

Time-Dependent Black Hole Lensing and the Unified Weak-to-Strong Deflection Framework

Ali Övgün ^{1,*} and Reggie C. Pantig ^{2,†}

¹*Physics Department, Eastern Mediterranean University,
Famagusta, 99628 North Cyprus via Mersin 10, Türkiye.*

²*Physics Department, School of Foundational Studies and Education,
Mapúa University, 658 Muralla St., Intramuros, Manila 1002, Philippines.*

We present a fully analytical framework that unifies weak-field and strong-deflection lensing of light in a time-dependent, perturbed Schwarzschild spacetime. The spacetime dynamics are modeled by a single, axisymmetric, even-parity quasinormal mode with $\ell = 2$, $m = 0$ and complex frequency ω . Working to first order in a small perturbation amplitude while keeping background null geodesics exact, we derive a time-dependent line-of-sight (Born) expression for the screen-plane deflection measured by a static observer at large radius. From the same integral, an asymptotic expansion yields the familiar weak-field $1/b$ law with a ringdown-frequency correction that drives a harmonic centroid wobble, whereas a near-photon-sphere expansion produces a time-dependent generalization of the logarithmic strong-deflection limit with modulated coefficients, including a small oscillation of the critical impact parameter. An observer tetrad built from the background static frame ensures that all screen-plane quantities—centroid motion, multi-image hierarchy and time delays, and photon-ring morphology—are gauge-safe at first order. We provide explicit matching across regimes, showing that the near-critical coefficients governing spacing and ring-radius modulations are encoded in the same Born kernel that controls the weak-field correction. The result is a coherent, purely theoretical account of how ringdown physics imprints on imaging observables without numerical ray tracing.

PACS numbers: 04.70.Bw; 04.25.Nx; 04.30.-w; 98.62.Sb; 95.30.Sf; 04.20.-q

Keywords: Gravitational lensing; quasinormal modes; Schwarzschild black hole; strong-deflection limit; photon ring.

I. INTRODUCTION

Gravitational lensing by compact objects remains a powerful probe of strong-field gravity and the nature of dark compact sources [1–13]. Foundational analyses of Schwarzschild lensing and its extensions to naked singularities established the basic phenomenology of multiple images, relativistic loops, and their observational signatures [14, 15]. Early work also highlighted how additional fields can imprint themselves on lensing observables [16] and connected these effects to broader questions such as Seifert’s conjecture [17]. The differential-geometric underpinning of photon surfaces furnished a rigorous framework for locating unstable light rings that control both strong lensing and shadow formation [18]. Building on this, detailed predictions for relativistic images, time delays, and magnification centroids were developed and refined [19, 20]. Recent progress has quantified image distortions in Schwarzschild lensing and uncovered conservation properties of distortion measures, providing robust diagnostics that are less sensitive to astrophysical systematics [21, 22]. In parallel, precise relations between the photon-sphere radius and black-hole shadow have been extended to include cosmological constant corrections, clarifying how large-scale curvature perturbs near-horizon optics [23]. These theoretical advances feed directly into constraints on the compactness of supermassive dark objects at galactic centers, sharpening tests of black-hole paradigms with high-resolution imaging and timing data [24]. Real-

istic black hole lensing is rarely vacuum: ambient plasma and magnetic fields induce frequency-dependent deflection, reshape caustics, and shift photon rings and shadows [25–27]. Such chromatic and dispersive effects persist across rotating and regular spacetimes, complicating image interpretation [25, 26]. Beyond GR, braneworld geometries predict systematic deviations in strong-deflection and retrolensing observables, offering complementary tests [28].

Gravitational lensing by black holes spans a wide dynamic range from weak deflections in the asymptotic region to the strong-field regime near the photon sphere [14, 29]. In parallel, the spacetime itself can be time-dependent, for instance, during black-hole ringdown when quasinormal modes (QNMs) drive metric perturbations with characteristic complex frequencies [30, 31]. These two features, wide spatial scales and intrinsic temporal modulation, are often modeled with separate tools that obscure their continuity. Our objective is to provide a single, purely analytical framework that treats both regimes within one description and that projects directly onto the observer’s screen in a gauge-safe manner.

We concentrate on light propagation in the Schwarzschild geometry of mass M , perturbed at first order by a single axisymmetric even-parity Zerilli mode with angular numbers $\ell = 2$, $m = 0$ and complex frequency $\omega = \omega_R + i\omega_I$ with $\omega_I < 0$ [32, 33]. This choice provides the simplest dynamical driver of ringdown that preserves axial symmetry, minimizes technical overhead, and still captures the essential physics of a decaying spacetime oscillation. Throughout, we work perturbatively in a small, dimensionless amplitude ε and keep the unperturbed null geodesics exact, so that all time dependence arises from the interaction of the QNM with the photon’s background trajectory [5, 34].

Our central construct is a time-dependent line-of-sight

* ali.ovgun@emu.edu.tr

† rcpantig@mapua.edu.ph

map that gives the deflection measured by a static observer at a large areal radius. The map is evaluated along the unperturbed null geodesic labeled by its impact parameter b . In the far field, it reproduces the familiar weak-bending law ($1/b$) [35–37] with an ε -suppressed, frequency-resolved correction that produces a harmonic motion of the image centroid on the screen. In the near-critical regime $b \rightarrow b_c^+$, with b_c the Schwarzschild critical impact parameter, the same map yields a time-dependent generalization of the logarithmic strong-deflection relation [6, 38]. The coefficients of this relation become slowly varying functions of the observer’s time, all oscillating at the QNM frequency. The critical scale itself acquires a small modulation that controls the most singular response near the photon ring [2, 39–42].

A key structural feature of our approach is the use of an observer tetrad fixed by the background static frame [2, 33]. Screen angles are constructed directly from locally measured photon momenta. At first order in ε , this guarantees that centroid shifts, image positions, and ring morphology are insensitive to gauge transformations that would otherwise act on the metric perturbation. In this way, the imaging diagnostics we develop are bona fide observables at the order of interest.

Recent work has begun to probe time-dependent gravitational lensing during black-hole relaxation. In particular, Zhong, Cardoso, and Chen [1] have used backward ray tracing in a relaxing Kerr background to show that the photon deflection inherits the characteristic ringdown pattern at intermediate times and exhibits a t_o^{-3} late-time decay, thereby establishing the qualitative imprint of QNM dynamics on light propagation [43]. Earlier analyses of time-dependent deflection by gravitational waves or localized, retarded sources also demonstrated oscillatory signatures in lensing and timing, albeit in regimes far from the photon sphere and without a black-hole ringdown setting [3, 4, 44–46]. At the same time, the strong-deflection limit (SDL) for stationary spacetimes is well understood, including the logarithmic divergence and its coefficients ($\bar{a}, \bar{b}, b_c$) that control image hierarchies and photon-ring morphology [6]. Classic studies of light propagation in nonstationary fields also delineate key limits: in particular, for localized sources, the wave-zone and intermediate-zone contributions cancel so that the leading time-dependent deflection is near-zone and falls as b^{-3} (contrasting with the $1/b$ Born imprint we obtain from a global QNM on a black-hole background), while stochastic or plane-wave backgrounds and moving lenses generate distinctive, velocity-dependent deflection and delay terms in the weak field [3–5]. Finally, oscillatory near-horizon structures can drive ring-scale astrometric modulations akin to ours, for example, superradiant bosonic clouds around spinning holes, which likewise predict QNM-frequency screen signatures in photon-ring observables, though sourced by matter fields rather than by the spacetime’s own even-parity QNM [47].

What is still missing is a unified, purely analytical framework that projects directly onto the observer’s screen [2], is gauge-safe at first order, and continuously connects the far-field Born law to a time-dependent SDL with QNM-modulated coefficients, while demonstrating explicit match-

ing between the two regimes. The present work fills this gap by deriving a single, line-of-sight Born kernel valid for all impact parameters and showing how the same kernel encodes the weak-field centroid wobble and the near-critical modulations, including the oscillation of the critical scale $b_c(t_o)$, thereby providing a coherent bridge between ringdown spectroscopy and imaging observables. For the gravitational-wave analogue—namely, the image-type-dependent distortions of lensed GW waveforms even in the geometric-optics limit—see [48].

We emphasize three theoretical payoffs of the unified treatment. First, weak-field centroid wobble, intermediate-field image multiplicity and delays, and strong-field photon-ring modulations are shown to be phase locked to the same QNM driver. Second, the link across regimes is explicit: the near-critical coefficients that govern the logarithmic law and the ring modulation are encoded in the near-critical expansion of the very amplitude that produces the weak-field $1/b$ correction. Third, all results follow from analytic asymptotics of a single line-of-sight kernel and therefore require no numerical ray tracing or data analysis.

The presentation is organized as follows. Section II specifies the background Schwarzschild geometry, the $\ell = 2, m = 0$ even-parity perturbation through the Zerilli master field, and the observer tetrad together with the screen construction and gauge considerations. Section III derives the time-dependent line-of-sight map and develops its far-field expansion, which produces the centroid wobble at the QNM frequency. Section IV extracts the near-photon-sphere behavior and obtains the time-dependent strong-deflection law together with the hierarchy of relativistic images and their arrival-time structure. Section V performs the matching between the weak and strong limits and assembles the screen-plane observables into a coherent set of diagnostics that are continuous across regimes. Section VI summarizes the framework and outlines natural extensions. Technical material is collected in three appendices: Appendix A reviews the Zerilli formalism and reconstructs the metric perturbation; Appendix B derives the master Born kernel and its equivalent representations; Appendix C develops the near-critical expansion and expresses the strong-deflection coefficients in terms of localized kernel data at the photon sphere.

II. BACKGROUND, PERTURBATION, AND OBSERVERS

In this section, we set up the spacetime, perturbation content, and observational frame used throughout. Our background is the Schwarzschild geometry of mass M in geometrized units $G = c = 1$. We then introduce a first-order, axisymmetric, even-parity QNM perturbation with angular numbers $(\ell, m) = (2, 0)$ and complex frequency $\omega = \omega_R + i\omega_I$ with $\omega_I < 0$. The perturbation is described by the Zerilli master function, whose dynamics and boundary conditions fix the time dependence that subsequently enters our lensing maps. The observer frame and screen construction will be given in Sec. II B.

A. Schwarzschild + even-parity $(\ell, m) = (2, 0)$ QNM

We take as background the Schwarzschild line element [39]

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2), \quad (1)$$

where

$$f(r) = 1 - \frac{2M}{r}, \quad (2)$$

Here, t is the Schwarzschild coordinate time, r is the areal coordinate radius, and (θ, ϕ) are the standard spherical coordinate angles. A null geodesic will be parameterized by an affine parameter λ with four-momentum $p_\mu = dx_\mu/d\lambda$, though in this subsection we focus on the spacetime rather than the geodesic equations.

We consider a linear perturbation of amplitude $\varepsilon \ll 1$,

$$g_{\mu\nu}(t, r, \theta, \phi) = g_{\mu\nu}^{(0)}(r) + \varepsilon h_{\mu\nu}(t, r, \theta, \phi), \quad (3)$$

where $g_{\mu\nu}^{(0)}$ is the Schwarzschild metric in Eq. (1) and $h_{\mu\nu}$ is an even-parity, axisymmetric QNM with $(\ell, m) = (2, 0)$. In the Regge-Wheeler-Zerilli (RWZ) gauge for $\ell \geq 2$, the nonvanishing components of an even-parity, axisymmetric perturbation can be written as [32, 49, 50]

$$\begin{aligned} h_{tt} &= f H_0(t, r) P_2(\cos\theta), \\ h_{tr} &= H_1(t, r) P_2(\cos\theta), \\ h_{rr} &= f^{-1} H_2(t, r) P_2(\cos\theta), \\ h_{\theta\theta} &= r^2 K(t, r) P_2(\cos\theta), \\ h_{\phi\phi} &= r^2 K(t, r) P_2(\cos\theta) \sin^2\theta, \end{aligned} \quad (4)$$

where H_0, H_1, H_2, K are functions of (t, r) and P_2 is the Legendre polynomial of degree $\ell = 2$. Axisymmetry implies no ϕ -dependence. Throughout the axisymmetric ($m = 0$) analysis here, we use the Legendre basis $P_\ell(\cos\theta)$ for angular dependence. In particular, for $\ell = 2$, $P_2(0) = -1/2$ for an equatorial observer. It is also convenient to package (H_0, H_1, H_2, K) into the gauge-invariant Zerilli master function $\Psi_Z(t, r)$, which fully characterizes the even-parity sector at first order.

Introducing the tortoise coordinate r_* defined by [39]

$$\frac{dr_*}{dr} = f(r)^{-1}, \quad r_* = r + 2M \ln\left(\frac{r}{2M} - 1\right), \quad (5)$$

the Zerilli function obeys the wave equation

$$(-\partial_t^2 + \partial_{r_*}^2 - V_Z(r)) \Psi_Z(t, r) = 0, \quad (6)$$

with the Zerilli potential [32, 33]

$$\begin{aligned} V_Z(r) &= \frac{2f(r)}{r^3} \frac{\lambda^2(\lambda+1)r^3 + 3\lambda^2 M r^2 + 9\lambda M^2 r + 9M^3}{(\lambda r + 3M)^2}, \\ \lambda &\equiv \frac{(\ell-1)(\ell+2)}{2}. \end{aligned} \quad (7)$$

For the $(\ell, m) = (2, 0)$ QNM, we impose homogeneous boundary conditions consistent with black-hole ringdown: purely ingoing at the event horizon and purely outgoing at null infinity. Writing $\Psi_Z(t, r) \propto e^{-i\omega t} \psi(r)$ with complex frequency $\omega = \omega_R + i\omega_I$ and $\omega_I < 0$, the asymptotics are [30]

$$\begin{aligned} \Psi_Z &\sim e^{-i\omega(t+r_*)} \quad \text{for } (r \rightarrow 2M), \\ \Psi_Z &\sim e^{-i\omega(t-r_*)} \quad \text{for } (r \rightarrow \infty). \end{aligned} \quad (8)$$

These conditions quantize ω to the discrete $\ell = 2$ even-parity spectrum and fix the temporal decay $e^{\omega_I t}$ inherent to ringdown.

At linear order in ε , the metric perturbation $h_{\mu\nu}$ can be reconstructed from Ψ_Z through algebraic and first-derivative relations standard in the RWZ formalism; we defer the explicit formulas to Appendix A. For definiteness, we normalize Ψ_Z so that the leading outgoing amplitude at large r is unity, and we absorb the physical smallness of the perturbation into ε . With these conventions, Eq. (3) provides the unique axisymmetric, even-parity $\ell = 2$ driving mode that will source the time dependence entering our unified lensing formalism in later sections.

B. Observer tetrad and gauge

We model detection at future null infinity by an asymptotic static observer located at areal radius $r_o \gg M$ and polar angle θ_o . The observer's four-velocity u^μ is aligned with the static Killing field at zeroth order and is corrected at $\mathcal{O}(\varepsilon)$ so that normalization holds in the perturbed geometry of Eq. (3). A convenient orthonormal tetrad $e^\mu_{\hat{a}}$ ($\hat{a} = \hat{t}, \hat{r}, \hat{\theta}, \hat{\phi}$) at the observer worldline is

$$\begin{aligned} e^\mu_{\hat{t}} &= f(r_o)^{-1/2} \delta^\mu_t, \\ e^\mu_{\hat{r}} &= f(r_o)^{1/2} \delta^\mu_r, \\ e^\mu_{\hat{\theta}} &= \frac{1}{r_o} \delta^\mu_\theta, \\ e^\mu_{\hat{\phi}} &= \frac{1}{r_o \sin\theta_o} \delta^\mu_\phi, \end{aligned} \quad (9)$$

with f given in Eq. (1) (here θ_o is the polar angle of the observer; the $\hat{\phi}$ -leg contains $\sin\theta_o$ as usual). These vectors satisfy

$$g_{\mu\nu}(x_o) e^\mu_{\hat{a}} e^\nu_{\hat{b}} = \eta_{\hat{a}\hat{b}} + \mathcal{O}(\varepsilon^2), \quad (10)$$

where x_o^μ denotes the observer event and $\eta_{\hat{a}\hat{b}} = \text{diag}(-1, 1, 1, 1)$. The $\mathcal{O}(\varepsilon)$ corrections implicit in Eq. (10) are uniquely fixed by enforcing orthonormality with the full metric of Eq. (3); their explicit forms are deferred to Appendix A.

