OPTIMAL QUANTITATIVE STABILITY ESTIMATES FOR ALEXANDROV'S SOAP BUBBLE THEOREM VIA GAGLIARDO-NIRENBERG-TYPE INTERPOLATION INEQUALITIES

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Abstract. The paper provides optimal quantitative stability estimates for the celebrated Alexandrov's Soap Bubble Theorem within the class of $C^{k,\alpha}$ domains, for any $k \geq 1$ and $0 < \alpha \leq 1$, by leveraging Gagliardo-Nirenberg-type interpolation inequalities. Optimal estimates of uniform closeness to a ball are established for L^r deviations of the mean curvature from being constant, for any $r \geq 2$ (more generally, for any $r > 1$ such that $r \geq (2N-2)/(N+1)$).

For $r > \frac{N-1}{2}$, the stability profile is linear, thus returning the existing results established in the literature through computations for nearly spherical sets. All the stability estimates for $r \leq \frac{N-1}{2}$, for which the profile is not linear, are new; even in the particular case $r = 2$ (which has been extensively studied, since it is a case of interest for several critical applications), the sharp stability profile that we obtain is new. Interestingly, we also prove that the (non-linear) profile for $r \leq \frac{N-1}{2}$ improves as k becomes larger to such an extent that it becomes formally linear as k goes to ∞.

Finally, for any $k \geq 1$ and $0 < \alpha \leq 1$, we show that our estimates are optimal within the class of $C^{k,\alpha}$ domains, by providing explicit examples.

1. Introduction

1.1. **State of the art and motivation.** We are interested in quantitative stability for the celebrated Alexandrov's Soap Bubble Theorem. Such a celebrated rigidity result deals with the mean curvature H of the boundary Γ of a bounded domain $\Omega \subset \mathbb{R}^N$. General quantitative results dealing with bubbling phenomena have been obtained in [6] (for uniform deviation of the mean curvature from being constant, i.e., $\|H - H_0\|_{L^\infty(\Gamma)}$, where H_0 is some reference constant), in [16] (for L^{N-1}-type deviations, i.e., $\|H - H_0\|_{L^{N-1}(\Gamma)}$), and finally in [23] (for L^1-type deviations[1] in any dimension). Under assumptions preventing bubbling phenomena (e.g., a uniform sphere condition or isoperimetric-type bounds), various sharp quantitative stability results of proximity to a single ball have been established. In [7] a sharp stability estimate (of Hausdorff-type closeness) was obtained for the uniform deviation $\|H - H_0\|_{L^\infty(\Gamma)}$. Weaker deviations were considered in [17, 18, 19, 20, 21, 11]. In particular, for L^2-deviations, [19, Theorem 4.6] gives the following (sharp) linear stability estimate:

$$\frac{|\Omega \Delta B_{1/H_0}(z)|}{|B_{1/H_0}(z)|} \leq C \, \|H - H_0\|_{L^2(\Gamma)}, \tag{1.1}$$

where z is a point in Ω, $H_0 := \frac{|\Gamma|}{N|\Omega|}$, and $B_{1/H_0}(z)$ denotes the ball centered at z with radius $1/H_0$. As shown in [22], the asymmetry in measure in (1.1) can be improved and replaced with a stronger L^2-type distance, that is,

$$\left\| |x - z| - \frac{1}{H_0} \right\|_{L^2(\Gamma)} \leq C \, \|H - H_0\|_{L^2(\Gamma)}.$$

Moreover, as shown in [21, Theorem 3.9, Formula (3.10)], linear stability can be obtained even for the (stronger) L^2-distance of the Gauss map, that is, the following (sharp) estimate holds true:

$$\left\| |x - z| - \frac{1}{H_0} \right\|_{L^2(\Gamma)} + \left\| \frac{1}{H_0} \, \nu - (x - z) \right\|_{L^2(\Gamma)} \leq C \, \|H - H_0\|_{L^2(\Gamma)}, \tag{1.2}$$

where ν denotes the exterior unit normal to Γ.

If on the left-hand side, we want to measure the closeness to a ball with a uniform norm, then, at least in high dimensions, we have to either pay something in the stability profile[2] or strengthen the L^2

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[1] More in general, for deviations such as $\int_\Gamma (H - H_0)^+ \, dS_x$.

[2] As we are going to show, in this case we cannot expect to obtain a linear profile for $N \geq 5$.

measure of the deviation on the right-hand side. On the one hand, [21, Theorem 3.9] established the following stability result for uniformly $C^{2,\alpha}$ domains:

$$\rho_e(z)-\rho_i(z)\le C\begin{cases}\|H-H_0\|_{L^2(\Gamma)} & \text{for } N=2,3,\\ \|H-H_0\|_{L^2(\Gamma)}\max\left[\log\left(\frac{1}{\|H-H_0\|_{L^2(\Gamma)}}\right),1\right] & \text{for } N=4,\\ \|H-H_0\|_{L^2(\Gamma)}^{4/N} & \text{for } N\ge 5,\end{cases}\tag{1.3}$$

where z is a point in Ω, and we have set

$$\rho_e(z):=\max_{x\in\Gamma}|x-z| \quad \text{and} \quad \rho_i(z):=\max_{x\in\Gamma}|x-z|.$$

On the other hand, in light of the computations for nearly spherical sets in [17, Theorem 1.10] (see also [11]), we know that linear stability estimates for the uniform closeness $\rho_e(z)-\rho_i(z)$ remain valid provided that we use deviations $\|H-H_0\|_{L^r(\Gamma)}$ with $r>(N-1)/2$ if $N\ge 5$, and $r\ge 2$ if $N\le 4$.

In the present paper, we establish the sharp stability profile concerning the uniform closeness $\rho_e(z)-\rho_i(z)$ and $\|H-H_0\|_{L^r(\Gamma)}$, for any $r\ge 2$ (more generally, for any $r>1$ such that $r\ge(2N-2)/(N+1)$); for any $k\ge 1$ and $0<\alpha\le 1$, our estimates are established for general uniformly $C^{k,\alpha}$ domains, without requiring further assumptions. The sharp general stability profile is presented in Theorem 1.5 below. For $r>\frac{N-1}{2}$ the stability profile is linear, thus returning the existing results established in the literature through computations for nearly spherical sets ([17, 11]). Surprisingly, all the stability estimates for $r\le\frac{N-1}{2}$, for which the profile is not linear, are new. Interestingly, we also prove that the (non-linear) profile for $r\le\frac{N-1}{2}$ improves as k becomes larger to such an extent that it becomes formally linear (in any dimension) as k goes to ∞.

In the present paper, we also show that the stability estimates obtained here are optimal for any $k\ge 1$ and $0<\alpha\le 1$. We stress that, while the optimality of the linear stability estimates (such as, (1.1) and (1.2) for any N, and (1.8) for $r>(N-1)/2$) can be readily checked by a simple calculation with ellipsoids, a careful analysis is needed to check that the stability exponent $\tau_{k,\alpha,N,r}$ in (1.8) is optimal, which we provide in the present paper.

We stress that the sharp profile we establish in Theorem 1.5 is new even in the particular case $r=2$. In fact, in the particular case where one considers $r=2$ and $C^{1,1}$ domains, the general result in Theorem 1.5 reduces to

$$\rho_e(z)-\rho_i(z)\le C\begin{cases}\|H-H_0\|_{L^2(\Gamma)} & \text{for } N=2,3,4,\\ \|H-H_0\|_{L^2(\Gamma)}\max\left[\frac{1}{2}\log\left(\frac{1}{\|H-H_0\|_{L^2(\Gamma)}}\right),1\right] & \text{for } N=5,\\ \|H-H_0\|_{L^2(\Gamma)}^{4/(N-1)} & \text{for } N\ge 6,\end{cases}$$

which clearly improves the stability profile in (1.3) for any $N\ge 4$. A key observation that allows such an improvement of the stability profile is to notice that if the deviation $\|H-H_0\|_{L^r(\Gamma)}$ is small enough – and we are working within a class of uniformly $C^{k,\alpha}$ ($k\ge 1$ and $0<\alpha\le 1$) domains to prevent bubbling phenomena –, then Ω must be nearly spherical. A similar remark was recently noticed in [11] by leveraging the elliptic regularity theory for almost minimal hypersurfaces; here, we avoid exploiting the elliptic regularity theory, but instead will explicitly deduce this qualitative information from (1.2). Such a remark allows to reduce the dimension from N to $N-1$, with consequent improvement of the profile when applying Sobolev embedding and interpolation inequalities.

1.2. Main results: optimal quantitative stability estimates for Alexandrov's Soap Bubble Theorem with L^r-type deviations.

Theorem 1.1. *Let $N\ge 4$ be an integer, $r\in\left[\frac{2N-2}{N+1},\frac{N-1}{2}\right)$, and $\Omega\subset\mathbb{R}^N$ a bounded domain with boundary Γ of class $C^{k,\alpha}$, where $k\ge 1$ and $0<\alpha\le 1$. If $k=1$, then we further assume that Γ is of class $W^{2,r}$. Set $H_0:=|\Gamma|/(N|\Omega|)$.*

Then, there exists a point $z\in\Omega$ such that

$$\rho_e(z):=\max_{x\in\Gamma}|x-z| \quad \textit{and} \quad \rho_i(z):=\max_{x\in\Gamma}|x-z|\tag{1.4}$$

satisfy

$$\rho_e(z)-\rho_i(z)\le C\|H_0-H\|_{L^r(\Gamma)}^{\tau_{k,\alpha,N,r}},\tag{1.5}$$

where

$$\tau_{k,\alpha,N,r}:=\frac{k+\alpha}{k+\alpha+\frac{N-1-2r}{r}},\tag{1.6}$$

and C is a constant only depending on N, k, α, r, the $C^{k,\alpha}$-regularity of Γ and the diameter of Ω.

Remark 1.2. (i) Notice that, as $k \to \infty$, (1.5) becomes (formally) linear

$$\lim_{k\to\infty} \tau_{k,\alpha,N,r} = 1.$$

(ii) The stability exponent (1.6) is continuous with respect to $k + \alpha$, that is,

$$\lim_{\alpha\to 0^+} \tau_{k+1,\alpha,N,r} = \tau_{k,1,N,r}$$

The next theorem deals with the limit case $r = \frac{N-1}{2}$.

Theorem 1.3 (The case $r = \frac{N-1}{2}$)**.** *Let $N \geq 4$, $k \geq 1$ an integer. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with boundary Γ of class $C^{k,\alpha}$, with $0 < \alpha \leq 1$, and set $H_0 := |\Gamma|/(N|\Omega|)$. If $k = 1$, further assume that Γ is of class $W^{2,\frac{N-1}{2}}$.*

Then, there exists a point $z \in \Omega$ such that the radii $\rho_e(z)$ and $\rho_i(z)$ defined in (1.4) satisfy

$$\rho_e(z) - \rho_i(z) \leq C\|H_0 - H\|_{L^{\frac{N-1}{2}}(\Gamma)} \max\left\{\frac{1}{k+\alpha} \log\left(\frac{1}{\|H_0 - H\|_{L^{\frac{N-1}{2}}(\Gamma)}}\right), 1\right\}, \tag{1.7}$$

where C is a constant only depending on N, k, the $C^{k,\alpha}$-regularity of Γ and the diameter of Ω.

Remark 1.4. Notice that in the case $r = \frac{N-1}{2}$, we still have that (1.7) becomes (formally) linear as $k \to \infty$, being as

$$\lim_{k\to\infty} \max\left\{\frac{1}{k+\alpha} \log\left(\frac{1}{\|H_0 - H\|_{L^{\frac{N-1}{2}}(\Gamma)}}\right), 1\right\} = 1.$$

With the two previous results, we are now able to present a complete stability profile. This is summarised in the following theorem.

Theorem 1.5. *Let $N \geq 2$ and $k \geq 1$ be two integers. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with boundary Γ of class $C^{k,\alpha}$, with $0 < \alpha \leq 1$, and set $H_0 = \frac{|\Gamma|}{N|\Omega|}$. Let $r > 1$ be such that $r \geq \frac{2N-2}{N+1}$. If $k = 1$, we further assume that Γ is of class $W^{2,r}$.*

Then, there exists a point $z \in \Omega$ such that the radii $\rho_e(z)$ and $\rho_i(z)$ defined in (1.4) satisfy

$$\rho_e(z) - \rho_i(z) \leq C \begin{cases} \|H - H_0\|_{L^r(\Gamma)} & \text{if } r > \frac{N-1}{2} \\ \|H_0 - H\|_{L^r(\Gamma)} \max\left\{\frac{1}{k+\alpha} \log\left(\frac{1}{\|H_0-H\|_{L^r(\Gamma)}}\right), 1\right\} & \text{if } r = \frac{N-1}{2} \\ \|H_0 - H\|_{L^r(\Gamma)}^{\tau_{k,\alpha,N,r}} & \text{if } r < \frac{N-1}{2}, \end{cases} \tag{1.8}$$

where C is a constant only depending on N, k, α, r, the diameter of Ω and the $C^{k,\alpha}$-regularity of Γ, and $\tau_{k,\alpha,N,r}$ is defined in (1.6).

