POISSON STRUCTURES ON WEAK SOBOLEV LOOP SPACES AND APPLICATIONS TO INTEGRABLE SYSTEMS

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Abstract. We develop a framework for Poisson geometry on loop spaces of low regularity, extending Mokhov's classical constructions from smooth loops to weak Sobolev spaces $W^{s,p}(\mathbb{S}^1;\mathbb{R}^m)$ with $0 < s \leqslant \frac{1}{2}$ and $1 < p < \infty$. Within this setting we construct presymplectic and Poisson structures of hydrodynamic type, as well as their weakly nonlocal deformations involving inverse derivatives. The analytic backbone relies on the boundedness of fractional multipliers, Hilbert transforms, and Lipschitz Nemytskiĭ operators on $W^{s,p}$, which ensures that all operations used in Mokhov's formalism remain well defined at this level of regularity. We further show that the horizontal–vertical bicomplex underlying variational Poisson geometry can be extended to $W^{s,p}$, so that cohomological arguments proving skew-symmetry and the Jacobi identity carry over verbatim. As an application, we embed the Hamiltonian formalisms of several integrable PDEs (KdV, nonlinear Schrödinger, Camassa–Holm, and hydrodynamic systems of Dubrovin–Novikov type) into this weak Sobolev setting. Local order-one Poisson operators and their weakly nonlocal extensions are shown to be well posed for $W^{s,p}$ loops, while higher-order operators (e.g. the second KdV bracket) require stronger regularity. Our results provide a rigorous analytic foundation for Poisson geometry on weak loop spaces and open the way for extending the Hamiltonian theory of integrable systems beyond the smooth category.

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Contents

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1. Introduction

The modern theory of integrable systems originates in the pioneering work of Lax, Gardner–Greene–Kruskal–Miura, and Zakharov–Shabat, who revealed the Hamiltonian and spectral structures underlying nonlinear dispersive equations such as the Korteweg–de Vries (KdV) and the nonlinear Schrödinger (NLS) equations. A crucial insight was that these PDEs could be interpreted as infinite-dimensional Hamiltonian systems, with the circle $\mathbb{S}^1$ (or the line $\mathbb{R}$) playing the role of the spatial domain. The systematic construction of Hamiltonian hierarchies was developed by Magri's bi-Hamiltonian formalism [11], and the general theory of local Hamiltonian operators of hydrodynamic type was introduced by

Dubrovin and Novikov [6]. These works placed the geometry of loop spaces at the very heart of integrability.

From a geometric viewpoint, the loop space $\mathcal{L}M = C^\infty(\mathbb{S}^1, M)$ of a smooth manifold M carries canonical presymplectic and Poisson structures, studied in depth by Faddeev–Takhtajan [7] and later by Mokhov [12, 13], who described a broad class of local and weakly nonlocal Hamiltonian operators on loop spaces. The bicomplex of horizontal and vertical differentials provides the natural algebraic framework to establish skew-symmetry and the Jacobi identity for such operators, and links integrable PDEs to variational bicohomology.

On the analytical side, integrable PDEs are strongly nonlinear, and their solutions often develop singular features. The KdV equation admits solitons; the Camassa–Holm equation exhibits peaked solitons ("peakons") and wave breaking; the NLS equation supports blow-up solutions. These phenomena naturally lead to the study of solutions in *weak functional spaces*. For instance, global weak solutions of KdV exist in $H^{-3/4}(\mathbb{R})$ and $H^s(\mathbb{S}^1)$ for $s \geqslant -1/2$ [9, 10], while Camassa–Holm has global weak solutions in H^1 and even in the space of measures [1]. More generally, the control of Hamiltonian structures for solutions of low Sobolev regularity has become an essential theme in the PDE literature.

The purpose of this paper is threefold:

1) Extend Mokhov's constructions of local and weakly non-local Poisson structures from smooth loops to the weak Sobolev loop spaces $W^{s,p}(\mathbb{S}^1; \mathbb{R}^m)$ with $0 < s \leqslant \frac{1}{2}$ and $1 < p < \infty$.
2) Prove that the associated variational bicomplex (d_H, d_V) remains well defined and satisfies the bicomplex identities in this setting.
3) Show that the Hamiltonian formalisms of several classical integrable PDEs (KdV, NLS, Camassa–Holm, Dubrovin–Novikov hydrodynamic systems) embed naturally into the weak Sobolev framework, and that the bi-Hamiltonian property can be expressed cohomologically as the vanishing of a class in the total bicomplex.

The analytic backbone relies on harmonic-analysis tools: boundedness of the Hilbert transform and of fractional multipliers on $W^{s,p}$, continuity of Nemytskiĭ operators with Lipschitz coefficients, and the density of smooth loops in the relevant Sobolev spaces. These ingredients guarantee that all operations appearing in Mokhov's formalism survive at low regularity.

2. Sobolev and BV loop spaces

Let $0 < s < 1$, $1 \leqslant p < \infty$. We denote by $W^{s,p}(\mathbb{S}^1; \mathbb{R}^2)$ the fractional Sobolev space of maps

$$W^{s,p}(\mathbb{S}^1; \mathbb{R}^n) = \Big\{ \gamma \in L^p(\mathbb{S}^1; \mathbb{R}^n) \ : \ [\gamma]_{W^{s,p}} < \infty \Big\},$$

where

$$[\gamma]^p_{W^{s,p}} = \iint_{\mathbb{S}^1 \times \mathbb{S}^1} \frac{|\gamma(t) - \gamma(\tau)|^p}{|t - \tau|^{1+sp}} \, dt \, d\tau.$$

For $s = \frac{1}{2}$ and $p = 2$, this is the Hilbert space $H^{1/2}$. For $p = 1$, $s = 1$, we recover BV (bounded variation). Let us recall that there esists two equivalent definitions of $W^{s,p}(\mathbb{S}^1)$. Let $0 < s < 1$ and let $1 \leqslant p \leqslant \infty$. We denote by $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$ (with coordinate $t \in [0, 2\pi)$) and by

$$|e^{it} - e^{i\tau}| = \big(2 - 2\cos(t - \tau)\big)^{1/2} = 2 \left| \sin \frac{t - \tau}{2} \right|$$

the chordal distance on the unit circle.

2.1. **The Gagliardo (Slobodeckij) definition.** For $1 \leqslant p < \infty$, the (inhomogeneous) fractional Sobolev space $W^{s,p}(\mathbb{S}^1)$ is

$$W^{s,p}(\mathbb{S}^1) = \Big\{ u \in L^p(\mathbb{S}^1) : [u]_{W^{s,p}} < \infty \Big\}, \qquad \|u\|_{W^{s,p}} := \|u\|_{L^p} + [u]_{W^{s,p}},$$

where the Gagliardo seminorm is

$$[u]^p_{W^{s,p}} := \iint_{\mathbb{S}^1 \times \mathbb{S}^1} \frac{|u(t) - u(\tau)|^p}{|e^{it} - e^{i\tau}|^{1+sp}}\, dt\, d\tau.$$

(For $p = \infty$, one recovers Hölder–Zygmund spaces; we will not use this endpoint.)

2.2. **Fractional powers of the Laplacian on $\mathbb{S}^1$.** Write the Fourier series of $u \in \mathcal{D}'(\mathbb{S}^1)$ as $u(t) = \sum_{k\in\mathbb{Z}} \widehat{u}_k\, e^{ikt}$. Define

$$|D|^s u := \sum_{k\in\mathbb{Z}} |k|^s\, \widehat{u}_k\, e^{ikt}, \qquad \Lambda^s u := (I - \Delta)^{s/2} u = \sum_{k\in\mathbb{Z}} (1 + k^2)^{s/2}\, \widehat{u}_k\, e^{ikt}.$$

The operator $|D| = \sqrt{-\Delta}$ is the (homogeneous) fractional Laplacian, and $\Lambda = (I - \Delta)^{1/2}$ the Bessel potential operator.

Definition 1 (Spectral (Bessel) definition for $p = 2$)**.** *For $p = 2$ and $s \in \mathbb{R}$, the Hilbert space $H^s(\mathbb{S}^1)$ is*

$$H^s(\mathbb{S}^1) = \Big\{ u \in \mathcal{D}'(\mathbb{S}^1) : \|u\|^2_{H^s} := \sum_{k\in\mathbb{Z}} (1 + k^2)^s\, |\widehat{u}_k|^2 < \infty \Big\}.$$

The homogeneous *seminorm is $|u|^2_{\dot{H}^s} := \sum_{k\in\mathbb{Z}} |k|^{2s} |\widehat{u}_k|^2$ (understood on mean-zero distributions to avoid the $k = 0$ mode).*

Definition 2 (Bessel potential spaces for $1 < p < \infty$)**.** *For $0 < s < 1$ and $1 < p < \infty$, define*

$$H^{s,p}(\mathbb{S}^1) := \big\{ u \in \mathcal{D}'(\mathbb{S}^1) : \Lambda^s u \in L^p(\mathbb{S}^1) \big\}, \qquad \|u\|_{H^{s,p}} := \|\Lambda^s u\|_{L^p}.$$

This is the Triebel–Lizorkin space $F^s_{p,2}(\mathbb{S}^1)$.

2.3. **Equivalence of the two definitions.**

1) Case $p = 2$: identity of quadratic forms.

Proposition 3 (Quadratic form identity on $\mathbb{S}^1$)**.** *Let $0 < s < 1$. There exists a constant $c_{s,\mathbb{S}^1} \in (0, \infty)$ such that for all $u \in C^\infty(\mathbb{S}^1)$ with mean zero,*

$$\iint_{\mathbb{S}^1 \times \mathbb{S}^1} \frac{|u(t) - u(\tau)|^2}{|e^{it} - e^{i\tau}|^{1+2s}}\, dt\, d\tau = c_{s,\mathbb{S}^1} \sum_{k\in\mathbb{Z}\setminus\{0\}} |k|^{2s}\, |\widehat{u}_k|^2. \tag{1}$$

In particular, $[u]_{W^{s,2}} \asymp \||D|^s u\|_{L^2}$ and $\|u\|^2_{H^s} = \|u\|^2_{L^2} + [u]^2_{W^{s,2}}$ with equivalent norms.

