

All-Gaussian State Discrimination Beyond the Coherent Helstrom Bound

Angus Walsh,^{1,*} Lorcán O. Conlon,^{2,3,†} Biveen Shajilal,² Özlem Erkiş,¹

Jiri Janousek,¹ Syed M. Assad,² Jie Zhao,¹ and Ping Koy Lam^{2,3}

¹*Center of Excellence for Quantum Computation and Communication Technology, The Department of Quantum Science and Technology, Research School of Physics, The Australian National University, Canberra, Australia*

²*Quantum Innovation Centre (Q.InC), Agency for Science Technology and Research (A*STAR), 2 Fusionopolis Way, Innovis #08-03, Singapore 138634, Republic of Singapore*

³*Centre for Quantum Technologies, National University of Singapore, 3 Science Drive 2, Singapore, Republic of Singapore*

A core problem in communications is the optimal discrimination of binary-phase-shift-keyed (BPSK) signals. A longstanding goal has been to reach the fundamental quantum limit, known as the Helstrom bound, for BPSK signals encoded in coherent states. However, due to technical constraints, proposals for reaching the bound remain impractical. In this letter we take an alternative approach: using only Gaussian optics - displaced squeezed states and homodyne detection - we achieve discrimination of BPSK signals with error rates below what can be achieved using coherent states and any quantum measurement.

The theory of quantum state discrimination provides tools to study the fundamental limits of communication systems [1, 2]. Notably, the Helstrom bound specifies the minimum probability of error in distinguishing non-orthogonal signals [3]. For a BPSK signal encoded in coherent states the Helstrom bound shows that a suitably designed receiver can surpass the standard quantum limit (SQL) for discriminating this signal.

A number of receiver designs have been proposed that approach, or in some limits attain, the Helstrom bound for the coherent BPSK (C-BPSK) signal [4]. Kennedy introduced the idea of a receiver that displaces the incoming signal prior to photon counting, and in doing so surpasses the SQL [5]. Variations on this theme have been able to improve upon its performance, and in some cases theoretically achieve the coherent Helstrom bound [6–9].

Displacement-based receivers have several problems with their implementations. It is implicitly assumed that the strength of the received signal is known perfectly, however if the amplitude varies with time - for example due to fluctuations in the channel loss - then the displacements in the receiver will be unoptimised leading to excess errors. Receiver designs that attain the coherent Helstrom bound require precise real-time adjustment of the displacement, and may only reach the bound in an asymptotic limit. Additionally, the requirement for single-photon or photon-number-resolving detectors makes these designs sensitive to detector inefficiencies and dark noise, leading to errors that can proliferate in multi-stage receivers [10–12].

The toolbox of Gaussian optics provides an experimentally accessible alternative to these approaches [13–15]. In contrast to single-photon detectors, homodyne detectors can be constructed with near-unity efficiency and large electronic noise clearance [16]. Recent advances in

quadrature squeezing have demonstrated high levels of noise reduction and long-term stability [16, 17], motivating the exploration of state discrimination using only Gaussian states and operations.

Rather than constructing an elaborate receiver, the states in which the BPSK signal is encoded can be optimised instead. By injecting quadrature squeezed light into the communications channel we can obtain a quadratic improvement in the signal-to-noise ratio (SNR) of the BPSK signal for the same mean energy of the states. This makes it possible to surpass the Helstrom bound for the C-BPSK signal using only Homodyne detection [18, 19]. In this manuscript we demonstrate this experimentally for the first time, and outline the importance of this task for quantum communication.

Theoretical Framework.—In optical communications the C-BPSK signal consists of symbols with equal amplitude and opposite phase, represented by the quantum states

$$|\psi_k\rangle = \hat{R}(k\pi)\hat{D}(\alpha)|0\rangle, k \in \{0, 1\} \quad (1)$$

where $\hat{D}$ and $\hat{R}$ are the displacement and rotation operators respectively. The coherent amplitude α is assumed to be real-valued. In a standard protocol these symbols are generated by a transmitter station and sent through an appropriate channel to a receiver station. The channel modifies the signal through the introduction of noise and optical losses. The performance of the protocol is quantified by the probability of error in estimating the transmitted symbol,

$$P_e = P(0)P(1|0) + P(1)P(0|1), \quad (2)$$

where $P(k)$ is the probability that the k state was sent, and $P(j|k)$ the probability that our decision strategy mistakes the k state for the j state. We aim to minimise this error for a given mean photon number, $\bar{n} = \langle \hat{N} \rangle$, of the signal states. We can simplify the error probability by assuming that each symbol is *a priori* equally likely - as this maximises the mutual information between the transmitter and receiver - then, noting that the BPSK