Remark 1.6. (i) The sharpness of the exponent 1 in the linear estimates $\big($i.e., for $r > \frac{N-1}{2}\big)$ can be readily checked by a simple calculation with ellipsoids.

(ii) For $k \geq 1$ and $0 < \alpha \leq 1$, the stability exponent $\tau_{k,\alpha,N,r}$ is sharp, as we prove in Section 4 and Appendix A.

As already mentioned, all the stability estimates for $r \leq \frac{N-1}{2}$, for which the profile is not linear, are new (even in the particular case where one considers L^2 deviations of the mean curvature from being constant). The linear estimates (i.e., the case $r > \frac{N-1}{2}$) essentially return those in [17, 11]. We also point out that in the case $r = \infty$, our stability estimates formally recover the sharp linear estimate in [7]; more precisely, while [7] provides a linear estimate where the constant depends on N, $|\Gamma|$, and the $C^{1,1}$ regularity[3], our result shows that a linear estimate remains valid even if we replace the dependence on the $C^{1,1}$ regularity with the dependence on the $C^{1,\alpha}$ regularity.

[3]More precisely, the regularity parameter used in [7] is the radius of the uniform interior and exterior sphere conditions, which is equivalent to the $C^{1,1}$ regularity of Γ (see, e.g., [2, Corollary 3.14]).

1.3. **Organization of the paper.** The rest of the paper is organized as follows.

In Section 2 we introduce the setting and recall some basic tools that will be used in the proof of our main results.

Section 3 is devoted to the proofs of Theorems 1.1, 1.3, and 1.5.

In Section 4, we prove the optimality of Theorem 1.1 for domains that are at least of class $C^{1,1}$.

Appendix A is devoted to proving the optimality of Theorem 1.1 for $C^{1,\alpha}$ domains, when $\alpha < 1$.

Appendices B and C are devoted to proving two auxiliary results that will be helpful in the proofs of Theorems 1.1, 1.3, and 1.5.

2. Preliminaries

We start by recalling the basic definitions of (possibly fractional) Sobolev spaces. Let $\Omega \subset \mathbb{R}^N$ be an open set in $\mathbb{R}^N$, $s \in (0, \infty)$ and $p \in [1, \infty]$. Write $s = m + \sigma$, where $m \in \mathbb{N}$ and $\sigma \in [0, 1)$ (if $m = 0$, then we assume that $\sigma \in (0, 1)$).
If $m = 0$ and $p < \infty$, the Sobolev space $W^{s,p}(\Omega)$ is defined as

$$W^{s,p}(\Omega) := \left\{ u \in L^p(\Omega) : |u|^p_{W^{s,p}(\Omega)} := \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x - y|^{N+\sigma p}} \, dy \, dx < \infty \right\},$$

normed with

$$\|u\|_{W^{s,p}(\Omega)} := \|u\|_{L^p(\Omega)} + |u|_{W^{s,p}(\Omega)}.$$

If $m = 0$ and $p = \infty$, the Sobolev space $W^{s,\infty}(\Omega)$ is the space of $s-$Hölder continuous functions C^s, normed with

$$\|u\|_{C^s(\Omega)} := \|u\|_{L^\infty(\Omega)} + \sup_{\substack{x,y\in\Omega \\ x\neq y}} \frac{|u(x) - u(y)|}{|x - y|^\sigma}$$

If $m > 0$ and $\sigma = 0$, then the Sobolev space $W^{s,p}(\Omega)$ is the standard integer Sobolev space $W^{m,p}(\Omega)$.
If $m > 0$ and $\sigma \in (0, 1)$, the Sobolev space $W^{s,p}(\Omega)$ is defined as

$$W^{s,p}(\Omega) := \{ u \in W^{m,p}(\Omega) : D^m u \in W^{\sigma,p}(\Omega) \},$$

normed with

$$\|u\|_{W^{s,p}(\Omega)} := \|u\|_{W^{m,p}(\Omega)} + |D^m u|_{W^{\sigma,p}(\Omega)}.$$

For an excellent reference on fractional Sobolev spaces see [8].

We adopt the following definition of a $C^{k,\alpha}$ domain, where $k \geq 0$ is an integer, and $0 \leq \alpha \leq 1$ (see e.g., [12, Page 94]).

Definition 2.1. *Given an integer $k \geq 0$, a real number $0 \leq \alpha \leq 1$ and a bounded domain $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) with boundary Γ. We say that Γ is of class $C^{k,\alpha}$, if there exist positive constants $\tilde{C}$ and $\tilde{r}$ such that for any $x \in \Gamma$, there exists a one-to-one map*

$$\psi_x : B_{\tilde{r}}(x) \to D \subset \mathbb{R}^N,$$

such that

$$\psi_x(B_r(x) \cap \Omega) \subset \mathbb{R}^N_+, \quad \psi_x(B_r(x) \cap \Gamma) \subset \partial\mathbb{R}^N_+,$$
$$\|\psi_x\|_{C^{k,\alpha}(B_r(x))} \leq \tilde{C} \quad \text{and} \quad \|\psi_x^{-1}\|_{C^{k,\alpha}(D)} \leq \tilde{C}.$$

Here above, $B_{\tilde{r}}(x)$ denotes the $N-$dimensional euclidean ball centered at $x \in \mathbb{R}^N$ with radius $\tilde{r}$.

In particular, for every $x \in \Gamma$, $B_{\tilde{r}}(x) \cap \Gamma$ can be written as the graph of a $C^{k,\alpha}$ function of $N - 1$ variables, whose $C^{k,\alpha}$ norm is bounded above by $\tilde{C}$. The converse is also true if $k \geq 1$.

Throughout the paper, when we say that a constant depends on the $C^{k,\alpha}$ regularity of Γ, it means that it can be estimated in terms of the constants $\tilde{C}$ and $\tilde{r}$ in Definition 2.1.

As mentioned in the Introduction, one of the steps in the proofs of Theorems 1.1, 1.3 and 1.5 is the observation that if the deviation $\|H - H_0\|_{L^r(\Gamma)}$ is small enough and Γ is of class $C^{k,\alpha}$ ($k \geq 1$ and $0 < \alpha \leq 1$), then Ω must be nearly spherical. This is done in Lemma 3.1 below and the key to the proof of Lemma 3.1 is the combination of the regularity properties of Γ with the following Lemma which provides a more general but rougher version of (1.2) which deals with L^r deviations of the mean curvature.

Lemma 2.2. *Let $r \in (1,\infty)$ and $\alpha \in (0,1]$, and let $\Omega \subset \mathbb{R}^N$ be a bounded domain with boundary Γ of class $C^{1,\alpha} \cap W^{2,r}$. Let $z \in \mathbb{R}^N$ be the center of mass of Ω, that is,*

$$z = \frac{1}{|\Omega|} \int_\Omega x \, dx.$$

Then

$$\left\| |x-z| - \frac{1}{H_0} \right\|_{L^2(\Gamma)} + \left\| \frac{\nu}{H_0} - (x-z) \right\|_{L^2(\Gamma)} \le C \, \|H - H_0\|_{L^r(\Gamma)}^{1/2}, \tag{2.1}$$

where ν denotes the exterior unit normal to Γ at x, and C is a positive constant only depending on N, r, the diameter of Ω and the $C^{1,\alpha}$ regularity of Γ.

For a proof of Lemma 2.2 we refer to Appendix B.

Note that, for $r \ge 2$, in the right-hand side of (2.1), we can replace $\|H - H_0\|_{L^r(\Gamma)}^{1/2}$ with $\|H - H_0\|_{L^r(\Gamma)}$, simply by applying Hölder's inequality to (1.2). We stress that while (1.2) is a sharp linear estimate for $r = 2$, here we decided to present a rougher estimate (with exponent $1/2$) which holds for any $r \in (1,\infty)$; the (non-sharp) exponent here is not an issue for our aim, as this estimate will be used only to obtain a qualitative information which allows to restrict the analysis to nearly spherical sets, whereas the sharp rate of stability will be established later in the proofs of the main theorems.

The next result we need is a Gagliardo-Nirenberg-type interpolation inequality that will play a crucial role in the proofs of Theorems 1.1, 1.3, and 1.5.

Theorem 2.3 (Theorem 1, [4]). *Let $D \subset \mathbb{R}^d$ $(d \ge 1)$ be a bounded Lipschitz domain. Let $p > 1$, $sp > d$, and $1 \le q \le p$. Then there exists a constant C only depending on d, s, p, q, D, such that for any $v \in W^{s,p}(D)$,*

$$\|v\|_{L^\infty(D)} \le C \|v\|_{W^{s,p}(D)}^{1-\theta} \|v\|_{L^q(D)}^{\theta}, \tag{2.2}$$

where

$$\theta := \frac{s - \frac{d}{p}}{s - \frac{d}{p} + \frac{d}{q}}. \tag{2.3}$$

This, in fact, is a particular case of [4, Theorem 1], where much more general inequalities are studied, and we refer to [4] for a proof of Theorem 2.3.

In particular, we will make use of the following particular case of Theorem 2.3 with $p = \infty$, $q = rd/(d-2r)$ and $d \ge 3$.

Corollary 2.4. *Let $D \subset \mathbb{R}^d$ $(d \ge 3)$ be a bounded Lipschitz domain, and let $s > 0$ and $r \in \left[1, \frac{d}{2}\right)$. If $v \in W^{s,\infty}(D)$, then*

$$\|v\|_{L^\infty(D)} \le C \|v\|_{W^{s,\infty}(D)}^{1-\theta} \|v\|_{L^{\frac{rd}{d-2r}}(D)}^{\theta}, \quad \text{with } \theta := \frac{s}{s + \frac{d-2r}{r}}, \tag{2.4}$$

where the constant C depends only on d, s, p, r and D.

We will also need the following technical lemma, which gives a bound on the Sobolev norm of a function defined in $\mathbb{R}^{N-1}$ in terms of the Sobolev norm of the same function transported to the sphere $\mathbb{S}^{N-1}$ via the stereographic projection.
We believe this is a well-known result in the literature, but since we did not find a reference containing it, we present it here and provide a detailed proof in Appendix C.

Lemma 2.5. *Let $N \ge 2$ be an integer and $p \in [1,\infty)$. Then, there exists positive constants R dependent only on N and C dependent only on N and p, such that*

$$\|v \circ \iota^{-1}\|_{W^{2,p}\left(B_R^{N-1}(0)\right)} \le C \|v\|_{W^{2,p}(\mathbb{S}^{N-1})}, \quad \text{for all} \quad v \in W^{2,p}(\mathbb{S}^{N-1}), \tag{2.5}$$

where $\iota : \mathbb{S}^{N-1} \setminus \{P\} \to \mathbb{R}^{N-1}$ denotes the (standard) stereographic projection from $P \in \mathbb{S}^{N-1}$, and $B_R^{N-1}(0) \subset \mathbb{R}^{N-1}$ is the $(N-1)$-dimensional euclidean ball centered at the origin with radius R.

In particular, (2.5) holds with

$$R = \frac{1}{8(N-1)^2}, \tag{2.6}$$

and $C = 3 \cdot 2^{\frac{N+5}{p} + \frac{5}{2}}$.

3. Proof of Theorems 1.1, 1.3 and 1.5

Before we proceed with the proofs of Theorems 1.1, 1.3 and 1.5, we show that we can restrict our analysis to nearly spherical sets.

Lemma 3.1. *Let $N \geq 2$ be a integer and $r \in (1, \infty)$. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with boundary of class $C^{1,\alpha} \cap W^{2,r}$ $(0 < \alpha \leq 1)$. Set the origin at the center of mass of Ω, that is,*

$$\frac{1}{|\Omega|} \int_\Omega x \, dx = 0, \tag{3.1}$$

which is always true up to a translation.