Let p^μ be the photon four-momentum at x_o^μ . The components measured in the observer frame are

$$p^{\hat{a}} = e^{\hat{a}}_{\mu} p^\mu, \quad e^{\hat{a}}_{\mu} e^\mu_{\hat{b}} = \delta^{\hat{a}}_{\hat{b}}, \quad (11)$$

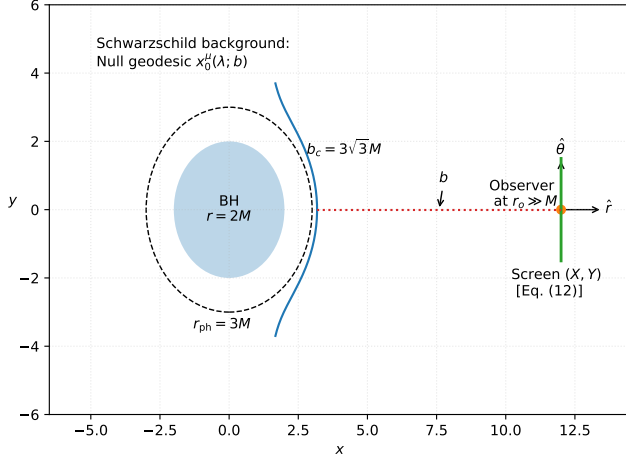


FIG. 1. Geometry and observer-screen schematic. The Schwarzschild background with horizon at $r = 2M$ (filled disk) and photon sphere at $r_{\text{ph}} = 3M$ (dashed circle) is shown together with a near-critical null trajectory with closest approach $r_0(b)$. A static observer at $r_o \gg M$ carries the tetrad of Eq. (9) and measures screen coordinates (X, Y) via Eq. (13). The critical impact parameter $b_c = 3\sqrt{3}M$ is indicated. This setup underlies the background bending integral in Eq. (17) and the time-dependent Born map (Eqs. (19)-(24)).

and the instantaneous celestial direction is encoded in the two-vector

$$\mathbf{n} \equiv \frac{\mathbf{p}}{p^{\hat{0}}} = \left(\frac{p^{\hat{r}}}{p^{\hat{0}}}, \frac{p^{\hat{\theta}}}{p^{\hat{0}}}, \frac{p^{\hat{\phi}}}{p^{\hat{0}}} \right), \quad (12)$$

where bold symbols denote spatial components in the tetrad. We define the observer screen as the plane orthogonal to $e^{\mu}_{\hat{r}}$ at x_o^{μ} and adopt the right-handed screen basis $(\hat{X}, \hat{Y}) = (\hat{\phi}, -\hat{\theta})$. The screen coordinates (X, Y) of the photon are obtained by the usual perspective map for a local inertial observer [2, 51],

$$X = -\frac{r_o p^{\hat{\phi}}}{p^{\hat{0}}}, \quad Y = \frac{r_o p^{\hat{\theta}}}{p^{\hat{0}}}, \quad b \equiv \sqrt{X^2 + Y^2}, \quad (13)$$

which defines the impact parameter b measured on the screen. Eq. (13) is equivalent to the common celestial-sphere parametrization by small angles $(\theta_X, \theta_Y) = (X/r_o, Y/r_o)$ at large r_o . We adopt the convention that $+X$ points toward decreasing ϕ (hence the minus sign in X), and $+Y$ toward increasing θ .

Figure 1 fixes the observer-plane conventions used throughout. The tetrad and the screen map make the measured angles and the impact parameter b explicit, while the photon sphere at $r_{\text{ph}} = 3M$ highlights the near-critical domain $b \rightarrow b_c^+$ central to Secs. IV-V. All weak- and strong-field results are projections onto this screen.

Because the perturbation is axisymmetric, we may, without loss of generality, take $\theta_o = \pi/2$ for most calculations. At zeroth order in ε the conserved energy $E = -p_t$ and angular

momentum $L = p_{\phi}$ of the photon in the Schwarzschild background Eq. (1) satisfy

$$\frac{X}{r_o} \simeq -\frac{L}{E r_o \sin \theta_o}, \quad \frac{Y}{r_o} \simeq \frac{p_{\theta}}{E r_o}, \quad b = \frac{L}{E} + \mathcal{O}\left(\frac{M}{r_o}\right), \quad (14)$$

so that b reduces to the usual ratio L/E in the asymptotically flat region. In later sections, we shall describe deflection through a two-dimensional vector $\alpha(b, t_o)$ defined on the screen; its weak- and strong-field forms follow from the master relation introduced in Eq. (1) of Sec. III.

We now address gauge issues at first order in ε . A linearized diffeomorphism generated by a vector field $\xi^{\mu} = \mathcal{O}(\varepsilon)$ acts on the metric perturbation as

$$h_{\mu\nu} \rightarrow h_{\mu\nu} - \nabla_{\mu} \xi_{\nu} - \nabla_{\nu} \xi_{\mu}, \quad (15)$$

and on tensorial quantities derived from $h_{\mu\nu}$ accordingly. We implement two safeguards to ensure that screen observables built from Eq. (13) are gauge-independent at $\mathcal{O}(\varepsilon)$. First, all dynamical content of the even-parity perturbation is encoded in the Zerilli gauge-invariant master field Ψ_Z of Eqs. (6)-(8), and the reconstruction used in Appendix A expresses $p^{\hat{a}}$ corrections in terms of Ψ_Z and its derivatives only. Second, the tetrad is fixed operationally by the normalization and orthogonality conditions in Eq. (10) at the physical observer event x_o^{μ} . With these choices, any change induced by Eq. (15) in the local frame is a pure $\mathcal{O}(\varepsilon)$ Lorentz transformation that leaves (X, Y) invariant at that order. Concretely, one finds that the gauge variation of the screen map is a boundary term proportional to ξ^{μ} evaluated at r_o , and our asymptotic conditions impose $\xi^{\mu} = \mathcal{O}(1/r_o)$, hence

$$\delta_{\varepsilon} X = \delta_{\varepsilon} Y = 0 + \mathcal{O}\left(\frac{\varepsilon}{r_o}\right). \quad (16)$$

Eq. (16) guarantees that the centroid motion, multi-image separations, and photon-ring modulations defined later are bona fide observables at first order.

Finally, we denote by t_o the Schwarzschild time coordinate of the detection event x_o^{μ} , and by λ_o the corresponding affine parameter value on the photon worldline. The association $t \mapsto t_o$ defines the phase with which the QNM signal in Eqs. (6)-(8) is sampled by the observer and will be the sole time variable entering the deflection $\alpha(b, t_o)$ in Secs. III-V.

III. UNIFIED BORN INTEGRAL AND WEAK-FIELD EXPANSION

We now derive a single, time-dependent line-of-sight expression for the lensing deflection that is valid to first order in the perturbation amplitude ε of Eq. (3) while keeping the Schwarzschild background of Eq. (1) exact. The resulting “master Born integral” will be evaluated along the unperturbed null geodesic with impact parameter b and will furnish, by asymptotic expansion, both the far-field Born law (Sec. III B) and the time-dependent strong-deflection form (Sec. IV). All observable angles are defined on the screen introduced in Sec. II B and are therefore gauge-safe to $\mathcal{O}(\varepsilon)$.

A. Master time-dependent Born integral

Let $x_0^\mu(\lambda; b)$ be the null geodesic of the Schwarzschild background (Eq. (1)) with conserved energy $E = -p_{0t}$ and angular momentum $L = p_{0\phi}$, and let $p_0^\mu(\lambda) = dx_0^\mu/d\lambda$ be its four-momentum. The impact parameter measured at infinity is $b = L/E$ to leading order, consistent with Eq. (14). The background bending angle $\alpha_0(b)$ is given exactly by the standard integral [36]

$$\alpha_0(b) = 2 \int_{r_0(b)}^{\infty} \frac{dr}{r^2} \left[\frac{1}{b^2} - \frac{f(r)}{r^2} \right]^{-1/2} - \pi, \quad (17)$$

where $r_0(b)$ is the distance of closest approach determined by $b^2 = r_0^2/f(r_0)$.

We switch on the first-order, axisymmetric, even-parity QNM perturbation of Eq. (3), encoded by the Zerilli master field Ψ_Z that satisfies Eqs. (6)-(8). In the ‘‘Born’’ or single-scattering approximation on the curved background, the photon follows $x_0^\mu(\lambda; b)$ while its momentum acquires an $\mathcal{O}(\varepsilon)$ correction from the perturbed Hamiltonian $H = (1/2)g^{\mu\nu}p_\mu p_\nu = 0$. Linearizing Hamilton’s equation $dp_\mu/d\lambda = -\partial_\mu H$ about the background solution and evaluating the source on x_0^μ and p_0^μ yields

$$\delta p_\mu(\lambda_o; b) = -\frac{1}{2} \int_{\lambda_s}^{\lambda_o} d\lambda \partial_\mu h_{\alpha\beta}(x_0(\lambda)) p_0^\alpha(\lambda) p_0^\beta(\lambda), \quad (18)$$

where λ_o and λ_s denote, respectively, the values at the observer and the source, and indices are raised/lowered with the background metric. Eq. (18) is the covariant Born response for the photon momentum on the fixed Schwarzschild trajectory.

Projecting δp_μ on the observer tetrad of Eq. (9) and converting to screen angles via Eq. (13) gives the master deflection map

$$\alpha(b, t_o) = \alpha_0(b) + \varepsilon \delta\alpha(b, t_o), \quad (19)$$

with $\alpha_0(b)$ determined by Eq. (17), and

$$\delta\alpha_A(b, t_o) = \int_{\lambda_s}^{\lambda_o} d\lambda \mathcal{K}_A[x_0(\lambda); p_0(\lambda)], \quad A \in X, Y. \quad (20)$$

The kernel $\mathcal{K}_A$ is linear in $h_{\mu\nu}$ and its first derivatives and depends only on background quantities and on the screen projector at the observer. A convenient representative is

$$\begin{aligned} \mathcal{K}_A = & -\frac{1}{2p_0^{\hat{0}}(\lambda_o)} \Pi_A{}^\mu(x_o) e^{\nu}_{\hat{0}}(x_o) \\ & \times \partial_\mu h_{\alpha\beta}(x_0(\lambda)) p_0^\alpha(\lambda) p_0^\beta(\lambda), \end{aligned} \quad (21)$$

where $e^{\nu}_{\hat{0}}$ is the time leg of the observer tetrad Eq. (9), $\Pi_A{}^\mu$ projects onto the (X, Y) screen directions at the observer Eq. (13), and hats denote tetrad components at x_o^μ . The above projector form is one convenient representative; Appendix B shows its equivalence to a manifestly covariant $\delta\Gamma$ line integral with cancelling boundary terms under our asymptotic

conditions. Different but equivalent kernel representatives follow from integrating Eq. (18) by parts and using the background geodesic equation; all yield the same $\delta\alpha$ at $\mathcal{O}(\varepsilon)$. Note that the screen deflection arises from varying X and Y in Eq. (13), yielding

$$\delta\alpha_A = \frac{1}{p_0^{\hat{0}}} \Pi_A{}^{\hat{b}} \delta p_{\hat{b}} \quad (22)$$

The QNM time dependence in Eqs. (6)-(8) implies that, along the background ray, the perturbation takes the form

$$\begin{aligned} h_{\mu\nu}(x_0(\lambda)) &= \text{Re} \left\{ \hat{h}_{\mu\nu}(r(\lambda), \theta(\lambda)) e^{-i\omega t_0(\lambda)} \right\}, \\ \omega &= \omega_R + i\omega_I \quad \text{for } \omega_I < 0, \end{aligned} \quad (23)$$

where $t_0(\lambda)$ is the Schwarzschild time along $x_0^\mu(\lambda)$. Writing $t_0(\lambda) = t_o - \tau(\lambda)$ with $\tau(\lambda) \geq 0$ the light-travel delay between the encounter point and the observer event, Eq. (20) can be cast as

$$\begin{aligned} \delta\alpha_A(b, t_o) &= \text{Re} \left\{ e^{-i\omega t_o} \mathcal{A}_A(b, \omega) \right\}, \\ \mathcal{A}_A(b, \omega) &= \int_{\lambda_s}^{\lambda_o} d\lambda \tilde{\mathcal{K}}_A[r(\lambda), \theta(\lambda)] e^{i\omega \tau(\lambda)}, \end{aligned} \quad (24)$$

where $\tilde{\mathcal{K}}_A$ is the complex kernel obtained from $\mathcal{K}_A$ after factoring out the harmonic time dependence (the explicit form follows from Appendix B together with the RWZ reconstruction in Appendix A). Eq. (24) shows that the measured deflection inherits the single QNM frequency ω ; its amplitude $\mathcal{A}_A$ depends on b only through the unperturbed trajectory and the delay profile $\tau(\lambda)$.

Eqs. (19)-(24) constitute the master, time-dependent Born formalism. Two regimes will be obtained from the same expression. In Sec. III B we expand Eq. (24) for $M/b \ll 1$ to recover the far-field Born law and the centroid wobble. In Sec. IV we analyze the contribution from segments of the trajectory that probe $r \approx 3M$, showing that Eq. (24) reproduces the logarithmic strong-deflection behavior once $b \rightarrow b_c^+ = 3\sqrt{3}M$, with the time dependence inherited from the same phase factor.

B. Far-field (Born) expansion and centroid wobble

We now evaluate Eqs. (19)-(24) in the weak-deflection domain $M/b \ll 1$, keeping the Schwarzschild background exact along the ray but expanding the integrals in powers of M/b . The background bending reduces to [35-37]

$$\alpha_0(b) = \frac{4M}{b} + \mathcal{O}\left(\frac{M^2}{b^2}\right), \quad (25)$$

as obtained from Eq. (17). For the perturbation, we use Eq. (24) with the unperturbed trajectory approximated by a straight line of impact parameter b through the asymptotically flat region, so that

$$r(\lambda) \simeq \sqrt{b^2 + z^2}, \quad t_0(\lambda) = t_o - \tau(\lambda), \quad \tau(\lambda) \simeq z, \quad (26)$$

where z is the affine proxy ($z = 0$ at the observer) for the longitudinal distance along the ray and t_o is the detection time

In the far zone the even-parity $\ell = 2$ QNM behaves as an outgoing quadrupolar wave with $h_{\mu\nu} \sim r^{-1} e^{-i\omega(t-r_*)}$, hence $\partial_\mu h_{\alpha\beta} \sim r^{-2} e^{-i\omega(t-r_*)}$. Substituting this falloff into Eq. (20) or, equivalently, Eq. (24), we obtain the leading Born correction

$$\delta\alpha_A(b, t_o) = \text{Re} \left\{ e^{-i\omega t_o} \frac{c_A(\omega)}{b} \mathcal{F}_A(\omega b) \right\} + \mathcal{O} \left(\frac{M}{b^2} \right), \quad A \in \quad (27)$$

where $c_A(\omega)$ are complex, frequency-dependent coefficients fixed by the RWZ reconstruction (Appendix A) and the kernel choice (Appendix B), and

$$\mathcal{F}_A(\omega b) \equiv \int_{-\infty}^{+\infty} \frac{dz}{b} \hat{\mathcal{K}}_A \left(\frac{\sqrt{b^2 + z^2}}{b} \right) e^{i\omega z} \quad (28)$$

is a dimensionless Fresnel-type factor encoding the finite-frequency sampling of the QNM along the nearly straight path. The explicit form of $\hat{\mathcal{K}}_A$ is given in Appendix B. Two limits of Eq. (27) are especially transparent:

$$\delta\alpha_A(b, t_o) \sim \begin{cases} \text{Re} \left\{ e^{-i\omega t_o} \frac{\tilde{c}_A(\omega)}{b} \right\}, & |\omega|, b \ll 1, \\ \text{Re} \left\{ e^{-i\omega t_o} \frac{\hat{c}_A(\omega)}{b} e^{-|\text{Im}\omega|b} \right\}, & |\omega|, b \gg 1, \end{cases} \quad (29)$$

with $\tilde{c}_A, \hat{c}_A$ calculable constants. Thus the leading time-dependent correction decays as $1/b$ and inherits the QNM phase $e^{-i\omega t_o}$, in agreement with the structure of Eq. (24).