* Angus.Walsh@anu.edu.au

† lorcanconlon@gmail.com

signal is symmetric, we can write the probability of error as

$$P_e = P(1|0) = P(0|1) \quad (3)$$

The probability of error for a specific protocol is a function of the signal-to-noise ratio (SNR) of the received signal. For Gaussian states, it is convenient to use the homodyne SNR, defined as

$$\text{SNR} := \frac{\langle \hat{X} \rangle^2}{\langle \Delta \hat{X}^2 \rangle}, \quad (4)$$

where $\hat{X}$ is a quadrature operator [20]. Coherent states have a SNR of $4\bar{n}$, however it is possible through quadrature squeezing to achieve a higher SNR for the same $\bar{n}$. Encoding a BPSK signal onto squeezed vacuum results in the displaced-squeezed-state BPSK (S-BPSK) signal

$$|\psi_k\rangle = \hat{R}(k\pi)\hat{D}(\beta)\hat{S}(r)|0\rangle, k \in \{0, 1\}, \quad (5)$$

where $\hat{S}$ is the squeezing operator. The parameter r is real-valued and positive so as to obtain amplitude squeezing, and we set the coherent amplitude $\beta = \sqrt{\alpha^2 - \sinh^2(r)}$ so that the S-BPSK and C-BPSK signals have the same energy and can justifiably be compared [21]. For states of a given mean photon number there is an optimal distribution of energy between squeezing and displacement that maximises the SNR. We can define the squeezing ratio as the contribution of squeezing to the total energy of the state, $\gamma = \sinh^2(r)/\bar{n}$. When the ratio is equal to

$$\gamma = \frac{\bar{n}}{2\bar{n} + 1} \quad (6)$$

a maximum SNR of $4(\bar{n}^2 + \bar{n})$ is obtained, this being the maximum SNR for a Gaussian state. The introduction of squeezing therefore leads to quadratic scaling of the SNR over the coherent signal [18, 19].

Using an optimal decision strategy the minimal probability of error when using a homodyne detector as a receiver is

$$P_{\text{HOM}} = \frac{1}{2} \text{Erfc}\left[\sqrt{\frac{\text{SNR}}{2}}\right]. \quad (7)$$

The fundamental lower limit on the discrimination of any two pure states, $|\psi_0\rangle$ and $|\psi_1\rangle$, is given by the Helstrom bound $P_{\text{HEL}} = \frac{1}{2}(1 - \sqrt{1 - |\langle \psi_0 | \psi_1 \rangle|^2})$ [3]. Using the fact that for any Gaussian BPSK signal $|\langle \psi_0 | \psi_1 \rangle|^2 = e^{-\text{SNR}}$ we can write the Helstrom bound as a function of the SNR

$$P_{\text{HEL}} = \frac{1}{2}(1 - \sqrt{1 - e^{-\text{SNR}}}), \quad (8)$$

which allows for direct comparison to the homodyne limit. Table 1 shows the homodyne limit and Helstrom

Encoding	Homodyne Error	Helstrom Error
Coherent	$\frac{1}{2} \text{Erfc}[\sqrt{2\bar{n}}]$	$\frac{1}{2}(1 - \sqrt{1 - e^{-4\bar{n}}})$
Squeezed	$\frac{1}{2} \text{Erfc}[\sqrt{2(\bar{n}^2 + \bar{n})}]$	$\frac{1}{2}(1 - \sqrt{1 - e^{-4(\bar{n}^2 + \bar{n})}})$

TABLE I. Error bounds for BPSK communication protocols as functions of the mean photon number of the signal states. The signal is encoded either in coherent states or displaced squeezed states, and received using either a Homodyne detector or a detector that saturates the Helstrom bound.

bound for both the C-BPSK and S-BPSK signals. The coherent-homodyne limit is commonly referred to as the standard quantum limit (SQL). The squeezed-homodyne limit is the lowest achievable error rate using only Gaussian states and operations. Despite this constraint, due to the quadratic scaling of the displaced squeezed state SNR, this bound can fall below the coherent Helstrom bound for signals with the same mean photon number.

By applying another squeezing operator, $\hat{S}(-r)$, to the S-BPSK signal at the receiver we can obtain a coherent state with the same SNR. Therefore any receiver that saturates the Helstrom bound for coherent states can be made to saturate the bound for squeezed states simply by including this operator as a preamplifier. However, it is important to note that the definition of the S-BPSK signal in Equation (5) is highly idealised. In practice the signal states will not be pure, due to excess noise in the anti-squeezed quadrature, and losses in the implementation of the displacement operation. Instead of squeezed vacuum the BPSK signal will be encoded onto squeezed thermal states characterised by a mean thermal photon number $\bar{n}_{th}$ which effectively represents wasted energy. Consequently the pure state Helstrom bound shown in Table (1) will not be applicable to any realistic scenario.

