same as $l_p T$ fixed. In terms

of RT , which is a good choice for $R < \infty$, and for $R \rightarrow \infty$ with $T \rightarrow 0$, one has from Eq. (7.17) that

$$4\pi RT = \frac{1}{\frac{r_+}{R}} \frac{1 + 3\frac{r_+^2}{R^2} \frac{R^2}{l^2}}{\sqrt{1 - \frac{r_+}{R}} \sqrt{1 + \frac{R^2}{l^2} \left(1 + \frac{r_+}{R} + \frac{r_+^2}{R^2}\right)}}. \quad (7.29)$$

In terms of lT , which is a good choice for T fixed and to consider the limit $R \rightarrow \infty$, one has from Eq. (7.17) that

$$4\pi lT = \frac{1}{\frac{r_+}{R} \frac{R}{l}} \frac{1 + 3\frac{r_+^2}{R^2} \frac{R^2}{l^2}}{\sqrt{1 - \frac{r_+}{R}} \sqrt{1 + \frac{R^2}{l^2} \left(1 + \frac{r_+}{R} + \frac{r_+^2}{R^2}\right)}}. \quad (7.30)$$

We can now analyze for each region of parameters the solutions of thermodynamic equilibrium.

7.4.3.2 Solutions and behaviour display for the Schwarzschild-anti-de Sitter thermodynamic black hole solutions with RT fixed

The first figure is shown in Fig. 7.2, describing how the black hole horizon radii $\frac{r_+}{R}$ behave in relation to $\frac{R}{l}$ for each $4\pi RT$.

The case $4\pi RT = \frac{\sqrt{27}}{2} = 2.598$, i.e., $RT = \frac{\sqrt{27}}{8\pi} = 0.207$, the equalities in decimal notation being approximate, is the first solution displayed in Fig. 7.2 by a black dot. This solution corresponds to the one with zero cosmological constant, $\frac{R}{l} = 0$, i.e., $\sqrt{\Lambda}R = 0$, which is the pure Schwarzschild solution found first by York. This solution is a coincident horizon solution with $\frac{r_{+1}}{R} = \frac{r_{+2}}{R} = \frac{2}{3} = 0.667$, the equality in decimal notation being approximate. For other larger $\frac{R}{l}$ there are no solutions.

The case $4\pi RT = 3.456$, the equality in decimal notation being approximate, i.e., $RT = 0.275$, is displayed by a blue curve in Fig. 7.2. There are two solutions in this case, and when $\frac{R}{l} = 14$, approximately, then $\frac{r_{+1}}{R} = \frac{r_{+2}}{R} = 0.04$, approximately. For other larger $\frac{R}{l}$, there are no solutions.

The case $4\pi RT = 12.57$, the equality in decimal notation being approximate i.e., $RT = 1$, is displayed by a yellow curve in Fig. 7.2. There are two solutions that exist through every $\frac{R}{l}$, that only meet at $\frac{R}{l} \rightarrow +\infty$.

The same behaviour of the case $4\pi RT = 12.57$ occurs to the cases $4\pi RT = 62.83$ with $RT = 5$, $4\pi RT = 125.66$ with $RT = 10$ and $4\pi RT = 1256.6$ with $RT = 100$. They are respectively displayed by a curve in red, green and purple in Fig. 7.2. Again there are two solutions, one that starts near $\frac{r_+}{R} = 1$ and decreases towards zero for increasing $\frac{R}{l}$, and another that starts near $\frac{r_+}{R} = 0$ and decreases towards zero. The solutions never meet.

7.4.3.3 Solutions and behaviour display for the Schwarzschild-anti-de Sitter thermodynamic black hole solutions with $\frac{R}{l}$ fixed

We display a snapshot for each $\frac{R}{l}$ of how the black hole horizon radii $\frac{r_+}{R}$ behave in relation to T , in Figs. 7.3 and 7.4. Specifically, Fig. 7.3 shows the behaviour in $4\pi RT$, while Fig. 7.4 shows the behaviour in $4\pi lT$.

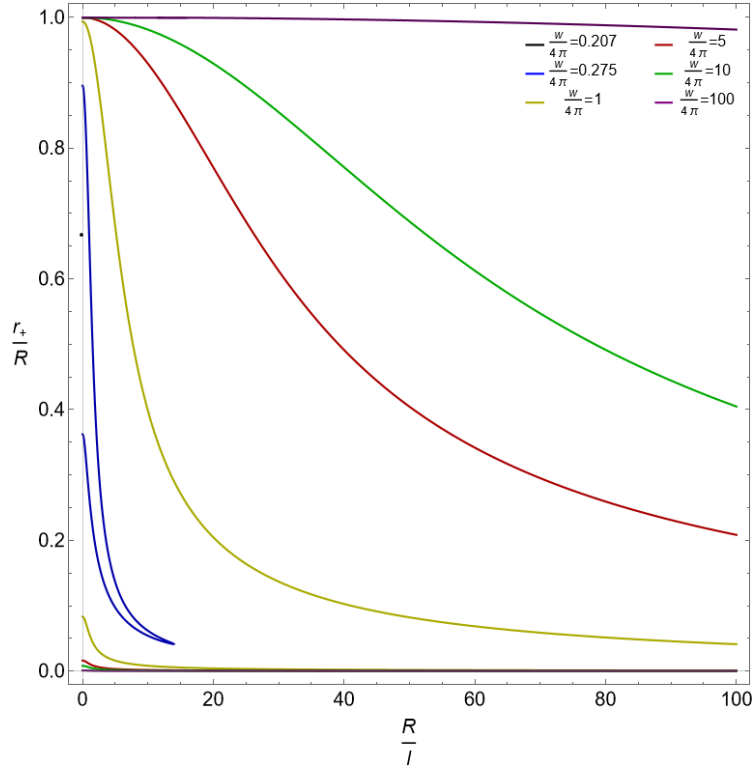


Figure 7.2: Plots of $\frac{r_+}{R}$ as a function of $\frac{R}{l}$ for six different values of $4\pi RT$: $4\pi RT = \frac{\sqrt{27}}{2} = 2.60$ with $RT = \frac{\sqrt{27}}{8\pi} = 0.207$ as a black dot, $4\pi RT = 3.46$ with $RT = 0.275$ as a blue curve, $4\pi RT = 12.57$ with $RT = 1$ as a yellow curve, $4\pi RT = 62.83$ with $RT = 10$ as a red curve, $4\pi RT = 125.7$ with $RT = 10$ as a green curve, and $4\pi RT = 1250$ with $RT = 100$ as a purple curve.

With respect to the curves $\frac{r_+}{R}$ in function of $4\pi RT$, the cases $\frac{R}{l} = \sqrt{10}$, $\frac{R}{l} = 10$ and $\frac{R}{l} = 100$ are displayed as green, blue and red curves, respectively in Fig. 7.3. For the three cases, there are no black hole solutions for $4\pi RT < 3.339$ for the green curve, $4\pi RT < 3.448$ for the blue curve and $4\pi RT < 3.463$ for the red curve, where the numerics are approximate. For larger temperatures, there are always two solutions. The solutions start from a bifurcating point where both solutions coincide and the small black hole decreases towards zero while the large black hole increases towards $\frac{r_+}{R} = 1$, for increasing temperature. It is important to note the similarities in behaviour of the solutions with the York's case, i.e. $\frac{R}{l} = 0$ or zero cosmological constant.

With respect to the curves $\frac{r_+}{R}$ in function of $4\pi lT$, the cases $\frac{R}{l} = \sqrt{10}$, $\frac{R}{l} = 10$ and $\frac{R}{l} = 100$ are displayed as green, blue and red curves, respectively in Fig. 7.4. For the three cases, there are no black hole solutions for $4\pi lT < 1.056$ for the green curve, $4\pi lT < 0.344$ for the blue curve and $4\pi lT < 0.034$ for the red curve, where the numerics are approximate. For larger temperatures, there are always two solutions. The solutions start from a bifurcating point where both solutions coincide and the small black hole decreases towards zero while the large black hole increases towards $\frac{r_+}{R} = 1$, for increasing temperature. While these cases can be described as a

rescale of the previous cases, the plot in function of $4\pi lT$ shows information about the limit of large cosmological constant, which we shall explore further below.

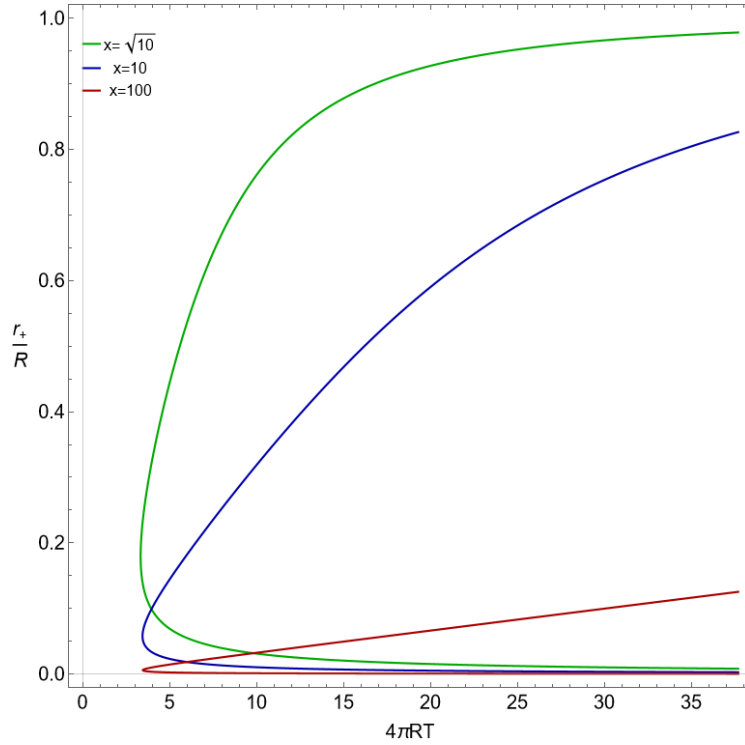


Figure 7.3: Plots of $\frac{r_+}{R}$ as a function of $4\pi RT$ for three different values of $\frac{R}{l}$: $\frac{R}{l} = \sqrt{10}$ as the green curve, $\frac{R}{l} = 10$ as the blue curve and $\frac{R}{l} = 100$ as the red curve.

7.4.4 Physical analysis of the solutions

We now make additional qualitative comments to the plots that have been displayed in Figs. 7.2-7.4.

One striking feature, that can be deduced from the plots, is that the space of black hole horizon radius solutions is enlarged as the reservoir temperature T , or rather $4\pi RT$, is increased. In fact, for very low temperatures there are no solutions for any Λ , or rather, for any ΛR^2 . At the temperature $4\pi RT = \frac{\sqrt{27}}{2} = 2.598$ there is only one solution, the pure Schwarzschild solution with zero cosmological constant, and it is the coincident solution. For higher $4\pi RT$ there are two solutions, one large, one small, up to a value of ΛR^2 . This value grows rapidly with increasing temperature. Also, with growing temperature, the large and small black holes tend to radius ∞ and radius 0 , for small ΛR^2 .

With the help of Figs. 7.2-7.4, we can give a qualitative explanation for the reason of why black hole solutions with some nonzero cosmological constant appear only for ever higher temperatures RT . The temperature of the reservoir defines a thermal length scale $\lambda = \frac{1}{T}$ for the system. There is also another length scale, the reservoir radius R , and the cosmological length $l = \frac{3}{\sqrt{|\Lambda|}}$. Thus, we can start from $|\Lambda| = 0$,

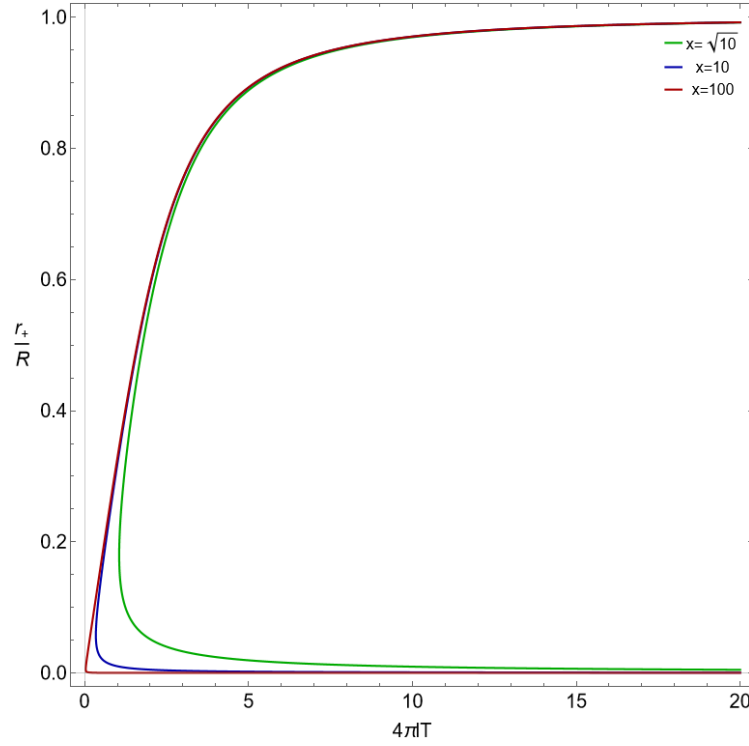


Figure 7.4: Plots of $\frac{r_+}{R}$ as a function of $4\pi l T$ for three different values of $\frac{R}{T}$: $\frac{R}{T} = \sqrt{10}$ as the green curve, $\frac{R}{T} = 10$ as the blue curve and $\frac{R}{T} = 100$ as the red curve.

so that the cosmological length scale $l = \frac{3}{\sqrt{|\Lambda|}}$ is infinite, $l = \infty$. In this case there is no coupling of this length scale with the other two, $\lambda = \frac{1}{T}$ and R . In this situation, we see that for low T , or high λ , one has $\lambda \gg R$. Since the thermal wavelength is very large compared to the reservoir radius R , then this wavelength is stuck to the reservoir and the corresponding energy cannot collapse to form a black hole in any circumstances. When T is sufficiently increased, i.e., $RT = \frac{R}{\lambda}$ is larger than approximately 0.2, the wavelength λ is sufficiently small, and the corresponding thermal energy can travel freely inside the reservoir and can collapse, so that formation of black holes is possible. The value $RT = \frac{R}{\lambda} = \frac{\sqrt{27}}{8\pi} = 0.206$, with the last equality approximate, divides no black hole from two black hole solutions. The existence of two black hole solutions for a given temperature T , i.e., a given thermal wavelength λ can also be explained. The small black hole form with an r_+ of the order of λ , and is unstable as the energy packets with length λ that escape from the black hole cannot be scattered back in enough time to maintain r_+ stable. The large black hole forms with an $R - r_+$ of the order of λ so the black hole and the reservoir exchange energy in a stable manner, as the energy packets with length λ that escape from the black hole are scattered back in enough time to maintain r_+ stable. Now, we do the analysis for the case of finite cosmological constant, i.e. l finite. For low enough l but slightly larger than R , the space inside the reservoir shrinks, due to the negative cosmological constant, and so in some way this inner space has less proper length along the radius. Although the reservoir

radius related to its area is still R , the radial length related to the volume is small, and so the volume is also small. This means that energy packets with same λ , same temperature, relatively to the cases with infinite l , cannot yet travel freely inside the cavity and cannot form black holes. As the temperature increases, λ decreases and one can have two black holes down to some finite l which gives the two coincident solutions. The value of r_+ for this finite l case decreases to smaller values as $\frac{R}{l}$ increases. This can be understood as well. As the temperature increases more, λ decreases more, and one can have a higher $\sqrt{\Lambda}R = 3\frac{R}{l}$ for the coincident solution. But higher $\sqrt{\Lambda}R$ means the space is further shrank and so the coincident solution has a value $r_{+1} = r_{+2}$ small. There is however a certain finite temperature at which the coincident solution only occurs for infinite cosmological constant, and the same happens for larger temperatures. We still haven't understood why this occurs.

This physical interpretation holds either for fixing RT or fixing $\frac{R}{l}$ as the existence or not of black hole solution is an interplay between R , T , and l , as we described.

7.4.5 Mathematical analysis of the solutions

7.4.5.1 Nomenclature

We now obtain through a mathematical analysis some important features displayed in the plots above, Figs. 7.2, 7.3 and 7.4. The important equation to analyze here is Eq. (7.29). The natural variables without units are $\frac{R}{l}$ and $\frac{r_+}{R}$. To shorten the notation, we define the variables x and y as

$$x \equiv \frac{R}{l}, \quad (7.31)$$

$$y \equiv \frac{r_+}{R}, \quad (7.32)$$

with the range of the variables being $0 \leq x < \infty$, and $0 \leq y \leq 1$. Furthermore, the variable w is additionally defined as

$$w \equiv 4\pi RT. \quad (7.33)$$

Then, with these definitions, Eq. (7.29) becomes

$$w = \frac{1 + 3x^2y^2}{y\sqrt{1-y}\sqrt{1+x^2(1+y+y^2)}}. \quad (7.34)$$

There are solutions for $w_0 \leq w < \infty$, where for convenience, $w_0 \equiv \frac{\sqrt{27}}{2} = 2.598$ is defined, with equality being approximate. Now, for a fixed temperature T , or fixed w , one has $dw = 0$, and so $\frac{dy}{dx} = -\frac{\frac{\partial w}{\partial x}}{\frac{\partial w}{\partial y}}$. After some calculations, one can obtain that $\frac{\partial y}{\partial x}$ at constant w is

$$\frac{\partial y}{\partial x} = \frac{2xy(1-y)Q(x,y)}{3R(x,y)}, \quad (7.35)$$

where

$$Q(x,y) \equiv (1+y+y^2)(1-3x^2y^2) - 6y^2, \quad (7.36)$$

and

$$R(x, y) \equiv -2(1 - 3x^2y^2)(1 - y)[1 + x^2(1 + y + y^2)] + (1 + 3x^2y^2)^2y. \quad (7.37)$$

In addition, we need in the analysis $\frac{\partial w}{\partial y}$ at constant x . One can obtain from Eq. (7.34) that

$$\frac{\partial w}{\partial y} = \frac{R(x, y)}{2y^2(1 - y)^{3/2} [1 + x^2(1 + y + y^2)]^{3/2}}. \quad (7.38)$$

We must recall that for each $\frac{R}{T}$ there are two solutions, r_{+1} , the small solution, and, r_{+2} , the large solution, which change as RT is changed, i.e., for each x , there are y_1 and y_2 , which change as w is changed. To summarize, the ranges of x , y , w for Eq. (7.34) can be written explicitly,

$$0 \leq x < \infty, \quad 0 \leq y \leq 1, \quad w_0 \leq w < \infty, \quad (7.39)$$

with $w_0 \equiv \frac{\sqrt{27}}{2} = 2.598$.

7.4.5.2 Analysis

With w fixed, we first analyze the coincident solutions $y_c = \frac{r_{+c}}{R}$, where $y_c = y_1 = y_2$, which are crucial for the analysis. We then analyze the solutions y_1 and y_2 .

With w fixed, we can look at the point $x = x_c$ where $y_1 = y_2 = y_c$. Unfortunately, we have not found a closed analytic solution. Nevertheless, we are able to obtain certain features. The point $x = x_c$ occurs when $\frac{\partial y}{\partial x} = \infty$, see Eq. (7.35), i.e., $R = 0$. So, the coincident solution satisfies the equation $2(1 - 3x^2y^2)(1 - y)[1 + x^2(1 + y + y^2)] - (1 + 3x^2y^2)^2y = 0$ together with Eq. (7.34). Since both equations are quadratic in x , one can subtract one by the other and obtain a formula for $x_c(y_c, w)$. One can further insert this relation into Eq. (7.34) and obtain the equation for $y_c(w)$

$$w^4y_c^6 + (2w^2 - 8)w^2y_c^3 - 12w^2y_c^2 + 48 - 4w^2 = 0, \quad (7.40)$$

which is a sixth order polynomial equation and cannot be solved numerically. The expression for x_c however can be simplified further using Eq. (7.40), yielding

$$x_c^2 = \frac{w^2(12 - 16y_c - w^2y_c^4)}{12(12 - w^2)}. \quad (7.41)$$

We note that there is an important value of w , which we denote here by $w_1 = 2\sqrt{3} = 3.4641$ approximately. The pole in Eq. (7.41) shows that the coincident solution y_c at x_c does not exist for $w > w_1$, since the numerator is positive for the solution y_c . In Fig. 7.5, we present the plot of the coincident solution $y_c(w)$, where $x = 0$ when $y_c = \frac{2}{3}$ and x tends to infinity when y_c tends to zero. As seen analytically, this last case happens when $w = w_1$.

To summarize, between $w_0 < w < w_1$, there is a y_c in the range $0 \leq y \leq \frac{2}{3}$ and a x_c where both solutions y_1 and y_2 coincide. In the limit $w = w_1$, the coincident solution becomes $y_c = 0$ with x_c going to infinity. For the range $w > w_1$, there is no coincident solution.

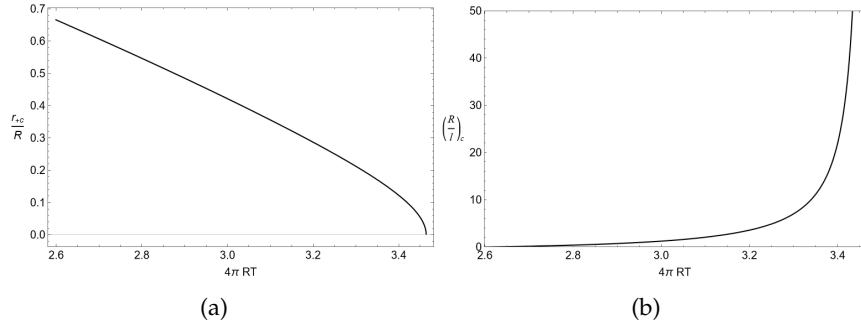


Figure 7.5: Plot of the coincident solution y_c in (a) and x_c in (b) in function of w . For $y_c = \frac{2}{3}$, $x_c = 0$ and x_c is then increased towards infinity, yielding $y_c = 0$ at $w = w_1 = 2\sqrt{3}$.

Considering the first solution y_1 from Eq. (7.34), we find that y_1 obeys an equation of the type

$$y_1 \sqrt{1 - y_1} \sqrt{1 + x^2(1 + y_1 + y_1^2)} w = 1 + 3x^2 y_1^2. \quad (7.42)$$

for a given fixed w in the range $w_0 \leq w \leq \infty$, $w_0 = \frac{\sqrt{27}}{2}$. An important property are the points where $\frac{\partial y_1}{\partial x} = 0$. From Eq. (7.35), this happens when $x = 0$ or $Q(x, y_1) = 0$. The point $x = 0$ can be point of a minimum, a saddle, or a maximum of y_1 depending on the value of w . The point given by the root $Q(x, y_1) = 0$ corresponds to a minimum of y_1 when it exists. The condition $Q(x, y_1) = 0$ can be reduced from Eq. (7.36) to the condition $x^2 = \frac{1 + y_1 - 5y_1^2}{3y_1^2(1 + y_1 + y_1^2)}$, where $x = x(y_1, w)$ according to Eq. (7.34). This latter equation has solutions for $x \geq 0$. When $x = 0$, one can find that the solution is $y_1 = \frac{1 + \sqrt{21}}{10}$, which putting back into Eq. (7.34), yields $w = \frac{10\sqrt{10}}{(1 + \sqrt{21})\sqrt{9 - \sqrt{21}}} \equiv w_*$, where w_* was defined and it has the value $w_* = 2.695$ approximately. For this temperature, there is thus a minimum of y_1 provided by $Q(x, y_1) = 0$. For $w > w_1$, there are no solutions of $Q(x, y_1) = 0$. This can be seen by understanding that there is no coincident solution for this range and the solution y_1 always decreases towards zero. For $w_* < w < w_1$, there are solutions of $Q(x, y_1) = 0$ for points $x > 0$, with the minimum of y_1 decreasing for larger w and x also increasing with larger w . On the other hand for $w < w_*$, there are also no solutions of $Q(x, y_1) = 0$.

Now, we are able to describe the solution y_1 in function of x with a fixed w . For $w_0 < w < w_*$, the solution at $x = 0$ gives a zero derivative of y_1 . We then conclude that for $w_0 < w < w_*$, the solution y_1 at $x = 0$ is a minimum of y_1 , i.e., y_1 starts from some value and then increases towards y_c . For $w = w_*$, the solution y_1 at $x = 0$ is a saddle point of y_1 but still it is the lowest value of y_1 , i.e. y_1 starts from $x = 0$ at some value and increases towards y_c . For $w_* < w < w_1$, the solution y_1 at $x = 0$ is a maximum of y_1 , and y_1 then decreases towards the minimum given by $Q(x, y_1) = 0$, and afterwards increases towards the coincident solution y_c . For $w \geq w_1$, the solution y_1 at $x = 0$ continues to be a maximum of y_1 and the solution decreases and tends to zero for larger x .

We now analyze the case of y_2 , the larger solution, in function of x with fixed w . From Eq. (7.34), one has

$$y_2 \sqrt{1 - y_2} \sqrt{1 + x^2(1 + y_2 + y_2^2)} w = 1 + 3x^2 y_2^2, \quad (7.43)$$

for a given fixed w in the range $w_0 \leq w \leq \infty$, $w_0 = \frac{\sqrt{27}}{2}$. An interesting property is the existence of the point $\frac{dy_2}{dx} = 0$. For the solution y_2 , the zero derivative occurs when $x = 0$, which corresponds to the only maximum of y_2 . One has that y_2 starts at $x = 0$ at a maximum, and then decreases for all x increasing. For $w_0 < w < w_1$, the solution y_2 decreases until it reaches the coincident solution y_c at some $x = x_c$. For $w_1 < w$, the solution y_2 decreases and tends to zero as x tends to infinity. As w increases from w_0 , the maximum of y_2 , which is at $x = 0$, increases. When $w \rightarrow \infty$ this maximum $y_2 \rightarrow 1$ for all x .

The solutions y_1 and y_2 in function of w with fixed x always follow the same pattern. For a certain value of w , the coincident solution $y_1 = y_2 = y_c$ appears and for increasing w , the solution y_1 decreases and tends to zero while y_2 increases and tends to $y = 1$. This happens for any value of $0 < x < \infty$.

7.5 THE PLANAR ADS BLACK HOLE AND THE HAWKING-PAGE BLACK HOLE SOLUTIONS: TAKING THE BOUNDARY TO INFINITY, $R \rightarrow \infty$

7.5.1 Preliminary analysis

We now consider the analysis of the solutions when the boundary goes to infinity $R \rightarrow \infty$. Due to the scaling property of the equations, we chose the temperature parameter as RT , but such parameter is not convenient for the limit $R \rightarrow \infty$ with T fixed for any $T \geq 0$. It turns out that we have to separate the cases $T > 0$ fixed, and $T = 0$ fixed in a correct manner. When $T > 0$, the limit to $R \rightarrow \infty$ is direct, the two black hole solutions r_{+1} and r_{+2} in this limit have a certain characteristic behavior. Indeed, the small unstable solution r_{+1} behaves as $r_{+1} = \frac{l}{4\pi RT}$, and so goes to zero $r_{+1} \rightarrow 0$, therefore $\frac{r_{+1}}{R} = 0$. The large solution r_{+2} behaves as $r_{+2} = cR$ for some $c > 0$, so $\frac{r_{+2}}{R} = c$ and this is the stable solution. The small unstable solution then tends to hot anti-de Sitter, while the large stable solution tends to a planar black hole, as we will see. When $T = 0$, the limit to $R \rightarrow \infty$ has to be taken with care. Indeed, the $R \rightarrow \infty$ in this case is such that RT should be finite, and so $T \rightarrow 0$ must be done in a definite manner. One can therefore parametrize $T = \frac{l}{R} T_*$ for some finite T_* . This case gives the two solutions of Hawking-Page, namely the unstable solution r_{+1} and the stable solution r_{+2} .

To see what type of geometry we obtain when performing the limit, we can write the Schwarzschild-AdS line element given in Eqs. (7.3) explicitly as

$$ds^2 = \frac{1}{(2\pi T_+^H)^2} \left(1 - \frac{r_+}{r} + \frac{r^2}{l^2} \left(1 - \left(\frac{r_+}{r} \right)^3 \right) \right) d\tau^2 + \frac{dr^2}{1 - \frac{r_+}{r} + \frac{r^2}{l^2} \left(1 - \left(\frac{r_+}{r} \right)^3 \right)} + r^2 d\Omega^2, \quad r_+ \leq r \leq R, \quad (7.44)$$

where $0 \leq \tau < 2\pi$, with $\frac{1}{T_+^H} = \frac{4\pi r_+}{1+3\frac{r_+^2}{l^2}}$, $T_+^H = \frac{1}{4\pi r_+} \left(1 + 3\frac{r_+^2}{l^2} \right)$. The space with the metric above is in thermal equilibrium at temperature T , the temperature of a reservoir placed at R , given by the imaginary proper time at the boundary. In agreement with the Tolman formula, the local temperature is then given by the $\tau\tau$ component of the metric. The relation between the temperature T and the event horizon radius in thermodynamic equilibrium is Eq. (7.6), which can be put in the form

$$4\pi RT = \frac{1}{\frac{r_+}{R}} \frac{1 + 3 \left(\frac{R}{l} \right)^2 \left(\frac{r_+}{R} \right)^2}{\sqrt{1 - \frac{r_+}{R}} \sqrt{\left(1 + \frac{R^2}{l^2} \left(1 + \frac{r_+}{R} + \left(\frac{r_+}{R} \right)^2 \right) \right)}}, \quad 0 \leq T < \infty, \quad r_+ \leq R < \infty, \quad (7.45)$$

so the reservoir temperature is fixed for each situation with $T \geq 0$. With Eqs. (7.44) and (7.45) we can now see the solutions that arise when $R \rightarrow \infty$ in the case the temperature of the reservoir is nonzero $T > 0$ and in the case the temperature of the reservoir is zero $T = 0$.

7.5.2 First limit: The planar AdS black hole solutions. Taking constant T with $T \geq 0$ first, and performing after the $R \rightarrow \infty$ limit

7.5.2.1 The planar AdS black hole solutions

For a given R , there can be up to two black hole solutions, if T is greater than a certain value. These are the small unstable solution r_{+1} and the large solution r_{+2} . For T less than this value, there is no black hole solution, but one can choose a different topology sector to obtain hot AdS space, i.e. pure AdS space with a temperature inside the cavity, which we regard here as a solution. For $T > 0$ and $R \rightarrow \infty$, one can show that the two black hole solutions still exist. Moreover, the small solution tends to $r_{+1} = 0$ and so degenerates to the hot AdS space in a sense, while the solution r_{+2} becomes a planar black hole in AdS.

Regarding the r_{+1} solution, for $R \rightarrow \infty$, the leading order expansion yields $\frac{r_{+1}}{R} = \frac{1}{4\pi RT}$ and thus $r_{+1} = \frac{1}{4\pi T}$ since T is a finite number. Therefore, we conclude that this is the Hawking small black hole or the Gibbons-Hawking small black hole. The leading order expansion gives precisely the relation of the Hawking temperature at spatial infinity.