Collecting Eqs. (25) and (27), the far-field deflection vector on the screen reads

$$\alpha(b, t_o) = \frac{4M}{b} \hat{e}_b + \varepsilon \text{Re} \left\{ e^{-i\omega t_o} \frac{\mathcal{A}(b, \omega)}{b} \right\} + \mathcal{O} \left(\frac{M^2}{b^2}, \varepsilon \frac{M}{b^2} \right), \quad (30)$$

where $\hat{e}_b$ is the outward radial unit vector in the (X, Y) plane and $\mathcal{A} = (c_X \mathcal{F}_X, c_Y \mathcal{F}_Y)$. That is, on the observer's screen we use the orthonormal basis $\{\hat{e}_X, \hat{e}_Y\}$ and define the screen radial unit vector $\hat{e}_b \equiv (X \hat{e}_X + Y \hat{e}_Y)/b$. The small $\mathcal{O}(M/b^2)$ terms arise from curvature corrections to Eq. (26) and from subleading pieces of the kernel in Eq. (20). As seen in Fig. 2, the $\mathcal{O}(\varepsilon)$ deflection decays like $1/b$, with gentle structure governed by $\mathcal{F}_A(\omega b)$. This is the weak-field imprint that, when averaged over a bundle, yields the centroid wobble in Eq. (32).

To connect with observables we project Eq. (30) onto the screen angles $(\theta_X, \theta_Y) = (X/r_o, Y/r_o)$ defined in Eq. (13). For a narrow bundle of rays of fixed b arriving at (X, Y) , the apparent position shift is

$$\Delta\theta(b, t_o) = \frac{\alpha(b, t_o)}{r_o} \quad (31)$$

up to the usual thin-deflection identification valid at $r_o \gg M$. We define the image centroid $\theta_c(t_o)$ as the intensity-weighted

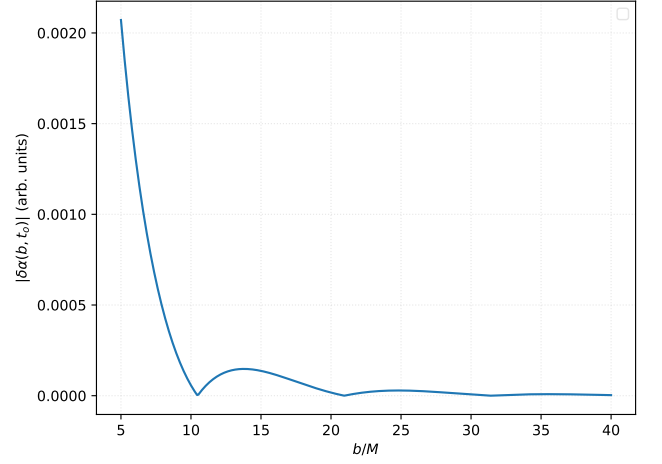


FIG. 2. Far-field, QNM-driven deflection amplitude vs impact parameter. The $\mathcal{O}(\varepsilon)$ correction from the Born map ($\omega_I < 0$) behaves as $|\delta\alpha(b, t_o)| \sim |c_A(\omega) \mathcal{F}_A(\omega b)|/b$, leading to the $1/b$ falloff summarized in Eq. (30). Mild oscillations from the Fresnel factor $\mathcal{F}_A$ encode the finite-frequency sampling of the QNM along the nearly straight path, Eq. (26).

average position of the bundle on the screen; within the present purely theoretical treatment, we capture its dynamics by the average of Eq. (31) over the bundle's narrow impact-parameter distribution. The centroid, therefore, exhibits a harmonic “wobble” at the QNM frequency:

$$\theta_c(t_o) = \theta_{c,0} + \varepsilon \text{Re} \left\{ e^{-i\omega t_o} \frac{\mathcal{B}(\omega)}{r_o} \right\} + \mathcal{O} \left(\frac{M}{r_o} \frac{M}{b^2} \right), \quad (32)$$

where $\theta_{c,0}$ is the static centroid from α_0 and $\mathcal{B}(\omega)$ is a bundle-dependent complex amplitude obtained from $\mathcal{A}(b, \omega)$ in Eq. (30). The phase of the wobble equals $\arg \mathcal{B}$ and is inherited from the travel-time weighting in Eq. (24). As seen in Fig. 3, the centroid motion is a clean, single-frequency probe of the perturbation, with decay governed by $\omega_I < 0$. The phase is inherited from the line-of-sight kernel in Eq. (24), anticipating the phase-locked relations with strong-field observables established in Sec. V. Eq. (32) provides the weak-field imaging diagnostic used later to match continuously to the strong-deflection modulations derived from the same master integral in Sec. IV.

IV. TIME-DEPENDENT STRONG-DEFLECTION LENSING

We now extract from the master Born representation in Eqs. (19)-(24) the strong-deflection limit *SDL* relevant to trajectories that skim the photon sphere. The analysis is controlled by the small parameter $\Delta \equiv b/b_c - 1$ with $0 < \Delta \ll 1$, where b is the impact parameter defined by Eq. (13) and b_c denotes the critical value associated with the unstable circular null orbit. The time dependence enters exclusively through the $(\ell, m) = (2, 0)$ QNM perturbation

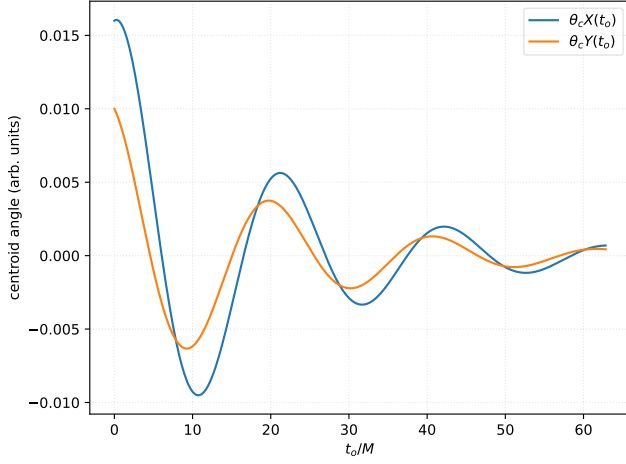


FIG. 3. Centroid wobble in the weak field from Eq. (32). The components $\theta_{c,X}(t_o)$ and $\theta_{c,Y}(t_o)$ execute a damped harmonic motion at the QNM frequency $\omega = \omega_R + i\omega_I$ with $\omega_I < 0$, with complex amplitude set by $\mathcal{B}(\omega)$. This panel visualizes the $\mathcal{O}(\varepsilon)$, gauge-safe centroid modulation predicted by the Born expansion in Eqs. (27)-(30).

of Sec. II, which modulates the SDL coefficients and, in particular, the critical scale b_c . Our goal in this section is to obtain a time-dependent generalization of the logarithmic SDL law from the same kernel that produced the Born expansion in Sec. III.

A. Near-photon-sphere expansion

On the Schwarzschild background, the circular photon orbit sits at the photon-sphere radius [39]

$$r_{\text{ph}} = 3M, \quad b_c = \frac{r_{\text{ph}}}{\sqrt{f(r_{\text{ph}})}} = 3\sqrt{3}M. \quad (33)$$

For $b \rightarrow b_c^+$ the bending angle $\alpha_0(b)$ obtained from Eq. (17) exhibits the well-known logarithmic divergence, [6, 38]

$$\alpha_0(b) = -\bar{a} \ln \left(\frac{b}{b_c} - 1 \right) + \bar{b} + \mathcal{O}((b - b_c) \ln |b - b_c|), \quad (34)$$

with the Schwarzschild values $\bar{a} = 1$, and $\bar{b} = \ln [216(7 - 4\sqrt{3})] - \pi$ if one adopts the standard normalization. The constant $\bar{b}$ depends on the subtraction convention for the $-\pi$ in the strong-deflection expansion; we follow the standard normalization commonly used for Schwarzschild. Eq. (34) follows by expanding the integral in Eq. (17) around the double root of the radial potential at r_{ph} .

We now switch on the perturbation of Eq. (3) and use the time-dependent Born representation, Eqs. (19)-(24), to determine how the coefficients in Eq. (34) are modulated. Physically, the integral in Eq. (24) becomes dominated by the segment of the unperturbed trajectory that executes a long “whirl” near r_{ph} , where small changes in the local

geometry feed into large changes of the total deflection. It is therefore convenient to introduce the near-photon-sphere coordinate

$$\rho \equiv \frac{r - r_{\text{ph}}}{r_{\text{ph}}}, \quad |\rho| \ll 1, \quad (35)$$

and to parameterize the background null ray by the angle φ accumulated during the whirl phase. In this regime the unperturbed relations implied by Eq. (17) reduce to the standard form

$$\varphi(\rho) \simeq -\frac{1}{\sqrt{\kappa}} \ln |\rho| + \varphi_0, \quad \kappa = \frac{1}{2} r_{\text{ph}}^2 (\partial_r^2 V_{\text{eff}})_{r_{\text{ph}}}, \quad (36)$$

where V_{eff} is the Schwarzschild null effective potential ($V_{\text{eff}} = f L^2 / r^2$ for equatorial motion), and $\kappa > 0$ encodes the instability exponent of the circular orbit. The logarithm in Eq. (36) is the origin of the SDL behavior in Eq. (34).

To first order in the perturbation amplitude ε , the metric perturbation $h_{\mu\nu}$ sampled along the whirl contributes to the kernel in Eq. (24) through $\partial_\mu h_{\alpha\beta} p_0^\alpha p_0^\beta$ evaluated at $r \simeq r_{\text{ph}}$. Exploiting Eq. (36) to trade the affine parameter for φ , the correction $\delta\alpha$ in Eq. (19) can be written, up to terms that remain finite as $b \rightarrow b_c^+$, as

$$\delta\alpha_A(b, t_o) = \text{Re} \left\{ e^{-i\omega t_o} \left[\mathcal{C}_A(\omega) \ln \left(\frac{b}{b_c} - 1 \right) + \mathcal{D}_A(\omega) \right] \right\} + \mathcal{O}((b - b_c) \ln |b - b_c|), \quad (37)$$

where the complex amplitudes $\mathcal{C}_A(\omega)$ and $\mathcal{D}_A(\omega)$ are determined by the near-photon-sphere part of the Born kernel and depend only on background quantities at r_{ph} and on the QNM frequency ω . The explicit expressions, obtained by evaluating the RWZ reconstruction (Appendix A) inside the line-of-sight integral (Appendix B), are not needed for the present section.

Combining Eqs. (34) and (37) with Eq. (19), we arrive at a compact time-dependent SDL law of the form

$$\alpha(b, t_o) = -\bar{a}(t_o) \ln \left(\frac{b}{b_c(t_o)} - 1 \right) + \bar{b}(t_o) + \mathcal{O}((b - b_c) \ln |b - b_c|), \quad (38)$$

where

$$\begin{aligned} \bar{a}(t_o) &= \bar{a} + \varepsilon \text{Re} \{ e^{-i\omega t_o} a_1(\omega) \}, \\ \bar{b}(t_o) &= \bar{b} + \varepsilon \text{Re} \{ e^{-i\omega t_o} b_1(\omega) \}, \\ b_c(t_o) &= b_c [1 + \varepsilon \text{Re} \{ e^{-i\omega t_o} \beta_1(\omega) \}], \end{aligned} \quad (39)$$

with a_1, b_1, β_1 complex response coefficients that follow from the same near-photon-sphere kernel that produced $\mathcal{C}_A, \mathcal{D}_A$ in Eq. (37). Eq. (38) is nothing but Eq. (34) with time-dependent coefficients. The modulation of b_c in Eq. (39) can be interpreted geometrically as a small shift of the photon-sphere location,

$$r_{\text{ph}}(t_o) = 3M + \varepsilon \text{Re} \{ e^{-i\omega t_o} \delta r_{\text{ph}}(\omega) \}, \quad (40)$$

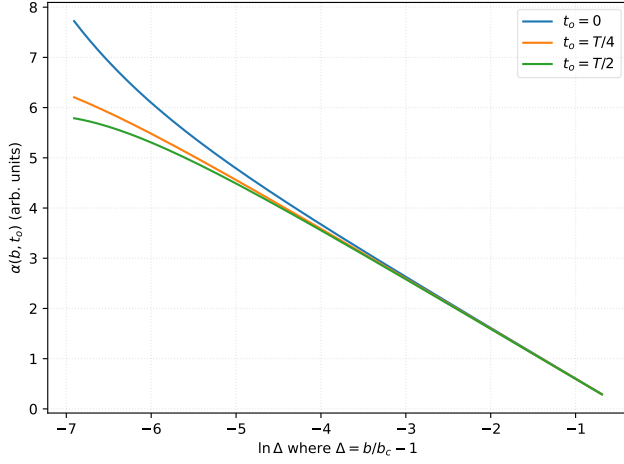


FIG. 4. Near-critical deflection from the time-dependent SDL law. Shown is $\alpha(b, t_o)$ versus $\ln \Delta$ with $\Delta = b/b_c - 1 \ll 1$, as predicted by Eq. (38) and its linearized form Eq. (41). The dominant $1/\Delta$ sensitivity arises from the modulation of $b_c(t_o)$ through $\beta_1(\omega)$, while the slope receives a small time-dependent correction from $a_1(\omega)$. Parallel, slightly offset curves at different t_o illustrate the coherent QNM-driven modulation.

propagated to $b_c(t_o)$ through the algebraic relation in Eq. (33). The quantity δr_{ph} is determined by the $\mathcal{O}(\varepsilon)$ change of the null effective potential extremum in the perturbed geometry and is therefore a gauge-invariant statement at the order of interest, given our use of the Zerilli master field.

It is often convenient to linearize Eq. (38) explicitly in ε . Writing $\Delta = b/b_c - 1$ and keeping only $\mathcal{O}(\varepsilon)$ terms, we obtain

$$\alpha(b, t_o) = \alpha_0(b) + \varepsilon \operatorname{Re} \left\{ e^{-i\omega t_o} \left[-a_1(\omega) \ln \Delta + \bar{a} \frac{\beta_1(\omega)}{\Delta} + b_1(\omega) \right] \right\} + \mathcal{O}(\varepsilon \delta \ln \Delta), \quad (41)$$

where $\alpha_0(b)$ is given by Eq. (34), and $\bar{a} = 1$ for the Schwarzschild case. By axisymmetry, the leading $\mathcal{O}(\varepsilon)$ near-critical deflection is radial on the screen; we therefore drop tangential components in what follows. The term proportional to β_1/Δ originates from expanding $\ln(b/b_c(t_o) - 1)$ and makes explicit that an oscillatory shift of b_c produces the largest time-dependent contribution as $b \rightarrow b_c^+$. With our P_2 normalization, $\beta_1(\omega) = \frac{1}{4} H_0(r_{\text{ph}}; \omega)$. Figure 4 makes explicit the near-critical amplification: the β_1/Δ piece dominates as $\Delta \rightarrow 0^+$, while the $-a_1 \ln \Delta$ term induces a mild slope variation. This behavior feeds directly into the image hierarchy and photon-ring modulations derived in Sec. IV B, and it will be matched to the weak-field kernel in Sec. V A. Eq. (41) is the central near-critical result that will feed, in Sec. IV B, the logarithmic time delay, the multiplicity of relativistic images, and their photon-ring modulations, all sharing the common QNM frequency inherited from Eq. (24).

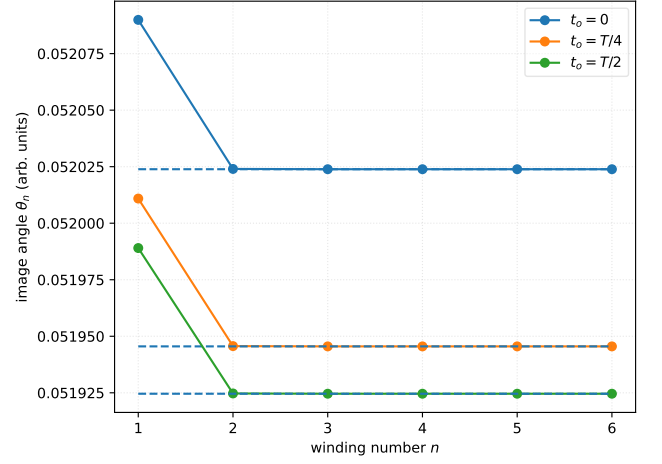


FIG. 5. Relativistic image hierarchy and photon ring. The angular positions $\theta_n(t_o)$ are shown versus winding number n for several snapshots in t_o , with the instantaneous ring radius $\theta_{\text{ring}}(t_o) = b_c(t_o)/r_o$ indicated. The geometric approach $\theta_n \rightarrow \theta_{\text{ring}}$ is governed by $s_n^{(0)} = \exp[(\bar{b} - 2\pi n)/\bar{a}]$, while the coherent, small-amplitude wobble of all θ_n is driven by the QNM via the response amplitudes in Eq. (44).