Experimental demonstration.—Figure 1 shows the setup of our experiment. Using a 1550 nm laser source we use an SHG cavity to produce light at 775 nm, which in turn pumps an optical parametric oscillator (OPO) to generate squeezed vacuum at 1550 nm. An electro-optic phase modulator is used to generate coherent states at the 3 MHz sideband of a 1550 nm carrier beam, which are then mixed with the squeezed vacuum on a highly biased beam splitter (nominal $R = 99.5$) to produce displaced squeezed states. A lossy channel is simulated by using a half wave plate and polarising beam splitter, and after passing through the channel the signal is measured on a homodyne detector with visibility $V = (98.75 \pm 0.4)\%$. A pair of acousto-optic modulators were used to produce a single sideband modulation at 20 MHz (not shown in the figure) which is injected into the OPO and co-propagates with the squeezed light, this serves as a common reference to lock the phases of the OPO pump, the coherent displacement, and the homodyne local oscillator (LO) [17]. In each run of the experiment the squeezing level is fixed and the coherent displacement is varied by increasing the amplitude of the 3 MHz modulation.

After demodulation of the 3 MHz sideband the homo-

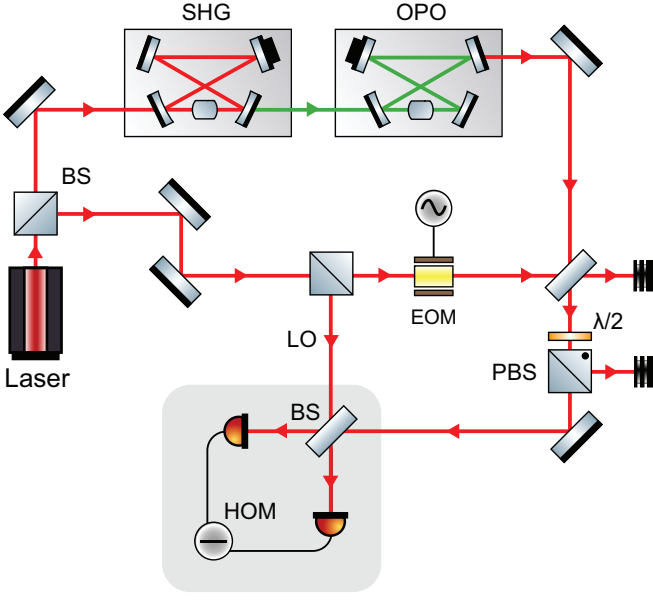


FIG. 1. Experimental setup for all-Gaussian state discrimination of BPSK signals. A continuous-wave 1550 nm laser output is separated into a pump, signal and local oscillator (LO) by a network of beam splitters (BS). The pump beam drives a second harmonic generation (SHG) cavity to produce light at 775 nm which in turn pumps an optical parametric oscillator (OPO) to generate squeezed vacuum at 1550 nm. Both cavities use periodically-poled potassium titanyl phosphate as a non-linear medium. An electro-optic modulator (EOM) displaces the 3 MHz sideband of the signal beam which is then mixed with the OPO output on a biased BS to make the S-BPSK signal. Channel loss is implemented using a half-wave plate ($\lambda/2$) and a polarising beam splitter (PBS) before the signal is measured at a homodyne receiver (HOM).

dyne current is recorded using an oscilloscope. With all the light blocked we first measure the electronic noise v_{en} of the Homodyne detector, we then measure the shot noise from the LO to which all other measurements are normalised. With the displacement beam blocked we measure the squeezing and anti-squeezing quadratures of the OPO output and calculate the squeezing parameter r and thermal noise v_{th} . Finally the displacement beam is unblocked and the modulation signal is incrementally increased, varying β while keeping the other parameters fixed. The measured state has an estimated mean photon number

$$\bar{n} = \frac{\beta^2 + \sinh^2(r) + (v_{th} - 1)/2}{V^2}, \quad (9)$$

from which the probability of error can be predicted by solving for β and substituting into the Homodyne error

$$P_e = \frac{1}{2} \text{Erfc} \left[\sqrt{\frac{2(\bar{n}V^2 - \sinh^2(r) - (v_{th} - 1)/2)}{(e^{-2r}v_{th} + v_{en})}} \right], \quad (10)$$

as shown in the dashed traces in Figure 2 (a). To directly estimate the probability of error from the data we simply

calculate the fraction of samples that are above (below) 0 when the $|\psi_1\rangle$ ($|\psi_0\rangle$) state is transmitted. To quantify the uncertainty in our estimate of the error rate we treat each sample as a Bernoulli trial, and using the Jeffrey's prior calculate the posterior distribution for P_e . Due to the large number of samples for each state ($n_s = 5 \times 10^6$) the resulting Beta distribution is to a good approximation normal, and its standard deviation is shown as error bars.

Figure 2 (a) shows the measured probability of error using displaced squeezed states. In each run of the experiment the squeezing level is fixed and the mean photon number of the state varies solely with the amplitude of the displacement. As there is a minimum amount of energy in the state due to the squeezing, the traces modelling the experiment do not begin at $\bar{n} = 0$. The limits for the probability of error using different protocols are shown for comparison, and we clearly demonstrate discrimination of the S-BPSK signal using homodyne detection with lower errors than possible for a protocol using a C-BPSK signal and an optimal quantum receiver.