Then, there exists a positive constant ε_0 dependent only on N, r, the diameter of Ω and the $C^{1,\alpha}$-regularity of Γ such that if $\|H - H_0\|_{L^r(\Gamma)} < \varepsilon_0$, then there exists $\omega : \mathbb{S}^{N-1} \to \mathbb{R}$ such that

$$\Gamma = \left\{ \left(\frac{1}{H_0} + \omega(x) \right) x : \, x \in \mathbb{S}^{N-1} \right\}. \tag{3.2}$$

Furthermore, for any $\eta > 0$ there exists $\varepsilon_1 > 0$ dependent only on N, r, η, the diameter of Ω and the $C^{1,\alpha}$-regularity of Γ such that, if $\|H - H_0\|_{L^r(\Gamma)} < \varepsilon_1$ then

$$\|\omega\|_{C^1(\mathbb{S}^{N-1})} \leq \eta. \tag{3.3}$$

Proof. By Lemma 2.2 and (3.1) we know that there exists a positive constant C, dependent on N, r, the diameter of Ω and the $C^{1,\alpha}$ regularity of Γ such that

$$\left\| |x| - \frac{1}{H_0} \right\|_{L^2(\Gamma)} + \left\| \frac{\nu}{H_0} - x \right\|_{L^2(\Gamma)} \leq C \, \|H - H_0\|_{L^r(\Gamma)}^{1/2}, \tag{3.4}$$

where ν denotes the exterior unit normal to Γ at x.

Discarding the second term in (3.4) we have

$$\int_\Gamma \left| |y| - \frac{1}{H_0} \right|^2 d\mathcal{H}_y^{N-1} \leq C^2 \|H - H_0\|_{L^r(\Gamma)}. \tag{3.5}$$

Let $\delta > 0$, and define the measurable sets $S_\delta^+, S_\delta^- \subset \mathbb{S}^{N-1}$ by setting

$$S_\delta^+ := \left\{ x \in \Gamma : \left| |y| - \frac{1}{H_0} \right|^2 > \delta \right\}, \quad \text{and} \quad S_\delta^- = \mathbb{S}^{N-1} \setminus S_\delta^+.$$

Then, we have that

$$|S_\delta^+| \leq \frac{1}{\delta} \int_\Gamma \left| |y| - \frac{1}{H_0} \right|^2 d\mathcal{H}_y^{N-1} \leq \frac{C^2}{\delta} \|H - H_0\|_{L^r(\Gamma)}, \tag{3.6}$$

where in the last inequality we used (3.5).

Let $y \in S_\delta^+$. Since Γ is of class $C^{1,\alpha}$, making use of Definition 2.1, we see that there exist r_1 dependent only on the $C^{1,\alpha}$ regularity of Γ, such that

$$\left(|w| - \frac{1}{H_0} \right)^2 > \frac{\delta}{2}, \quad \text{for all } w \in \{ z \in \Gamma : \, |z - y| < r_1 \},$$

and

$$|\{ z \in \Gamma : \, |z - y| < r_1 \}| \geq C_1 \, r_1^{N-1},$$

where $C_1 = \left| B_1^{N-1} \right|$ is the volume of the unit $(N-1)$-dimensional Euclidean ball in $\mathbb{R}^{N-1}$. The last inequality follows directly from Definition 2.1, which allows us to write $\{ z \in \Gamma : \, |z - y| < r_1 \}$ as the graph of a $C^{1,\alpha}$ function defined on a $(N-1)$-dimensional euclidean ball in $\mathbb{R}^{N-1}$ with radius r_1, and the formula for the area of a surface (see, e.g., [10]).

But then we conclude that

$$|S_{\delta/2}^+| \geq C_1 r_1^{N-1}.$$

Recalling (3.6), we obtain

$$C_1 r_1^{N-1} \leq |S_{\delta/2}^+| \leq \frac{2C^2}{\delta} \|H - H_0\|_{L^r(\Gamma)}.$$

So, setting $\tilde{\varepsilon}_1(\delta) := \frac{\delta C_1}{2C^2} r_1^{N-1}$, we see that if

$$\|H - H_0\|_{L^r(\Gamma)} < \tilde{\varepsilon}_1(\delta)$$

then we conclude that $S_\delta^+ = \emptyset$.

Similarly, we conclude that for any $\epsilon > 0$ there exists $\tilde{\varepsilon}_2(\epsilon) > 0$ dependent on the $C^{1,\alpha}$ regularity of Γ, N, r and ϵ such that, if $\|H - H_0\|_{L^r(\Gamma)} < \tilde{\varepsilon_2}(\epsilon)$, then

$$\left\{ y \in \Gamma : \left| \frac{\nu(y)}{H_0} - y \right|^2 > \epsilon \right\} = \emptyset.$$

In particular, if we take $\epsilon < \frac{1}{H_0}$ then, if $\|H - H_0\|_{L^r(\Gamma)} < \tilde{\varepsilon}_2(\epsilon)$, we have that $\nu(y) \cdot y > 0$ for all $y \in \Gamma$. This implies that Ω is star-shaped about the origin, and therefore (up to a translation), we can write

$$\Gamma = \left\{ \left(\frac{1}{H_0} + \omega(x) \right) x : \, x \in \mathbb{S}^{N-1} \right\}, \tag{3.7}$$

for some function $\omega : \mathbb{S}^{N-1} \to \mathbb{R}$ satisfying

$$\omega(x) \geq -\frac{1}{H_0}. \tag{3.8}$$

So, taking $\varepsilon_0 := \tilde{\varepsilon}_2 \left(\frac{1}{2H_0} \right)$, (3.2) follows. We note that the dependence of ε_0 on H_0 can be removed using Lemma B.3 below.[4] In this way, ε_0 depends on N, r, the diameter of Ω and the $C^{1,\alpha}$ regularity of Γ.

Notice that since Γ is of class $C^{1,\alpha}$, then ω is in $C^{1,\alpha}(\mathbb{S}^{N-1})$ and $\|\omega\|_{C^{1,\alpha}(\mathbb{S}^{N-1})}$ depends only on the $C^{1,\alpha}$ regularity of Γ.

Now we prove the second part of the statement. To that end, let $\sigma > 0$ be such that

$$\sigma << \frac{1}{H_0}, \tag{3.9}$$

and let $\varepsilon_1 = \min\{\tilde{\varepsilon}_1(\sigma), \tilde{\varepsilon}_2(\sigma)\} > 0$. Then, by the arguments above, if $\|H - H_0\|_{L^r(\Gamma)} < \varepsilon_1$, then

$$\left\{ y \in \Gamma : \left| |y| - \frac{1}{H_0} \right|^2 > \sigma \right\} = \emptyset, \quad \text{and} \quad \left\{ y \in \Gamma : \left| \frac{\nu(y)}{H_0} - y \right|^2 > \sigma \right\} = \emptyset. \tag{3.10}$$

Note that, due to the arguments presented above, we know that ε_1 will depend only on the $C^{1,\alpha}$ regularity of Γ, N, r, and σ. Furthermore, (3.7) holds for some $C^{1,\alpha}$ function $\omega : \mathbb{S}^{N-1} \to \mathbb{R}$.

Using (3.7), given $y = \left(\frac{1}{H_0} + \omega(x) \right) x$, the outward unit normal at y, $\nu(y)$, can be written as (see [24])

$$\nu(y) = \nu\left(\left(\frac{1}{H_0} + \omega(x) \right) x \right) = \frac{\left(\frac{1}{H_0} + \omega(x) \right) x - \nabla_{\mathbb{S}^{N-1}} \omega(x)}{\sqrt{\left(\frac{1}{H_0} + \omega(x) \right)^2 + |\nabla_{\mathbb{S}^{N-1}} \omega(x)|^2}}. \tag{3.11}$$

Combining (3.7) and (3.10) we see that

$$\left| \left| \frac{1}{H_0} + \omega(x) \right| - \frac{1}{H_0} \right| \leq \sigma^{1/2}, \quad \text{for all}\, x \in \mathbb{S}^{N-1}, \tag{3.12}$$

and this implies that

$$|\omega(x)| \leq \sigma^{1/2}, \quad \text{for all}\, x \in \mathbb{S}^{N-1}, \quad \text{or} \quad |\omega(x) + \frac{2}{H_0}| \leq \sigma^{1/2}, \quad \text{for all}\, x \in \mathbb{S}^{N-1}.$$

Combining this with (3.8) and (3.9) yields

$$|\omega(x)| \leq \sigma^{1/2}, \quad \text{for all}\, x \in \mathbb{S}^{N-1}. \tag{3.13}$$

Combining (3.7), (3.10),(3.11) and (3.13) and the fact that $x \cdot \nabla_{\mathbb{S}^{N-1}} \omega(x) = 0$ for all $x \in \mathbb{S}^{N-1}$, we obtain

$$\frac{1}{H_0^2} + \left(\frac{1}{H_0} + \omega(x) \right)^2 - \frac{2}{H_0} \frac{\left(\frac{1}{H_0} + \omega(x) \right)^2}{\sqrt{\left(\frac{1}{H_0} + \omega(x) \right)^2 + |\nabla_{\mathbb{S}^{N-1}} \omega(x)|^2}} \leq \sigma. \tag{3.14}$$

This combined with (3.13) leads to

$$|\nabla_{\mathbb{S}^{N-1}} \omega(x)| \leq C_2 \sigma^{1/4}, \tag{3.15}$$

where C_2 is a positive constant dependent only on H_0. As usual, we can remove the dependence on H_0 recalling Lemma B.3.

[4] In fact, by Lemma B.3 and the definition of H_0, we have that $\kappa \leq \frac{1}{H_0}$, where κ is the constant appearing in the statement of Lemma B.3, which only depends on the $C^{1,\alpha}$ regularity of Γ.

Combining (3.13) and (3.15), (3.3) easily follows by adjusting σ.

□

We are now ready for the

Proof of Theorem 1.1. We set the origin at the center of mass of Ω, that is,

$$\frac{1}{|\Omega|}\int_\Omega x\,dx = 0,$$

which is always true up to a translation.

Start by assuming that $\|H-H_0\|_{L^r(\Gamma)} < \varepsilon_0$, where ε_0 is given by the previous lemma. Then, there exists $\omega : \mathbb{S}^{N-1} \to \mathbb{R}$ such that

$$\Gamma = \left\{ \left(\frac{1}{H_0} + \omega(x)\right) x : x \in \mathbb{S}^{N-1} \right\},$$

$\omega \in C^{k,\alpha}(\mathbb{S}^{N-1}) \cap W^{2,r}(\mathbb{S}^{N-1})$ and $\omega(x) \geq -\frac{1}{H_0}$, for all $x \in \mathbb{S}^{N-1}$. Then, we have that

$$\rho_e(0) - \rho_i(0) \leq 2\|\omega\|_{L^\infty(\mathbb{S}^{N-1})}. \tag{3.16}$$

Let $x_0 \in \mathbb{S}^{N-1}$ be such that $|\omega(x_0)| = \|\omega\|_{L^\infty(\mathbb{S}^{N-1})}$. Assume that $x_0 = (0, ..., 0, -1)$, which, up to rotations, is true. In this way, we have that $\iota(x_0) = 0$, where ι denotes the standard stereographic projection from $P = (0, ..., 0, 1)$ (see (C.1) for an explicit formula). Let R be the constant defined in (2.6) (which depends only on N). Applying Corollary 2.4 (in $\mathbb{R}^{N-1}$) with $D = B_R^{N-1}(0)$, $s = k+\alpha$ and $v = \omega \circ \iota^{-1}$ we obtain

$$\|\omega\|_{L^\infty(\mathbb{S}^{N-1})} = \|\omega\circ\iota^{-1}\|_{L^\infty\left(B_R^{N-1}(0)\right)} \leq C_1 \|\omega\circ\iota^{-1}\|^{1-\theta}_{W^{k+\alpha,\infty}\left(B_R^{N-1}(0)\right)} \|\omega\circ\iota^{-1}\|^{\theta}_{L^{\frac{r(N-1)}{N-1-2r}}\left(B_R^{N-1}(0)\right)}, \tag{3.17}$$

where C_1 only depends[5] on N, k, α and r and where

$$\theta := \frac{k+\alpha}{k+\alpha+\frac{N-1-2r}{r}}. \tag{3.18}$$

By the Sobolev embedding Theorem (see [1, Chapter 4]) we have that

$$\|\omega\circ\iota^{-1}\|_{L^{\frac{r(N-1)}{N-1-2r}}\left(B_R^{N-1}(0)\right)} \leq C_2 \left\|\omega\circ\iota^{-1}\right\|_{W^{2,r}\left(B_R^{N-1}(0)\right)},$$

where C_2 is a positive constant dependent only on N and r. This leads to

$$\|\omega\|_{L^\infty(\mathbb{S}^{N-1})} \leq C_3 \|\omega\circ\iota^{-1}\|^{1-\theta}_{W^{k+\alpha,\infty}\left(B_R^{N-1}(0)\right)} \|\omega\circ\iota^{-1}\|^{\theta}_{W^{2,r}\left(B_R^{N-1}(0)\right)}, \tag{3.19}$$

where $C_3 = C_1 C_2^\theta$.