B. Time delays, multiplicity, and photon-ring modulation

Eq. (38) implies that, for each winding number $n \in \mathbb{N}$, relativistic images arise from solutions of the lens equation with total bending $\alpha(b, t_o) \approx 2\pi n + \Delta\alpha_n$ where $|\Delta\alpha_n| \ll 1$. Using Eq. (38) at fixed observer time t_o and solving for b near $b_c(t_o)$ gives the instantaneous sequence of critical impact parameters

$$b_n(t_o) = b_c(t_o) [1 + s_n(t_o)], \quad s_n(t_o) \equiv \exp \left(\frac{\bar{b}(t_o) - 2\pi n}{\bar{a}(t_o)} \right), \quad (42)$$

which is the time-dependent generalization of the standard static SDL hierarchy. The corresponding image angles on the screen are $\theta_n(t_o) = b_n(t_o)/r_o$, where r_o is the observer radius introduced in Sec. II B.

To expose the imprint of the $(\ell, m) = (2, 0)$ QNM, we linearize Eq. (42) in ε using Eq. (39). Writing $\theta_c \equiv b_c/r_o$ and $s_n^{(0)} \equiv \exp[(\bar{b} - 2\pi n)/\bar{a}]$, we obtain

$$\theta_n(t_o) = \theta_c \left(1 + s_n^{(0)} \right) + \varepsilon \operatorname{Re} \left\{ e^{-i\omega t_o} \delta\theta_n \right\} + \mathcal{O}(\varepsilon s_n^{(0)2}), \quad (43)$$

with the complex response amplitude

$$\delta\theta_n = \theta_c \beta_1 \left(1 + s_n^{(0)} \right) + \theta_c s_n^{(0)} \left(\frac{b_1}{\bar{a}} - \frac{\bar{b} - 2\pi n}{\bar{a}^2} a_1 \right). \quad (44)$$

Eqs. (43)-(44) show that all relativistic images wobble coherently at the QNM frequency ω , with an amplitude that decays geometrically with n through $s_n^{(0)}$. Figure 5 displays the expected geometric accumulation of images

toward the ring, with a phase-locked modulation at frequency ω inherited from Eq. (24). The β_1 contribution shifts the ring itself, while the a_1 and b_1 terms imprint the subleading n -dependent offsets in Eq. (44).

The same near-photon-sphere control that yields Eq. (38) provides the coordinate travel time for a near-critical ray. Denote by $T(b, t_o)$ the total coordinate time elapsed between emission and detection for a ray of impact parameter b . By standard SDL analysis, one finds a logarithmic form

$$T(b, t_o) = -\tilde{a}(t_o) \ln \left(\frac{b}{b_c(t_o)} - 1 \right) + \tilde{b}(t_o) + n T_{\text{whirl}}(t_o) + \dots, \quad (45)$$

where n counts the net azimuthal winds accumulated during the whirl phase, $T_{\text{whirl}}(t_o)$ is the period of one circular photon orbit measured in Schwarzschild time, and $\tilde{a}(t_o), \tilde{b}(t_o)$ are time-dependent analogues of $\bar{a}(t_o), \bar{b}(t_o)$ governed by the same Born kernel. For Schwarzschild, in the static limit, we have [39]

$$T_{\text{whirl}} = \frac{2\pi}{\Omega_{\text{ph}}}, \quad \Omega_{\text{ph}} = \frac{1}{3\sqrt{3}M}, \quad (46)$$

so that successive relativistic images are separated, to leading order, by $2\pi/\Omega_{\text{ph}}$ in arrival time. Allowing for $\mathcal{O}(\varepsilon)$ modulations analogous to Eq. (39), we write

$$\begin{aligned} \tilde{a}(t_o) &= \bar{a} + \varepsilon \text{Re} \left\{ e^{-i\omega t_o} \tilde{a}_1 \right\}, \\ \tilde{b}(t_o) &= \bar{b} + \varepsilon \text{Re} \left\{ e^{-i\omega t_o} \tilde{b}_1 \right\}, \\ T_{\text{whirl}}(t_o) &= T_{\text{whirl}} \left[1 + \varepsilon \text{Re} \left\{ e^{-i\omega t_o} \gamma_1 \right\} \right], \end{aligned} \quad (47)$$

with complex response coefficients $\tilde{a}_1, \tilde{b}_1, \gamma_1$ fixed by the same near-photon-sphere kernel that determines a_1, b_1, β_1 in Eq. (39). Evaluating Eq. (45) at $b = b_n(t_o)$ and forming differences, the inter-image delay becomes

$$\begin{aligned} \Delta T_{n+1,n}(t_o) &\equiv T(b_{n+1}(t_o), t_o) - T(b_n(t_o), t_o) \\ &= T_{\text{whirl}}(t_o) + \frac{2\pi}{\bar{a}(t_o)} \tilde{a}(t_o) + \dots, \end{aligned} \quad (48)$$

where the second term arises from the logarithmic piece through $\ln s_{n+1} - \ln s_n = -2\pi/\bar{a}(t_o)$. Linearizing Eq. (48) using Eqs. (39) and (47) shows that $\Delta T_{n+1,n}$ acquires a harmonic modulation $\propto \text{Re} e^{-i\omega t_o}$ with an amplitude set by $\gamma_1, \tilde{a}_1, a_1$. Figure 6 makes the phase-locked behavior explicit: variations in $\Delta T_{n+1,n}(t_o)$ track the same QNM phase that controls $q(t_o)$ and $\theta_{\text{ring}}(t_o)$. Eliminating t_o between Eqs. (48) and (50) yields the differential correlation summarized in Eq. (62), further underscoring the unified kernel control.

Finally, the photon ring corresponds to the accumulation of the sequence $\theta_n(t_o)$ as $n \rightarrow \infty$ [52]. From Eq. (43) we obtain the instantaneous ring radius

$$\begin{aligned} \theta_{\text{ring}}(t_o) &= \lim_{n \rightarrow \infty} \theta_n(t_o) = \frac{b_c(t_o)}{r_o} \\ &= \theta_c \left[1 + \varepsilon \text{Re} \left\{ e^{-i\omega t_o} \beta_1 \right\} \right], \end{aligned} \quad (49)$$

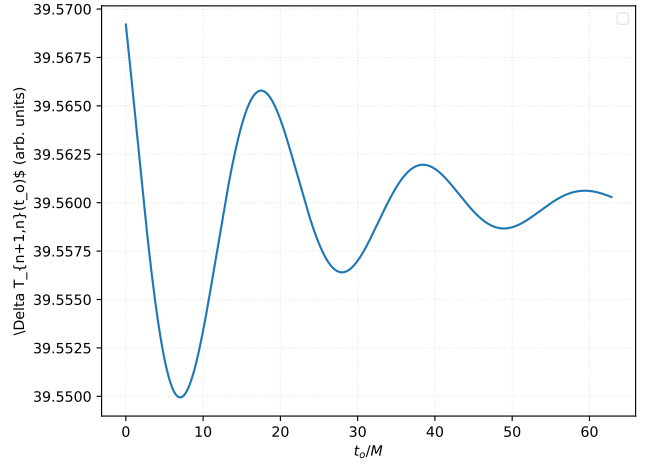


FIG. 6. Inter-image time delays $\Delta T_{n+1,n}(t_o)$ from Eq. (46). The baseline separation is set by the whirl period $T_{\text{whirl}} = 2\pi/\Omega_{\text{ph}}$ with $\Omega_{\text{ph}} = (3\sqrt{3}M)^{-1}$, while the weak modulation arises from the time-dependent coefficients in Eq. (47) and the $\tilde{a}(t_o)$ dependence. The delays oscillate in phase with the spacing modulation $q(t_o)$, reflecting their common origin in the near-photon-sphere kernel.

so the fractional modulation of the ring radius equals $\varepsilon \text{Re} e^{-i\omega t_o} \beta_1$. The exponential spacing of successive relativistic images is governed by

$$\begin{aligned} q(t_o) &\equiv \frac{\theta_{n+1}(t_o) - \theta_{\text{ring}}(t_o)}{\theta_n(t_o) - \theta_{\text{ring}}(t_o)} = \exp \left(-\frac{2\pi}{\bar{a}(t_o)} \right) \\ &= e^{-2\pi} \left[1 + \varepsilon \text{Re} \left\{ e^{-i\omega t_o} \frac{2\pi a_1}{\bar{a}^2} \right\} \right] + \mathcal{O}(\varepsilon^2), \end{aligned} \quad (50)$$

where we used $\bar{a} = 1$ for Schwarzschild in the static limit. Thus, both the ring radius and the logarithmic spacing inherit the QNM frequency and phase from Eq. (24). The same analysis yields the magnifications $\mu_n(t_o)$, which scale as $\mu_n \propto q(t_o)^n$ up to a slowly varying prefactor; we refrain from writing the prefactor explicitly because it depends on the source configuration, while the time-dependent exponential factor follows directly from Eq. (50). Related visibility-domain signatures and detectability of higher-order rings in realistic accretion models are discussed in [53]; an analytic “gap parameter” characterization of higher-order rings in spherically symmetric metrics is given in [54]. As shown in Fig. 7, the spacing ratio oscillates weakly around the Schwarzschild value $q_0 = e^{-2\pi}$, with phase fixed by the QNM. Together with $\theta_{\text{ring}}(t_o)$, this quantity isolates the pair (a_1, β_1) that encode the near-photon-sphere response of the Born kernel.

Eqs. (43)-(50), together with the deflection law in Eq. (38), provide a self-contained, analytic description of how a single even-parity $\ell = 2, m = 0$ QNM coherently imprints on the arrival times, multiplicity, and morphology of strong-field images, culminating in a phase-locked modulation of the photon ring.

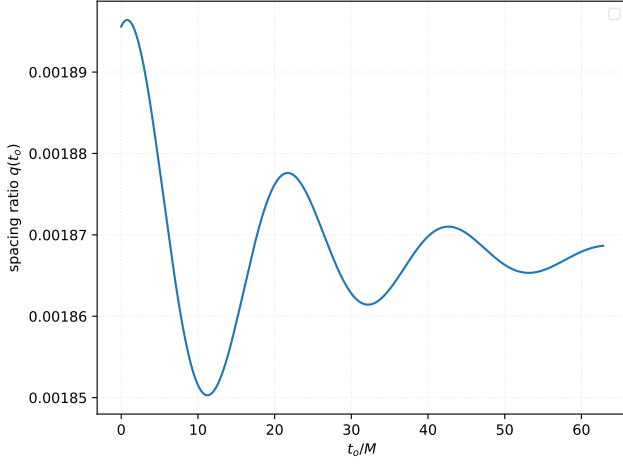


FIG. 7. Instantaneous spacing ratio $q(t_o)$ between successive relativistic images. For Schwarzschild, $q_0 = e^{-2\pi/\bar{a}} = e^{-2\pi}$ provides the static baseline, while the small QNM-driven modulation is $\propto \text{Re}\{e^{-i\omega t_o} a_1\}$. Because $q(t_o)$ controls both the inter-image angular spacing and (up to a prefactor) magnification ratios, it supplies a compact strong-field diagnostic tied to the same kernel coefficients as the ring modulation.

V. IMAGING DIAGNOSTICS AND CONTINUITY ACROSS REGIMES

We now assemble a single, observer-based description of lensing observables that is valid from the weak field to the time-dependent strong-deflection limit. The central object is the Born line-of-sight map in Eq. (24), which yielded the far-field expansion of Eq. (30) and the strong-field law of Eq. (38). Our task is twofold: first, to show that these limits admit a common overlap where they agree term by term once expressed on the screen defined in Sec. II B; second, to translate the matched deflection into gauge-safe imaging diagnostics for centroid motion, multi-image structure, and photon-ring modulations.

A. The unified matching

The matching proceeds directly at the level of the master integral in Eq. (24). We split the integral at an arbitrary radius r_m chosen so that $3M < r_m \ll r_o$, writing

$$\mathcal{A}_A(b, \omega) = \int_{\lambda_s}^{\lambda(r_m)} d\lambda \tilde{\mathcal{K}}_A e^{i\omega\tau} + \int_{\lambda(r_m)}^{\lambda_o} d\lambda \tilde{\mathcal{K}}_A e^{i\omega\tau} \equiv \mathcal{A}_A^{\text{in}} + \mathcal{A}_A^{\text{out}}, \quad (51)$$

where $A \in X, Y$ and all quantities are evaluated along the background ray labeled by b . The inner piece $\mathcal{A}_A^{\text{in}}$ is dominated by the whirl segment near $r_{\text{ph}} = 3M$; the outer piece $\mathcal{A}_A^{\text{out}}$ samples the nearly straight leg through the asymptotic domain. Both contributions depend on r_m , but their sum does not.

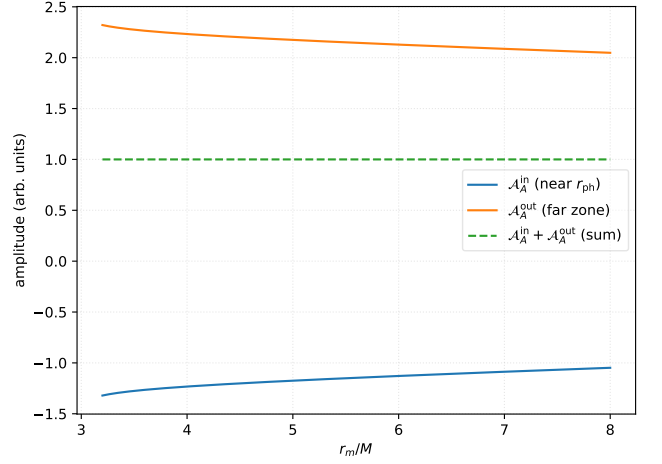


FIG. 8. Kernel split and r_m independence of the sum. The master line-of-sight amplitude is decomposed as in Eq. (51) into the inner (near-photon-sphere) piece and the outer (far-zone) piece. Each contribution depends on the arbitrary split radius r_m and carries complementary logarithmic/finite parts, yet their sum is independent of r_m . This illustrates that physical observables derived from the Born kernel are unique and do not depend on the cutoff.

For $b \rightarrow b_c^+$ at fixed r_m , the inner integral reproduces the logarithmic structure derived in Eq. (37), hence

$$\delta\alpha_A^{\text{in}}(b, t_o) = \text{Re} \left\{ e^{-i\omega t_o} \left[\mathcal{C}_A(\omega) \ln \left(\frac{b}{b_c} - 1 \right) + \mathcal{D}_A(\omega; r_m) \right] \right\} + \dots, \quad (52)$$

with $\mathcal{C}_A$ independent of r_m and $\mathcal{D}_A$ carrying the r_m -dependence required to cancel that of the outer piece. Conversely, for $M/b \ll 1$ at fixed r_m , the outer contribution matches the far-field form in Eq. (27),

$$\delta\alpha_A^{\text{out}}(b, t_o) = \text{Re} \left\{ e^{-i\omega t_o} \frac{c_A(\omega; r_m)}{b} \mathcal{F}_A(\omega b) \right\} + \mathcal{O} \left(\frac{M}{b^2} \right), \quad (53)$$

where $c_A(\omega; r_m)$ depends on the cutoff but $\mathcal{F}_A$ does not. Adding Eqs. (52)-(53) yields a representation that is uniformly valid for $b > b_c$. Figure 8 shows how the r_m -dependence cancels between the inner and outer integrals in Eq. (51). This cancellation underlies the robustness of the matched coefficients in Eq. (54) and, ultimately, the overlap relation in Eq. (55).

To make contact with the time-dependent SDL coefficients of Eq. (39), we expand the outer factor $\mathcal{F}_A(\omega b)$ for b close to b_c . Using the standard near-critical relation between the longitudinal coordinate and the azimuthal angle, Eq. (36), one finds that $\mathcal{F}_A$ carries a logarithmic dependence identical to that of the inner piece. The r_m -dependence cancels between $\mathcal{D}_A(\omega; r_m)$ and $c_A(\omega; r_m)$, and the sum can be cast as

$$\delta\alpha_A(b, t_o) = \text{Re} \left\{ e^{-i\omega t_o} \left[-a_{1,A}(\omega) \ln \left(\frac{b}{b_c} - 1 \right) \right] \right\}$$

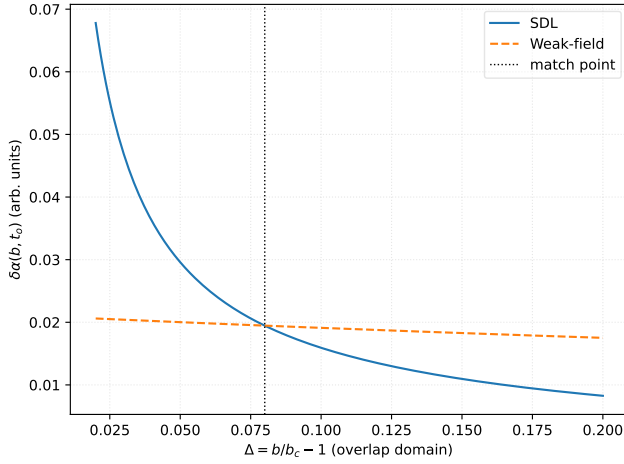


FIG. 9. Matching weak and strong limits in the overlap domain. The weak-field Born expression (Eq. (30), dashed) and the linearized SDL law (Eq. (41), solid) are compared at $b = b_c(1+\Delta)$ for small $\Delta > 0$. Around an intermediate reference Δ the two curves coincide, and they agree within a band across the overlap window, illustrating Eq. (55): the near-critical expansion of the $1/b$ Born amplitude encodes the SDL coefficients, in particular the $1/\Delta$ residue that fixes $\beta_1(\omega)$.

$$+ \frac{\beta_{1,A}(\omega)}{\frac{b}{b_c} - 1} + b_{1,A}(\omega) \Big] \Big\} + \dots, \quad (54)$$

which is the vectorial version of Eq. (41) and fixes, component by component, the response coefficients $a_{1,A}, \beta_{1,A}, b_{1,A}$ in terms of the same kernel $\tilde{\mathcal{K}}_A$ used in the far-field calculation.