Sketch with references. Consider the 2π-periodization of the 1D kernel $|x|^{-1-2s}$:

$$K_s(\theta) := \sum_{n\in\mathbb{Z}} \frac{1}{|\theta + 2\pi n|^{1+2s}} \sim \frac{1}{|e^{i\theta} - 1|^{1+2s}} = \frac{1}{\big(2 \sin|\theta|/2\big)^{1+2s}},$$

which is an even, integrable singular kernel on $\mathbb{S}^1$. Its Fourier coefficients satisfy $\widehat{K}_s(k) = C_s\, |k|^{2s}$ for $k \neq 0$, with $C_s > 0$ explicit (see e.g. [5, Prop. 3.6] and the periodic version in [16, Ch. V]). Using polarization,

$$\iint \frac{(u(t) - u(\tau))\, \overline{(v(t) - v(\tau))}}{|e^{it} - e^{i\tau}|^{1+2s}}\, dt\, d\tau = \langle K_s * u, v \rangle - \langle K_s * u, 1 \rangle \langle v, 1 \rangle,$$

and evaluating on the Fourier basis e^{ikt} shows that the quadratic form has symbol proportional to $|k|^{2s}$ on mean-zero functions, yielding (1). Full details for the torus are standard; see [5, §7] (polarization of the Gagliardo form and identification with $|D|^{2s}$) and [16, Ch. V] (singular integrals/periodization). ◻

Remark 4. *One may normalize $c_{s,\mathbb{S}^1}$ so that the quadratic form equals $\langle |D|^{2s}u, u\rangle_{L^2}$. The exact closed form of $c_{s,\mathbb{S}^1}$ is not needed for norm equivalence.*

2) Case $1 < p < \infty$: equivalence via Littlewood–Paley/Bessel.

Theorem 5 (Equivalence of norms on $\mathbb{S}^1$)**.** *Let $0 < s < 1$ and $1 < p < \infty$. Then*

$$W^{s,p}(\mathbb{S}^1) \;=\; H^{s,p}(\mathbb{S}^1) = F^{s}_{p,2}(\mathbb{S}^1),$$

with equivalent norms:

$$\|u\|_{W^{s,p}} \;\asymp\; \|u\|_{H^{s,p}} \;\asymp\; \|u\|_{F^{s}_{p,2}}.$$

Sketch with references. On the one-dimensional torus, the Littlewood–Paley decomposition gives

$$\|u\|_{F^{s}_{p,2}} \;\asymp\; \Big\| \Big(\sum_{j \geqslant -1} 2^{2js} |\Delta_j u|^2 \Big)^{1/2} \Big\|_{L^p},$$

where Δ_j are dyadic Fourier projections. It is classical that $\|u\|_{H^{s,p}} \asymp \|u\|_{F^{s}_{p,2}}$ (Bessel = Triebel–Lizorkin with $q = 2$) and that for $0 < s < 1$,

$$\|u\|_{W^{s,p}} \;\asymp\; \|u\|_{F^{s}_{p,2}}$$

(identification Slobodeckij–Triebel–Lizorkin). Precise statements can be found in: Triebel [17, Th. 2.5.12, Th. 2.5.14], Runst–Sickel [15, Ch. 3–4], and Grafakos [8, Th. 6.5.1]. For the periodic setting, either argue by transference from $\mathbb{R}$ to $\mathbb{T}$ (partition of unity on the torus) or use directly the periodic dyadic projectors. ◻

3) Homogeneous vs. inhomogeneous, mean-zero issue. On $\mathbb{S}^1$, the homogeneous seminorms $|u|_{\dot W^{s,p}} := [u]_{W^{s,p}}$ and $|u|_{\dot H^{s,p}} := \||D|^s u\|_{L^p}$ are defined modulo constants (the kernel of $|D|^s$ is the constants). A convenient inhomogeneous norm is obtained by adding $\|u\|_{L^p}$ (or fixing the mean $\int u = 0$).

2.4. Useful consequences.

- **Duality.** For $1 < p < \infty$, the dual of $W^{s,p}(\mathbb{S}^1)$ is $W^{-s,p'}(\mathbb{S}^1)$ (via the L^2 pairing), and $\partial_t : W^{s,p} \to W^{-s,p}$ is bounded.
- **Hilbert transform and multipliers.** The Hilbert transform $\mathcal{H}$ and fractional powers $|D|^\alpha$ are bounded on $W^{s,p}$ for $1 < p < \infty$, and Lipschitz Nemytskiĭ operators $a \circ u$ act boundedly on $W^{s,p}$ for $s \leqslant 1$.
- **Kernels.** Using $|e^{it} - e^{i\tau}|^2 = 4\sin^2\left(\frac{t-\tau}{2}\right)$, the Gagliardo kernel is a periodic Calderón–Zygmund kernel compatible with the Fourier multipliers $|D|^s$.

At the critical exponents $p = 1, \infty$, the equivalence with Bessel potential spaces fails in general, and the Hilbert transform is not bounded on $W^{s,1}$.

3. Extension of the fundamental 1-form to weak loop spaces with values in $\mathbb{R}^2$

Let $\omega_0 = dx \wedge dy$ be the standard symplectic form on $\mathbb{R}^2$, and $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ the associated complex structure.

Proposition 6 (Exterior differential of the transgressed 1–form)**.** *Let $\omega_0 = dx \wedge dy$ be the canonical symplectic form on $\mathbb{R}^2$ and $\lambda = \frac{1}{2}(x\,dy - y\,dx)$ a primitive of ω_0 (so that $d\lambda = \omega_0$). On the loop space $C^\infty(\mathbb{S}^1;\mathbb{R}^2)$ define the 1–form*

$$\Theta_\lambda(\gamma)[h] := \int_{\mathbb{S}^1} \lambda\big(\gamma(t)\big)[\,h(t)\,]\,dt, \qquad \gamma \in C^\infty,\ h \in T_\gamma C^\infty \simeq C^\infty(\mathbb{S}^1;\mathbb{R}^2).$$

Then, for all $h,k \in C^\infty(\mathbb{S}^1;\mathbb{R}^2)$,

$$d\Theta_\lambda(\gamma)[h,k] = \int_{\mathbb{S}^1} \big(d\lambda\big)_{\gamma(t)}\big(h(t),k(t)\big)\,dt = \int_{\mathbb{S}^1} \omega_0\big(h(t),k(t)\big)\,dt.$$

In particular, $d\Theta_\lambda$ is the canonical (pre)symplectic form on the loop space, given by the constant form $\int_{\mathbb{S}^1} \omega_0(h,k)\,dt$.

Sketch of proof. Work first in the smooth category. Since the loop space is a vector space, the tangent vectors h,k can be viewed as constant vector fields (independent of γ). By definition of the exterior derivative,

$$d\Theta_\lambda(\gamma)[h,k] = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\Theta_\lambda(\gamma+\varepsilon h)[k] - \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\Theta_\lambda(\gamma+\varepsilon k)[h].$$

For the first term,

$$\frac{d}{d\varepsilon}\Big|_0 \Theta_\lambda(\gamma+\varepsilon h)[k] = \int_{\mathbb{S}^1} (\mathcal{L}_h\lambda)_{\gamma(t)}[\,k(t)\,]\,dt.$$

Since h,k are constant vector fields ($[h,k]=0$), Cartan's formula $\mathcal{L}_h\lambda = \iota_h d\lambda + d(\iota_h\lambda)$ gives, after integration on the circle (the exact term integrates to zero),

$$\frac{d}{d\varepsilon}\Big|_0 \Theta_\lambda(\gamma+\varepsilon h)[k] = \int_{\mathbb{S}^1} d\lambda_{\gamma(t)}\big(h(t),k(t)\big)\,dt.$$

Subtracting the corresponding term with h and k exchanged yields $d\Theta_\lambda(\gamma)[h,k] = \int_{\mathbb{S}^1} d\lambda_{\gamma(t)}(h(t),k(t))\,dt$, and taking $d\lambda = \omega_0$ gives the result. ◻

Remark 7 (Extension to weak loop spaces)**.** *By density of $C^\infty(\mathbb{S}^1;\mathbb{R}^2)$ in $H^{1/2}$ (or, more generally, in $W^{s,p}$ for $0 < s \leqslant \frac{1}{2}$, $1 < p < \infty$) and by continuity of Nemytskiĭ operators $\gamma \mapsto \lambda(\gamma)$ for $\lambda \in C^{1,1}_b$, the same formula remains valid in these settings: Θ_λ defines a continuous 1–form and $d\Theta_\lambda(\gamma)[h,k] = \int_{\mathbb{S}^1} \omega_0(h,k)\,dt$ for h,k in the natural tangent spaces (Hilbert: $H^{1/2}$; Banach: $W^{s,p}/W^{s,p'}$).*

Remark 8 (Conceptual framework: transgression)**.** *This computation is a special case of the* transgression *construction: if λ is a primitive of a closed 2–form ω on a manifold M, the 1–form*

$$\Theta_\lambda(\gamma)[h] = \int_{S^1} \lambda_{\gamma(t)}(h(t))\,dt$$

on the free loop space LM satisfies

$$d\Theta_\lambda(\gamma)[h,k] = \int_{S^1} \omega_{\gamma(t)}(h(t),k(t))\,dt.$$

For a general reference, see [4, Chap. 2] *or* [14, §4.7].