Regarding the case of r_{+2} , the large and stable solution, for $T \geq 0$, and $R \rightarrow \infty$, Eq. (7.45) yields $4\pi lT = 3 \frac{\frac{r_+}{R}}{\sqrt{1-\frac{r_+}{R}} \sqrt{1+\frac{r_+}{R}+(\frac{r_+}{R})^2}}$ with $T \geq 0$ and $R = \infty$, where it was assumed that $r_+ = cR$, with c being some number. Note that we can do this since this is the behaviour of r_{+2} . Then, this suggests the following coordinate change and definitions

$$\begin{aligned}\bar{\tau} &= \tau, \quad \bar{r} = c \frac{r}{R} l, \\ \bar{r}_+ &= c \frac{r_+}{R} l, \quad \bar{R} = cl, \quad \bar{l} = l,\end{aligned}\tag{7.46}$$

with c a real number, $c > 0$. Note that the new reservoir coordinate radius is $\bar{R} = cl$. Then, we find $ds^2 = \frac{1}{(2\pi\bar{T}_+^H)^2} \frac{1}{\bar{l}^2} \left(\bar{r}^2 - \frac{\bar{r}_+^3}{\bar{r}} \right) d\bar{\tau}^2 + \frac{\bar{l}^2 d\bar{r}^2}{\bar{r}^2 - \frac{\bar{r}_+^3}{\bar{r}}} + \frac{\bar{r}^2}{\bar{l}^2} \left(\frac{R^2}{c^2} d\Omega^2 \right)$ where $0 < \bar{\tau} < 2\pi$, and now $\frac{1}{\bar{T}_+^H} = \frac{4\pi\bar{l}^2}{3\bar{r}_+}$. We note that the metric is blowing because it is covering the whole sphere with its radius R tending to infinity. This can be cured by precisely selecting a very small section of the sphere, which locally is flat. Choose a precise point with coordinates $\theta = \theta_0$, $\phi = \phi_0$, and expand the spherical metric around those points with arbitrarily small $\Delta\theta$ and $\Delta\phi$ but such that $\bar{x} = \frac{R}{c}\Delta\theta$ and $\bar{y} = \frac{R}{c}\sin\theta_0\Delta\phi$. Then the metric around such patch is

$$\begin{aligned}ds^2 &= \frac{1}{(2\pi\bar{T}_+^H)^2} \frac{1}{\bar{l}^2} \left(\bar{r}^2 - \frac{\bar{r}_+^3}{\bar{r}} \right) d\bar{\tau}^2 + \frac{\bar{l}^2 d\bar{r}^2}{\bar{r}^2 - \frac{\bar{r}_+^3}{\bar{r}}} \\ &\quad + \frac{\bar{r}^2}{\bar{l}^2} (d\bar{x}^2 + d\bar{y}^2), \quad \bar{r}_+ \leq \bar{r} \leq \bar{R},\end{aligned}\tag{7.47}$$

with $0 \leq \bar{\tau} < 2\pi$, $\bar{r}_+ \leq \bar{R} < \infty$, $-\infty < \bar{x} < \infty$, $-\infty < \bar{y} < \infty$, which is the planar black hole line element. The Hawking temperature is $\bar{T}_+^H = \frac{3\bar{r}_+}{4\pi\bar{l}^2}$. One can verify that there is no conical singularity in the $\bar{\tau} \times \bar{r}$ plane at $\bar{r} = \bar{r}_+$. The condition is $\sqrt{\frac{\partial_{\bar{r}} g_{\bar{\tau}\bar{\tau}}}{g_{\bar{r}\bar{r}}}} = 1$. If we write the metric as $ds^2 = \frac{1}{\bar{l}^2} \left(\frac{2\bar{l}}{3\bar{r}_+} \right)^2 \left(\bar{r}^2 - \frac{\bar{r}_+^3}{\bar{r}} \right) d\bar{\tau}^2 + \frac{\bar{l}^2 d\bar{r}^2}{\bar{r}^2 - \frac{\bar{r}_+^3}{\bar{r}}}$, with $0 < \bar{\tau} < 2\pi$, we can see that $\sqrt{\frac{\partial_{\bar{r}} g_{\bar{\tau}\bar{\tau}}}{g_{\bar{r}\bar{r}}}} = \left(\frac{1}{3\bar{r}_+} (2\bar{r} + \frac{\bar{r}_+^3}{\bar{r}^2}) (\bar{r}^2 - \frac{\bar{r}_+^3}{\bar{r}})^{-\frac{1}{2}} (\bar{r}^2 - \frac{\bar{r}_+^3}{\bar{r}})^{\frac{1}{2}} \right)_{\bar{r}_+} = 1$, as it should.

The temperature T of the reservoir is the inverse of the Euclidean time length given by

$$T = \frac{3\bar{r}_+}{4\pi\bar{l}\sqrt{\bar{R}^2 - \frac{\bar{r}_+^3}{\bar{R}}}},\tag{7.48}$$

where $\bar{R}$ is the new coordinate radial position of the reservoir, given by $\bar{R} = cl$ for some c , and is precisely the equation $4\pi lT = \frac{3 \frac{r_+}{R}}{\sqrt{1-\frac{r_+}{R}} \sqrt{1+\frac{r_+}{R}+(\frac{r_+}{R})^2}}$ with $T > 0$ and $R = \infty$, from which the analysis was started but with new definitions. Now, Eq. (7.48) can be put in the form $4\pi\bar{l}T = \frac{3 \frac{\bar{r}_+}{\bar{R}}}{\sqrt{1-\frac{\bar{r}_+}{\bar{R}} \sqrt{1+\frac{\bar{r}_+}{\bar{R}}+(\frac{\bar{r}_+}{\bar{R}})^2}}}$, which shows that $\frac{\bar{r}_+}{\bar{R}}$ is a function of $\bar{l}T$ alone, i.e. $\frac{\bar{r}_+}{\bar{R}} = \frac{\bar{r}_+}{\bar{R}}(\bar{l}T)$. From Eq. (7.48), when it exists, there is only one solution for $\frac{\bar{r}_+}{\bar{R}}(\bar{l}T)$ as expected from the limit we took. Moreover, there is always

a solution for any T , contrarily to the spherical case where for T below a certain value there are not solutions.

7.5.2.2 *The case of zero cosmological constant $\Lambda = 0$, i.e., $l = \infty$: The Gibbons-Hawking small black hole and Rindler boundary at infinity with flat space inside*

It is interesting to consider the case of zero cosmological constant $\Lambda = 0$, i.e., $l = \infty$. The small unstable black hole r_{+1} in this limit reduces to the Schwarzschild solution given by Gibbons and Hawking in [67]. We have seen that the large stable solution r_{+2} in the infinite cavity limit gives a planar black hole solution. Going further with the limit of zero cosmological constant, we must proceed with care. Indeed, from Eq. (7.47), we can deduce that the space becomes now Rindler space with the boundary at infinity receding with appropriate temperature T .

The limit that allows one to obtain the Rindler space from the planar black hole solution can be seen as follows. Having the planar black hole line element, Eq. (7.47), written as $ds^2 = \left(\frac{2l^2}{3\bar{r}_+}\right)^2 \frac{1}{l^2\bar{r}}(\bar{r}^3 - \bar{r}_+^3)d\tau^2 + \frac{l^2\bar{r}d\bar{r}^2}{\bar{r}^3 - \bar{r}_+^3} + \frac{\bar{r}^2}{l^2} \frac{l^2}{\bar{R}^2}(dx^2 + dy^2)$, one can employ the limit $\bar{r}_+ \rightarrow \bar{R}$ and $l \rightarrow +\infty$, but such that $l\sqrt{\bar{R}^3 - \bar{r}_+^3}$ is finite. In order to do this, one can evaluate the proper radial length as $\tilde{r}(\bar{r}) = l \int_{r_+}^{\bar{r}} \frac{\sqrt{\bar{\rho}d\rho}}{\sqrt{\rho^3 - \bar{r}_+^3}}$, which gives

$$\tilde{r}(\bar{r}) = \frac{2l}{3\bar{r}_+^{\frac{3}{2}}} \sqrt{\bar{r}^3 - \bar{r}_+^3}, \quad (7.49)$$

which is valid for very small $\bar{r} - \bar{r}_+$. The planar black hole metric then becomes $ds^2 = \frac{\bar{r}_+}{\bar{r}} \tilde{r}^2 d\tau^2 + d\tilde{r}^2 + \frac{\bar{r}_+^2}{\bar{R}^2}(dx^2 + dy^2)$. Consider that from Eq. (7.49), one has $\bar{r}^3 = \left(\frac{3\bar{r}_+^{3/2}\tilde{r}}{2l}\right)^2 + \bar{r}_+^3$ and also that $\bar{r}_+ \rightarrow \bar{R}$, then $l \rightarrow +\infty$ implies $\frac{\bar{r}_+}{\bar{r}} = 1 - \mathcal{O}\left(\frac{1}{l}\right)$ and $\frac{\bar{r}_+^2}{\bar{R}^2} = 1 - \mathcal{O}\left(\frac{1}{l}\right)$. Therefore, we find that the line element in this limit $l \rightarrow \infty$ is

$$ds^2 = \tilde{r}^2 d\tau^2 + d\tilde{r}^2 + dx^2 + dy^2, \quad (7.50)$$

which is the Rindler line element in the $\tau \times r$ plane times a flat plane. Note that the Rindler horizon is at $\tilde{r} = \tilde{r}_+ = 0$, which corresponds to the old black hole horizon $\bar{r} = \bar{r}_+$, see Eq. (7.49). Thus, the situation is the following after the limit. There is a reservoir at $\bar{R}$ at temperature T accelerating away with acceleration $a = \frac{1}{2\pi T}$, with T corresponding to the Unruh temperature.

7.5.3 *Second limit: The Hawking-Page spherical black hole solutions. Taking the $T \rightarrow 0$ limit, and concomitantly taking $R \rightarrow \infty$, with constant RT*

7.5.3.1 *The Hawking-Page spherical black holes*

When $R \rightarrow \infty$ is taken first, we see from Eq. (7.45) that the Hawking-Page solutions can be recovered by performing the limit $T \rightarrow 0$ such that $RT = \text{constant}$. We therefore have r_+ finite, although $\frac{r_+}{R} = 0$ since we are taking $R \rightarrow \infty$.

The limit can be seen with more care. From $4\pi RT = \frac{1}{\frac{r_+}{R}} \frac{1+3\left(\frac{R}{l}\right)^2\left(\frac{r_+}{R}\right)^2}{\sqrt{1-\frac{r_+}{R}} \sqrt{\left(1+\frac{R^2}{l^2}\left(1+\frac{r_+}{R}+\left(\frac{r_+}{R}\right)^2\right)\right)}}$,

see Eq. (7.45), the limit $R \rightarrow \infty$ gives $4\pi RT = \frac{1+(\frac{r_+}{l})^2}{r_+}l$. The idea is to define a new conformal temperature such that

$$T = T_* \frac{l}{R}, \quad T \rightarrow 0, \quad R \rightarrow \infty, \quad (7.51)$$

and so the thermodynamic equilibrium equation becomes

$$T_* = \frac{1}{4\pi} \frac{1+3\left(\frac{r_+}{l}\right)^2}{r_+}. \quad (7.52)$$

This is the limit $T \rightarrow 0$ and $R \rightarrow \infty$. Note that T_* is essentially T_+^H of Eq. (7.44). The equation in Eq. (7.52) yields the two Hawking-Page r_+ solutions. One is the solution

$$\frac{r_{+1}}{l} = \frac{2\pi}{3}lT - \sqrt{\left(\frac{2\pi}{3}lT_*\right)^2 - \frac{1}{3}}, \quad (7.53)$$

which is the small solution and it is unstable. The other solution is

$$\frac{r_{+2}}{l} = \frac{2\pi}{3}lT + \sqrt{\left(\frac{2\pi}{3}lT_*\right)^2 - \frac{1}{3}}, \quad (7.54)$$

which is the large solution and it is stable. These two black hole solutions exist for temperatures obeying $T_* \geq \frac{\sqrt{3}}{2\pi l}$. When there is equality $T_* = \frac{\sqrt{3}}{2\pi l}$, the two solutions merge into one given by $\frac{r_{+1}}{l} = \frac{r_{+2}}{l} = \frac{2\pi}{3}lT$. When $T_* < \frac{\sqrt{3}}{2\pi l}$, i.e., for low enough temperatures, there are no black hole solutions, one is in the presence of pure hot AdS space, also called classical hot space. Thus, the Hawking-Page solutions inherit from the R finite solutions the same properties.

7.5.3.2 The case of zero cosmological constant $\Lambda = 0$, i.e., $l = \infty$: The Gibbons-Hawking black hole and hot flat planar space

In the limit of infinite cavity, while keeping RT constant, we have the two Hawking-Page solutions. It is also interesting to proceed with the limit of zero cosmological constant in these two solutions. When $l = \infty$, the solution r_{+1} becomes the Gibbons-Hawking unstable Schwarzschild black hole solution, while the solution r_{+2} becomes the Rindler solution, but now the cavity resides at infinity.

Taking the limit $l \rightarrow \infty$ with T_* constant, one gets from Eq. (7.53) that $r_{+1} = \frac{1}{4\pi T_*}$ which is the Gibbons-Hawking black hole solution. At spatial infinity, the temperature is T_* . The thermal energy of this solution is equal to its mass $E = m$. The heat capacity is negative $C = -2\pi r_+^2$, therefore the solution is unstable.

Also, from Eq. (7.54), we find $r_{+2} = \frac{4\pi T_*}{3}l^2$, i.e., $r_{+2} = \infty$ when $l \rightarrow \infty$. It may seem that there is no way to understand this limit. However, we can look into the

line element and see the consequences of doing $l \rightarrow \infty$. From Eqs. (7.3) and (7.4), the line element is

$$ds^2 = \left(\frac{2r_+}{1 + 3\frac{r_+^2}{l^2}} \right)^2 V(r) d\tau^2 + \frac{dr^2}{V(r)} + r^2 d\Omega^2 ,$$

$$V(r) = 1 + \frac{r^2}{l^2} - \left(1 + \frac{r_+^2}{l^2} \right) \frac{r_+}{r} . \quad (7.55)$$

The idea is now to substitute $r_{+2} = \frac{4\pi T_*}{3} l^2$ and perform the coordinate transformation $r = \frac{4\pi T_*}{3} l^2 \bar{x}$, with $\bar{x} \in]1, +\infty[$. Due to the limit $l \rightarrow \infty$, the function $V(r)$ has leading order terms $V(r) \rightarrow \left(\frac{4}{3} \pi T_* \right)^2 l^2 \left(\bar{x}^2 - \frac{1}{\bar{x}} \right)$. Moreover, the 2-sphere line element $r^2 d\Omega^2$ becomes $\left(\frac{4\pi T_*}{3} \right)^2 \bar{x}^2 l^4 d\Omega^2$. Similar to the case of the planar black hole, we can regularize the 2-sphere line element by considering very small angles around a specific point (θ_0, ϕ_0) such that we have new coordinates $dy = \frac{4\pi T_*}{3} l^2 \bar{x} d\theta$ and $dz = \frac{4\pi T_*}{3} l^2 \bar{x} \sin(\theta_0) d\phi$. Hence, the leading order line element in the limit $l \rightarrow \infty$ becomes

$$ds^2 = l^2 \left(\frac{4}{9} \left(\bar{x}^2 - \frac{1}{\bar{x}} \right) d\tau^2 + \frac{d\bar{x}^2}{\bar{x}^2 - \frac{1}{\bar{x}}} \right) + dy^2 + dz^2 , \quad (7.56)$$

which has the form of the hot planar black hole geometry. Note that the appearance of this geometry here is not surprising. The length l can be regarded as the radius of a natural cavity in AdS and we already have seen that the limit of infinite cavity in AdS gives precisely the hot planar black hole. But here, we still must perform the limit $l \rightarrow \infty$ in Eq. (7.56). In some sense, performing the limit $l \rightarrow \infty$ to the r_{+2} solution of Hawking and Page is the same as performing the limit $l \rightarrow \infty$ to the hot planar black hole, which arises from the limit of infinite cavity.

From Eq. (7.56), we can see that the limit $l \rightarrow \infty$ gives an infinite line element without any further considerations. That means only that all the points outside the neighbourhood of $\bar{x} = 1$ are at infinite distance from points at $\bar{x} = 1$. In order to regularize the metric, we must thus expand in the neighbourhood of $\bar{x} = 1$. The proper radius length $\epsilon = l \int_1^{\bar{x}} \frac{\sqrt{\bar{x}} d\bar{x}}{\sqrt{\bar{x}^3 - 1}}$ is given at leading order by $\epsilon = l \frac{2}{\sqrt{3}} \sqrt{\bar{x} - 1}$. We now must perform the limit $l \rightarrow \infty$ with $\sqrt{\bar{x} - 1}$ being very small, such that ϵ is finite. Then, $l^2 \bar{x}^3 - \frac{l^2}{\bar{x}} \rightarrow \frac{9}{4} \epsilon^2$. The line element of the space in the limit of $l \rightarrow \infty$ becomes

$$ds^2 = \epsilon^2 d\tau^2 + d\epsilon^2 + dy^2 + dz^2 , \quad (7.57)$$

with $\epsilon \in]0, +\infty[$. This is again the Rindler metric but now the boundary of space is at infinity. Indeed, the inverse temperature at the boundary of Rindler is infinite, i.e. the temperature is zero, which agrees with the limit $l \rightarrow \infty$ in $T_* = \frac{3r_+}{4\pi l^2}$ while keeping $\frac{r_+}{l}$ finite. One can further make a coordinate transformation to obtain the Euclidean flat space

$$ds^2 = dq^2 + dw^2 + dy^2 + dz^2 , \quad (7.58)$$

where $q = \epsilon \cos(\tau)$ and $w = \epsilon \sin(\tau)$. One can think of this space as the hot flat planar space which has topology $R^2 \times R^2 = R^4$. This solution cannot be found from the original Gibbons-Hawking action because it has different boundary conditions and the class of spherically symmetric metrics chosen does not cover this solution. Indeed, the choice of writing the metric in terms of the compactified imaginary time means that the hot flat planar space can only be achieved by finding the Rindler space first. However, the Rindler space is not spherically symmetric. This can be seen by transforming Eq. (7.57) into spherical coordinates, having then $ds^2 = r^2 \cos^2(\theta) d\tau^2 + dr^2 + r^2 d\Omega_2^2$.

While the Rindler metric obtained and the hot flat planar space are related by a coordinate transformation, we note that the physical situation described here is the one of an accelerated observer at infinity. This is so because the local temperature is defined by the length along orbits of the imaginary proper time, which in this case correspond in the physical space to the trajectories of constant accelerated observers. In other terms, the temperature is measured by constant accelerated observers and the fixed temperature of space corresponds to the temperature measured by the constant accelerated observer at infinity. And so, the Rindler metric describes explicitly the physical situation, although being equivalent to hot flat planar space.

A property of these limits in this subsection is that procedures somehow commute. For the case of the small black hole solution, both procedures give an endpoint described by the Gibbons-Hawking solution. Regarding the large solution, if one starts from the planar solution and takes zero cosmological constant, one obtains the planar Rindler solution with the reservoir accelerated. If one instead starts with the large Hawking-Page solution and takes zero cosmological constant, the solution also becomes the Rindler solution with some coordinates accelerated.

7.5.4 *The limits visualized*

The figures displayed in Figs. 7.3 and 7.4 are helpful to visualize the limits described in this section. Indeed, in Fig. 7.3, where the solutions are plotted in function of $w = 4\pi RT$, one can see that for large $\frac{R}{l}$ the distance between the solutions starts to narrow. It was checked in fact that for larger and larger $\frac{R}{l}$, the solutions tend to merge towards zero. These are the two Hawking-Page solutions which can be recovered if one rescales r_+ by l instead of R .

On the other hand, in Fig. 7.4, where the solutions are plotted in function of $4\pi lT$, the two solutions have different behaviours for large $\frac{R}{l}$. The small black hole solution seems to tend towards zero while the large black hole solution tends to a smooth curve. The smooth curve corresponds to the solution of the planar AdS black hole. An interesting point is $lT = 0$ in the limit of infinite $\frac{R}{l}$. The planar solution at $lT = 0$ has $r_+ = 0$, meaning that the location of event horizon's plane is pulled towards an infinite proper length, i.e. to infinity. But at $lT = 0$, there are also the Hawking-Page solutions as the limits above show.

7.6 CONCLUSIONS

In this chapter, we have analyzed the canonical ensemble of a Schwarzschild-AdS black hole inside a cavity, with particular focus on the limits of the horizon radius solutions that are in thermodynamic equilibrium with the cavity.

We have shown that, with $\Lambda R^2 \ll 1$, York's solution for pure Schwarzschild is automatically incorporated when $\Lambda R^2 = 0$, appearing first for $RT = \frac{\sqrt{27}}{8\pi}$, with a coincident black hole horizon radius $r_{+1} = r_{+2} = \frac{2}{3}R$. For higher ΛR^2 , the coincident black hole horizon radius gets decreased values for some higher RT . The value of RT for the coincident black hole solution saturates to a particular value $RT = \frac{2\sqrt{3}}{4\pi}$ for infinite ΛR^2 and it has zero event horizon radius. We gave a heuristic understanding of this behavior. Changing the values of ΛR^2 and RT , we obtain either two thermodynamics solutions, one which is a small solution, r_{+1} , and one which is large, r_{+2} . The solution r_{+1} is thermodynamically unstable, while the solution r_{+2} is stable.

We have shown that for $|\Lambda|R^2 \rightarrow \infty$, unexpected solutions also arise. There are two different classes of solutions in this limit. One class is obtained by keeping a constant finite T and by performing the limit of $R \rightarrow \infty$. For this class, the small unstable black hole solution r_{+1} disappears, whereas the large stable black hole solution r_{+2} turns into a planar black hole. The second class is obtained by making the limit $R \rightarrow \infty$ but also by putting the temperature to zero, such that RT is constant. For this class, the two black hole solutions yield the Hawking-Page spherical solutions in AdS. The $|\Lambda| = 0$ case in this limit was also considered, where the small black hole solution r_{+1} becomes the Gibbons-Hawking solution and the large black hole solution r_{+2} becomes a Rindler space with accelerated boundary.

Our work in this chapter establishes the connection between the existing solutions in the literature in a unifying way through the limits performed. It would be interesting to expand this analysis to a larger family of ensembles with more parameters.

THE CANONICAL ENSEMBLE OF A SELF-GRAVITATING MATTER THIN SHELL IN ADS

8.1 INTRODUCTION

With the previous chapters being based on configurations with black holes either with a Maxwell field or a negative cosmological constant, we now turn our attention towards spacetimes containing self-gravitating matter.

As discussed in the previous chapter, when considering asymptotically anti-de Sitter (AdS) spacetimes, it was shown [69] that for Schwarzschild-AdS there would be two black hole solutions, with the largest being stable. Hence, asymptotically AdS spacetimes stabilize thermodynamically black hole configurations as the negative cosmological constant makes the spacetime being described as a box. Moreover, it was found in [69] the existence of a phase transition between the hot thermal AdS, i.e. pure AdS containing nonself-gravitating gravitons, and the stable black hole, the so called Hawking-Page phase transition. Other ensembles in asymptotically AdS were also further studied, see [131, 135].

The application of the formalism to self-gravitating matter is of great interest to explore the effects of thermodynamics in curved spacetime and uncover also the connection between thermodynamics and gravity. The inclusion of matter shells as simple descriptions of matter surrounding a black hole has been done in the construction of ensembles with curved space [136], where it was shown that the total entropy of the system is the sum of matter entropy with the black hole entropy. A more thorough analysis was done in [137], while keeping the radius of the shell fixed.

In this chapter, we consider the canonical ensemble of matter shell in asymptotically AdS space, using the Euclidean path integral approach. The objective is to analyze the phase transitions between a black hole and a self-gravitating configuration which may mimic hot thermal AdS. We consider a matter action which is approximated by a fluid description, which is motivated from the path integral over the self-gravitating matter fields. We impose the zero loop approximation, and analyze the equilibrium and stability conditions describing this approximation. The thermodynamic quantities of the system are then obtained from the partition function. A characteristic of the system is that the condition corresponding to the mechanical stability of the shell is not accessible by thermodynamics. We choose a specific equation of state, corresponding to a matter gas with mass, or alternatively,

to a graviton gas restricted to a thin shell. We find four solutions for the shell, with only one being fully stable. We analyze the favorability between the thin shell and the black hole solutions and we find the phase transition between the two phases, which possesses a behaviour analogous to the Hawking-Page phase transition.

This chapter is organized as follows. In Sec. 8.2, we construct the canonical ensemble of matter thin shell in AdS. In Sec. 8.4, we apply the zero loop approximation, and we obtain the equilibrium and the stability conditions an arbitrary equation of state. In Sec. 8.4, we obtain the thermodynamics of the thin shell in AdS from the partition function. In Sec. 8.5, we choose a specific equation of state and we study the solutions of the ensemble. In Sec. 8.6, we compare the AdS black hole solutions to the matter thin shell solutions, and we find a phase transition. In Sec. 8.7, we present the conclusions. This chapter is based on [6].

8.2 CANONICAL ENSEMBLE OF A SELF-GRAVITATING MATTER THIN SHELL IN ASYMPTOTICALLY ADS SPACE

8.2.1 The partition function

The canonical ensemble of a four dimensional curved space with negative cosmological constant and with matter fields can be given by $Z = \int Dg_{\alpha\beta} D\psi e^{-I[g_{\mu\nu}, \psi]}$, where $g_{\alpha\beta}$ represents the Euclidean metric, ψ describes the matter fields, and I is the Euclidean action. Due to the difficulties in performing the full path integral, we perform here the zero loop approximation of the path integral, but we do it in steps. We assume that the path integral over the matter fields can be put inside the path integral over metrics in the sense of $Z = \int Dg_{\alpha\beta} e^{-I_{gl}} \int D\psi e^{-I_\psi}$, where $I_{gl} = I_{gl}[g_{\mu\nu}]$ is the Euclidean gravitational action with negative cosmological constant and $I_\psi = I_\psi[g_{\mu\nu}, \psi]$ is the Euclidean matter action of any field ψ . We assume minimal coupling between the field ψ and the metric $g_{\alpha\beta}$. While for the general case one cannot perform the path integral on matter, for the case of a matter thin shell in spherical symmetry one can perform the path integral exactly, if the action is quadratic in the field. This is because the metric components are seen as constants in the action of the matter thin shell and the path integral becomes an integration over gaussian functions, yielding $\int D\psi e^{-I_\psi[g_{\mu\nu}, \psi]} = e^{-I_m[g_{\mu\nu}]}$, where I_m is an effective matter action. Therefore, the partition function considered here is

$$Z = \int Dg_{\alpha\beta} e^{-I_{gl} - I_m}, \quad (8.1)$$

Since a matter thin shell, denoted by $\mathcal{C}$, is considered, the asymptotically AdS space M is split into two spaces M_1 and M_2 . The outer boundary of M is represented as ∂M . The gravitational action is then given by

$$\begin{aligned} I_{gl} = & -\frac{1}{16\pi l_p^2} \int_{M \setminus \{\mathcal{C}\}} \left(R + \frac{6}{l^2} \right) \sqrt{g} d^4x + \int_{\mathcal{C}} \frac{[K]}{8\pi l_p^2} \sqrt{\gamma} d^3y \\ & - \frac{1}{8\pi l_p^2} \int_{\partial M} K \sqrt{\gamma} d^3y - I_{\text{AdS}}, \end{aligned} \quad (8.2)$$

where l_p is the Planck length, R is the Ricci scalar, g is the metric determinant, $l = \sqrt{-\frac{3}{\Lambda}}$ is defined as the AdS length, with Λ being the negative cosmological constant, γ_{ab} is the induced metric from the space metric $g_{\alpha\beta}$ on the hypersurface in analysis, γ is the determinant of γ_{ab} , K_{ab} is the extrinsic curvature of the hypersurface in analysis, with trace K given by $K = n^\alpha{}_{;\alpha}$, n^α being the normal vector to the hypersurface in analysis, i.e., either $\mathcal{C}$ or ∂M , the bracket $[K] = K|_{M_2} - K|_{M_1}$ means the difference between K evaluated at M_2 and K evaluated at M_1 , and I_{AdS} is the action of pure AdS which is the reference space with negative cosmological constant. In relation to the matter part of the Euclidean action, the Lagrangian density is taken as the matter free energy per unit area $\mathcal{F}_m$. This stems from the fact the canonical ensemble is being considered which is connected to the thermodynamic Helmholtz free energy. Then,

$$I_m = \int_{\mathcal{C}} \mathcal{F}_m[\gamma_{ab}] \sqrt{\gamma} d^3x, \quad (8.3)$$

where $\mathcal{F}_m$ is a functional of the induced metric γ_{ab} on the shell, with γ being the determinant of h_{ab} . The Euclidean action of the system $I = I_{gl} + I_m$ is then given by

$$I = -\frac{1}{16\pi l_p^2} \int_{M \setminus \{\mathcal{C}\}} \left(R + \frac{6}{l^2} \right) \sqrt{g} d^4x + \int_{\mathcal{C}} \left(\frac{[K]}{8\pi l_p^2} + \mathcal{F}_m[\gamma_{ab}] \right) \sqrt{\gamma} d^3y - \frac{1}{8\pi l_p^2} \int_{\partial M} K \sqrt{\gamma} d^3y - I_{\text{AdS}}, \quad (8.4)$$

with all quantities having been properly defined.

8.2.2 Geometry and boundary conditions

In the analysis, we only consider paths which are spherically symmetric. We also assume that the spaces are static. The metric for the space M_1 is written as

$$ds_{M_1}^2 = b_1^2(u) \frac{b_2^2(u_m)}{b_1^2(u_m)} d\tau^2 + a_1^2(u) dy^2 + r(u)^2 d\Omega^2, \quad 0 \leq u < u_m. \quad (8.5)$$

For the thin shell $\mathcal{C}$, the induced metric is written as

$$ds_{\mathcal{C}}^2 = b_2^2(u_m) d\tau^2 + r^2(u_m) d\Omega^2, \quad u = u_m. \quad (8.6)$$

For space M_2 , the metric is

$$ds_{M_2}^2 = b_2^2(u) d\tau^2 + a_2^2(u) dy^2 + r^2(u) d\Omega^2, \quad u_m < u < 1. \quad (8.7)$$

Here b_1 , b_2 , a_1 , a_2 , and r are functions of the coordinate u . The Euclidean time coordinate τ is chosen to be an angular coordinate in the interval $0 < \tau < 2\pi$ on M , the radial coordinate u takes values as above, $d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$ is the line element of the 2-sphere with surface area $\Omega = 4\pi$, and the coordinates θ and ϕ are the usual spherical coordinates. The points at the thin shell are located at

$u = u_m$ and u_m is exhibited as a label that is fixed, while the radius of the shell $r(u_m)$ depends on the arbitrary function $r(u)$.

We must further impose regularity conditions and boundary conditions on the metrics that are being summed in the path integral. In the region M_1 , the interior region, we impose regularity conditions at $u = 0$ corresponding to flat conditions at the origin, i.e.,

$$\begin{aligned} r|_{y=0} &= 0, & b_1|_{y=0} &\text{finite and positive,} \\ \frac{r'}{a_1}\Big|_{y=0} &= 1, & \frac{1}{r'}\left(\frac{r'}{a_1}\right)'\Big|_{y=0} &= 0, & \frac{b'_1}{a_1}\Big|_{y=0} &= 0, \end{aligned} \quad (8.8)$$

where a primed quantity means derivative with respect to u , e.g., $r' = \frac{dr}{du}$.

In the region M_2 , the exterior region, we impose that the space behaves as asymptotically AdS, when $u \rightarrow 1$. As seen in Chapter 3, the AdS boundary conditions are summarized as

$$\begin{aligned} \frac{b_2(u)}{r(u)}\Big|_{u \rightarrow 1} &= \frac{\bar{\beta}}{2\pi l}, \\ \frac{a_2(u)r(u)}{r'(u)}\Big|_{u \rightarrow 1} &= l, \end{aligned} \quad (8.9)$$

where the parameter $\bar{\beta}$ is defined to be the fixed quantity of the ensemble. In some sense, the parameter $\bar{\beta}$ is proportional to the total proper length of the conformal boundary with induced metric $(\frac{l^2}{r(y)^2}ds^2)|_{u \rightarrow 1}$, with conformal factor $\frac{l}{r(y)}$, and the $\bar{\beta}$ is identified to the inverse of the local temperature $\bar{T}$ of the conformal boundary, such that $\bar{\beta} = \frac{1}{\bar{T}}$. It must be pointed out that fixing the inverse temperature in this conformal boundary as $\bar{\beta}$ is a choice of the formalism. Here, the choice coincides with the usual Euclidean proper time approach formalism, e.g., the way the temperature is defined at infinity for a black hole yields the same as the Hawking-Page definition. One could have chosen a different conformal transformation as long as the asymptotic AdS behaviour is imposed. This indeed leads to possible different choices of the fixed inverse temperature $\bar{\beta}$. However, one can view the different fixed temperatures as being related to the choice of conformal observer that measures the temperature. In order to obtain a nonsingular conformal metric, the conformal transformation must behave asymptotically as $\frac{c}{r(y)}$, with c being a constant. This leads to a $\bar{\beta}$ only differing by a constant multiplication factor which can be thought of as a change of scale for the temperature and energy. The physical results do not alter from such choice.