A practical matching condition follows by selecting an overlap domain where both asymptotics are valid. Let $b = b_c(1 + \Delta)$ with $0 < \Delta \ll 1$ and keep $M/b \ll 1$. Expanding Eq. (30) to first order in Δ and equating to Eq. (54) yields

$$\frac{\mathcal{A}(b_c, \omega)}{b_c} = \lim_{\Delta \rightarrow 0^+} \left[-a_1(\omega) \ln \Delta + \frac{\beta_1(\omega)}{\Delta} + b_1(\omega) \right] \hat{e}_b + \Xi(\omega), \quad (55)$$

where $\mathcal{A} = (c_X \mathcal{F}_X, c_Y \mathcal{F}_Y)$ from Eq. (30), $\hat{e}_b$ is the screen radial unit vector, and $\Xi(\omega)$ collects finite terms that are independent of the arbitrary split radius r_m . Expanding $A(b, \omega)$ about $b = b_c(1 + \Delta)$ generates the $\ln \Delta$ and $1/\Delta$ structures that match the strong-deflection expansion on the inner side, making the regulator independence manifest. Eq. (55) states that the $1/b$ Born amplitude evaluated at b_c encodes, through its singular Δ -expansion, the three time-dependent SDL responses in Eq. (39). In particular, the residue of the $1/\Delta$ pole fixes $\beta_1(\omega)$ and therefore the photon-ring modulation in Eq. (49), while the coefficient of $\ln \Delta$ fixes $a_1(\omega)$ and hence the spacing modulation in Eq. (50). Figure 9 visualizes the content of Eq. (55): the same kernel controls both limits, and their expansions meet smoothly for small but finite Δ . The agreement pins down the triplet (β_1, a_1, b_1) that governs ring-radius and spacing modulations (Sec. (IV)).

Finally, since Eqs. (24), (30), and (38) are all constructed using the observer tetrad of Eq. (9), the matching preserves the gauge safety guaranteed by Eq. (16). The result is a single kernel-based description in which weak-field centroid wobble, intermediate-field image shifts, and strong-field ring modulations are different faces of the same frequency-resolved response.

B. Screen-plane observables: centroid, multi-images, and ring

All observables are defined on the screen introduced in Eq. (13) and inherit their time dependence from the master kernel in Eq. (24). We collect here the gauge-safe quantities that diagnose the weak, intermediate, and strong lensing regimes and emphasize how they are continuously related by the matching established in Sec. V A.

We first package the screen coordinates into a single complex angle

$$\zeta \equiv \frac{X + iY}{r_o}, \quad (56)$$

so that a ray of impact parameter b arriving at (X, Y) corresponds to $\zeta(b, t_o) = \alpha_X(b, t_o)/r_o + i\alpha_Y(b, t_o)/r_o$, with α given by Eq. (19). For a narrow bundle around b the centroid angle is the intensity-weighted mean,

$$\zeta_c(t_o) = \zeta_{c,0} + \varepsilon \operatorname{Re} \{ e^{-i\omega t_o} \mathcal{B}(\omega) \} + \mathcal{O} \left(\frac{M}{r_o} \frac{M}{b^2} \right), \quad (57)$$

which is the complex form of Eq. (32). The complex amplitude $\mathcal{B}(\omega)$ depends on the bundle's impact-parameter distribution through $\mathcal{A}_A(b, \omega)$ in Eq. (24). Since both the phase and modulus of $\mathcal{B}$ are determined by the same kernel that appears in the strong-field coefficients of Eq. (39), the centroid “wobble” is phase-locked to the modulations of the relativistic images and of the photon ring discussed below. In particular, defining

$$\Phi_{c-\text{ring}} \equiv \arg \mathcal{B} - \arg \beta_1, \quad (58)$$

the relative phase between the weak-field centroid wobble and the strong-field ring-radius oscillation, Eq. (49), is an overlap-domain diagnostic of the common QNM driver. Figure 10 exhibits the anticipated phase locking: the centroid and ring phases are parallel lines with a constant offset $\Phi_{c-\text{ring}}$. This provides a direct, gauge-safe confirmation that the Born kernel controls both observables, as argued in Sec. V A and encoded in Eq. (55).

Relativistic images are labeled by the winding number $n \in \mathbb{N}$ and obey Eq. (43). Their instantaneous angular radii $\theta_n(t_o)$ separate naturally into a ring-centered offset,

$$\Delta\theta_n(t_o) \equiv \theta_n(t_o) - \theta_{\text{ring}}(t_o) = \theta_c s_n^{(0)} + \varepsilon \operatorname{Re} \{ e^{-i\omega t_o} \delta\theta_n^{\text{off}} \} + \mathcal{O} \left(s_n^{(0)2} \right), \quad (59)$$

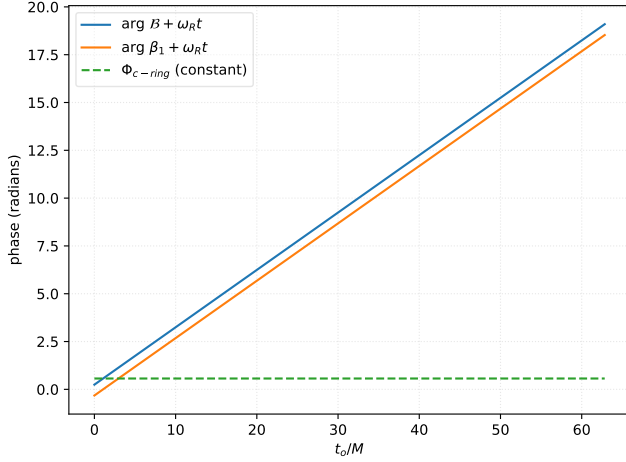


FIG. 10. Phase-lock diagnostic across regimes. We show the phases associated with the weak-field centroid wobble (from Eq. (32)) and the strong-field ring-radius modulation (from Eq. (49)), together with their difference $\Phi_{c-\text{ring}}$ defined in Eq. (58). The two phases advance in parallel at frequency ω_R while their difference remains constant, demonstrating that weak- and strong-field screen observables are driven by the same QNM kernel.

with $\theta_{\text{ring}}(t_o)$ given by Eq. (49). The complex offset amplitude,

$$\delta\theta_n^{\text{off}} = \theta_c s_n^{(0)} \left(\frac{b_1}{\bar{a}} - \frac{\bar{b} - 2\pi n}{\bar{a}^2} a_1 \right), \quad (60)$$

is the second term in Eq. (44). The geometric decay $\propto s_n^{(0)}$ ensures that higher-order images accumulate at the ring while remaining coherently modulated at the QNM frequency ω .

The logarithmic spacing is governed by $q(t_o)$ in Eq. (50). Writing $q(t_o) = q_0 [1 + \varepsilon \text{Re} e^{-i\omega t_o} \delta_q]$ with $q_0 = e^{-2\pi/\bar{a}}$ and $\bar{a} = 1$ for Schwarzschild, we have

$$\delta_q = \frac{2\pi a_1}{\bar{a}^2}, \quad (61)$$

so that a measurement of the instantaneous ratio $(\Delta\theta_{n+1}/\Delta\theta_n)(t_o)$ yields $\text{Re} e^{-i\omega t_o} \delta_q$ directly. In combination with Eq. (49), which determines $\text{Re} e^{-i\omega t_o} \beta_1$, this provides two independent strong-field observables tied to the same kernel coefficients a_1 and β_1 .

Arrival times follow from the travel-time function $T(b, t_o)$ in Eq. (45). Evaluated at $b_n(t_o)$ from Eq. (42), the inter-image delays $\Delta T_{n+1,n}(t_o)$ are given by Eq. (48) and thus oscillate at the QNM frequency through the modulations in Eq. (47). Eliminating t_o between Eqs. (48) and (50) gives a purely geometric correlation,

$$\frac{d}{dt_o} \ln q(t_o) = \frac{2\pi}{\bar{a}^2} \frac{d}{dt_o} \bar{a}(t_o) = \mathcal{C}_T \frac{d}{dt_o} \Delta T_{n+1,n}(t_o) + \mathcal{O}(\varepsilon^2), \quad (62)$$

where $\mathcal{C}_T$ is a constant fixed by the static Schwarzschild values of $\bar{a}$ and $\bar{a}$. Eq. (62) expresses the common origin of spacing and delay modulations in the near-photon-sphere

kernel and, in our framework, is the time-domain analogue of the matching relation (55).

Magnifications $\mu_n(t_o)$ depend on the source configuration, but their exponential hierarchy is controlled solely by $q(t_o)$: for a small, fixed source displacement, the leading scaling is

$$\mu_{n+1}(t_o) \simeq q(t_o) \mu_n(t_o), \quad (63)$$

up to a slowly varying prefactor that is independent of n . Using Eq. (50), the fractional modulation of μ_n at fixed n is therefore $\delta\mu_n/\mu_n \simeq \text{Re} e^{-i\omega t_o} \delta_q$, while at fixed t_o the ratio μ_{n+1}/μ_n isolates $q(t_o)$. Together with Eqs. (49)-(50), this yields a closed set of strong-field observables $\theta_{\text{ring}}, q, \mu_{n+1}/\mu_n$ that are all phase-locked to ω .

Finally, the weak- and strong-field diagnostics can be combined without reference to the arbitrary matching radius used in Eq. (51). A convenient pair is

$$(\zeta_c(t_o) - \zeta_{c,0}) \quad \text{and} \quad \theta_{\text{ring}}(t_o) - \theta_c, \quad (64)$$

whose complex phases are $\arg \mathcal{B}$ and $\arg \beta_1$, respectively. The constancy of $\Phi_{c-\text{ring}}$ in Eq. (58) across the overlap domain provides a direct test of continuity from Eq. (30) to Eq. (38). Because all quantities in Eqs. (57)-(64) are defined through the tetrad map Eq. (13), the gauge-independence argument of Eq. (16) applies verbatim: centroid motion, inter-image spacing, ring radius, and magnification ratios are bona fide $\mathcal{O}(\varepsilon)$ observables tied to a single, unified response encoded in the Born kernel (24).

VI. CONCLUSION

We have developed a unified, fully analytical framework that links weak-field lensing to the time-dependent strong-deflection limit for light propagating in a perturbed Schwarzschild spacetime. The backbone of our construction is the line-of-sight Born map in Eq. (24), derived from the first-order response of null geodesics to an axisymmetric even-parity $(\ell, m) = (2, 0)$ QNM perturbation. By evaluating the same kernel on background rays we obtained, on one hand, the far-field expansion of Eq. (30), which produces a ringdown-frequency centroid wobble on the observer's screen, and, on the other hand, the near-photon-sphere expansion of Eq. (38), which yields a time-dependent generalization of the logarithmic strong-deflection law. The linearized form in Eq. (41) exposes the dominance of the critical-scale modulation as $b \rightarrow b_c^+$. This single-kernel perspective ensures that weak-field and strong-field observables are phase-locked to the same QNM frequency and are related by controlled asymptotics.

The imaging layer built on the observer tetrad of Eq. (9) renders all diagnostics gauge-safe at $\mathcal{O}(\varepsilon)$. In the weak field, the centroid $\theta_c(t_o)$ follows Eq. (32) and provides a clean harmonic signature. In the strong field, the hierarchy of relativistic images $\theta_n(t_o)$ in Eq. (43) and the photon-ring radius $\theta_{\text{ring}}(t_o)$ in Eq. (49) exhibit coherent, frequency-resolved modulations. The logarithmic spacing $q(t_o)$ in Eq. (50) and the inter-image delays in Eq. (48) trace the same near-photon-sphere kernel that controls the modulation of $b_c(t_o)$

via Eq. (39). The matching procedure of Sec. V A, summarized by Eq. (55), makes the continuity across regimes explicit: the residue that governs the photon-ring modulation and the coefficient that governs spacing variations are encoded in the near-critical expansion of the very $1/b$ amplitude that drives the centroid wobble. For a global, quasi-normal oscillation on a black-hole background, the leading Born imprint scales as $1/b$; for localized radiative sources, near-zone terms dominate and yield net b^{-3} scaling after intermediate-wave zone cancellations.

Several extensions suggest themselves while preserving the purely theoretical character of this work. First, one may include additional even-parity multipoles or odd-parity counterparts within the same Born kernel; the algebra expands, but the logic remains the same as in Eqs. (24), (30), and (38) remains unchanged. Second, although we have focused on Schwarzschild, the strategy is designed to be ported to slowly rotating backgrounds by treating spin as a controlled perturbation; the essential step is to replace the Zerilli sector with its Teukolsky analogue while keeping the observer-plane definitions fixed. For spin, the same kernel applies once the metric is reconstructed from Teukolsky variables (e.g., via the Chrzanowski-Cohen-Kegeles or Hertz potential approaches), and the observer-tetrad projection carries through unchanged. Third, the present construction invites a systematic study of phase relations among weak-field wobble, spacing, and ring modulations, such as $\Phi_{\text{c-ring}}$ in Eq. (58), as robust, kernel-level diagnostics independent of source morphology. Finally, our appendices provide the algebraic infrastructure for reconstructing $h_{\mu\nu}$ from gauge-invariant master fields and for evaluating the Born kernel in both asymptotic regimes; these derivations can be adapted to explore higher-order effects without resorting to numerical implementations.

To end, by anchoring all lensing observables to a single time-dependent Born representation and by tracing their weak-to-strong continuity, we have established an analytic, gauge-controlled framework for QNM-driven imaging phenomenology around Schwarzschild black holes. The resulting relations among centroid motion, multi-image structure, and photon-ring modulations supply a compact theoretical toolbox that is readily extensible and that clarifies how ringdown physics imprints across the entire screen.

ACKNOWLEDGMENTS

A. O and R. P. would like to acknowledge networking support of the COST Action CA21106 - COSMIC WISPerS in the Dark Universe: Theory, astrophysics and experiments (CosmicWISPerS), the COST Action CA22113 - Fundamental challenges in theoretical physics (THEORY-CHALLENGES), the COST Action CA21136 - Addressing observational tensions in cosmology with systematics and fundamental physics (CosmoVerse), the COST Action CA23130 - Bridging high and low energies in search of quantum gravity (BridgeQG), and the COST Action CA23115 - Relativistic Quantum Information (RQI) funded by COST (European Cooperation

in Science and Technology). A. O and R. P. would also like to acknowledge the funding support of SCOAP3.

Appendix A: Zerilli framework and reconstruction to $h_{\mu\nu}$

In this appendix, we collect the even-parity RWZ machinery used in Secs. II-V. We keep the background Schwarzschild metric $g_{\mu\nu}^{(0)}$ and notation of Eq. (1), the perturbative split of Eq. (3), and the Zerilli master equation and potential of Eqs. (6)-(7), together with the QNM boundary conditions in Eq. (8). Our goal is twofold: to define a gauge-invariant even-parity master field Ψ_Z appropriate for $(\ell, m) = (2, 0)$, and to reconstruct the RWZ-gauge metric amplitudes H_0, H_1, H_2, K and the combination $h_{\alpha\beta}p^\alpha p^\beta$ that appears in the Born kernel of Eqs. (20)-(24).