By [11, Theorem 1.6][6] we know that there exists a positive constant C_4 dependent only N, r and the $C^{1,\alpha}$ regularity of Γ such that

$$\|\omega\|_{W^{2,r}(\mathbb{S}^{N-1})} \leq C_4 \|H-H_0\|_{L^r(\Gamma)}. \tag{3.20}$$

Combining (3.16), (3.19)[7], Lemma 2.5 (with $v = \omega$ and $p = r$) and (3.20), we obtain (1.5)- (1.6), provided that $\|H-H_0\|_{L^r(\Gamma)} < \varepsilon_0$.

If $\|H-H_0\|_{L^r(\Gamma)} \geq \varepsilon_0$, the result trivially follows noting that, for any $z \in \Omega$,

$$\rho_e(z) - \rho_i(z) \leq d_\Omega = \frac{d_\Omega}{\varepsilon_0^\theta}\varepsilon_0^\theta \leq \frac{d_\Omega}{\varepsilon_0^\theta}\|H-H_0\|^\theta_{L^r(\Gamma)},$$

where d_Ω is the diameter of Ω. □

[5]We point out that the dependence of the constant C_1 on $D = B_R^{N-1}(0)$ can be dropped because $B_R^{N-1}(0)$ does not depend on Ω, but it is a ball in $\mathbb{R}^{N-1}$ with radius R which only depends on N.

[6]Note that by (3.1) and (3.3) in Lemma 3.1 we can apply [11, Theorem 1.6].

[7]We point out that $\|\omega\circ\iota^{-1}\|_{W^{k+\alpha,\infty}\left(B_R^{N-1}(0)\right)}$ depends solely on the regularity of Γ. In the case $\alpha = 1$ we point out that since $\omega\circ\iota^{-1} \in C^{k,1}\left(B_R^{N-1}(0)\right)$, then we also have $\omega\circ\iota^{-1} \in W^{k+1,\infty}\left(B_R^{N-1}(0)\right)$ and the $W^{k+1,\infty}$ norm of $\omega\circ\iota^{-1}$ will depend only on the $C^{k,1}$ norm of $\omega\circ\iota^{-1}$, see [9] for additional details.

Proof of Theorem 1.3. We set the origin at the center of mass of Ω, that is,

$$\frac{1}{|\Omega|}\int_\Omega x\,dx = 0,$$

which is always true up to a translation.

Now we assume that $\|H-H_0\|_{L^{\frac{N-1}{2}}(\Gamma)} < \varepsilon_0$, where ε_0 is given by Lemma 3.1. Then there exists $\omega : \mathbb{S}^{N-1} \to \mathbb{R}$ such that

$$\Gamma = \left\{ \left(\frac{1}{H_0} + \omega(x) \right) x : x \in \mathbb{S}^{N-1} \right\},$$

$\omega \in C^{k,\alpha}(\mathbb{S}^{N-1}) \cap W^{2,r}(\mathbb{S}^{N-1})$ and $\omega(x) \geq -\frac{1}{H_0}$, for all $x \in \mathbb{S}^{N-1}$. As a consequence, we have that

$$\rho_e(0) - \rho_i(0) \leq 2\|\omega\|_{L^\infty(\mathbb{S}^{N-1})}. \tag{3.21}$$

Let $x_0 \in \mathbb{S}^{N-1}$ be such that

$$|\omega(x_0)| = \|\omega\|_{L^\infty(\mathbb{S}^{N-1})}.$$

Similarly to what we did in the proof of Theorem 1.1, assume that $x_0 = (0, ..., 0, -1)$, which, up to rotations, is true. In this way, we have $\iota(x_0) = 0$, where ι denotes the standard stereographic projection from $P = (0, ..., 0, 1)$ (see (C.1) for an explicit formula). Let R be the constant defined in (2.6) (which only depends on N). Then, applying [21, Lemma 2.1] and [21, Lemma 2.5 (ii)] (in $\mathbb{R}^{N-1}$) with $f = \omega \circ \iota^{-1}$, $p = N-1$, $q = \infty$ and $\mathcal{C}$ a cone in $\mathbb{R}^{N-1}$ with vertex $0 \in \mathbb{R}^{N-1}$, opening width $\pi/4$ and height $R/2$ we obtain the following

$$\left|\omega \circ \iota^{-1}(0) - (\omega \circ \iota^{-1})_{\mathcal{C}}\right| \leq \frac{R}{N|\mathcal{C}|^{1/(N-1)}} \left\|\nabla\left(\omega \circ \iota^{-1}\right)\right\|_{L^{N-1}(\mathcal{C})} \log\left(e\, \frac{|\mathcal{C}|^{\frac{1}{N-1}} \left\|\nabla\left(\omega \circ \iota^{-1}\right)\right\|_{L^\infty(\mathcal{C})}}{\left\|\nabla\left(\omega \circ \iota^{-1}\right)\right\|_{L^{N-1}(\mathcal{C})}} \right),$$

where $(\omega \circ \iota^{-1})_{\mathcal{C}} = \frac{1}{|\mathcal{C}|}\int_{\mathcal{C}} \omega \circ \iota^{-1}(y)\,dy$.

By the triangle inequality, and using that $\omega \circ \iota^{-1}(0) = \omega(x_0) = \|\omega\|_{L^\infty(\mathbb{S}^{N-1})}$, we thus have that

$$\|\omega\|_{L^\infty(\mathbb{S}^{N-1})} \leq |(\omega \circ \iota^{-1})_{\mathcal{C}}| + \frac{R}{N|\mathcal{C}|^{1/(N-1)}} \left\|\nabla\left(\omega \circ \iota^{-1}\right)\right\|_{L^{N-1}(\mathcal{C})} \log\left(e\, \frac{|\mathcal{C}|^{\frac{1}{N-1}} \left\|\nabla\left(\omega \circ \iota^{-1}\right)\right\|_{L^\infty(\mathcal{C})}}{\left\|\nabla\left(\omega \circ \iota^{-1}\right)\right\|_{L^{N-1}(\mathcal{C})}} \right).$$

Using that $\mathcal{C} \subset B_R^{N-1}(0)$, the monotonicity, for any $A > 0$ and $t > 0$, of the function $t \to t\max\{\log(A/t), 1\}$, and the fact that $(\omega \circ \iota^{-1})_{\mathcal{C}} \leq \frac{1}{|\mathcal{C}|^{1/(N-1)}} \left\|\omega \circ \iota^{-1}\right\|_{L^{N-1}\left(B_R^{N-1}(0)\right)}$, we readily obtain

$$\|\omega\|_{L^\infty(\mathbb{S}^{N-1})} \leq C_1 \|\omega \circ \iota^{-1}\|_{W^{1,N-1}\left(B_R^{N-1}(0)\right)} \max\left\{ \log\left(e\, \frac{|\mathcal{C}|^{\frac{1}{N-1}} \left\|\nabla\left(\omega \circ \iota^{-1}\right)\right\|_{L^\infty\left(B_R^{N-1}(0)\right)}}{\|\omega \circ \iota^{-1}\|_{W^{1,N-1}\left(B_R^{N-1}(0)\right)}} \right), 1 \right\}, \tag{3.22}$$

where C_1 is a positive constant dependent only on N (recall that $|\mathcal{C}|$ depends only on N, by construction).

Set $\ell := \frac{\nabla\left(\omega \circ \iota^{-1}\right)(x_M)}{|\nabla(\omega \circ \iota^{-1})(x_M)|}$ where $x_M \in \overline{B_R^{N-1}(0)}$ is a point where $|\nabla\left(\omega \circ \iota^{-1}\right)|$ attains its maximum in $\overline{B_R^{N-1}(0)}$; applying Theorem 2.3 (in $\mathbb{R}^{N-1}$) with $D = B_R^{N-1}(0)$, $v = \langle \nabla\left(\omega \circ \iota^{-1}\right), \ell\rangle$, $p = \infty$ and $q = N-1$ gives that, for any $s > 0$,

$$\left\|\nabla\left(\omega \circ \iota^{-1}\right)\right\|_{L^\infty\left(B_R^{N-1}(0)\right)} \leq C_2 \left\|\nabla\left(\omega \circ \iota^{-1}\right)\right\|_{W^{s,\infty}\left(B_R^{N-1}(0)\right)}^{\frac{1}{s+1}} \left\|\nabla\left(\omega \circ \iota^{-1}\right)\right\|_{L^{N-1}\left(B_R^{N-1}(0)\right)}^{\frac{s}{s+1}},$$

and hence

$$\left\|\nabla\left(\omega \circ \iota^{-1}\right)\right\|_{L^\infty\left(B_R^{N-1}(0)\right)} \leq C_2 \left\|\omega \circ \iota^{-1}\right\|_{W^{s+1,\infty}\left(B_R^{N-1}(0)\right)}^{\frac{1}{s+1}} \left\|\omega \circ \iota^{-1}\right\|_{W^{1,N-1}\left(B_R^{N-1}(0)\right)}^{\frac{s}{s+1}}, \tag{3.23}$$

where the constant C_2 depends[8] only on s and N.

Combining (3.21), (3.22) and (3.23) easily leads to

$$\rho_e(0) - \rho_i(0) \leq C_3 \left\|\omega \circ \iota^{-1}\right\|_{W^{1,N-1}\left(B_R^{N-1}(0)\right)} \max\left\{ \frac{1}{s+1} \log\left(\frac{\left\|\left(\omega \circ \iota^{-1}\right)\right\|_{W^{s+1,\infty}\left(B_R^{N-1}(0)\right)}}{\|\omega \circ \iota^{-1}\|_{W^{1,N-1}\left(B_R^{N-1}(0)\right)}} \right), 1 \right\}, \tag{3.24}$$

where the constant C_3 depends only on s and N.

[8]As already noticed, we can drop the dependence on $D = B_R^{N-1}(0)$.

Applying the Sobolev embedding Theorem (see [1, Chapter 4]), Lemma 2.5 and [11, Theorem 1.6][9], we obtain

$$\|\omega\circ\iota^{-1}\|_{W^{1,N-1}\left(B_R^{N-1}(0)\right)} \leq C_4\|\omega\circ\iota^{-1}\|_{W^{2,\frac{N-1}{2}}(\mathbb{S}^{N-1})} \leq C_5\|H-H_0\|_{L^{\frac{N-1}{2}}(\Gamma)}, \tag{3.25}$$

where C_4 is a positive constant dependent only on N, and C_5 is a positive constant dependent on N and the $C^{1,\alpha}$ regularity of Γ.

So, combining (3.24), (3.25) and the monotonicity, for any $A>0$ and $t>0$, of the function $t\mapsto t\max\{\log(A/t),1\}$, we obtain

$$\rho_e(0)-\rho_i(0)\leq C_6\|H_0-H\|_{L^{\frac{N-1}{2}}(\Gamma)}\max\left\{\frac{1}{s+1}\log\left(\frac{\|\omega\circ\iota^{-1}\|_{W^{s+1,\infty}\left(B_R^{N-1}(0)\right)}}{\|H_0-H\|_{L^{\frac{N-1}{2}}(\Gamma)}}\right),1\right\}, \tag{3.26}$$

where $s>0$, and the constant C_6 only depends on s, N and the $C^{1,\alpha}$-regularity of Γ.

Finally, we apply (3.26) with $s+1=k+\alpha$ to obtain (1.7), provided that $\|H-H_0\|_{L^{\frac{N-1}{2}}(\Gamma)}<\varepsilon_0$. This finishes the proof under the assumption that $\|H-H_0\|_{L^{\frac{N-1}{2}}(\Gamma)}<\varepsilon_0$.

If $\|H-H_0\|_{L^{\frac{N-1}{2}}(\Gamma)}\geq\varepsilon_0$, then it is clear that (1.7) still holds since, for any $z\in\Omega$,

$$\rho_e(z)-\rho_i(z)\leq d_\Omega\leq\frac{d_\Omega}{\varepsilon_0}\|H-H_0\|_{L^{\frac{N-1}{2}}(\Gamma)},$$

where d_Ω denotes the diameter of Ω. □

We finalise this section with the proof of Theorem 1.5.

Proof of Theorem 1.5. We start by noting that the only case left to prove is the one where $r>\frac{N-1}{2}$. Indeed, if $N\leq 3$ then we have $r>1>\frac{N-1}{2}$. If $N\geq 4$, the case $r\leq\frac{N-1}{2}$ has already been covered by Theorems 1.1 and 1.3, so that we are left with the case $r>\frac{N-1}{2}$.