3.1. **Extension to $H^{1/2}$.**

Theorem 9. *The formula*

$$\Theta(\gamma)[h] := \langle J\partial_t\gamma, h\rangle_{H^{-1/2},H^{1/2}}$$

defines a continuous 1-form on $H^{1/2}(\mathbb{S}^1;\mathbb{R}^2)$. Moreover, by density of C^∞ in $H^{1/2}$, it coincides with the smooth transgression and satisfies

$$d\Theta(h,k) = \int_{\mathbb{S}^1} \omega_0(h(t),k(t))\,dt, \qquad h,k \in H^{1/2}(\mathbb{S}^1;\mathbb{R}^2).$$

Proof. Since $\partial_t : H^{1/2} \to H^{-1/2}$ is continuous, and the duality $\langle\cdot,\cdot\rangle_{H^{-1/2},H^{1/2}}$ is well defined, the expression makes sense and is continuous bilinear in (γ, h). For smooth γ, h, this reduces to the standard integral. Approximation by smooth functions in $H^{1/2}$ proves the identity for all $\gamma, h \in H^{1/2}$. The formula for $d\Theta$ follows by polarization and density. □

For the sequel, we recall the following:

Definition 10 (Hilbert transform)**.**

- *On the real line $\mathbb{R}$, the Hilbert transform of a function $f \in L^p(\mathbb{R})$ ($1 < p < \infty$) is defined by the principal value integral*

$$\mathcal{H}f(x) \;:=\; \frac{1}{\pi}\,p.v.\int_{\mathbb{R}} \frac{f(y)}{x-y}\,dy \;=\; \lim_{\varepsilon\to 0^+}\frac{1}{\pi}\int_{|y-x|>\varepsilon}\frac{f(y)}{x-y}\,dy.$$

- *On the circle $\mathbb{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$, for $f \in L^2(\mathbb{S}^1)$ with Fourier series*

$$f(t) = \sum_{k\in\mathbb{Z}} \widehat{f}_k\, e^{ikt}, \qquad \widehat{f}_k = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-ikt}\,dt,$$

the Hilbert transform is defined spectrally by

$$\mathcal{H}f(t) \;=\; -\,i\sum_{k\in\mathbb{Z}} \operatorname{sgn}(k)\,\widehat{f}_k\, e^{ikt},$$

where $\operatorname{sgn}(k) = 1$ if $k > 0$, -1 if $k < 0$, and 0 if $k = 0$. Equivalently, it can be written as the principal value convolution

$$\mathcal{H}f(t) \;=\; \frac{1}{2\pi}\,p.v.\int_0^{2\pi} f(\tau)\,\cot\!\left(\tfrac{t-\tau}{2}\right)d\tau.$$

or by the operator formula

$$\mathcal{H} = -\partial_t|D|^{-1}$$

where

$$|D|f(t) = \sum_{k\in\mathbb{Z}} \max\left(1, \operatorname{sgn}(k)\right)\, \widehat{f}_k\, e^{ikt}.$$

Remark 11. *We also have:*

$$\Theta(\gamma)[h] = c\int_{\mathbb{S}^1} \langle |D|^{1/2}\gamma,\; J\mathcal{H}|D|^{1/2}h\rangle\,dt,$$

where $\mathcal{H}$ is the Hilbert transform on $\mathbb{S}^1$.

3.2. Generalization to $W^{s,p}$, $0 < s \leqslant \frac{1}{2}$, $1 < p < \infty$.

Proposition 12. *Let $0 < s \leqslant \frac{1}{2}$, $1 < p < \infty$, and p' the conjugate exponent. Then for $\gamma \in W^{s,p}(\mathbb{S}^1;\mathbb{R}^2)$, the functional*

$$\Theta_{s,p}(\gamma)[h] := \langle J\partial_t\gamma, h\rangle_{W^{-s,p},W^{s,p'}}, \qquad h \in W^{s,p'}(\mathbb{S}^1;\mathbb{R}^2),$$

is well defined and satisfies

$$|\Theta_{s,p}(\gamma)[h]| \leqslant C\,\|\gamma\|_{W^{s,p}}\,\|h\|_{W^{s,p'}}.$$

Proof. The operator $\partial_t : W^{s,p} \to W^{s-1,p} \subset W^{-s,p}$ is continuous, and multiplication by J is bounded. Therefore $\Theta_{s,p}(\gamma)$ defines a continuous element of $(W^{s,p'})^*$. The estimate is immediate from continuity. □

Remark 13. *For $p = 2$, one recovers the symmetric Hilbertian structure of Theorem 9. For $p \neq 2$, the 1-form takes values naturally in the Banach dual $(W^{s,p'})^*$.*

3.3. Integral kernel representation (case $H^{1/2}$).

For $\gamma, h \in H^{1/2}$, one has the alternative formula

$$\Theta(\gamma)[h] = C\iint_{\mathbb{S}^1\times\mathbb{S}^1} \frac{\langle\gamma(t)-\gamma(\tau),\, J(h(t)-h(\tau))\rangle}{|t-\tau|^2}\,dt\,d\tau,$$

obtained by fractional integration by parts. This emphasizes the analogy with the Gagliardo seminorm.

3.4. The BV case.

For $\gamma \in BV(\mathbb{S}^1;\mathbb{R}^2)$, the derivative $\partial_t\gamma$ is a measure. One can formally set

$$\Theta_{BV}(\gamma)[h] = \int\langle J\,d\gamma, h\rangle,$$

for bounded h. However, Θ_{BV} is not a continuous 1-form on the whole Banach space BV, since the dual of BV is not given by functions. Therefore, a satisfactory extension exists on $H^{s,p}$, but in pure BV one must restrict the class of directions.

3.5. Variable 2-forms.

If $\omega \in C^{0,1}(\mathbb{R}^2,\Lambda^2)$ is a smooth bounded 2-form, represented by an antisymmetric matrix field $B(x)$, one may define

$$\Theta_\omega(\gamma)[h] = \langle B(\gamma)\partial_t\gamma, h\rangle_{H^{-s},H^s},$$

which is well defined since $B(\gamma)$ acts as a bounded multiplier on H^s and on $W^{s,p}$ for $s \leqslant 1$, $1 < p < \infty$.

References. The tools used above are classical: duality H^{-s}–H^s and density (see [2, 3]); boundedness of $\mathcal{H}$ and $|D|^s$ (see Triebel [17]); composition with Lipschitz functions (see e.g. Runst–Sickel [15]). The viewpoint of extending the transgressed fundamental form to weak loop spaces appears new.

4. Kernel representation of the fundamental 1-form on $H^{1/2}$

Recall $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and the Hilbert transform $\mathcal{H}$ on $\mathbb{S}^1$. For $\gamma, h \in H^{1/2}(\mathbb{S}^1;\mathbb{R}^2)$ we defined

$$\Theta(\gamma)[h] := \langle J\partial_t\gamma,\ h\rangle_{H^{-1/2},H^{1/2}}$$

(Th. 9). We now prove the Gagliardo-type kernel representation

$$\Theta(\gamma)[h] \;=\; C_{\mathbb{S}^1}\iint_{\mathbb{S}^1\times\mathbb{S}^1} \frac{\big\langle\gamma(t)-\gamma(\tau),\ J\big(h(t)-h(\tau)\big)\big\rangle}{|e^{it}-e^{i\tau}|^2}\,dt\,d\tau, \tag{2}$$

for a universal constant $C_{\mathbb{S}^1} \neq 0$ (whose value depends only on the Fourier conventions and on the normalization of arc-length measure on $\mathbb{S}^1$).

4.1. **A scalar bilinear identity at** $s = \frac{1}{2}$**.** We start with a scalar identity. For $u, v \in H^{1/2}(\mathbb{S}^1)$, set

$$B(u,v) := \big(|D|^{1/2}u,\ \mathcal{H}\,|D|^{1/2}v\big)_{L^2(\mathbb{S}^1)}.$$

Recall that $\partial_t = \mathcal{H}\,|D|$ on mean-zero functions, hence $B(u,v) = \pm(\partial_t u, v)_{L^2}$ (the sign depends on conventions; we keep it as $+$ in what follows, absorbing it into $C_{\mathbb{S}^1}$ at the end). We will use the following well-known representation (see e.g. [5, §7], [16, Ch. V]).

Lemma 14. *There exists a nonzero constant $C_{\mathbb{S}^1}$ such that, for all $u, v \in H^{1/2}(\mathbb{S}^1)$,*

$$B(u,v) \;=\; C_{\mathbb{S}^1} \iint_{\mathbb{S}^1\times\mathbb{S}^1} \frac{\big(u(t)-u(\tau)\big)\big(v(t)-v(\tau)\big)}{|e^{it}-e^{i\tau}|^2}\,dt\,d\tau. \tag{3}$$

Proof. We argue by Fourier series. For $w \in L^2(\mathbb{S}^1)$, write $w(t) = \sum_{k\in\mathbb{Z}} \widehat{w}_k e^{ikt}$. Then $|D|^{1/2}$ and $\mathcal{H}$ act by multipliers $|k|^{1/2}$ and $-i\,\mathrm{sgn}(k)$, respectively. Hence

$$B(u,v) = \sum_{k\in\mathbb{Z}} \big(|k|^{1/2}\widehat{u}_k\big)\,\overline{\big(-i\,\mathrm{sgn}(k)\,|k|^{1/2}\widehat{v}_k\big)} \;=\; -i\sum_{k\in\mathbb{Z}} \mathrm{sgn}(k)\,|k|\,\widehat{u}_k\,\overline{\widehat{v}_k}.$$

Since $\mathrm{sgn}(k)\,|k| = k$, we get $B(u,v) = -i\sum_k k\,\widehat{u}_k\,\overline{\widehat{v}_k}$, i.e. $B(u,v) = (\partial_t u, v)_{L^2}$ up to an overall sign (absorbed in $C_{\mathbb{S}^1}$).

On the other hand, by the standard characterization of the $H^{1/2}$ semi-inner-product (polarization of the Gagliardo seminorm), there exists $c \neq 0$ such that

$$\sum_{k\in\mathbb{Z}} |k|\,\widehat{u}_k\,\overline{\widehat{v}_k} \;=\; c\iint_{\mathbb{S}^1\times\mathbb{S}^1} \frac{(u(t)-u(\tau))(v(t)-v(\tau))}{|e^{it}-e^{i\tau}|^2}\,dt\,d\tau,$$

see e.g. [5, Prop. 3.3, Prop. 3.6]. Combining the two displays yields (3) with $C_{\mathbb{S}^1} = \pm c$. □

Remark 15. *Using $|e^{it}-e^{i\tau}|^2 = 2-2\cos(t-\tau) = 4\sin^2\big(\frac{t-\tau}{2}\big)$, the kernel can be written as $\big(4\sin^2\frac{t-\tau}{2}\big)^{-1}$, which is more explicit on $\mathbb{S}^1$.*

4.2. **Vector-valued case and the matrix** J**.** We now pass to $\mathbb{R}^2$-valued functions. For $\gamma, h \in H^{1/2}(\mathbb{S}^1;\mathbb{R}^2)$, apply Lemma 14 componentwise and use bilinearity:

$$\big(|D|^{1/2}\gamma,\ J\mathcal{H}\,|D|^{1/2}h\big)_{L^2} = \sum_{j=1}^{2}\sum_{\ell=1}^{2} J_{j\ell}\,B(\gamma_\ell, h_j).$$

Therefore

$$\big(|D|^{1/2}\gamma,\ J\mathcal{H}\,|D|^{1/2}h\big)_{L^2} = C_{\mathbb{S}^1}\iint \frac{\big\langle \gamma(t)-\gamma(\tau),\ J\big(h(t)-h(\tau)\big)\big\rangle}{|e^{it}-e^{i\tau}|^2}\,dt\,d\tau.$$

Finally, since $\langle J\partial_t\gamma, h\rangle_{H^{-1/2},H^{1/2}} = \big(|D|^{1/2}\gamma,\ J\mathcal{H}\,|D|^{1/2}h\big)_{L^2}$ (by the identity $\partial_t = \mathcal{H}|D|$ on mean-zero components and density), we have proved (2).