Harmonic decomposition and RWZ gauge: We expand first-order even-parity perturbations in scalar spherical harmonics $Y_{\ell m}(\theta, \phi)$,

$$\begin{aligned} h_{tt} &= f H_0^{\ell m}(t, r) Y_{\ell m}, \\ h_{tr} &= H_1^{\ell m}(t, r) Y_{\ell m}, \\ h_{rr} &= f^{-1} H_2^{\ell m}(t, r) Y_{\ell m}, \\ h_{tA} &= h_{rA} = 0, \\ h_{AB} &= r^2 K^{\ell m}(t, r) \Omega_{AB} Y_{\ell m}, \end{aligned} \quad (\text{A1})$$

with $A, B \in \theta, \phi$, $\Omega_{AB} = \text{diag}(1, \sin^2 \theta)$, and $f = 1 - 2M/r$ as in Eq. (1). The RWZ gauge conditions set the even-parity vector and tensor harmonics (j_a, G) to zero. For axisymmetry ($m = 0$) we have $Y_{\ell 0}(\theta) = \sqrt{\frac{2\ell+1}{4\pi}} P_\ell(\cos \theta)$. All amplitudes H_0, H_1, H_2, K used in the main text (Secs. II-V) are defined in the P_ℓ basis. When needed, Y -basis amplitudes and P -basis amplitudes are related by $H_a^{(Y)} = \sqrt{\frac{2\ell+1}{4\pi}} H_a^{(P)}$ for $a \in \{0, 1, 2, K\}$. It is also convenient to adopt the tortoise coordinate r_* of Eq. (5), so that the wave operator acting on master fields takes the flat-space form in (t, r_*) .

Gauge-invariant Zerilli-Moncrief master field: Gauge invariants for the even-parity sector can be built à la Moncrief. A convenient choice is the Zerilli-Moncrief master field $\Psi_Z(t, r)$ defined by

$$\begin{aligned} \Psi_Z &= \frac{r}{\lambda r + 3M} \left[K + \frac{r - 2M}{\lambda r + 3M} (H_2 - r \partial_r K) \right], \\ \lambda &\equiv \frac{(\ell - 1)(\ell + 2)}{2} = 2. \end{aligned} \quad (\text{A2})$$

By construction, Ψ_Z is invariant under linearized even-parity gauge transformations [cf. Eq. (15)]. In vacuum it satisfies the Zerilli equation already quoted in Eq. (6),

$$(-\partial_t^2 + \partial_{r_*}^2 - V_Z(r)) \Psi_Z = 0, \quad (\text{A3})$$

with V_Z given in Eq. (7). For QNMs, we impose the homogeneous boundary conditions of Eq. (8). In the frequency domain, $\Psi_Z(t, r) = \psi(r) e^{-i\omega t}$ with complex $\omega = \omega_R + i\omega_I$, $\omega_I < 0$.

A normalization convention is needed to separate the physical smallness of the perturbation from the shape of Ψ_Z . Throughout we take the outgoing amplitude of Ψ_Z at large r to be unity and collect the overall smallness in the book-keeping parameter ε of Eq. (3); this is the convention used in Secs. III-V.

Reconstruction of H_0, H_1, H_2, K from Ψ_Z : In vacuum, the Einstein equations relate (H_0, H_1, H_2, K) algebraically to Ψ_Z and its first derivatives. One convenient RWZ-gauge reconstruction, obtained by inverting Eq. (A2) and using the field equations, reads

$$\begin{aligned} K &= \frac{\mathcal{N}_K(r)}{r^2(\lambda r + 3M)} \Psi_Z + \frac{f}{r} \partial_r \Psi_Z, \\ H_2 &= \frac{\mathcal{N}_2(r)}{r^2 f(\lambda r + 3M)} \Psi_Z + \frac{\mathcal{D}_2(r)}{r f} \partial_r \Psi_Z, \\ H_0 &= H_2, \\ H_1 &= -\frac{i\omega \mathcal{N}_1(r)}{f(\lambda r + 3M)} \Psi_Z + i\omega r, \partial_r \Psi_Z, \end{aligned} \quad (\text{A4})$$

where ω is the Fourier frequency introduced above and the radial polynomials are

$$\begin{aligned} \mathcal{N}_K &= \lambda(\lambda + 1)r^2 + 3\lambda M r + 6M^2, \\ \mathcal{N}_2 &= \lambda(\lambda + 1)r^2 - 3\lambda M r - 6M^2, \\ \mathcal{D}_2 &= \lambda r - 3M, \\ \mathcal{N}_1 &= \lambda r^2 + 3\lambda M r + 6M^2. \end{aligned} \quad (\text{A5})$$

For Schwarzschild with $\ell = 2$ one has $\lambda = 2$. Eq. A.4 is written in a form that makes the large- r behavior transparent: $K \sim r^{-1}$, $H_2 \sim r^{-1}$, $H_1 \sim r^{-1}$ once $\Psi_Z \sim r^{-1} e^{-i\omega(t-r_*)}$.

We remark that (i) The equality $H_0 = H_2$ follows from the vacuum field equations in RWZ gauge; (ii) Other algebraically equivalent reconstructions exist; any such choice leads to the same gauge-invariant observables; (iii) The particular coefficients in Eq. (A5) are tailored to the definition in Eq. (A2). If a different master variable is adopted, the polynomials change accordingly without affecting physical content.

Asymptotics and horizon behavior: From Eq. (8), Ψ_Z behaves as an outgoing wave at infinity and as an ingoing wave at the horizon,

$$\Psi_Z \sim \begin{cases} A_\infty e^{-i\omega(t-r_*)}, & r \rightarrow \infty, \\ A_H e^{-i\omega(t+r_*)}, & r \rightarrow 2M, \end{cases} \quad (\text{A6})$$

with $A_\infty = 1$ by our convention and A_H determined by the solution of Eq. (A3). Substituting A.6 into A.4 and using $r_* \sim r + 2M \ln(r/2M - 1)$, we find

$$\begin{aligned} H_{0,1,2} K &= \mathcal{O}(r^{-1}) e^{-i\omega(t-r)} \quad \text{for } (r \rightarrow \infty), \\ H_{0,1,2}, K &= \mathcal{O}(1) e^{-i\omega(t+r_*)} \quad \text{for } (r \rightarrow 2M), \end{aligned} \quad (\text{A7})$$

which ensures that the perturbed metric decays as r^{-1} at infinity and is regular on the future horizon in ingoing Eddington-Finkelstein coordinates.

The contraction $h_{\alpha\beta} p^\alpha p^\beta$ along a background ray: The Born kernel in Eqs. (20)-(24) involves $\partial_\mu h_{\alpha\beta} p_0^\alpha p_0^\beta$ evaluated on the Schwarzschild null geodesic $x_0^\mu(\lambda; b)$. Using Eq. (A1) with $m = 0$ and the axisymmetric harmonic $Y_{20}(\theta) = \sqrt{5/(4\pi)} P_2(\cos \theta)$, we obtain

$$\begin{aligned} h_{\alpha\beta} p_0^\alpha p_0^\beta &= Y_{20}(\theta) \left[f H_0 (p_0^t)^2 + 2 H_1 p_0^t p_0^r \right. \\ &\quad \left. + f^{-1} H_2 (p_0^r)^2 + r^2 K \Omega_{AB} \hat{p}_0^A \hat{p}_0^B \right], \end{aligned} \quad (\text{A8})$$

where $\hat{p}_0^A = (p_0^\theta, p_0^\phi)$ are the angular components of the background momentum and $\Omega_{AB} \hat{p}_0^A \hat{p}_0^B = (p_0^\theta)^2 + \sin^2 \theta (p_0^\phi)^2 = L^2/r^4$ with L the conserved angular momentum of the background geodesic [cf. Eq. (14)]. Substituting Eq. (A4) into Eq. (A8) and simplifying gives the compact representation

$$h_{\alpha\beta} p_0^\alpha p_0^\beta = Y_{20}(\theta) [\mathcal{P}_0(r; E, L) \Psi_Z + \mathcal{P}_1(r; E, L) \partial_r \Psi_Z], \quad (\text{A9})$$

with

$$\begin{aligned} \mathcal{P}_0(r; E, L) &= \frac{\mathcal{A}_0(r)}{r^2(\lambda r + 3M)} (E^2 - f L^2/r^2) \\ &\quad - \frac{2i\omega \mathcal{N}_1(r)}{f(\lambda r + 3M)} E \sqrt{E^2 - f L^2/r^2}, \mathcal{P}_1(r; E, L) \\ &= \frac{f}{r} (E^2 - f L^2/r^2) + 2i\omega r E \sqrt{E^2 - f L^2/r^2}, \end{aligned} \quad (\text{A10})$$

where $\mathcal{A}_0(r) \equiv \mathcal{N}_K(r) + \mathcal{N}_2(r)$ and we have expressed the radial momentum as $p_0^r = \pm \sqrt{E^2 - f L^2/r^2}$ with $E = -p_{0t}$ the conserved energy (the sign flips at turning points). Eqs. (A9)-(A10) make explicit that only Ψ_Z and $\partial_r \Psi_Z$ enter the kernel, and that the dependence on the orbital constants (E, L) factorizes algebraically.

Taking a derivative and using the background geodesic equations, the source term in Eq. (18) can be written in the equivalent forms

$$\begin{aligned} \partial_\mu (h_{\alpha\beta} p_0^\alpha p_0^\beta) dx_0^\mu &= (\partial_t - \dot{r}_0 \partial_r) [h_{\alpha\beta} p_0^\alpha p_0^\beta] d\lambda \\ &= \left(-i\omega + \frac{dr_*}{d\lambda} \partial_{r_*} \right) [h_{\alpha\beta} p_0^\alpha p_0^\beta] d\lambda \end{aligned} \quad (\text{A11})$$

which is useful for casting the line-of-sight integrals (20)-(24) in terms of the tortoise coordinate r_* and the delay τ used in Eq. (24).

Large- (r) limit and normalization check: With the outgoing normalization $A_\infty = 1$ in Eq. (A6), the asymptotic behavior of Eq. (A4) implies

$$\begin{aligned} h_{\alpha\beta} p_0^\alpha p_0^\beta &\sim \frac{Y_{20}(\theta)}{r} e^{-i\omega(t-r)} \\ &\quad \times [\mathfrak{c}_0(\omega; E, L) + \mathfrak{c}_1(\omega; E, L) \mathcal{O}(1/r)], \end{aligned} \quad (\text{A12})$$

for some algebraic coefficients $\mathfrak{c}_{0,1}$. Substitution of Eq. (A12) into the kernel Eq. (21) reproduces the far-zone scaling used in Sec. III B and leads directly to the $1/b$

dependence in Eq. (27), providing a consistency check of the normalization choices above.

Near-photon-sphere reduction: Near the photon sphere $r_{\text{ph}} = 3M$ [Eq. (33)], the master equation in Eq. (A3) can be approximated by a Schrödinger problem with an inverted harmonic potential around the unstable maximum of V_Z . Expanding Ψ_Z and its derivative in that region and inserting into Eq. (A4) and then (A.8) yields

$$h_{\alpha\beta} p_0^\alpha p_0^\beta \simeq Y_{20}(\theta) \left[\mathfrak{C}_0(\omega; E, L) \Psi_Z + \mathfrak{C}_1(\omega; E, L) \partial_r \Psi_Z \right]_{r=3M} + \dots, \quad (\text{A13})$$

with constants $\mathfrak{C}_{0,1}$ determined by the polynomials in Eq. (A5). This reduction underlies the logarithmic structures extracted in Sec. IV, culminating in Eqs. (37)-(41).

Eqs. (A2)-(A5) provide a self-contained RWZ-gauge reconstruction from the Zerilli-Moncrief master field to the metric amplitudes (H_0, H_1, H_2, K) . The contraction formulas in Eqs. (A8)-(A11) deliver the precise input required by the Born kernel of Eq. (21), and their far-zone and near-photon-sphere limits, Eqs. (A12)-(A13), are the only ingredients needed to derive the weak-field expansion (Sec. III B) and the time-dependent SDL laws (Sec. IV). All steps are consistent with the QNM boundary conditions in Eq. (8), and all observable statements inherit gauge safety from the use of Ψ_Z and the observer tetrad defined in Eq. (9).

Appendix B: Derivation of the master Born kernel

Our goal is to derive the line-of-sight expression that underlies Eqs. (19)-(24) for the screen-plane deflection produced by a first-order metric perturbation $h_{\mu\nu}$ on the Schwarzschild background of Eq. (1). We keep the perturbative split of Eq. (3), use the observer tetrad of Eq. (9), and map momenta to screen angles via Eq. (13). Throughout, we follow a Born, or single-scattering, treatment: the null ray follows the unperturbed geodesic $x_0^\mu(\lambda; b)$ while its momentum p^μ is corrected at $\mathcal{O}(\varepsilon)$.

From Hamilton's equations to a line-of-sight integral: Start from the Hamiltonian for null motion

$$H(x, p) = \frac{1}{2} g^{\mu\nu}(x) p_\mu p_\nu = 0, \quad (\text{B1})$$

and expand $g^{\mu\nu} = g^{(0)\mu\nu} - \varepsilon h^{\mu\nu} + \mathcal{O}(\varepsilon^2)$, where indices on h are raised with $g^{(0)\mu\nu}$. Let $x^\mu = x_0^\mu + \varepsilon \delta x^\mu$ and $p_\mu = p_{0\mu} + \varepsilon \delta p_\mu$. To first order,

$$\dot{x}_0^\mu = \frac{\partial H_0}{\partial p_{0\mu}} = g^{(0)\mu\nu} p_{0\nu} \equiv p_0^\mu, \quad \dot{p}_{0\mu} = -\frac{\partial H_0}{\partial x_0^\mu} = 0, \quad (\text{B2})$$

along x_0^μ , where a dot is $d/d\lambda$ and $H_0 = \frac{1}{2} g^{(0)\mu\nu} p_{0\mu} p_{0\nu} = 0$. The first-order Hamilton equation for δp_μ is

$$\dot{\delta p}_\mu = -\frac{\partial}{\partial x^\mu} \left(\frac{1}{2} h^{\alpha\beta} p_{0\alpha} p_{0\beta} \right) = -\frac{1}{2} \partial_\mu h_{\alpha\beta} p_0^\alpha p_0^\beta, \quad (\text{B3})$$

which integrates to Eq. (18) once evaluated between source λ_s and observer λ_o :

$$\delta p_\mu(\lambda_o; b) = -\frac{1}{2} \int_{\lambda_s}^{\lambda_o} d\lambda \partial_\mu h_{\alpha\beta}(x_0(\lambda)) p_0^\alpha(\lambda) p_0^\beta(\lambda). \quad (\text{B4})$$

Eq. (B4) is the covariant Born response on a fixed background path.

Two comments are useful. First, the “force” driving δp_μ can equivalently be written in terms of the perturbed Christoffel symbols, since

$$\begin{aligned} \partial_\mu h_{\alpha\beta} p_0^\alpha p_0^\beta &= 2 \delta \Gamma_{\mu\alpha\beta} p_0^\alpha p_0^\beta + \partial_\lambda (h_{\mu\nu} p_0^\nu), \\ \delta \Gamma_{\mu\alpha\beta} &= \frac{1}{2} (\nabla_\alpha h_{\mu\beta} + \nabla_\beta h_{\mu\alpha} - \nabla_\mu h_{\alpha\beta}), \end{aligned} \quad (\text{B5})$$

so that integrating by parts turns B.4 into a $\delta\Gamma$ -driven integral plus boundary terms. Second, the contraction $h_{\alpha\beta} p_0^\alpha p_0^\beta$ used below is given in (A8)-(A11) in terms of the Zerilli master field Ψ_Z .