Now let $r>\frac{N-1}{2}$. Similarly to what we did in the previous proofs, we set the origin at the center of mass of Ω, that is,

$$\frac{1}{|\Omega|}\int_\Omega x\,dx=0,$$

which is always true up to a translation. We also assume that $\|H-H_0\|_{L^r(\Gamma)}<\varepsilon_0$, where ε_0 is given by Lemma 3.1. Then, there exists $\omega:\mathbb{S}^{N-1}\to\mathbb{R}$ such that

$$\Gamma=\left\{\left(\frac{1}{H_0}+\omega(x)\right)x : x\in\mathbb{S}^{N-1}\right\},$$

$\omega\in C^{k,\alpha}(\mathbb{S}^{N-1})\cap W^{2,r}(\mathbb{S}^{N-1})$ and $\omega(x)\geq-\frac{1}{H_0}$. Then, we have that

$$\rho_e(0)-\rho_i(0)\leq 2\|\omega\|_{L^\infty(\mathbb{S}^{N-1})}. \tag{3.27}$$

Let $x_0\in\mathbb{S}^{N-1}$ be such that $|\omega(x_0)|=\|\omega\|_{L^\infty(\mathbb{S}^{N-1})}$. Assume that $x_0=(0,...,0,-1)$, which, up to rotations, is true. In this way, we have that $\iota(x_0)=0$, where ι denotes the standard stereographic projection from $P=(0,...,0,1)$ (see (C.1) for an explicit formula). Let R be the constant defined in (2.6) (which depends only on N).

Since $r>\frac{N-1}{2}$, we can apply the Sobolev embedding Theorem (see [1, Chapter 4]) to conclude that there exists a positive constant C_1 dependent only on N and r, such that

$$\|\omega\|_{L^\infty(\mathbb{S}^{N-1})}=\|\omega\circ\iota^{-1}\|_{L^\infty\left(B_R^{N-1}(0)\right)}\leq C_1\|\omega\circ\iota^{-1}\|_{W^{2,r}\left(B_R^{N-1}(0)\right)}.$$

This, combined with (3.27), Lemma 2.5 and [11, Theorem 1.6], leads to

$$\rho_e(0)-\rho_i(0)\leq C_2\|H-H_0\|_{L^r(\Gamma)},$$

where C_2 is a positive constant dependent only N, r and the $C^{k,\alpha}$ regularity of Γ. This concludes the proof of (1.8) under the assumption that $\|H-H_0\|_{L^r(\Gamma)}<\varepsilon_0$.

If $\|H-H_0\|_{L^r(\Gamma)}\geq\varepsilon_0$, then (1.8) still holds since, for any $z\in\Omega$,

$$\rho_e(z)-\rho_i(z)\leq\frac{d_\Omega}{\varepsilon_0}\|H-H_0\|_{L^r(\Gamma)},$$

where d_Ω denotes the diameter of Ω. □

[9] As before, we note that by (3.1) and (3.3) in Lemma 3.1 we can apply [11, Theorem 1.6].

4. Optimality of Theorem 1.1 with $k+\alpha \geq 2$

In this section, we show the optimality of Theorem 1.1 for $C^{k,\alpha}$ domains, when $k+\alpha \geq 2$ ($0<\alpha\leq 1$). Since the estimates to check the optimality of Theorems 1.1 are slightly different and more technical for $C^{1,\alpha}$ ($\alpha \in (0,1)$) domains, we leave the $C^{1,\alpha}$ case for the Appendix (see Section A).

Throughout this section, B_r^{N-1} denotes the $(N-1)$-dimensional ball of radius r with center at the origin and orthogonal to $e_N = (0,...,0,1) \in \mathbb{R}^N$. Let $N \geq 4$, $k \geq 1$ be nonnegative integers and $\alpha \in (0,1]$ be, such that $k+\alpha \geq 2$ (if $k=1$, the only option is $\alpha = 1$).

Define $\Gamma_t \subset \mathbb{R}^N$ (for $t>0$) by setting

$$\Gamma_t \cap \left(B_1^{N-1} \times (0,+\infty)\right) = \{(x, \varphi_t(x)) : x \in B_1^{N-1}\},$$
$$\Gamma_t \setminus \left(B_1^{N-1} \times (0,+\infty)\right) = \mathbb{S}^{N-1} \setminus \left(B_1^{N-1} \times (0,+\infty)\right),$$

where $\varphi_t : B_1^{N-1} \to (0,\infty)$ is a function of class $C^{k,\alpha}$.
Now we define φ_t. First, let $\varphi_0 : B_1^{N-1} \to \mathbb{R}$ be given by

$$\varphi_0(x) = \sqrt{1-|x|^2}.$$

Take $\psi \in C_c^\infty(\mathbb{R}^{N-1})$ to be a nonnegative compactly supported smooth function such that

$$\mathrm{supp}(\psi) \subset B_1^{N-1},\ 0 \leq \psi_t \leq 1 \text{ and } \psi_{|_{B_{1/2}^{N-1}}} \equiv 1.$$

Given $t>0$, we define $\psi_t(x) := \psi\left(\frac{x}{t}\right)$. Finally, for $0 < t \leq t_1$ (where $t_1 := t_1(k,N) < 1$ will be set later), we define $\varphi_t : B_1^{N-1} \to \mathbb{R}$ by setting

$$\varphi_t(x) = \varphi_0(x) + \psi_t(x) \sum_{i=1}^{N-1} |x_i|^{k+\alpha}. \tag{4.1}$$

Let $\Omega_t \subset \mathbb{R}^N$ be the only bounded, open, connected set such that $\partial\Omega_t = \Gamma_t$. Note that, by construction, the regularity of Γ_t matches the regularity of φ_t, which is clearly $C^{k,\alpha}$. Furthermore, the family of functions $\{\varphi_t\}$ is in $C^{k,\alpha}$ but not in $C^{k,\alpha'}$ for any $\alpha' > \alpha$. To simplify the presentation, we write $\Psi_t(x) := \varphi_t(x) - \varphi_0(x)$.

To check the optimality of Theorem 1.1 we need some estimates regarding Ψ_t, $|\nabla\Psi_t|$ and $|D^2\Psi_t|$. From now on, we assume that

$$t_1 \leq \frac{1}{10(N-1)(k+1)}.$$

Note that since $\mathrm{supp}\psi_t \subset B_t^{N-1}$ and $0 \leq \psi_t \leq 1$, we have

$$|\Psi_t(x)| \leq \sum_{i=1}^{N-1} |x_i|^{k+\alpha} \leq (N-1)t^{k+\alpha} \tag{4.2}$$

Using the fact that $t \leq t_1$, we have the uniform (in t) bound

$$|\Psi_t(x)| \leq \frac{1}{2}. \tag{4.3}$$

Regarding $|\nabla\Psi_t|$ we have

$$\begin{aligned} |\nabla\Psi_t(x)| &\leq \left|\psi_t(x)\nabla\left(\sum_{i=1}^{N-1}|x_i|^{k+\alpha}\right) + \left(\sum_{i=1}^{N-1}|x_i|^{k+\alpha}\right)\nabla\psi_t(x)\right| \\ &\leq (k+\alpha)\sum_{i=1}^{N-1}|x_i|^{k+\alpha-1} + \frac{\|\nabla\psi\|_{L^\infty(B_1^{N-1})}}{t}\sum_{i=1}^{N-1}|x_i|^{k+\alpha} \\ &\leq (N-1)\left(k+\alpha+\|\nabla\psi\|_{L^\infty\left(B_1^{N-1}\right)}\right)t^{k+\alpha-1}. \end{aligned} \tag{4.4}$$

Here, in the third inequality, we used the fact that $x \in B_t^{N-1}$.

The same argument leads to

$$\left|D^2\Psi_k(x)\right| \leq C_1 t^{k+\alpha-2}, \tag{4.5}$$

with C_1 being a positive constant dependent only on N, k, α and $\|\psi\|_{C^2(\mathbb{R}^{N-1})}$.

From now on, whenever we make use of (4.4) or (4.5) we will omit the dependence on $\|\psi\|_{C^2(\mathbb{R}^{N-1})}$ since it is a fixed quantity that does not change with any of the relevant parameters.
Now let $z_t \in \Omega_t$ be such that

$$\inf_{z\in\Omega_t} (\rho_e(z) - \rho_i(z)) = \rho_e(z_t) - \rho_i(z_t).$$

Simple geometric considerations imply that:

$$2\rho_e(z_t) \geq d_{\Omega_t} \geq 1 + \sqrt{|x_t|^2 + \psi_t(x_t)^2} \text{ and } \rho_i(z_t) \leq 1,$$

where by d_{Ω_t} we mean the diameter of Ω_t, and where $x_t \in B_1^{N-1}$ is such that $\max_{x\in B_1^{N-1}} \Psi_t(x) = \Psi_t(x_t)$. So, the following chain holds:

$$\begin{aligned} 2\rho_e(z_t) - 2\rho_i(z_t) &\geq 1 + \sqrt{|x_t|^2 + \psi_t(x_t)^2} - 2 = \sqrt{|x_t|^2 + \psi_t(x_t)^2} - 1 \\ &= \frac{|x_t|^2 + \varphi_t(x_t)^2 - 1}{\sqrt{|x_t|^2 + \varphi_t(x_t)^2} + 1} = \frac{2\varphi_0(x_t)\Psi_t(x_t) + \Psi_t(x_t)^2}{\sqrt{1 + 2\varphi_0(x_t)\Psi_t(x_t) + \Psi_t(x_t)^2} + 1} \\ &\geq \frac{2}{3}\Psi_t(x_t), \end{aligned}$$

where we used (4.3) and $\Psi_t \geq 0$. Hence, recalling (4.1) and that $\Psi_t = \varphi_t - \varphi_0$, we obtain

$$\rho_e(z_t) - \rho_i(z_t) \geq \frac{2}{3}\frac{t^{k+\alpha}}{2^{k+\alpha}}. \tag{4.6}$$

Now, all that is left for us to do is study the behaviour of the L^2-deviation of the mean curvature of the sets Ω_t from the reference constant $H_0 := \frac{|\Gamma_t|}{N|\Omega_t|}$ as a function of t.

To do this, we break $\|H_t - H_0\|_{L^r(\Gamma_t)}$ into three terms:

$$\begin{aligned} \|H_t - H_0\|_{L^r(\Gamma_t)} &\leq \|H_t - 1\|_{L^r(\Gamma_t)} + |\Gamma_t|^{\frac{1}{r}}|H_0 - 1| \\ &\leq \|H_t - 1\|_{L^r(\Gamma_t)} + |\Gamma_t|^{\frac{1}{r}}\frac{\big||\Gamma_t| - |\mathbb{S}^{N-1}|\big|}{N|\Omega_t|} + \frac{|\Gamma_t|^{\frac{1}{r}}\,|\mathbb{S}^{N-1}|}{N|B_1||\Omega_t|}\big||\Omega_t| - |B_1|\big| \\ &\leq \|H_t - 1\|_{L^r(\Gamma_t)} + C_2\big||\Gamma_t| - |\mathbb{S}^{N-1}|\big| + C_2\big||\Omega_t| - |B_1|\big|, \end{aligned} \tag{4.7}$$

where C_2 is a positive constant dependent on N and r, and where we have used the fact that because $t \leq t_1(k,N) \leq \frac{1}{10(N-1)(k+1)}$, we have $|\Omega_t| \geq \frac{|B_1|}{2}$ and $|\Gamma_t| \leq 2|\mathbb{S}^{N-1}|$.

Now, we estimate $\|H_t - 1\|_{L^r(\Gamma_t)}$, $\big||\Gamma_t| - |\mathbb{S}^{N-1}|\big|$ and $\big||\Omega_t| - |B_1|\big|$, individually.

First, for the deviation of the measures, using (4.2) we see that there exists a positive constant C_9 dependent only on N, such that:

$$\big||\Omega_t| - |B_1|\big| = \left|\int_{B_1^{N-1}} (\varphi_t - \varphi_0)\,dx\right| \leq \int_{B_t^{N-1}} |\Psi_t(x)|dx \leq \left|\mathbb{S}^{N-2}\right|\, t^{k+\alpha+N-1}. \tag{4.8}$$

Second, the deviation of the perimeters

$$\begin{aligned} \big||\Gamma_t| - |\mathbb{S}^{N-1}|\big| &= \left|\int_{B_1^{N-1}} \left(\sqrt{1+|\nabla\varphi_t|^2} - \sqrt{1+|\nabla\varphi_0|^2}\right)dx\right| \\ &= \left|\int_{B_1^{N-1}} \frac{|\nabla\varphi_t|^2 - |\nabla\varphi_0|^2}{\sqrt{1+|\nabla\varphi_t|^2} + \sqrt{1+|\nabla\varphi_0|^2}}\,dx\right| \leq \int_{B_1^{N-1}} \left||\nabla\varphi_t|^2 - |\nabla\varphi_0|^2\right| dx \\ &= \int_{B_1^{N-1}} \left||\nabla\Psi_t|^2 + 2\langle\nabla\Psi_t, \nabla\varphi_0\rangle\right| dx \leq \int_{B_t^{N-1}} |\nabla\Psi_t|^2 + 2|\nabla\Psi_t||\nabla\varphi_0|\,dx \\ &\leq C_3 t^{k+\alpha+N-1}, \end{aligned} \tag{4.9}$$

where C_3 is a positive constant dependent only on N, and where we have used (4.4) and the fact that $|\nabla\varphi_0(x)| \leq 2t$ on B_t^{N-1}.

