OPEN NEIGHBORHOOD IDEALS OF WELL TOTALLY DOMINATED TREES ARE COHEN-MACAULAY

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Abstract. We introduce and investigate the open neighborhood ideal $\mathcal{N}(G)$ of a finite simple graph G. We describe the minimal primary decomposition of $\mathcal{N}(G)$ in terms of the minimal total dominating sets (TD-sets) of G. Then we prove that the open neighborhood ideal of a tree is Cohen-Macaulay if and only if the tree is unmixed (well totally dominated) and calculate the Cohen-Macaulay type. We also give a descriptive characterization of all unmixed trees which takes polynomial time to verify.

Contents

1. Introduction

Commutative algebra and combinatorics have a rich history of intersections. In this paper, we focus on connections between commutative algebra and graph theory. This approach begins with work of Villarreal [18] who introduced and investigated the *edge ideal* $\mathcal{I}(G)$ of a finite, simple graph G, which is the ideal generated by the edges of G in an appropriate polynomial ring. He described the minimal primary decomposition of $\mathcal{I}(G)$ in terms of the minimal vertex covers of G and proved that the edge ideal of a tree is Cohen-Macaulay if and only if the tree is unmixed (well covered). He also gave explicit descriptive and constructive characterizations of all unmixed trees. Much subsequent research has expanded our understanding of the relations between the algebraic properties of $\mathcal{I}(G)$ and the graph-theoretic properties of G; see, e.g., [17, 19, 9, 11, 7].

Variations on this algebraic construction relate to different properties of the graph. For instance, Conca and De Negri [6] introduced the r-path ideal $\mathcal{I}_r(G)$ which is generated by the paths of length r in G and is connected to r-path covers; the edge ideal is the special case $r = 1$, i.e., $\mathcal{I}_1(G) = \mathcal{I}(G)$. Campos, et al. [5] characterized the Cohen-Macaulay r-path ideals of trees. Sharifan and Moradi [15] introduced the closed neighborhood ideal of G which is generated by the closed neighborhoods in G and is connected to domination sets. Honeycutt and Sather-Wagstaff [12] characterized the trees whose closed neighborhood ideals are Cohen-Macaulay as those that are unmixed (well dominated tree).

This paper investigates a similar construction based on open neighborhoods: the *open neighborhood ideal* $\mathcal{N}(G)$ of G is the ideal generated by the open neighborhoods of G. We define this formally in Definition 2.2.4 below. Section 2 contains some foundational information. It includes Theorem 2.2.6 which describes the minimal primary decomposition of $\mathcal{N}(G)$ in terms of the minimal total dominating sets (TD-sets) of G.

Section 3 consists of graph-theoretic tools, our descriptive characterization of unmixed trees in Theorem 3.4.9, and the constructive characterization of unmixed balanced trees in Theorem 3.5.6. Then in Section 4, we show that the open neighborhood ideals of unmixed trees are Cohen-Macaulay; see Corollary 4.2.16 which follows from the fact that the corresponding Stanley-Reisner simplicial complex is shellable, proved in Theorem 4.2.15. The current paper ends with Section 5 where we use our results to compute the Cohen-Macaulay type of open neighborhood ideal of unmixed trees in Theorem 5.2.7.

Assumptions. Throughout, we assume that G is a finite, simple graph, $\Bbbk$ is a field, and A is a nonzero commutative ring with unity. The vertex set and the edge set of G are $V(G)$ and $E(G)$, respectively. If the graph we consider is clear in context, then we write $V := V(G)$ and $E := E(G)$. Since G is simple, we denote each edge $e \in E$ as $uv = vu$ where $u, v \in V$ are the distinct vertices incident to e. Let $A[V(G)] = A[V]$ be the polynomial ring over A whose variables are the vertices of G. We set $R := \Bbbk[V]$ or $R := A[V]$ depending on the context.

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2. Total Dominating Sets and Open Neighborhood Ideals

This section contains some graph-theoretical, foundational information and Theorem 2.2.6 which describes the minimal primary decomposition of $\mathcal{N}(G)$ in terms of the minimal total dominating sets (TD-sets) of G.

2.1. Foundations.

Definition 2.1.1. For $v \in V(G)$, the *open neighborhood of* v *in* G is the set of vertices

$$N_G(v) := \{u \in V \mid uv \in E\}.$$

For any subset $S \subseteq V(G)$, we define

$$N_G(S) := \bigcup_{v \in S} N_G(v).$$

If the graph G is clear in the context, we set $N(v) := N_G(v)$ and $N(S) := N_G(S)$. For $U \subseteq V$, define the *monomial of* U in $A[V]$ by $X_U := \prod_{v \in U} v$.

Definition 2.1.2. Fix $S \subseteq V$. A set $D \subseteq V$ is an S*-totally dominating set* of G (S*-TD-set* of G) if $N(D) \supseteq S$. An S-TD-set D of G is *minimal* if there is no proper subset D' of D such that $N_G(D') \supseteq S$, i.e., if D does not properly contain another S-TD-set of G. We say that G is S*-unmixed* if every minimal S-TD-set of G has the same size. The S-open neighborhood ideal of G is the following ideal in $R = A[V]$

$$\mathcal{N}_S(G) := \left\langle X_{N(v)} \mid v \in S \right\rangle.$$

Example 2.1.3. Consider the path P_5 with 5 edges and set $R = A[V]$ and $S = \{v_2, v_4, v_6\}$. Then the S-open neighborhood ideal of P_5 is

$$\mathcal{N}_S(P_5) = \left\langle X_{N(v_2)}, X_{N(v_4)}, X_{N(v_6)} \right\rangle = \langle v_1v_3, v_3v_5, v_5 \rangle = \langle v_1v_3, v_5 \rangle$$

which decomposes irredundantly as

$$\mathcal{N}_S(P_5) = \langle v_1v_3, v_5 \rangle = \langle v_1, v_5 \rangle \cap \langle v_3, v_5 \rangle .$$

It is straightforward to see that the sets $\{v_1, v_5\}$ and $\{v_3, v_5\}$ corresponding to the ideals in this decomposition are exactly the minimal S-TD-sets of P_5. In particular, P_5 is S-unmixed.

The following fact is used in the proof of Theorem 2.2.2 which gives a general decomposition of $\mathcal{N}_S(G)$.

Fact 2.1.4. Let $S \subseteq V$. The following statements are straightforward to verify.

(1) For any $V' \subseteq V'' \subseteq V$, if V' is an S-TD-set of G, then V'' is an S-TD-set of G.
(2) Every S-TD-set of G contains a minimal S-TD-set of G.

2.2. **Decompositions.**

Lemma 2.2.1. *Let $S, V' \subseteq V$ and $R = A[V]$. Then V' is an S-TD-set of G if and only if $\mathcal{N}_S(G) \subseteq \langle V' \rangle$.*

Proof. For the forward implication, assume that V' is an S-TD-set. Consider a generator $X_{N(s)} \in \mathcal{N}_S(G)$ with $s \in S$. Since V' is an S-TD-set, there exists some vertex $v \in V'$ such that $v \in N(s)$. Hence we get $v | X_{N(s)}$, so $X_{N(s)} \in (v)R \subseteq \langle V' \rangle$. Thus $\mathcal{N}_S(G) \subseteq \langle V' \rangle$.

For the converse implication, assume that $\mathcal{N}_S(G) \subseteq \langle V' \rangle$. Let $u \in S$. Then $X_{N(u)} \in \mathcal{N}_S(G) \subseteq \langle V' \rangle$. Hence there exists some $v \in V'$ such that $v | X_{N(u)}$. By definition of $X_{N(u)}$, we must have $v \in N(u)$. Thus $u \in N(v) \subseteq N(V')$. Since u is an arbitrary vertex in S, we get $S \subseteq N(V')$, so V' is an S-TD-set. □

The following theorem describes the decomposition of S-open neighborhood ideals using S-totally dominating sets.

Theorem 2.2.2. *Let $S \subseteq V$, and $R = A[V]$. The S-open neighborhood ideal has the following decomposition*

$$\mathcal{N}_S(G) = \bigcap_D \langle D \rangle = \bigcap_{D \ min} \langle D \rangle$$

where the first intersection is taken over all S-TD-sets of G, and the second intersection is taken over all minimal S-TD-sets of G. Moreover, the second decomposition is irredundant.

Proof. For any $A, B \subseteq V$, we have $\langle A \rangle \subseteq \langle B \rangle$ if and only if $A \subseteq B$. Hence the second intersection is irredundant. Now let $V' \subseteq V$ be an S-TD-set which is not minimal. Then V' contains a minimal S-TD-set by Fact 2.1.4. So, we have $\bigcap_D \langle D \rangle = \bigcap_{D \neq V'} \langle D \rangle$. Since V is finite, by repeating the same argument finitely many times, we can conclude that $\bigcap_D \langle D \rangle = \bigcap_{D \ min} \langle D \rangle$.

By Lemma 2.2.1, we get $\mathcal{N}_S(G) \subseteq \bigcap_D \langle D \rangle$. For the containment $\mathcal{N}_S(G) \supseteq \bigcap_D \langle D \rangle$, since $\mathcal{N}_S(G)$ is a square-free monomial, there are subsets $D_1, \ldots, D_k \subseteq V$ such that $\mathcal{N}_S(G) = \bigcap_{i=1}^k \langle D_i \rangle$. For any index $j \in [k]$, we have $\mathcal{N}_S(G) \subseteq \langle D_j \rangle$, which implies that D_j is an S-TD-set by Lemma 2.2.1. Thus we get $\mathcal{N}_S(G) = \bigcap_{i=1}^k \langle D_i \rangle \supseteq \bigcap_D \langle D \rangle$. □

Now we consider a specific case when $S = V$, the entire vertex set.

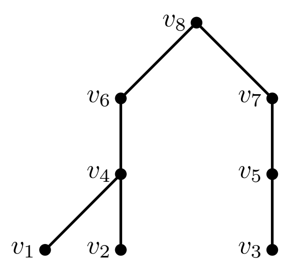
Definition 2.2.3. A subset $D \subseteq V(G)$ is a *total dominating set (TD-set) of* G if $N(D) = V(G)$. We say D is a *minimal TD-set* if no proper subset of D is a TD-set. We say G is *unmixed* if every minimal TD-set of G has the same size.

The minimal TD-set problem on graphs is a well studied subject in graph theory [10], and finding the size of the smallest minimal TD-set of a graph turns out to be an NP-complete problem [13]. Also, finding and characterising unmixed graphs (also called *well totally dominated graphs*) is an interesting problem [8, 1]. In this paper, we focus on the unmixedness property of trees to establish our goal.

Definition 2.2.4. The *open neighborhood ideal of* G in $R = A[V]$ is the ideal generated by the monomials of the open neighborhoods of the vertices of G

$$\mathcal{N}(G) := \mathcal{N}_V(G) = \left\langle X_{N_G(v)} \mid v \in V \right\rangle .$$

Example 2.2.5. Let T be the following tree with vertex set $V = \{v_1, v_2, \dots, v_8\}$.



Then $R = A[V] = A[v_1, \dots, v_8]$, and we have

$$\mathcal{N}(T) = \langle v_4, v_5, v_1v_2v_6, v_3v_7, v_4v_8, v_5v_8, v_6v_5\rangle = \langle v_4, v_5, v_1v_2v_6, v_3v_7, v_6v_5\rangle .$$

We get the following decomposition theorem of open-neighborhood ideals as a corollary of Theorem 2.2.2. When A is a field, this gives the (minimal) primary decomposition of $\mathcal{N}(G)$.

Theorem 2.2.6. *The open neighborhood ideal of G in $R = A[V]$ has the following decomposition*

$$\mathcal{N}(G) = \bigcap_{V'} \langle V' \rangle = \bigcap_{V'\ min} \langle V' \rangle$$

where the first intersection is taken over all TD-sets of G, and the second intersection is taken over all minimal TD-sets of G. Moreover, the second decomposition is irredundant.

Example 2.2.7. Consider the tree T and its open neighborhood ideal from Example 2.2.5. One can show that the following sets are the minimal TD-sets of T:

$$\{v_4, v_5, v_6, v_7\} \qquad \{v_1, v_4, v_5, v_7\} \qquad \{v_2, v_4, v_5, v_7\} \qquad \{v_4, v_5, v_6, v_3\} .$$

Thus T is unmixed, and by Theorem 2.2.6, we get

$$\begin{aligned} \mathcal{N}(T) &= \langle v_4, v_5, v_1v_2v_6, v_3v_7, v_6v_5\rangle \\ &= \langle v_4, v_5, v_6, v_7\rangle \cap \langle v_1, v_4, v_5, v_7\rangle \cap \langle v_2, v_4, v_5, v_7\rangle \cap \langle v_4, v_5, v_6, v_3\rangle . \end{aligned}$$

On the other hand it is straightforward to show that the path P_4

$$\ell_1 \text{ ——— } s_1 \text{ ——— } u \text{ ——— } s_2 \text{ ——— } \ell_2$$

is mixed because

$$\mathcal{N}(P_4) = \langle s_1, \ell_1 u, s_1 s_2, u\ell_2, s_2 \rangle = \langle s_1, s_2, \ell_1, \ell_2 \rangle \cap \langle s_1, s_2, u \rangle .$$

3. Characterizations of unmixed trees

Our goal in this section is to give two characterizations of unmixed trees. The first one is descriptive, allowing one to detect whether a given tree T is unmixed; see Theorem 3.4.9, which is the main result of Subsection 3.4. Essentially, T has two noteworthy "interior" subgraphs, $T_{\mathcal{R}}$ and $T_{\mathcal{B}}$, which are induced subforests that arise from a red-blue coloring of T, and such that T is unmixed if and only if $T_{\mathcal{R}}$ and $T_{\mathcal{B}}$ are unmixed; see Corollary 3.4.8.