Projection to the observer's screen: Let $e^\mu_{\hat{a}}(x_o)$ be the tetrad of Eq. (9) at the observer event x_o^μ . The measured momentum components are

$$p^{\hat{a}} = e^\mu_{\hat{a}} p^\mu = p_0^{\hat{a}} + \varepsilon \delta p^{\hat{a}}, \quad \delta p^{\hat{a}} = e^\mu_{\hat{a}} \delta p^\mu, \quad (\text{B6})$$

to first order, since the $\mathcal{O}(\varepsilon)$ tetrad correction only induces $\mathcal{O}(\varepsilon^2)$ changes in $p^{\hat{a}}$ when combined with δp^μ [the $\mathcal{O}(\varepsilon)$ orthonormality is enforced in Eq. (10)]. The screen basis is $(\hat{X}, \hat{Y}) = (\hat{\phi}, -\hat{\theta})$ and the mapping to angles uses Eq. (13):

$$\begin{aligned} X &= -\frac{r_o p^{\hat{\phi}}}{p^{\hat{\theta}}}, & Y &= \frac{r_o p^{\hat{\theta}}}{p^{\hat{\theta}}}, \\ \alpha_A &\equiv \frac{A}{r_o} = \frac{p^{\hat{A}}}{p^{\hat{\theta}}}, & A &\in X, Y. \end{aligned} \quad (\text{B7})$$

Varying Eq. (B7) and keeping terms linear in ε gives

$$\delta \alpha_A = \frac{\delta p^{\hat{A}}}{p_0^{\hat{\theta}}} - \frac{p_0^{\hat{A}}}{(p_0^{\hat{\theta}})^2} \delta p^{\hat{\theta}} = \frac{1}{p_0^{\hat{\theta}}} \Pi_A^{\hat{b}} \delta p_{\hat{b}}, \quad (\text{B8})$$

where $\Pi_A^{\hat{b}}$ is the projector onto the screen directions at fixed x_o^μ ,

$$\Pi_A^{\hat{b}} = \delta_A^{\hat{b}} - \alpha_{0,A} \delta^{\hat{b}}_{\hat{\theta}}, \quad \alpha_{0,A} = \frac{p_0^{\hat{A}}}{p_0^{\hat{\theta}}}, \quad (\text{B9})$$

and hatted indices are raised/lowered with $\eta_{\hat{a}\hat{b}}$. Using $\delta p_{\hat{b}} = e_{\hat{b}}^\mu \delta p_\mu$ and substituting Eq. (B4) then yields

$$\begin{aligned} \delta \alpha_A(b, t_o) &= -\frac{1}{2 p_0^{\hat{\theta}}(\lambda_o)} \int_{\lambda_s}^{\lambda_o} d\lambda \Pi_A^{\hat{b}} e_{\hat{b}}^\mu(x_o) \\ &\quad \times \partial_\mu h_{\alpha\beta}(x_0(\lambda)) p_0^\alpha(\lambda) p_0^\beta(\lambda), \end{aligned} \quad (\text{B10})$$

which is Eq. (20) with the explicit kernel

$$\mathcal{K}_A[x_0(\lambda); p_0(\lambda)] = -\frac{1}{2 p_0^{\hat{\theta}}(\lambda_o)} \Pi_A^{\hat{b}} e_{\hat{b}}^\mu(x_o)$$

$$\times \partial_\mu h_{\alpha\beta}(x_0(\lambda)) p_0^\alpha p_0^\beta. \quad (\text{B11})$$

Eq. (B11) is identical to Eq. (21) once we identify the tetrad time leg $e^\nu_{\hat{0}}$ and note that $\Pi_A^\mu = \Pi_A^{\hat{b}} e_{\hat{b}}^\mu$.

Equivalent kernel forms and boundary terms: Using the identity Eq. (B5) and integrating by parts, Eq. (B10) becomes

$$\delta\alpha_A = \int_{\lambda_s}^{\lambda_o} d\lambda \mathcal{K}_A^{(\Gamma)} + \mathcal{B}_A \Big|_{\lambda_s}^{\lambda_o}, \quad (\text{B12})$$

with

$$\mathcal{K}_A^{(\Gamma)} = -\frac{1}{p_0^{\hat{0}}(\lambda_o)} \Pi_A^{\hat{b}} e_{\hat{b}}^\mu(x_o) \delta\Gamma_{\mu\alpha\beta}(x_0(\lambda)) p_0^\alpha p_0^\beta,$$

$$\mathcal{B}_A = \frac{1}{2p_0^{\hat{0}}(\lambda_o)} \Pi_A^{\hat{b}} e_{\hat{b}}^\mu(x_o) h_{\mu\nu}(x_0(\lambda)) p_0^\nu. \quad (\text{B13})$$

The boundary term $\mathcal{B}_A$ vanishes under our asymptotic conditions: at the observer, the screen projector eliminates any $\hat{0}$ contamination and $h_{\mu\nu}(x_o) = \mathcal{O}(r_o^{-1})$ [Eq. (A7)]; at the source, we assume compact support or sufficiently rapid falloff. Hence, all kernel representatives obtained by redistributing derivatives are equivalent at $\mathcal{O}(\varepsilon)$.

A further useful form follows by changing variables from affine parameter λ to the tortoise coordinate r_* or to the delay τ of Eq. (24). Using Eq. (A11),

$$\delta\alpha_A = \int_{\tau=0}^{\infty} d\tau \tilde{\mathcal{K}}_A[r(\tau), \theta(\tau)] e^{i\omega\tau} \times e^{-i\omega t_o} + \text{c.c.}, \quad (\text{B14})$$

which is the factorized form quoted in Eq. (24) once the QNM time dependence in Eq. (23) is inserted.

Gauge behavior at $\mathcal{O}(\varepsilon)$: Under a linearized diffeomorphism generated by ξ^μ , Eq. (15) gives $\delta h_{\alpha\beta} = -\nabla_\alpha \xi_\beta - \nabla_\beta \xi_\alpha$. Substituting this into Eq. (B10) and integrating by parts yields

$$\delta\xi\alpha_A = \frac{1}{p_0^{\hat{0}}(\lambda_o)} \Pi_A^{\hat{b}} e_{\hat{b}}^\mu(x_o) \xi_\mu \Big|_{\lambda_s}^{\lambda_o} + \mathcal{O}\left(\frac{\varepsilon}{r_o}\right), \quad (\text{B15})$$

where we used the background geodesic equation and $\dot{p}_{0\mu} = 0$. With our asymptotic conditions $\xi^\mu = \mathcal{O}(1/r)$, Eq. (B15) reproduces the gauge-independence statement in Eq (16): $\delta\xi\alpha_A = 0 + \mathcal{O}(\varepsilon/r_o)$. Thus, the kernel Eq. (B11) produces gauge-safe screen observables at first order, provided the observer tetrad is fixed by Eq. (10).

Far-zone reduction and the $(1/b)$ law: Insert the large- r asymptotics A.12 into B.11, parametrize the background ray by a straight line with longitudinal coordinate z as in Eq. (26), and note that $p_0^{\hat{0}}(\lambda_o) = E$ to leading order. One finds

$$\delta\alpha_A(b, t_o) = \text{Re} \left\{ e^{-i\omega t_o} \frac{c_A(\omega)}{b} \mathcal{F}_A(\omega b) \right\} + \mathcal{O}\left(\frac{M}{b^2}\right), \quad (\text{B16})$$

with c_A and $\mathcal{F}_A$ defined in Eqs. (27)-(28). This reproduces the weak-field Born expansion used in Sec. III B and provides

the link between the kernel normalization and the centroid wobble of Eq. (32).

Near-photon-sphere reduction and the logarithmic law: In the near-critical regime $b \rightarrow b_c^+$, reparametrize the background ray by the azimuth φ accumulated during the whirl [Eq. (36)]. Use the near-photon-sphere reduction A.13 inside B.11 and integrate over the long dwell time $\Delta\varphi \sim -\ln \Delta$, with $\Delta = b/b_c - 1$. The integral yields

$$\delta\alpha_A(b, t_o) = \text{Re} \left\{ e^{-i\omega t_o} [\mathcal{C}_A(\omega) \ln \Delta + \mathcal{D}_A(\omega)] \right\} + \dots, \quad (\text{B17})$$

which is Eq. (37). When combined with the background divergence in Eq. (34) and reorganized as in Eqs. (38)-(41), Eq. (B17) fixes the time-dependent SDL coefficients a_1, b_1, β_1 from the same kernel.

Starting from Hamilton's equations, we derived the Born response (B4) for the photon momentum, projected it to the observer's screen to obtain the master kernel (B11), and exhibited equivalent $\delta\Gamma$ and delay-space forms, Eqs. (B13)-(B14). The kernel is gauge-safe at $\mathcal{O}(\varepsilon)$ by Eq. (B15). Its far-zone and near-photon-sphere reductions, Eqs. B.16 and B.17, reproduce the weak-field $1/b$ behavior Sec. III B and the logarithmic SDL structure Sec. IV, respectively. In this sense, all imaging diagnostics discussed in Sec. V are different asymptotic faces of a single line-of-sight kernel driven by the Zerilli master field.

Appendix C: Near-critical expansion and SDL coefficients

In this appendix, we provide a self-contained derivation of the near-critical expansion underlying Eqs. (34) and (38)-(41), and we express the time-dependent strong-deflection-limit *SDL* response coefficients $a_1(\omega), b_1(\omega), \beta_1(\omega)$ in terms of quantities localized at the photon sphere and of moments of the Born kernel introduced in Appendix B. We keep all conventions of Secs. II-IV, in particular the Schwarzschild background (Eq. (1)), the perturbative split (Eq. (3)), and the master Born representation (Eqs. (19)-(24)).

Static near-critical recap: For null geodesics on a static, spherically symmetric background $ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\Omega^2$, the impact parameter as a function of the radial turning point r_0 is

$$b^2(r_0) = \frac{r_0^2}{A(r_0)}. \quad (\text{C1})$$

The circular photon orbit r_{ph} solves

$$\mathcal{F}(r) \equiv r A'(r) - 2A(r) = 0, \quad r = r_{\text{ph}}, \quad (\text{C2})$$

and the critical impact parameter is $b_c = r_{\text{ph}}/\sqrt{A(r_{\text{ph}})}$. For Schwarzschild, $A = f = 1 - 2M/r$ gives $r_{\text{ph}} = 3M$ and $b_c = 3\sqrt{3}M$ as stated in Eq. (33). Expanding the background bending integral 16 around $b \rightarrow b_c^+$ yields Eq. (34),

$$\alpha_0(b) = -\bar{a} \ln \left(\frac{b}{b_c} - 1 \right) + \bar{b} + \mathcal{O}((b - b_c) \ln |b - b_c|), \quad (\text{C3})$$

with $\bar{b}$ is a constant depending on the chosen normalization, and $\bar{a} = 1$ for the Schwarzschild case.

Linear response of the critical scale $b_c(t_o)$: We now compute the $\mathcal{O}(\varepsilon)$ modulation of b_c induced by the even-parity $(\ell, m) = (2, 0)$ perturbation described in Sec. II A. We adopt an “instantaneous snapshot” characterization at the detection time t_o : the time dependence appears as a harmonic factor $e^{-i\omega t_o}$ multiplying complex amplitudes evaluated at r_{ph} . This adiabatic reading is equivalent, to first order, to the near-photon-sphere reduction of the Born kernel and provides a compact route to $\beta_1(\omega)$.

Let $A(r) \rightarrow A(r) + \varepsilon a(r, t)$, where $A = f$ for the background and a is the even-parity perturbation of g_{tt} (we keep B unmodified since, to first order in ε , b_c is controlled by A alone through Eq. (C1)). The photon-sphere condition (C2) becomes $\mathcal{F}(r_{\text{ph}} + \varepsilon \delta r) = 0$ with

$$\delta r \mathcal{F}'(r_{\text{ph}}) + [r a'(r) - 2a(r)]_{r=r_{\text{ph}}} = 0,$$

$$\mathcal{F}'(r_{\text{ph}}) = r_{\text{ph}} A''(r_{\text{ph}}) - A'(r_{\text{ph}}). \quad (\text{C4})$$

Solving Eq. (C4) gives

$$\begin{aligned} \delta r &= \frac{2a(r_{\text{ph}}) - r_{\text{ph}} a'(r_{\text{ph}})}{\mathcal{F}'(r_{\text{ph}})} \\ &= -\frac{3M}{2} [2a(r_{\text{ph}}) - r_{\text{ph}} a'(r_{\text{ph}})]. \end{aligned} \quad (\text{C5})$$

The fractional change of b_c follows from $\ln b_c = \ln r_{\text{ph}} - \frac{1}{2} \ln A(r_{\text{ph}})$:

$$\frac{\delta b_c}{b_c} = \frac{\delta r}{r_{\text{ph}}} - \frac{1}{2} \left[\frac{a(r_{\text{ph}})}{A(r_{\text{ph}})} + \frac{A'(r_{\text{ph}})}{A(r_{\text{ph}})} \delta r \right]. \quad (\text{C6})$$

Substituting $A(r_{\text{ph}}) = 1/3$, $A'(r_{\text{ph}}) = 2/(9M)$, and Eq. (C5) leads to a remarkable cancellation of the $a'(r_{\text{ph}})$ terms and yields

$$\frac{\delta b_c}{b_c} = \frac{3}{2} a(r_{\text{ph}}). \quad (\text{C7})$$

Thus, the leading near-critical modulation of the critical scale depends only on the instantaneous value of $a = \delta A$ at the photon sphere. Since $A = -g_{tt}$, we have $a = -h_{tt}$. Using the RWZ reconstruction in Eq. (A4) with $m = 0$, $h_{tt} = f, H_0, Y_{20}(\theta)$; evaluating at the equator $\theta = \pi/2$ and at r_{ph} with $f = 1/3$ gives (equator, P_2 basis)

$$\beta_1(\omega) \equiv \frac{\delta b_c}{b_c} = \frac{1}{4} H_0(r_{\text{ph}}; \omega) \quad (\text{C8})$$

In the Y -harmonic normalization one has $Y_{20} = \sqrt{\frac{5}{4\pi}} P_2$, so $\beta_1 = \frac{1}{4} H_0^{(P)} = \frac{1}{4} \left(\sqrt{\frac{4}{\pi}} 5 H_0^{(Y)} \right)$; we use the P_2 convention throughout the main text. The time dependence $\propto \text{Re} e^{-i\omega t_o} \beta_1(\omega)$ then reproduces the $1/\Delta$ term in Eq. (41). Eq. (C) is equivalent to extracting the residue of the $1/\Delta$ pole directly from the Born kernel Appendix B, but makes explicit the quasi-local character of β_1 .

We comment that the cancellation of $a'(r_{\text{ph}})$ in Eq. (C6) is a special feature of Schwarzschild and the particular combination entering b_c . It reflects the fact that b_c depends on A through r_{ph} and the normalization $A(r_{\text{ph}})$ only; at linear order these two contributions conspire to eliminate the derivative at r_{ph} .

Logarithmic and finite response: $a_1(\omega)$ and $b_1(\omega)$: We next relate the coefficients $a_1(\omega)$ and $b_1(\omega)$ in Eq. (39) to moments of the near-photon-sphere Born kernel. Let $\hat{e}_b$ be the radial unit vector on the screen. Axisymmetry implies that, to leading order in $\Delta = b/b_c - 1$, the deflection correction $\delta\alpha$ is radial, so we write $\delta\alpha = \delta\alpha_{\parallel} \hat{e}_b$. Comparing Eq. (37) to the scalar linearization (41) gives

$$a_1(\omega) = -\hat{e}_b \cdot \mathcal{C}(\omega), \quad b_1(\omega) = \hat{e}_b \cdot \mathcal{D}(\omega), \quad (\text{C9})$$

where $\mathcal{C} = (\mathcal{C}_X, \mathcal{C}_Y)$ and $\mathcal{D} = (\mathcal{D}_X, \mathcal{D}_Y)$ are the complex vector amplitudes in Eq. (37). In turn, $\mathcal{C}_A$ and $\mathcal{D}_A$ follow from the near-photon-sphere reduction of the kernel $B.11$. Introducing the azimuth φ during the whirl phase (Eq. (36)) and the delay $\tau(\varphi)$, we can write

$$\mathcal{C}_A(\omega) = \int_{\varphi_{\min}}^{\varphi_{\max}} d\varphi W_A^{(0)}(\varphi),$$

$$\mathcal{D}_A(\omega) = \int_{\varphi_{\min}}^{\varphi_{\max}} d\varphi \left[W_A^{(1)}(\varphi) + i\omega U_A^{(1)}(\varphi) \right], \quad (\text{C10})$$

where the weights $W_A^{(0)}, W_A^{(1)}, U_A^{(1)}$ are smooth functions of the background quantities at $r \simeq r_{\text{ph}}$ and of the RWZ amplitudes $\Psi_Z \partial_{r_*} \Psi_Z$ evaluated there (see Eqs. (A13) and (B17)). Their explicit algebraic forms are lengthy but completely fixed by Eqs. (A4), (A8)-(A11), and (B11). The limits $(\varphi_{\min}, \varphi_{\max})$ scale as $|\ln \Delta|$ and generate the logarithm in Eq. (37); the finite parts define b_1 once the inner and outer contributions are combined as in Eq. (52).

Eqs. (C9)-(C10) summarize the content of a_1 and b_1 : both are localized, frequency-resolved moments of the Zerilli field and its first radial derivative at r_{ph} , with a_1 capturing the universal logarithmic weight and b_1 the finite near-photon-sphere remainder after matching to the outer integral (Sec. VA).