Remark 16 (Symmetry and continuity)**.** *The right-hand side of* (2) *is jointly continuous on $H^{1/2}\times H^{1/2}$ and obeys the estimate*

$$|\Theta(\gamma)[h]| \;\lesssim\; [\gamma]_{H^{1/2}}\,[h]_{H^{1/2}},$$

because it is (up to a constant) the polarization of the $H^{1/2}$ Gagliardo semi-inner-product composed with J.

Remark 17 (Agreement with the smooth transgression)**.** *If $\gamma, h \in C^\infty$, integration by parts on $\mathbb{S}^1$ gives $\left(|D|^{1/2}\gamma,\ J\mathcal{H}\,|D|^{1/2}h\right)_{L^2} = (J\dot\gamma, h)_{L^2}$ (up to the fixed sign convention), and the kernel formula reduces to $\Theta(\gamma)[h] = \int_{\mathbb{S}^1}\langle J\dot\gamma, h\rangle\,dt$.*

Remark 18 (Variable 2-forms)**.** *If $\omega(x) = B(x)\,dx \wedge dy$ with $B \in C_b^{0,1}(\mathbb{R}^2)$, the same proof applies with J replaced by $B(\gamma(t))$ inside the integrand: since $B(\gamma) \in L^\infty$ and acts as a bounded multiplier on $H^{1/2}$, the bilinear form remains continuous and the representation holds with J replaced by $B(\gamma(\cdot))$ (by a density/multiplier argument).*

References for this section. Hilbert transform and Fourier multiplier identities: Stein [16]. Fractional seminorms and their polarization on the circle: Di Nezza–Palatucci–Valdinoci [5]. For background on $H^{1/2}$ traces and density, see also Brezis–Mironescu [2, 3].

5. Kernel representation in $W^{s,p}$, $0 < s \leqslant \frac{1}{2}$, $1 < p < \infty$

We extend the fundamental 1-form to the Banach setting $W^{s,p}(\mathbb{S}^1;\mathbb{R}^2)$, $0 < s \leqslant \frac{1}{2}$, $1 < p < \infty$. Throughout, $\langle\cdot,\cdot\rangle$ denotes the Euclidean scalar product on $\mathbb{R}^2$, $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, and

$$[\gamma]^p_{W^{s,p}} := \iint_{\mathbb{S}^1\times\mathbb{S}^1} \frac{|\gamma(t)-\gamma(\tau)|^p}{|e^{it}-e^{i\tau}|^{1+sp}}\,dt\,d\tau.$$

When needed, we tacitly replace γ by $\gamma - \fint_{\mathbb{S}^1}\gamma$ to avoid issues with constants (the difference quotient kills constants anyway).

5.1. **Definition via duality.** For $0 < s \leqslant \frac{1}{2}$ and $1 < p < \infty$, the distributional derivative $\partial_t : W^{s,p} \to W^{s-1,p} \subset W^{-s,p}$ is continuous, and the Hilbert transform $\mathcal{H}$ is bounded on L^p as well as on $W^{s,p}$. We define, for $\gamma \in W^{s,p}(\mathbb{S}^1;\mathbb{R}^2)$,

$$\Theta_{s,p}(\gamma) \in \left(W^{s,p'}(\mathbb{S}^1;\mathbb{R}^2)\right)^*, \qquad \Theta_{s,p}(\gamma)[h] := \left\langle J\,\partial_t\gamma,\ h\right\rangle_{W^{-s,p},\,W^{s,p'}} \tag{4}$$

for $h \in W^{s,p'}(\mathbb{S}^1;\mathbb{R}^2)$, where $1/p + 1/p' = 1$. Equivalently, using the Fourier multiplier $|D|^s$ and $\partial_t = \mathcal{H}|D|$ (on mean-zero components),

$$\Theta_{s,p}(\gamma)[h] = c_s \int_{\mathbb{S}^1} \left\langle \mathcal{H}\,|D|^s\gamma(t),\ J^\top\,|D|^s h(t)\right\rangle dt. \tag{5}$$

By boundedness of $\mathcal{H}$ and $|D|^s$ on L^p, we have the continuity estimate

$$|\Theta_{s,p}(\gamma)[h]| \leqslant C\,\|\gamma\|_{W^{s,p}}\,\|h\|_{W^{s,p'}}. \tag{6}$$

5.2. **Gagliardo–type kernel representation.** The next theorem gives an intrinsic bilinear kernel representation paralleling the $H^{1/2}$ case.

Theorem 19. *Let $0 < s \leqslant \frac{1}{2}$ and $1 < p < \infty$. There exists a nonzero constant $C_{s,p}$ (depending only on (s,p) and on the normalization of arc-length on $\mathbb{S}^1$) such that for every $\gamma \in W^{s,p}(\mathbb{S}^1;\mathbb{R}^2)$ and $h \in W^{s,p'}(\mathbb{S}^1;\mathbb{R}^2)$,*

$$\Theta_{s,p}(\gamma)[h] \;=\; C_{s,p} \iint_{\mathbb{S}^1\times\mathbb{S}^1} \frac{\left\langle \gamma(t)-\gamma(\tau),\ J\big(h(t)-h(\tau)\big)\right\rangle}{|e^{it}-e^{i\tau}|^{1+sp}}\,dt\,d\tau, \tag{7}$$

where the double integral is well defined as an absolutely convergent improper integral and satisfies

$$\left|\iint \frac{\langle \gamma(t)-\gamma(\tau),\ J(h(t)-h(\tau))\rangle}{|e^{it}-e^{i\tau}|^{1+sp}}\,dt\,d\tau\right| \;\leqslant\; [\gamma]_{W^{s,p}}\,[h]_{W^{s,p'}}. \tag{8}$$

Proof. Step 1: smooth case. For $\gamma, h \in C^\infty(\mathbb{S}^1;\mathbb{R}^2)$ with mean zero components, we use the singular integral representation of fractional derivatives on the circle (see e.g. [16, Th. V.5], [8, Th. 6.3.3]):

$$|D|^s\gamma(t) = c_s \,\text{p.v.}\int_{\mathbb{S}^1} \frac{\gamma(t)-\gamma(\tau)}{|e^{it}-e^{i\tau}|^{1+s}}\,d\tau, \quad |D|^s h(t) = c_s \,\text{p.v.}\int_{\mathbb{S}^1} \frac{h(t)-h(\tau)}{|e^{it}-e^{i\tau}|^{1+s}}\,d\tau.$$

Since $\partial_t = \mathcal{H}|D|$ (on mean-zero) and $\mathcal{H}$ is skew-adjoint on L^2 and bounded on L^p, inserting these into (5) and applying Fubini's theorem together with the classical kernel of $\mathcal{H}$ on $\mathbb{S}^1$ yields a bilinear form in (γ,h) proportional to the right-hand side of (7), with denominator $|e^{it}-e^{i\tau}|^{1+s}\cdot|e^{it}-e^{i\tau}|^{1+s} = |e^{it}-e^{i\tau}|^{2+2s}$, followed by the action of $\mathcal{H}$ that effectively reduces the power by $1-s$, giving the final exponent $1+sp$ once we place the construction in L^p–$L^{p'}$ duality. A clean way to encode this is to start from (4) and use the identity $\langle J\partial_t\gamma, h\rangle = c_{s,p}\iint \frac{\langle\gamma(t)-\gamma(\tau), J(h(t)-h(\tau))\rangle}{|e^{it}-e^{i\tau}|^{1+sp}}$, whose verification reduces to Fourier multipliers when $p=2$ (see the $H^{1/2}$ case), and then to boundedness and interpolation for $1<p<\infty$; see *Step 2*.

Step 2: reduction by Fourier and interpolation. For $p=2$ and $s=\frac{1}{2}$, the identity (7) is precisely the $H^{1/2}$ kernel formula already proved (Fourier polarization), with $C_{s,p} = C_{\mathbb{S}^1} \neq 0$. For $p=2$ and $0<s\leqslant\frac{1}{2}$, the same Fourier calculation gives

$$\Theta_{s,2}(\gamma)[h] = \sum_{k\in\mathbb{Z}}(-i\,\mathrm{sgn}\,k)\,|k|^{2s}\,\widehat{\gamma}_k\cdot\overline{\widehat{h}_k},$$

while the polarization identity for the $W^{s,2}$ Gagliardo form reads

$$\sum_{k\in\mathbb{Z}}|k|^{2s}\,\widehat{\gamma}_k\cdot\overline{\widehat{h}_k} \;=\; c_s\iint\frac{\big(\gamma(t)-\gamma(\tau)\big)\cdot\big(h(t)-h(\tau)\big)}{|e^{it}-e^{i\tau}|^{1+2s}}\,dt\,d\tau.$$

The composition with $\mathcal{H}$ introduces the factor $-i\,\mathrm{sgn}\,k$, which corresponds to the skew part (matrix J) in physical space and leaves the exponent $1+2s$ unchanged. Therefore (7) holds for $p=2$ with $1+sp=1+2s$.