Before estimating $\|H_t - 1\|_{L^r(\Gamma_t)}$ we first give a pointwise estimate on $|H_t - 1|$. For that, note that, by construction, we have $H_t(x) = 1$ for all $x \in \Gamma_t \setminus \left(B_t^{N-1} \times (0,+\infty)\right)$. Therefore, to estimate the deviation of the mean curvature from 1 we only need to look at the mean curvature of the portion of Γ_t inside $B_t^{N-1} \times (0,+\infty)$. To that end, recall that for the sets under consideration, the mean curvature of the portion of Γ_t inside $B_t^{N-1} \times (0,+\infty)$ is given by the formula (see, for instance, [12, Chapter 14]):

$$H_t(x,\varphi_t(x)) = -\frac{\Delta\varphi_t(x)}{(N-1)(1+|\nabla\varphi_t(x)|^2)^{1/2}} + \frac{(\nabla\varphi_t(x))^T D^2\varphi_t(x)\nabla\varphi_t(x)}{(N-1)(1+|\nabla\varphi_t(x)|^2)^{3/2}}.$$

Expanding the formula above for the mean curvature, we have (we omit the dependence on the $x-$variable to simplify the presentation)

$$
\begin{aligned}
&(N-1)\left(H_t(x,\varphi_t(x))-1\right)=-\frac{\Delta\varphi_0+\Delta\Psi_t}{(1+|D\varphi_t|^2)^{1/2}}+\frac{(\nabla\varphi_0+\nabla\Psi_t)^T\left(D^2\varphi_0+D^2\Psi_t\right)(\nabla\varphi_0+\nabla\Psi_t)}{(1+|\nabla\varphi_t|^2)^{3/2}}\\
&+\frac{\Delta\varphi_0}{(1+|\nabla\varphi_0|^2)^{1/2}}-\frac{\nabla\varphi_0^TD^2\varphi_0\nabla\varphi_0}{(1+|\nabla\varphi_0|^2)^{3/2}}\\
&=-\frac{\Delta\varphi_0+\Delta\Psi_t}{(1+|\nabla\varphi_t|^2)^{1/2}}+\frac{\nabla\varphi_0^TD^2\varphi_0\nabla\varphi_0}{(1+|\nabla\varphi_t|^2)^{3/2}}+\frac{\nabla\varphi_0^TD^2\varphi_0\nabla\Psi_t}{(1+|\nabla\varphi_t|^2)^{3/2}}+\frac{\nabla\varphi_0^TD^2\Psi_t\nabla\varphi_0}{(1+|\nabla\varphi_t|^2)^{3/2}}\\
&+\frac{\nabla\varphi_0^TD^2\Psi_t\nabla\Psi_t}{(1+|\nabla\varphi_t|^2)^{3/2}}+\frac{\nabla\Psi_t^TD^2\varphi_0\nabla\varphi_0}{(1+|\nabla\varphi_t|^2)^{3/2}}+\frac{\nabla\Psi_t^TD^2\varphi_0\nabla\Psi_t}{(1+|\nabla\varphi_t|^2)^{3/2}}+\frac{\nabla\Psi_t^TD^2\Psi_t\nabla\varphi_0}{(1+|\nabla\varphi_t|^2)^{3/2}}\\
&+\frac{\nabla\Psi_t^TD^2\Psi_t\nabla\Psi_t}{(1+|\nabla\varphi_t|^2)^{3/2}}+\frac{\Delta\varphi_0}{(1+|\nabla\varphi_0|^2)^{1/2}}-\frac{\nabla\varphi_0^TD^2\varphi_0\nabla\varphi_0}{(1+|\nabla\varphi_0|^2)^{3/2}}\\
&=\Big((1+|\nabla\varphi_0|^2)^{-1/2}-(1+|\nabla\varphi_t|^2)^{-1/2}\Big)\Delta\varphi_0\\
&+\Big((1+|\nabla\varphi_t|^2)^{-\frac32}-(1+|\nabla\varphi_0|^2)^{-\frac32}\Big)\nabla\varphi_0^TD^2\varphi_0\nabla\varphi_0\\
&-\frac{\Delta\Psi_t}{(1+|\nabla\varphi_t|^2)^{1/2}}+2\frac{\nabla\varphi_0^TD^2\varphi_0\nabla\Psi_t}{(1+|\nabla\varphi_t|^2)^{3/2}}+\frac{\nabla\varphi_0^TD^2\Psi_t\nabla\varphi_0}{(1+|\nabla\varphi_t|^2)^{3/2}}\\
&+2\frac{\nabla\varphi_0^TD^2\Psi_t\nabla\Psi_t}{(1+|\nabla\varphi_t|^2)^{3/2}}+\frac{\nabla\Psi_t^TD^2\varphi_0\nabla\Psi_t}{(1+|\nabla\varphi_t|^2)^{3/2}}+\frac{\nabla\Psi_t^TD^2\Psi_t\nabla\Psi_t}{(1+|\nabla\varphi_t|^2)^{3/2}}.
\end{aligned}
\tag{4.10}
$$

Note that for $j=1,3$, we have

$$
\begin{aligned}
\left|(1+|\nabla\varphi_t|^2)^{-j/2}-(1+|\nabla\varphi_0|^2)^{-j/2}\right|&=\left|\int_{|\nabla\varphi_t|}^{|\nabla\varphi_0|}\frac{js}{(1+s^2)^{\frac{j}{2}+1}}\,ds\right|\le 2\left||\nabla\varphi_0|^2-|\nabla\varphi_t|^2\right|\\
&=2\left||\nabla\Psi_t|^2+2\langle\nabla\Psi_t,\nabla\varphi_0\rangle\right|\le 2|\nabla\Psi_t|^2+4|\nabla\Psi_t||\nabla\varphi_0|.
\end{aligned}
\tag{4.11}
$$

Recall the formulas for $\nabla\varphi_0$ and $D^2\varphi_0$:

$$\nabla\varphi_0(x)=-\frac{x}{\sqrt{1-|x|^2}},$$

and

$$\left(D^2\varphi_0(x)\right)_{ij}=-\frac{1}{\sqrt{1-|x|^2}}\delta_{ij}+\frac{1}{(1-|x|^2)^{3/2}}x_ix_j=-\frac{1}{\varphi_0(x)}\delta_{ij}+\frac{1}{\varphi_0(x)^3}x_ix_j,$$

We see that for $t\le t_1$,

$$|\nabla\varphi_0(x)|\le 2|x|,\text{ and }|D^2\varphi_0(x)|\le 3,\quad\text{for all }x\in B_t^{N-1}.\tag{4.12}$$

From (4.10) we obtain

$$
\begin{aligned}
\left|(N-1)(H_t-1)+\frac{\Delta\Psi_t}{(1+|\nabla\varphi_t|^2)^{1/2}}\right|&\le\left|(1+|\nabla\varphi_0|^2)^{-1/2}-(1+|\nabla\varphi_t|^2)^{-1/2}\right||\Delta\varphi_0|\\
&+\left|(1+|\nabla\varphi_t|^2)^{-\frac32}-(1+|\nabla\varphi_0|^2)^{-\frac32}\right|\left|\nabla\varphi_0^TD^2\varphi_0\nabla\varphi_0\right|\\
&+2\left|\frac{\nabla\varphi_0^TD^2\varphi_0\nabla\Psi_t}{(1+|\nabla\varphi_t|^2)^{3/2}}\right|+\left|\frac{\nabla\varphi_0^TD^2\Psi_t\nabla\varphi_0}{(1+|\nabla\varphi_t|^2)^{3/2}}\right|\\
&+2\left|\frac{\nabla\varphi_0^TD^2\Psi_t\nabla\Psi_t}{N(1+|\nabla\varphi_t|^2)^{3/2}}\right|+\left|\frac{\nabla\Psi_t^TD^2\varphi_0\nabla\Psi_t}{(1+|\nabla\varphi_t|^2)^{3/2}}\right|+\left|\frac{\nabla\Psi_t^TD^2\Psi_t\nabla\Psi_t}{(1+|\nabla\varphi_t|^2)^{3/2}}\right|\\
&\le 2\left(|\nabla\Psi_t|^2+|\nabla\Psi_t||\nabla\varphi_0|\right)\left(|\Delta\varphi_0|+|\nabla\varphi_0|^2|D^2\varphi_0|\right)\\
&+|\nabla\varphi_0||D^2\varphi_0||\nabla\Psi_t|+|\nabla\varphi_0|^2|D^2\Psi_t|+|\nabla\varphi_0||D^2\Psi_t||\nabla\Psi_t|\\
&+|\nabla\Psi_t|^2|D^2\varphi_0|+|\nabla\Psi_t|^2|D^2\Psi_t|\\
&\le C_4\left(\left(t^{2(k+\alpha))-2}+t^{k+\alpha}\right)(1+t^2)+t^{k+\alpha}+t^{k+\alpha+1}\right)\\
&+C_4\left(t^{3(k+\alpha))-2}+t^{2(k+\alpha))-2}+t^{3(k+\alpha))-4}\right)\\
&\le 9C_4t^{k+\alpha},
\end{aligned}
\tag{4.13}
$$

where C_4 is a positive constant dependent on N, k and α. Here, in the first inequality, we used (4.10). In the second inequality, we used (4.11). In the third inequality, we used (4.12), (4.4) and (4.5) (note that $x \in B_t^{N-1}$). Finally, for the fourth inequality, we used $t < 1$ together with $k \geq 2$. So, there is a positive constant C_5 dependent only on N, k and α such that

$$|H_t - 1| \leq |\Delta\Psi_t| + C_5 t^{k+\alpha}.$$

Using this in combination with (4.4), (4.5) and recalling that $t \leq t_1$, we see that

$$\begin{aligned}\|H_t - 1\|^r_{L^r(\Gamma_t)} &\leq \int_{B_t^{N-1}} \left(|\Delta\Psi_t| + C_5 t^{k+\alpha}\right)^r \sqrt{1 + |\nabla\varphi_t(x)|^2}\, dx \\ &\leq C_6 \left(1+t^2\right)^r t^{r(k+\alpha-2)+N-1} \\ &\leq 2C_6\, t^{r(k+\alpha)-2r+N-1},\end{aligned}$$

for some positive constant C_6 dependent only on N.

Combining this with (4.7), (4.8) and (4.9), we conclude (for $t \leq t_1$) that

$$\|H_t - H_0\|_{L^r(\Gamma_t)} \leq C_7 t^{k+\alpha+\frac{N-1-2r}{r}}, \tag{4.14}$$

for some positive constant C_7 that depend solely on N, k, α and r.

To check the optimality of Theorem 1.1 we combine (4.6) and (4.14) to obtain

$$\|H_t - H_0\|^{\tau_{k,\alpha,N,r}}_{L^r(\Gamma_t)} \leq C_7^{\tau_{k,\alpha,N,r}} t^{\left(k+\alpha+\frac{N-1-2r}{r}\right)\tau_{k,\alpha,N,r}} \leq C_7^{\tau_{k,\alpha,N,r}} (\rho_e(t) - \rho_i(t)),$$

where $\tau_{k,\alpha,N,r}$ is given by (1.6). Thus, the optimality of Theorem 1.1 is confirmed.

Remark 4.1. We point out that, using (4.13), one can also find a constant C_8 dependent only on N, k, α and p, such that

$$\|H_t - 1\|_{L^r(\Gamma_t)} \geq \frac{C_8}{2^{k+\alpha}} t^{k+\alpha+\frac{N-1-2r}{r}},$$

for $t \leq t_1$, provided that t_1 is sufficiently small.

Then, noting that, by the inverse triangle inequality, we also have

$$\|H_t - H_0\|_{L^r(\Gamma_t)} \geq \|H_t - 1\|_{L^r(\Gamma_t)} - |H_0 - 1| \geq \|H_t - 1\|_{L^r(\Gamma_t)} - C_2\left||\Gamma_t| - \left|\mathbb{S}^{N-1}\right|\right| - C_2\left||\Omega_t| - |B_1|\right|,$$

where C_2 is the constant in (4.7). We also see (for $t \leq t_1$) that

$$\|H_t - H_0\|_{L^r(\Gamma_t)} \geq \frac{C_9}{2^{k+\alpha}} t^{k+\alpha+\frac{N-1-2r}{r}},$$

for some constant C_9 dependent only on N, k, α and r. This shows that the asymptotic behaviour of $\|H_t - H_0\|_{L^r(\Gamma_t)}$, as $t \to 0^+$, is the same (up to constants dependent on N, k, α and r) as $t^{k+\alpha+\frac{N-1-2r}{r}}$.