Time-of-flight counterparts: The coordinate travel time for near-critical rays has the structure as Eq. (45),

$$\begin{aligned} T(b, t_o) &= -\tilde{a}(t_o) \ln \left(\frac{b}{b_c(t_o)} - 1 \right) \\ &\quad + \tilde{b}(t_o) + n T_{\text{whirl}}(t_o) + \dots, \end{aligned} \quad (\text{C11})$$

where n counts azimuthal winds. Repeating the steps that led to C.9 with the temporal kernel (obtained by replacing the screen projector in Eq. (B11)) with the unit covector $u_\mu = e^{\hat{0}}_\mu$ gives

$$\tilde{a}_1(\omega) = -\hat{t} \cdot \mathcal{C}^{(T)}(\omega), \quad \tilde{b}_1(\omega) = \hat{t} \cdot \mathcal{D}^{(T)}(\omega),$$

$$\gamma_1(\omega) = \frac{\delta T_{\text{whirl}}}{T_{\text{whirl}}}, \quad (\text{C12})$$

where $\hat{t}$ denotes projection onto the time leg of the tetrad and the superscript T reminds us that the weights are those appropriate to the time functional. The same near-photon-sphere data $\Psi_Z \partial_{r_*} \Psi_Z|_{r_{\text{ph}}}$ enter, so the inter-image delay modulation (Eq. (48)) and the spacing/ring modulations (Eqs. (49)-(50)) are tied to the same localized kernel, as emphasized in Sec. VB.

Consistency with the $1/\Delta$ pole and matching: Expanding the time-dependent law 36 to linear order using $b_c(t_o) = b_c, [1 + \varepsilon \text{Re} e^{-i\omega t_o} \beta_1(\omega)]$ and $\bar{a}(t_o) = \bar{a} + \varepsilon \text{Re} e^{-i\omega t_o} a_1(\omega)$ reproduces Eq. (41) with the identifications

$$\begin{aligned} \text{(i)} \quad & \beta_1(\omega) = \frac{\delta b_c}{b_c} \quad \text{from (C.8),} \\ \text{(ii)} \quad & a_1(\omega), b_1(\omega) \quad \text{from (C.9)-(C.10).} \end{aligned} \quad (\text{C13})$$

The $1/\Delta$ term $\bar{a}, \beta_1/\Delta$ is fixed entirely by (C) and is independent of the arbitrary split radius used in Eq. (51); the logarithmic and finite pieces are determined by the inner kernel moments and are rendered unique by the matching cancellations between Eqs. (52)-(53). This is the precise content of the overlap relation (55): the near-critical asymptotics of the $1/b$ Born amplitude encodes the triplet (β_1, a_1, b_1) .

Near criticality, all time-dependent SDL coefficients are controlled by local data at the photon sphere together with

universal whirl-integrals of the Born kernel:

- The critical-scale modulation is quasi-local:

$$\beta_1(\omega) = \frac{1}{4} H_0(r_{\text{ph}}; \omega), \quad (\text{C14})$$

equivalently $\delta b_c/b_c = (3/2), a(r_{\text{ph}})$ with $a = -h_{tt}$, and $P_2(0) = -1/2$.

- The logarithmic and finite coefficients $a_1(\omega)$ and $b_1(\omega)$ are given by kernel moments in Eq. (C10) built from Ψ_Z and $\partial_{r_*} \Psi_Z$ at r_{ph} , projected radially according to Eq. (C9).
- The time-of-flight counterparts $\tilde{a}_1, \tilde{b}_1, \gamma_1$ arise from the same local ingredients via Eq. (C12), explaining the phase-locked modulations of spacing and delays (Sec. IV B) and their correlation with ring-radius oscillations (Sec. V B).

These results complete the derivation of Eqs. (38)-(41) from the kernel formalism, making explicit how the Zerilli master field, evaluated at the photon sphere, controls all near-critical, time-dependent imaging phenomenology in our unified framework.

-
- [1] Zhen Zhong, Vitor Cardoso, and Yifan Chen, “Dynamical Lensing Tomography of Black Hole Ringdowns,” *Phys. Rev. Lett.* **134**, 211402 (2025), [arXiv:2408.10303 \[gr-qc\]](#).
 - [2] Michael D. Johnson *et al.*, “Universal interferometric signatures of a black hole’s photon ring,” *Sci. Adv.* **6**, eaaz1310 (2020), [arXiv:1907.04329 \[astro-ph.IM\]](#).
 - [3] Nick Kaiser and Andrew H. Jaffe, “Bending of light by gravity waves,” *Astrophys. J.* **484**, 545–554 (1997), [arXiv:astro-ph/9609043](#).
 - [4] Thibault Damour and Gilles Esposito-Farese, “Light deflection by gravitational waves from localized sources,” *Phys. Rev. D* **58**, 044003 (1998), [arXiv:gr-qc/9802019](#).
 - [5] Sergei M. Kopeikin and Gerhard Schaefer, “Lorentz covariant theory of light propagation in gravitational fields of arbitrary moving bodies,” *Phys. Rev. D* **60**, 124002 (1999), [arXiv:gr-qc/9902030](#).
 - [6] Naoki Tsukamoto, “Deflection angle in the strong deflection limit in a general asymptotically flat, static, spherically symmetric spacetime,” *Phys. Rev. D* **95**, 064035 (2017), [arXiv:1612.08251 \[gr-qc\]](#).
 - [7] Sunny Vagnozzi *et al.*, “Horizon-scale tests of gravity theories and fundamental physics from the Event Horizon Telescope image of Sagittarius A,” *Class. Quant. Grav.* **40**, 165007 (2023), [arXiv:2205.07787 \[gr-qc\]](#).
 - [8] Alireza Allahyari, Mohsen Khodadi, Sunny Vagnozzi, and David F. Mota, “Magnetically charged black holes from non-linear electrodynamics and the Event Horizon Telescope,” *JCAP* **02**, 003 (2020), [arXiv:1912.08231 \[gr-qc\]](#).
 - [9] Mohsen Khodadi, Alireza Allahyari, Sunny Vagnozzi, and David F. Mota, “Black holes with scalar hair in light of the Event Horizon Telescope,” *JCAP* **09**, 026 (2020), [arXiv:2005.05992 \[gr-qc\]](#).
 - [10] Niyaz Uddin Molla, Himanshu Chaudhary, Salvatore Capozziello, Farruh Atamurotov, G. Mustafa, and Ujjal Debnath, “Observable signatures of RN black holes with dark matter halos via strong gravitational lensing and constraints from EHT observations,” *Phys. Dark Univ.* **47**, 101804 (2025), [arXiv:2501.09439 \[gr-qc\]](#).
 - [11] Allah Ditta, Abdelmalek Bouzenada, G. Mustafa, Faisal Javed, Fakhranda Afandi, and Asif Mahmood, “Impact of gravitational collapse exhibiting loop quantum black holes on thermodynamical features and weak gravitational lensing,” *Phys. Dark Univ.* **47**, 101818 (2025).
 - [12] Ziyodulla Turakhonov, Husanboy Hoshimov, Farruh Atamurotov, Sushant G. Ghosh, and Ahmadjon Abdujabbarov, “Observational signatures of strong gravitational lensing in GUP-modified Schwarzschild black holes,” *Phys. Dark Univ.* **46**, 101716 (2024).
 - [13] Mirzabek Alloqulov, Hrishikesh Chakrabarty, Daniele Malafarina, Bobomurat Ahmedov, and Ahmadjon Abdujabbarov, “Gravitational lensing of neutrinos in parametrized black hole spacetimes,” *JCAP* **02**, 070 (2025), [arXiv:2408.12916 \[gr-qc\]](#).
 - [14] K. S. Virbhadra and George F. R. Ellis, “Schwarzschild black hole lensing,” *Phys. Rev. D* **62**, 084003 (2000), [arXiv:astro-ph/9904193](#).
 - [15] K. S. Virbhadra and G. F. R. Ellis, “Gravitational lensing by naked singularities,” *Phys. Rev. D* **65**, 103004 (2002).
 - [16] K. S. Virbhadra, D. Narasimha, and S. M. Chitre, “Role of the scalar field in gravitational lensing,” *Astron. Astrophys.* **337**, 1–8 (1998), [arXiv:astro-ph/9801174](#).
 - [17] K. S. Virbhadra, “Naked singularities and Seifert’s conjecture,” *Phys. Rev. D* **60**, 104041 (1999), [arXiv:gr-qc/9809077](#).

- [18] Clarissa-Marie Claudel, K. S. Virbhadra, and G. F. R. Ellis, "The Geometry of photon surfaces," *J. Math. Phys.* **42**, 818–838 (2001), [arXiv:gr-qc/0005050](#).
- [19] K. S. Virbhadra, "Relativistic images of Schwarzschild black hole lensing," *Phys. Rev. D* **79**, 083004 (2009), [arXiv:0810.2109 \[gr-qc\]](#).
- [20] K. S. Virbhadra and C. R. Keeton, "Time delay and magnification centroid due to gravitational lensing by black holes and naked singularities," *Phys. Rev. D* **77**, 124014 (2008), [arXiv:0710.2333 \[gr-qc\]](#).
- [21] K. S. Virbhadra, "Distortions of images of Schwarzschild lensing," *Phys. Rev. D* **106**, 064038 (2022), [arXiv:2204.01879 \[gr-qc\]](#).
- [22] K. S. Virbhadra, "Conservation of distortion of gravitationally lensed images," *Phys. Rev. D* **109**, 124004 (2024), [arXiv:2402.17190 \[gr-qc\]](#).
- [23] Stephen L. Adler and K. S. Virbhadra, "Cosmological constant corrections to the photon sphere and black hole shadow radii," *Gen. Rel. Grav.* **54**, 93 (2022), [arXiv:2205.04628 \[gr-qc\]](#).
- [24] K. S. Virbhadra, "Compactness of supermassive dark objects at galactic centers," *Can. J. Phys.* **102**, 512 (2024), [arXiv:2204.01792 \[gr-qc\]](#).
- [25] V. S. Morozova, B. J. Ahmedov, and A. A. Tursunov, "Gravitational lensing by a rotating massive object in a plasma," *Astrophys. Space Sci.* **346**, 513–520 (2013).
- [26] Jan Schee, Zdeněk Stuchlík, Bobomurat Ahmedov, Ahmadjon Abdujabbarov, and Bobir Toshmatov, "Gravitational lensing by regular black holes surrounded by plasma," *Int. J. Mod. Phys. D* **26**, 1741011 (2017).
- [27] Bobur Turimov, Bobomurat Ahmedov, Ahmadjon Abdujabbarov, and Cosimo Bambi, "Gravitational lensing by a magnetized compact object in the presence of plasma," *Int. J. Mod. Phys. D* **28**, 2040013 (2019), [arXiv:1802.03293 \[gr-qc\]](#).
- [28] Ahmadjon Abdujabbarov, Bobomurat Ahmedov, Naresh Dadhich, and Farruh Atamurotov, "Optical properties of a braneworld black hole: Gravitational lensing and retrolensing," *Phys. Rev. D* **96**, 084017 (2017).
- [29] Volker Perlick and Oleg Yu. Tsupko, "Calculating black hole shadows: Review of analytical studies," *Phys. Rept.* **947**, 1–39 (2022), [arXiv:2105.07101 \[gr-qc\]](#).
- [30] Emanuele Berti, Vitor Cardoso, and Andrei O. Starinets, "Quasinormal modes of black holes and black branes," *Class. Quant. Grav.* **26**, 163001 (2009), [arXiv:0905.2975 \[gr-qc\]](#).
- [31] Kostas D. Kokkotas and Bernd G. Schmidt, "Quasinormal modes of stars and black holes," *Living Rev. Rel.* **2**, 2 (1999), [arXiv:gr-qc/9909058](#).
- [32] F. J. Zerilli, "Gravitational field of a particle falling in a schwarzschild geometry analyzed in tensor harmonics," *Phys. Rev. D* **2**, 2141–2160 (1970).
- [33] V. Moncrief, "Gravitational perturbations of spherically symmetric systems. I. The exterior problem," *Annals Phys.* **88**, 323–342 (1974).
- [34] Antony Lewis and Anthony Challinor, "Weak gravitational lensing of the CMB," *Phys. Rept.* **429**, 1–65 (2006), [arXiv:astro-ph/0601594](#).
- [35] A. Einstein, "Erklärung der perihelbewegung des merkur aus der allgemeinen relativitätstheorie," *Albert Einstein: Akademie-Vorträge*, 78–87 (2005).
- [36] Steven Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (John Wiley and Sons, New York, 1972).
- [37] Clifford M. Will, "The Confrontation between General Relativity and Experiment," *Living Rev. Rel.* **17**, 4 (2014), [arXiv:1403.7377 \[gr-qc\]](#).
- [38] V. Bozza, "Gravitational lensing in the strong field limit," *Phys. Rev. D* **66**, 103001 (2002), [arXiv:gr-qc/0208075](#).
- [39] Subrahmanyan Chandrasekhar, *The mathematical theory of black holes* (Oxford University Press, 1985).
- [40] Samuel E. Gralla, Alexandru Lupasca, and Daniel P. Marone, "The shape of the black hole photon ring: A precise test of strong-field general relativity," *Phys. Rev. D* **102**, 124004 (2020), [arXiv:2008.03879 \[gr-qc\]](#).
- [41] Kazunori Akiyama *et al.* (Event Horizon Telescope), "First M87 Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole," *Astrophys. J. Lett.* **875**, L1 (2019), [arXiv:1906.11238 \[astro-ph.GA\]](#).
- [42] Kazunori Akiyama *et al.* (Event Horizon Telescope), "First Sagittarius A* Event Horizon Telescope Results. I. The Shadow of the Supermassive Black Hole in the Center of the Milky Way," *Astrophys. J. Lett.* **930**, L12 (2022), [arXiv:2311.08680 \[astro-ph.HE\]](#).
- [43] Richard H. Price, "Nonspherical perturbations of relativistic gravitational collapse. 1. Scalar and gravitational perturbations," *Phys. Rev. D* **5**, 2419–2438 (1972).
- [44] Mingzhi Wang, Songbai Chen, and Jiliang Jing, "Effect of gravitational wave on shadow of a Schwarzschild black hole," *Eur. Phys. J. C* **81**, 509 (2021), [arXiv:1908.04527 \[gr-qc\]](#).
- [45] Reggie C. Pantig and Ali Övgün, "Imprints of a gravitational wave through the weak field deflection of photons*," *Chin. Phys. C* **48**, 085104 (2024), [arXiv:2406.05782 \[gr-qc\]](#).
- [46] Mohsen Khodadi, Gaetano Lambiase, Leonardo Mastrototaro, and Tanmay Kumar Poddar, "Primordial gravitational waves from spontaneous Lorentz symmetry breaking," *Phys. Lett. B* **867**, 139597 (2025), [arXiv:2501.14395 \[astro-ph.CO\]](#).
- [47] Yifan Chen, Xiao Xue, Richard Brito, and Vitor Cardoso, "Photon Ring Astrometry for Superradiant Clouds," *Phys. Rev. Lett.* **130**, 111401 (2023), [arXiv:2211.03794 \[gr-qc\]](#).
- [48] Liang Dai and Tejaswi Venumadhav, "On the waveforms of gravitationally lensed gravitational waves," (2017), [arXiv:1702.04724 \[gr-qc\]](#).
- [49] Tullio Regge and John A. Wheeler, "Stability of a Schwarzschild singularity," *Phys. Rev.* **108**, 1063–1069 (1957).
- [50] Karl Martel and Eric Poisson, "Gravitational perturbations of the Schwarzschild spacetime: A Practical covariant and gauge-invariant formalism," *Phys. Rev. D* **71**, 104003 (2005), [arXiv:gr-qc/0502028](#).
- [51] C. T. Cunningham and J. M. Bardeen, "The optical appearance of a star orbiting an extreme kerr black hole," *The Astrophysical Journal* **173**, L137 (1972).
- [52] Samuel E. Gralla, Daniel E. Holz, and Robert M. Wald, "Black Hole Shadows, Photon Rings, and Lensing Rings," *Phys. Rev. D* **100**, 024018 (2019), [arXiv:1906.00873 \[astro-ph.HE\]](#).
- [53] Frederic H. Vincent, Samuel E. Gralla, Alexandru Lupasca, and Maciek Wielgus, "Images and photon ring signatures of thick disks around black holes," *Astron. Astrophys.* **667**, A170 (2022), [arXiv:2206.12066 \[astro-ph.HE\]](#).
- [54] Fabio Aratore, Oleg Yu. Tsupko, and Volker Perlick, "Constraining spherically symmetric metrics by the gap between photon rings," *Phys. Rev. D* **109**, 124057 (2024), [arXiv:2402.14733 \[gr-qc\]](#).