The connected components of $T_{\mathcal{R}}$ and $T_{\mathcal{B}}$ have a property that we call "balanced." The general properties of balanced trees are the focus of Subsection 3.2, and their unmixedness is the topic of Subsection 3.3, the main result of which is our descriptive characterization of them in Theorem 3.3.12.

The second characterization of unmixed trees is constructive, allowing us to build unmixed trees. It is the main result of Subsection 3.5; see Theorem 3.5.6.

3.1. **Domination selector.** The goal of this subsection is to define the notion of a *domination selector* of G, which is a useful tool for demonstrating the minimality of a given set of vertices. The main result of this subsection is Lemma 3.1.11, which we use subsequently in several key places. We begin with some fundamental graph-theoretic terminology.

Notation 3.1.1. Recall that a forest is a finite disjoint union of trees. Let $x, y \in V$. A *path* from x to y is a sequence of distinct vertices $(v_0, \ldots, v_d)$ where $d \in \mathbb{N}$, $v_0 = x$, $v_d = y$, and $v_i v_{i+1} \in E$ for all $i = 0, \ldots, d-1$. The *length* of a path $(v_0, \ldots, v_d)$ is d also written as $d =: |(v_0, \ldots, v_d)|$, and the *distance* from x to y is the length of the shortest path from x to y, denoted $\operatorname{dist}(x, y)$. In case the given graph is a tree (or a forest), denote the (shortest) path from x to y by $\operatorname{path}(x, y)$. If a vertex $z \in V$ equals to one of the v_j in $\operatorname{path}(x, y)$, we write $z \in \operatorname{path}(x, y)$.

Definition 3.1.2. A vertex $v \in V$ is *isolated* if $\deg(v) = 0$, it is a *leaf* if $\deg(v) = 1$, and it is a *support vertex* if it is adjacent to a leaf. Every vertex adjacent to a support vertex is called a *supported vertex*.

Definition 3.1.3. Let G be a graph with at least one leaf. The *height* of a vertex $v \in V$ is given by

$$\operatorname{height}(v) := \min \{\operatorname{dist}(v, \ell) \mid \ell \text{ is a leaf in } G\} .$$

If v is an isolated vertex, then we set $\operatorname{height}(v) = 0$ as a convention. We denote

$$V_k = V_k(G) := \{v \in V \mid \operatorname{height}(v) = k\}$$

and the *height* of G is the integer

$$\operatorname{height}(G) := \max \{k \in \mathbb{N} \mid V_k \neq \emptyset\} .$$

For instance, every leaf of a graph has height 0, and every non-leaf support vertex has height 1.

Lemma 3.1.4. *Let G be a graph with a support vertex s. Then for any minimal TD-set D, we have $|D \cap N(s) \cap V_0(G)| \leq 1$.*

Proof. Assume not. Then there are two vertices $x, y \in D \cap N(s) \cap V_0(G)$. Then $N(D \setminus \{x\}) = V(G)$ as $s \in N(y)$, which contradicts the minimality of D. $\square$

Lemma 3.1.5 is the first application of Lemma 3.1.4. It states that given a graph G, adding a leaf to a support vertex or deleting a leaf adjacent to a support vertex with more than one leaf adjacent does not change the mixedness of G.

Lemma 3.1.5. *Let G be a connected graph with at least one leaf. Let G' be a graph obtained from G by attaching any number of leaves to its suppport vertices. Then G is unmixed if and only if G' is unmixed.*

Proof. Let $s \in V(G)$ be a support vertex adjacent to a leaf $\ell \in V(G)$. Let H be a graph with $V(H) = V(G) \cup \{\ell'\}$ and $E(H) = E(G) \cup \{s\ell'\}$. It suffices to prove the claim for G and H.

Let $S \subseteq V(G)$ be a minimal TD-set of G. Since $N_G(S) = V(G) \ni \ell$, we have $s \in S$ as s is the only vertex adjacent to ℓ. Now consider $S \subseteq H$. Since $s \in S$, we have $N_H(S) = V(H)$. To show that S is a minimal TD-set of H, assume by way of contradiction that there is a proper subset $S' \subsetneq S$ with $N_H(S') = V(H)$. Then the set $S' \subseteq V(G)$ has $N_G(S') = V(G)$, contradicting the minimality of S in G. Hence S is a minimal TD-set of H.

Next, let $S \subseteq V(H)$ be a minimal TD-set of H. We show that there is a minimal TD-set of G of the same size as S. If $\ell' \notin S$, then $S \subseteq V(G)$ is also a minimal TD-set of G by the preceding paragraph. So suppose that $\ell' \in S$. Then by Lemma 3.1.4, ℓ' is the only leaf in S adjacent to s. By the minimality of S, we have $N_H(S \setminus \{\ell'\}) = V(H) \setminus \{s\}$. Hence the set $S' := (S \setminus \{\ell'\}) \cup \{\ell\}$ is a minimal TD-set of H. Since $S' \subseteq V(G)$, the set $S' \subseteq V(G)$ has $N_G(S') = V(G)$. To show that S' is minimal in G, asumme by way of contradiction that there is a proper subset $S'' \subsetneq S'$ with $N_G(S'') = V(G)$. Then by the previous argument, S'' is a minimal TD-set of H contained in S', contradicting the minimality of S' in H. Thus S' is a minimal TD-set of G with $|S'| = |S|$.

So far, we have shown that any minimal TD-set of G is a minimal TD-set of H, and for any minimal TD-set of H, there is a minimal TD-set of G of the same size. This implies that G is unmixed if and only if H is unmixed. □

We next extend Definitions 2.1.2 and 2.2.3 to any subset of vertices.

Definition 3.1.6. A subset $D \subseteq V(G)$ is *minimal* (with respect to open neighborhoods) if there is no proper subset $D' \subsetneq D$, such that $N(D') = N(D)$.

From now on, a set being minimal will always refer to Definition 3.1.6. The following lemma gives an equivalent definition of a minimal set.

Lemma 3.1.7. *Let $D \subseteq V$. Then D is minimal if and only if for every vertex $v \in D$, there is a vertex $u \in N(D)$ such that $N(u) \cap D = \{v\}$.*

Proof. First, suppose that D is minimal. By way of contradiction, assume that there exists a vertex $c \in D$ such that for all $x \in N(c)$ we have $(N(x) \cap D) \setminus \{c\} \neq \emptyset$. We claim that $N(D \setminus \{c\}) = N(D)$, which contradicts the minimality of D. The containment $N(D \setminus \{c\}) \subseteq N(D)$ follows from Definition 2.1.1. For the other containment, let $v \in N(D)$. Then there exists $u \in D$ such that $vu \in E$. If $u \neq c$, then $u \in D \setminus \{c\}$, hence $v \in N(D \setminus \{c\})$. So suppose that $u = c$. Then we have $(N(v) \cap D) \setminus \{c\} \neq \emptyset$. So, there exists $c' \in D$ with $vc' \in E$ and $c' \neq c$. Thus $v \in N(D \setminus \{c\})$.

Now suppose that for every vertex $v \in D$, there is a vertex $u \in N(D)$ such that $N(u) \cap D = \{v\}$. Assume for a contradiction that D is not minimal. Hence there is $c \in D$ such that $N(D \setminus \{c\}) = N(D)$. By hypothesis, there is some vertex $y \in N(D)$ such that $N(y) \cap D = \{c\}$. Hence $N(D \setminus \{c\}) \not\ni y$, a contradiction. □

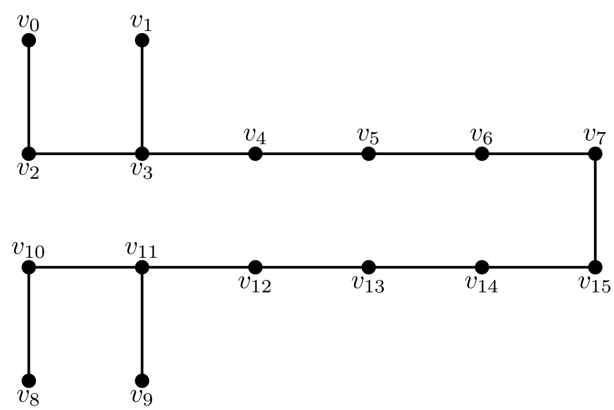
Using Lemma 3.1.7, we can now define domination selectors.

Definition 3.1.8. Let $D \subseteq V(G)$ be a minimal set. Let $\mathcal{D} : D \to N(D)$ be a function such that for all $v \in D$, we have $N(\mathcal{D}(v)) \cap D = \{v\}$. We call such a function a *domination selector* of D in G.

Domination selectors are well defined, injective functions by Lemma 3.1.7. Note that domination selectors of a minimal set might not be unique. The following is a corollary of Lemma 3.1.7 and Definition 3.1.8.

Corollary 3.1.9. *Let $D \subseteq V$ such that $N(D) = V$. Then D is a minimal TD-set if and only if D has a domination selector.*

Example 3.1.10. Consider the following graph Y.



One can check that $D := \{v_2, v_3, v_5, v_6, v_{10}, v_{11}, v_{14}, v_{15}\}$ is a TD-set in Y, and that the function $\mathcal{D} : D \to V(Y)$ given by

$$\mathcal{D}(x) = \begin{cases} v_0, & x = v_2 \\ v_1, & x = v_3 \\ v_6, & x = v_5 \\ v_5, & x = v_6 \\ v_8, & x = v_{10} \\ v_9, & x = v_{11} \\ v_{15}, & x = v_{14} \\ v_{14}, & x = v_{15}\ . \end{cases}$$

is a domination selector of D in Y. Hence by Corollary 3.1.9, D is minimal.

Another way of constructing a minimal set is described in the following two results.

Lemma 3.1.11. *Let $D', D'' \subseteq V$ be disjoint minimal sets, and let $I = N(D') \cap N(D'')$. If there are domination selectors $\mathcal{D}' : D' \to N(D)$ and $\mathcal{D}'' : D'' \to N(D'')$ such that $\mathcal{D}'(D') \cap I = \emptyset$ and $\mathcal{D}''(D'') \cap I = \emptyset$, then $D = D' \cup D''$ is also minimal.*

Proof. Define $\mathcal{D} : D \to N(D)$ by

$$\mathcal{D}(x) := \begin{cases} \mathcal{D}'(x) & x \in D' \\ \mathcal{D}''(x) & x \in D''\ . \end{cases}$$

We show that $\mathcal{D}$ is a domination selector of D. Let $v \in D$. By symmetry of D' and D'', suppose that $v \in D'$. Then we have

$$N(\mathcal{D}(v)) \cap D = N(\mathcal{D}'(v)) \cap D.$$

Since $\mathcal{D}'(v) \notin I$, we have $N(\mathcal{D}'(v)) \cap D'' = \emptyset$. Hence we get

$$N(\mathcal{D}'(v)) \cap D = (N(\mathcal{D}'(v)) \cap D') \cup (N(\mathcal{D}'(v)) \cap D'') = \{v\} \cup \emptyset = \{v\}\,.$$

Therefore, $\mathcal{D}$ is a domination selector of D. □

Corollary 3.1.12. *Let $D', D'' \subseteq V$ be disjoint minimal sets such that $N(D') \cap N(D'') = \emptyset$. Then $D = D' \cup D''$ is also a minimal set.*

3.2. **Balanced trees, RD-, and BD-sets.** In this subsection, we define balanced trees and two types of S-TD-sets which we then use to give our first characterization of unmixed trees in the main result of this subsection, Corollary 3.2.11.

Assumptions. In this subsection, we assume that $T = (V, E)$ is a tree with a 2-coloring $\chi : V \to \{\mathcal{R}, \mathcal{B}\}$ where $\mathcal{R}$ and $\mathcal{B}$ stands for red and blue color, respectively.

Definition 3.2.1. The tree T is *balanced* if no two vertices of the same height are adjacent.

Proposition 3.2.4 gives an equivalent definition of balanced trees. We next give a lemma used in its proof.

Lemma 3.2.2. *Let T be a balanced tree, and let $u, v \in V$ be distinct vertices that are adjacent to each other. Then height(u) − height$(v) = \pm 1$.*

Proof. Set $\text{height}(u) = h$ and $\text{height}(v) = h'$. Since T is a balanced tree, we must have $h \neq h'$ as $uv \in E$. Without loss of generality, assume that $h > h'$. We claim that $h' = h - 1$. By way of contradiction, suppose that $h' < h - 1$. Then there exists a leaf $v_0 \in V_0$ so that $\text{path}(v_0, v) = (v_0, \ldots, v_{h'})$ with $v_{h'} = v$. Now consider $\text{path}(v_0, u) = (v_0, \ldots, v_{h'}, u)$; the equality holds by the uniqueness of paths in trees. Since we have

$$|\text{path}(v_0, u)| = h' + 1 < h$$

$\text{height}(u)$ is less than h, a contradiction. □

Corollary 3.2.3. *Let T be a balanced tree and let $u, v \in V_h$ for some $0 \leq h \leq$ height(T). Then dist(u, v) is even.*

Proof. If $u = v$, then $\text{dist}(u, v) = 0$ which is even. So assume that $u \neq v$. By way of contradiction, assume that $\text{dist}(u, v) = 2k + 1$ for some $k \in \mathbb{N}$. Then by inductively applying Lemma 3.2.2 on k, one can show that the parity of $\text{height}(u)$ and $\text{height}(v)$ are different, hence $\text{height}(u) \neq \text{height}(v)$, a contradiction. □

Proposition 3.2.4. *The following conditions on a tree T are equivalent:*

(1) T is balanced.
(2) Any two vertices of the same height have the same color under χ.
(3) Every leaf has the same color.