8.2.3 Matter free energy and stress-energy tensor

The matter Lagrangian density can be identified to the thermodynamic Helmholtz free energy density since we are dealing with the canonical ensemble. The Helmholtz free energy potential F is described by $F = E - TS$, where E is the thermodynamic energy, S the entropy, and T the temperature of the reservoir. We must then analyze

the associated density quantities, and so the free energy density $\mathcal{F}_m$ can be written as

$$\mathcal{F}_m[\gamma_{ab}] = \epsilon_m[\gamma_{ab}] - T_m[\gamma_{ab}]s_m[\gamma_{ab}], \quad (8.10)$$

where ϵ_m is the total energy density of the matter, T_m is the local temperature of the shell, and s_m is the entropy density of the matter. All these quantities are functionals of the induced metric γ_{ab} . Since from Eq. (8.6) one has that γ_{ab} depends on two arbitrary quantities that are seen as constants at the shell, $b_2(y_m)$ and $r(y_m)$, the matter free energy density $\mathcal{F}_m[\gamma_{ab}]$ depends only locally on these two quantities, and so a dependence on derivatives of γ_{ab} is ruled out.

The radius α of the shell is defined as

$$\alpha = r(u_m). \quad (8.11)$$

Although in order to keep a consistent nomenclature we should have defined the radius of the shell as $r_m = r(u_m)$, it is preferred to stick with $\alpha = r(u_m)$ to not overcrowd the symbols ahead. We can define a local temperature at some point u as $T(u) = \frac{1}{2\pi b_2(u)}$. So the local temperature of the shell is

$$T_m = \frac{1}{2\pi b_2(u_m)}. \quad (8.12)$$

The rationale for this definition comes from continuity, since in the canonical ensemble one fixes the Euclidean proper time length at the boundary and assigns to it the meaning of an inverse temperature. One must keep in mind however that this definition does not give information about the specific expression of the temperature since $b_2(u_m)$ is arbitrary.

The free energy per unit area $\mathcal{F}_m[\gamma_{ab}] = \mathcal{F}_m[b_2(u_m), r(u_m)]$ can then be put in the form

$$\mathcal{F}_m[h_{ab}] = \mathcal{F}_m[\alpha, T_m], \quad (8.13)$$

upon using Eq. (8.11) and (8.12). Now, we assume the first law to describe the matter energy density as $d\epsilon_m = T_m ds_m - 2(\epsilon_m - T_m s_m + p_m) \frac{d\alpha}{\alpha}$, where p_m is the matter tangential pressure at the shell. Thus, from Eq. (8.10), the free energy density has the differential

$$d\mathcal{F}_m = -s_m dT_m - 2(\mathcal{F}_m + p_m) \frac{d\alpha}{\alpha}. \quad (8.14)$$

With the known differential of the free energy density regarding its dependence on the metric components, one can compute the surface stress-energy tensor S^{ab} as the functional derivative, $S^{ab} = -\frac{2}{\sqrt{\gamma}} \frac{\delta(\sqrt{\gamma}\mathcal{F}_m)[\gamma_{ab}]}{\delta\gamma_{ab}}$. From Eq. (8.6), one has $h_{\tau\tau} = b_2^2(u_m)$ and $h_{\theta\theta} = \frac{h_{\phi\phi}}{\sin^2(\theta)} = r^2(u_m)$. Then, from $\alpha = r(u_m)$ and $T_m = \frac{1}{2\pi b_2(u_m)}$, see Eqs. (8.11) and (8.12), one finds that the variation yields $S^\tau_\tau = -\mathcal{F}_m + T_m \frac{\partial \mathcal{F}_m}{\partial T_m}$ and $S^\theta_\theta = S^\phi_\phi = -\frac{1}{2}\alpha \frac{\partial \mathcal{F}_m}{\partial \alpha} - \mathcal{F}_m$, where the partial derivatives are done keeping the hidden variable constant, and $\delta\gamma = \gamma\gamma^{ab}\delta\gamma_{ab}$ has been used. Some care is

needed while performing the variational derivative to obtain S^θ_θ and S^ϕ_ϕ , as one must calculate $d\alpha$ in $d\mathcal{F}_m$ as $d\alpha = d\left(\sqrt[4]{\frac{\gamma_{\theta\theta}\gamma_{\phi\phi}}{\sin^2\theta}}\right)$. From Eqs. (8.10) and (8.14), one has $\epsilon_m = \mathcal{F}_m - T_m \frac{\partial \mathcal{F}_m}{\partial T_m}$ and $p_m = -\frac{1}{2}\alpha \frac{\partial \mathcal{F}_m}{\partial \alpha} - \mathcal{F}_m$. Thus, the stress-energy tensor S^a_b has components

$$S^\tau_\tau = -\epsilon_m, \quad S^\theta_\theta = S^\phi_\phi = p_m. \quad (8.15)$$

The fluid is thus isotropic, more specifically, it is a perfect fluid. Note that $\epsilon_m = \epsilon_m(\alpha, T_m)$ and $p_m = p_m(\alpha, T_m)$. The rest mass m of the shell is important for the analysis below and it is defined as

$$m = 4\pi\alpha^2\epsilon_m. \quad (8.16)$$

Since $\epsilon_m = \epsilon_m(\alpha, T_m)$, one has $m = m(\alpha, T_m)$. The dependence of the thermodynamic quantities in α and T_m is helpful when one makes variations of the action on the metric components to find the Hamiltonian constraint, however it is also helpful to invert the first law of thermodynamics to get $ds_m = \frac{1}{T_m}d\epsilon_m + 2(\epsilon_m - T_ms_m + p_m)\frac{d\alpha}{\alpha}$. One can integrate over the area to obtain the first law of thermodynamics in the form,

$$T_mdS_m = dm + p_mdA_m, \quad (8.17)$$

where

$$A_m = 4\pi\alpha^2, \quad (8.18)$$

$$S_m = s_mA_m, \quad (8.19)$$

are the area of the shell and the entropy of the matter in the shell, respectively. Written like this, also the quantities S_m , T_m , and p_m become functions of m and α . The dependencies used below shall be explicitly indicated.

8.2.4 Euclidean action in spherical symmetry

Since only spherically symmetric metrics are considered, we can write the action explicitly in terms of its components. The gravitational action with negative cosmological constant, written in Eq. (8.2), for a C^0 metric is

$$\begin{aligned} I_{gl} = & \left(\frac{2\pi b_2 r}{l_p^2} \left(\left(\frac{r'}{a_2} \right)_{\text{AdS}} - \frac{r'}{a_2} \right) \right) \Big|_{u \rightarrow 1} - \frac{\pi}{l_p^2} \left(\frac{b'_1 b_2(u_m) r^2}{a_1 b_1(u_m)} \right) \Big|_{u=0} \\ & + \frac{1}{8\pi l_p^2} \int_{M_1} a_1 b_1 \frac{b_2(u_m)}{b_1(u_m)} r^2 \left(G_1^\tau{}_\tau - \frac{3}{l^2} \right) d^4x \\ & + \frac{1}{8\pi l_p^2} \int_{M_2} a_2 b_2 r^2 \left(G_2^\tau{}_\tau - \frac{3}{l^2} \right) d^4x \\ & - \frac{1}{8\pi l_p^2} \int_C ([K^\tau{}_\tau] - [K]) \sqrt{\gamma} d^3x, \end{aligned} \quad (8.20)$$

where $I_{\text{AdS}} = - \left(\frac{2\pi b_2 r}{l_p^2} \left(\frac{r'}{a_2} \right)_{\text{AdS}} \right) \Big|_{u \rightarrow 1}$ with $\left(\frac{r'}{a_2} \right)_{\text{AdS}}$ being the redshift factor of AdS, see Chapter 3, the Einstein tensor component $G_1^{\tau}_{\tau}$ and $G_2^{\tau}_{\tau}$ are given by

$$\begin{aligned} G_1^{\tau}_{\tau} &= \frac{1}{r' r^2} \left(r \left(\frac{r'^2}{a_1^2} - 1 \right) \right)' , \\ G_2^{\tau}_{\tau} &= \frac{1}{r' r^2} \left(r \left(\frac{r'^2}{a_2^2} - 1 \right) \right)' , \end{aligned} \quad (8.21)$$

and with the terms depending on the extrinsic curvature being given as

$$[K] - [K^{\tau}_{\tau}] = \frac{2}{r} \left(\frac{r'}{a_2} - \frac{r'}{a_1} \right) \Big|_{u=u_m} . \quad (8.22)$$

We now can use the regularity and boundary conditions in Eqs. (8.8) and (8.9), respectively, to further simplify the gravitational action as

$$\begin{aligned} I_{gl} &= \left(\frac{\bar{\beta} r^2}{l_p^2 l} \left(\left(\frac{r'}{a} \right)_{\text{AdS}} - \frac{r'}{a_2} \right) \right) \Big|_{u \rightarrow 1} \\ &+ \frac{1}{8\pi l_p^2} \int_{M_1} a_1 b_1 \frac{b_2(u_m)}{b_1(u_m)} r^2 \left(G_1^{\tau}_{\tau} - \frac{3}{l^2} \right) d^4 x \\ &+ \frac{1}{8\pi l_p^2} \int_{M_2} a_2 b_2 r^2 \left(G_2^{\tau}_{\tau} - \frac{3}{l^2} \right) d^4 x \\ &- \frac{1}{8\pi l_p^2} \int_{\mathcal{C}} ([K^{\tau}_{\tau}] - [K]) \sqrt{\gamma} d^3 x , \end{aligned} \quad (8.23)$$

Finally, we must look towards the thin shell matter action in Eq. (8.3). Through the definition of the matter free energy $\mathcal{F}_m = \epsilon_m - T_m s_m$, one can rewrite the matter action as

$$I_m = \int_{\mathcal{C}} \epsilon_m \sqrt{\gamma} d^3 x - S_m , \quad (8.24)$$

where it was used that $T_m = \frac{1}{2\pi b_2(u_m)}$, and $S_m = 4\pi \alpha^2 s_m$. The full action can then be written as

$$\begin{aligned} I &= \left(\frac{\bar{\beta} r^2}{l_p^2 l} \left(\left(\frac{r'}{a} \right)_{\text{AdS}} - \frac{r'}{a_2} \right) \right) \Big|_{u \rightarrow 1} - S_m \\ &+ \frac{1}{8\pi l_p^2} \int_{M_1} a_1 b_1 \frac{b_2(u_m)}{b_1(u_m)} r^2 \left(G_1^{\tau}_{\tau} - \frac{3}{l^2} \right) d^4 x \\ &+ \frac{1}{8\pi l_p^2} \int_{M_2} a_2 b_2 r^2 \left(G_2^{\tau}_{\tau} - \frac{3}{l^2} \right) d^4 x \\ &- \frac{1}{8\pi l_p^2} \int_{\mathcal{C}} ([K^{\tau}_{\tau}] - [K] - 8\pi l_p^2 \epsilon_m) \sqrt{\gamma} d^3 x . \end{aligned} \quad (8.25)$$

Further details can be found on Chapter 3 on the construction of the path integral, on the regularity conditions, on the boundary conditions, and on the expression of the action for spherically symmetric spaces.

8.3 THE ZERO LOOP APPROXIMATION

8.3.1 The constrained path integral and reduced action

Having the action in spherical symmetry, we now proceed with the zero loop approximation. We make this approximation in steps, starting by imposing the Hamiltonian and momentum constraint equations, so the path integral is along the constraint paths. We don't apply the zero loop approximation straight away since we want to perform a stability analysis, or rather to see the validity of the zero loop approximation. Only afterwards we perform the full zero-loop approximation. We start with the Hamiltonian constraint, consisting of one equation for each region M_1 and M_2 , and a junction condition on the matter shell $\mathcal{C}$. We analyze then the momentum constraint.

The Hamiltonian constraint in the regions M_1 and M_2 makes use of the Einstein tensor component G^τ_τ , in Eq. (8.21) for each M_1 and M_2 . The Hamiltonian constraint is the Einstein equation $G^\tau_\tau = \frac{3}{l^2}$, which can be integrated in both spaces M_1 and M_2 to yield

$$\left(\frac{r'}{a_1}\right)^2 = 1 + \frac{r^2}{l^2} \equiv f_1(r), \quad (8.26)$$

$$\left(\frac{r'}{a_2}\right)^2 = 1 + \frac{r^2}{l^2} - \frac{\tilde{r}_+ + \frac{\tilde{r}_+^3}{l^2}}{r} \equiv f_2(r, \tilde{r}_+), \quad (8.27)$$

where the regularity condition $\frac{r'}{a_1}|_{y=0} = 1$ in Eq. (8.8) was used, $\tilde{r}_+$ is the gravitational radius of the system and it is featured as an integration constant obeying $\tilde{r}_+ < \alpha$, and the functions $f_1(r)$ and $f_2(r, \tilde{r}_+)$ have been defined for convenience. Due to the order of the differential equation in the Hamiltonian constraint equation, the regularity condition $\frac{1}{r'} \left(\frac{r'}{a_1}\right)'|_{y=0} = 0$ in Eq. (8.8) was not used but it is naturally satisfied. The same thing happens for the function $\frac{r'}{a_2}$ which obeys naturally the boundary condition Eq. (8.9).

The Hamiltonian constraint in the hypersurface $\mathcal{C}$ is described by the junction condition $[K^\tau_\tau] - [K] = -8\pi l_p^2 S^\tau_\tau$, where S^τ_τ is the $\tau\tau$ component of the surface stress-energy tensor. The extrinsic curvature term $[K^\tau_\tau] - [K]$ is given by Eq. (8.22). The surface stress-energy tensor is the functional derivative S^{ab} , with $S^\tau_\tau = -\epsilon_m$, see Eq. (8.15). Then, for the mass $m = 4\pi\alpha^2\epsilon_m$, Eq. (8.16), one finds that the Hamiltonian constraint at the shell is

$$m = \frac{\alpha}{l_p^2} (\sqrt{f_1(\alpha)} - \sqrt{f_2(\alpha, \tilde{r}_+)}), \quad (8.28)$$

with $f_1(\alpha) = \left(\frac{r'}{a_1}\right)^2|_\alpha = 1 + \frac{\alpha^2}{l^2}$ and $f_2(\alpha, \tilde{r}_+) = \left(\frac{r'}{a_2}\right)^2|_\alpha = 1 + \frac{\alpha^2}{l^2} - \frac{\tilde{r}_+ + \frac{\tilde{r}_+^3}{l^2}}{\alpha}$, see Eqs. (8.26) and (8.27), respectively. While the dependence of m in the metric components is described by $m = m(\alpha, T_m)$, one can invert in order to T_m and get

$T_m = T_m(m, \alpha)$. Using now the junction condition Eq. (8.28), one obtains the temperature of the shell as a function of $\tilde{r}_+$ and α as $T_m = T_m(m(\tilde{r}_+, \alpha), \alpha)$, as long as the equation of state $T_m(m, \alpha)$ is provided.

In relation to the momentum constraints, due to the spherical symmetry of the metrics in Eqs. (8.5) and (8.7) and the symmetry on translations in τ , the momentum constraints in the regions M_1 and M_2 are satisfied a priori. Moreover, the momentum constraint at the shell is satisfied since the matter shell stress tensor is diagonal as $\mathcal{F}_m$ is a functional only of b_2 and α .

Imposing the Hamiltonian constraints in both spaces M_1 and M_2 , together with the junction condition at the shell, the bulk terms in the action Eq. (8.25) vanish. The term that remains to be calculated is the limit $\left(r^2 \left(\frac{r'}{a} \right)_{\text{AdS}} - r^2 \frac{r'}{a_2} \right) \Big|_{u \rightarrow 1}$. Through the Hamiltonian constraints, one has that $\left(\frac{r'}{a} \right)_{\text{AdS}} = \sqrt{f_1(r)}$ since it is the redshift factor of pure *AdS* and $\frac{r'}{a_2} = \sqrt{f_2(r, \tilde{r}_+)}$. Hence, the limit yields

$$\left(r^2 \sqrt{f_1(r)} - r^2 \sqrt{f_2(r, \tilde{r}_+)} \right) \Big|_{u \rightarrow 1} = \frac{l}{2} \left(\tilde{r}_+ + \frac{\tilde{r}_+^3}{l^2} \right). \quad (8.29)$$

The action in Eq. (8.25) then becomes the reduced action

$$I_*[\tilde{\beta}; \tilde{r}_+, \alpha] = \frac{\tilde{\beta}}{2l_p^2} \left(\tilde{r}_+ + \frac{\tilde{r}_+^3}{l^2} \right) - S_m(m(\tilde{r}_+, \alpha), \alpha), \quad (8.30)$$

where $m(\tilde{r}_+, \alpha)$ is given by the right-hand side of Eq. (8.28). The partition function of Eq. (8.1) with its path integral reduces thus to the following expression

$$Z[\tilde{\beta}] = \int D\tilde{r}_+ D\alpha e^{-I_*[\tilde{\beta}; \tilde{r}_+, \alpha]}, \quad (8.31)$$

as the sum over different metrics with spherical symmetry reduces to the sum over metrics with different $\tilde{r}_+$ and different α . For clarification, the integration over α arises due to the sum over metric functions $r(y)$. Although the Hamiltonian constraint ensures that the metric in the bulk has the same form for any arbitrary function $r(y)$, through a coordinate transformation $r = r(y)$, the value $\alpha = r(y_m)$ that separates the regions M_1 and M_2 depends on the specific function $r(y)$, and so one must sum over the possible values of α .

8.3.2 The zero-loop approximation from the reduced action and stationary conditions

With the reduced action of the system being given by Eq. (8.30), we now minimize it to find the action in the zero-loop approximation. To find the minimum of the action, we need to find its stationary conditions which are given by

$$\frac{\partial I_*[\tilde{\beta}; \tilde{r}_+, \alpha]}{\partial \alpha} = 0, \quad (8.32)$$

$$\frac{\partial I_*[\tilde{\beta}; \tilde{r}_+, \alpha]}{\partial \tilde{r}_+} = 0. \quad (8.33)$$

The stationary conditions given in Eqs. (8.32) and (8.33) can be understood as the remaining Einstein equations whose solutions minimize the action. Since the reduced action is essentially the Einstein-Hilbert action together with the matter action, having the Hamiltonian constraint being imposed, the minimization of the action in relation to α and to $\tilde{r}_+$ is equivalent to the minimization in relation to the metric components $g_{\theta\theta}$ and to g_{uu} . And so, these conditions yield the Dirac delta terms of $G_{\theta\theta} = 8\pi T_{\theta\theta}$, and the equation $G_{uu} = 0$, where $G_{\theta\theta}$ and G_{uu} are the corresponding components of the Einstein tensor, and $T_{\theta\theta}$ is the corresponding component of the stress-energy tensor. In order to further develop the derivatives of Eqs. (8.32) and (8.33), one has to find $\frac{\partial S_m}{\partial \alpha}$ and $\frac{\partial S_m}{\partial \tilde{r}_+}$. From the first law of thermodynamics given in Eq. (8.17), the matter entropy has the differential form $dS_m = \frac{dm(\tilde{r}_+, \alpha)}{T_m} + \frac{p_m}{T_m} dA_m$, and so to find the derivatives of S_m one has to find $\frac{\partial A_m}{\partial \alpha}$, $\frac{\partial A_m}{\partial \tilde{r}_+}$, $\frac{\partial m}{\partial \alpha}$, and $\frac{\partial m}{\partial \tilde{r}_+}$. From the expression of A_m , Eq. (8.18), one has

$$\frac{\partial A_m}{\partial \alpha} = 8\pi\alpha, \quad \frac{\partial A_m}{\partial \tilde{r}_+} = 0. \quad (8.34)$$

From the expression of $m(\tilde{r}_+, \alpha)$, Eq. (8.28), one has

$$\begin{aligned} \frac{\partial m}{\partial \alpha} &\equiv -8\pi\alpha p_g, & \frac{\partial m}{\partial \tilde{r}_+} &= \frac{1 + \frac{3\tilde{r}_+^2}{l^2}}{2l_p^2 \sqrt{f_2(\alpha, \tilde{r}_+)}} , \\ p_g &\equiv \frac{1}{8\pi\alpha l_p^2} \left(\frac{1 + 2\frac{\alpha^2}{l^2} - \frac{\tilde{r}_+ + \frac{\tilde{r}_+^3}{l^2}}{2\alpha}}{\sqrt{f_2(\alpha, \tilde{r}_+)}} - \frac{1 + 2\frac{\alpha^2}{l^2}}{\sqrt{f_1(\alpha)}} \right), \end{aligned} \quad (8.35)$$

where p_g is defined as the gravitational pressure. Using then the chain rule on the first law of thermodynamics, one gets the following derivatives for the matter entropy

$$\frac{\partial S_m}{\partial \alpha} = \frac{8\pi\alpha}{T_m} (p_m - p_g), \quad \frac{\partial S_m}{\partial \tilde{r}_+} = \frac{1 + 3\frac{\tilde{r}_+^2}{l^2}}{2l_p^2 T_m \sqrt{f_2(\alpha, \tilde{r}_+)}}. \quad (8.36)$$

Since the differential of the entropy has been recast in terms of α and $\tilde{r}_+$, the temperature of the shell and the pressure of the shell also have that dependence as $T_m = T_m(m(\alpha, \tilde{r}_+), \alpha) = T_m(\alpha, \tilde{r}_+)$ and $p_m = p_m(m(\tilde{r}_+, \alpha), \alpha) = p_m(\alpha, \tilde{r}_+)$. From now on, we abbreviate this dependence to avoid cluttering, however, the dependence must be assumed.

Then, using the reduced action given in Eq. (8.30) and the derivatives of the matter entropy in Eq. (8.36), we find the stationary conditions. The stationary condition of Eq. (8.32) yields

$$p_g = p_m. \quad (8.37)$$

This equation gives the condition for mechanical equilibrium of the shell. We call Eq. (8.37) as the balance of pressure equation. The stationary condition of Eq. (8.33) yields

$$\bar{\beta} = \frac{1}{T_m \sqrt{f_2(\alpha, \tilde{r}_+)}}. \quad (8.38)$$

This equation gives the condition for thermodynamic equilibrium of the shell. We call Eq. (8.38) as the balance of temperature equation. Thus, these two equations above give, as expected, the conditions for equilibrium.

One can verify that Eq. (8.37) only depends on $\tilde{r}_+$ and α , which means the solutions to this equation can be expressed as

$$\alpha = \alpha(\tilde{r}_+). \quad (8.39)$$

Then, one can input such solutions into Eq. (8.38) to obtain an equation only dependent on $\tilde{r}_+$ and $\tilde{\beta}$, which can be solved by a function $\tilde{r}_+(\tilde{\beta})$, i.e.,

$$\tilde{\beta} = \frac{\iota(\tilde{r}_+)}{1 + 3\frac{\tilde{r}_+^2}{l^2}}, \quad \text{implying} \quad \tilde{r}_+ = \tilde{r}_+(\tilde{\beta}), \quad (8.40)$$

where $\iota(\tilde{r}_+)$ is a function of $\tilde{r}_+$ that appears for convenience and is given for the thin shell by

$$\iota(\tilde{r}_+) = \frac{1 + 3\frac{\tilde{r}_+^2}{l^2}}{T_m(\tilde{r}_+, \alpha(\tilde{r}_+)) \sqrt{f_2(\alpha(\tilde{r}_+))}}. \quad (8.41)$$

In comparison, for the Hawking-Page black hole one has $\iota(\tilde{r}_+) = 4\pi r_+$. Of course, the expression of Eq. (8.41) means we are dealing with a thin shell.

Since, from Eq. (8.39), one has $\alpha = \alpha(\tilde{r}_+)$, the reduced action $I_*[\tilde{\beta}; \alpha, \tilde{r}_+]$ of Eq. (8.30) under the mechanical stationary condition can be written as an effective reduced action of the form $I_*[\tilde{\beta}; \tilde{r}_+]$. This in turn implies that the partition function given in Eq. (8.31) is now $Z[\tilde{\beta}] = \int D\tilde{r}_+ e^{-I_*[\tilde{\beta}; \tilde{r}_+]}$, as the zero-loop approximation in the path integral over α as been done, i.e., using the reduced action evaluated at the stationary point provided by Eq. (8.37), or what amounts to the same thing, by Eq. (8.39). It is interesting to note that this behavior can be deduced from the structure of the reduced action in Eq. (8.30) together with Eq. (8.31). In fact, the path integral over α in the partition function, $\int D\alpha e^{S_m}$, corresponds to the partition function of the microcanonical ensemble. Therefore, this indicates that the canonical ensemble of the full system can be described by an effective reduced action determined by the microcanonical ensemble of a hot self-gravitating matter thin shell with fixed $\tilde{r}_+$, $I_*[\tilde{\beta}; \tilde{r}_+, \alpha(\tilde{r}_+)] = I_*[\tilde{\beta}; \tilde{r}_+]$, while the solutions $\alpha(\tilde{r}_+)$ are but a consequence of performing the zero-loop approximation on the path integral over α , i.e., of performing the zero loop approximation on the microcanonical ensemble. Given Eqs. (8.30), (8.39), and (8.40), the solutions $\tilde{r}_+(\tilde{\beta})$ of the canonical ensemble give the path that extremizes the reduced action. Having $\tilde{r}_+(\tilde{\beta})$, one finds $\alpha = \alpha(\tilde{r}_+(\tilde{\beta}))$, and then the action $I_*[\tilde{\beta}; \tilde{r}_+(\tilde{\beta})]$, which is the action of the stationary points. This action is the zero-loop approximation action $I_0(\tilde{\beta})$. Indeed,

$$I_0[\tilde{\beta}] \equiv I_*[\tilde{\beta}; \tilde{r}_+(\tilde{\beta})], \quad (8.42)$$

i.e., the zero-loop action $I_0[\tilde{\beta}]$ is found by evaluating the reduced action around its stationary points with $\alpha(\tilde{r}_+(\tilde{\beta}))$ and $\tilde{r}_+(\tilde{\beta})$ being found from the stationary

conditions. The partition function of the canonical ensemble can then be obtained as

$$Z(\bar{\beta}) = e^{-I_0[\bar{\beta}]}, \quad (8.43)$$

in the zero loop approximation, and the thermodynamic properties of the system can be extracted.

8.3.3 The stability criteria from the reduced action of a hot self-gravitating thin shell in asymptotically AdS space

Going a step further within this formalism, we can apply the zero-loop approximation of the path integral in Eq. (8.31), and go one order up to first order approximation by evaluating the reduced action around its stationary points up until second order and write

$$I_*[\bar{\beta}; \tilde{r}_+, \alpha] = I_0[\bar{\beta}] + \sum_{ij} H_{ij} \delta r^i \delta r^j, \quad (8.44)$$

where $I_0[\bar{\beta}] = I_*[\bar{\beta}; \tilde{r}_+(\bar{\beta}), \alpha(\bar{\beta})]$ is the reduced action evaluated at the stationary points given in Eq. (8.42), with $\tilde{r}_+(\bar{\beta})$ and $\alpha(\bar{\beta})$ being found from the stationary conditions of I_* , and $H_{ij} = \left. \frac{\partial^2 I_*}{\partial r^i \partial r^j} \right|_0$ is the Hessian of the reduced action I_* evaluated at the stationary points, with the parameters $r^i = (\alpha, \tilde{r}_+)$, with $i = 1, 2$. The partition function can then be written in the saddle point approximation as

$$Z[\bar{\beta}] = e^{-I_0[\bar{\beta}]} \int D r^i e^{-\sum_{jk} H_{jk} \delta r^j \delta r^k}, \quad (8.45)$$

where the first and second factors are the zero and first-loop contributions. Although we only consider the zero-loop contribution, i.e., the zero-loop approximation, we also take into account the first-loop contribution in the sense that it gives some information about the stability of the approximation. For the path integral to converge and so for the formalism to be stable, the Hessian H_{ij} must be positive definite, i.e., the stationary points must correspond to a local minimum of the reduced action.

The components of the Hessian are

$$H_{\alpha\alpha} = \frac{8\pi\alpha}{T_m} \left(\left(\frac{\partial p_g}{\partial \alpha} \right)_{\tilde{r}_+} - \left(\frac{\partial p_m}{\partial \alpha} \right)_m + 8\pi\alpha p_m \left(\frac{\partial p_m}{\partial m} \right)_\alpha \right), \quad (8.46)$$

$$H_{\alpha\tilde{r}_+} = \left(\frac{1 + \frac{3\tilde{r}_+^2}{l_p^2}}{2T_m \sqrt{f_2} l_p^2} \right) \left(\frac{\frac{\alpha}{l_p^2} + \frac{\tilde{r}_+ + \frac{\tilde{r}_+^3}{l_p^2}}{2\alpha^2}}{f_2} - 8\pi\alpha \left(\frac{\partial p_m}{\partial m} \right)_\alpha \right), \quad (8.47)$$

$$H_{\tilde{r}_+\tilde{r}_+} = \left(\frac{1 + \frac{3\tilde{r}_+^2}{l_p^2}}{2T_m \sqrt{f_2} l_p} \right)^2 \left[\frac{1}{l_p^2} \left(\frac{\partial T_m}{\partial m} \right)_\alpha - \frac{T_m}{\alpha \sqrt{f_2}} \right], \quad (8.48)$$

where

$$\left(\frac{\partial p_g}{\partial \alpha}\right)_{\tilde{r}_+} = \frac{1}{8\pi\alpha^2 l_p^2} \left(\frac{3 \frac{\tilde{r}_+ + \frac{\tilde{r}_+^3}{l^2}}{2\alpha} \left(1 - \frac{\alpha^2}{l^2}\right) - 1 - 3 \frac{\left(\tilde{r}_+ + \frac{\tilde{r}_+^3}{l^2}\right)^2}{4\alpha^2}}{f_2^{3/2}} + \frac{1}{f_1^{3/2}} \right), \quad (8.49)$$

and since here the function depends on three variables, the variable that is kept constant while performing the partial derivative is written in the subscript of the parenthesis of the partial derivative. We choose the sufficient conditions for the positive definiteness of the Hessian to be

$$H_{\alpha\alpha} > 0, \quad (8.50)$$

$$H_{\tilde{r}_+\tilde{r}_+} - \frac{H_{\alpha\tilde{r}_+}^2}{H_{\alpha\alpha}} > 0. \quad (8.51)$$

Applying Eqs. (8.46)-(8.48) to Eqs. (8.50) and (8.51), and including the marginal case, one has

$$\left(\frac{\partial p_g}{\partial \alpha}\right)_{\tilde{r}_+} - \left(\frac{\partial p_m}{\partial \alpha}\right)_m + 8\pi\alpha p_m \left(\frac{\partial p_m}{\partial m}\right)_\alpha \geq 0, \quad (8.52)$$

$$\frac{d\tilde{r}_+}{d\tilde{T}} \geq 0, \quad (8.53)$$

respectively. Indeed, one can obtain the derivative of the solution $\tilde{r}_+(\tilde{\beta})$ by applying the derivative of $\tilde{\beta}$ to Eqs. (8.37) and (8.38), obtaining

$$\frac{d\tilde{r}_+}{d\tilde{\beta}} = -\frac{1}{2l_p^2} \left(1 + 3 \frac{\tilde{r}_+^2}{l^2}\right) \left(H_{\tilde{r}_+\tilde{r}_+} - \frac{H_{\tilde{r}_+\alpha}^2}{H_{\alpha\alpha}}\right)^{-1}. \quad (8.54)$$

And so Eq. (8.51) implies that $\frac{d\tilde{r}_+}{d\tilde{\beta}} < 0$, or in terms of temperature $\frac{d\tilde{r}_+}{d\tilde{T}} > 0$, leading to Eq. (8.53), when one includes the marginal case. Regarding the meaning of these stability conditions, one can verify that Eq. (8.52) is precisely the mechanical stability condition for a static shell in AdS with constant $\tilde{r}_+$. Regarding the other condition in Eq. (8.53), in some sense it is a thermal stability condition, which shall be seen in the thermodynamic analysis.