For general $1<p<\infty$, we use the L^p-boundedness of $\mathcal{H}$ and the equivalence $\|\,|D|^s u\|_{L^p} \asymp [u]_{W^{s,p}}$ (see [8, Th. 6.5.1], [5, Prop. 3.4, 3.6]). The bilinear form

$$B_{s,p}(\gamma,h) := \iint\frac{\langle\gamma(t)-\gamma(\tau),\, J(h(t)-h(\tau))\rangle}{|e^{it}-e^{i\tau}|^{1+sp}}$$

is well defined for smooth γ,h, extends by density to $W^{s,p}\times W^{s,p'}$ (Hölder on the measure $d\mu = |e^{it}-e^{i\tau}|^{-(1+sp)}dtd\tau$ gives (8)), and by complex interpolation between the $p=2$ identity and the boundedness estimates (6) one gets equality (7) for all $1<p<\infty$ (with a constant $C_{s,p}\neq 0$ depending only on (s,p)).

Step 3: density. Finally, $C^\infty(\mathbb{S}^1)$ is dense in $W^{s,p}$, $W^{s,p'}$. Both sides of (7) are continuous in (γ,h) with respect to the $W^{s,p}\times W^{s,p'}$ topology, hence the identity extends to all $\gamma\in W^{s,p}$, $h\in W^{s,p'}$. □

Remark 20 (Continuity and bounds)**.** *The bound* (8) *follows directly from Hölder's inequality:*

$$|B_{s,p}(\gamma,h)| \leqslant \Big(\iint\frac{|\gamma(t)-\gamma(\tau)|^p}{|e^{it}-e^{i\tau}|^{1+sp}}\Big)^{1/p}\Big(\iint\frac{|h(t)-h(\tau)|^{p'}}{|e^{it}-e^{i\tau}|^{1+sp}}\Big)^{1/p'} = [\gamma]_{W^{s,p}}\,[h]_{W^{s,p'}}.$$

Thus $\Theta_{s,p}$ is a continuous bilinear map $W^{s,p}\times W^{s,p'}\to\mathbb{R}$.

Remark 21 (Endpoint $p = 1$). *Our argument uses the L^p-boundedness of $\mathcal{H}$ and the equivalence $\| |D|^s u\|_{L^p} \asymp [u]_{W^{s,p}}$, which both fail at the endpoint $p = 1$. Hence we restrict to $1 < p < \infty$.*

References for this section. Boundedness of $\mathcal{H}$ and singular integral representations for $|D|^s$: Stein [16], Grafakos [8]. Fractional Sobolev spaces and Gagliardo forms: Di Nezza–Palatucci–Valdinoci [5]. For $W^{s,p}$ traces and density, see also Brezis–Mironescu [2, 3].

6. The canonical presymplectic form via the L^2 pairing

Let $\langle \cdot, \cdot \rangle$ be the Euclidean inner product on $\mathbb{R}^m$ (here $m = 2$), and write ∂_t for the distributional derivative on $\mathbb{S}^1$.

6.1. Definition of the 1-form Θ_0 on weak loop spaces.

Definition 22 (The L^2-primitive Θ_0).

- *For $\gamma, h \in H^1(\mathbb{S}^1; \mathbb{R}^m)$, set*

$$\Theta_0(\gamma)[h] \;:=\; \int_{\mathbb{S}^1} \langle \gamma'(t),\, h(t) \rangle \, dt.$$

- *For $\gamma, h \in H^{1/2}(\mathbb{S}^1; \mathbb{R}^m)$, define the continuous extension*

$$\boxed{\Theta_0(\gamma)[h] \;:=\; \langle \partial_t \gamma,\, h \rangle_{H^{-1/2},\,H^{1/2}}}\,.$$

- *More generally, for $0 < s \leqslant \frac{1}{2}$ and $1 < p < \infty$, define*

$$\boxed{\Theta_{s,p}(\gamma)[h] \;:=\; \langle \partial_t \gamma,\, h \rangle_{W^{-s,p},\,W^{s,p'}}\,, \qquad \gamma \in W^{s,p},\ h \in W^{s,p'},}$$

  where p' is the conjugate exponent.

Remark 23. *The definitions coincide on the overlaps: if $\gamma \in H^1$ then $\partial_t \gamma \in L^2 \subset H^{-1/2}$ and $\Theta_0(\gamma)[h] = (\gamma', h)_{L^2}$ for $h \in L^2$. Continuity follows from the bounded map $\partial_t : H^{1/2} \to H^{-1/2}$ (resp. $W^{s,p} \to W^{-s,p}$).*

6.2. The differential $d\Theta_0$: the canonical presymplectic form.

Theorem 24 (Canonical presymplectic form on $H^{1/2}$). *On the Hilbert loop space $H^{1/2}(\mathbb{S}^1; \mathbb{R}^m)$, the differential of the 1-form Θ_0 is the continuous bilinear, skew form*

$$\boxed{d\Theta_0(\gamma)[h,k] \;=\; \langle \partial_t h,\, k \rangle_{H^{-1/2},\,H^{1/2}} - \langle \partial_t k,\, h \rangle_{H^{-1/2},\,H^{1/2}}\,,}$$

which may be written equivalently as

$$d\Theta_0(\gamma)[h,k] \;=\; 2\,\langle \partial_t h,\, k \rangle_{H^{-1/2},\,H^{1/2}} \;=\; 2c \int_{\mathbb{S}^1} \langle |D|^{1/2} h,\, \mathcal{H}\, |D|^{1/2} k \rangle \, dt,$$

where $\mathcal{H}$ is the Hilbert transform and $|D|^{1/2}$ the fractional derivative. In particular, $d\Theta_0$ is independent of γ (translation-invariant) and defines the canonical presymplectic form *on the loop space.*

Proof. First assume $\gamma, h, k \in C^\infty(\mathbb{S}^1; \mathbb{R}^m)$. Since the underlying space is linear, we may regard h, k as constant vector fields. Then

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \Theta_0(\gamma + \varepsilon h)[k] = \int_{\mathbb{S}^1} \langle h'(t), k(t) \rangle \, dt, \qquad \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \Theta_0(\gamma + \varepsilon k)[h] = \int_{\mathbb{S}^1} \langle k'(t), h(t) \rangle \, dt.$$

Thus

$$d\Theta_0(\gamma)[h,k] = \int \langle h', k\rangle - \int \langle k', h\rangle = 2\int \langle h', k\rangle = -2\int \langle h, k'\rangle,$$

where we used integration by parts on $\mathbb{S}^1$. This shows skew-symmetry and independence of γ.

For general $h, k \in H^{1/2}$, approximate in $H^{1/2}$ by smooth sequences h_n, k_n; use the boundedness of $\partial_t : H^{1/2} \to H^{-1/2}$ and continuity of the duality to pass to the limit, which yields

$$d\Theta_0(\gamma)[h,k] = \langle \partial_t h,\ k\rangle_{H^{-1/2},H^{1/2}} - \langle \partial_t k,\ h\rangle_{H^{-1/2},H^{1/2}}.$$

Finally, the Fourier identity $\partial_t = \mathcal{H}\,|D|$ on mean-zero components gives the last expression with $|D|^{1/2}$ and $\mathcal{H}$. ◻

Proposition 25 (Banach version)**.** *Let $0 < s \leqslant \frac{1}{2}$ and $1 < p < \infty$. On $W^{s,p}(\mathbb{S}^1;\mathbb{R}^m)$, the differential of $\Theta_{s,p}$ is*

$$\boxed{d\Theta_{s,p}(\gamma)[h,k] \ = \ \langle \partial_t h,\ k\rangle_{W^{-s,p},\, W^{s,p'}} - \langle \partial_t k,\ h\rangle_{W^{-s,p},\, W^{s,p'}}\ ,}$$

a continuous skew bilinear form on $W^{s,p} \times W^{s,p'}$, independent of γ. When $p = 2$ this coincides with Theorem 24.

Proof. Same computation as in the smooth case, then pass to the limit using $\partial_t : W^{s,p} \to W^{-s,p}$ and the duality with $W^{s,p'}$. ◻

Remark 26 (Non-degeneracy modulo constants)**.** *On $H^{1/2}$ (and on $W^{s,2}$), $d\Theta_0$ is weakly non-degenerate modulo constant loops: if $d\Theta_0(\gamma)[h,\cdot] \equiv 0$ then $\partial_t h = 0$ in $H^{-1/2}$, hence h is constant. One usually quotients by constants to obtain a genuine symplectic form.*

Remark 27 (Kernel representation in $H^{1/2}$)**.** *Using $\partial_t = \mathcal{H}|D|$ and polarization of the Gagliardo form,*

$$d\Theta_0(\gamma)[h,k] = 2c\int_{\mathbb{S}^1} \langle |D|^{1/2}h,\ \mathcal{H}|D|^{1/2}k\rangle\, dt = C\iint_{\mathbb{S}^1\times\mathbb{S}^1} \frac{\big(h(t)-h(\tau)\big)\cdot\big(\mathcal{H}k(t)-\mathcal{H}k(\tau)\big)}{|e^{it}-e^{i\tau}|^2}\, dt\, d\tau,$$

which makes sense as a continuous bilinear form on $H^{1/2} \times H^{1/2}$.

7. The canonical presymplectic form via the L^2 pairing

Let $\langle\cdot,\cdot\rangle$ denote the Euclidean inner product on $\mathbb{R}^m$ (here $m = 2$). For $\gamma \in H^{1/2}(\mathbb{S}^1;\mathbb{R}^m)$ we define the 1-form

$$\Theta_0(\gamma)[h] := \big\langle J\partial_t\gamma,\ h\big\rangle_{H^{-1/2},\, H^{1/2}}, \qquad h \in H^{1/2}(\mathbb{S}^1;\mathbb{R}^m). \tag{4.1}$$

Theorem 28 (Canonical presymplectic form on $H^{1/2}$)**.** *The differential of Θ_0 is the continuous skew-symmetric bilinear form*

$$d\Theta_0(\gamma)[h,k] = \big\langle \partial_t h,\ k\big\rangle_{H^{-1/2},\, H^{1/2}} - \big\langle \partial_t k,\ h\big\rangle_{H^{-1/2},\, H^{1/2}}.$$

Equivalently,

$$d\Theta_0(\gamma)[h,k] = 2\big\langle \partial_t h,\ k\big\rangle_{H^{-1/2},\, H^{1/2}} = 2c_{\mathbb{S}^1}\int_{\mathbb{S}^1} \big\langle |D|^{1/2}h,\ \mathcal{H}|D|^{1/2}k\big\rangle\, dt, \tag{9}$$

where $c_{\mathbb{S}^1} \neq 0$ is the constant from Lemma 14. In particular, $d\Theta_0$ does not depend on γ and defines the canonical presymplectic form on the loop space.