Proof. (1 ⇒ 2) Assume T is balanced, and let $u, v \in V_h$ for some $0 \leq h \leq \text{height}(T)$. Then by Corollary 3.2.3, $\text{dist}(u, v)$ is even, hence $\chi(u) = \chi(v)$.

(2 ⇒ 3) Since all leaves have height 0, they have the same color.

($3 \Rightarrow 1$) Assume that every leaf in T has the same color. Let $u, v \in V_h$ for some $0 \leq h \leq \operatorname{height}(T)$. If $h = 0$, then u and v are leaves, so they have the same color by assumption, implying they are not adjacent. If $h > 0$, then there are leaves $u_0, v_0 \in V_0$ so that $\operatorname{path}(u_0, u) = (u_0, \ldots, u_h)$ and $\operatorname{path}(v_0, v) = (v_0, \ldots, v_h)$ where $u_h = u$ and $v_h = v$. Since u_0 and v_0 have the same color, u and v must have the same color, implying that $uv \notin E$. $\square$

Example 3.2.5. Consider the following trees with 2-coloring.

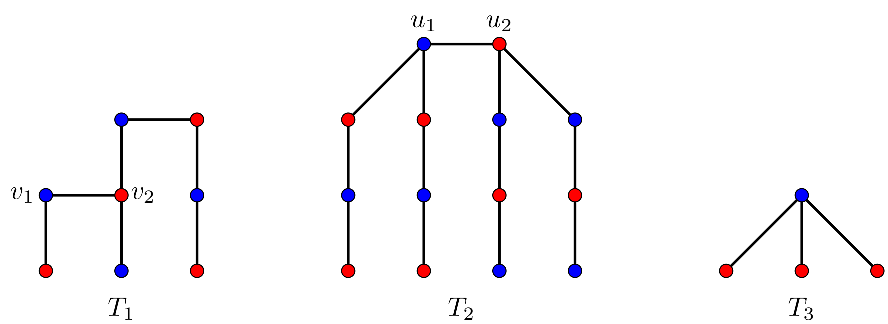


FIGURE 1. Trees with 2-colorings

The tree T_1 is not balanced since $\operatorname{height}(v_1) = 1 = \operatorname{height}(v_2)$ but $v_1v_2 \in E(T_1)$. Similarly, u_1 and u_2 show that T_2 is not balanced. We can also deduce this from Proposition 3.2.4 since the leaves in each tree do not have the same color. On the other hand, T_3 is a balanced tree by definition, or since all of its leaves share the same color.

The following corollary is a quick consequence of Proposition 3.2.4.

Corollary 3.2.6. *Let T be a balanced tree. Then all vertices of even height have the same color, and all vertices of odd height have the same color.*

Now we give a way to partition a TD-set of a tree into 2 types of S-TD-sets. This is used to construct two minimal TD-sets with different sizes.

Notation 3.2.7. Denote the set of red and blue vertices of T by $V_{\mathcal{R}}$ and $V_{\mathcal{B}}$, respectively; i.e., for all $C \in \{\mathcal{R}, \mathcal{B}\}$, define $V_C := \chi^{-1}(C)$.

Definition 3.2.8. Let $D \subseteq V$. Then D is a *red-dominating set* (RD-set) if $N(D) = V_{\mathcal{R}}$. And D is a *blue-dominating set* (BD-set) if $N(D) = V_{\mathcal{B}}$.

Note that by the definition of 2-coloring, if D is an RD-set, then $D \subseteq V_{\mathcal{B}}$, and $D \subseteq V_{\mathcal{R}}$ if D is a BD-set. Hence we can state the following as a fact.

Fact 3.2.9. Let $D' \subseteq V_{\mathcal{B}}$, $D'' \subseteq V_{\mathcal{R}}$, and $D = D' \cup D''$. Then

(1) D is a TD-set if and only if D' and D'' are red-dominating and blue-dominating, respectively.
(2) D is a minimal TD-set if and only if D' and D'' are minimal red-dominating and minimal blue-dominating, respectively.

Example 3.2.10. Consider the tree Y with vertices colored red and blue in Figure 2. The set $D := \{v_2, v_3, v_5, v_6, v_{10}, v_{11}, v_{14}, v_{15}\}$ is a minimal TD-set of Y by Example 3.1.10. Now,

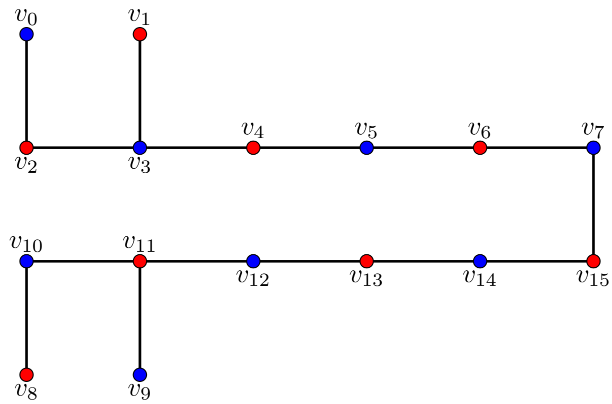


FIGURE 2. Graph Y with a 2-coloring

we partition the set D by the coloring: set $D' := \{v_3, v_5, v_{10}, v_{14}\}$ and $D'' := \{v_2, v_6, v_{11}, v_{15}\}$. Then it is straightforward to show that D' and D'' are red-dominating and blue dominating, respectively. Now, define $\mathcal{D}' : D' \to V(Y)$ and $\mathcal{D}'' : D'' \to V(Y)$ by

$$\mathcal{D}'(x) = \begin{cases} v_1, & x = v_3 \\ v_6, & x = v_5 \\ v_8, & x = v_{10} \\ v_{15}, & x = v_{14} \end{cases} \qquad \text{and} \qquad \mathcal{D}''(x) = \begin{cases} v_0, & x = v_2 \\ v_5, & x = v_6 \\ v_9, & x = v_{11} \\ v_{14}, & x = v_{15} \end{cases}.$$

Notice that $\mathcal{D}'$ and $\mathcal{D}''$ are obtained from the function $\mathcal{D}$ from Example 3.1.10 by restricting its domain to D' and D'', respectively. Hence the functions $\mathcal{D}'$ and $\mathcal{D}''$ are domination selectors for D' and D'' respectively, hence both sets are minimal.

Corollary 3.2.11. *A tree T is unmixed if and only if every minimal RD-set has the same size and every minimal BD-set has the same size.*

Proof. Apply Fact 3.2.9 with $D' = D \cap V_{\mathcal{B}}$ and $D'' = D \cap V_{\mathcal{R}}$. □

3.3. **Descriptive characterization of unmixed balanced trees.** Theorem 3.3.12 is the main result of this subsection. It allows us to detect when a balanced tree is unmixed. The case of an arbitrary tree is handled in Subsection 3.4 below. We begin with a definition.

Definition 3.3.1. Let $x \in V(G)$. The *radar of x of distance d* (in G) is the set of vertices

$$R(x, d) = R_G(x, d) := \{y \in V \mid \operatorname{dist}(x, y) = d\}.$$

Now let T be a tree with $x, r \in V(T)$. The *branch spanned from r to x* is the set of vertices

$$\operatorname{branch}_r(x) := \{y \in V(T) \mid x \in \operatorname{path}(y, r)\}.$$

Example 3.3.2. Let T be the tree given in Figure 3. Then the radar $R(a, 2)$ is the set of green and orange vertices. The branch $\operatorname{branch}_c(d)$ is the set of all cyan vertices; note that $c \notin \operatorname{branch}_c(d)$. Let H be the subgraph induced by the vertex set $\operatorname{branch}_b(a)$. Then the radar $R_H(a, 2)$ is the set of orange vertices.

The next few results provide conditions that force a balanced tree to be mixed by constructing two RD-sets of different sizes and using Corollary 3.2.11.

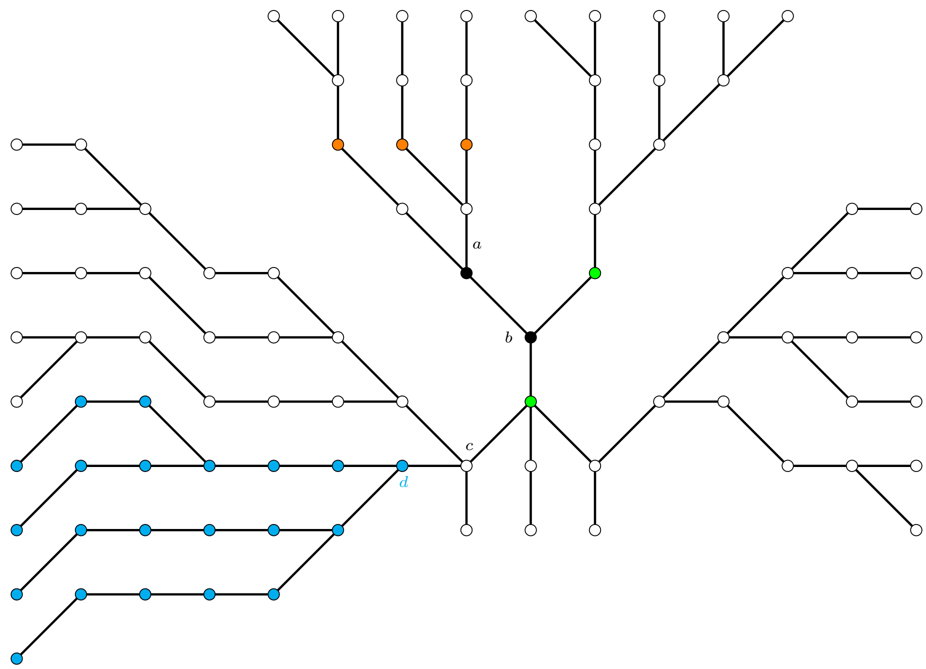


FIGURE 3. Branch and radar example on G

Assumptions. For the remainder of this section, given a balanced tree assume that the even height vertices are blue and odd height vertices are red using Corollary 3.2.6.

Lemma 3.3.3. *Let T be a balanced tree of height $d > 0$. Then $|V_{d-1}| > |V_d|$.*

Proof. By way of contradiction, assume that $|V_{d-1}| \leq |V_d|$. Since no vertex in $V_d(T)$ is a leaf, every vertex in $V_d(T)$ is adjacent to at least 2 vertices in $V_{d-1}(T)$. Hence there are at least $2|V_d|$ edges between V_d and V_{d-1}. Thus the subgraph induced by $V_d(T) \cup V_{d-1}(T)$ is not a forest; any forest H with k connected components has exactly $|V(H)| - k$ number of edges, but the subgraph induced by $V_d \cup V_{d-1}$ has more edges than the number of vertices as $2|V_d| \geq |V_d| + |V_{d-1}|$. Since an induced subgraph of T is not a forest, T is not a tree, a contradiction. $\square$

The next result extends the mixedness of P_4 noted in Example 2.2.7.

Theorem 3.3.4. *Let T be a balanced tree. If T has two leaves l_1 and l_2 such that $dist(l_1, l_2) = 4$, then T is mixed.*

Proof. Let s_1 and s_2 be the support vertices adjacent to l_1 and l_2 respectively, and let $v \in V_2$ be the vertex which is adjacent to both s_1 and s_2. By Corollary 3.2.11, it suffices to show that there are two minimal RD-sets of different sizes. By Lemma 3.1.5, we assume that there is a unique leaf for each support vertex in T. Let $u_1, u_2, \cdots, u_k$ be vertices which are adjacent to v which are not support vertices (hence height$(u_i) = 3$ by Lemma 3.2.2 for all i). Let $s_1, s_2, \cdots, s_m$ be the support vertices adjacent to v, and let l_i be the leaf adjacent to s_i for each i. Note that $u_1, \cdots, u_k \in V_{\mathcal{R}}$ as $v \in V_{\mathcal{B}}$ since T is a balanced tree (see Figure 4).

For $i = 1, 2, \ldots, k$, define G_{u_i} to be the subgraph of G induced by $\text{branch}_v(u_i) =: B_{u_i}$. For each G_{u_i}, consider the union of radars $R_{u_i} := R_{G_{u_i}}(u_i, 1) \cup \left(\bigcup_{j=1}^{\infty} R_{G_{u_i}}(u_i, 3 + 2j)\right)$. Since u_i is red color, the set R_{u_i} is colored blue by Lemma 3.2.2 and Corollary 3.2.6. Also, one can

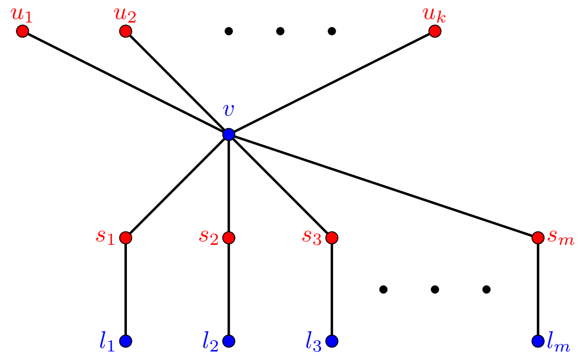


FIGURE 4. Tree T with vertices labeled and colored

verify that R_{u_i} is an RD-set in G_{u_i}. Since R_{u_i} is an RD-set in G_{u_i}, there is some minimal RD-set $D_{u_i} \subseteq R_{u_i}$ in G_{u_i}. Set $D_u := \cup_{i=1}^{k} D_{u_i}$. By Corollary 3.1.12, D_u is minimal.