In principle the experimental traces should be tangent to the Gaussian limit when they have an optimal squeezing ratio, however due to impurities in the squeezing, and the presence of electronic noise, they are not. In Figure 2 (b) the error probabilities for three S-BPSK signals with approximately the same energy but with different levels of squeezing are shown. Even for very weak signals there is always some advantage to be gained through squeezing, consequently there is always a S-BPSK signal that can outperform the SQL for any $\bar{n}$. However in order to beat the coherent Helstrom bound the signal energy must be above a certain value, found by numerical evaluation of the coherent Helstrom and Gaussian bounds to be $\bar{n} \approx 0.67$.

Figure 3 shows the performance of the same S-BPSK signals for three channels with different loss. The ideal performance of equivalent energy C-BPSK signals are shown as a comparison. It is clear that the performance of the S-BPSK signal degrades faster than in the C-BPSK case, as there are some states that no longer surpass the coherent state limit when loss is present. There is in principle always some improvement of the SNR that can be obtained regardless of the channel loss, however the improvement may not be worth the cost of allocating signal energy to squeezing.

The performance of squeezed states degrades rapidly when optical losses are present in the channel. The adjusted error rate model for a channel with loss $L \in [0, 1]$ is

$$P_e = \frac{1}{2} \text{Erfc} \left[\sqrt{\frac{2(\bar{n}V^2 - \sinh^2(r) - (v_{th} - 1)/2)(1 - L)}{(e^{-2r}v_{th}(1 - L) + L + v_{en})}} \right]. \quad (11)$$

An alternative figure of merit to the probability of error is the mutual information

$$I = 1 - P_e \log_2 \left(\frac{1}{P_e} \right) - (1 - P_e) \log_2 \left(\frac{1}{1 - P_e} \right) \quad (12)$$

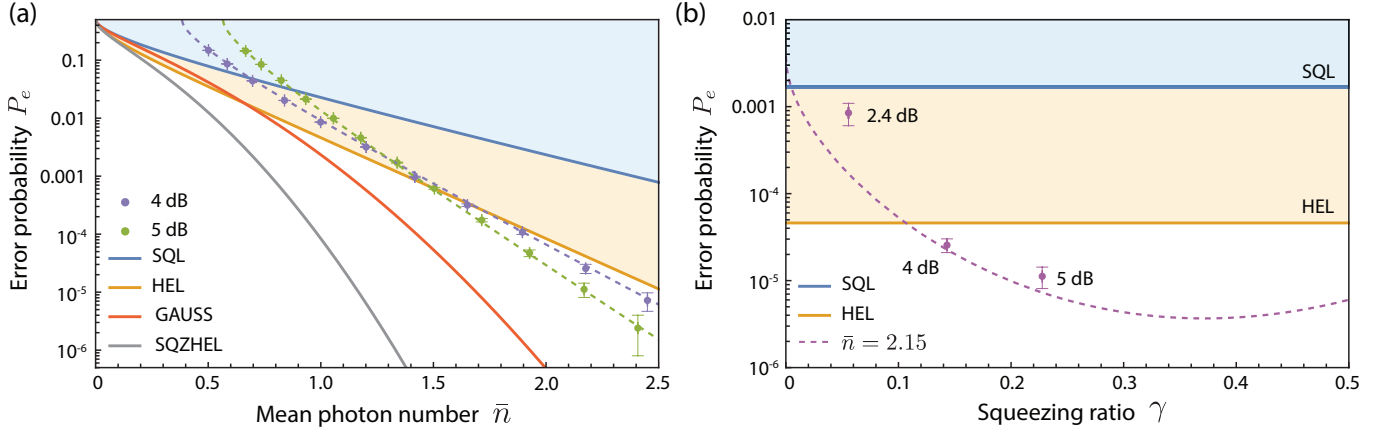


FIG. 2. (a) Error probability in BPSK communication against the mean photon number of the signal. (b) Error probability in BPSK communication against the proportion of energy allocated to squeezing the signal, for states with mean photon numbers close to $\bar{n} = 2.15$. Markers show experimental results from our implementation of a BPSK protocol with displaced squeezed states and homodyne detection. Dashed curves show the expected error given the experimental parameters, and solid curves represent the theoretical limits: the standard quantum limit of coherent states and homodyne detection (SQL), the Helstrom bound for coherent states (HEL), the limit of Gaussian states and measurements (GAUSS), and the Helstrom bound for squeezed states (SQZHEL).