We must comment about the quantity $\frac{d\alpha}{d\tilde{\beta}}$. One can also obtain the derivative of the solution $\alpha(\tilde{\beta})$ by applying the derivative of $\tilde{\beta}$ to Eqs. (8.37) and (8.38), obtaining

$$\frac{d\alpha}{d\tilde{\beta}} = -\frac{H_{\tilde{r}_+\alpha}}{H_{\alpha\alpha}} \frac{d\tilde{r}_+}{d\tilde{\beta}}, \quad (8.55)$$

i.e., $\frac{d\alpha}{d\tilde{T}} = -\frac{H_{\tilde{r}_+\alpha}}{H_{\alpha\alpha}} \frac{d\tilde{r}_+}{d\tilde{T}}$. Thus, if mechanical stability holds, $H_{\alpha\alpha} > 0$, Eq. (8.50), then the radius of the shell α decreases with ensemble temperature if $H_{\tilde{r}_+\alpha} > 0$, and increases if $H_{\tilde{r}_+\alpha} < 0$. The sign of $H_{\tilde{r}_+\alpha}$ depends on the particular shell one is studying.

8.4 THERMODYNAMICS OF THE HOT SELF-GRAVITATING THIN SHELL IN THE ZERO-LOOP APPROXIMATION

In the statistical mechanics formalism of the canonical ensemble, the partition function is given by the free energy F as $Z = e^{-\beta F}$, while the zero-loop approximation gives a partition function $Z = e^{-I_0}$. By connecting both, one has $F = \bar{T} I_0$, where $\bar{T}$ is the temperature of the system, $\bar{T} = \frac{1}{\beta}$. Then, from Eq. (8.30), the free energy is

$$F = \frac{\tilde{r}_+ (l^2 + \tilde{r}_+^2)}{2l^2 l_p^2} - \bar{T} S_m, \quad (8.56)$$

with $\tilde{r}_+$ given by the solution $\tilde{r}_+ = \tilde{r}_+(\bar{T})$ of Eq. (8.40), and α given by the solution $\alpha = \alpha(\tilde{r}_+(\bar{T}))$ of Eq. (8.39) together with Eq. (8.40).

We can now obtain the thermodynamic quantities for the system from the derivatives of the free energy. In terms of the thermodynamic energy E , the temperature $\bar{T}$, and the entropy S_m , the free energy of a system and its differential are given by

$$F = E - \bar{T} S, \quad dF = -S d\bar{T}, \quad (8.57)$$

respectively. From the Eqs. (8.56) and (8.57), we obtain that the entropy of the system is

$$S = S_m, \quad (8.58)$$

and the mean energy is

$$E = \frac{1}{2l_p^2} \tilde{r}_+ \left(1 + \frac{\tilde{r}_+^2}{l^2} \right), \quad (8.59)$$

with $\tilde{r}_+ = \tilde{r}_+(\bar{T})$. One can identify the Schwarzschild-AdS mass M as the right-hand side of Eq. (8.59), i.e., $E = M$.

Regarding thermodynamic stability, we must verify if the heat capacity C is positive. If

$$C \geq 0, \quad (8.60)$$

the system is thermodynamically stable, where the limiting case was included, otherwise it is unstable. The heat capacity is defined as $C = \frac{dE}{d\bar{T}}$. Using Eq. (8.59) together with Eq. (8.40), we find

$$C = \frac{1}{l_p^2} \frac{\left(1 + \frac{3\tilde{r}_+^2}{l^2} \right) l^2(\tilde{r}_+)}{\frac{12\tilde{r}_+ \iota(\tilde{r}_+)}{l^2} - 2 \left(1 + \frac{3\tilde{r}_+^2}{l^2} \right) \frac{\partial \iota(\tilde{r}_+)}{\partial \tilde{r}_+}}, \quad (8.61)$$

where $\iota(\tilde{r}_+) = \frac{1+3\frac{\tilde{r}_+^2}{l^2}}{T_m(\tilde{r}_+, \alpha(\tilde{r}_+)) \sqrt{f_2(\alpha(\tilde{r}_+))}}$, see Eq. (8.41), and $\tilde{r}_+ = \tilde{r}_+(\bar{T})$. One can see that the heat capacity is positive if

$$6 \frac{\tilde{r}_+}{l^2} - \iota'(\tilde{r}_+) \bar{T} \geq 0, \quad (8.62)$$

where the limiting case was included. Using Eq. (8.40), we find that Eq. (8.62) is equivalent to $\frac{d\tilde{r}_+}{dT} \geq 0$ which is Eq. (8.53) of the ensemble theory. One can now see that the remaining stability condition Eq. (8.50) is not present or cannot be accessed by the thermodynamics of the system. It is moreover interesting to see that the thermodynamic stability of the canonical ensemble given by Eqs. (8.60)-(8.62) is also given by the saddle point approximation of the effective reduced action $I_*[\tilde{\beta}; \tilde{r}_+]$, only dependent in the parameter $\tilde{r}_+$. This may be due to the fact that $\tilde{r}_+$ is associated to the quasilocal energy and so the effective reduced action $I_*[\tilde{\beta}; \tilde{r}_+]$ plays the role of the appropriate generalized free energy that when minimized yields indeed the thermodynamic equilibrium and stability of the canonical ensemble. It is important to note also that the thermodynamic stability of the canonical ensemble is different from the intrinsic stability of the system in the sense of Callen, as intrinsic stability requires more conditions on the concavity of the free energy.

8.5 SPECIFIC CASE OF MATTER THIN SHELL WITH BAROTROPIC EQUATION OF STATE

In order to proceed with the analysis of the canonical ensemble of a self-gravitating matter thin shell, we must now give the equations of state for the matter in the shell. Here, we give an equation of state for the pressure in the form of a barotropic equation, i.e.,

$$p_m(m, \alpha) = \frac{1}{3} \frac{m}{4\pi\alpha^2}. \quad (8.63)$$

We choose the equation of state for the temperature of the matter as

$$T_m = \frac{4}{3C_0} \frac{m^{\frac{1}{4}}}{(4\pi\alpha^2)^{\frac{1}{4}}}, \quad (8.64)$$

where C_0 is a constant with units. Then, integrating the first law of thermodynamics yields that the matter entropy has the equation

$$S_m = C_0 m^{\frac{3}{4}} (4\pi\alpha^2)^{\frac{1}{4}}. \quad (8.65)$$

A more general equation for $p_m(m, \alpha)$ in Eq. (8.63) could be chosen, e.g., $p_m(m, \alpha) = \lambda \frac{m}{4\pi\alpha^2}$, where λ is a constant, with $\lambda = \frac{1}{2}$ corresponding to the barotropic equation of state of a two-dimensional ultrarelativistic gas. The mechanical stability condition, Eq. (8.52), requires that $\lambda < \frac{1}{2}$. Since the dominant energy condition requires $-1 < \lambda < 1$, a reasonable range for λ is $0 < \lambda < \frac{1}{2}$, as we require the pressure to be positive. If this general expression for the pressure is integrated, using from the first law of thermodynamics that $p_m = -\left(\frac{\partial m}{\partial A_m}\right)_{S_m}$, the partial derivative in relation the area A_m being defined at constant matter entropy, one would obtain the expression for S_m in the form $S_m(m, \alpha) = S_m(m(4\pi\alpha^2)^\lambda)$, i.e., S_m is an arbitrary function of $m(4\pi\alpha^2)^\lambda$. Here, we choose a power-law expression for the entropy of the form $S_m(m, \alpha) = C_0 m^\delta (4\pi\alpha^2)^{\delta\lambda}$, with C_0 being a constant and

δ a number. Then, the temperature equation of state could be deduced using the first law of thermodynamics, i.e., $\frac{1}{T_m} = \left(\frac{\partial S_m}{\partial m} \right)_{A_m}$ giving $T_m = \frac{m^{1-\delta}}{\delta C_0 (4\pi\alpha^2)^{\delta\lambda}}$. We could instead have picked up this equation of state for T_m from the necessity of the equality of the second order cross derivatives to have an exact S_m , and then integrate the first law to find S_m itself. We narrow further the analysis to a specific λ and δ , using Eq. (8.63)-(8.65). In particular, we choose $\lambda = \frac{1}{3}$, with the pressure given in Eq. (8.63). We also choose $\delta = \frac{3}{4}$, so that the temperature and the entropy have equations given in Eq. (8.64) and Eq. (8.65), respectively.

For numerical purposes, it is best to write the pressure as $l_p^2 p_m(m, \alpha) = \frac{1}{3} \frac{m^{\frac{1}{2}} l_p^2}{4\pi \alpha^2}$,

the temperature as $T_m = \frac{1}{l} \frac{4}{3c_0} \left(\frac{l}{l_c} \right)^{\frac{1}{4}} \left(\frac{l}{l_p} \right)^{\frac{1}{2}} \left(\frac{m l_p^2}{l} \right)^{\frac{1}{4}} \left(4\pi \frac{\alpha^2}{l^2} \right)^{-\frac{1}{4}}$, and the entropy as

$S_m = c_0 \left(\frac{l_c}{l} \right)^{\frac{1}{4}} \left(\frac{l}{l_p} \right)^{\frac{3}{2}} \left(\frac{m l_p^2}{l} \right)^{\frac{3}{4}} \left(4\pi \frac{\alpha^2}{l^2} \right)^{\frac{1}{4}}$, i.e., $C_0 = c_0 l_c^{\frac{1}{4}}$, where c_0 has no units and l_c can be understood as the Compton wavelength associated to the rest mass of the constituents of the shell. The motivation for the matter equations of state given above is both physical and mathematical. Physically, the equations of state resemble the equations of state of a radiation gas. Namely, the equation of state for the pressure $p_m(m, \alpha)$ can be thought of as the equation of state of a three-dimensional radiation gas confined in a very thin shell of small width l_c . It can also be thought of as a two-dimensional gas of a fundamental field with some Compton wavelength l_c . Mathematically, it allows for an analytical treatment of the balance of the pressure which facilitates the search for the solutions of the shell and the analysis of its stability.

With the equations of state described, we can solve numerically Eq. (8.38), to obtain two solutions for the radius α of the shell which is written as $\alpha_u(\tilde{r}_+)$ and $\alpha_s(\tilde{r}_+)$, see Fig. 8.1, the meaning of the subscripts u and s will turn up shortly. The solution $\alpha_u(\tilde{r}_+)$ is monotonically increasing, while $\alpha_s(\tilde{r}_+)$ is monotonically

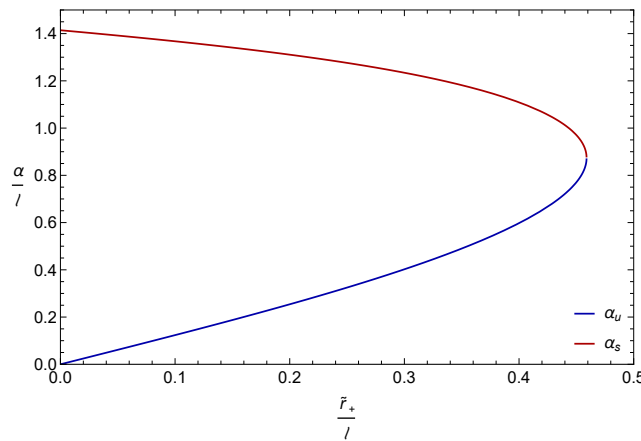


Figure 8.1: Solutions of the balance of pressure $\frac{\alpha_u}{l}$ and $\frac{\alpha_s}{l}$ as function of $\frac{\tilde{r}_+}{l}$.

decreasing, until both meet a common point. By evaluating the stability condition in Eq. (8.52), it turns out that the solution $\alpha_u(\tilde{r}_+)$ is mechanically unstable, while

$\alpha_s(\tilde{r}_+)$ is mechanically stable, hence the nature of the subscripts for the solutions. This mechanical behavior of the thin shell is rather like the radius-mass behavior of a white dwarf or a neutron star. The two solutions translate into two possible radii for the shell for a given energy, one which is large and another which is small. The small radius solution has very high pressure and is unstable, the large radius solution has low pressure and is stable. Physically, this is similar to the two solutions appearing in models of astrophysical objects, such as white dwarfs, neutron stars with polytropic-type equations of state, one solution being unstable while the other being stable.

Knowing the solutions $\alpha_u(\tilde{r}_+)$ and $\alpha_s(\tilde{r}_+)$ for the radius of the shell, we can put them into Eq. (8.38) in order to obtain the solutions for the gravitational radius $\tilde{r}_+(\bar{T})$. We find that there are four solutions in total, two solutions $\tilde{r}_{+u1}(\bar{T})$ and $\tilde{r}_{+u2}(\bar{T})$ with shell radius α_u , i.e., $\alpha_u(\tilde{r}_{+u1}(\bar{T}))$ and $\alpha_u(\tilde{r}_{+u2}(\bar{T}))$, respectively, and other two solutions $\tilde{r}_{+s1}(\bar{T})$ and $\tilde{r}_{+s2}(\bar{T})$ with shell radius α_s , i.e., with $\alpha_s(\tilde{r}_{+s1}(\bar{T}))$ and $\alpha_s(\tilde{r}_{+s2}(\bar{T}))$, respectively. These solutions are shown in Fig. 8.2. Regarding stability, the solutions are thermodynamically stable if the gravitational radius increases with the temperature $\bar{T}$. From the figure, $\tilde{r}_{+u1}(\bar{T})$ and $\tilde{r}_{+s2}(\bar{T})$ are ther-

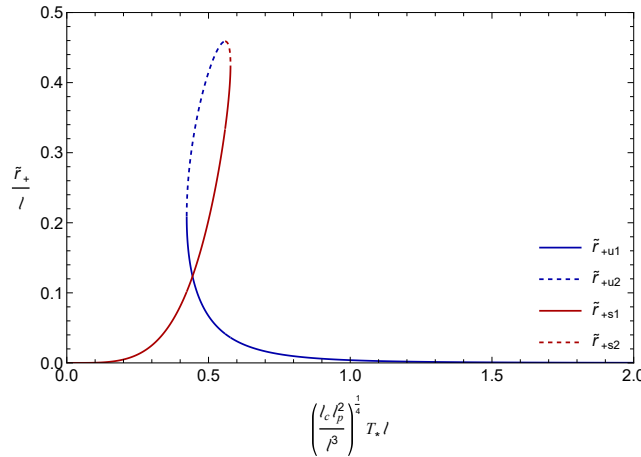


Figure 8.2: Solutions of the ensemble $\frac{\tilde{r}_{+u1}}{l}$, $\frac{\tilde{r}_{+u2}}{l}$, $\frac{\tilde{r}_{+s1}}{l}$, and $\frac{\tilde{r}_{+s2}}{l}$, as functions of $\bar{T}l \left(\frac{l_c}{l}\right)^{\frac{1}{4}} \left(\frac{l_p}{l}\right)^{\frac{1}{2}}$. Both $\frac{\tilde{r}_{+u1}}{l}$ and $\frac{\tilde{r}_{+u2}}{l}$ have shell radius α_u , while both $\frac{\tilde{r}_{+s1}}{l}$ and $\frac{\tilde{r}_{+s2}}{l}$ have shell radius α_s .

modynamically unstable, and $\tilde{r}_{+u2}(\bar{T})$ and $\tilde{r}_{+s1}(\bar{T})$ are thermodynamically stable.

We now analyze the entropy S_m , see Fig. 8.3. We performed a polynomial fit to the matter entropy in order to understand its leading power of $\tilde{r}_+$. The fit for S_m given in Eq. (8.65) for the solution α_u as a function of $\tilde{r}_{+u}$ is described by the function $\left(\frac{l^3}{l_p^2 l_c}\right)^{\frac{1}{4}} \frac{l_p^2}{l^2} S_m = 1.54662 \left(\frac{\tilde{r}_{+u}}{l}\right)^{1.2323}$ with a coefficient of determination $R^2 = 0.999992$. The fit of S_m given in Eq. (8.65) for the solution α_s as a function of $\tilde{r}_{+s}$ is described by the function $\left(\frac{l^3}{l_p^2 l_c}\right)^{\frac{1}{4}} \frac{l_p^2}{l^2} S_m = 0.898912 \left(\frac{\tilde{r}_{+s}}{l}\right)^{0.755675} + 0.867397 \left(\frac{\tilde{r}_{+s}}{l}\right)^{2.91424}$ with a coefficient of determination $R^2 = 1$, with this equality being approximate. We attempted another fit of S_m for the solution α_s with just one power, giving

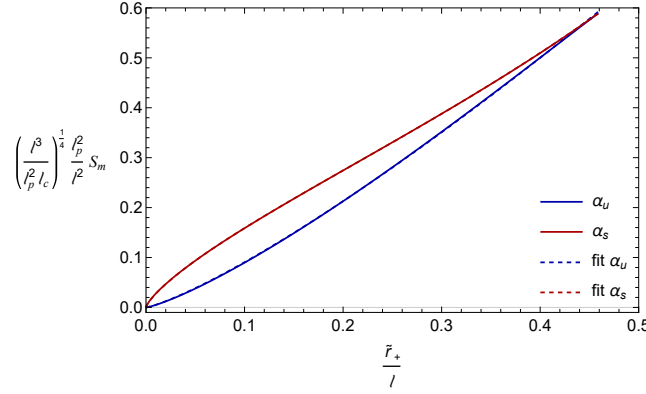


Figure 8.3: Matter entropy $\left(\frac{l^3}{l_c^2}\right)^{\frac{1}{4}} \frac{l_p^2}{l^2} S_m$ in function of the gravitational radius $\frac{\tilde{r}_+}{l}$ for the two shell radius solutions $\alpha_u(\tilde{r}_+)$ and $\alpha_s(\tilde{r}_+)$. A fit was performed for each branch, with $\left(\frac{l^3}{l_c^2}\right)^{\frac{1}{4}} \frac{l_p^2}{l^2} S_m = 1.54662 \left(\frac{\tilde{r}_+}{l}\right)^{1.2323}$ for the case of $\alpha_u(\tilde{r}_+)$ and $\left(\frac{l^3}{l_c^2}\right)^{\frac{1}{4}} \frac{l_p^2}{l^2} S_m = 0.898912 \left(\frac{\tilde{r}_+}{l}\right)^{0.755675} + 0.867397 \left(\frac{\tilde{r}_+}{l}\right)^{2.91424}$ for $\alpha_s(\tilde{r}_+)$, with respective coefficients of determination $R^2 = 0.999992$ and $R^2 = 1$, with this last equality being approximate.

$\left(\frac{l^3}{l_c^2}\right)^{\frac{1}{4}} \frac{l_p^2}{l^2} S_m = 1.12374 \left(\frac{\tilde{r}_{+s}}{l}\right)^{0.867504}$, with $R^2 = 0.999607$, however the differences of the fit are visible in the plot and we considered instead the fit with two powers. The fits we obtained are very close to the numerical results for the S_m , which is surprising and one could wonder if there might be an analytic solution. But in order to obtain the expression of S_m , one needs to solve a quintic polynomial equation and we were not able to find an analytic solution. A feature that the fits do not capture is the fact that S_m is only defined in the interval $0 < \frac{\tilde{r}_+}{l} < 0.4589$ with this last number being approximate.

Another equivalent indicator of thermal stability is given by the positivity of the heat capacity, see Fig. 8.4. Yet, it must be noticed that $\tilde{r}_{+u2}(\bar{T})$ has a shell radius $\alpha_u(\tilde{r}_{+u2}(\bar{T}))$, which is mechanically unstable. Therefore, the only fully stable solution is $\tilde{r}_{+s1}(\bar{T})$ with shell radius $\alpha_s(\tilde{r}_{+s1}(\bar{T}))$.

8.6 HOT THIN SHELL VERSUS BLACK HOLE IN ADS

8.6.1 The black hole

We now present the relevant quantities of the canonical ensemble of a Schwarzschild-AdS black hole for completeness and also because it is important in the analysis of phase transitions regarding the hot thin shell. In order to obtain the reduced action and then the zero-loop action, we can carry on in a similar way the calculations above but now with black hole boundary conditions rather than thin shell ones, or we can simply use the Hawking-Page results.

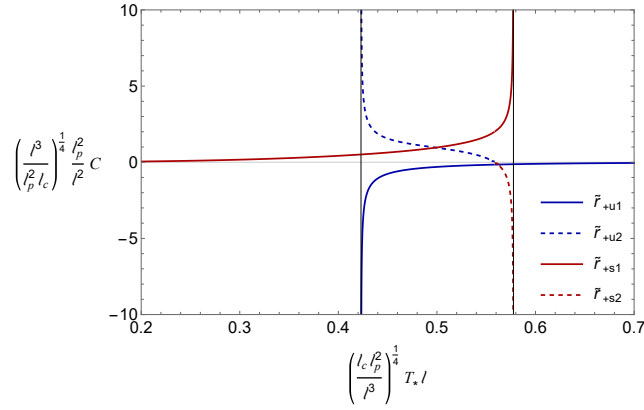


Figure 8.4: Adimensional heat capacity for the solutions $\tilde{r}_{+u1}$, $\tilde{r}_{+u2}$, $\tilde{r}_{+s1}$, and $\tilde{r}_{+s2}$ as functions of $\tilde{T}l \left(\frac{l_c}{l}\right)^{\frac{1}{4}} \left(\frac{l_p}{l}\right)^{\frac{1}{2}}$, where the solutions α_u and α_s are also assumed. The solutions $\tilde{r}_{+u1}$ and $\tilde{r}_{+s2}$ are thermodynamically unstable, while $\tilde{r}_{+u2}$ and $\tilde{r}_{+s1}$ are thermodynamically stable.

The action of the Hawking-Page black hole solutions is

$$I_{\text{bh}} = \frac{1}{2l_p^2 \bar{T}} \left(r_+ + \frac{r_+^3}{l^2} \right) - \frac{\pi r_+^2}{l_p^2}, \quad (8.66)$$

with the radius r_+ being a function of $\bar{T}$. Indeed here, the gravitational radius r_+ , which is also a horizon radius, is given by the equation $\bar{\beta} \equiv \frac{1}{\bar{T}} = \frac{\iota(r_+)}{1+3\frac{r_+^2}{l^2}}$, see also Eq. (8.40) with $\iota(r_+) = 4\pi r_+$.

Therefore, we have to solve for r_+ the equation $\bar{T} = \frac{1+3\frac{r_+^2}{l^2}}{4\pi r_+}$. The solutions $\frac{r_+}{l}$ as a function of $\bar{T}l$ are given by

$$\frac{r_+}{l} = \frac{2\pi l \bar{T}}{3} \pm \frac{1}{3} \sqrt{(2\pi l \bar{T})^2 - 3}. \quad (8.67)$$

Thus, for $l\bar{T} \geq \frac{\sqrt{3}}{2\pi}$, there are two black hole solutions, $r_{+1}(\bar{T})$, the solution with the minus sign, which is thermodynamically unstable, and $r_{+2}(\bar{T})$, the solution with the plus sign, which is stable. When equality holds, $l\bar{T} = \frac{\sqrt{3}}{2\pi}$, one has a degenerate solution, $r_{+1}(\bar{T}) = r_{+2}(\bar{T}) = \frac{1}{\sqrt{3}}$. For $l\bar{T} < \frac{\sqrt{3}}{2\pi}$ there are no solutions, see Fig. 8.5. The numerical value of this critical temperature is $l\bar{T} = \frac{\sqrt{3}}{2\pi} = 0.276$ approximately, with the corresponding horizon radius $\frac{r_+}{l} = \frac{1}{\sqrt{3}} = 0.577$.

The black hole entropy can be obtained from the action as

$$S_{\text{bh}} = \frac{\pi r_+^2}{l_p^2}, \quad (8.68)$$

i.e., the Bekenstein-Hawking entropy, with r_+ standing for r_{+1} or r_{+2} . The entropy describes the usual parabola, see Fig. (8.6).

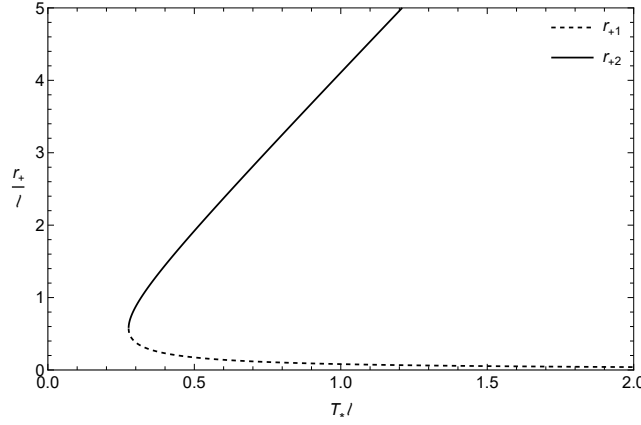


Figure 8.5: Solutions of the ensemble $\frac{r_{+1}}{l}$ and $\frac{r_{+2}}{l}$ for the black hole in asymptotically AdS.

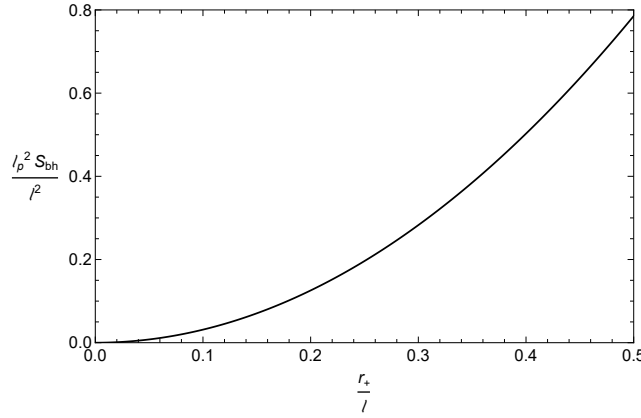


Figure 8.6: Black hole entropy $\frac{l_p^2}{l^2} S_{\text{bh}}$ as a function of the horizon radius $\frac{r_+}{l}$, which stands either for $\frac{r_{+1}}{l}$ or for $\frac{r_{+2}}{l}$.

The heat capacity for the Schwarzschild-AdS black hole is

$$C_{\text{bh}} = \frac{2\pi r_+^2 \left(1 + 3\frac{r_+^2}{l^2}\right)}{3 \left(\frac{r_+^2}{l^2} - \frac{1}{3}\right)}, \quad (8.69)$$

for each solution $r_{+1}(\bar{T})$ and $r_{+2}(\bar{T})$, see Fig. 8.7. The heat capacity is positive for $\frac{r_+}{l} > \frac{\sqrt{3}}{3}$, and so r_{+1} is thermodynamic unstable and r_{+2} is thermodynamic stable.

We must add that pure hot AdS and black hole in AdS compete to be the prominent thermodynamic phase, with the phase that has the minimum action being the one that is favored. Pure hot AdS has zero action, $I_{\text{PAdS}} = 0$, so if $I_{\text{bh}} > 0$ then AdS is favored, if $I_{\text{bh}} = 0$ the two phases coexist equally, and if $I_{\text{bh}} < 0$ then the black hole is favored. From the black hole action, Eq. (8.66), one finds that as one increases $l\bar{T}$ from zero, there is a first order phase transition from thermal AdS to the stable black hole state, the transition happening at $l\bar{T} = \frac{1}{\pi} = 0.318$, the latter number being approximate, to the stable black hole with horizon radius $\frac{r_{+2}}{l} = 1$.

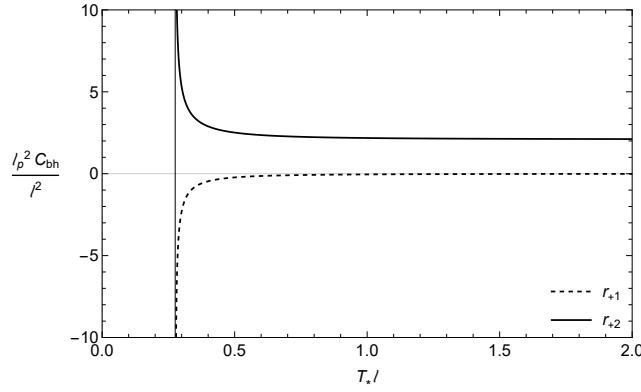


Figure 8.7: Adimensional heat capacity $\frac{l_p^2 C_{bh}}{l^2}$ of the black hole solutions r_{+1} and r_{+2} in function of $\bar{T}l$.

8.6.2 Hot thin shell versus black hole and favorable states

8.6.2.1 Gravitational radii, entropies, and heat capacities

We are interested in the comparison between the properties obtained for the hot thin shell in AdS and the properties of the black hole in AdS. More specifically, we can compare the gravitational radii of each thin shell with the gravitational radius of the black hole, examine their entropies, and analyze their heat capacities.

First, we compare the two possible gravitational radii of each thin shell and the gravitational radii of the black hole. For the mechanically unstable hot thin shell, which is given by the shell radius α_u , we have found that there are two branches for the gravitational radius. One of the branches, $\tilde{r}_{+u1}$ is thermodynamically unstable, while the other branch $\tilde{r}_{+u2}$ is thermodynamically stable. As well, for the Hawking-Page black hole, the horizon radius r_+ has one branch r_{+1} which is thermodynamically unstable, and another branch r_{+2} which is thermodynamically stable. It is clear from Fig. 8.1 and Fig. 8.5 that the two solutions for the gravitational radius with thin shell radius α_u , which is mechanically unstable, share similarities with the two solutions for the horizon radius of the black hole. In detail, the thermodynamically unstable branch of the thin shell solution, $\tilde{r}_{+u1}$, follows the same behavior as the thermodynamically unstable black hole solution, r_{+1} . As well, the thermodynamically stable branch of the thin shell solution, $\tilde{r}_{+u2}$, follows the same behavior as the thermodynamically stable black hole solution, r_{+2} . These similarities could perhaps be expected since the mechanically unstable shell is bound to collapse into a black hole and can be understood as a black hole precursor. For the mechanically stable hot thin shell, which is given by the shell radius α_s , we have found that there are also two branches for the gravitational radius. One of the branches, $\tilde{r}_{+s1}$ is thermodynamically unstable, while the other branch $\tilde{r}_{+s2}$ is thermodynamically stable. From Fig. 8.1 and Fig. 8.5, it can be seen that the two solutions for the gravitational radius with thin shell radius α_s , which is mechanically stable, share no similarities with the two solutions for the horizon radius of the black hole. However, the solutions $\tilde{r}_{+s1}$ and $\tilde{r}_{+s2}$ appear to have similarities with

the behavior of the Davies black hole solutions, which correspond to an electrically charged black hole [3] in the canonical ensemble. These black hole solutions have a stable branch that start at zero temperature with a horizon radius given by the electric charge and then the horizon radius increases with the temperature up until a maximum temperature. The same happens with the mechanically and thermodynamically stable hot matter thin shell, starting at zero temperature with zero gravitational radius instead of a non-zero value. This behavior is expected from a solution of hot self-gravitating matter that models hot AdS space with radiation at the same order of approximation.