Next, for $j = 1, 2, \ldots, m$, let G_{s_j} be the subgraph of G induced by $B_{s_j} := \mathrm{branch}_v(s_j)\setminus\{l_j\}$. For each G_{s_j}, define the set $R_{s_j} = \bigcup_{i=0}^{\infty} R_{G_{s_j}}(s_j, 3+2i)$ (see Figure 5). Then $R_{s_j} \subseteq V_{\mathcal{B}}(T)$, and R_{s_j} dominates all red vertices in G_{s_j} but s_j. Let $D_{s_j} \subseteq R_{s_j}$ be a minimal set of R_{s_j}, and set $D_s := \bigcup_{j=1}^{m} D_{s_j}$. Again by Corollary 3.1.12, D_s is minimal.

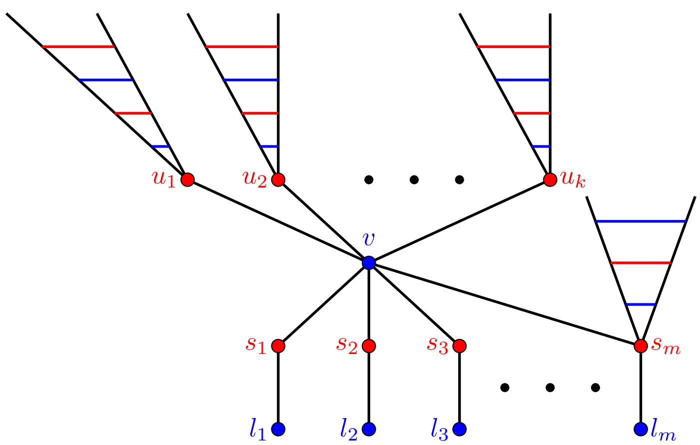


FIGURE 5. Tree T with branches (there are branches growing from all s_i's like s_m)
(View the horizontal lines as the set of vertices)

Let $D = D_u \cup D_s \cup \{v\}$ and let $D' = D_u \cup D_s \cup \{l_1, l_2, \cdots, l_m\}$. We claim that D and D' are minimal RD-sets in G. Since $N(D_u \cup D_s) = V_{\mathcal{R}}(T) \setminus \{s_1, s_2, \cdots, s_k\}$, we have $N(D) = N(D') = V_{\mathcal{R}}(T)$ as $\{s_1, \ldots, s_m\} \subseteq N(v)$ and $\{s_1, \ldots, s_m\} \subseteq N(l_1, \ldots, l_m)$.

Now we check the minimality of D and D'. Set $D_{u,s} := D_u \cup D_s$. Then $D_{u,s}$ is minimal by Corollary 3.1.12, hence there exists a domination selector $\mathcal{D}_{u,s} : D_{u,s} \to N(D_{u,s})$. Note that for each i and for all $x \in D_{u_i}$, we may choose $\mathcal{D}_{u,s}(x) \in B_{u_i} \setminus \{u_i\}$ since $R_{G_{u_i}}(u_i, 3) \cap D_{u_i} = \emptyset$. Also, for all $x \in D_{s_j}$ for each j, we have $\mathcal{D}_{u,s}(x) \neq s_j$ since $s_j \notin N_{G_{s_j}}(D_{s_j})$.

As $\{v\}$ is minimal with domination selector $\mathcal{D}_v : \{v\} \to N(v)$ given by $\mathcal{D}_v(v) = s_1$, setting

$$I := N(D_{u,s}) \cap N(v) = \{u_1, \ldots, u_k\}$$

we have $\mathcal{D}_{u,s}(D_{u,s}) \cap I = \emptyset$. So, D is minimal by Lemma 3.1.11.

Since $N(D_{u,s}) \cap N(s_1, \ldots, s_m) = \emptyset$, D' is minimal by Corollary 3.1.12. Since $m > 1$, D and D' are two minimal RD-sets of different sizes, so T is mixed, as desired. □

Example 3.3.5. We demonstrate the construction of two minimal RD-sets of different sizes shown in Theorem 3.3.4. Consider the balanced tree in Figure 6. We use the notation

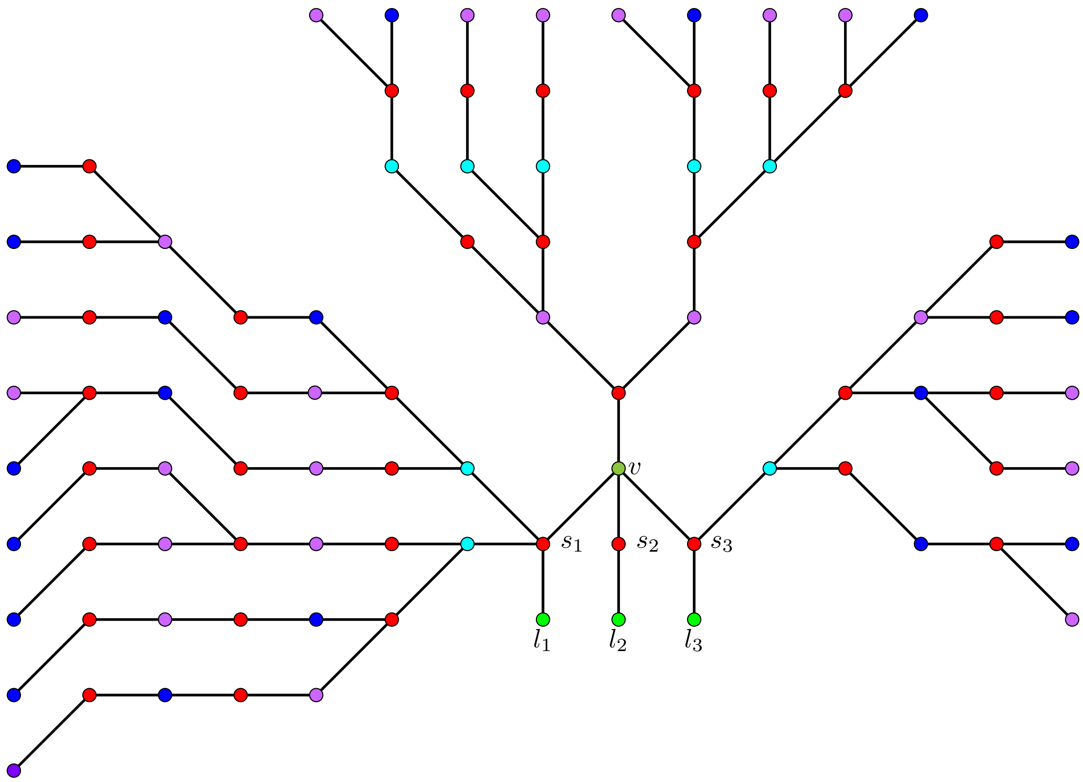


FIGURE 6. Balanced tree with distance 4 apart leaves

from the proof of Theorem 3.3.4. The cyan vertices represent the even height vertices (blue vertices) that were excluded when we constructed R_{u_i} and R_{s_i}. Let P be the set of all purple vertices (P corresponds to the set $D_u \cup D_s$). Notice that P is minimal, and that $N(P) = V_{\mathcal{R}} \backslash \{s_1, s_2, s_3\}$. One can compute that the sets $D := P \cup \{v\}$ and $D' := P \cup \{l_1, l_2, l_3\}$ are minimal RD-sets.

Also, one can choose a different purple set (minimal, dominates all red vertices but the s_i's, and does not include any cyan vertices) and see that the union of the new purple set with either $\{v\}$ or $\{l_1, l_2, l_3\}$ forms a minimal RD-set; minimality of the union is preserved by not picking the cyan vertices.

The mixedness of P_4 makes another appearance in our next result.

Lemma 3.3.6. *Let T be a balanced tree. Then for all leaves $\ell_1, \ell_2 \in V_0$, we have dist$(\ell_1, \ell_2) \neq 4$ if and only if $|N(v) \cap V_1| = 1$ for all $v \in V_2$.*

Proof. ($\Rightarrow$) Assume that $\text{dist}(\ell_1, \ell_2) \neq 4$ for all leaves ℓ_1, ℓ_2, and suppose by way of contradiction that there is a vertex $v \in V_2$ such that $|N(v) \cap V_1| \neq 1$. By definition of height, we must have $|N_T(v) \cap V_1| \geq 1$. So, there are distinct $s_1, s_2 \in N(v) \cap V_1$. Then there are leaves $\ell_1, \ell_2 \in V_0$ such that $\ell_1 s_1, \ell_2 s_2 \in E$. This forms a path between ℓ_1 and ℓ_2, namely $\ell_1 s_1 v s_2 \ell_2$, so $\mathrm{dist}(\ell_1, \ell_2) \leq 4$, hence $\mathrm{dist}(\ell_1, \ell_2) < 4$ by assumption. Since s_i is the unique neighbor of ℓ_i for $i = 1, 2$, it follows that s_1 and s_2 are adjacent, contradicting the fact that T is balanced.

($\Leftarrow$) Assume that $|N(v) \cap V_1| = 1$ for all $v \in V_2$. Let $\ell_1, \ell_2 \in V_0$ be two leaves. Then there are support vertices $s_1, s_2 \in V_1$ such that $\ell_1 s_1, \ell_2 s_2 \in E$. If $s_1 = s_2$, then $\ell_1 s_1 \ell_2$ forms a path of length 2 between ℓ_1 and ℓ_2. Thus by the uniqueness of path between two vertices in a tree, we get $\mathrm{dist}(\ell_1, \ell_2) = 2 \neq 4$. Now suppose that $s_1 \neq s_2$. Since T is a balanced tree, there is no edge between s_1 and s_2. Thus we must have $\mathrm{dist}(\ell_1, \ell_2) > 3$. Assume by way of contradiction that $\mathrm{dist}(\ell_1, \ell_2) = 4$. Then there exists $v \in V_2$ such that $\ell_1 s_1 v s_2 \ell_2$ forms a path of length 4. This implies that $s_1, s_2 \in N(v)$, so $|N(v) \cap V_1| \geq 2$, contradicting the assumption that $|N(v) \cap V_1| = 1$ as $v \in V_2$. Thus $\mathrm{dist}(\ell_1, \ell_2) \neq 4$. □

Toward our descriptive characterization of unmixed balanced trees we next show that all balanced trees of height at least 4 are mixed.

Theorem 3.3.7. *Let T be a balanced tree with blue leaves and* height$(T) \geq 4$. *Then T has two minimal RD-sets of different sizes, and T is mixed.*

Proof. Suppose by way of contradiction that all minimal RD-sets of T have the same size. Since $\mathrm{height}(T) \geq 4$, there is some vertex v of height 4. Hence there are vertices w_0, x, s, ℓ where ℓ is a leaf such that $\mathrm{path}(v, \ell) = (v, w_0, x, s, \ell)$. By Corollary 3.2.11, Theorem 3.3.4, and Lemma 3.3.6, for all $a \in V_2(T)$, we have $|N(a) \cap V_1(T)| = 1$. Let $w_0, w_1, w_2, \cdots, w_k$ be the distinct neighbors of v. Choose v to be the root of T. For each $i = 1, \ldots, k$, let G_i be the subgraph induced by $\mathrm{branch}_v(w_i) =: B_i$, and let G_0 to be the subgraph induced by $B_0 := \mathrm{branch}_v(w_0) \setminus \{x, s, \ell\}$.

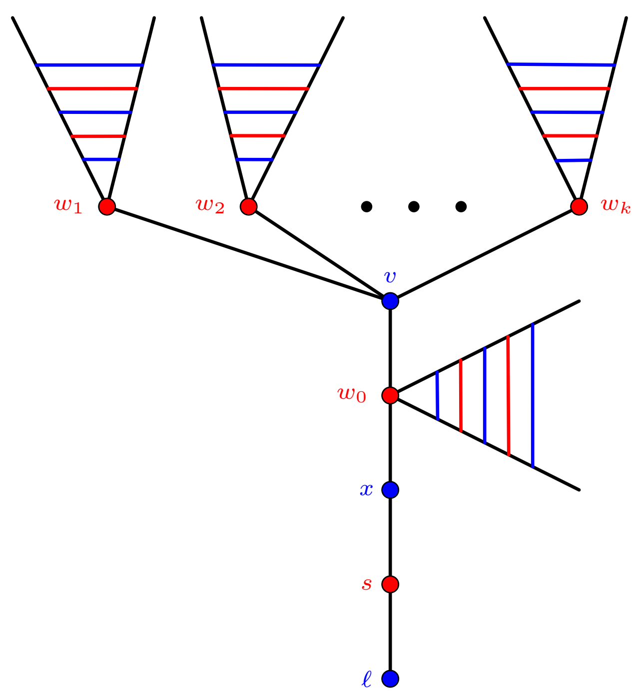


Figure 7. Tree T with branches B_i

For each $i = 1, \ldots, k$, consider the union of radars

$$R_i := R_{G_i}(w_i, 1) \cup \left(\bigcup_{j=1}^{\infty} R_{G_i}(w_i, 3+2j) \right)$$

and

$$R_0 := R_{G_0}(w_0, 3) \cup \left(\bigcup_{j=1}^{\infty} R_{G_0}(w_0, 5+2j) \right).$$

Next, notice that each R_i with $i > 0$ is an RD-set of G_i, and R_0 dominates all red vertices in G_0 but w_0. Since R_i is an RD-set in G_i for $i > 0$, there exists a minimal RD-set $D_i \subseteq R_i$ for each $i > 0$. Also, there is a minimal set $D_0 \subseteq R_0$ which dominates all red vertices in G_0 but w_0. Set $D_w := \cup_{i=0}^{k} D_i$.