Proof. For smooth loops the computation is elementary:

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\Theta_0(\gamma+\varepsilon h)[k]=\int_{\mathbb{S}^1}\langle h',k\rangle\,dt,\qquad \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}\Theta_0(\gamma+\varepsilon k)[h]=\int_{\mathbb{S}^1}\langle k',h\rangle\,dt.$$

Subtracting yields (28). Passing to the limit for $h,k\in H^{1/2}$ uses the continuity of $\partial_t : H^{1/2}\to H^{-1/2}$ and of the duality pairing. The Fourier identity $\partial_t=\mathcal{H}|D|$ on mean-zero functions gives (9) after invoking Lemma 14. ◻

Proposition 29 (Banach-space version)**.** *Let $0<s\leqslant\frac{1}{2}$ and $1<p<\infty$. Define*

$$\Theta_{s,p}(\gamma)[h]:=\big\langle J\partial_t\gamma,\ h\big\rangle_{W^{-s,p},\,W^{s,p'}},\qquad \gamma\in W^{s,p},\ h\in W^{s,p'}. \tag{10}$$

Then

$$d\Theta_{s,p}(\gamma)[h,k]=\big\langle\partial_t h,\ k\big\rangle_{W^{-s,p},\,W^{s,p'}}-\big\langle\partial_t k,\ h\big\rangle_{W^{-s,p},\,W^{s,p'}} \tag{11}$$

defines a continuous skew-symmetric bilinear form on $W^{s,p}\times W^{s,p'}$, independent of γ. For $p=2$ this coincides with Theorem 28.

Proof. Exactly the same argument as in the proof of Theorem 28, using the boundedness of $\partial_t : W^{s,p}\to W^{-s,p}$ (see Appendix A), yields (11). ◻

Remark 30 (Non-degeneracy modulo constants)**.** *On both $H^{1/2}$ and $W^{s,2}$ the form $d\Theta_0$ (resp. $d\Theta_{s,2}$) is weakly non-degenerate modulo constant loops: if $d\Theta_0(\gamma)[h,\cdot]\equiv 0$ then $\partial_t h=0$ in $H^{-1/2}$, hence h is constant. Quotienting by the subspace of constant loops yields a genuine symplectic form.*

8. Mokhov-type bicomplex and Poisson structures on weak Sobolev loop spaces

Throughout this section we fix $0<s\leqslant\frac{1}{2}$ and $1<p<\infty$. Set

$$E:=W^{s,p}(\mathbb{S}^1;\mathbb{R}^m).$$

8.1. Admissible functionals.

Definition 31 (Admissible functionals)**.** *$\mathcal{F}$ denotes the class of functionals*

$$F(\gamma)=\int_{\mathbb{S}^1}f\big(\gamma(t)\big)\,dt,\qquad f\in C_b^{1,1}(\mathbb{R}^m),$$

defined on E. Their variational derivative is $\delta F(\gamma)=\nabla f(\gamma)\in W^{s,p}(\mathbb{S}^1;\mathbb{R}^m)$.

8.2. Local (hydrodynamic-type) Poisson operators.

Theorem 32 (Local Mokhov operator)**.** *Let $J:\mathbb{R}^m\to\mathfrak{so}(m)$ be Lipschitz bounded ($J\in C_b^{0,1}$). Define for $\gamma\in E$ and $\xi\in W^{s,p'}(\mathbb{S}^1;\mathbb{R}^m)$*

$$\mathcal{P}_{\mathrm{loc}}(\gamma)\,\xi:=J(\gamma)\,\partial_t\xi. \tag{5.1}$$

Then

(i) $\mathcal{P}_{\mathrm{loc}}(\gamma):W^{s,p'}\to W^{-s,p'}$ is bounded and skew-adjoint.

(ii) For $F,G\in\mathcal{F}$ the bracket

$$\{F,G\}_{\mathrm{loc}}(\gamma):=\big\langle\mathcal{P}_{\mathrm{loc}}(\gamma)\,\delta G(\gamma),\ \delta F(\gamma)\big\rangle_{W^{-s,p'},\,W^{s,p}} \tag{5.2}$$

is well defined and skew-symmetric.

(iii) If J satisfies Mokhov's algebraic conditions (the vanishing of the Schouten bracket $[\mathcal{P}_{\mathrm{loc}},\mathcal{P}_{\mathrm{loc}}]=0$ in the smooth category), then $\{\cdot,\cdot\}_{\mathrm{loc}}$ fulfills the Jacobi identity and defines a Poisson bracket on $\mathcal{F}$.

Sketch. For smooth loops the statement is precisely Mokhov's construction [12, 13]. The map $\gamma \mapsto J(\gamma)$ is a bounded Nemytskiĭ operator on $W^{s,p}$ (Proposition 46), while $\partial_t : W^{s,p'} \to W^{-s,p'}$ is continuous. Hence $\mathcal{P}_{\mathrm{loc}}(\gamma)$ extends by density to all $\gamma \in E$. Skew-adjointness follows from the antisymmetry of J and of ∂_t in the duality $W^{-s,p'} \times W^{s,p'}$. The algebraic Jacobi identity is preserved under the limit because all operations are continuous; thus the bracket satisfies the Jacobi identity. ◻

8.3. **Weakly non-local Poisson operators.**

Theorem 33 (Weakly non-local Mokhov operator)**.** *Let $J \in C_b^{0,1}(\mathbb{R}^m, \mathfrak{so}(m))$ and $A_r, B_r \in C_b^{0,1}(\mathbb{R}^m, \mathbb{R}^{m\times m})$ for $r = 1, \ldots, R$. Define*

$$\mathcal{P}_{\mathrm{nl}}(\gamma)\,\xi := J(\gamma)\,\partial_t\xi + \sum_{r=1}^{R} A_r(\gamma)\,\partial_t^{-1}\big(B_r(\gamma)^\top\xi\big), \tag{12}$$

where $\partial_t^{-1} := \mathcal{H}|D|^{-1}$ is the inverse derivative on mean-zero functions. Then

(i) $\mathcal{P}_{\mathrm{nl}}(\gamma) : W^{s,p'} \to W^{-s,p'}$ is bounded.

(ii) The bracket

$$\{F,G\}_{\mathrm{nl}}(\gamma) := \big\langle \mathcal{P}_{\mathrm{nl}}(\gamma)\,\delta G(\gamma),\, \delta F(\gamma)\big\rangle_{W^{-s,p'},\,W^{s,p}} \tag{13}$$

is well defined and skew-symmetric.

(iii) Under Mokhov's algebraic constraints on the collection (J, A_r, B_r) (ensuring skew-adjointness and the vanishing of the Schouten bracket), the bracket $\{\cdot,\cdot\}_{\mathrm{nl}}$ satisfies the Jacobi identity.

Sketch. For $\xi \in W^{s,p'}$ we have $B_r(\gamma)^\top\xi \in W^{s,p'}$ by the Lipschitz multiplier property (Proposition 46). The operator ∂_t^{-1} maps $W^{s,p'}$ into $W^{1+s,p'} \hookrightarrow W^{s,p'}$ (Appendix A), and multiplication by the bounded matrix $A_r(\gamma)$ preserves $W^{s,p'}$. Hence each term in (12) belongs to $W^{-s,p'}$. Skew-symmetry follows from the antisymmetry of J and the fact that ∂_t^{-1} is skew-adjoint on L^2 and bounded on $W^{s,p'}$. The Jacobi identity is proved exactly as in Mokhov's smooth setting, using density of smooth loops and continuity of all operators involved. ◻

8.4. **Canonical constant case.**

Corollary 34. *If $J_0 \in \mathfrak{so}(m)$ is constant, then*

$$\{F,G\}(\gamma) := \big\langle J_0\,\partial_t\delta G(\gamma),\, \delta F(\gamma)\big\rangle_{W^{-s,p'},\,W^{s,p}}$$

defines a Poisson bracket on $\mathcal{F}$. The associated presymplectic form is

$$\Omega(h,k) = \int_{\mathbb{S}^1} \big\langle J_0 h(t), k(t)\big\rangle\,dt.$$

Remark 35 (Analytic ingredients)**.** *The key analytical facts used above are:*

- *$\partial_t : W^{s,p} \to W^{-s,p}$ is continuous.*
- *The Hilbert transform $\mathcal{H}$ and the fractional powers $|D|^\alpha$ are bounded on $W^{s,p}$ for $0 \leqslant \alpha \leqslant 1$ and $1 < p < \infty$.*
- *Lipschitz functions act as multipliers on $W^{s,p}$ for $s \leqslant 1$ (Proposition 46).*

Remark 36 (Invariance under reparametrisations)**.** *The brackets $\{\cdot,\cdot\}$ are invariant under the natural right action of $\mathrm{Diff}^+(\mathbb{S}^1)$ or $\mathrm{BiLip}^+(\mathbb{S}^1)$ on loops, since admissible functionals are integrals of densities of weight 1 and the operators ∂_t, ∂_t^{-1} transform covariantly.*

8.5. **Horizontal and vertical differentials.**

Definition 37 (Horizontal differential)**.** *For an admissible density $\mathcal{L}(\gamma)$ we set*

$$(d_H\mathcal{L})(\gamma) := \partial_t\big(\mathcal{L}(\gamma)\big),$$

where ∂_t acts on each factor according to the Leibniz rule. Integration by parts is understood in the duality $W^{-s,p} \times W^{s,p'}$.

Definition 38 (Vertical differential)**.** *Let $F \in \mathcal{F}$ be as above. Its vertical differential is the 1-form*

$$(d_VF)(\gamma)[h] := \big\langle \delta F(\gamma),\, h\big\rangle_{W^{s,p},\, W^{s,p'}}, \qquad h \in W^{s,p'}(\mathbb{S}^1;\mathbb{R}^m),$$

where $\delta F(\gamma) \in W^{s,p}(\mathbb{S}^1;\mathbb{R}^m)$ denotes the variational derivative of F (the Fréchet derivative with respect to the loop variable). In coordinates,

$$\delta F(\gamma)(t) = \nabla f\big(\gamma(t)\big) \quad \textit{if} \quad F(\gamma) = \int_{\mathbb{S}^1} f\big(\gamma(t)\big)\, dt.$$

For higher-degree admissible forms the vertical differential is defined by the usual alternating polarization: if $\Omega \in \Omega^{p,q}$ is a (p,q)-form, then

$$(d_V\Omega)(\gamma)[h_0,\ldots,h_q] = \sum_{i=0}^{q}(-1)^i\, \Omega(\gamma)[h_0,\ldots,\widehat{h_i},\ldots,h_q],$$

where the hat indicates omission of the i-th argument.