Second, we now compare the matter thin shell entropy with the black hole entropy. Both entropies depend on their own gravitational radius, which in the black hole case is also a horizon radius. For the mechanical unstable and stable thin shells, we have seen that the matter entropy S_m can be described approximately by a power law. For the unstable shell α_u , we found that $S_m = \zeta \tilde{r}_+^{1.2323}$, for some ζ . For the stable shell α_s , we found that $S_m = \zeta \tilde{r}_+^{0.867504}$, for some other ζ . For the black hole, the entropy is also given by a power law $S_{bh} = \chi r_+^2$, for some χ . Both the mechanical unstable shell α_u and the black hole have exponents γ satisfying $\gamma > 1$, while the stable shell has an exponent $\gamma < 1$. This behavior reinforces the similarity properties between the mechanical unstable shell and the black hole, as advocated above. To understand this, we can resort to a Bekenstein argument [43]. It is stated in it, without making calculations, that one should expect that black holes have an entropy with an exponent γ obeying $\gamma > 1$. Suppose that they had an exponent $\gamma < 1$, then one would have that two isolated black holes that will merge into one should have a final mass lower than the sum of the initial mass, part of the initial mass being lost in gravitational radiation. Concomitantly the entropy of the final black hole should be greater than the sum of the entropies of the initial black hole, to have the second law of thermodynamics obeyed. The two conditions cannot be met simultaneously when $\gamma < 1$. Suppose for these purposes that the black hole entropy is proportional to a power of the gravitational radius, $S_{bh} = \chi r_+^\gamma$, for some χ and γ . For black hole a and black hole b merging into a third black hole c , one has from the first condition $r_{+c} < r_{+a} + r_{+b}$ and from the second condition $r_{+c}^\gamma > r_{+a}^\gamma + r_{+b}^\gamma$, i.e., one has the range for the horizon radius of the black hole c obeying $(r_{+a}^\gamma + r_{+b}^\gamma)^{\frac{1}{\gamma}} < r_{+c} < r_{+a} + r_{+b}$. This inequality can only be fulfilled for $\gamma > 1$. From thermodynamic arguments then it was chosen $\gamma = 2$, the correct value for black holes. The point here is that the unstable shell α_u has an exponent $\gamma = 1.2323$ that is indeed greater than one, and following the arguments above it is black hole like, i.e., the shell behaves as the black hole that it can originate from gravitational collapse. The stable shell α_s has an exponent $\gamma = 0.867504$ that is less than one, and thus it does not behave as a black hole that it could originate upon collapse. Another remark in relation to the entropy is that for the black hole $S_{bh} = \pi \frac{l_p^2}{\tilde{r}_+^2} (\frac{r_+}{\tilde{r}_+})^2$, the exponent is $\gamma = 2$, and it grows much faster than for the matter for large gravitational radius, as expected, since it is known that black holes have the maximum entropy. However, for small gravitational radius, the matter entropy S_m is larger than the black hole entropy, so this might indicate

that there are no stable black holes for small gravitational radius, see Fig. 8.5 noting that r_{+1} is unstable.

Third, we can compare the heat capacities for the hot thin shell and black hole, C and C_{bh} , respectively, with the help of Figs. 8.4 and 8.7. The heat capacity C as a function of the temperature for the mechanically unstable shell $\alpha_u(\tilde{r}_+)$ behaves in the same manner as the heat capacity of the black hole C_{bh} as a function of temperature. There are parts that are thermodynamically unstable, $\tilde{r}_{+u1}$ for the shell and r_{+1} for the black hole, and parts that are thermodynamically stable $\tilde{r}_{+u2}$ for the shell, and r_{+2} for the black hole. The heat capacity C as a function of the temperature for the mechanically stable shell α_u behaves differently from the heat capacity of the black hole C_{bh} as a function of temperature. However, it shows similarities to the heat capacity of the electrically charged black hole in the canonical ensemble [3]. In particular, the thermodynamically stable $\tilde{r}_{+s1}$ branch of the heat capacity is similar to the heat capacity of the stable branch of an electrically charged black hole. These similarities displayed here for the heat capacity are the same as the similarities that we found above when comparing the gravitational radii, and indeed they come from the fact that the heat capacity is related to the first derivative in temperature of the gravitational radius, and so the similarities from the gravitational radius solutions are carried into the heat capacity.

8.6.2.2 *Favorable states: Comparison between hot thin shell, black hole, and pure hot AdS thermodynamic states*

We now make the identification of the favorable states of the ensemble, i.e., given a fixed temperature, we analyze if the hot thin shell is favored in relation to the black hole or if the contrary happens.

In order to do this, to identify the favorable states at a fixed temperature of the ensemble, we must compare the action of the stable hot self-gravitating matter thin shell with the action of the stable black hole solution of Schwarzschild-AdS. This is so because the sector with a self-gravitating matter shell may compete with the black hole sector and the pure hot AdS sector in the path integral. From thermodynamics, it is known that the preferred configuration is the one with the least free energy. This also means the one with the least free action, since $I_0 = \bar{\beta}F$. This can be seen because the partition function is $Z = e^{-I_0}$, and thus the configuration with less I_0 is the more probable one. Now, if at a certain temperature, the configuration with the least action changes, then this marks a first order phase transition as the action is continuous but not differentiable there. In [69], hot thermal AdS, i.e., hot AdS in one-loop approximation, and the stable black hole solution were discussed, where it was discovered the Hawking-Page phase transition from hot thermal AdS to the stable black hole. Here, the stable self-gravitating matter thin shell can be understood as one possible description of hot AdS with thermal self-gravitating matter, i.e., hot curved AdS.

To help in the comparison of the actions, we can write the action of the matter thin shell given in Eq. (8.31) as $\frac{l_p^2}{l^2} I_0 = \frac{\tilde{r}_+ \left(1 + \frac{\tilde{r}_+^2}{l^2}\right)}{2\tilde{T}l} - c_0 \left(\frac{l_p^2 l_c}{l^3}\right)^{\frac{1}{4}} \left(\frac{m l_p^2}{l}\right)^{\frac{3}{4}} \left(4\pi \frac{\alpha^2}{l^2}\right)^{\frac{1}{4}},$

with $\tilde{r}_+$ given by the stable solution $\tilde{r}_+ = \tilde{r}_+(\bar{T})$ of Eq. (8.40), i.e., $\tilde{r}_{+s2}$, and α given by the stable solution $\alpha = \alpha(\tilde{r}_+(\bar{T}))$ of Eq. (8.39) together with Eq. (8.40), i.e., $\alpha_s = \alpha_s(\tilde{r}_{+s2}(\bar{T}))$, and the action has been set without units. It is useful to define $z = \left(\frac{l^3}{l_p^2 l_c}\right)^{\frac{1}{4}}$ as the parameter without units that establishes the relevant scale ratios between the Planck length, AdS length, and the Compton length in the case of the hot matter thin shell with the chosen equations of state above. Then, we can write the adimensional action of the matter thin shell as

$$\begin{aligned} I_0 &= \frac{1}{z} \bar{I}_0 \left(\frac{\bar{T}l}{z} \right), & z &\equiv \left(\frac{l^3}{l_p^2 l_c} \right)^{\frac{1}{4}}, \\ \bar{I}_0 \left(\frac{\bar{T}l}{z} \right) &\equiv \frac{l^2}{l_p^2} \frac{\tilde{r}_+ \left(1 + \frac{\tilde{r}_+^2}{l^2} \right)}{2 \frac{\bar{T}l}{z}} - \frac{l^2}{l_p^2} c_0 \left(\frac{m l_p^2}{l} \right)^{\frac{3}{4}} \left(4\pi \frac{\alpha^2}{l^2} \right)^{\frac{1}{4}}. \end{aligned} \quad (8.70)$$

This property of the thin shell action is also useful for numerical purposes, as one can compare actions with a given parameter z . The behavior of the action of the matter thin shell, Eq. (8.70), evaluated at the solutions of the shell, can be summarized as follows. The action starts at zero at $l\bar{T} = 0$ and decreases for increasing $l\bar{T}$. After a final temperature lT_{*f} , the stable thin shell solution ceases to exist, which can be interpreted as the matter having larger thermal agitation than the permitted to have a shell, implying that the shell can collapse to a black hole or disperse to infinity. This lT_{*f} depends on the scale ratio parameter $z = \left(\frac{l^3}{l_p^2 l_c}\right)^{\frac{1}{4}}$, which itself depends on the natural gravitational scale ratio $\frac{l}{l_p}$ and on the matter scale ratio $\frac{l}{l_c}$.

In relation to the action of the stable black hole, one has the Hawking-Page action given by

$$\frac{l_p^2}{l^2} I_{bh} = \frac{\frac{r_+}{l} \left(1 + \frac{r_+^2}{l^2} \right)}{2 \bar{T}l} - \frac{\pi r_+^2}{l^2}, \quad (8.71)$$

with r_+ being the stable black hole solution, i.e., r_{+2} , given as a function of $l\bar{T}$ by $\frac{r_+}{l} = \frac{2\pi l\bar{T}}{3} + \frac{1}{3} \sqrt{(2\pi l\bar{T})^2 - 3}$. We have seen that the stable solution only exists for $l\bar{T} \geq \frac{\sqrt{3}}{2\pi} = 0.276$, with last equality being approximate.

Now, we must compare the matter action with the black hole action for each possible parameter z and c_0 , and also with pure hot AdS space characterized by $I_{pAdS} = 0$. Here, we made the choice $c_0 = 1$ since c_0 can be in some sense absorbed by the parameter z . Then, the comparison of the actions can be made only on z . The plot of the actions is shown in Fig. 8.8, where for the matter shell three values of z are considered, namely, $z = 0.1$, $z = 0.581$, and $z = 1$. The action of matter thin shell is zero at $l\bar{T} = 0$ and decreases to negative values for increasing temperature, until the final temperature lT_{*f} is reached with a corresponding minimum negative action. Note that the maximum temperature depends on the parameter z as $lT_{*f} = 0.577z$, where 0.577 is approximate. Above this temperature lT_{*f} the shell stops to exist and probably collapses. With respect to the black hole case, the action I_{bh} only starts to

exist for $l\bar{T} \geq \frac{\sqrt{3}}{2\pi} = 0.276$, and decreases with increasing temperature. The black hole action is positive in the range $\frac{\sqrt{3}}{2\pi} \leq lT_* \leq 0.318$ where 0.318 is approximate, is zero at $lT_* = 0.318$, and is negative for $l\bar{T} > 0.318$. The point of intersection

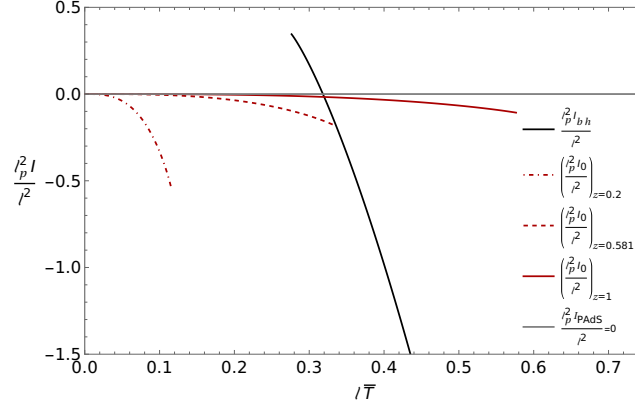


Figure 8.8: Plot of the actions I_{bh} , I_0 , and I_{PAdS} as functions of the temperature $l\bar{T}$. The solution that has lower action between stable black hole, hot shell, and pure hot AdS is the one that is favored. It is chosen $z = \left(\frac{l}{l_c}\right)^{\frac{1}{4}} \left(\frac{l}{l_p}\right)^{\frac{1}{2}} = 0.2, 0.581, 1$ to compare the actions. For $z = 0.2$ the hot shell ceases to exist at temperature $lT_{*f} = 0.115$, for $z = 0.581$ at temperature $lT_{*f} = 0.335$, and for $z = 1$ at temperature $lT_{*f} = 0.577$.

between the two actions is given by the equality $I_{bh}(lT_*) = \frac{1}{z} \bar{I}_0\left(\frac{lT_*}{z}\right)$ for each z . For example, in the case of $z = 1$, one has that the actions intersect at $l\bar{T} = 0.320$, and so as one increases the temperature around this point, there is a first order phase transition from the hot matter thin shell to the stable black hole. This is analogous to the case of the Hawking-Page phase transition, where the matter is treated in a one-loop approximation, rather than in zero loop. It can be found that the intersection between the matter thin shell action and the black hole action only happens for a range of z . As one decreases z , the maximum temperature of the thin shell also decreases, while the black hole action is unaltered. And so, there must be a minimum value of z for which the intersection occurs. The minimum value can be found by considering that the two actions intersect exactly at the maximum temperature of the shell, i.e., $I_{bh}(0.577z) = \frac{1}{z} \bar{I}_0(0.577)$, which numerically can be solved and gives $z = 0.581$ approximately and the first order phase transition occurs for this case at $l\bar{T} = 0.336$, approximately. As a consequence, the action of the matter thin shell intersects the action of the black hole only if $0.581 \leq z < \infty$, with first number being approximate. If $0 < z < 0.581$, the matter shell solution ceases to exist before it intersects the curve of the action of the black hole. Therefore, there is only a first order phase transition from the matter thin shell to the black hole when $0.581 \leq z < \infty$.

We must comment about the phase of pure hot AdS space, i.e., AdS space in zero loop, in comparison with the hot thin shell and black hole configurations. For $0.581 < z < \infty$, the matter thin shell action always intersects the black hole action and it is always negative, and so the pure hot AdS space is always the least

favorable. For $0.547 \leq z < 0.581$, with 0.547 being approximate, the matter thin shell does not intersect the black hole action. However, when the matter thin shell ceases to exist at the maximum temperature, there is a black hole solution with negative action and so both the thin shell and the black hole solutions are still more favorable than pure hot AdS space. For the range $0 \leq z \leq 0.547$, there is an interval of temperatures, between the maximum temperature of the shell and the temperature at which the black hole solution has zero action, where pure hot AdS space is favorable.

It is worth stressing that the parameter z can be restricted from validity arguments of the zero-loop approximation. The zero-loop approximation should be valid for the cases $l \gg l_p$ but with l not that large and since α is comparable to l , one must have $l \gg l_c$ also so that matter at the thin shell can be judged thermodynamic. Moreover, all scales must be much greater than the Planck scale l_p , as the zero-loop approximation is being used here. Therefore, one must have $l \gg l_c \gg l_p$, which means a large value of $z = \left(\frac{l^3}{l_p^3 l_c}\right)^{\frac{1}{4}}$. In this regime, the first order phase transition can always occur, and both the matter thin shell and the black hole solutions are more favorable than pure hot AdS space $I_{\text{PAdS}} = 0$. This strengthens the interpretation that the matter thin shell with the chosen equation of state models hot AdS space with self-gravitating radiation matter at low temperatures.

8.6.2.3 *Favorable states: Comparison between thin shell, black hole, and hot thermal AdS thermodynamic states*

We have seen how the action of a stable self-gravitating matter system in AdS, which is a realization of hot curved AdS, compares with the action of the stable AdS black hole solution, and the action of pure AdS, describing classical AdS space devoid of any matter.

It is also interesting to substitute pure AdS for hot thermal AdS, i.e., AdS space with nonself-gravitating radiation obtained from the one-loop approximation, and compare with the action for black hole and the thin shell. The action I_{TAdS} for hot thermal AdS is given by

$$I_{\text{TAdS}} = -\frac{\pi^4 (l\bar{T})^3}{45}, \quad (8.72)$$

where hot thermal AdS is assumed to be made of particles, each with effective number of spin states equal to two, such as gravitons do. We can then compare the first order phase transition treated above with the Hawking-Page phase transition, which is a transition between hot thermal AdS with action given in Eq. (8.72) and the black hole action given in Eq. (8.66) [69]. Moreover, when the temperature of the radiation is sufficiently high, the radiation forms a singularity of the Buchdahl type in the center and then it presumably collapses to a black hole. We find that this maximum Buchdahl radiation temperature for which radiation ceases to exist is given by $lT_{\text{Buch}} = 0.4234 \left(\frac{2\pi^2}{30}\right)^{-\frac{1}{4}} \left(\frac{l}{l_p}\right)^{\frac{1}{2}}$ [69], see also [170].

In Fig. 8.9, we plot the actions for the stable black hole, the hot matter thin shell, and hot thermal AdS. In the figure, we consider the parameters $z = \left(\frac{l}{l_c}\right)^{\frac{1}{4}} \left(\frac{l}{l_p}\right)^{\frac{1}{2}} = 1$ and $\frac{l_p}{l} = 1$ to compare the actions. First, we can comment on the transitions between

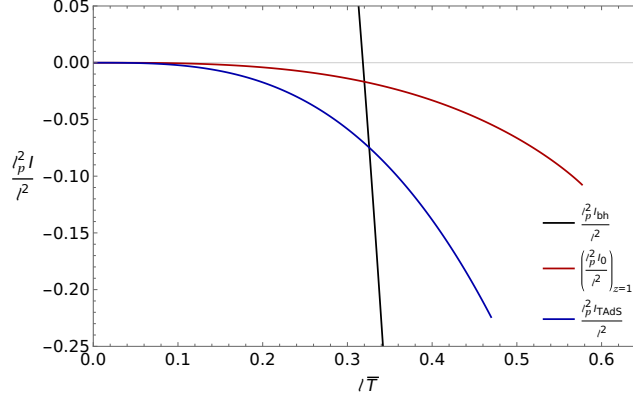


Figure 8.9: Plot of the actions I_{bh} , I_0 , and I_{TAdS} as functions of the temperature $l\bar{T}$. The solution that has lower action between stable black hole, hot shell, and thermal hot AdS, i.e., AdS with nonself-gravitating radiation, is the one that is favored.

It is chosen $z = \left(\frac{l}{l_c}\right)^{\frac{1}{4}} \left(\frac{l}{l_p}\right)^{\frac{1}{2}} = 1$ and $\frac{l_p}{l} = 1$ to compare the actions. For $z = 1$, the hot shell ceases to exist at temperature $lT_{*f} = 0.577$. Thermal hot AdS ceases to exist at temperature $lT_{Buch} = 0.4701$.

stable black hole and hot thermal AdS thermodynamic states and, second, we can compare the results for the hot matter thin shell and hot thermal AdS. In relation to the first point, we have seen that there are no black holes, and therefore no stable black hole, in the range of temperatures $0 \leq l\bar{T} < \frac{\sqrt{3}}{2\pi} \approx 0.276$, the last number being approximate. From $\frac{\sqrt{3}}{2\pi} \leq l\bar{T} < 0.325$, hot thermal AdS is favored in relation to a black hole state. At $l\bar{T} = 0.325$ hot thermal AdS and black hole coexist equally, and this is the temperature at which a first order phase transition occurs. This is the Hawking-Page phase transition. For $0.325 < l\bar{T} < lT_{Buch}$, where $lT_{Buch} = 0.4701$, the black hole is favored over hot thermal AdS, meaning that is more probable to find the system in a black hole state. For $lT_{Buch} < l\bar{T} < \infty$, the system is in a collapsed black hole state in AdS, meaning that at these temperatures it is not possible to find the system in a hot thermal AdS state. In relation to the second point, we can see from the figure that the first order phase transition between the matter thin shell and the black hole has the same behavior as the Hawking-Page phase transition between hot thermal AdS and the black hole. However, it seems that hot thermal AdS, i.e., AdS with nonself-gravitating radiation, is more favorable than the matter thin shell, with the differences being very small as one increases z and $\frac{l}{l_p}$. Since the action for hot thermal AdS does not include gravitation, as it corresponds to nonself-gravitating radiation, it is clear that the hot thin shell should mimic self-gravitating radiation due to its similar behavior around the phase transition. In addition, they share the feature of a maximum temperature,

$lT_{*f} = 0.577$ for the hot thin shell and $lT_{\text{Buch}} = 0.4701$ for hot thermal AdS, the values being for $z = 1$ and $l = l_p$.

8.7 CONCLUSIONS

In this chapter, we studied the canonical ensemble of a hot self-gravitating matter thin shell in AdS by finding the partition function of the system via the Euclidean path integral approach to quantum gravity. Our study was restricted to spherically symmetric metrics and we established the boundary conditions. Imposing the Hamiltonian constraint, we obtained the reduced action of the matter thin shell in AdS, and from the reduced action, we obtained the stationary and stability conditions. There are two equations for the stationary condition, i.e., the balance of pressure and the balance of temperature, and two stability conditions, i.e., the mechanical stability condition and the thermodynamic stability condition. The fact that one can obtain the two stability conditions shows the power of the reduced action in the Euclidean path integral approach, in that it gives not only information about thermodynamics but also of mechanics.

We have shown that for the case of the matter thin shell in AdS, one can obtain the canonical ensemble of the thin shell by establishing an effective reduced action only dependent on the gravitational radius of the thin shell. This eases the analysis of the canonical ensemble and further shows that one can build an effective reduced action dependent only on the gravitational radius for this case, hiding the description of matter in the form of the effective entropy. We established here the link between the effective entropy and the specific description of the shell with an equation of state, in the zero-loop approximation.

The thermodynamics of the system follows directly from zero-loop approximation consisting of the reduced action evaluated at the stationary points, which is equivalent to finding the action for the specific solution of Einstein equation. In this approximation, we obtained directly the relevant thermodynamic quantities, namely, the mean free energy, the entropy, the mean energy, and the heat capacity. We found there is a correspondence between thermodynamic stability in the ensemble theory and positive heat capacity in the derived thermodynamics, as it should. On the other hand, within thermodynamics itself, we cannot determine mechanical stability by varying the thermodynamic quantity fixed at the conformal boundary, i.e., the temperature. This fact seems to be a consequence of applying the zero-loop approximation to the internal degree of freedom, the radius of the shell. It also means that the zero-loop approximation of the effective reduced action yields the expected thermodynamic stability condition, since it can be seen as a generalized free energy function.

We introduced an equation of state for the matter and we obtained the solutions of the canonical ensemble for the matter thin shell. We found that there are in total four solutions, with only one of them being stable both mechanically and thermodynamically. We compared the action of the stable matter shell solution with the Hawking-Page AdS black hole stable solution and we verified the existence of a first order phase transition in a physically reasonable range of scale lengths. We

have shown that this first order phase transition follows an analogous behaviour to the Hawking-Page phase transition, and that the hot matter thin shell can mimic self-gravitating radiation.

It will be interesting to uncover the sector with a black hole and a shell together to fully understand the space of configurations and respective phase transitions. Moreover, it would be interesting to explore additional fixed parameters of the ensemble, such as the chemical potential, to understand if one is able to access the mechanical stability condition from varying these fixed parameters, in the sense that they are needed for thermodynamic stability. This line of research motivates Chapter 9.

THERMODYNAMIC ENSEMBLES OF A BLACK HOLE AND A SELF-GRAVITATING MATTER THIN SHELL WITH A FIXED CHEMICAL POTENTIAL: EQUILIBRIUM, STABILITY AND LE CHATELIER-BRAUN PRINCIPLE

9.1 INTRODUCTION

The study of the canonical ensemble including self-gravitating matter thin shell and a black hole was first done in [136], using the Euclidean path integral approach to quantum gravity [67]. The analysis of this system was further deepened in [137], by keeping the radius of the shell fixed. In Chapter 8, we considered the canonical ensemble of a matter thin shell in anti-de Sitter with a chosen equation of state, revealing that the mechanical stability of the shell appeared as a condition for the validity of the zero loop approximation but it was not needed for the thermodynamic stability.

In this chapter, we progress further in the study of self-gravitating matter thin shells to study how the formalism handles the stability of composed systems. In that regard, we construct the grand canonical ensemble of a self-gravitating matter thin shell with a black hole inside, with the system surrounded by a finite cavity. We calculate the partition function through the Euclidean path integral approach in the zero loop approximation. We introduce the chemical potential in the description of matter in order to understand its implications to thermodynamic stability and the system at hand. For convenience, the partition function is written in terms of the another partition function for an ensemble with cavity at infinity, with fixed energy E and fixed logarithm of the fugacity $\beta\mu$, which we call here the $(E, \beta\mu)$ ensemble. Note that this ensemble is a modification of the microcanonical ensemble. We show that the Le Chatelier-Braun principle follows from the validity of the zero loop approximation and also that the conditions for the validity of the zero loop approximation, including the mechanical stability of the shell, must be considered to infer thermodynamic stability, due to the presence of the chemical potential.

This chapter is organized as follows. In Sec. 9.2, we construct the partition function of the ensemble, where we obtain that the partition function of the grand canonical ensemble can be described in terms of another partition function related to the $(E, \beta\mu)$ ensemble. In Sec. 9.4.1, we perform the zero loop approximation to the partition function of the $(E, \beta\mu)$ ensemble. In Sec. 9.4.3, the zero loop approximation is performed to the partition function of the grand canonical ensemble. In Sec. 9.5,

we obtain the thermodynamics of the system composed by a black hole with a thin shell surrounding, from the $(E, \beta\mu)$ ensemble. In Sec. 9.6, we obtain the thermodynamics from the grand canonical ensemble. In Sec. 9.7, we consider the Martinez equation of state and we obtain another fundamental equation of state. In Sec. 9.8, we display the Hessian of the relevant actions. In Sec. 9.9, we review the mechanical stability of a thin shell surrounding a black hole. In Sec. 9.10, we present the conclusions.

9.2 THE GRAND CANONICAL ENSEMBLE AND THE $(E, \beta\mu)$ ENSEMBLE THROUGH THE PATH INTEGRAL APPROACH

9.2.1 The grand canonical statistical partition function

The construction of the grand canonical ensemble of a curved space M with matter can be done with the Euclidean path integral approach to quantum gravity, with the partition function given by

$$Z = Dg_{\alpha\beta} D\psi e^{-I[g_{\gamma\nu}, \psi]}, \quad (9.1)$$

where $g_{\alpha\beta}$ is the Euclidean metric, ψ represents a matter field, $I[g_{\gamma\nu}, \psi]$ is the Euclidean action, and $Dg_{\alpha\beta}$ and $D\psi$ are the integration measures for $g_{\alpha\beta}$ and ψ , respectively. In the canonical ensemble, the integration is done over periodic fields $g_{\alpha\beta}$ and ψ , if ψ is bosonic. However, this condition can suffer modifications according to the type of ensemble one considers.

In the case of this chapter, we are interested in including and fixing the chemical potential of the matter field, hence the ensemble we are considering is the grand canonical ensemble. While we explained the Euclidean path integral approach in Chapter 3, here it is important to explain how we can introduce the chemical potential in the action $I[g_{\gamma\nu}, \psi]$. We can first trace back the partition function in the formal representation $Z = \text{Tr}(e^{-\beta H})$ for the canonical ensemble, where β is the fixed inverse temperature defined by $\beta = \int_{\partial M} (g^{\tau\tau})^{-1/2} d\tau$, with τ being the imaginary Euclidean time having period 2π . In order to consider the grand canonical ensemble, one must modify the partition function as $Z = \text{Tr}(e^{-\beta H + \beta\mu N})$, where μ is the fixed chemical potential and N is the mean particle number. The operator βH can be defined by the mean Euclidean Hamiltonian as $\beta H = \int_M \mathfrak{h} d^3x d\tau$, where $\mathfrak{h}$ is the Hamiltonian tensor density with respect to $g_{\alpha\beta}$ of the Euclideanized space, i.e. $\mathfrak{h}$ transforms as the determinant of the metric $\sqrt{g}$. Depending on the Hamiltonian of the field, one can find the functional version of the particle number as $N(\tau) = \int \mathfrak{n} d^3x$ for a slice of constant τ , where $\mathfrak{n}$ is the particle number tensor density that transforms like the determinant of the induced metric of hypersurfaces with constant τ . From here, it seems non-trivial to include such operator in the trace without assumptions, since it is an integral dependent on the slice of constant τ . A simple way to avoid such dependence is to use $N = \frac{1}{2\pi} \int N(\tau) d\tau$, that is the mean particle number over the slices of constant τ . In principle, if the particle number is conserved over the slices, then $N = N(\tau)$, giving the right result. One could then build the term $\beta\mu N$ as $\beta\mu N = \int (g^{\tau\tau})^{-1/2} \mu(x) \mathfrak{n} d^3x d\tau$, where $\mu(x)$ is

a scalar and it is the local chemical potential defined by $\frac{\beta\mu}{2\pi} = (g^{\tau\tau})^{-1/2}\mu(\tau, x)$. The introduction of this local chemical potential allows to construct a covariant integral. The full trace can then be transformed into the Euclidean path integral in Eq. (9.1), where $I[g_{\alpha\beta}, \psi]$ is the Euclidean action of the metric space and matter fields with a modification that depends on the local chemical potential $\mu(x)$. For complex matter fields with a kinetic part which is quadratic, such modification can be obtained by a simpler manner. One can consider the typical matter field lagrangian but with a transformation of the field, for example $\psi = e^{-\frac{\beta\mu}{2\pi}\tau}\hat{\psi}$ and $\psi^\dagger = e^{\frac{\beta\mu}{2\pi}\tau}\hat{\psi}^\dagger$ for complex bosonic fields, where $\dagger$ means complex conjugate. This gives a modified lagrangian for the fields $\hat{\psi}$ and $\hat{\psi}^\dagger$ which include the chemical potential. The periodic conditions are satisfied not by ψ and $\psi^\dagger$ but for $\hat{\psi}$ and $\hat{\psi}^\dagger$. This agrees with the fact that μ , although being a constant of the ensemble, is a scalar density with respect to a one dimensional metric. The fact we are fixing μ means that we are choosing a specific foliation of space in hypersurfaces of constant τ . Nevertheless, we can proceed assuming the identity $\frac{\beta\mu}{2\pi} = (g^{\tau\tau})^{-1/2}\mu(\tau, x)$. It is also convenient to define a notion of local temperature as $\frac{1}{T(\tau, x)} = 2\pi(g^{\tau\tau})^{-1/2}$, although it must be emphasized that only β and $\beta\mu$ have definite meanings as they are fixed in the ensemble. The result $\beta\mu = \mu(\tau, x)/T(\tau, x)$ then comes naturally.

In this paper, we consider a spherically symmetric cavity with a matter thin shell inside together with a black hole. The partition function for the system is assumed to be given by $Z = \int Dg_{\alpha\beta} e^{-I_g[g_{\alpha\beta}]} \int D\psi e^{-I_m[g_{\alpha\beta}, \psi]}$, where the gravitational action I_g is given by the Einstein Hilbert action plus the Gibbons-Hawking-York term, and the matter action I_m depends on the type of matter considered. In general, it is not possible to obtain a closed form for the matter path integral, even more it is not possible to guarantee its convergence after the techniques of regularization and renormalization. Yet, we can assume that such path integral, if it is convergent, yields a general expression $e^{-\int \mathcal{W}[g_{\alpha\beta}] \sqrt{g} d^4x} = \int D\psi e^{-I_m[g_{\alpha\beta}, \psi]}$, where $\mathcal{W}$ is the matter grand potential density. For the case of a matter thin shell inside a spherically symmetric cavity, the induced metric in the thin shell is described by the metric of a 2-sphere with constant radius plus the metric of a ring with constant radius parametrized by Euclidean imaginary time. Therefore, the path integral in principle can be integrated. If we take γ_{ab} to be the induced metric on the shell, we can assume the matter grand potential being described by the functional $\mathcal{W} = \mathcal{F}(T_m, n_m) - \mu_m n_m$, where T_m is the local temperature at the shell defined as $\frac{1}{T_m} = 2\pi(\sqrt{\gamma^{\tau\tau}})^{-1}$, n_m is the particle number density defined as $n_m = (\sqrt{\gamma}\sqrt{\gamma^{\tau\tau}})^{-1}n_m$ with n_m being the particle number tensor density, and μ_m is the local chemical potential conjugate to the particle number, where $\frac{\mu_m}{T_m}$ must be constant.

9.2.2 Grand canonical action for a black hole and a matter thin shell

Here, we consider spherically symmetric configurations which are stationary and so these have a connection to a physical spacetime, in particular a black hole with a matter thin shell in equilibrium inside a cavity. The Euclideanized spacetime M

is split into two parts, M_1 and M_2 , by the thin shell described by a hypersurface $\mathcal{C}$. The space M_1 designates the Euclidean space with boundary $\mathcal{C}$ only, while M_2 designates the Euclidean space with two disjoint boundaries $\mathcal{C}$ and ∂M , with the latter being the boundary of the cavity. In some sense, M_1 is the inner Euclidean space while M_2 is the outer Euclidean space.