Next, let G_x and G_s be the subgraphs of G induced by the sets $\mathrm{branch}_v(x) \setminus \mathrm{branch}_v(s)$ and $\mathrm{branch}_v(s) \setminus \{\ell\}$, respectively (see Figure 8) and set

$$R_x := R_{G_x}(x, 2) \cup \left(\bigcup_{i=0}^{\infty} R_{G_x}(v, 6+2i) \right) \quad \text{and} \quad R_s := \bigcup_{i=0}^{\infty} R_{G_s}(y, 3+2i).$$

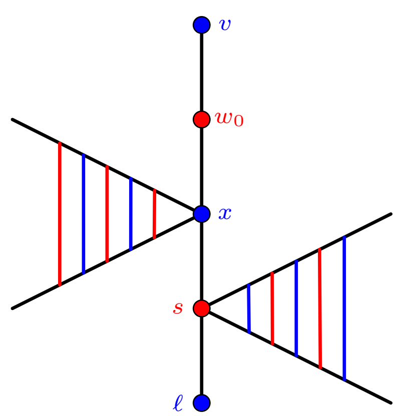


FIGURE 8. Branches B_x and B_s in T

Then R_x dominates all red vertices in G_x, and R_s dominates all red vertices in G_s but s. Let $D_x \subseteq R_x$ and $D_s \subseteq R_s$ be minimal sets of R_x and R_s. We claim that the sets

$$D := D_w \cup D_x \cup D_y \cup \{v, \ell\}$$

and

$$D' := D_w \cup D_x \cup D_y \cup \{x\}$$

are minimal RD-sets in T. Since $N_T(D_w \cup D_x \cup D_s) = V_{\mathcal{R}}(T) \setminus \{w_0, s\}$, $N_T(\{v, \ell\}) \supseteq \{w_0, s\}$, and $N_T(\{x\}) \supseteq \{w_0, s\}$, the sets D and D' are RD-sets in T.

Now we show that D and D' are minimal. First note that the open neighborhoods of D_w, D_x, and D_y are all mutually disjoint. Hence by Corollary 3.1.12, the set $Z := D_w \cup D_x \cup D_y$ is minimal in T. Thus there exists a domination selector $\mathcal{D}_Z : Z \to N_T(Z)$. Set $W := \{w_0, \ldots, w_k\}$. For each $i = 1, \ldots, k$, we may choose $\mathcal{D}_Z(D_{w_i})$ so that $\mathcal{D}_Z(D_{w_i}) \cap W = \emptyset$ since

$R_{G_i}(w_i, 3) \cap D_{w_i} = \emptyset$. Also, as $w_0 \notin N_T(D_{w_0})$, we have $\mathcal{D}_Z(D_w) \cap W = \emptyset$. The set $\{v, \ell\}$ is minimal with domination selector $\mathcal{D}_{v\ell} : \{v, \ell\} \to N_T(\{v, \ell\})$ given by

$$\mathcal{D}_{v\ell}(v') := \begin{cases} w_0 & v' = v \\ s & v' = \ell \ . \end{cases}$$

Since we have $N_T(Z) \cap N_T(\{v, \ell\}) = W \setminus \{w_0\}$, the conditions $\mathcal{D}_Z(Z) \cap (W \setminus \{w_0\}) = \emptyset$ and $\mathcal{D}_{v\ell}(\{v, \ell\}) \cap (W \setminus \{w_0\}) = \emptyset$ show that D is minimal by Lemma 3.1.11.

To show that D' is minimal, we choose $\mathcal{D}_Z$ such that for each $\alpha \in D_x \cap R_{G_x}(x, 2)$, pick $\beta \in N_{G_x}(\alpha) \cap R_{G_x}(x, 3)$ and set $\mathcal{D}_Z(\alpha) = \beta$ (which is possible since $D_x \cap R_{G_x}(x, 4) = \emptyset$). This choice of $\mathcal{D}_Z$ gives us $\mathcal{D}_Z(D_x) \cap R_{G_x}(x, 1) = \emptyset$. Note that $\{x\}$ is minimal with domination selector $\mathcal{D}_x : \{x\} \to N_T(x)$ given by $\mathcal{D}_x(x) = w_0$. Since $N_T(Z) \cap N_T(x) \subseteq R_{G_x}(x, 1)$, $\mathcal{D}_Z(Z) \cap R_{G_x}(x, 1) = \emptyset$, and $\mathcal{D}_x(\{x\}) \cap R_{G_x}(x, 1) = \emptyset$, D' is minimal by Lemma 3.1.11. Since $|D| > |D'|$, the tree T has two minimal RD-sets of difference sizes as desired. This contradicts our initial supposition. Finally, Corollary 3.2.11 implies that T is mixed. $\square$

Example 3.3.8. Consider the tree T in Figure 9. We show that there are two minimal

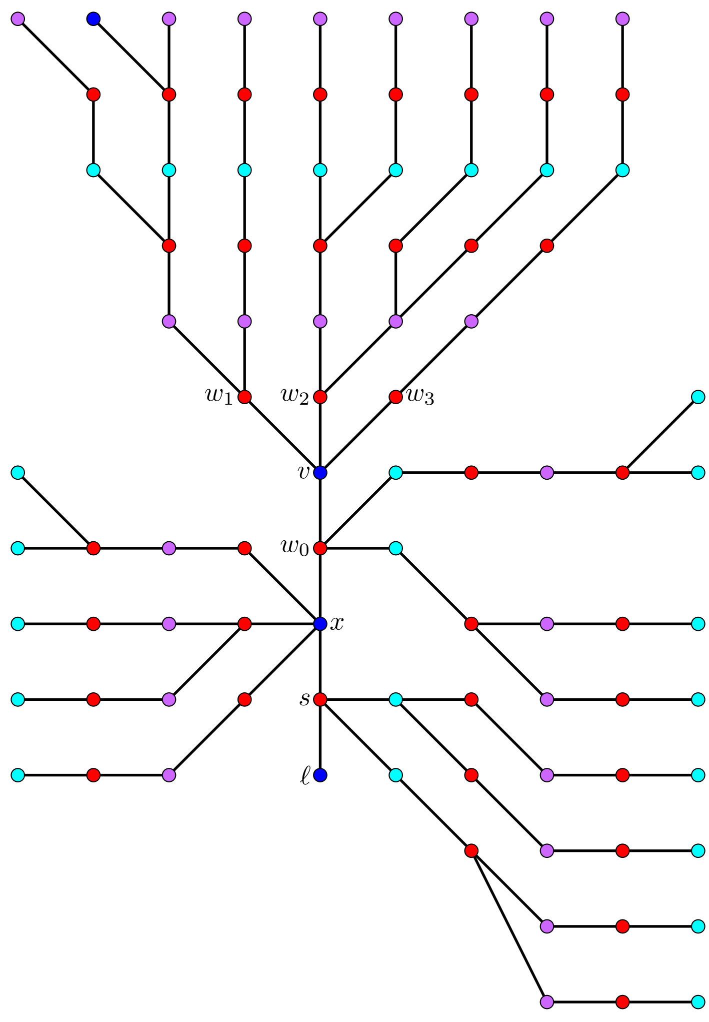


FIGURE 9. a balanced tree T of height greater than 3

RD-sets of different sizes in T. We use the notation as in Theorem 3.3.7. Let Z be the set of all purple vertices ($Z = D_w \cup D_x \cup D_s$). The cyan vertices represent the blue vertices (even height vertices) excluded when we construct R_x, R_s, and R_i for $i = 1, 2, 3$ in this example. Then the sets $D := Z \cup \{v, \ell\}$ and $D' := Z \cup \{x\}$ form minimal RD-sets.

With Theorem 3.3.7, our next result shows that an unmixed balanced tree must have height 0,1, or 3.

Theorem 3.3.9. *Let T be a balanced tree with height$(T) = 2$. Then T has two leaves of distance 4 apart, hence T is mixed.*

Proof. Since height$(T) = 2$, there exists a vertex $v \in V_2$. Then there is a leaf $l \in V_0$ and a support vertex $s \in V_1$ such that path$(v, l) = (v, s, l)$. Since height$(v) = 2 =$ height(T), the vertex v is only adjacent to height 1 vertices. Also, the degree of v must be at least 2 (else v is a leaf). So, there is a vertex $s' \neq s$ of height 1 where $vs' \in E(V)$. Since height$(s') = 1$, there is a leaf $l' \neq l$ which is adjacent to s'. Since path$(l, l') = (l, s, v, s', l')$ has length 4, we have dist$(l, l') = 4$. Therefore, by Theorem 3.3.4, the tree T is mixed. □

Theorem 3.3.10. *Let T be a balanced tree with blue leaves. If there is a support vertex $s \in V_1$ such that $|N(s) \cap V_2| \geq 2$, then T has two minimal RD-sets of different sizes, hence T is mixed.*

Proof. If height$(T) \geq 4$, then T has two minimal RD-sets of different sizes by Theorem 3.3.7. So we assume that height$(T) \leq 3$. Since $V_2 \neq \emptyset$ by assumption, height$(T) \geq 2$. If height$(T) = 2$, then T has two minimal RD-sets of different sizes and is mixed by Corollary 3.2.11 and Thoerem 3.3.9. Hence we assume that height$(T) = 3$.

If there is a vertex $x \in V_2$ with $|N(x) \cap V_1| \geq 2$, then T has two minimal RD-sets of different sizes and is mixed by Corollary 3.2.11, Theorem 3.3.4, and Lemma 3.3.6. Thus we also assume that for all $x \in V_2$, we have $|N(x) \cap V_1| = 1$.

Let l be a leaf adjacent to s. Since $|N(s) \cap V_2| \geq 2$, there are two distinct vertices v_1, v_2 in $N(s) \cap V_2$. Since height$(v_1) =$ height$(v_2) = 2$, v_1 and v_2 are not leaves. Hence deg(v_1) and deg(v_2) are at least 2. By assumption, we have $|N(v_i) \cap V_1| = 1$ for $i = 1, 2$. Thus v_i must be adjacent to a height 3 vertex u_i for $i = 1, 2$. Similarly, since u_1 and u_2 are not leaves, there are distinct vertices $v_1', v_2' \in V_2$, distinct support vertices $s_1, s_2 \in V_1$, and distinct leaves $l_1, l_2 \in V_0$ such that path$(u_i, l_i) = (u_i, v_i', s_i, l_i)$ for $i = 1, 2$. Let T' be the subgraph of T induced by the set $\{l, l_1, l_2, s, s_1, s_2, v_1, v_1', v_2, v_2', u_1, u_2\}$ (Figure 10).

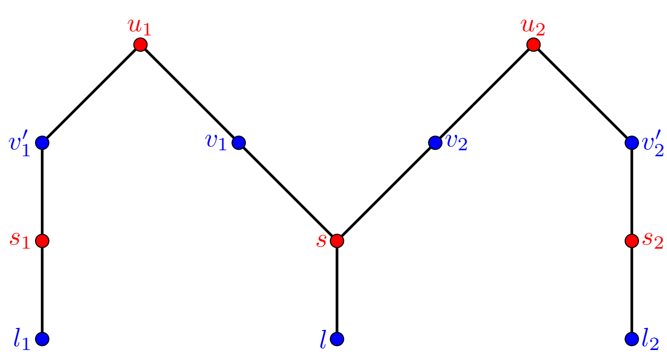


FIGURE 10. Induced subgraph T' of T

Next, we construct a set dominating every red (odd height) vertex that is not in T'. Set

$$X_3 := (N_T(V_2(T')) \cap V_3(T)) \setminus V_3(T') \qquad X_2 := (N_T(X_3) \cap V_2(T)) \setminus V_2(T')$$

$$X_1 := (N_T(X_2) \cap V_1(T)) \setminus V_1(T') \qquad X_0 := (N_T(X_1) \cap V_0(T)) \setminus V_0(T')$$
$$Y_2 := (N_T(X_1) \cap V_2(T)) \setminus (X_2 \cup V_2(T'))$$

see Figure 11. We claim that the set

$$U := (V_0(T) \cup V_2(T)) \setminus (X_0 \cup Y_2 \cup N_T(V_1(T') \cup V_3(T')))$$

has $N_T(U) = (V_1(T) \cup V_3(T)) \setminus (V_1(T') \cup V_3(T'))$.