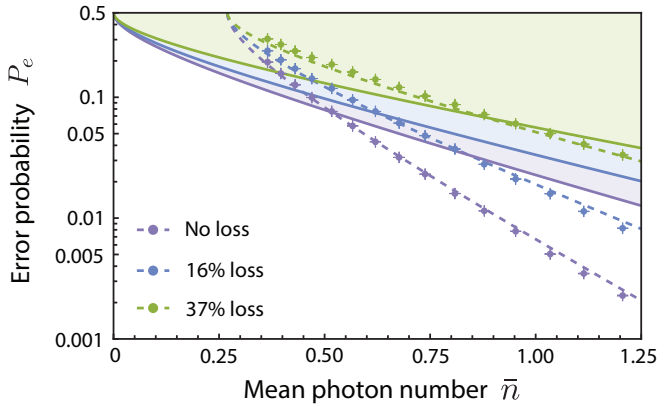


FIG. 3. Error probability as a function of mean photon number for different channel loss. Markers show measured values using a S-BPSK signal with 3.8 dB squeezing. Dashed curves are our model of the experiment, and solid curves show the C-BPSK limits. Error bars are obscured by the marker size.

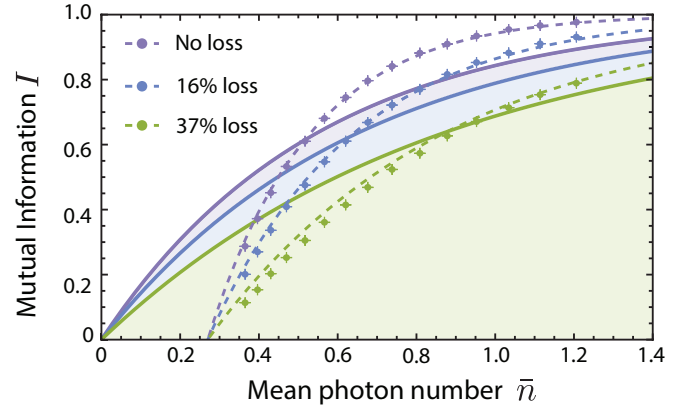


FIG. 4. Mutual information as a function of mean photon number for different channel loss. Markers show values calculated from the measured probability of error using a S-BPSK signal with 3.8 dB squeezing. Dashed curves are our model of the experiment, and solid curves show the C-BPSK limits. Error bars are obscured by the marker size.

between the transmitter and receiver. In our case due to the symmetry of the protocol this is equivalent to the capacity of a binary-symmetric channel, and has a theoretical maximum of one bit per use when $P_e \rightarrow 0$. Figure 4 shows the mutual information calculated from the error rates of the S-BPSK signal in lossy channels. We can consider the introduction of squeezing to be a modification of the channel itself [18]; specifically we are lowering the additive noise of the channel. The channel capacity is an asymptotic limit that can be approached by using error correction codes, and due to the lower probability of error the S-BPSK signal could potentially achieve the same data rates as C-BPSK signals while using shorter error correction codes.

Discussion.—The introduction of quadrature squeezing greatly enhances the SNR of BPSK signals, allowing the coherent Helstrom bound to be passed with only readily-available Gaussian optics. We have demonstrated this experimentally for the first time by generating quantum states with an optimised ratio of squeezing to displacement. The impact of such states for quantum enhanced communication has also been analysed.

On a practical level there remain several issues that limit the immediate applicability of our results. Realistic communications systems are almost always limited by channel losses, and the improvement from squeezing can quickly be diminished, as shown in Figure 3. Addition-

ally, larger signal alphabets are required to attain higher channel capacity, so it would be desirable to extend the BPSK signal alphabet to higher-order amplitude-shift-keyed (ASK) or phase-shift-keyed (PSK) alphabets. Because ASK signals can always be decoded by measuring only a single quadrature we predict a similar improvement from using squeezed states instead of coherent states, as is found for BPSK. However PSK signals, which are more energy efficient, require both quadratures to be measured when there are more than two signal states. Doing so involves splitting the received signal in two and consequently incurring a 3 dB loss to the signal amplitude. For squeezed states the SNR penalty is even greater as vacuum noise is coupled into the squeezed quadrature. Consequently we find that for M-PSK with $M > 2$ there is no advantage to be obtained from using squeezed signals and dual homodyne detection [22]. Interestingly it has been shown elsewhere that for the specific case of $M = 3$ the Helstrom bound of the PSK signal can be lowered through squeezing [23]; combined with our re-

sult this indicates that a non-Gaussian measurement is required to see this improvement.

Despite these issues, we believe that the higher distinguishability of displaced squeezed states is of fundamental interest. The advantage offered by squeezed states may translate to other types of quantum information processing where the energy of signal states is constrained and optical losses can be kept to a minimum.

Finally, we wish to draw a comparison with quantum sensing. For many years, squeezed states of light have been used to enhance the sensitivity of quantum measuring devices, most notably in gravitational wave detection [24], and in achieving Heisenberg-limited sensing [25]. We believe we have demonstrated an analogous advantage for state discrimination in this work.