The action for the system inside the cavity is given by

$$\begin{aligned} I = & - \int_{M \setminus \{\mathcal{C}\}} \frac{1}{16\pi l_p^2} R \sqrt{g} d^4x \\ & + \int_{\mathcal{C}} \left(\frac{[K]}{8\pi l_p^2} + \mathcal{F}(T_m, n_m, \sqrt{\gamma}) - \mu_m n_m \right) \sqrt{\gamma} d^3s \\ & - \frac{1}{8\pi l_p^2} \int_{\partial M} (K - K_0) \sqrt{\gamma} d^3s, \end{aligned} \quad (9.2)$$

where R is the Ricci tensor, $g_{\alpha\beta}$ is the Euclidean metric, $K = r^\alpha_{;\alpha}$ is the trace of the extrinsic curvature with r_a being the outward unit normal to the considered hypersurface, γ_{ab} represents both the induced metric on $\mathcal{C}$ and the induced metric on ∂M with determinant γ , depending on the context, written in a chosen coordinate system $s^i = (\tau, \theta, \phi)$, $\mathcal{F}$ is the free energy density of the thin shell, $\frac{1}{T_m} = 2\pi(\sqrt{\gamma^{\tau\tau}})^{-1}$ is the local inverse temperature at the shell, n_m is the particle number density of the 2-surface of the thin shell which is a functional $n_m = (\sqrt{\gamma^{\tau\tau}}\sqrt{\gamma})^{-1}n_m$ with n_m being the particle number scalar density, μ_m is the local chemical potential at the shell, and K_0 is the extrinsic curvature of the hypersurface considered embedded in flat space. Also, $[K]$ means the difference $K|_{M_2} - K|_{M_1}$ evaluated at the hypersurface, where K can be any tensor. For convenience, in connection to Chapter 3, one can split the action $I = I_{\text{gf}} + I_m$ into the gravitational action I_{gf} expressed by

$$\begin{aligned} I_{\text{gf}} = & - \int_{M \setminus \{\mathcal{C}\}} \frac{1}{16\pi l_p^2} R \sqrt{g} d^4x \\ & + \int_{\mathcal{C}} \left(\frac{[K]}{8\pi l_p^2} \right) \sqrt{\gamma} d^3s \\ & - \frac{1}{8\pi l_p^2} \int_{\partial M} (K - K_0) \sqrt{\gamma} d^3s, \end{aligned} \quad (9.3)$$

and the matter action I_m with the expression

$$I_m = \int_{\mathcal{C}} (\mathcal{F}(T_m, n_m, \sqrt{\gamma}) - \mu_m n_m) \sqrt{\gamma} d^3s. \quad (9.4)$$

9.2.3 Geometry and the matter thin shell description

We fix the geometry of the boundary of the cavity ∂M as a spherically symmetric hypersurface, with topology $\mathbb{S}^1 \times \mathbb{S}^2$. Because of this fixing, we assume that

spherically symmetric spaces contribute the most for the path integral due to a spherically symmetric space M . The metrics considered on M_1 are

$$ds_{M_1}^2 = b_1^2(u) \frac{b_2^2(u_m)}{b_1^2(u_m)} d\tau^2 + a_1^2(u) du^2 + r(u)^2 d\Omega^2, \quad (9.5)$$

and the metrics considered on M_2 are

$$ds_{M_2}^2 = b_2^2(u) d\tau^2 + a_2^2(u) du^2 + r^2(u) d\Omega^2, \quad (9.6)$$

where b_1, b_2, a_1, a_2 , and r are functions of u only, $d\Omega^2$ is the line element of the 2-sphere, S^2 , and also the coordinates are chosen so that $\tau \in]0, 2\pi[$ on M , $u \in]0, y_m[$ on M_1 and $u \in]y_m, 1[$ on M_2 , and θ and ϕ are the spherical coordinates on M .

The hypersurface $\mathcal{C}$ is described by the condition $u = u_m$, with induced metric

$$ds_{\mathcal{C}}^2 = b_2^2(u_m) d\tau^2 + \alpha^2 d\Omega^2, \quad (9.7)$$

where α is defined as $r(u_m) \equiv \alpha$, i.e. the radius of the matter thin shell. Notice that the choices for the metrics on M_1 and on M_2 ensure that the metric is continuous on $\mathcal{C}$, i.e. the metric on M is C^0 . Relatively to the quantities at the shell, we assume that matter is in equilibrium. These considerations allow us to describe the differential of the free energy as $d\mathcal{F} = -s_m dT_m + \mu_m dn_m - \frac{\chi}{\sqrt{\gamma}} d\sqrt{\gamma}$, with the quantity χ defined as $\chi = \epsilon_m + p_m - s_m T_m - \mu n_m$, with s_m being the entropy per area and ϵ_m being the energy density. The quantity χ is present to include the possibility of having degrees of homogeneity different from unity, such is the case for black hole like equations of state, see Chapter 2. For a degree of homogeneity of one, the quantity χ should be zero. Moreover, the matter shell obeys several equations of state, i.e. an expression for $\mathcal{F}(T_m, n_m, \sqrt{\gamma})$ must be known apriori as it is derived from the matter path integral. Notice that by integrating $\mathcal{F}(T_m, n_m, \sqrt{\gamma})$ along the shell, one obtains the mean free energy which obeys the typical thermodynamic differential.

The hypersurface ∂M describing the boundary of the cavity is given by the condition $u = 1$, with induced metric

$$ds_{\partial M}^2 = b_2^2(1) d\tau^2 + R^2 d\Omega^2, \quad (9.8)$$

where R is defined as $r(1) = R$.

9.2.4 Grand canonical boundary conditions

We must impose boundary conditions to select the topology of the spaces that are summed on the path integral and to establish the quantities that are fixed on the ensemble.

At $u = 0$, we impose black hole regularity conditions which are summarized by

$$b(0) = 0 , \quad (9.9)$$

$$\left. \frac{b'_1}{a_1} \right|_{u=0} \frac{b_2(u_m)}{b_1(u_m)} = 1 , \quad (9.10)$$

$$\left. \frac{1}{a_1} \left(\frac{b'_1}{a_1} \right)' \right|_{u=0} \frac{b_2(u_m)}{b_1(u_m)} = 0 , \quad (9.11)$$

$$r(0) = r_+ , \quad (9.12)$$

$$\left. \frac{r'}{a_1} \right|_{u=0} = 0 , \quad (9.13)$$

where a primed quantity means derivative over u , i.e. $b'_1 = \frac{db_1}{du}$, see Chapter 3 for more details.

At $u = 1$, the boundary conditions are specific to the fixed quantities of the ensemble. In this case, we fix the geometry of ∂M , having a topology $S^1 \times S^2$, with the metric components

$$r(1) = R , \quad (9.14)$$

$$2\pi b_2(1) = \beta , \quad (9.15)$$

i.e. the radius of the boundary of the cavity is fixed to be R and the Euclidean time length corresponds to the inverse temperature as $\int (\gamma^{\tau\tau})^{-1/2} d\tau = \beta = T^{-1}$. Finally, we also fix the chemical potential μ at ∂M which obeys the relation

$$\beta\mu = \frac{\mu_m}{T_m} , \quad (9.16)$$

where $\beta\mu$ can be understood as the logarithm of the fugacity, and we shall consider it instead of μ for convenience.

9.2.5 Constraint equations

In order to simplify the path integral, we perform the zero loop approximation. An intermediate step for this approximation, which also avoids metrics whose action is arbitrarily negative, is to impose the constraint equations that are obeyed by the stationary points of the action. In some sense, one is integrating over metrics that are physically relevant but do not necessarily obey the evolution equations. In this case, the constraint equations consist on the Hamiltonian and momentum constraints for the Euclidean space and the Gauss constraint to the Maxwell field. We here impose these constraints in M_1 , M_2 and $\mathcal{C}$. Notice that the momentum constraints are apriori satisfied since we have a static spacetime with matter.

The Hamiltonian constraint for spaces M_1 and M_2 are given by $G^\tau_\tau = 8\pi l_p^2 T^\tau_\tau$, where G^τ_τ is the $\tau\tau$ -component of the Einstein tensor given in this case by

$$G^\tau_\tau|_{M_1} = \frac{2}{2r'r^2} \left(r \left[\left(\frac{r'}{a_1} \right)^2 - 1 \right] \right)' , \quad (9.17)$$

$$G^\tau_\tau|_{M_2} = \frac{2}{2r'r^2} \left(r \left[\left(\frac{r'}{a_2} \right)^2 - 1 \right] \right)' , \quad (9.18)$$

for M_1 and M_2 , respectively, and $T^a_b = 0$ since one has vacuum space. Therefore, the Hamiltonian constraints given for M_1 and M_2 are respectively

$$\frac{2}{r'r^2} \left(r \left[\left(\frac{r'}{a_1} \right)^2 - 1 \right] \right)' = 0, \quad (9.19)$$

$$\frac{2}{r'r^2} \left(r \left[\left(\frac{r'}{a_2} \right)^2 - 1 \right] \right)' = 0 . \quad (9.20)$$

For $\mathcal{C}$, one has the terms of the Hamiltonian constraint depending on a Dirac delta positioned at the shell, which leads to the junction condition $[K^\tau_\tau] - h^\tau_\tau[K] = -8\pi l_p^2 S^\tau_\tau$, where $S^\tau_\tau = -\epsilon_m$ is the $\tau\tau$ component of the surface stress-energy tensor of the shell. Notice that this stress-energy tensor is diagonal, with the other diagonal components being $S^\theta_\theta = S^\phi_\phi = p_m$, i.e. the tangential pressure. This stress-energy tensor is the same as if one considered the variational principle of a perfect fluid, see [171]. In our case, it comes from the fact that the term $\mu_m n_m \sqrt{g}$ gives $\frac{\beta\mu}{2\pi} n_m$ and so does not depend on the metric, while the variation of $\mathcal{F}$ in order to the metric is

$$\delta\mathcal{F} = -s_m \delta \left(\frac{\sqrt{\gamma^{\tau\tau}}}{2\pi} \right) - \frac{\epsilon_m + p_m - T_m s_m}{\sqrt{\gamma^{\tau\tau}\gamma}} \delta \left(\sqrt{\gamma^{\tau\tau}\gamma} \right) . \quad (9.21)$$

It is useful to explicitly express the extrinsic curvatures for constant u hypersurfaces as

$$K_{ab} dx^a dx^b|_{M_1} = \frac{b'_1 b_1}{a_1} \left(\frac{b_2^2(u_m)}{b_1^2(u_m)} \right) d\tau^2 + \frac{r'}{a_1 r} d\Omega^2 , \quad (9.22)$$

$$K_{ab} dx^a dx^b|_{M_2} = \frac{b'_2 b_2}{a_2} d\tau^2 + \frac{r'}{a_2 r} d\Omega^2 , \quad (9.23)$$

with

$$K|_{M_1} = \frac{b'_1}{a_1 b_1} + 2 \frac{r'}{a_1 r} , \quad (9.24)$$

$$K|_{M_2} = \frac{b'_2}{a_2 b_2} + 2 \frac{r'}{a_2 r} , \quad (9.25)$$

for M_1 and M_2 respectively. And so the junction condition from the Hamiltonian constraint is

$$\frac{\alpha}{l_p^2} \left[\frac{r'}{a_1} - \frac{r'}{a_2} \right] = m , \quad (9.26)$$

where the mass of the shell $m = 4\pi\alpha^2\epsilon_m$ has been defined.

We can then integrate the Hamiltonian constraints in Eqs. (9.19) and (9.20). In particular, the Hamiltonian constraints are satisfied with the following expressions for a_1 and a_2

$$\left(\frac{r'}{a_1}\right)^2 \equiv f(r_+, r) \equiv f_1(r) = 1 - \frac{r_+}{r}, \quad (9.27)$$

$$\left(\frac{r'}{a_2}\right)^2 \equiv f(\tilde{r}_+, r) \equiv f_2(r) = 1 - \frac{\tilde{r}_+}{r}, \quad (9.28)$$

where the boundary condition Eq. (9.12) was used to find the integration constant of f_1 , which corresponds to the horizon radius given by r_+ , and the integration constant of f_2 was parametrized with the total gravitational radius of the system given by $\tilde{r}_+$. The function f_1 and f_2 are actually the same function $f_1 = f_2 = f$ if the arguments are the same, but we make the distinction here to treat $f_1(r)$ as f parametrized by r_+ , while $f_2(r)$ is treated as f parametrized by $\tilde{r}_+$, i.e. to avoid bloating.

Notice that the mass of the shell m is determined by the total gravitational radius $\tilde{r}_+$, the horizon radius r_+ and the shell radius α , through the junction condition in Eq. (9.26), i.e.

$$m = m(\tilde{r}_+, r_+, \alpha) = \frac{\alpha}{l_p^2} \left[\sqrt{f_1(\alpha)} - \sqrt{f_2(\alpha)} \right]. \quad (9.29)$$

From now on, we abbreviate the dependence of $\tilde{r}_+$, r_+ and α throughout the chapter, except when explicitly stated otherwise.

9.2.6 Grand canonical reduced action

With the boundary conditions established, the geometry chosen and with the constraint equations in mind, we can express the action in Eq. (9.2) for the metrics obeying the constraints. We can split the action as the sum $I = I_{\text{gf}} + I_m$, where I_{gf} is the gravitational action which in spherical symmetry gives

$$\begin{aligned} I_{\text{gf}} = & \left(\frac{2\pi b_2 r}{l_p^2} \left(1 - \frac{r'}{a_2} \right) \right) \Big|_{u \rightarrow 1} - \frac{\Omega}{4l_p^2} \left(\frac{b'_1 b_2(u_m) r^2}{a_1 b_1(u_m)} \right) \Big|_{u=0} \\ & + \frac{1}{8\pi l_p^2} \int_{M_1} a_1 b_1 \frac{b_2(u_m)}{b_1(u_m)} r^2 G_1^\tau{}_\tau d^4x + \frac{1}{8\pi l_p^2} \int_{M_2} a_2 b_2 r^2 G_2^\tau{}_\tau d^4x \\ & - \frac{1}{8\pi l_p^2} \int_{\mathcal{C}} ([K^\tau{}_\tau] - [K]) \sqrt{\gamma} d^3s, \end{aligned} \quad (9.30)$$

see Chapter 3 for more details, and the matter action can be written using the property $\mathcal{F} = \epsilon_m - T_m s_m$ as

$$I_m = \int_{\mathcal{C}} (\epsilon_m - T_m s_m, -\mu_m n_m) \sqrt{\gamma} d^3s. \quad (9.31)$$

Using the properties of the local temperature and the local chemical potential, i.e. $2\pi T_m = (\sqrt{\gamma^{\tau\tau}})^{-1}$ and $\frac{\mu_m}{T_m} = \beta\mu$, one can further reduce the matter action as

$$I_m = \int_{\mathcal{C}} \epsilon_m \sqrt{\gamma} d^3s - S_m - \beta\mu N_m , \quad (9.32)$$

where it was defined $S_m = 4\pi\alpha^2 s_m$ and $N_m = 4\pi\alpha^2 n_m$. Putting now together the actions, one gets

$$\begin{aligned} I = & \left(\frac{2\pi b_2 r}{l_p^2} \left(1 - \frac{r'}{a_2} \right) \right) \Big|_{u \rightarrow 1} - \frac{\Omega}{4l_p^2} \left(\frac{b'_1 b_2(u_m) r^2}{a_1 b_1(u_m)} \right) \Big|_{u=0} - S_m - \beta\mu N_m \\ & + \frac{1}{8\pi l_p^2} \int_{M_1} a_1 b_1 \frac{b_2(u_m)}{b_1(u_m)} r^2 G_1^{\tau} d^4x + \frac{1}{8\pi l_p^2} \int_{M_2} a_2 b_2 r^2 G_2^{\tau} d^4x \\ & - \frac{1}{8\pi l_p^2} \int_{\mathcal{C}} ([K^{\tau}_{\tau}] - [K] - \epsilon_m) \sqrt{\gamma} d^3s . \end{aligned} \quad (9.33)$$

By applying the boundary conditions in Eqs. (9.9) -(9.16) and the Hamiltonian constraints with the junction condition, one obtains finally the expression for the reduced action as

$$\begin{aligned} I^*(R, T, \beta\mu; \tilde{r}_+, r_+, \alpha) = & \beta \frac{R}{l_p^2} \left(1 - \sqrt{f_2(R)} \right) \\ & - \frac{\pi r_+^2}{l_p^2} - S_m - \beta\mu N_m , \end{aligned} \quad (9.34)$$

where $S_m + \beta\mu N_m$ must be a function of m given by Eq. (9.29), the area of the shell $A(\alpha) = 4\pi\alpha^2$, the chemical potential over temperature $\beta\mu$, and, the variables to the left of ; are fixed. The dependence of the matter terms can be seen by inverting the first law of thermodynamics applied to $m(S_m, A(\alpha), N_m)$, i.e. $dm = T_m dS_m - p_m dA(\alpha) + \mu_m dN_m$, to get the function $S_m(m, A(\alpha), N_m)$ and then add $\beta\mu N_m$. The differential is then $d(S_m + \beta\mu N_m) = \frac{dm}{T_m} + \frac{p_m}{T_m} dA(\alpha) + N_m d(\beta\mu)$, and the reduced action is then fully determined by giving equations of state that describe this differential plus the expression for m in Eq. (9.29).

For convenience, we define the function $\mathcal{S}$ by the quantity

$$\begin{aligned} \mathcal{S}(\beta\mu; \tilde{r}_+, r_+, \alpha) = & \frac{\pi r_+^2}{l_p^2} + S_m(m(\tilde{r}_+, r_+, \alpha), A(\alpha), \beta\mu) \\ & + \beta\mu N_m(m(\tilde{r}_+, r_+, \alpha), A(\alpha), \beta\mu) . \end{aligned} \quad (9.35)$$

9.2.7 The constrained path integral for the grand canonical ensemble

The constrained path integral over configurations obeying the boundary conditions above becomes now

$$Z = \int D\omega e^{-I^*(z; \omega)} , \quad (9.36)$$

where I^* is the reduced action depending on the vector $z = (R, T, \beta\mu)$ whose components are the fixed parameters z^i with $i \in 1, 2, 3$, which correspond to a fixed radius of the cavity, a fixed temperature and a fixed logarithm of the fugacity at the cavity, but it also depends on the vector of variables integrated over the path integral, ω , with components ω^i corresponding to $\omega^i = (\tilde{r}_+, r_+, \alpha)$. We can see how the constraint equations reduce the path integral in this way. Initially, the variables to be integrated on the path integral are b_1, b_2, a_1, a_2 and r , on the space of physical metrics. The Hamiltonian constraints imply that the dependence on b_1 and b_2 disappear from the action, in particular the junction condition removes the dependence of $b_2(u_m)$ from the action. Due to these types of constraints, the functions r'/a_1 and r'/a_2 are functionals of the variable r_+ and $\tilde{r}_+$, respectively. This means there can be a change of integration element $Da_1 Da_2$ to $D\tilde{r}_+ Dr_+$. Moreover r'/a_1 and r'/a_2 are functions of r alone. This means one can invert the function $r(u)$ to perform an arbitrary change of coordinates $u = u(r)$, which will not change the metric except on the location of the shell $\alpha = r(u_m)$. Therefore, the integration element Dr becomes $D\alpha$. Notice here that the Jacobian of these transformations on the integration element were not considered since we are only interested on the zero loop approximation. An equation of state for $S_m + \beta\mu N_m$ in function of m given by Eq. (9.29), A and $\mu_m/T_m = \beta\mu$ is still required to determine the partition function.

9.2.8 The partition function of the $(E, \beta\mu)$ ensemble and its relation to the grand canonical ensemble

Interestingly, it is possible to rewrite the constrained path integral in Eq. (9.36) in the following way

$$Z = \int D\tilde{r}_+ e^{-\beta \frac{R}{l_p^2} (1 - \sqrt{f_2(R)})} Z_S(\beta\mu; \tilde{r}_+) , \quad (9.37)$$

where the functional Z_S is

$$Z_S(\beta\mu; \tilde{r}_+) = \int D\hat{\omega} e^{S(\beta\mu; \tilde{r}_+, \hat{\omega})} , \quad (9.38)$$

where $\hat{\omega}$ is defined as the vector with components $\hat{\omega}^A = (r_+, \alpha)$, with indices $A \in \{2, 3\}$. Therefore, the partition function of the grand canonical ensemble is given by a Laplace-like transform [117] of the functional Z_S . If we did not consider the chemical potential, then Z_S would describe the microcanonical ensemble of the black hole with a self-gravitating matter shell, as it is the path integral with the action without the gravitational boundary term. However, with the chemical potential, the functional Z_S does not fit into the partition functions of the usual ensembles, as it describes a partition function of the black hole and self-gravitating shell with fixed $\tilde{r}_+$, or fixed energy, and $\beta\mu$ fixed. As already stated, for simplicity, we call this ensemble the $(E, \beta\mu)$ ensemble of a black hole with a self-gravitating shell. It would be interesting to obtain the partition function Z_S from first principles as we did for the grand canonical ensemble with a finite cavity. However, the

calculations are similar as the grand canonical ensemble. One would have to consider the action without the Gibbons-Hawking-York boundary term, which is consistent with fixing the quasilocal energy at the cavity rather than the temperature of the cavity in the boundary conditions. Notice that having a cavity at finite or infinite radius in the $(E, \beta\mu)$ ensemble is equivalent to fixing either the quasilocal energy at the cavity at a radius R or the gravitational radius $\tilde{r}_+$. Since here we fix $\tilde{r}_+$, we can consider Z_S as the partition function of the $(E, \beta\mu)$ ensemble with a cavity at infinity, indeed Z_S does not depend on R .

Assuming that we could determine Z_S , either by performing the path integral or the zero loop approximation, we can define the function $\tilde{S}$ as

$$e^{\tilde{S}(\beta\mu; \tilde{r}_+)} = \int D\hat{\omega} e^{\mathcal{S}(\beta\mu; \tilde{r}_+, \hat{\omega})} = Z_S, \quad (9.39)$$

and the grand canonical partition function can be given by

$$Z = \int D\tilde{r}_+ e^{-\tilde{I}(z; \tilde{r}_+)}, \quad (9.40)$$

with

$$\tilde{I}(z; \tilde{r}_+) = \beta R \left(1 - \sqrt{f_2(R)} \right) - \tilde{S}(\beta\mu; \tilde{r}_+), \quad (9.41)$$

being the effective action of the grand canonical ensemble. The result of Eq. (9.40) together with Eq. (9.39) allows for a better understanding of the full zero loop approximation applied to Eq. (9.36), as we shall see below. Moreover, Eq. (9.40) means that the system of a black hole and self-gravitating matter thin shell can be described by an effective action, and the freedom of choosing the equations of state for the shell turns into some freedom on the expression of the function $\tilde{S}$.

9.3 $(E, \beta\mu)$ ENSEMBLE IN THE ZERO LOOP APPROXIMATION

9.3.1 Expansion around the stationary points

Here, we treat the zero loop approximation applied to the path integral in Eq. (9.39), i.e. to the partition function of the black hole plus a thin shell with fixed $\tilde{r}_+$ and $\beta\mu$. This means that the function $\mathcal{S}$ must be expanded around its stationary points $\hat{\omega}_0^A = (r_+(\beta\mu; \tilde{r}_+), \alpha(\beta\mu; \tilde{r}_+))$ defined by $\frac{\partial \mathcal{S}}{\partial \hat{\omega}^A} |_{\hat{\omega}=\hat{\omega}_0} = 0$. The path integral in Eq. (9.39) can be expanded up to second order as

$$Z_S = e^{\mathcal{S}(\beta\mu; \tilde{r}_+, \hat{\omega}_0)} \int D\delta\hat{\omega} e^{-\hat{H}_{\hat{\omega}^A \hat{\omega}^B} \delta\hat{\omega}^A \delta\hat{\omega}^B}, \quad (9.42)$$

with $\hat{H}_{\hat{\omega}^A \hat{\omega}^B}$ being the negative of the second derivatives of $\mathcal{S}$ as $\hat{H}_{\hat{\omega}^A \hat{\omega}^B} = -\frac{\partial^2 \mathcal{S}}{\partial \hat{\omega}^A \partial \hat{\omega}^B}$ evaluated at $\hat{\omega} = \hat{\omega}_0$ and $\delta\hat{\omega}^A = \hat{\omega}^A - \hat{\omega}_0^A$. In order for the zero loop approximation to be well-defined, the path integral in Eq. (9.42) should be convergent. This means that the matrix $\hat{H}_{\hat{\omega}^A \hat{\omega}^B}$ must be positive definite and so the stationary points must be a maximum of the function $\mathcal{S}$. By truncating the expansion at zeroth order, we

obtain $Z_{\mathcal{S}} = e^{\mathcal{S}(\beta\mu; \tilde{r}_+, \hat{\omega}_0)}$ and so, from the definition of the function $\tilde{\mathcal{S}}$ in Eq. (9.39), we have

$$\tilde{\mathcal{S}}(\beta\mu; \tilde{r}_+) = \mathcal{S}(\beta\mu; \tilde{r}_+, \hat{\omega}_0) . \quad (9.43)$$

9.3.2 Stationary equations

The stationary conditions follow from finding the stationary points of the function $\mathcal{S}$, with fixed $\tilde{r}_+$ and $\beta\mu$. And so the stationary points $\hat{\omega}_0 = (r_+(\hat{z}), \alpha(\hat{z}))$, with $\hat{z} = (\tilde{r}_+, \beta\mu)$, are such that $(\frac{\partial \mathcal{S}}{\partial \hat{\omega}^A})|_{\hat{\omega}=\hat{\omega}_0} = 0$. The derivatives of $\mathcal{S}$ are

$$\begin{aligned} \frac{\partial \mathcal{S}}{\partial r_+} &= \left(1 - \frac{T(r_+, \alpha)}{T_m}\right) \frac{2\pi r_+}{l_p^2} , \\ \frac{\partial \mathcal{S}}{\partial \alpha} &= \frac{2\alpha}{T_m} (4\pi p_m - 4\pi p(\alpha)) , \end{aligned} \quad (9.44)$$

where the following definitions for the temperature and pressure functions, for simplicity, were used

$$T(r_+, \alpha) = \frac{1}{4\pi r_+ \sqrt{f(r_+; \alpha)}} , \quad (9.45)$$

$$4\pi p(\alpha) = \frac{1}{4\alpha l_p^2} \left(\frac{1 + f_2(\alpha)}{\sqrt{f_2(\alpha)}} - \frac{1 + f_1(\alpha)}{\sqrt{f_1(\alpha)}} \right) . \quad (9.46)$$

Then, the stationary conditions become

$$T_m = T(r_+, \alpha) , \quad (9.47)$$

$$4\pi p_m = 4\pi p(\alpha) . \quad (9.48)$$

Therefore, the stationary conditions imply that the temperature of the shell must be at the temperature given by the Tolman formula and that there must be an equilibrium of pressures at the shell.

9.3.3 Stability conditions and their relation to the behaviour of the solutions

The zero loop approximation in the context of the $(E, \beta\mu)$ ensemble is valid if the stationary points obtained from solving Eqs. (9.47) and (9.48) are local maxima of $\mathcal{S}$. To such stationary points, we designate them as stable solutions. As we have seen, this only happens if the matrix $\hat{H}_{\hat{\omega}^A \hat{\omega}^B}$ is positive definite. Since the matrix is 2×2 , we can use Sylvester's criterion to obtain the two sufficient conditions for stability as

$$\hat{H}_{\alpha\alpha} > 0 , \quad (9.49)$$

$$\frac{|\hat{H}|}{\hat{H}_{\alpha\alpha}} > 0 , \quad (9.50)$$

where $|\hat{H}|$ is the determinant of the matrix $\hat{H}_{\hat{\omega}^A \hat{\omega}^B}$. The components of the hessian $\hat{H}_{\hat{\omega}^A \hat{\omega}^B}$ are presented in Sec. 9.8. Namely, the component $\hat{H}_{\alpha\alpha}$ is related to the mechanical stability of the shell. Indeed, $\hat{H}_{\alpha\alpha}$ corresponds to the derivative of the difference of pressures, and the condition in Eq. (9.49) is precisely the condition that must be obeyed for the shell to be mechanically stable, as we show in Sec. 9.9. This is quite interesting as the validity of the zero loop approximation through the path integral approach gives precisely the mechanical stability condition of the shell.

We can write the stability conditions in a different way which may help to understand the behaviour of the solutions under stability. The system in Eqs. (9.49) and (9.50) does not give an explicit connection of the stability conditions with the behaviour of the solutions. However, we can establish this connection by considering the following. The stationary solutions $\hat{\omega}_0^A$ are described by Eqs. (9.47) and (9.48). One can now perform on Eqs. (9.47) and (9.48) the total derivative in $\hat{z}^A = (\tilde{r}_+, \beta\mu)$, which are the quantities that are fixed in the $(E, \beta\mu)$ ensemble. One can obtain the following relations $\hat{\xi}_{\hat{z}^C \hat{\omega}^A} + \hat{H}_{\hat{\omega}^A \hat{\omega}^B} \frac{\partial \hat{\omega}_0^B}{\partial \hat{z}^C} = 0$, where $\hat{\xi}_{\hat{z}^C \hat{\omega}^A} = - \left. \frac{\partial^2 S}{\partial \hat{\omega}^A \partial \hat{z}^C} \right|_{\hat{\omega}=\hat{\omega}_0}$. And so, these relations can be inverted to yield the derivatives of the solutions of the $(E, \beta\mu)$ ensemble as

$$\frac{\partial \hat{\omega}_0^A}{\partial \hat{z}^C} = -(\hat{H}^{-1})^{\hat{\omega}^A \hat{\omega}^B} \hat{\xi}_{\hat{z}^A \hat{\omega}^B}, \quad (9.51)$$

Now, we can build a matrix $\hat{\mathcal{H}}_{\hat{z}^D \hat{z}^C} = -\hat{\xi}_{\hat{z}^D \hat{\omega}^A} \frac{\partial \hat{\omega}_0^A}{\partial \hat{z}^C}$, which is related to the inverse of the hessian by Eq. (9.51), or explicitly $\hat{\mathcal{H}}_{\hat{z}^D \hat{z}^C} = \hat{\xi}_{\hat{z}^D \hat{\omega}^A} (\hat{H}^{-1})^{\hat{\omega}^A \hat{\omega}^B} \hat{\xi}_{\hat{z}^C \hat{\omega}^B}$. The vectors $\hat{\xi}_{\hat{z}^A \hat{\omega}^A}$ are

$$\hat{\xi}_{\tilde{r}_+ \hat{\omega}^A} = -\frac{\partial}{\partial \hat{\omega}^A} \left(\frac{T(\tilde{r}_+, \alpha)}{T_m} \frac{2\pi \tilde{r}_+}{l_p^2} \right), \quad (9.52)$$

$$\hat{\xi}_{\beta\mu \hat{\omega}^A} = -\frac{\partial N_m}{\partial \hat{\omega}^A}. \quad (9.53)$$

Therefore, we can write the matrix $\mathcal{H}_{\hat{z}^C \hat{z}^D}$ as

$$\mathcal{H} = \begin{pmatrix} \left. \frac{\partial}{\partial \tilde{r}_+} \left(\frac{T(\tilde{r}_+, \alpha)}{T_m} \frac{2\pi \tilde{r}_+}{l_p^2} \right) \right|_{\hat{\omega}=\hat{\omega}_0} & \left. \frac{\partial N_m}{\partial \tilde{r}_+} \right|_{\hat{\omega}=\hat{\omega}_0} \\ \left. \frac{\partial N_m}{\partial \tilde{r}_+} \right|_{\hat{\omega}=\hat{\omega}_0} & \left. \frac{\partial N_m}{\partial \beta\mu} \right|_{\hat{\omega}=\hat{\omega}_0} \end{pmatrix}, \quad (9.54)$$

where $\frac{\partial}{\partial \hat{z}^C} = \frac{\partial \hat{\omega}_0^A}{\partial \hat{z}^C} \frac{\partial}{\partial \hat{\omega}^A}$ represents the partial derivative over the implicit dependence of $\hat{z}^C$, and also $\frac{\partial N_m}{\partial \beta\mu} = \frac{\partial}{\partial \beta\mu} \left(\frac{T(\tilde{r}_+, \alpha)}{T_m} 2\pi \tilde{r}_+ \right)$ due to the hessian being symmetric. The stability conditions stipulate that the matrix $\hat{H}_{\hat{\omega}^A \hat{\omega}^B}$ must be positive definite, which means that $(\hat{H}^{-1})^{\hat{\omega}^A \hat{\omega}^B}$ must also be positive definite. Since the vectors $\hat{\xi}_{\hat{z}^C \hat{\omega}^A}$ for each $\hat{z}^C$ are in principle independent, they can be represented as a nonsingular matrix, the matrix $\mathcal{H}_{\hat{z}^A \hat{z}^B}$ can be seen as $\mathcal{H}_{\hat{z}^A \hat{z}^B} = \hat{\xi}_{\hat{z}^A \hat{\omega}^C} (\hat{H}^{-1})^{\hat{\omega}^C \hat{\omega}^D} \hat{\xi}_{\hat{\omega}^D \hat{z}^B}^T$, where $\hat{\xi}_{\hat{\omega}^D \hat{z}^B}^T$ are the transpose components of the matrix $\hat{\xi}_{\hat{z}^B \hat{\omega}^D}$. Therefore, $\mathcal{H}_{\hat{z}^A \hat{z}^B}$ must be positive definite if and only if $\hat{H}_{\hat{\omega}^A \hat{\omega}^B}$ is positive definite. This statement is related to the thermodynamics of the system, which we discuss below.