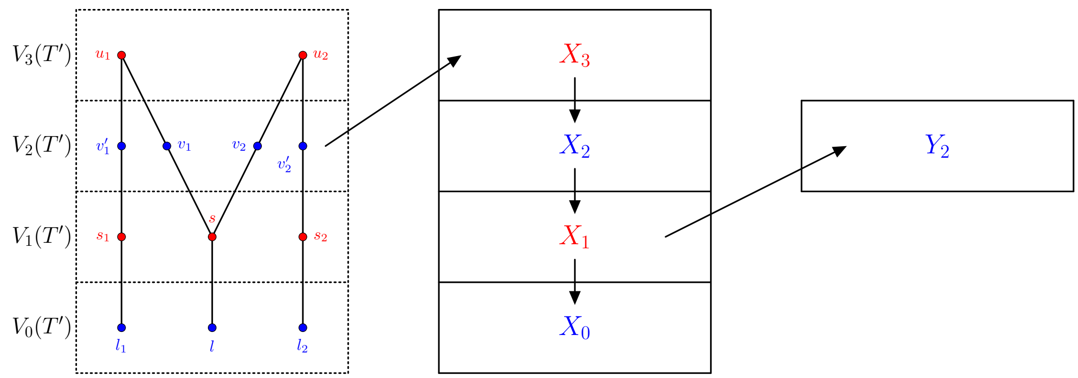


FIGURE 11. Vertex sets X_3, X_2, X_1, X_0, and Y_2
(Arrows indicate the adjacency relation between the sets; for instance, $Y_2 \subseteq N_T(X_1)$)

The containment $N_T(U) \subseteq (V_1(T) \cup V_3(T)) \setminus (V_1(T') \cup V_3(T'))$ is straightforward since $U \subseteq V_0(T) \cup V_2(T)$ implies $N_T(U) \subseteq V_1(T) \cup V_3(T)$ because T is a balanced tree, and $N_T(V_1(T') \cup V_3(T')) \cap U = \emptyset$ implies $N_T(U) \cap (V_1(T') \cup V_3(T')) = \emptyset$.

Before we prove the reverse containment, we show that $X_2 \subseteq U$. Since $X_2 \subseteq V_2(T)$, we need to show that $X_2 \cap (X_0 \cup Y_2 \cup N_T(V_1(T') \cup V_3(T'))) = \emptyset$. By the constructions of X_0 and Y_2, we get $X_2 \cap (X_0 \cup Y_2) = \emptyset$. To show that $X_2 \cap N_T(V_1(T') \cup V_3(T')) = \emptyset$, assume by way of contradiction that there is a vertex $c \in X_2 \cap N_T(V_1(T') \cup V_3(T'))$. We construct a cycle containing the vertex c, contradicting that T is a tree. First, suppose that $c \in N_T(V_1(T'))$. Then there is a vertex $s' \in V_1(T')$ that is adjacent to c. By construction of X_2 and X_3, there are vertices $x_3 \in X_3$ and v' in $V_2(T')$ such that $cx_3 \in E(T)$ and $v'x_3 \in E(T)$. Since T' is a tree and s' and v' are vertices in T', there is a path P in T' from v' to s'. Thus the paths P and (s', c, x_3, v') form a cycle in T. Similarly, we can construct a cycle when $c \in N_T(V_3(T'))$.

Now we show that $N_T(U) \supseteq (V_1(T) \cup V_3(T)) \setminus (V_1(T') \cup V_3(T'))$. Let $x \in (V_1(T) \cup V_3(T))$ such that $x \notin (V_1(T') \cup V_3(T'))$. We show that there is a vertex in U that is adjacent to x. We have three cases to consider.

Case 1. $x \in X_3 \cup X_1$. If $x \in X_1$, then there is a vertex $x_2 \in X_2 \subseteq U$ that is adjacent to x. If $x \in X_3$, then $\deg(x) \geq 2$ as x is not a leaf. Hence x is adjacent to at least 2 vertices in $V_2(T)$. By construction of X_3, x is adjacent to a vertex in $V_2(T')$. Notice that x cannot be adjacent to 2 vertices in $V_2(T')$ since this will form a cycle in T. Thus x must be adjacent to a vertex $y \in V_2(T)$ such that $y \notin V_2(T')$. By construction of X_2, $y \in X_2 \subseteq U$ as desired.

Case 2. $x \in V_1(T) \setminus X_1$. Then there is a leaf $l_x \in V_1(T)$ that is adjacent to x. Since $x \notin X_1$, $l_x \notin X_0$ as x is the only vertex l_x is adjacent to. We also have $l_x \notin Y_2 \cup N_T(V_3(T'))$ since vertices in $Y_2 \cup N_T(V_3(T'))$ have height 2. Finally, $l_x \notin N_T(V_1(T'))$ since we assumed that $x \notin V_1(T')$. Thus we have $l_x \in U$ as desired.

Case 3. $x \in V_3(T) \setminus X_3$. Then x is not a leaf, hence $\deg_T(x) \geq 2$. Also, as $\mathrm{height}(T) = 3 = \mathrm{height}(x) = 3$ and T is balanced, x is adjacent to at least two height 2 vertices $t_1, t_2 \in V_2(T)$.

To show that at least one of t_1 or t_2 is in U, assume by way of contradiction that $t_1, t_2 \notin U$. Then $t_1, t_2 \in X_0 \cup Y_2 \cup N_T(V_1(T') \cup V_3(T'))$ by definition of U. Since height of t_1 and t_2 is 2, we have $t_1, t_2 \in Y_2 \cup N_T(V_1(T') \cup V_3(T'))$. As in the proof of $X_2 \subseteq U$ there is a path from t_1 to t_2 only using the vertices in $V(T') \cup \left(\bigcup_{i=1}^{3} X_i\right) \cup Y_2 \cup N_T(V_1(T') \cup V_3(T'))$. But (t_1, x, t_2) is also a path from t_1 to t_2, with x not belonging to $V(T') \cup \left(\bigcup_{i=1}^{3} X_i\right) \cup Y_2 \cup N_T(V_1(T') \cup V_3(T'))$. Thus we have a cycle in T, a contradiction.

Therefore, we have $N_T(U) = (V_1(T) \cup V_3(T)) \setminus (V_1(T') \cup V_3(T'))$ as desired.

Now let $D \subseteq U$ be a minimal set of U. Let $D_1 := \{l_1, l_2, v_1, v_2\}$ and $D_2 := \{l, v_1', v_2'\}$. We claim that the sets $D \cup D_1$ and $D \cup D_2$ are minimal RD-sets. For $i = 1, 2$, since T is balanced and $N_T(D_i) \supseteq \{u_1, u_2, s, s_1, s_2\} = V_1(T') \cup V_3(T')$, we get $N_T(D \cup D_i) = V_1(T) \cup V_3(T)$.

To show the minimality of $D \cup D_1$ and $D \cup D_2$, define domination selectors for D_1 and D_2

$$\mathcal{D}_1(v) := \begin{cases} s_1 & v = l_1 \\ s_2 & v = l_2 \\ u_1 & v = v_1 \\ u_2 & v = v_2 \end{cases} \qquad \mathcal{D}_2(v) := \begin{cases} s_1 & v = v_1' \\ s_2 & v = v_2' \\ s & v = l \end{cases}.$$

For D, we claim that there is a domination selector $\mathcal{D} : D \to N_T(D)$ such that for all $x \in D \cap X_2$, we have $\mathcal{D}(x) \in X_1$. To this end, we need to show that for all $x \in D \cap X_2$, there is a vertex $y \in X_1$ such that $N_T(y) \cap D = \{x\}$. So, assume by way of contradiction that there is a vertex $x \in D \cap X_2$ such that for all $y \in X_1$, we have $N_T(y) \cap D \neq \{x\}$. Since $\text{height}(x) = 2$, there is a support vertex $s' \in V_1(T)$ that is adjacent to x. By construction of X_2, either $s' \in X_1$ or $s' \in V_1(T')$. To show that $s' \in X_1$, assume by way of contradiction that $s' \in V_1(T)'$. Then one can show that this assumption creates a cycle containing s' and x using the vertices in $V(T') \cup X_2 \cup X_3$, a contradiction. Thus $s' \in X_1$. Since $N_T(s') \cap D \neq \{x\}$ but $x \in N_T(s') \cap D$, there is a vertex $x' \in D \cap X_2$ such that $N_T(s') \cap D \supseteq \{x, x'\}$. By the construction of X_2 and X_3, there is a path P from x to x' only using the vertices in $V(T') \cup X_3$. On the other hand, (x, s', x') is a path from x to x' which is not P as $s' \notin V(T') \cup X_3$. Hence we have a cycle in T, a contradiction, and the claim is established.

Furthermore, since the only vertices in U that are adjacent to X_3 are in X_2, we get $\mathcal{D}(D) \cap X_3 = \emptyset$ as $\mathcal{D}(x) \in X_1$ for all $x \in D \cap X_2$.

Now consider $N_T(D) \cap D_i$ for $i = 1, 2$. By the above computation of $N_T(U) = N_T(D)$, we have $N_T(D) \cap D_i = \emptyset$ for $i = 1, 2$. Since $\mathcal{D}_i(D_i) \cap X_3 = \emptyset$ for $i = 1, 2$, and $\mathcal{D}(D) \cap X_3 = \emptyset$, the sets $D \cup D_1$ and $D \cup D_2$ are minimal by Lemma 3.1.11. □

The following result is a converse to some previous results when $\text{height}(T) = 3$.

Theorem 3.3.11. *Let T be a balanced tree of height 3. Suppose that*

(1) for all $u \in V_2$, $|N(u) \cap V_1| = 1$, and
(2) for all $s \in V_1$, $|N(s) \cap V_2| \leq 1$.

Then T is unmixed.

Proof. First, we show some structural properties of T. By Lemma 3.1.5, the unmixedness of T is independent of the numbers of leaves attached to the support vertices. Hence we assume that every support vertex is adjacent to exactly one leaf, which implies that $|V_0| = |V_1|$. So, for each $s \in V_1$, denote the unique leaf adjacent to s by $\ell(s)$, and for each leaf $\ell \in V_0$, denote the unique support vertex adjacent to ℓ by $s(\ell)$.

Next, since T is connected and balanced with height$(T) = 3$, every support vertex must be adjacent to at least one height-2 vertex because there is a path between every support vertex and height-3 vertex. Hence the inequality in (2) becomes an equality: for all $s \in V_1$, $|N(s) \cap V_2| = 1$.

Now we claim that $|V_1| = |V_2|$. To show the claim, assume by way of contradiction that $|V_2| \neq |V_1|$. First, suppose that $|V_2| > |V_1|$. By (1) and the pigeonhole principle, there exists a support vertex $c \in V_1$ that is adjacent to at least two height-2 vertices. This contradicts (2). Now suppose that $|V_2| < |V_1|$. Then by (2) and the pigeonhole principle, there exists a vertex $c \in V_2$ that is adjacent to at least two support vertices, contradicting (1). Therefore, we get $|V_2| = |V_1|$.

The hypotheses (1)–(2) and the condition $|V_2| = |V_1|$ implies that the set of edges

$$E(V_1, V_2) := \{uv \in E \mid u \in V_1, v \in V_2\}$$

forms a bijection (perfect matching) between V_1 and V_2. Hence for each $v \in V_2$, let $s(v)$ be the unique support vertex adjacent to v; i.e., $s(v)v \in E(V_1, V_2)$.

To show that T is unmixed, we use Corollary 3.2.11. First, we show that every minimal BD-set of T has the same size. More specifically, we show that V_1 is the only minimal BD-set of T. Let D be a minimal BD-set of T. Then $N(D) = V_0 \cup V_2$. Since $V_0 \subset N(D)$ and the support vertices are the only vertices that are adjacent to the leaves, we have $V_1 \subseteq D$. But, we also have $N(V_1) = V_0 \cap V_2$. By mininmality of D, we have $V_1 = D$. Thus V_1 is a minimal BD-set contained in any BD-set, showing that V_1 is the only minimal BD-set in T.

Next, we show that every minimal RD-set has the same size. First, notice that the set V_2 is a minimal RD-set because it dominates $V_1 \cup V_3$ and each $s \in V_1$ is dominated by a unique $x \in V_2$. Let D be a minimal RD-set. We show that $|D| = |V_2|$. Since there are no vertices $c \in D \subseteq V_0 \cup V_2$ with $|N(c) \cap V_1| \geq 2$, we have $|D| \geq |V_1|$. Next, we show that $|D| \leq |V_1|$. Let $s \in V_1$. Then there are vertices $l \in V_0$ and $u \in V_2$ such that $N(s) = \{l, u\}$ by the assumptions on T. Since $N(l) = \{s\} \subseteq N(u)$, the minimality of D implies that $|D \cap N(s)| \leq 1$. Since $D \subseteq V_0 \cup V_2 = \bigcup_{t \in V_1} N(t)$ where the union is disjoint, we get

$$|D| = \left| \bigcup_{t \in V_1} (D \cap N(t)) \right| = \sum_{t \in V_1} |D \cap N(t)| \leq \sum_{t \in V_1} 1 = |V_1| \ .$$

This gives $|D| = |V_1|$ hence every minimal RD-set of T has the same size. □

Now we present our descriptive characterization of unmixed balanced trees.

Theorem 3.3.12. *Let T be a balanced tree. Then T is unmixed if and only if*

(1) height$(T) \leq 3$,
(2) for all $v \in V_2$, $|N(v) \cap V_1| = 1$, and
(3) for all $v \in V_1$, $|N(v) \cap V_2| \leq 1$.

Proof. ($\Rightarrow$) Suppose that T is unmixed. Then Theorem 3.3.7 gives (1), Theorem 3.3.4 and Lemma 3.3.6 give (2), and Theorem 3.3.10 and gives (3).