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Supplementary Material for “All-Gaussian State Discrimination Beyond the Coherent Helstrom Bound”

S1. DERIVATION OF THE OPTIMAL GAUSSIAN PROTOCOL

Demonstrating that displaced squeezed states followed by homodyne detection forms the optimal Gaussian scheme for discriminating a binary-phase-shift-keyed (BPSK) signal requires two steps. First, subject to the constraint of fixed mean photon number $\bar{n}$, we maximise the signal-to-noise ratio (SNR) of an arbitrary Gaussian state and show that this results in a displaced squeezed state. We then demonstrate that after the signal is defined, but prior to its measurement by a homodyne receiver, no Gaussian operations can improve the SNR. It's worth emphasising a subtlety: applying a squeezing operation to a coherent signal does not change the SNR, however it does change the mean energy of the state, and compared to a coherent state with the changed energy, the displaced squeezed state may have better SNR.

An arbitrary Gaussian state can be constructed as follows: starting with a thermal state $\hat{\rho}_{th}$ with mean photon number n_{th} we apply a squeezing operation $\hat{S}(r)$. We assume r is real apply a rotation $\hat{R}(\theta)$ to obtain squeezing in an arbitrary quadrature. Finally we apply a displacement $\hat{D}(\alpha)$ to obtain the state

$$\hat{\rho}_0 = \hat{D}(\alpha)\hat{R}(\theta)\hat{S}(r)\hat{\rho}_{th}\hat{S}^\dagger(r)\hat{R}^\dagger(\theta)\hat{D}^\dagger(\alpha). \quad (13)$$

It can be assumed without loss of generality that α is real, as changing the angle between squeezing and displacement can be done with θ , and a rotation of the entire state can be achieved by measuring a rotated quadrature operator. The homodyne SNR

$$\text{SNR} = \frac{\langle \hat{X} \rangle^2}{\langle \Delta \hat{X}^2 \rangle} \quad (14)$$

of the state is

$$\text{SNR} = \frac{4\alpha^2}{(2n_{th} + 1)(e^{-2r} \cos^2(\theta) + e^{2r} \sin^2(\theta))}, \quad (15)$$

and it is obvious by inspection that the SNR is maximised when $\theta = 0$ and $n_{th} = 0$. The resulting SNR can be written out in terms of the mean photon number contribution from squeezing $\bar{n}_s = \sinh^2(r)$ and the total mean photon number $\bar{n} = \alpha^2 + \sinh^2(r)$:

$$\text{SNR} = 4(\bar{n} - \bar{n}_s)(\sqrt{\bar{n}_s} + \sqrt{\bar{n}_s + 1})^2, \quad (16)$$

which obtains a maximum value of $4(\bar{n}^2 + \bar{n})$ when $\bar{n}_s = \bar{n}^2/(2\bar{n} + 1)$ [19].

We can now show that any Gaussian unitary applied prior to homodyne detection does not decrease the probability of error. It is trivial to see that rotating the received states by an angle ϕ can only be detrimental, as not only will the amplitude decrease but noise from the anti-squeezed quadrature will be introduced:

$$\text{SNR}(\phi) = \frac{4\alpha^2 \cos^2(\phi)}{e^{-2r} \cos^2(\phi) + e^{2r} \sin^2(\phi)}. \quad (17)$$

Applying further squeezing will change both the signal and the noise by a factor of $e^{-2r'}$ resulting in no net change to the SNR. Finally we note that while displacement by an amplitude α' will increase the SNR (for one of the signal states), in order for the protocol to remain symmetric the decision threshold will also have to shift from $0 \rightarrow \alpha'$, consequently the error probability is unchanged. As any Gaussian unitary can be decomposed into the above transformations, these considerations encompass the totality of Gaussian operations that may be performed on the state.

Therefore we can conclude that a displaced squeezed state signal with $\gamma = \bar{n}/(2\bar{n} + 1)$ followed by homodyne detection constitutes the optimal scheme for all-Gaussian BPSK in loss-free channels. (In a lossy channel the above argument still holds, however the optimal squeezing ratio γ will be reduced.) Finally we note that because the SNR

is unchanged by any squeezing operation after transmission, any receiver that reaches the coherent Helstrom bound can be made to reach the squeezed state Helstrom bound

$$P = \frac{1}{2}(1 - \sqrt{1 - e^{-4(\bar{n}^2 + \bar{n})}}), \quad (18)$$

simply by the inclusion of an anti-squeezing operation that transforms the incoming squeezed states into coherent states.

S2. THREE STATE DISCRIMINATION

A. Amplitude shift keying

The three state amplitude shift keyed signal is

$$|\psi_k\rangle = \hat{D}(k\alpha)\hat{S}(r)|0\rangle, k \in \{-1, 0, 1\}, \quad (19)$$

which results in three squeezed states in a line along the amplitude quadrature. As only two signal states have coherent displacements the energy allocation is uneven. The average signal energy is

$$\bar{n} = \frac{2}{3}\alpha^2 + \sinh^2(r). \quad (20)$$

Again we set $\gamma = \sinh^2(r)/\bar{n}$ leading to the amplitude of the displacement being

$$\alpha = \sqrt{\frac{3}{2}(1 - \gamma)\bar{n}} \quad (21)$$

Using homodyne detection the decision threshold between the $k = 0$ and $k = 1$ states is $\alpha/2$, and the error rates can be found by simply integrating the homodyne distributions for each state. The average probability of error is

$$P = \frac{2}{3}\text{Erfc}\left[\frac{e^r\alpha}{\sqrt{2}}\right]. \quad (22)$$