8.6. **The bicomplex identities.** The two differentials commute up to sign and satisfy the familiar nilpotency relations. The proof relies only on the boundedness of the operators listed in $\mathcal{A}$ and on the density of smooth loops in $W^{s,p}$.

Theorem 39 (Mokhov bicomplex on $W^{s,p}$)**.** *Let $0 < s \leqslant \frac{1}{2}$ and $1 < p < \infty$. For every admissible form $\Omega \in \Omega^{p,q}$ the operators d_H and d_V satisfy*

$$d_H^2 = 0, \qquad d_V^2 = 0, \qquad d_Vd_H + d_Hd_V = 0. \tag{14}$$

Consequently $(\Omega^{\bullet,\bullet}, d_H, d_V)$ is a double complex.

Sketch. For smooth loops the identities are proved in Mokhov's original work [12, 13]. Let Ω be an admissible form and choose a sequence $(\Omega_n)_{n\geqslant 1}$ of smooth admissible forms such that $\Omega_n \to \Omega$ in the topology of $W^{s,p}$. All building blocks of $\mathcal{A}$—the derivative ∂_t, its inverse ∂_t^{-1}, the fractional multipliers $|D|^\alpha$, the Hilbert transform $\mathcal{H}$, and multiplication by Lipschitz maps—are bounded linear operators on $W^{s,p}$ (see Appendix A). Hence d_H and d_V are continuous maps

$$d_H, d_V : \Omega^{p,q} \longrightarrow \Omega^{p+1,q} \text{ or } \Omega^{p,q+1}.$$

Since the identities hold for each smooth Ω_n, passing to the limit yields (14) for Ω. The mixed relation $d_Vd_H + d_Hd_V = 0$ is precisely the integration-by-parts formula $\langle \partial_t u, v\rangle = -\langle u, \partial_t v\rangle$ valid in the duality $W^{-s,p} \times W^{s,p'}$. ◻

8.7. **Horizontal, vertical and total cohomology.** Because $(\Omega^{\bullet,\bullet}, d_H, d_V)$ is a double complex we can define the usual cohomology groups.

Definition 40 (Cohomology groups)**.** *Let $\Omega^{p,q}$ denote the space of admissible (p,q)-forms.*

- *The **horizontal cohomology** is*

$$H_H^p := \frac{\ker\big(d_H : \Omega^{p,\bullet} \to \Omega^{p+1,\bullet}\big)}{\operatorname{im}\big(d_H : \Omega^{p-1,\bullet} \to \Omega^{p,\bullet}\big)}.$$

- *The **vertical cohomology** is*

$$H_V^q := \frac{\ker\big(d_V : \Omega^{\bullet,q} \to \Omega^{\bullet,q+1}\big)}{\operatorname{im}\big(d_V : \Omega^{\bullet,q-1} \to \Omega^{\bullet,q}\big)}.$$

- *The **total (variational) cohomology** is obtained from the total differential $d := d_H + d_V$:*

$$H_{\mathrm{tot}}^k := \frac{\ker\big(d : \Omega^k \to \Omega^{k+1}\big)}{\operatorname{im}\big(d : \Omega^{k-1} \to \Omega^k\big)}, \qquad \Omega^k := \bigoplus_{p+q=k} \Omega^{p,q}.$$

9. Application: hydrodynamic-type order-1 Poisson structures via bicohomology

We illustrate how the bicomplex $(\Omega^{\bullet,\bullet}, d_H, d_V)$ on $W^{s,p}(\mathbb{S}^1; \mathbb{R}^m)$ controls the classification of hydrodynamic-type, order-1 Poisson structures (Mokhov-type) through the total bicohomology H_{tot}^2.

9.1. **Hydrodynamic-type operators and their class.** Fix $0 < s \leqslant \frac{1}{2}$, $1 < p < \infty$. Consider operators of the form

$$\mathcal{P}(\gamma)\,\xi \;=\; J(\gamma)\,\partial_t \xi, \qquad J : \mathbb{R}^m \to \mathfrak{so}(m),\ J \in C_b^{0,1}, \tag{15}$$

acting on variational derivatives $\xi \in W^{s,p'}$. They induce brackets on admissible functionals $F, G \in \mathcal{F}$ by

$$\{F, G\}(\gamma) \;=\; \big\langle \mathcal{P}(\gamma)\,\delta G(\gamma),\ \delta F(\gamma) \big\rangle_{W^{-s,p'},\,W^{s,p}}.$$

Definition 41 (Bivector represented by $\mathcal{P}$)**.** *Define a (vertical) 2-form with values in horizontal densities*

$$\Pi_\gamma(h,k) \;:=\; \big\langle J(\gamma)\,\partial_t k,\ h \big\rangle_{W^{-s,p'},\,W^{s,p}}, \qquad h \in W^{s,p},\ k \in W^{s,p'}.$$

Up to admissible integrations by parts, Π determines a class $[\Pi] \in \Omega^{1,2}/d(\cdot)$ that will represent a class in H_{tot}^2.

9.2. **Cocycle condition and Jacobi identity.**

Theorem 42 (Cocycle $\Leftrightarrow$ Jacobi)**.** *Let $\mathcal{P}$ be as in* (15)*, and let Π be its associated bivector. The following are equivalent:*

(1) The induced bracket $\{\cdot,\cdot\}$ on $\mathcal{F}$ satisfies the Jacobi identity.
(2) The class $[\Pi] \in H_{\mathrm{tot}}^2$ is d-closed, i.e. $d\Pi = 0$ in the total complex $(\Omega^\bullet, d = d_H + d_V)$.

Moreover, if $\Pi = d\Upsilon$ for some admissible 1-vector Υ (i.e. $[\Pi] = 0$ in H_{tot}^2), then $\mathcal{P}$ differs from a trivial operator by a variational coboundary and induces the zero bracket on $\mathcal{F}$.

Sketch. As in Mokhov's smooth theory, the Jacobi identity is equivalent to the vanishing of the Schouten bracket $[\Pi, \Pi] = 0$. In the bicomplex formalism, $[\Pi, \Pi] = 0$ is encoded by $d\Pi = 0$ in the total complex: d_V captures variational antisymmetry and d_H the total derivatives in t. Continuity of d_H, d_V on the admissible algebra and density of smooth loops in $W^{s,p}$ transfer Mokhov's proof verbatim. Exactness $\Pi = d\Upsilon$ corresponds to a change of representative by a variational coboundary, hence to a trivial deformation of the bracket (cohomologically zero). □

9.3. Invariance and reduction.

Proposition 43 (Invariance under reparametrizations)**.** *Let G be* $\mathrm{Diff}^+(\mathbb{S}^1)$ *or* $\mathrm{BiLip}^+(\mathbb{S}^1)$ *acting by precomposition. If J in* (15) *is independent of t and depends only on γ, then* Π *and its class* $[\Pi] \in H^2_{\mathrm{tot}}$ *are G-invariant.*

Proof. Admissible functionals in $\mathcal{F}$ are integrals of densities of weight 1; ∂_t and the duality transform covariantly under $t \mapsto \phi(t)$. Hence the bivector and the class are preserved. □

Remark 44 (Nondegeneracy and constants)**.** *The presymplectic form associated with ∂_t is weakly nondegenerate modulo constant loops. One may pass to the reduced space (quotient by constants) to obtain a genuine symplectic form, leaving the classification unchanged.*

9.4. **Weakly non-local extensions.** The same scheme applies to weakly non-local operators

$$\mathcal{P}_{\mathrm{nl}}(\gamma)\,\xi = J(\gamma)\partial_t\xi + \sum_{r=1}^{R} A_r(\gamma)\,\partial_t^{-1}\big(B_r(\gamma)^\top\xi\big),$$

with $A_r, B_r \in C^{0,1}_b$. The associated bivector Π_{nl} defines a class $[\Pi_{\mathrm{nl}}] \in H^2_{\mathrm{tot}}$, and Theorems 42–45 remain valid under Mokhov's algebraic constraints on (J, A_r, B_r).

9.5. Classification modulo coboundaries.

Theorem 45 (Classification of order-1 Poisson structures)**.** *Let $\mathcal{P}$ and $\widetilde{\mathcal{P}}$ be two Poisson operators of the form* (15) *with admissible matrices J and $\widetilde{J}$. If the associated bivectors satisfy* $[\Pi] = [\widetilde{\Pi}]$ *in* H^2_{tot}*, then the brackets* $\{\cdot,\cdot\}$ *and* $\{\cdot,\cdot\}^\sim$ *coincide on the space of admissible functionals $\mathcal{F}$. Conversely, non-cohomologous classes give distinct Poisson brackets.*

Proof. If $[\Pi] = [\widetilde{\Pi}]$ then $\widetilde{\Pi} - \Pi = d\Upsilon$ for some admissible vertical 1-vector Υ. The change of Poisson operator amounts to adding the coboundary term $\langle d\Upsilon\,\delta G, \delta F\rangle$, which vanishes after integration by parts (the duality pairing is skew-symmetric). Hence the brackets agree. The converse follows by testing the brackets against cylindrical functionals (functionals depending only on finitely many Fourier modes), which detect the cohomology class. □

10. Applications to classical integrable systems

In this section we illustrate how the Hamiltonian formalisms of several well-known integrable PDEs fit naturally into the weak Sobolev framework developed above. Throughout we identify a periodic field $u : \mathbb{S}^1 \to \mathbb{R}^m$ with a loop $\gamma \in W^{s,p}(\mathbb{S}^1; \mathbb{R}^m)$. Local Hamiltonians of an integrable PDE have the form

$$H(\gamma) = \int_{\mathbb{S}^1} \mathcal{H}\big(\gamma, \partial_x\gamma, \ldots, \partial_x^N\gamma\big)\,dx,$$

with $\mathcal{H}$ differential-polynomial or smooth in its arguments. In our $W^{s,p}$ setting ($0 < s \leqslant \frac{1}{2}$, $1 < p < \infty$) we interpret ∂_x as the distributional derivative ∂_t and *restrict* to densities built from bounded operators (∂_t, ∂_t^{-1}, $|D|^\alpha$, $\mathcal{H}$), so that $\delta H(\gamma)$ lies in the correct dual Sobolev class (Def. 31).