($\Leftarrow$) Suppose that (1), (2), and (3) hold. By (2), we have height$(T) \neq 2$ as height 2 vertices cannot be leaves. Hence by (1), we have height$(T) = 0, 1$, or 3. If height$(T) = 0$, then T is a graph with a single vertex, which is vacuously unmixed. If height$(T) = 1$, then it is straightforward to show that T is a graph with a single support vertex s with at least 2 leaves attached (such graphs are called star graphs, see Figure 12). In this case, every minimal TD-set of T is of the form $\{s, l\}$ where l is a leaf, hence T is unmixed. If height$(T) = 3$,

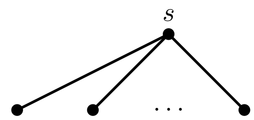


FIGURE 12. Balanced tree of height 1 (star graph)

then by Theorem 3.3.11, T is unmixed. □

Theorem 3.3.12 and the proof of Theorem 3.3.11 show that if T is an unmixed balanced tree with $\operatorname{height}(T) = 3$, then the inequality in (3) becomes an equality.

3.4. **Descriptive characterization of arbitrary trees via interior graphs.** In this subsection, we define the interior graphs of a tree T and prove that T is unmixed if and only if its interior graphs are unmixed; see Corollary 3.4.8. We combine this result with Theorem 3.3.12 to obtain our descriptive characterization of unmixed trees in Theorem 3.4.9. The interior graphs are also used to compute the Cohen-Macaulay type of open neighborhood ideals in Section 5.

Assumptions. In this subsection, we let T be a tree with a 2-coloring $\chi : V(T) \to \{\mathcal{R}, \mathcal{B}\}$.

Definition 3.4.1. We define the *blue interior graph* of T to be the subgraph $T_{\mathcal{B}}$ of T induced by the set

$$V \setminus N[V_1 \cap \chi^{-1}(\mathcal{B})]$$

where $N[U] := N(U) \cup U$ is the *closed neighborhood of* U for any $U \subseteq V$; i.e., T_B is the subgraph of T induced by deleting all blue support vertices together with their neighbors. Similarly, we define $T_{\mathcal{R}}$ to be the *red interior graph* of T induced by the set

$$V \setminus N[V_1 \cap \chi^{-1}(\mathcal{R})].$$

Example 3.4.2. Figure 13 shows a 2-colored tree T, and Figure 14 below shows its blue and red interior graphs. This example shows that the interior graphs can be forests.

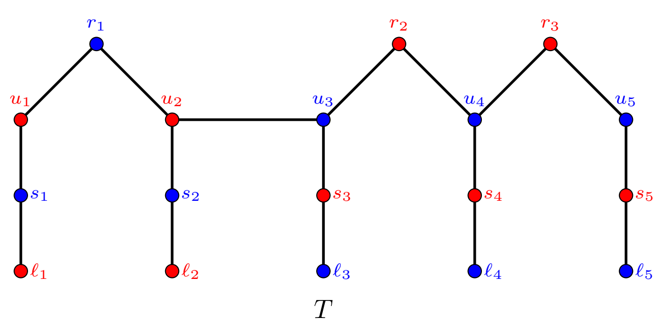


FIGURE 13. A 2-coloring of a tree T

Example 3.4.3. Consider the tree T in Figure 15. Its interior graphs $T_{\mathcal{B}}$ and $T_{\mathcal{R}}$ are shown in Figure 16. For this specific tree T, its interior graphs are isomorphic.

Each connected component of the interior graphs in Examples 3.4.3 and 3.4.2 is a balanced tree. This is true in general and is proved in the following lemma.

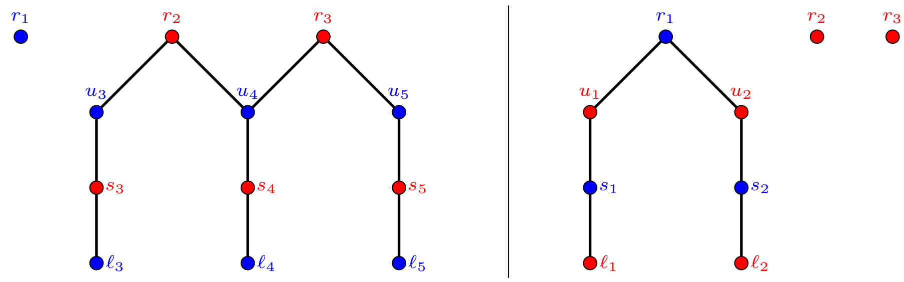


FIGURE 14. Interior graphs $T_{\mathcal{B}}$ (left) and $T_{\mathcal{R}}$ (right) of T from Figure 13

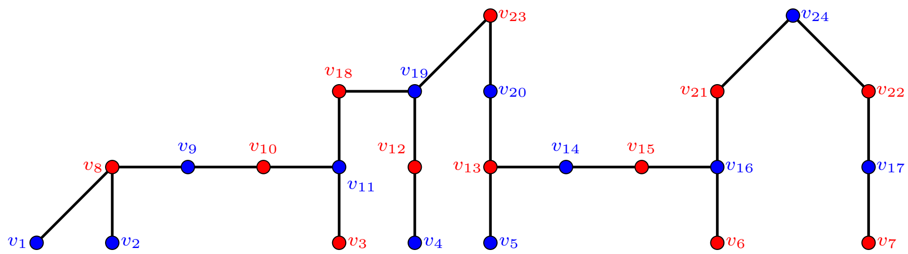


FIGURE 15. A tree T

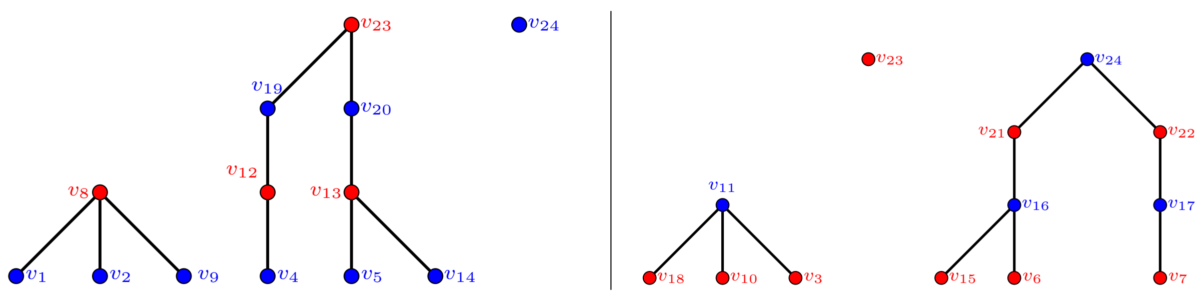


FIGURE 16. $T_{\mathcal{B}}$ (left) and $T_{\mathcal{R}}$ (right) of T from Figure 15

Lemma 3.4.4. *Every connected component of $T_{\mathcal{B}}$ and $T_{\mathcal{R}}$ is a balanced tree. Hence we say that $T_{\mathcal{B}}$ and $T_{\mathcal{R}}$ are balanced forests.*

Proof. Without loss of generality, consider the blue interior graph $T_{\mathcal{B}}$. By Definition 3.4.1, we delete all blue support vertices with their neighboring vertices which are red when we construct $T_{\mathcal{B}}$. Hence the leaves and the isolated vertices in $T_{\mathcal{B}}$ must be blue, implying that the connected components of $T_{\mathcal{B}}$ must be balanced trees by Proposition 3.2.4. $\square$

The following results demonstrate the importance of studying the unmixedness of the interior graphs of T to understand the unmixedness of T.

Lemma 3.4.5. *Let $D \subseteq V(T)$. Then D is a minimal BD-set in T if and only if*

$$D = (V_1(T) \cap V_{\mathcal{R}}(T)) \cup D'$$

where D' is a minimal BD-set in $T_{\mathcal{R}}$. Similarly, D is a minimal RD-set of T if and only if

$$D = (V_1(T) \cap V_{\mathcal{B}}(T)) \cup D'$$

where D' is a minimal RD-set in $T_{\mathcal{B}}$.

Proof. It suffices to prove the case for BD-sets. Write $V_{1,\mathcal{R}} := (V_1(T) \cap V_{\mathcal{R}}(T))$.

For one implication, we assume that D' is a minimal BD-set in $T_{\mathcal{R}}$ and show that $D := V_{1,\mathcal{R}} \cup D'$ is a minimal BD-set in T. The set of blue vertices of T that are not in $T_{\mathcal{R}}$ is $N_T(V_{1,\mathcal{R}})$ by Definition 3.4.1, so D is a BD-set in T. Let $\mathcal{D}'_{\mathcal{R}} : D' \to N_{T_{\mathcal{R}}}(D')$ be a domination selector of D' in $T_{\mathcal{R}}$, and let $\mathcal{D}'' : V_{1,\mathcal{R}} \to N_T(V_{1,\mathcal{R}})$ be the function defined by

$$\mathcal{D}''(s) = l_s$$

for some choice of $l_s \in V_0(T) \cap N_T(s)$. Since $T_{\mathcal{R}}$ is an induced subgraph, one can show that the function $\mathcal{D}' : D' \to N_T(D')$ given by

$$\mathcal{D}'(x) = \mathcal{D}'_{\mathcal{R}}(x)$$

is also a domination selector of D' in T. Next, note that we have

$$N_T(D') \cap N_T(V_{1,\mathcal{R}}) \subseteq N_T(V_{1,\mathcal{R}}) \setminus V_0(T).$$

Since $\mathcal{D}'(D') = \mathcal{D}'_{\mathcal{R}}(D') \subseteq V(T_{\mathcal{R}})$, we get $\mathcal{D}'(D') \cap (N_T(V_{1,\mathcal{R}}) \setminus V_0(T)) = \emptyset$ by construction of $T_{\mathcal{R}}$. On the other hand, since $\mathcal{D}''(V_{1,\mathcal{R}}) \subseteq V_0(T)$, we get $\mathcal{D}''(V_{1,\mathcal{R}}) \cap (N_T(V_{1,\mathcal{R}}) \setminus V_0(T)) = \emptyset$. Therefore, D is minimal by Lemma 3.1.11.

For the converse, suppose that D is a minimal BD-set of T. Then $V_{1,\mathcal{R}} \subseteq D$ since all blue leaves in D must be dominated. Set $D' := D \setminus V_{1,\mathcal{R}}$. We show that D' is a minimal BD-set of $T_{\mathcal{R}}$. Indeed we have $N_{T_{\mathcal{R}}}(D') = V_{\mathcal{B}}(T_{\mathcal{R}})$ since $N_T(V_{1,\mathcal{R}}) \cap V(T_{\mathcal{R}}) = \emptyset$ by Definition 3.4.1, and $N_T(D) = V_{\mathcal{B}}(T)$. If D' is not minimal in $T_{\mathcal{R}}$, then there exists a minimal BD-set $D'' \subsetneq D'$ in $T_{\mathcal{R}}$. But by the previous implication, $V_{1,\mathcal{R}} \cup D'' \subsetneq D$ becomes a minimal BD-set in T, contradicting the minimality of D. Thus D' must be a minimal BD-set in $T_{\mathcal{R}}$. $\square$

Here is a corollary of Lemma 3.4.5.

Corollary 3.4.6. *Every minimal RD-set in T has the same size if and only if every minimal RD-set in $T_{\mathcal{B}}$ has the same size. Similarly, every minimal BD-set in T has the same size if and only if every minimal BD-set in $T_{\mathcal{R}}$ has the same size.*

Lemma 3.4.7. *Let T be a balanced tree with blue leaves. Then T is unmixed if and only if every minimal RD-set in T has the same size.*

Proof. The forward implication is by Corollary 3.2.11. So, suppose that every minimal RD-set of T has the same size. Again by Corollary 3.2.11, it suffices to show that every BD-set of T has the same size; in particular, we show that T has a unique minimal BD-set. Let $D \subseteq V$ be a minimal BD-set. Since the leaves of T are blue, $V_1 \subseteq D$. Also, as every RD-set of T has the same size, we get height$(T) \leq 3$ by Theorem 3.3.7. Thus we have $V_{\mathcal{B}} = V_0 \cup V_2$. Thus $N(V_1) = V_{\mathcal{B}}$. By the minimality of D, we get $D = V_1$. $\square$

Corollaries 3.2.11 and 3.4.6 combine with Lemma 3.4.7 to establish our next result.

Corollary 3.4.8. *Let T be a tree with a 2-coloring. Then T is unmixed if and only if its interior graphs $T_{\mathcal{B}}$ and $T_{\mathcal{R}}$ are unmixed.*

We conclude this subsection with our descriptive characterization of unmixed trees. It follows directly from Corollary 3.4.8 and Theorem 3.3.12.

Theorem 3.4.9. *A tree T is unmixed if and only if every connected component T' of $T_{\mathcal{B}}$ and $T_{\mathcal{R}}$ has*

(1) height$(T') \leq 3$,

(2) for all $v \in V_2(T')$, we have $|N_{T'}(v) \cap V_1(T')| = 1$, and
(3) for all $v \in V_1(T')$, we have $|N_{T'}(v) \cap V_2(T')| \leq 1$.

3.5. **Constructing unmixed balanced trees.** The main result of this subsection, Theorem 3.5.6, gives a constructive characterization of unmixed balanced trees.