For coherent states this reduces to

$$P_c = \frac{2}{3}\text{Erfc}\left[\frac{\sqrt{3\bar{n}}}{2}\right], \quad (23)$$

while for squeezed states we once again find the the optimal splitting ratio is $\gamma = \bar{n}/(2\bar{n} + 1)$ This gives a probability of error

$$P_s = \frac{2}{3}\text{Erfc}\left[\frac{\sqrt{3(\bar{n}^2 + \bar{n})}}{2}\right], \quad (24)$$

which shows similar improvement to what is found for binary signals.

B. Phase shift keying

The three state phase shift keyed signal is

$$|\psi_k\rangle = \hat{R}(k\frac{2\pi}{3})\hat{D}(\beta)\hat{S}(r)|0\rangle, k \in \{-1, 0, 1\}. \quad (25)$$

This divides phase space into three wedges and the error rate is the probability that the result of a dual-homodyne measurement is outside the appropriate wedge. By symmetry we can consider just the error probability of the $k = 0$ state that is both displaced and squeezed along the amplitude quadrature. The joint probability density function for the dual-homodyne measurement is then

$$P(x, p) = \mathcal{N}(x; \frac{\alpha}{\sqrt{2}}, \frac{1 + e^{-2r}}{8}) \times \mathcal{N}(p; 0, \frac{1 + e^{2r}}{8}), \quad (26)$$

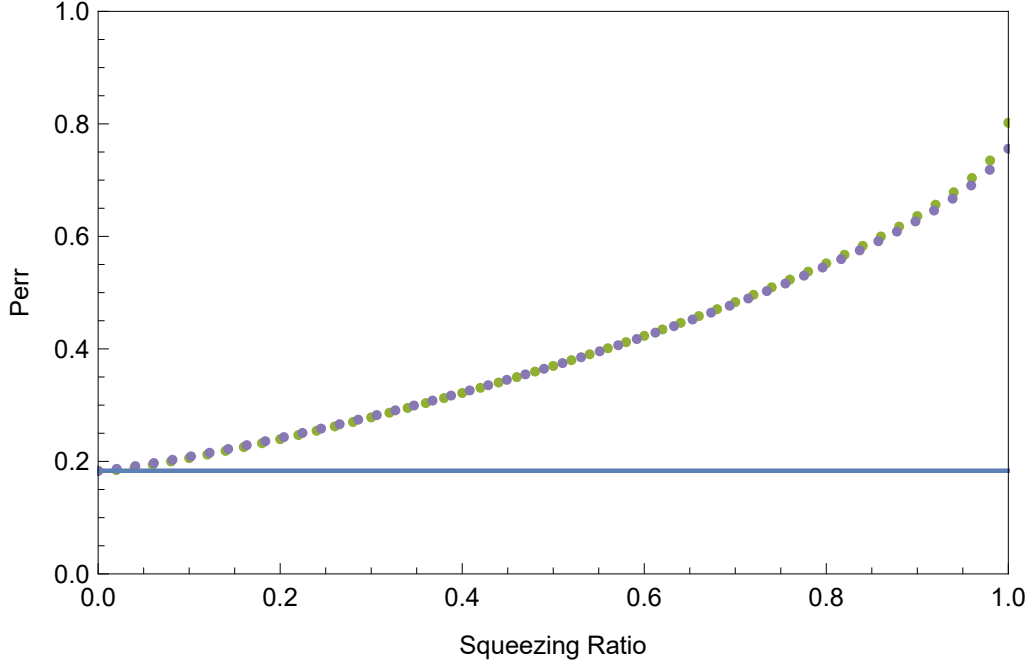


FIG. S1. Probability of error for ternary-phase-shift-keyed signals and dual-homodyne detection for states with a mean photon number of 1, against the proportion of energy allocated to squeezing. Purple and green markers are calculated by numerically integrating over the phase space coordinates or the phase angle respectively. The solid blue line is the coherent state error.

where $\mathcal{N}(z; \mu_z, \sigma_z^2)$ is a normal distribution. Note that we are using the convention that $\hat{X} = (\hat{a} + \hat{a}^\dagger)/2$. We can see that both the amplitude has been reduced and vacuum noise has been introduced, due to the necessity of a balanced beam-splitter in the dual-homodyne receiver. The one-third of phase space that we need to integrate over is defined by the bounds $p = \tan(\pm\pi/3)x$, which translates to the constraints $\sqrt{3}x > |p|$ and $x > 0$, so the probability of error is given by the integral

$$P = \int_0^\infty dx \int_{-\sqrt{3}x}^{\sqrt{3}x} dp P(x, p), \quad (27)$$

for which we do not have an analytic expression. However numerical integration shows that the probability of error can not be lowered through the introduction of squeezing, as seen in Figure S1.