Appendix A. Optimality of Theorem 1.1: The $C^{1,\alpha}$ case

We now show how to adapt the estimates in Section 4 to show the optimality of Theorem 1.1 in the case where the domain is of class $C^{1,\alpha}$ with $\alpha \in (0,1)$. In fact, we will take $\alpha \in \left(\frac{r-1}{r}, 1\right)$ for a (technical) reason that will be made clear later.

To do this, we make the same construction as in Section 4, where in (4.1) we set $k = 1$.
Note that, as in Section 4, for $t < 1$, the $C^{1,\alpha}$ regularity of the family of sets $\{\Gamma_t\}$ is uniformly bounded. Furthermore, for $t < t_1$ (where $t_1 := t_1(N) \leq \frac{1}{10(N-1)}$), (4.2), (4.3) and (4.4) still hold. Consequently, (4.6) and (4.8) remain valid. However, (4.9) needs to be updated as follows:

$$\begin{aligned}\left||\Gamma_t| - \left|\mathbb{S}^{N-1}\right|\right| &\leq \int_{B_t^{N-1}} |\nabla\Psi_t|^2 + 2|\nabla\Psi_t||\nabla\varphi_0|\, dx \\ &\leq C_1 t^{2\alpha+N-1},\end{aligned} \tag{A.1}$$

where C_1 is a positive constant dependent on N. Here, in the first inequality, we proceeded as in the first three lines of (4.9) and for the second inequality, we used (4.4) (with $k = 1$) together with $\alpha < 1$.

Another difference that one encounters when adapting the estimates in Section 4 to the $C^{1,\alpha}$ case is when we need to take into account second-order terms. This is because, when taking two derivatives of Ψ_t, singularities appear. For this reason, to avoid the points where singularities appear, we will do our (pointwise) estimates for $x \in \left\{x \in B_{4t}^{N-1} : \forall i \in \{1, ..., N-1\}, x_i \neq 0\right\}$. Indeed, when estimating a pointwise upper-bound for $|D^2\Psi_t(x)|$ ($x \in \left\{x \in B_{4t}^{N-1} : \forall i \in \{1, ..., N-1\},\, x_i \neq 0\right\}$), we now obtain:

$$|D^2\Psi_t(x)| \leq C_2 \sum_{i=1}^{N-1} |x_i|^{\alpha-1}, \tag{A.2}$$

where C_2 is a positive constant dependent only on N.
By proceeding just as in Section 4 we obtain:

$$\begin{aligned}\left|(N-1)(H_t-1)+\frac{\Delta\Psi_t}{(1+|\nabla\varphi_t|^2)^{1/2}}\right| &\le 2\left(|\nabla\Psi_t|^2+|\nabla\Psi_t||\nabla\varphi_0|\right)\left(|\Delta\varphi_0|+|\nabla\varphi_0|^2|D^2\varphi_0|\right)\\ &+|\nabla\varphi_0||D^2\varphi_0||\nabla\Psi_t|+|\nabla\varphi_0|^2|D^2\Psi_t|+|\nabla\varphi_0||D^2\Psi_t||\nabla\Psi_t|\\ &+|\nabla\Psi_t|^2|D^2\varphi_0|+|\nabla\Psi_t|^2|D^2\Psi_t|\\ &=2\left(|\nabla\Psi_t|^2+|\nabla\Psi_t||\nabla\varphi_0|\right)\left(|\Delta\varphi_0|+|\nabla\varphi_0|^2|D^2\varphi_0|\right)\\ &+|\nabla\varphi_0||D^2\varphi_0||\nabla\Psi_t|+|\nabla\Psi_t|^2|D^2\varphi_0|\\ &+\left(|\nabla\varphi_0|^2+|\nabla\varphi_0||\nabla\Psi_t|+|\nabla\Psi_t|^2\right)\left|D^2\Psi_t\right|\\ &\le C_3\left(t^{2\alpha}+t^{1+\alpha}\right)(1+t^2)+C_3t^{1+\alpha}+C_3t^{2\alpha}\\ &+C_3\left(t^2+t^{1+\alpha}+t^{2\alpha}\right)\sum_{i=1}^{N-1}|x_i|^{\alpha-1}\\ &\le C_4t^{2\alpha}\sum_{i=1}^{N-1}|x_i|^{\alpha-1},\end{aligned}$$

where C_3 and C_4 are positive constants dependent only on N. In the chain of estimates above, the first inequality follows from (4.10) and the first two inequalities in (4.13), the second inequality follows from (4.4), (4.12) and (A.2), and the third inequality is a consequence of the facts that $t \le t_1$ and $\alpha < 1$.

Thus, we see that

$$|H_t-1|\le|\Delta\Psi_t|+C_4t^{2\alpha}\sum_{i=1}^{N-1}|x_i|^{\alpha-1}.$$

Noting that by construction, we have

$$\Delta\Psi_t=(1+\alpha)\alpha\sum_{i=1}^{N-1}\frac{|x_i|^{\alpha-1}}{4^{1+\alpha}}.$$

This leads to the following pointwise estimate for the deviation of the mean curvature:

$$|H_t-1|\le C_5\sum_{i=1}^{N-1}|x_i|^{\alpha-1},$$

where C_5 is a positive constant dependent only on N. So, we have

$$\begin{aligned}\|H_t-1\|^r_{L^r(\Gamma_t)}&=\int_{B_t^{N-1}}|H_t-1|^r\sqrt{1+|\nabla\varphi_t|^2}\,dx\\ &=\int_{\left\{x\in B_t^{N-1}:\forall i\in\{1,\dots,N-1\},x_i\neq0\right\}}|H_t-1|^r\sqrt{1+|\nabla\varphi_t|^2}\,dx\\ &\le C_6\int_{\left\{x\in B_t^{N-1}:\forall i\in\{1,\dots,N-1\},x_i\neq0\right\}}\left(\sum_{i=1}^{N-1}|x_i|^{\alpha-1}\right)^r dx\\ &\le C_6\int_{-t}^{t}\cdots\int_{-t}^{t}\left(\sum_{i=1}^{N-1}|x_i|^{\alpha-1}\right)^r dx_1\cdots dx_{N-1}\\ &=C_6 2^{N-1}\int_0^t\cdots\int_0^t\left(\sum_{i=1}^{N-1}x_i^{(\alpha-1)r}\right)dx_1\cdots dx_{N-1}\\ &\le\frac{C_7}{r\alpha+1-r}t^{(1+\alpha)r+N-1-2r}\end{aligned}\tag{A.3}$$

where C_6 and C_7 are positive constants dependent only on N. On the second line, we used the fact that $\left|B_t^{N-1}\setminus\left\{x\in B_t^{N-1}:\forall i\in\{1,\dots,N-1\},x_i\neq0\right\}\right|=0$. On the last inequality, we used the assumption that $\alpha>\frac{r-1}{r}$ (without this assumption, note that $H_t\notin L^r(\Gamma_t)$).

Also note that combining (A.2) and the last four lines in (A.3) show that $D^2\Psi_t\in L^r(B_t^{N-1})$, with $\|D^2\Psi_t\|_{L^r(B_t^{N-1})}$ being uniformly bounded with respect to $t<t_1$. As a consequence, we have that the family of sets $\{\Gamma_t\}$ is in $W^{2,r}$.

Now, to conclude that the estimates of Theorem 1.1 for $C^{1,\alpha}$ domains are optimal, we proceed like we did in Section 4 to confirm the optimality of Theorem 1.1 for more regular domains.

Appendix B. Proof of Lemma 2.2

Throughout this section, let $r \in (1,\infty)$ and $\alpha \in (0,1]$ and $\Omega \subset \mathbb{R}^N$ be a bounded domain whose boundary Γ is of class $C^{1,\alpha} \cap W^{2,r}$.

To prove Lemma 2.2, we consider a solution, u, to the so-called torsion problem, that is, the following boundary value problem:

$$\begin{cases} \Delta u = N & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma. \end{cases} \tag{B.1}$$

Since u is the solution to (B.1), by standard elliptic regularity (see, e.g., [12]) we have that $u \in C^{1,\alpha}(\overline{\Omega})$ and

$$\|u\|_{C^{1,\alpha}(\overline{\Omega})} \le C, \tag{B.2}$$

for a constant C only depending on N, α, and the $C^{1,\alpha}$-regularity of Γ. Given a point $z \in \mathbb{R}^N$, for $x \in \overline{\Omega}$ we set

$$h_z(x) := q_z(x) - u(x), \qquad \text{where} \qquad q_z(x) := \frac{|x-z|^2}{2}. \tag{B.3}$$

The following three lemmas are essentially contained in [19], except for the fact that the uniform sphere condition used there is replaced by the $C^{1,\alpha}$-regularity of Γ (with $0 < \alpha \le 1$) here. This requires some technical modifications that we collect here for the reader's convenience. This is in analogy with what was done in [5, Appendix] for the stability results obtained in [19, 20, 21] concerning Serrin's overdetermined problem.

We are working with the $C^{1,\alpha}$-regularity of Γ; nevertheless, we mention that in some results such a parameter may be weakened by exploiting the notion of interior pseudo-ball condition in [2].

The first lemma adapts to our setting [19, Lemma 3.1], whereas the second lemma adapts to our setting [19, Lemma 3.4]. The third lemma provides an explicit upper bound for $|\Gamma|$, in terms of N, $|\Omega|$ and the $C^{1,\alpha}$ regularity of Γ; this will be useful to simplify the dependencies of the constants in our estimates.

Lemma B.1. *Given $N \ge 2$, let $\Omega \subset \mathbb{R}^N$ be a bounded domain in $\mathbb{R}^N$ with boundary Γ of class $C^{1,\alpha}$ and let u be the solution to* (B.1).

Then,

$$-u(x) \ge C\,\delta_\Gamma(x) \qquad \text{for any } x \in \overline{\Omega}, \tag{B.4}$$

where $\delta_\Gamma(x)$ is the distance function from Γ, and where C is a positive constant dependent only on N and the $C^{1,\alpha}$-regularity of Ω.

Proof. By comparison, we easily obtain the following rough estimate

$$-u(x) \ge \frac{\delta_\Gamma(x)^2}{2} \qquad \text{for any } x \in \overline{\Omega}. \tag{B.5}$$

The last inequality easily follows by comparing u with $w(y) := \left(|y-x|^2 - \delta_\Gamma(x)^2\right)/2$ (that is the solution of (B.1) in the ball $B_{\delta_\Gamma(x)}(x)$ centered at x with radius $\delta_\Gamma(x)$). We stress that we have not used the $C^{1,\alpha}$-regularity of Γ yet, and in fact (B.5) remains true without such an assumption (see, e.g, [19, Lemma 3.1]).

By the Hopf-Olenik lemma for $C^{1,\alpha}$ domains (see, e.g., [13] and [2, Theorem 4.4]), for any $y \in \Gamma$ we have that

$$-u(y - t\,\nu(y)) > \kappa t \qquad \text{for any } 0 < t < \underline{\delta}, \tag{B.6}$$

where κ and $\underline{\delta}$ are two constants only depending on N and the $C^{1,\alpha}$-regularity of Γ. Here, as usual, $\nu(x)$ denotes the outward unit normal to Γ at x. Let x be any point in $\overline{\Omega}$. If $\delta_\Gamma(x) < \underline{\delta}$, then (B.6) with $t := \delta_\Gamma(x)$ and $y \in \Gamma$ such that $|y - x| = \delta_\Gamma(x)$ (so that $y - t\,\nu(y) = x$) gives that (B.4) holds true with $C := \kappa$. On the other hand, if $\delta_\Gamma(x) \ge \underline{\delta}$, (B.5) easily gives (B.4) with $:= \underline{\delta}/(2N)$.

In any case, for any $x \in \overline{\Omega}$, (B.4) always holds true with $C := \max\{\kappa, \underline{\delta}/(2N)\}$. □

Lemma B.2. *Given $N \ge 2$, let $\Omega \subset \mathbb{R}^N$ be a bounded domain in $\mathbb{R}^N$ with boundary Γ of class C^2. Let z be the center of mass of Ω, that is,*

$$z = \frac{1}{|\Omega|} \int_\Omega x\, dx.$$

Let u be the solution to (B.1) *and h_z be as defined in* (B.3)*. Then, we have that*

$$\int_\Gamma |\nabla h_z|^2 dS_x \le C \int_\Omega (-u)|D^2 h_z|^2 dx,$$

where C is a constant only depending on N, the diameter of Ω, and the $C^{1,\alpha}$-regularity of Γ.