9.4 GRAND CANONICAL ENSEMBLE IN THE ZERO LOOP APPROXIMATION

9.4.1 Grand canonical path integral expansion around the stationary points

We now proceed with the zero loop approximation of the statistical path integral for the grand canonical ensemble in Eq. (9.36), with the reduced action given by Eq. (9.34) and with the relations given by Eq. (9.29). We then perform the zero loop approximation by expanding the reduced action around its stationary points. These stationary points are the solutions $\omega_0 = (\tilde{r}_+(z), r_+(z), \alpha(z))$ such that $\frac{\partial I^*}{\partial \omega^i} \Big|_{\omega=\omega_0} = 0$. The path integral around the stationary point can then be rewritten as

$$Z = e^{-I_0(z)} \int D\delta\omega e^{-(H_{\omega^i\omega^j})\delta\omega^i\delta\omega^j}, \quad (9.55)$$

where $I_0(z) = I^*(z; \omega_0)$ is the reduced action evaluated at the stationary point, $H_{\omega^i\omega^j} = \frac{\partial^2 I^*}{\partial \omega^i \partial \omega^j} \Big|_{\omega=\omega_0}$ is the hessian of the reduced action over the variables ω , evaluated at the stationary point, and $\delta\omega$ is the difference vector of the variables to the solutions of the stationary points, i.e. $\delta\omega^i = \omega^i - \omega_0^i$. We adopt the notation that the partial derivatives are done while keeping the variables of the definition of the function constant. The zero loop approximation is then valid if the hessian is positive definite i.e. for solutions that are minima of the action. If the solutions of the ensemble are minima, then the solutions are stable, otherwise they are maxima and unstable or saddle points and so marginally stable. The partition function for the stable solutions is then $Z = e^{-I_0(z)}$.

However, with the zero loop approximation done to the partition function of the $(E, \beta\mu)$ ensemble, it is better to envision the zero loop approximation of the path integral describing the grand canonical ensemble through the identity in Eq. (9.37). We can apply the zero loop approximation in parts, starting by the functional Z_S , which was obtained in Sec. 9.4.1, and the partition function of the grand canonical ensemble becomes

$$Z = \int D\tilde{r}_+ \left[e^{-\tilde{I}(z; \tilde{r}_+)} \int D\delta\hat{\omega} e^{-\hat{H}_{\hat{\omega}^A\hat{\omega}^B}\delta\hat{\omega}^A\delta\hat{\omega}^B} \right], \quad (9.56)$$

where the effective action $\tilde{I}(z, \tilde{r}_+)$ is given by Eq. (9.41) and the function $\tilde{S}$ is provided by Eq. (9.43), i.e. it is determined in this case through the zero loop approximation of the path integral describing the $(E, \beta\mu)$ ensemble of the black hole and the self-gravitating thin shell, $\tilde{S} = \mathcal{S}(\beta\mu; \tilde{r}_+, \hat{\omega}_0)$. Note that this assignment is done by deprecating the path integral over the fluctuations of the parameters $\hat{\omega}^A$, since we are interested in the zero loop. Now we can proceed with the expansion over the stationary points of the effective action, $\frac{\partial \tilde{I}}{\partial \tilde{r}_+} \Big|_{\tilde{r}_+=\tilde{r}_+(z)} = 0$, obtaining

$$Z = e^{-I_0(z)} \int D\delta\tilde{r}_+ \left[e^{-\tilde{H}_{\tilde{r}_+\tilde{r}_+}\delta\tilde{r}_+\delta\tilde{r}_+} \times \int D\delta\hat{\omega} e^{-\hat{H}_{\hat{\omega}^A\hat{\omega}^B}\delta\hat{\omega}^A\delta\hat{\omega}^B} \right], \quad (9.57)$$

where $\tilde{H}_{\tilde{r}_+} = \frac{\partial^2 \tilde{I}}{\partial \tilde{r}_+^2} |_{\tilde{r}_+ = \tilde{r}_+(z)}$, and $I_0(z) = \tilde{I}(z; \tilde{r}_+(z))$. In connection with the expansion in Eq. (9.55), the zeroth order action is the same, i.e.

$$I_0(z) = \tilde{I}(z; \tilde{r}_+(z)) = I^*(z; \tilde{r}_+(z), \hat{\omega}_0 |_{\tilde{r}_+ = \tilde{r}_+(z)}) , \quad (9.58)$$

and the stationary points in Eq. (9.55) are the same, i.e.

$$\omega_0^i = (\tilde{r}_+(z), \hat{\omega}_0^A |_{\tilde{r}_+ = \tilde{r}_+(z)}) . \quad (9.59)$$

Considering the second order perturbations of the action in the two expansions, i.e. in Eqs. (9.55) and (9.57), we can prove the equivalence between the two by using the transformation $\delta\omega^A = \delta\hat{\omega}^A + \frac{\partial\hat{\omega}_0^A}{\partial\tilde{r}_+} \delta\tilde{r}_+$, with $\frac{\partial\hat{\omega}_0^A}{\partial\tilde{r}_+} = -H_{\tilde{r}_+ \hat{\omega}^B} (\hat{H}^{-1})^{\hat{\omega}^A \hat{\omega}^B} |_{\tilde{r}_+ = \tilde{r}_+(z)}$, see Sec. 9.8 for the expression of the Hessians. Note that the Hessians in this case behave as tensors since the first derivatives of the respective actions vanish due to the stationary conditions. Therefore, the conditions for the validity of the zero loop approximation in both expansions are equivalent. However, the expansion in Eq. (9.57) gives a more clear interpretation of the zero loop approximation of the path integral describing the grand canonical ensemble and the meaning of the conditions for its validity. It is thus convenient to work with Eqs. (9.56) and (9.57) and the effective action given in Eq. (9.41), i.e. $\tilde{I}(z; \tilde{r}_+) = \beta R \left(1 - \sqrt{f_2(R)}\right) - \tilde{\mathcal{S}}(\beta\mu; \tilde{r}_+)$, with the identification in Eq. (9.43), i.e. $\tilde{\mathcal{S}}(\beta\mu; \tilde{r}_+) = \mathcal{S}(\beta\mu; \tilde{r}_+, \hat{\omega}_0)$.

9.4.2 Stationary equation

The stationary equation describing the minimum of the effective action is determined by $\frac{\partial \tilde{I}}{\partial \tilde{r}_+} |_{\tilde{r}_+ = \tilde{r}_+(z)} = 0$, with the minimum $\tilde{r}_+ = \tilde{r}_+(z)$. Knowing the expression of the effective action in Eq. (9.41), we obtain the stationary equation as

$$\beta = B(\beta\mu; \tilde{r}_+) \sqrt{f_2(R)} , \quad (9.60)$$

where $B(\beta\mu; \tilde{r}_+) = 2 \frac{\partial \tilde{\mathcal{S}}}{\partial \tilde{r}_+}$. Using the fact that $\tilde{\mathcal{S}}(\beta\mu; \tilde{r}_+) = \mathcal{S}(\beta\mu; \tilde{r}_+, \hat{\omega}_0)$ for the black hole and thin shell in the zero loop approximation, the function $B(\beta\mu; \tilde{r}_+)$ can be written in terms of the quantities of the black hole and shell evaluated at the stationary points of the $(E, \beta\mu)$ ensemble, yielding

$$B(\beta\mu; \tilde{r}_+) = \frac{1}{\sqrt{f_2(\alpha)} T_m(m(\tilde{r}_+, \hat{\omega}_0), A(\alpha(\hat{z})), \beta\mu)} . \quad (9.61)$$

It is also interesting to consider the number of particles $\frac{\partial \tilde{\mathcal{S}}}{\partial \beta\mu} = \tilde{N}(\beta\mu; \tilde{r}_+)$ which for the case of the black hole and thin shell is $\tilde{N}(\beta\mu; \tilde{r}_+) = N_m(m(\tilde{r}_+, \hat{\omega}_0), A(\alpha(\hat{z})), \beta\mu)$.

9.4.3 Stability condition

For the validity of the zero loop approximation of the path integral describing the grand canonical ensemble, we must require that $\tilde{H}_{\tilde{r}_+} = \left. \frac{\partial^2 \tilde{I}}{\partial \tilde{r}_+^2} \right|_{\tilde{r}_+ = \tilde{r}_+(z)} > 0$, which reduces to the condition

$$\tilde{H}_{\tilde{r}_+} = \left(\frac{B}{2f_2(R)R} - \frac{\partial B}{\partial \tilde{r}_+} \right) \Big|_{\tilde{r}_+ = \tilde{r}_+(z)} \geq 0. \quad (9.62)$$

This condition can be tied to the behaviour of the solution $\tilde{r}_+ = \tilde{r}_+(z)$. Indeed, by using Eq. (9.60), the derivative of the solution $\tilde{r}_+(z)$ is given by

$$\frac{\partial \tilde{r}_+}{\partial T} = \frac{2RB^2 f_2^3(R)}{B - 2f_2(R)R \frac{\partial B}{\partial \tilde{r}_+}} \Big|_{\tilde{r}_+ = \tilde{r}_+(z)}, \quad (9.63)$$

where the partial derivative in T is done by keeping R and $\beta\mu$ constant. And so the stability condition in Eq. (9.62) leads to the condition that $\frac{\partial \tilde{r}_+}{\partial T} > 0$, the gravitational radius must increase with the temperature of the ensemble. It is also convenient to write the other derivative of the solution from applying the derivative over $\beta\mu$ on the stationary condition, giving

$$\frac{\partial \tilde{r}_+}{\partial \beta\mu} = \frac{T}{B} \frac{\partial \tilde{r}_+}{\partial T} \frac{\partial \tilde{N}}{\partial \tilde{r}_+} \Big|_{\tilde{r}_+ = \tilde{r}_+(z)}. \quad (9.64)$$

Here, the sign of the derivative $\frac{\partial \tilde{r}_+}{\partial \beta\mu}$ depends on the sign of $\frac{\partial \tilde{N}}{\partial \tilde{r}_+}$, and ultimately depends on the choice of equation of state for the shell.

Note however that the stability conditions of the $(E, \beta\mu)$ ensemble, Eqs. (9.49) and (9.50), must be satisfied simultaneously with Eq. (9.62), yielding precisely the positive definiteness condition of $H_{\omega^i \omega^j}$ in Eq. (9.55). The reason for these stability conditions is the identification in Eq. (9.43), which comes from applying the zero loop approximation to $\mathcal{S}$. If one did not perform the zero loop approximation to $\mathcal{S}$ but performed the zero loop approximation on the effective action, one would only have the stability condition in Eq. (9.62). This means that the stability condition in Eq. (9.62) is the one inherent to the grand canonical ensemble.

9.5 THERMODYNAMICS OF A SELF-GRAVITATING MATTER THIN SHELL AND A BLACK HOLE IN THE $(E, \beta\mu)$ ENSEMBLE WITH CAVITY AT INFINITY

9.5.1 The $(E, \beta\mu)$ ensemble from statistical mechanics

The general idea to build the $(E, \beta\mu)$ ensemble from statistical arguments is to start by constructing the partition function of system with a number of discrete states that exchanges particles with the reservoir. For that, we can make use of the microcanonical ensemble of the system A_s and the reservoir A_r . The system plus the reservoir only exchange the number of particles but such that the total

number of particles is conserved $N^{(0)} = N_s + N_r$, where N_s is the number of particles of system A_s and N_r is the total number of particles of the reservoir. If the system finds itself in just one state with N_s particles, the reservoir will find itself with possible $\Omega'(N^{(0)} - N_s)$ number of states with the number of particles $N_r = N^{(0)} - N_s$. Meaning that the probability of the system to be at exactly one state with the number of particles N_r is $P_r = c\Omega'(N^{(0)} - N_r)$, which comes from the postulate of equal probability between states and c is a normalization constant to be determined by the sum of probabilities being unity. Since A_r is a reservoir, the number of particles N_s must be much smaller than N_r . This means one can expand $\Omega'(N^{(0)} - N_s)$ as $\ln(\Omega'(N^{(0)} - N_s)) = \ln(\Omega'(N^{(0)})) - \partial_{N^{(0)}} \ln(\Omega'(N^{(0)})) N_s$. With the definition of $\beta\mu$ being $\beta\mu = -\partial_{N^{(0)}} \ln(\Omega'(N^{(0)}))$, one gets the probability $P_s = \frac{1}{Z_S} e^{\beta\mu N_s}$, where Z_S is the partition function of the $(E, \beta\mu)$ ensemble. Since Z_S is determined by normalization of the probability, one obtains

$$Z_S = \sum_{N_s} e^{S_s + \beta\mu N_s} , \quad (9.65)$$

where $\sum_{N_s}$ is done over the possible number of particles of the system, S_s is the entropy of the system with number of particles N_s correspondent to the logarithm of the number of states of the system with N_s particles. Now Z_S is a function of the energy of the system E and $\beta\mu$ as $Z_S(E, \beta\mu)$. From the definition of the inverse temperature as $\beta_C = \partial_E \ln(Z_S(E, \beta\mu))$, together with the fact that the mean number of particles is given by $N_C = \partial_{\beta\mu} \ln(Z_S(E, \beta\mu))$, the differential of the logarithm of the partition function is

$$d \ln(Z_S) = \beta_C dE - N_C d\beta\mu . \quad (9.66)$$

But, from the first law of thermodynamics, one has that $d(S_C + \beta\mu N_C) = \beta_C dE - N_C d\beta\mu$, where S_C is the entropy of the system. Therefore, the partition function can be related to the thermodynamic quantity $S_C + \beta\mu N_C$ as

$$Z_S = e^{S_C + \beta\mu N_C} , \quad (9.67)$$

as a function of the energy and $\beta\mu$.

Such ensemble is not used frequently as it seems difficult to realize a reservoir that only exchanges particles but not energy. However, for our purposes, it is convenient to consider it as a step towards the grand canonical ensemble. The arguments to obtain the partition function are based from Reif's book, but we adapted them here to the number of particles.

9.5.2 Connection between the action and thermodynamics

The $(E, \beta\mu)$ partition function with fixed total gravitational radius $\tilde{r}_+$ and a fixed $\beta\mu$ should be described by $Z_S = e^{\tilde{S}(\beta\mu; \tilde{r}_+)}$, with $\tilde{S}(\beta\mu; \tilde{r}_+) = S(\beta\mu; \tilde{r}_+, \dot{\omega}_0)$. From statistical mechanics, the partition function of a system with constant energy $\tilde{r}_+/2$ and constant $\beta\mu$ is given by $Z = e^{S_C + \beta\mu N_C}$, where S_C and N_C are the entropy and

the mean particle number, respectively, of the $(E, \beta\mu)$ ensemble, with the subscript C standing for chemical. By connecting the two partition functions, we obtain that

$$S_C + \beta\mu N_C = \mathcal{S}(\beta\mu; \tilde{r}_+, \hat{\omega}_0) . \quad (9.68)$$

From here, we can compute the relevant thermodynamic quantities of the $(E, \beta\mu)$ ensemble.

9.5.3 Entropy, temperature and particle number

From the thermodynamic quantity $S_C + \beta\mu N_C$, one has the differential

$$d(S_C + \beta\mu N_C) = \frac{1}{T_C} d\left(\frac{\tilde{r}_+}{2}\right) + N_C d\beta\mu , \quad (9.69)$$

where the derivatives of the Legendre transform of the entropy are given by $\frac{1}{T_C} = 2 \frac{\partial(S_C + \beta\mu N_C)}{\partial \tilde{r}_+}$ and $N_C = \frac{\partial(S_C + \beta\mu N_C)}{\partial \beta\mu}$. From the expression of $\mathcal{S}$, we have then that the temperature of the $(E, \beta\mu)$ ensemble is

$$T_C = T_m(m(\tilde{r}_+, \hat{\omega}_0), A(\alpha(\hat{z})), \beta\mu) \sqrt{f_2(\alpha(\hat{z}))} , \quad (9.70)$$

while the particle number is

$$N_C = N_m(m(\tilde{r}_+, \hat{\omega}_0), A(\alpha(\hat{z})), \beta\mu) . \quad (9.71)$$

Finally, we can compute the entropy of the system as $S_C = \mathcal{S}(\beta\mu; \tilde{r}_+, \hat{\omega}_0) - \beta\mu N$, yielding

$$S_C = \pi r_+^2(\hat{z}) + S_m(m(\tilde{r}_+, \hat{\omega}_0), A(\alpha(\hat{z})), \beta\mu) . \quad (9.72)$$

9.5.4 Thermodynamic stability of the $(E, \beta\mu)$ ensemble with the reservoir

In order to analyze the thermodynamic stability of the $(E, \beta\mu)$ ensemble, we must use the total entropy functional of the system plus the reservoir at infinity, which only fixes $\bar{\beta}\mu$ of the system. This functional is $\bar{\mathcal{S}} = S_C + \bar{\beta}\mu N_C$, whose variation represents the variation of the total entropy of the system and the reservoir together, as one has $d\bar{\mathcal{S}} = dS_C + dS_{CMr}$, with the variation of the entropy of the reservoir being $dS_{CMr} = -\bar{\beta}\mu dN_{CMr}$. Since the variation on the particle number of the reservoir is $dN_{CMr} = -dN_C$ due to number particle conservation, and since the variation is done with fixed energy, then one has $d\bar{\mathcal{S}} = (\bar{\beta}\mu - \beta\mu)dN_C$. Now, the total entropy must be at its maximum, leading to $d\bar{\mathcal{S}} = 0$, i.e. $\beta\mu = \bar{\beta}\mu$, and leading to the stability condition

$$\frac{\partial N_C}{\partial \beta\mu} > 0 , \quad (9.73)$$

where N_C is given by Eq. (9.71).

We can now establish the connection of the stability or validity of the zero loop approximation with the thermodynamic stability of the ensemble. The stability condition in Eq. (9.73) can be expanded into $\left. \frac{\partial N_m}{\partial \beta\mu} \right|_{\hat{\omega}=\hat{\omega}_0} + \frac{\partial \hat{\omega}_0^A}{\partial \beta\mu} \left. \frac{\partial N_m}{\partial \omega^A} \right|_{\hat{\omega}=\hat{\omega}_0} > 0$. From the matrix in Eq. (9.54), one of the conditions for the positive definiteness of the matrix $\mathcal{H}$ is $\left. \frac{\partial N_m}{\partial \beta\mu} \right|_{\hat{\omega}=\hat{\omega}_0} = \frac{\partial \hat{\omega}_0^A}{\partial \beta\mu} \left. \frac{\partial N_m}{\partial \omega^A} \right|_{\hat{\omega}=\hat{\omega}_0} > 0$, which is not enough to ensure Eq. (9.73).

We must then consider the positivity of $\left. \frac{\partial N_m}{\partial \beta\mu} \right|_{\hat{\omega}=\hat{\omega}_0}$, which depends heavily on the choice of the equations of state for the shell as $dS_m = \frac{1}{T_m} dm + \frac{p_m}{T_m} d(4\pi\alpha^2) - \beta\mu dN_m$. Moreover, its intrinsic stability must require that $\left. \frac{\partial N_m}{\partial \beta\mu} \right|_{\hat{\omega}=\hat{\omega}_0} > 0$. Therefore, if one chooses a thermodynamically intrinsically stable shell, the maximization of $\mathcal{S}$ indicates that the ensemble is thermodynamically stable, as Eq. (9.73) is satisfied.

It is also interesting to explore the relation between the thermodynamic stability in Eq. (9.73) and the mechanical stability of the shell in Eq. (9.49). From the relation $\hat{\mathcal{H}}_{\hat{z}^D \hat{z}^C} = \hat{\zeta}_{\hat{z}^D \hat{\omega}^A} (\hat{H}^{-1})^{\hat{\omega}^A \hat{\omega}^B} \hat{\zeta}_{\hat{z}^C \hat{\omega}^B}$ in Sec. 9.3.3, one obtains explicitly

$$\left. \frac{\partial N_C}{\partial \beta\mu} \right|_{\hat{\omega}=\hat{\omega}_0} = \frac{\hat{H}_{\alpha\alpha}}{|\hat{H}|} \left(\left. \frac{\partial N_m}{\partial r_+} - \frac{\partial N_m}{\partial \alpha} \frac{H_{r_+\alpha}}{H_{\alpha\alpha}} \right)^2 \right|_{\hat{\omega}=\hat{\omega}_0} + \frac{1}{H_{\alpha\alpha}} \left(\left. \frac{\partial N_m}{\partial \alpha} \right)^2 \right|_{\hat{\omega}=\hat{\omega}_0} + \left. \frac{\partial N_m}{\partial \beta\mu} \right|_{\hat{\omega}=\hat{\omega}_0}, \quad (9.74)$$

and so mechanical stability is not sufficient to guarantee thermodynamic stability, however to have thermodynamic stability one needs mechanical stability. This effect is due to the ensemble we are considering with $\beta\mu$ fixed.

The conditions for the stability of the zero loop approximation seem to be more restrictive than the thermodynamic stability condition of the $(E, \beta\mu)$ ensemble. One must remember that the system describes actually two subsystems in equilibrium. Therefore, we must analyze the thermodynamics of the $(E, \beta\mu)$ ensemble as the interaction of two systems at constant total energy and constant $\beta\mu$.

9.5.5 The $(E, \beta\mu)$ ensemble describing two systems in equilibrium and Le Chatelier-Braun principle

9.5.5.1 The equilibrium of the two systems plus the reservoir and the recovery of the thermodynamic quantities of the black hole inside a cavity

In order to treat the ensemble as two systems in equilibrium rather than a system as a whole, we must expand the entropy functional in terms of variables of both systems. We can choose the entropy of the black hole, $S_{bh} = \pi r_+^2$ and the area of the cavity $4\pi\alpha^2$. The total energy is held constant, which means that the energy of the shell and the energy of the black hole inside the cavity obey a certain relation, which is precisely $m = m(\tilde{r}_+, r_+, \alpha)$. Then, the differential of the functional $\bar{\mathcal{S}}$ is given by

$$d\bar{\mathcal{S}} = \left(1 - \frac{T_{bh}}{T_m} \right) dS_{bh} + \frac{1}{T_m} (p_m - p_{bh}) d(4\pi\alpha^2) + (\bar{\beta}\mu - \beta\mu) dN_m, \quad (9.75)$$

where T_{bh} and p_{bh} are the temperature and the mean pressure of the black hole inside a cavity. But the two terms are precisely the differential of $\mathcal{S}$ in Eq. (9.44), meaning that $T_{bh} = T(r_+, \alpha)$ and $p_{bh} = p(\alpha)$. One thus recovers the thermodynamic quantities of the black hole inside a cavity of radius α .

From the principle that the functional must be a maximum, this means that the first derivatives must vanish, i.e. $T_{bh} = T_m$ and $p_m = p(\alpha)$, which are exactly the equilibrium conditions found by the zero loop approximation, and also one has $\beta\mu = \tilde{\beta}\mu$.

9.5.5.2 Le Chatelier-Braun principle

Since the functional $\bar{\mathcal{S}}$ must be a maximum, and with the vanishing first derivatives, we must also consider the condition that the hessian of the functional must be negative definite, i.e. $d^2\bar{\mathcal{S}} < 0$. This precisely yields that the matrix

$$d^2\bar{\mathcal{S}} = \begin{pmatrix} -\Lambda^T \cdot \hat{H} \cdot \Lambda & 0_2 \\ 0_2^T & -\frac{\partial N_m}{\partial \beta\mu} \Big|_{\hat{\omega}=\hat{\omega}_0} \end{pmatrix}, \quad (9.76)$$

$$\Lambda = \begin{pmatrix} \left(\frac{\partial S_{bh}}{\partial r_+}\right)^{-1} \Big|_{\hat{\omega}=\hat{\omega}_0} & 0 \\ 0 & \left(\frac{\partial(4\pi\alpha^2)}{\partial \alpha}\right)^{-1} \Big|_{\hat{\omega}=\hat{\omega}_0} \end{pmatrix}, \quad (9.77)$$

must be negative definite, where 0_2 is the two-dimensional zero vector and 0_2^T its transpose. This means that precisely $\hat{H}$ must be positive definite and that the shell must be intrinsically stable as $\frac{\partial N_m}{\partial \beta\mu} > 0$.

The thermodynamic meaning of the positive definiteness of $\hat{H}$ is exactly the Le Chatelier-Braun principle of the two subsystems in equilibrium, i.e. the black hole inside a cavity made by the matter shell and the matter shell itself. Indeed, one can use the variable of the black hole S_{bh} , which has a conjugate variable $\left(1 - \frac{T(r_+, \alpha)}{T_m}\right)$, being zero if the black hole is in equilibrium with the cavity. The other variable α , which is a variable of matter shell, has a conjugate variable $\frac{1}{T_m}(p_m - p(\alpha))$, which is zero if the black hole inside the cavity and the matter shell are in mechanical equilibrium. Now, if one assumes that the black hole seizes to be in thermal equilibrium, the black hole entropy increases and its conjugate variable varies as $\Delta \left(1 - \frac{T(r_+, \alpha)}{T_m}\right) = \left(\left(\frac{\partial S_{bh}}{\partial r_+}\right)^{-1} \frac{\partial}{\partial r_+} \left(1 - \frac{T(r_+, \alpha)}{T_m}\right)\right) \Big|_{\hat{\omega}=\hat{\omega}_0} \Delta S_{bh}$, while maintaining $4\pi\alpha^2$ constant. These variations result in a violation also of the equilibrium condition $\frac{1}{T_m}(p_m - p(\alpha))$. As equilibrium is restored, one has a variation of the equilibrium condition $\Delta \left(1 - \frac{T(r_+, \alpha(\tilde{r}_+, \tilde{\beta}\mu, r_+))}{T_m}\right) = \left(\left(\frac{\partial S_{bh}}{\partial r_+}\right)^{-1} \frac{\partial}{\partial r_+} \left(1 - \frac{T(r_+, \alpha(\tilde{r}_+, \tilde{\beta}\mu, r_+))}{T_m}\right)\right) \Big|_{\hat{\omega}=\hat{\omega}_0} \Delta S_{bh}$,

where $p_m = p(\alpha)$ is assumed, yielding solutions to $\alpha = \alpha(\tilde{r}_+, \beta\mu, r_+)$. One can write this last derivative as

$$\begin{aligned} & \left(\left(\frac{\partial S_{bh}}{\partial r_+} \right)^{-1} \frac{\partial}{\partial r_+} \left(1 - \frac{T(r_+, \alpha(r_+, \tilde{r}_+, \beta\mu))}{T_m} \right) \right) \Big|_{\hat{\omega}=\hat{\omega}_0} \\ &= \left(\left(\frac{\partial S_{bh}}{\partial r_+} \right)^{-1} \frac{\partial}{\partial \alpha} \left(1 - \frac{T(r_+, \alpha)}{T_m} \right) \right) \Big|_{\hat{\omega}=\hat{\omega}_0} - \frac{T_m \left(\frac{\partial}{\partial \alpha} \left(1 - \frac{T(r_+, \alpha)}{T_m} \right) \right)^2}{8\pi\alpha \frac{\partial}{\partial \alpha} (p_m - p(\alpha))} \Big|_{\hat{\omega}=\hat{\omega}_0}, \end{aligned} \quad (9.78)$$

where it was used

$$\frac{\partial \alpha(\tilde{r}_+, \beta\mu, r_+)}{\partial r_+} = - \frac{2\pi r_+ T_m \partial_\alpha \left(1 - \frac{T(r_+, \alpha)}{T_m} \right)}{\partial_\alpha (p_m - p_\alpha) 8\pi\alpha}. \quad (9.79)$$

Now, we can identify the right-hand side of Eq. (9.78) as being proportional to $|\hat{H}|/\hat{H}_{\alpha\alpha}$, which is positive by the stability condition of the minimization of the reduced action. Therefore, we obtain that

$$\frac{\partial}{\partial S_{bh}} \left(1 - \frac{T(r_+, \alpha)}{T_m} \right) \Big|_{\hat{\omega}=\hat{\omega}_0} > \frac{\partial}{\partial S_{bh}} \left(1 - \frac{T(r_+, \alpha(r_+, \tilde{r}_+, \beta\mu))}{T_m} \right) \Big|_{\hat{\omega}=\hat{\omega}_0} > 0,$$

This is precisely the Le Chatelier-Braun principle for the black hole inside a cavity and the shell, arising from the stability of the zero loop approximation.

Curiously, the positive definiteness of $\hat{H}$ implies that $\mathcal{H}$ is positive definite as we have shown in Sec. 9.3.3. The positive definiteness of $\hat{H}$ represents the Le Chatelier-Braun principle as we have shown above, but the meaning of the positive definiteness of $\mathcal{H}$ seems to be still illusive. Indeed, one of the conditions of positive definiteness of $\mathcal{H}$ contributes to the thermodynamic stability of the ensemble, however another condition remains. This condition may yield a statement in the sense of Le Chatelier-Braun principle, since it only involves implicit derivatives of $\tilde{r}_+$ and $\beta\mu$. Indeed, one can rewrite the matrix as

$$\mathcal{H} = \mathcal{H}_t - \mathcal{H}_p, \quad (9.80)$$

$$\mathcal{H}_t = \begin{pmatrix} \frac{\partial}{\partial \tilde{r}_+} \left(\frac{1}{2T_C} \right) & \frac{\partial N_C}{\partial \tilde{r}_+} \\ \frac{\partial N_C}{\partial \tilde{r}_+} & \frac{\partial N_C}{\partial \beta\mu} \end{pmatrix}, \quad (9.81)$$

$$\mathcal{H}_p = \begin{pmatrix} \frac{\partial}{\partial \tilde{r}_+} \left(\frac{1}{2T_m \sqrt{f_2(\alpha)}} \right) \Big|_{\hat{\omega}=\hat{\omega}_0} & \frac{\partial N_m}{\partial \tilde{r}_+} \Big|_{\hat{\omega}=\hat{\omega}_0} \\ \frac{\partial N_m}{\partial \tilde{r}_+} \Big|_{\hat{\omega}=\hat{\omega}_0} & \frac{\partial N_m}{\partial \beta\mu} \Big|_{\hat{\omega}=\hat{\omega}_0} \end{pmatrix}, \quad (9.82)$$

where $\mathcal{H}_t$ has components with total derivatives at the solutions of the inner system and $\mathcal{H}_p$ are the partial derivatives. Then, the positive definiteness of the matrix $\mathcal{H}$ seems to indicate that, in some sense, when more energy and $\beta\mu$ are available

to the system with the system remaining in the previous state, the two systems respond in order to have

$$\frac{\partial}{\partial \tilde{r}_+} \left(\frac{1}{2T_C} \right) > \frac{\partial}{\partial \tilde{r}_+} \left(\frac{1}{2T_m \sqrt{f_2(\alpha)}} \right) \Big|_{\tilde{\omega}=\tilde{\omega}_0} + \frac{\left(\frac{\partial N_C}{\partial \tilde{r}_+} - \frac{\partial N_m}{\partial \tilde{r}_+} \right)^2}{\left(\frac{\partial N_C}{\partial \beta\mu} - \frac{\partial N_m}{\partial \beta\mu} \right)} \Big|_{\tilde{\omega}=\tilde{\omega}_0}, \quad (9.83)$$

$$\frac{\partial N_C}{\partial \beta\mu} > \frac{\partial N_m}{\partial \beta\mu} \Big|_{\tilde{\omega}=\tilde{\omega}_0}. \quad (9.84)$$

For the first condition, the difference of the inverse temperature is further increased by the response of the system. For the second condition, the difference in the mean number of particles is also further increased by the response of the system, and it leads to the thermodynamic stability of the $(E, \beta\mu)$ ensemble if the shell has $\frac{\partial N_m}{\partial \beta\mu} > 0$, as seen previously.