We verify this numerical result by calculating the same quantity from a different integral. The probability distribution over x and p can be cast in terms of the phase angle $\theta = \arctan(\frac{p}{x})$; Aalo et al. [26] have shown that the resulting distribution is

$$P(\theta) = \frac{e^{-\mu_x^2/2\sigma_x^2}}{4\pi\sigma_x\sigma_p\mathcal{A}(\theta)} \left(\frac{\sqrt{\pi}\mathcal{B}(\theta)}{2\sqrt{\mathcal{A}(\theta)}} * e^{\frac{\mathcal{B}(\theta)}{4\mathcal{A}(\theta)}} * \text{Erfc}\left[\frac{-\mathcal{B}(\theta)}{2\sqrt{\mathcal{A}(\theta)}}\right] + 1 \right), \quad (28)$$

where

$$\mathcal{A}(\theta) = \frac{\cos^2(\theta)}{2\sigma_x^2} + \frac{\sin^2(\theta)}{2\sigma_p^2} \quad (29)$$

and

$$\mathcal{B}(\theta) = \frac{\mu_x \cos(\theta)}{\sigma_x^2}. \quad (30)$$

Numerical integration of this function over the range $\theta \in [-\pi/3, \pi/3]$ gives a result in agreement with the integration over the phase space coordinates, as shown in Figure S1. Slightly different values of γ were used in each case so that the two scatter plots might be distinguished. As $\gamma \rightarrow 1$ there is a small discrepancy which we attribute to errors in the numerical integration process, however this does not affect the findings. Finally we use MATLAB to run a Monte Carlo simulation of the communication protocol and find a result, shown in Figure S2, that is in agreement with the numerical integration.

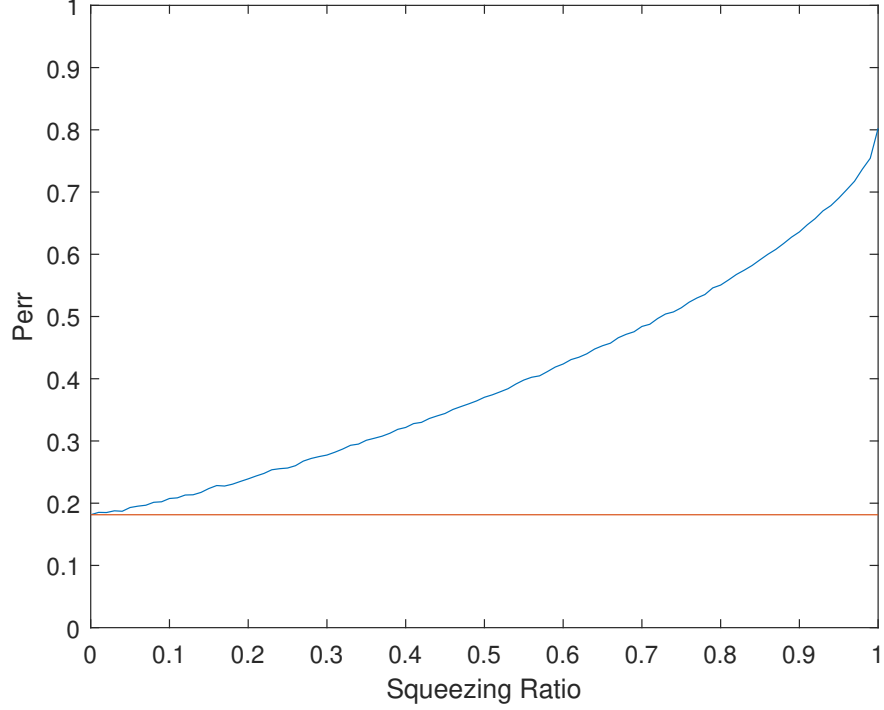


FIG. S2. Probability of error for ternary-phase-shift-keyed signals and dual-homodyne detection for states with a mean photon number of 1, against the proportion of energy allocated to squeezing. The blue trace is the error for a displaced squeezed state, and the orange trace is the coherent state error.

S3. FOUR STATE DISCRIMINATION

The four state phase shift keyed signal is

$$|\psi_k\rangle = \hat{R}\left(\frac{\pi(2k+1)}{4}\right)\hat{D}(\beta)\hat{S}(r)|0\rangle, k \in \{0, 1, 2, 3\}. \quad (31)$$

The additional rotation by $\pi/4$ gives the states symmetric variance in both quadratures

$$V = (e^{2r} + e^{-2r})/8, \quad (32)$$

which is minimised for $r = 0$. As the coherent amplitude of a state with fixed energy is maximised for $r = 0$ squeezing does not improve the performance of this signal. The SNR of the signal is

$$\text{SNR} = \frac{\bar{n}(1 - \gamma)}{\bar{n}\gamma + 1} \quad (33)$$

which is minimised for $\gamma = 0$.