Proof. Note that by the choice of z and the definition of h_z, it follows that

$$\int_\Omega \nabla h_z \, dx = 0.$$

The result easily follows using the same argument of the proof of [20, (ii) of Lemma 2.5] (with $v := h_z$), but replacing [20, (1.15)] with (B.4) and [18, Theorem 3,10] with

$$u_\nu \ge \kappa, \quad \text{where } \kappa \text{ is that appearing in (B.6)},$$

which immediately follows from (B.6). □

Lemma B.3. *Given $N \ge 2$, let $\Omega \subset \mathbb{R}^N$ with boundary Γ of class $C^{1,\alpha}$. Then, we have that*

$$|\Gamma| \le \frac{N|\Omega|}{\kappa},$$

where κ is the constant appearing in (B.6).

Proof. Let $u \in C^{1,\alpha}(\overline{\Omega})$ be the solution of (B.1). Using the divergence theorem, we obtain the following identity

$$N|\Omega| = \int_\Gamma u_\nu \nu \, d\mathcal{H}^{N-1}.$$

Combining this with the fact that $u_\nu \ge \kappa$, which follows immediately from (B.6), we obtain

$$N|\Omega| \ge \kappa|\Gamma|,$$

from which the result follows immediately. □

Proof of Lemma 2.2. Start by assuming that Γ is of class C^2. Recall the following fundamental identity, which was proven in [18]

$$\frac{1}{N-1}\int_\Omega |D^2 h_z|^2 \, dx + H_0 \int_\Gamma \left(u_\nu - \frac{1}{H_0}\right)^2 d\mathcal{H}^{N-1} = \int_\Gamma (H - H_0)\,(u_\nu)^2 \, d\mathcal{H}^{N-1}.$$

Combining this identity, (B.2) and Hölder's inequality, we find that

$$\frac{1}{N-1}\int_\Omega |D^2 h_z|^2 \, dx + H_0 \int_\Gamma \left(u_\nu - \frac{1}{H_0}\right)^2 d\mathcal{H}^{N-1} \le C\|H - H_0\|_{L^r(\Gamma)}, \tag{B.7}$$

where C is a positive constant dependent on r, N, $|\Gamma|$ and the $C^{1,\alpha}$ regularity of Γ.

Using the triangle inequality, we see that

$$\left\| |x-z| - \frac{1}{H_0} \right\|_{L^2(\Gamma)} \le \left\| \frac{1}{H_0}\nu - (x-z) \right\|_{L^2(\Gamma)} \tag{B.8}$$

and

$$\left\| \frac{1}{H_0}\nu - (x-z) \right\|_{L^2(\Gamma)} \le \left\| \frac{1}{H_0} - u_\nu \right\|_{L^2(\Gamma)} + \|\nabla h_z\|_{L^2(\Gamma)}. \tag{B.9}$$

Combining (B.2), Lemma B.2,(B.7), (B.8) and (B.9) we conclude that

$$\left\| \frac{1}{H_0}\nu - (x-z) \right\|_{L^2(\Gamma)} + \left\| |x-z| - \frac{1}{H_0} \right\|_{L^2(\Gamma)} \le C\|H - H_0\|_{L^r(\Gamma)}^{1/2},$$

where C is a positive constant dependent on N, α, r, H_0, $|\Gamma|$ and the $C^{1,\alpha}$ regularity of Γ; the dependence on $|\Gamma|$ and H_0 can be removed and replaced with the dependence on $|\Omega|$, by leveraging Lemma B.3; in turn, the dependence on $|\Omega|$ can be replaced by the dependence on the diameter of Ω.

This concludes the proof of the result under the assumption that Γ is of class C^2. For Γ of class $C^{1,\alpha} \cap W^{2,r}$, we simply leverage the approximation result in [3]. □

Appendix C. Proof of Lemma 2.5

Proof. We start by describing the (standard) stereographic projection. Let $P = (0, ..., 1) \in \mathbb{S}^{N-1}$, the stereographic projection from P is given by $\iota : \mathbb{S}^{N-1} \setminus \{P\} \to \mathbb{R}^{N-1}$ be given by

$$\iota(x_1, ..., x_N) = \frac{1}{1 - x_N}(x_1, ..., x_{N-1}). \tag{C.1}$$

The (standard) metric on $\mathbb{S}^{N-1}$, $G(x) = (g_{ij}(x))$ in the local coordinates induced by ι is given by (see [15, Chapter 1])

$$g_{ij}(x) = \frac{4}{(|x|^2+1)^2}\delta_{ij}, \quad x \in \mathbb{R}^{N-1}, \tag{C.2}$$

and $G^{-1}(x) = (g^{ij}(x))$ is given by

$$g^{ij}(x) = \frac{(|x|^2+1)^2}{4}\delta_{ij}, \quad x \in \mathbb{R}^{N-1}. \tag{C.3}$$

By definition, we have (see [14, Chapter 2])

$$\|v\|_{W^{2,p}(\mathbb{S}^{N-1})} = \left(\int_{\mathbb{S}^{N-1}} |v|^p + |\nabla_{\mathbb{S}^{N-1}} v|^p + |D^2_{\mathbb{S}^{N-1}} v|^p \, dS_x \right)^{1/2}, \tag{C.4}$$

where $\nabla_{\mathbb{S}^{N-1}} v$ and $D^2_{\mathbb{S}^{N-1}} v$ denote the first and second derivatives on the sphere. Since $v \circ \iota^{-1}$ denotes the function v written in the coordinates induced by the stereographic projection, we will use the slight abuse of notation and also denote by v the function $v \circ \iota^{-1}$.

In the coordinates induced by the stereographic projection, we have

$$|\nabla_{\mathbb{S}^{N-1}} v(x)|^2 = g^{ij}(x)\partial_i v(x)\partial_j v(x) = \frac{(|x|^2+1)^2}{4}|\nabla v|^2, \tag{C.5}$$

$$\begin{aligned} \left|D^2_{\mathbb{S}^{N-1}} v\right|^2 &= g^{ia}(x)g^{jb}(x)\left(\partial_{ij}v(x) - \Gamma^k_{ij}\partial_k v(x)\right)\left(\partial_{ab}v(x) - \Gamma^k_{ab}\partial_k v(x)\right) \\ &= \frac{(|x|^2+1)^4}{16}\left(\partial_{ij}v(x) - \Gamma^k_{ij}\partial_k v(x)\right)^2 \end{aligned} \tag{C.6}$$

where we employed the Einstein summation convention, and where Γ^k_{ij} denote the Christoffel symbols, which are given by

$$\Gamma^k_{ij} = \frac{1}{2}g^{nk}\left(\partial_j g_{in}(x) + \partial_i g_{jn}(x) - \partial_n g_{ij}(x)\right).$$

Using (C.2) and (C.3) we see that

$$\Gamma^k_{ij} = -\frac{2}{|x|^2+1}\left(x_j\delta_{ik} + x_i\delta_{jk} - x_k\delta_{ij}\right),$$

which, in particular, yields

$$\left|\Gamma^k_{ij}\right| \le \frac{2|x|}{|x|^2+1}, \quad \text{for all } x \in \mathbb{R}^{N-1}.$$

Combining this, (C.5) and (C.6), we see that

$$\begin{aligned} \left|D^2_{\mathbb{S}^{N-1}} v\right|^2 &= \frac{(|x|^2+1)^4}{16}\left(\partial_{ij}v(x) - \Gamma^k_{ij}\partial_k v(x)\right)^2 \\ &= \frac{(|x|^2+1)^4}{16}\left((\partial_{ij}v(x))^2 - 2\partial_{ij}v(x)\Gamma^k_{ij}\partial_k v(x) + \left(\Gamma^k_{ij}\partial_k v(x)\right)^2\right) \\ &\ge \frac{(|x|^2+1)^4}{16}\left((\partial_{ij}v(x))^2\left(1 - \left|\Gamma^k_{ij}\right|\right) + \left(\left(\Gamma^k_{ij}\right)^2 - \left|\Gamma^k_{ij}\right|\right)(\partial_k v(x))^2\right) \\ &\ge \frac{(|x|^2+1)^4}{16}\left((\partial_{ij}v(x))^2\left(1 - \frac{2(N-1)|x|}{|x|^2+1}\right) - \frac{2(N-1)^2|x|}{|x|^2+1}(\partial_k v(x))^2\right) \\ &= \frac{(|x|^2+1)^4}{16}\left(1 - \frac{2(N-1)|x|}{|x|^2+1}\right)|D^2 v|^2 - \frac{1}{2}(N-1)^2|x|(|x|^2+1)\left|\nabla_{\mathbb{S}^{N-1}} v\right|^2. \end{aligned}$$

Letting $R = \frac{1}{8(N-1)^2}$, we see that, for $|x| \le R$ we have that

$$\begin{aligned} \left|D^2_{\mathbb{S}^{N-1}} v(x)\right|^2 + |\nabla_{\mathbb{S}^{N-1}} v(x)|^2 &\ge \frac{(|x|^2+1)^4}{32}|D^2 v|^2 + \frac{1}{2}\left|\nabla_{\mathbb{S}^{N-1}} v\right|^2 \\ &= \frac{(|x|^2+1)^4}{32}|D^2 v|^2 + \frac{(|x|^2+1)^2}{8}|\nabla v|^2. \end{aligned} \tag{C.7}$$

Using this and recalling (C.4), we obtain the following chain

$$\begin{aligned}
\|v\|^p_{W^{2,p}(B_R^{N-1}(0))} &= \int_{B_R^{N-1}(0)} \left(|v(x)|^p + |\nabla v(x)|^p + |D^2 v(x)|^p\right)\, dx \\
&\le \int_{B_R^{N-1}(0)} \left(|v(x)|^p + 2\left(|\nabla v(x)|^2 + |D^2 v(x)|^2\right)^{p/2}\right)\, dx \\
&\le \int_{B_R^{N-1}(0)} \left(|v(x)|^p + 2\frac{32^{p/2}}{(|x|^2+1)^p}\left(\frac{(|x|^2+1)^4}{32}|D^2 v|^2 + \frac{(|x|^2+1)^2}{8}|\nabla v|^2\right)^{p/2}\right)\, dx \\
&\le 2^{1+\frac{5p}{2}} \int_{B_R^{N-1}(0)} \left(|v|^p + \left(|\nabla_{\mathbb{S}^{N-1}} v|^2 + |D^2_{\mathbb{S}^{N-1}} v|^2\right)^{p/2}\right) dx \\
&\le 2^{N+5+\frac{5p}{2}} \int_{B_R^{N-1}(0)} \left(|v|^2 + \left(|\nabla_{\mathbb{S}^{N-1}} v| + |D^2_{\mathbb{S}^{N-1}} v|\right)^p\right) \frac{(|x|^2+1)^{N-1}}{2^{N-1}} dx \\
&\le 2^{N+5+\frac{5p}{2}} \int_{\mathbb{S}^{N-1}} \left(|v|^p + \left(|\nabla_{\mathbb{S}^{N-1}} v| + |D^2_{\mathbb{S}^{N-1}} v|\right)^p\right) d\mathcal{H}^{N-1} \\
&= 2^{N+5+\frac{5p}{2}} \left(\|v\|^p_{L^p(\mathbb{S}^{N-1})} + \||\nabla_{\mathbb{S}^{N-1}} v| + |D^2_{\mathbb{S}^{N-1}} v|\|^p_{L^p(\mathbb{S}^{N-1})}\right),
\end{aligned}$$

where, in the first inequality, we used the pointwise inequality $a^p + b^p \le 2(a+b)^p$ with $a, b \ge 0$, in the second inequality, we used (C.7), in the fourth inequality, we used the fact that $(a^2+b^2)^{p/2} \le (a+b)^p$ with $a, b \ge 0$. From this, we now conclude the result using the triangle inequality

$$\|v\|_{W^{2,p}\left(B_R^{N-1}(0)\right)} \le 2^{\frac{N+5}{p}+\frac{5}{2}} \left(\|v\|^p_{L^p(\mathbb{S}^{N-1})} + \||\nabla_{\mathbb{S}^{N-1}} v| + |D^2_{\mathbb{S}^{N-1}} v|\|^p_{L^p(\mathbb{S}^{N-1})}\right)^{1/p} \le C\|v\|_{W^{2,p}(\mathbb{S}^{N-1})},$$

where $C = 3 \cdot 2^{\frac{N+5}{p}+\frac{5}{2}}$. □

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