9.6 THERMODYNAMICS IN THE GRAND CANONICAL ENSEMBLE OF A BLACK HOLE AND A SELF-GRAVITATING MATTER THIN SHELL INSIDE A CAVITY

9.6.1 The grand potential of a black hole and a self-gravitating matter thin shell inside a cavity

We now obtain the thermodynamic properties from the grand canonical ensemble. The grand canonical ensemble of the outer system inside a cavity is described by the partition function $Z = e^{-\beta W}$, where W is the grand potential of the outer system inside a cavity. From the path integral approach in the zero loop approximation, the partition function is related to the reduced action evaluated at the solutions of the ensemble, i.e. $Z = e^{-\tilde{I}_0}$, where $\tilde{I}_0 = \tilde{I}(z; \tilde{r}_+(z))$, with $\tilde{I}$ given in Eq. (9.41) and $\tilde{r}_+ = \tilde{r}_+(z)$ being the solution to Eq. (9.60). By combining the two expressions for the partition function, one obtains the relation

$$W(T, A(R), \beta\mu) = T\tilde{I}_0(z), \quad (9.85)$$

where $A(R) = 4\pi R^2$. Therefore, the expression for the grand potential can be written explicitly as

$$W = R(1 - \sqrt{f(\tilde{r}_+(z), R)}) - T\tilde{\mathcal{S}}(\beta\mu; \tilde{r}_+(z)), \quad (9.86)$$

where $\tilde{\mathcal{S}}(\beta\mu; \tilde{r}_+)$ is the functional given by the zero loop approximation of the path integral describing $(E, \beta\mu)$ ensemble in Eq. (9.43), i.e. $\tilde{\mathcal{S}}(\beta\mu; \tilde{r}_+) = \mathcal{S}(\beta\mu; \tilde{r}_+, \tilde{\omega}_0)$ and $\tilde{\omega}_0$ are the solutions to the system in Eqs. (9.47) and (9.48).

The grand potential is described thermodynamically as the Legendre transform of the energy as

$$W = E - TS - \mu N, \quad (9.87)$$

where the energy is written in terms of $E = E(S, A(R), N)$, and where T is the temperature, S is the total entropy, μ is the chemical potential and N is the mean particle number. From here and from the derivatives of the grand potential, we can obtain the thermodynamic mean quantities.

9.6.2 Mean energy, entropy and mean particle number

The differential of the grand potential can be written as $W = W(T, A(R), \beta\mu)$ is

$$dW = -(S + N\beta\mu)dT - p dA(R) - TN d(\beta\mu) , \quad (9.88)$$

where the derivatives of the grand potential can then be read out as $S + N\beta\mu = \frac{\partial W}{\partial T}$, $p = -\frac{\partial W}{\partial A(R)}$ and $TN = -\frac{\partial W}{\partial \beta\mu}$. Now, due to the fact that the stationary points obey the condition $\left. \frac{\partial \tilde{I}}{\partial \tilde{r}_+} \right|_{\tilde{r}_+ = \tilde{r}_+(z)} = 0$, we can perform the derivatives of the grand potential using the chain rule to obtain thermodynamic quantities of the total gravitational system as $S + N\beta\mu = -\left. \frac{\partial(T\tilde{I})}{\partial T} \right|_{\tilde{r}_+ = \tilde{r}_+(z)}$, $p = -\left. \frac{\partial(T\tilde{I})}{\partial A(R)} \right|_{\tilde{r}_+ = \tilde{r}_+(z)}$, and $TN = -\left. \frac{\partial(T\tilde{I})}{\partial \beta\mu} \right|_{\tilde{r}_+ = \tilde{r}_+(z)}$. Therefore, we obtain that $S + N\beta\mu = \tilde{\mathcal{S}}(\beta\mu; \tilde{r}_+(z))$, and that $TN = T\tilde{N}(\beta\mu, \tilde{r}_+(z))$. In terms of the quantities for the black hole and the shell, the entropy is

$$S = \pi r_+^2(z) + S_m(m(\omega_0), A(\alpha(z)), \beta\mu) , \quad (9.89)$$

the mean particle number is

$$N = N_m(m(\omega_0), A(\alpha(z)), \beta\mu) , \quad (9.90)$$

and the mean pressure is

$$p = \frac{1}{16\pi R \sqrt{f_2(R)}} \left(1 - \sqrt{f_2(R)} \right)^2 . \quad (9.91)$$

Finally, from Eqs. (9.89)-(9.91) and Eq. (9.87), the mean energy can be computed to be

$$E = R(1 - \sqrt{f_2(R)}) . \quad (9.92)$$

9.6.3 Thermodynamic stability of the grand canonical ensemble with the reservoir

In order to study the thermodynamic stability of the grand canonical ensemble, we need to consider the total entropy of the ensemble plus the reservoir. A small difference in this entropy is given by $dS + dS_{res}$, where $dS_{res} = \frac{1}{\bar{T}} dE_{res} - \bar{\beta}\mu dN_{res}$ and $\bar{T}$ and $\bar{\beta}\mu$ are fixed quantities of the ensemble. Since one has $dE_{res} = -dE$ and $dN_{res} = -dN$ due to conservation of energy and particle number, then the difference in the sum of entropies becomes $dS - \frac{1}{\bar{T}} dE + \bar{\beta}\mu dN = -\frac{d\bar{W}}{\bar{T}}$, where $\bar{W}$ is the grand potential functional given by

$$\bar{W}(T, \beta\mu) = E - \bar{T}S - \bar{T}\bar{\beta}\mu N , \quad (9.93)$$

where the dependence on R is omitted since the quantity must be constant. To have stability, the sum of entropies must be a maximum for the equilibrium configurations. Since the difference of the sum of entropies is $-\frac{d\bar{W}}{\bar{T}}$, then this

means that the grand potential functional should be at a minimum in stable equilibrium configurations. The differential of the grand potential functional is given by $d\bar{W} = (T - \bar{T})dS + (T\beta\mu - \bar{T}\bar{\beta}\mu)dN$, and at equilibrium it must vanish, yielding the equilibrium conditions $T = \bar{T}$ and $\beta\mu = \bar{\beta}\mu$. Now the condition of the stable configurations being at a minimum of the grand potential functional translates into having a positive definite hessian of the grand potential functional, i.e.

$$d^2\bar{W} = \begin{pmatrix} \frac{\partial(S+\beta\mu N)}{\partial T} & T\frac{\partial N}{\partial T} \\ T\frac{\partial N}{\partial T} & T\frac{\partial N}{\partial \beta\mu} \end{pmatrix}, \quad (9.94)$$

must be positive definite. In terms of thermodynamic coefficients, the stability conditions can be related to the condition $C_{A,N} > 0$ and $\frac{\partial N}{\partial \beta\mu} > 0$, where $C_{A,N}$ is the heat capacity at constant area and particle number $C_{A,N} = T \left(\frac{\partial S}{\partial T} \right)_{N,A}$ given by

$$C_{A,N} = T \frac{\partial(S + \beta\mu N)}{\partial T} - T^2 \left(\frac{\partial N}{\partial \beta\mu} \right)^{-1} \left(\frac{\partial N}{\partial T} \right)^2 > 0. \quad (9.95)$$

In order to connect the thermodynamic stability conditions in Eq. (9.94) with the stability conditions of the zero loop approximation regarding the effective action, we can rewrite the components of Eq. (9.94) as

$$\frac{\partial(S + \beta\mu N)}{\partial T} = \frac{\partial \tilde{r}_+}{\partial T} \frac{\partial \tilde{S}}{\partial \tilde{r}_+} \Big|_{\tilde{r}_+ = \tilde{r}_+(z)}, \quad (9.96)$$

$$T \frac{\partial N}{\partial T} = T \frac{\partial \tilde{r}_+}{\partial T} \frac{\partial \tilde{N}}{\partial \tilde{r}_+} \Big|_{\tilde{r}_+ = \tilde{r}_+(z)}, \quad (9.97)$$

$$T \frac{\partial N}{\partial \beta\mu} = T \left(\frac{\partial \tilde{N}}{\partial \beta\mu} + \frac{\partial \tilde{N}}{\partial \beta\mu} \frac{\partial \tilde{r}_+}{\partial \beta\mu} \right) \Big|_{\tilde{r}_+ = \tilde{r}_+(z)}. \quad (9.98)$$

The thermodynamic stability conditions for the grand canonical ensemble then simplify, using the relation $\frac{\partial \tilde{r}_+}{\partial \beta\mu} = \frac{T}{B} \frac{\partial \tilde{r}_+}{\partial T}$, as

$$\frac{\partial \tilde{r}_+}{\partial T} > 0, \quad (9.99)$$

$$\frac{\partial \tilde{N}}{\partial \beta\mu} \Big|_{\tilde{r}_+ = \tilde{r}_+(z)} > 0. \quad (9.100)$$

The first condition, Eq. (9.99), is exactly the same as the stability condition in Eq. (9.62) of the zero loop approximation. The second condition, Eq. (9.100), is precisely the condition of thermodynamic stability of the $(E, \beta\mu)$ ensemble.

Therefore, there is thermodynamic stability if the zero loop approximation of effective action is valid and moreover, if the zero loop approximation of the path integral describing the $(E, \beta\mu)$ ensemble is valid. This relation is captured because of the fixed parameter $\beta\mu$ which is also the intrinsic parameter of the shell since $\beta\mu$ is constant throughout the space. This does not happen for example in [6], where thermodynamic stability is completely disconnected from mechanical stability. We

must note however that the connection between mechanical stability and thermodynamic stability is thin, because the condition is a sum of the mechanical condition with another term involved in the Le Chatelier-Braun principle. Additionally, the shell must have $\frac{\partial N_m}{\partial \beta\mu} > 0$, which depends on the choice of the equation of state for the shell.

9.7 FUNDAMENTAL EQUATIONS OF STATE

9.7.1 The Martinez pressure equation of state

Here, we evaluate the possibility of giving the pressure equation of state from general relativity, as was done in [93]. The differential of the functional $\mathcal{S}$ for the shell can be written as

$$d\mathcal{S} = \frac{1}{T_m} dm + \frac{p_m}{T_m} dA + N_m d\beta\mu . \quad (9.101)$$

Now, the energy of the shell is given by Eq. (9.29) and the pressure is assumed to be given by the equation of state $p_m = p(\alpha)$, where $p(\alpha)$ is given in Eq. (9.46). But for this to be true, $p(\alpha)$ must be a function of m , α and $\beta\mu$. This was true for the case of a shell only, since only $\tilde{r}_+$ appeared and so the dependence on m or $\tilde{r}_+$ is equivalent through a transformation of variables. However, for the case of a thin shell with a black hole inside, one also has the dependence on r_+ . From Eq. (9.29), the function $p(\alpha)$ must then be a function $P(\sqrt{f_2} - \sqrt{f_1}, \alpha, \beta\mu)$ for it to be valid as an equation of state. The pressure equation of state can be rewritten as being proportional to $m \left(\frac{1}{\sqrt{f_1 f_2}} - 1 \right)$. Taking $\sqrt{f_1}$ and $\sqrt{f_2}$ as independent variables, it can be seen that $\sqrt{f_1} \sqrt{f_2}$ can never be written as $\sqrt{f_1} - \sqrt{f_2}$. Hence, the equilibrium of pressures obtained from the Einstein equations, or more specifically the junction conditions, cannot be used as an equation of state.

We can, however, extract the equation of state from a self-gravitating matter thin shell and impose here. From [93], the pressure equation of state is

$$p_m = \frac{l_p^2 m^2}{16\pi\alpha^3 \left(1 - \frac{l_p^2 m}{\alpha} \right)} . \quad (9.102)$$

Note that p_m only depends here on the mass of the shell m and the area of the shell $A = 4\pi\alpha^2$. Using the integrability conditions, we can further obtain that the inverse temperature must satisfy

$$\frac{1}{T_m} = \left(1 - \frac{l_p^2 m}{\alpha} \right) g \left[m \left(2 - \frac{l_p^2 m}{\alpha} \right), \beta\mu \right] , \quad (9.103)$$

where g is some function of $m \left(2 - \frac{l_p^2 m}{\alpha} \right)$ and $\beta\mu$. The functional $\mathcal{S}$ can be described as

$$\mathcal{S} = \frac{1}{2} \int_0^m \left(2 - \frac{l_p^2 s}{\alpha} \right) g(s, \beta\mu) ds , \quad (9.104)$$

which depends on the choice of the function g . Preliminary analysis of this equation of state with the choice $b(x, \beta\mu) \propto x^{\frac{3}{2}}$ indicate that there are no stable shell solutions with a black hole inside. Rather, we should consider $\mathcal{S}$ evaluated at the limits of the parameter space, i.e. when there is no black hole or when the black hole meets the shell, or when the black hole sits inside shell but the gravitational radius of the system meets the shell. The largest value of $\mathcal{S}$ for these cases seems to vary with the fixed parameters of the ensemble and with the chosen function g .

9.7.2 A fundamental equation of state for the shell with a black hole inside

There is however a fundamental equation of state for the configuration of a shell in equilibrium with a black hole inside that we briefly explore here. This equation of state must be seen from the equilibrium equations for the pressure and temperature. Namely, one has

$$T_m^{-1} = 4\pi r_+ \sqrt{f_1(\alpha)} , \quad (9.105)$$

$$p_m = \frac{1}{16\pi\alpha l_p^2} \left(\frac{1+f_2(\alpha)}{\sqrt{f_2(\alpha)}} - \frac{1+f_1(\alpha)}{\sqrt{f_1(\alpha)}} \right) , \quad (9.106)$$

with the expression for the shell mass $m = \alpha l_p^{-2} (\sqrt{f_1(\alpha)} - \sqrt{f_2(\alpha)})$. The idea is to substitute $f_2(\alpha)$ by m , α and $f_1(\alpha)$ as

$$\sqrt{f_2(\alpha)} = \sqrt{f_1(\alpha)} - \frac{l_p^2 m}{\alpha} . \quad (9.107)$$

The two equilibrium equations become

$$\begin{aligned} \frac{\partial \mathcal{S}}{\partial m} &= 4\pi\alpha(1-f_1(\alpha))\sqrt{f_1(\alpha)} , \\ \frac{\frac{\partial \mathcal{S}}{\partial A}}{\frac{\partial \mathcal{S}}{\partial m}} &= \frac{m}{16\pi\alpha^2} \left[\frac{1}{\sqrt{f_1(\alpha)}(\sqrt{f_1(\alpha)} - \frac{l_p^2 m}{\alpha})} - 1 \right] , \end{aligned} \quad (9.108)$$

where it was used $\frac{1}{T_m} = \frac{\partial \mathcal{S}}{\partial m}$ and $\frac{p_m}{T_m} = \frac{\partial \mathcal{S}}{\partial A}$. The question we are trying to answer here is if there is an equation of state such that one can eliminate the dimensions of the system in Eq. (9.108). The answer seems to be positive. Indeed, we can choose an equation of state of the form

$$\mathcal{S} = \frac{\alpha^2}{l_p^2} \varphi \left(\frac{l_p^2 m}{\alpha} \right) , \quad (9.109)$$

where $\varphi(\frac{l_p^2 m}{\alpha})$ is a function to be determined in terms of $\frac{l_p^2 m}{\alpha}$. Putting this equation of state into Eq. (9.108), one obtains

$$\begin{aligned} \varphi' &= 4\pi(1-f_1)\sqrt{f_1} , \\ \frac{4\varphi}{\varphi'} &= \frac{l_p^2 m}{\alpha} \left(\frac{1}{\sqrt{f_1}(\sqrt{f_1} - \frac{l_p^2 m}{\alpha})} + 1 \right) , \end{aligned} \quad (9.110)$$

where the dependence on α was dropped and φ' is the derivative on the argument of φ . Notice now that Eq. (9.110) can be further simplified by solving the first equation in Eq. (9.110) to obtain $f_1 = f_1(\varphi')$ and substitute it into the second equation to obtain a differential equation for $\varphi(\frac{l_p^2 m}{\alpha})$ as

$$\frac{4\varphi}{\varphi'} = \frac{l_p^2 m}{\alpha} \left(\frac{1}{\sqrt{f_1(\varphi')}(\sqrt{f_1(\varphi')} - \frac{l_p^2 m}{\alpha})} + 1 \right). \quad (9.111)$$

One can have multiple solutions for φ , since both equations need to be inverted to finally obtain an expression for φ' in function of $\frac{l_p^2 m}{\alpha}$ and φ . If there is indeed such a φ that solves the differential equation, then both equilibrium equations are satisfied if and only if the first equation in Eq. (9.110) is satisfied. By using $\frac{l_p^2 m}{\alpha} = \sqrt{f_1} - \sqrt{f_2}$, this means that the solution is some $\sqrt{f_1}$ in function of $\sqrt{f_2}$, i.e. $\frac{r_{\pm}}{\alpha}$ in function of $\frac{\tilde{r}_{\pm}}{\alpha}$. With fixed $\tilde{r}_{+}$, one is then free to pick an α such that the solution is still valid, obtaining a value of r_{+} for each α . For each fixed $\tilde{r}_{+}$, there is then a non-countable collection of solutions described by a curve in the $\alpha \times r_{+}$ plane. This accomplishes the same functionality of the Martinez' equation of state, however it is much more involved for this case. Unfortunately, for some solutions we analyzed numerically, these solutions seem to be minima of $\mathcal{S}$.

9.8 HESSIANS RELATED TO THE ACTIONS

In order to analyze stability, we need to evaluate the hessian of the reduced action on the stationary points. In this section, the six components of the hessian for the case of a black hole inside a self-gravitating matter thin shell in a cavity are presented. The stationary points are given by solving the simultaneous vanishing of the components of the gradient of the reduced action.

The components of the hessian respective to at least one derivative on $\tilde{r}_{+}$, i.e. $H_{\tilde{r}_{+}\omega^A}$, are

$$H_{\tilde{r}_{+}\tilde{r}_{+}} = \frac{1}{4l_p^2 f_2(R)} \left[\frac{1}{l_p^2 T_m^2} \frac{\partial T_m}{\partial m} \frac{f_2(R)}{f_2(\alpha)} - \frac{1}{(f_2(\alpha))^{\frac{3}{2}} T_m} \left(\frac{1}{\alpha} - \frac{1}{R} \right) \right], \quad (9.112)$$

$$H_{\tilde{r}_{+}r_{+}} = -\frac{1}{4l_p^4 f_2(\alpha) T_m^2} \frac{\partial T_m}{\partial m}, \quad (9.113)$$

$$H_{\tilde{r}_{+}\alpha} = \frac{4\pi\alpha}{l_p^2} \left[\left(\frac{\partial T_m}{\partial A(\alpha)} - p(\alpha) \frac{\partial T_m}{\partial m} \right) \frac{1}{T_m^2 \sqrt{f_2(\alpha)}} + \frac{\tilde{r}_{+}}{16\pi\alpha^3 (f_2(\alpha))^{\frac{3}{2}} T_m} \right]. \quad (9.114)$$

The components to at least a derivative in r_{+} are

$$\hat{H}_{r_{+}r_{+}} = H_{r_{+}r_{+}} = \frac{4\pi^2 r_{+}^2}{l_p^2} \left[\frac{1 - 3f_1(\alpha)}{4\pi r_{+}^2 f_1(\alpha)} + \frac{1}{l_p^2} \frac{\partial T_m}{\partial m} \right], \quad (9.115)$$

$$\hat{H}_{r_{+}\alpha} = H_{r_{+}\alpha} = -\frac{16\pi^2 r_{+}\alpha}{l_p^2} \left[\frac{r_{+}}{16\pi\alpha^3 f_1(\alpha)} + \frac{1}{T_m} \left(\frac{\partial T_m}{\partial A(\alpha)} - \frac{\partial T_m}{\partial m} p(\alpha) \right) \right], \quad (9.116)$$

And finally the last component of the hessian is

$$\hat{H}_{\alpha\alpha} = H_{\alpha\alpha} = \frac{64\pi^2\alpha^2}{T_m} \left[\frac{1}{8\pi\alpha} \frac{\partial p(\alpha)}{\partial \alpha} - \frac{\partial p_m}{\partial A(\alpha)} + p(\alpha) \frac{\partial p_m}{\partial m} \right], \quad (9.117)$$

where $\frac{\partial p(\alpha)}{\partial \alpha}$ is given by

$$\frac{\partial p(\alpha)}{\partial \alpha} = \frac{1}{16\pi l_p^2} \left(\frac{3f_1^2 + 1}{2\alpha^2 f_1^{\frac{3}{2}}} - \frac{3f_2^2 + 1}{2\alpha^2 f_2^{\frac{3}{2}}} \right). \quad (9.118)$$

The hessian of the effective action in Eq. (9.41) is

$$\tilde{H}_{\tilde{r}_+\tilde{r}_+} = \left(\frac{B}{2f_2(R)R} - \frac{\partial B}{\partial \tilde{r}_+} \right) \Big|_{\tilde{r}_+=\tilde{r}_+(z)}, \quad (9.119)$$

which by using the transformation $\delta\omega^A = \delta\hat{\omega}^A + \frac{\partial\hat{\omega}_0^A}{\partial\tilde{r}_+}\delta\tilde{r}_+$, with $\frac{\partial\hat{\omega}_0^A}{\partial\tilde{r}_+}\Big|_{\tilde{r}_+=\tilde{r}_+(z)} = -H_{\tilde{r}_+\hat{\omega}^B}(\hat{H}^{-1})^{\hat{\omega}^A\hat{\omega}^B}\Big|_{\tilde{r}_+=\tilde{r}_+(z)}$ and the consistency relations of the inverse matrix $(H^{-1})^{\omega^i\omega^j}$, one obtains the relation between $\tilde{H}_{\tilde{r}_+\tilde{r}_+}$ with $H_{\omega^i\omega^j}$ as

$$\tilde{H}_{\tilde{r}_+\tilde{r}_+} = \frac{|H|}{|H|_{\{\tilde{r}_+\},\{\tilde{r}_+\}}}. \quad (9.120)$$

9.9 MECHANICAL STABILITY OF A SHELL AROUND A BLACK HOLE

We relate here the derivative of the difference of the pressures with the mechanical stability condition of a shell with a black hole inside. Following [172], the equations of motion for a shell are

$$\ddot{r} = \frac{8\pi r k_2 k_1}{l_p m} \left[p_m + \frac{m}{8\pi r^2} - \frac{\tilde{r}_+ k_1 - r_+ k_2}{l_p^2 r^2 k_1 k_2} \right], \quad (9.121)$$

$$m = \frac{r}{l_p^2} (k_1 - k_2). \quad (9.122)$$

$$k_1 = \sqrt{f_1 + \dot{r}^2}, \quad k_2 = \sqrt{f_2 + \dot{r}^2}. \quad (9.123)$$

For a shell in equilibrium at radius $r = \alpha$, it is required that $\dot{r} = \ddot{r} = 0$, which gives the shell pressure equilibrium equation $p_m = p(\alpha)$, with $p(\alpha)$ described in Eq. (9.46). For small perturbation in the radius, one has $r = \alpha + \delta r$, where the equation for the perturbations is given by

$$\delta\ddot{r} = \frac{8\pi\alpha f_2 f_1}{l_p m} \frac{\partial}{\partial \alpha} [p_m - p(\alpha)] \delta r. \quad (9.124)$$

To have a mechanically stable shell, δr must have an oscillatory motion and not an exponential one. And so mechanical stability means $\frac{\partial}{\partial \alpha} (p(\alpha) - p_m) > 0$, i.e. $H_{\alpha\alpha} > 0$.

9.10 CONCLUSIONS

In this chapter, we constructed the grand canonical ensemble of a thin shell with a black hole inside a cavity and also the $(E, \beta\mu)$ ensemble of a black hole and a thin shell, which is an ensemble with fixed energy and fixed chemical potential. We construct the $(E, \beta\mu)$ ensemble because it gives a better understanding and serves as a first step towards the construction of the grand canonical ensemble. We apply the zero loop approximation, leading to equilibrium equations and stability conditions for the validity of the approximation. To obtain the solutions of the ensembles, we still need to make a choice of equations of state.

In this chapter, we have shown the power of the Euclidean path integral approach in the construction of the ensembles of self-gravitating systems. In the zero loop approximation, the formalism provides the equilibrium equations and the stability conditions. These last conditions include the mechanical stability of the shell and also lead to the Le Chatelier-Braun principle. In connection to the thermodynamics of the system in the grand canonical ensemble and the $(E, \beta\mu)$ ensemble, the stability conditions lead to thermodynamic stability, but the stability conditions are more restrictive. This means one cannot infer that the stability conditions are satisfied if there is thermodynamic stability. This may be due to the nature of the zero loop approximation, since the stability conditions are tied to the expansion of the integral over the minima of the action. Indeed, if one was able to obtain the path integral in a convergent way, the stability conditions would not exist. As we have seen in Chapter 8, the mechanical stability is disconnected from thermodynamic stability of the canonical ensemble of a shell in AdS, but this is because the chemical potential was not included. Since the chemical potential times the inverse temperature is constant throughout the space, in some way, we have some limited access to the properties of the thin shell in the ensemble.

The task of finding an equation of state such that yields stable solutions for a thin shell in equilibrium with a black hole still remains. While the fundamental equation of state from Martinez [93] does not give solutions of a shell with a black hole inside, we found another fundamental equation of state. However, it seems that the solutions are unstable. It was argued in [172] that the configuration of a shell with a black hole inside would always be unstable. However, a linear equation of state was used for the thin shell and there were no thermodynamic considerations in the analysis. A more thorough study of stability is still needed to ascertain the families of equations of state that could yield a stable configuration.

CONCLUDING REMARKS

In this thesis, we explored the thermodynamics of curved spacetimes and self-gravitating matter via two methods, by imposing the first law of thermodynamics and by constructing statistical ensembles through the Euclidean path integral approach in quantum gravity. There are however a few caveats to the analysis we took here and there are also future research lines.

In the first Part, composed only by the Chapter 2, we considered an electrically charged self-gravitating matter thin shell and we imposed the first law of thermodynamics to such shell. Furthermore, we imposed the Martinez fundamental pressure equation of state, allowing the shell to be in mechanical equilibrium for every radius of the shell. We chose the temperature equation of state so that it allowed the black hole limit, in order to recover black hole thermodynamics. We further analyzed the intrinsic thermodynamic stability. We showed that the shell can be put at the brink of becoming a black hole and also that the shell becomes marginally stable when doing so. The Martinez equation of state gives special characteristics to the shell. However, what is the extent of the existence of such fundamental equation of state for an isolated thin shell? It is not clear if the fundamental pressure equation of state is specific to general relativity and to matter thin shells. A further study must be done for alternative theories of gravity, where the junction conditions are different. For example, for $f(R, T)$ theories, i.e. theories with a lagrangian which is dependent on the Ricci scalar and the trace of the stress energy tensor, it was shown in [173] that a self-gravitating thin shell must be at a specific radius for mechanical equilibrium, but however its thermodynamic implications still need to be explored.

In the second Part, we used the Euclidean path integral approach to compute the partition function of several curved spacetime configurations. In Chapter 3, we reviewed the formalism for the case of spherically symmetric metrics, which established a basis for the rest of the Chapters. Additionally, in all Chapters of this Part, we considered the zero loop approximation, where in some cases the Hamiltonian and momentum constraints were imposed to the Euclidean action to obtain a reduced action. While the choice of topology for the path integral has been motivated by the topology of the boundary of the space, it would be of interest to extend the study in a more thorough way to axisymmetric complex spaces. Kerr-Newmann complexified spaces [132] were considered already, but a deeper study of the reduced action for these spaces has not been done yet. Furthermore, an issue that is transversal to this Part is that we analyzed the validity of the zero

loop approximation using the reduced action, which already assumes the field constraints. A further study regarding the equivalence between the existence of negative modes of Riemannian or pseudo-Riemannian spaces and the analysis of the reduced action must still be done.

In particular, in Chapters 4, 5 and 6, we considered ensembles of Reissner-Nordström black holes in arbitrary dimensions, in the zero loop approximation. Namely, in Chapter 4, we considered the grand canonical ensemble inside a cavity. In Chapter 5, we considered the canonical ensemble of a charged black hole inside a cavity with infinite radius. And in Chapter 6, we considered the canonical ensemble of a charged black hole inside a finite cavity. Note that in Chapter 5, we have shown explicitly that the results from imposing the first law of thermodynamics agree with the Euclidean path integral approach. An innovation that we introduced was the modelling of a configuration corresponding to hot flat space in each ensemble. In the grand canonical ensemble, we modelled the configuration by a charged sphere with no gravity in the limit of very small radius, which allowed the existence of an electric potential difference. In some sense, it described hot flat space with an electric potential difference. In the canonical ensemble, we modelled the configuration by a charged shell with no gravity in the limit that the shell approached the boundary of space. In some sense, it described hot flat space with electric charge at the boundary of the cavity. These configurations yielded a zero action in their respective ensembles. We were able to study the phase transitions between the charged black hole and these configurations, something that was missing in the literature. However, these configurations are heavily simplified. In order to further improve the analysis in this thesis, the study of the matter section with electric charge is required, which may give a better description of these configurations.

In Chapter 7, we considered the limits of the solutions of the zero loop approximation of Schwarzschild-AdS black holes inside a cavity. Indeed, we have shown that these limits unify the existing black hole solutions described in [67], [68], the planar AdS black hole solution and the Rindler solution.

In Chapter 8, we studied the canonical ensemble for a matter thin shell in anti-de Sitter (AdS). We gave an equation of state for the shell that is similar to an equation of state of a graviton gas trapped in a shell. Such equation of state allowed the existence of a stable solution for the shell. While one could choose another equation of state, our main goal was to analyze the phase transitions between the matter thin shell and the black hole, where we have shown that the phase transition is similar to the Hawking-Page phase transition. It is expected that such phase transition occurs between self-gravitating matter and black holes in AdS and also in asymptotically flat, but further analysis must be done. An interesting avenue is to consider self-gravitating fluids in the formalism to model exactly a gas of gravitons and photons with backreaction, which we started to study but did not include it in the thesis since it is very preliminary.

In Chapter 9, we have constructed the grand canonical ensemble of a self-gravitating matter thin shell with a black hole inside, all within a cavity, including the chemical potential of the shell. We have shown that the analysis of the stationary

points of the reduced action yields precisely the Le Chatelier-Braun principle between the thin shell and the black hole, and also yields thermodynamic stability of the full system. Related to the work in Chapter 2, the Martinez equation of state does not seem to give a stable solution for a shell with a black hole inside and only the limiting cases of a thin shell alone or a black hole alone are permitted. There seems to be another fundamental equation state different from Martinez equation of state allowing a non-countable number of equilibrium configurations, however they seem unstable thermodynamically. Further research is needed to understand if there is always a fundamental equation of state for shells in equilibrium with other systems.

We conclude by stating that main message of the thesis. The formalism of the Euclidean path integral approach is quite powerful in describing the thermodynamics of curved spacetimes, including self-gravitating matter. However, there are still many angles that need to be explored, not only on the limitations of the formalism but also regarding the connection to other semiclassical descriptions.

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