10.1. **KdV.** The Korteweg–de Vries equation

$$u_t = 6uu_x - u_{xxx}, \qquad x \in \mathbb{S}^1,$$

possesses two compatible Poisson structures

$$\mathcal{P}_0 = \partial_x, \qquad \mathcal{P}_1 = \partial_x^3 + 2u\,\partial_x + u_x.$$

The first operator $\mathcal{P}_0$ is of the local hydrodynamic type $\mathcal{P}_0 = J_0\partial_x$ with $J_0 \equiv 1$. Hence, by Theorem 32, $\mathcal{P}_0$ defines a Poisson bracket on $W^{s,p}$ for any $0 < s \leqslant \frac{1}{2}$, $1 < p < \infty$. The second operator $\mathcal{P}_1$ is of order three; to realise it one needs at least H^1 regularity, which lies beyond the minimal Sobolev range considered here. Nevertheless, the *recursion operator*

$$\mathcal{R} := \mathcal{P}_1\mathcal{P}_0^{-1} = \partial_x^2 + 2u + u_x\partial_x^{-1}$$

is weakly non-local. Its non-local term involves ∂_x^{-1}, which coincides with $\partial_t^{-1} = \mathcal{H}|D|^{-1}$ on the circle. Therefore $\mathcal{R}$ fits the framework of Theorem 33, and the whole KdV hierarchy can be interpreted on $W^{s,p}$ via the weakly non-local Poisson operators.

10.2. **Nonlinear Schrödinger (NLS).** Writing $\psi = q_1 + iq_2$ and $\gamma = (q_1, q_2)$ the focusing NLS

$$i\psi_t + \psi_{xx} + 2|\psi|^2\psi = 0$$

has the Hamiltonian structure

$$\mathcal{P}_{\mathrm{NLS}} = J_0\partial_x, \qquad J_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Since J_0 is constant, Theorem 32 applies directly and yields a well-defined Poisson bracket on any $W^{s,p}$ with $0 < s \leqslant \frac{1}{2}$. The energy functional

$$H_{\mathrm{NLS}}(\gamma) = \frac{1}{2}\int_{\mathbb{S}^1} |\partial_x\gamma|^2 - \frac{1}{2}\int_{\mathbb{S}^1} |\gamma|^4$$

belongs to $\mathcal{F}$, and the Hamiltonian flow generated by $\mathcal{P}_{\mathrm{NLS}}$ reproduces the NLS equation in the weak Sobolev setting.

10.3. **Camassa–Holm.** The Camassa–Holm equation

$$m_t + um_x + 2u_xm = 0, \qquad m = u - u_{xx},$$

is bi-Hamiltonian with the pair

$$\mathcal{P}_0 = \partial_x - \partial_x^3, \qquad \mathcal{P}_1 = m\partial_x + \partial_x m.$$

Both operators contain the inverse Helmholtz operator $(1 - \partial_x^2)^{-1}$, which is a Fourier multiplier of order -2 and therefore bounded on $W^{s,p}$ for any $s \geqslant 0$. Moreover, ∂_x^{-1} appears explicitly in the weakly non-local formulation of $\mathcal{P}_1$. Consequently, Theorem 33 guarantees that the Camassa–Holm Poisson structures are well defined on $W^{s,p}$ with $0 < s \leqslant \frac{1}{2}$, $1 < p < \infty$.

10.4. **Dubrovin–Novikov hydrodynamic systems.** Consider a system of hydrodynamic type

$$u^i_t = v^i_j(u)\, u^j_x, \qquad i = 1, \dots, m,$$

with a Dubrovin–Novikov Poisson bracket

$$\{u^i(x), u^j(y)\} = g^{ij}(u)\, \delta'(x-y) + b^{ij}_k(u)\, u^k_x\, \delta(x-y).$$

In the loop-space language this corresponds to the operator

$$\mathcal{P}^{ij} = g^{ij}(\gamma)\, \partial_t + b^{ij}_k(\gamma)\, \partial_t\gamma^k.$$

If the metric g^{ij} and the connection coefficients b^{ij}_k are $C^{0,1}$ functions of γ, then $\mathcal{P}$ belongs to the class of local Mokhov operators (Theorem 32). The classical algebraic conditions for the Jacobi identity (flatness of g and compatibility of b) translate into the cocycle condition $d\Pi = 0$ in the total bicomplex, so the Poisson bracket is valid on $W^{s,p}$.

10.5. **Summary table – Poisson structures for integrable systems.** Classical integrable systems on the circle embed into the $W^{s,p}$ Mokhov framework as follows:

- *Local order-1* Poisson brackets (NLS/AKNS, hydrodynamic type) are exactly the constant or state-dependent $J(\gamma)\partial_t$ brackets.
- *Recursion operators* and many bi-Hamiltonian pairs introduce *weakly nonlocal* pieces (∂_t^{-1}).
- The *bicomplex* (d_H, d_V) interprets Jacobi/compatibility as d-closedness classes in H^2_{tot}, recovering Magri's scheme in the weak Sobolev setting.

In all cases above, the *bi-Hamiltonian* property ($\mathcal{P}_0, \mathcal{P}_1$ compatible) is equivalent to the d-closure of the corresponding total bivectors in the variational bicomplex. The generating sequence of Hamiltonians $\{H_n\}$ in Magri's scheme satisfies

$$\mathcal{P}_1\, \delta H_n = \mathcal{P}_0\, \delta H_{n+1},$$

which, in our $W^{s,p}$ setting, is well posed as an identity in $W^{-s,p'}$ and propagates regularity thanks to the boundedness of $\partial_t^{\pm 1}$, $\mathcal{H}$, and multipliers $C^{0,1}$.

System	Poisson operator(s)	Type	Regularity
KdV	$\mathcal{P}_0 = \partial_x$, $\mathcal{R} := \mathcal{P}_1\mathcal{P}_0^{-1}$ $= \partial_x^2 + 2u + u_x\partial_x^{-1}$	local (order 1) + weakly non-local (rank 1)	$0 < s \leqslant \frac{1}{2}$, $1 < p < \infty$
NLS / AKNS	$\mathcal{P} = J_0\,\partial_x$, $J_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	local (order 1)	$0 < s \leqslant \frac{1}{2}$, $1 < p < \infty$
Camassa–Holm	$\mathcal{P}_0 = \partial_x - \partial_x^3$, $\mathcal{P}_1 = m\,\partial_x + \partial_x m$, $m = (1-\partial_x^2)^{-1}u$	mixed: order 1 + non-local operator $(1-\partial_x^2)^{-1}$	$0 < s \leqslant \frac{1}{2}$, $1 < p < \infty$
Dubrovin–Novikov	$\mathcal{P}^{ij} = g^{ij}(\gamma)\,\partial_x$ $+b^{ij}_k(\gamma)\,\partial_x\gamma^k$	local (order 1)	$0 < s \leqslant \frac{1}{2}$, $1 < p < \infty$
General Mokhov	$\mathcal{P}(\gamma)\xi = J(\gamma)\,\partial_x\xi$ $+\sum_{r=1}^{R} A_r(\gamma)\,\partial_x^{-1}\big(B_r(\gamma)^\top\xi\big)$	local + weakly non-local (rank 1)	$0 < s \leqslant \frac{1}{2}$, $1 < p < \infty$

TABLE 1. **Embedding of classical integrable-system Poisson structures into the $W^{s,p}$ framework.** Order-1 operators (and their weakly non-local deformations) are well defined for any Sobolev regularity $0 < s \leqslant \frac{1}{2}$ and any exponent $1 < p < \infty$. Higher-order local operators (e.g. the second KdV bracket) require stronger regularity (typically H^1 or $W^{1,p}$). All matrix-valued coefficients J, A_r, B_r, g^{ij}, b^{ij}_k are assumed Lipschitz-bounded ($C^{0,1}_b$).

APPENDIX A. ANALYTIC BACKGROUND ON $W^{s,p}$

For completeness we collect the analytic facts repeatedly used in the text.

A.1. **Boundedness of the Hilbert transform.** For $0 \leqslant s \leqslant 1$ and $1 < p < \infty$ the Hilbert transform $\mathcal{H}$ is a bounded linear operator on $W^{s,p}(\mathbb{S}^1)$ (see [16, Th. III.3]).

A.2. **Fractional derivatives.** For $0 \leqslant s \leqslant 1$ and $1 < p < \infty$ the Fourier multiplier $|D|^s$ defines an isomorphism

$$|D|^s : W^{\sigma,p}(\mathbb{S}^1) \longrightarrow W^{\sigma-s,p}(\mathbb{S}^1), \qquad |D|^{-s} : (W^{\sigma-s,p}) \to W^{\sigma,p}.$$

A.3. **Inverse derivative.** We set

$$\partial_t^{-1} := \mathcal{H}|D|^{-1},$$

which is bounded $W^{s,p} \to W^{1+s,p} \hookrightarrow W^{s,p}$ for $0 < s \leqslant \frac{1}{2}$, $1 < p < \infty$.

A.4. **Nemytskiĭ operators.**

Proposition 46. *If $a : \mathbb{R}^m \to \mathbb{R}$ is Lipschitz and bounded, then the operator $u \mapsto a \circ u$ maps $W^{s,p}(\mathbb{S}^1; \mathbb{R}^m)$ continuously into itself for any $0 < s \leqslant 1$, $1 < p < \infty$.*

Proof. See Runst–Sickel [15, Ch. 4] or Triebel [17, Th. 2.5.12]. □

A.5. **Duality.** For $1 < p < \infty$ the dual space of $W^{s,p}(\mathbb{S}^1)$ is $W^{-s,p'}(\mathbb{S}^1)$ with the pairing $\langle \xi, h \rangle_{W^{-s,p}, W^{s,p'}}$. If $\xi \in W^{-s,p}$ and $h \in W^{s,p'}$ then $|\langle \xi, h \rangle| \leqslant \|\xi\|_{W^{-s,p}} \|h\|_{W^{s,p'}}$.

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