Assumptions. Set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For $n \in \mathbb{N}_0$, define $[n] := \{1, 2, \dots, n\}$ and $[n]_0 := \{0, 1, \dots, n\}$. For every $n \in \mathbb{N}_0$, we denote P_n to be the path graph with $V(P_n) = [n]_0$ and $E(P_n) = \{(0,1), (1,2), \dots, (n-1, n)\}$.

We apply the following operation to balanced trees below.

Definition 3.5.1. For two graphs $G_1 = (V', E')$ and $G_2 = (V'', E'')$ with vertices $v' \in V'$ and $v'' \in V''$, we set $e := v'v''$ and define the *edge join* of G_1 and G_2 by e to be the graph

$$G_1 +_e G_2 := (V' \cup V'', E' \cup E'' \cup \{v'v''\}).$$

We define an operation $\mathcal{O}$ as follows: Let $T = (V, E)$ be a tree, and let $v \in \bigcup_{i=1}^{3} V_i$.

(1) If $\text{height}(v) = 1$, then
$$\mathcal{O}(T, v) := T +_{(v,0)} P_0.$$
(2) If $\text{height}(v) = 2$, then
$$\mathcal{O}(T, v) := T +_{(v,3)} P_3.$$
(3) If $\text{height}(v) = 3$, then
$$\mathcal{O}(T, v) := T +_{(v,2)} P_2.$$

In other words, the operator $\mathcal{O}$ adds a "whisker" of length 1, 4, or 3 to T, depending on $\text{height}(v)$. In particular $\mathcal{O}(T, v)$ is a tree, so for $v_1 \in \bigcup_{i=1}^{3} V_i$ after appropriate relabeling so that $V(\mathcal{O}(T, v_1)) \cap \mathbb{N}_0 = \emptyset$, let $v_2 \in \bigcup_{i=1}^{3} V_i(\mathcal{O}(T, v_1))$, and set

$$\mathcal{O}(T, (v_1, v_2)) := \mathcal{O}(\mathcal{O}(T, v_1), v_2).$$

Inductively, given $n \in \mathbb{N}$ and $v_j \in \bigcup_{i=1}^{3} V_i\left(\mathcal{O}(T, (v_1, \dots, v_{j-1}))\right)$ for $j \in [n]$, we set

$$\mathcal{O}(T, (v_1, \dots, v_n)) := \mathcal{O}(\mathcal{O}(T, v_1, \dots, v_{n-1}), v_n).$$

Example 3.5.2. Consider the following graph T in Figure 17 that is isomorphic to P_6. The graph $\mathcal{O}(T, s_1)$ is isomorphic to the graph in Example 2.2.5. The graphs $\mathcal{O}(T, u_2)$, and

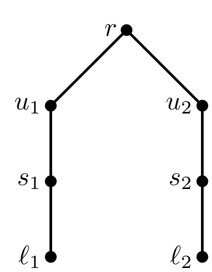


FIGURE 17. Graph $T \cong P_6$

$\mathcal{O}(T, r)$ are given in Figure 18.

Fact 3.5.3. Let T be a balanced tree of height 3. Let $v \in V \setminus V_0$. Then T is unmixed if and only if $\mathcal{O}(T, v)$ is unmixed, by Theorem 3.3.12, since $\mathcal{O}$ preserves the height, and conditions (2) and (3) of Theorem 3.3.12

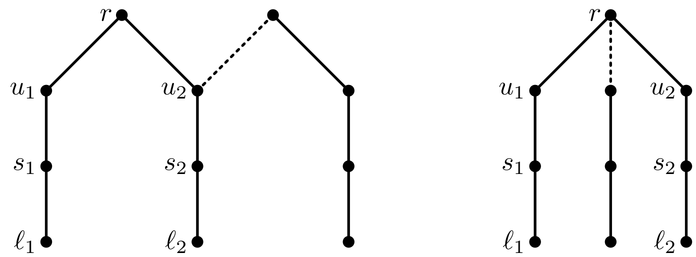


FIGURE 18. Graphs $\mathcal{O}(T, u_2)$ and $\mathcal{O}(T, r)$.

The proof of Theorem 3.3.12 shows that P_0 is the unique height-0 unmixed balanced tree, and every height-1 unmixed balanced tree is of the form $\mathcal{O}(P_2, (s, \dots, s))$ where $V_1(P_2) = \{s\}$. Since Theorems 3.3.7 and 3.3.9 show that an unmixed balanced tree must have height 0,1, or 3, it remains to construct every height-3 unmixed balanced tree. We begin this process with the following lemma; the graph in Figure 10 shows why T needs to be unmixed and balanced for the conclusion to hold.

Lemma 3.5.4. *Let $T = (V, E)$ be an unmixed balanced tree of height 3. Let $T' = (V', E')$ be the subgraph of T induced by the set $V_3 \cup V_2$. Then T' is a tree, i.e., T' is connected.*

Proof. Recall the following property of trees: Let G be a graph with a leaf ℓ. Then G is a tree if and only if $G \setminus \ell$ is a tree. Next, consider the subgraph T'' of T induced by the set $V_1 \cup V_2 \cup V_3$. Then T'' is obtained by only deleting the leaves of T, hence T'' is a tree. By Theorem 3.3.12, the vertices in $V_1(T)$ must be leaves in T''. Since T' can also be obtained by deleting the leaves of T'' which are the vertices in $V_1(T)$, T' is also a tree. □

The following corollary is critical for the proof of Theorem 3.5.6.

Corollary 3.5.5. *Let T be an unmixed balanced tree of height 3. Then there exists $u \in V_2$ such that $|N(u) \cap V_3| = 1$.*

Proof. First, we show that for all $u \in V_2$, we have $|N(u) \cap V_3| \geq 1$. Assume by way of contradiction that there is a vertex $c \in V_2$ with $|N(c) \cap V_3| = 0$. Since T balanced and height$(c) = 2$, we have $N(c) \subset V_1 \cup V_3$. But since T is unmixed, $|N(c) \cap V_1| = 1$ by Theorem 3.3.12. Thus $\deg(c) = 1$, so c is a leaf, contradicting that height$(c) = 2$.

Next, assume by way of contradiction that for all $u \in V_2$, we have $|N_T(u) \cap V_3| \geq 2$. Let $T' = (V', E')$ be the subgraph of T induced by the set $V_2 \cap V_3$. Since T is a balanced tree, there is no edge between vertices of same height. Hence every edge in T' is incident to a vertex in $V_2(T)$ and a vertex in $V_3(T)$; hence T' is a bipartite graph with partite sets $V_2(T)$ and $V_3(T)$. Thus we get

$$|E'| = \sum_{v \in V_2(T)} \deg_{T'}(v) = \sum_{v \in V_2(T)} |N_T(v) \cap V_3(T)|.$$

Then by assumption, we get

$$\begin{aligned} |E'| &= \sum_{v \in V_2(T)} |N_T(v) \cap V_3(T)| \\ &\geq \sum_{v \in V_2(T)} 2 \end{aligned}$$

$$\begin{aligned} &= 2 \cdot |V_2(T)| \\ &> |V_2(T)| + |V_3(T)|. && \text{(Lemma 3.3.3)} \end{aligned}$$

By Lemma 3.5.4, T' must be a tree. Hence we have

$$|E'| = |V'| - 1 = |V_2(T)| + |V_3(T)| - 1$$

contradicting the inequality $|E'| > |V_2(T)| + |V_3(T)|$. ☐

Now we are ready to give our constructive characterization of unmixed balanced trees of height 3.

Theorem 3.5.6. *Let T be an unmixed balanced tree of height 3. Then either $T \cong P_6$, or there exists a sequence of vertices $v_1, \dots, v_k$ in V such that $T \cong \mathcal{O}(P_6, (v_1, \dots, v_k))$.*

Proof. We prove the claim by minimal counterexample. So assume that T is an unmixed balanced tree of height 3 that can not be obtained by applying $\mathcal{O}$ to any other unmixed balanced tree of height 3. By Lemma 3.1.5, we may assume that every support vertex of T has exactly one leaf attach to it. By Corollary 3.5.5, there exists some vertex $u \in V_2$ such that $|N(u) \cap V_3| = 1$. Let $r \in N(u) \cap V_3$.

Suppose that $\deg(r) > 2$, and set T' to be the subgraph of T induced by $V \setminus \text{branch}_r(u)$. By assumption, the subgraph of T induced by the set $\text{branch}_r(u)$ is isomorphic to P_2, so we get $T \cong \mathcal{O}(T', r)$. But, T' is also an unmixed balanced tree of height 3 by Fact 3.5.3, a contradiction.

Now suppose that $\deg(r) = 2$. If $|V_3| > 1$, then set $N(r) =: \{u, u'\}$. Now let T' be a subgraph of T induced by $V \setminus \text{branch}_{u'}(r)$ (this time, the subgraph induced by $\text{branch}_{u'}(r)$ is isomorphic to P_3), we get $T \cong \mathcal{O}(T', u')$, a contradiction since T' is an unmixed balanced tree of height 3 by Fact 3.5.3. If $|V_3| = 1$, then $T \cong P_6$ by the assumption made by using Lemma 3.1.5. ☐

Example 3.5.7. Consider the following tree in Figure 19. Noticing that the cyan edges

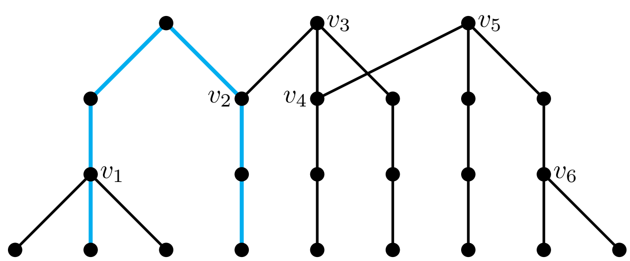


FIGURE 19. An unmixed balanced tree of height 3, T.

form a P_6, one can see that $T \cong \mathcal{O}(P_6, (v_1, v_1, v_2, v_3, v_4, v_5, v_6))$.

4. Shellability of Stable complexes

Our goal in this section is to prove that the Stanley-Reisner complex [16] of open neighborhood ideals of unmixed trees, called stable complexes, is shellable; see Theorem 4.2.15. In subsection 4.1, we define stable complexes of graphs and even-stable complexes of balanced trees, which correspond to the Stanley-Reisner complexes of open neighborhood ideals of graphs and odd-open neighborhood ideals of balanced trees, respectively. Then we show that stable complexes of trees are the join of even-stable complexes of its interior graphs in

Theorem 4.1.9. In subsection 4.2, we show that the even-stable complexes of unmixed balanced trees are shellable in Theorem 4.2.12, which implies the main theorem of this section.

4.1. **Stable complexes of graphs.** In this subsection, we define the stable complexes of graphs and the even-stable complexes of balanced trees; see Definitions 4.1.1 and 4.1.7. We then show that the stable complexes of trees are the join of the even-stable complexes of its interior graphs; see Theorem 4.1.9. To start, we review some basic notions of simplicial complexes.

Let V be a finite set. We say $\Delta \subseteq 2^V$ (where 2^V denotes the power set of V) is a *simplicial complex* if $F \in \Delta$ and $G \subset F$ implies $G \in \Delta$. Elements in Δ are called *faces*, and the maximal elements (under inclusion) in Δ are called *facets*. We denote the set of all facets of Δ by $\mathcal{F}(\Delta)$. If $\mathcal{F}(\Delta) = \{F_1, \ldots, F_m\}$, then we also write $\Delta = \langle F_1, \ldots, F_m \rangle$. In this case, we say Δ is *generated* by $F_1, \ldots, F_m$. The *dimension* of a face $F \in \Delta$ is $\dim(F) := |F| - 1$, and the *dimension* of Δ is $\dim(\Delta) := \max\{\dim(F) \mid F \in \Delta\}$. We say Δ is *pure* if every facet in Δ has the same dimension.

Definition 4.1.1. Let $G = (V, E)$ be a graph. A set $V' \subseteq V$ is *stable* if $V \setminus V'$ is a TD-set. We say V' is *maximal* if $V \setminus V'$ is minimal. The *stable complex* of G is the simplicial complex on V generated by the maximal stable sets of G:

$$\mathcal{S}(G) := \langle F \subseteq V \mid F \text{ is a maximal stable set} \rangle = \langle F \subseteq V \mid V \setminus F \text{ is a minimal TD-set} \rangle .$$

Stable sets and their properties appear in the transversal theory of hypergraphs; see [2] for more details.

Let Δ be a simplicial complex on $V = \{x_1, \ldots, x_n\}$. The *Stanley-Reisner ideal* associated with Δ is the square-free monomial ideal in $R := \Bbbk[V]$

$$I_\Delta := \langle X_F \mid F \notin \Delta \rangle .$$

On the other hand, let $I \subseteq R$ be a square-free monomial ideal. Then the *Stanley-Reisner (simplicial) complex* associated to I is the simplicial complex on V

$$\Delta_I := \{F \subseteq V \mid X_F \notin I\}$$

where we assume the product of elements in the empty set to be $1 \in R$. The following fact is a basic property of stable complexes. Its proof is a routine application of Theorem 2.2.6.

Fact 4.1.2. Let $G = (V, E)$ be a graph. Then $I_{\mathcal{S}(G)} = \mathcal{N}(G)$ and $\Delta_{\mathcal{N}(G)} = \mathcal{S}(G)$.

The following definition and fact are standard results in the theory of simplicial complexes.

Definition 4.1.3. Let Δ' and Δ'' be simplicial complexes on V' and V'', respectively, and assume that $V' \cap V'' = \emptyset$. The *join* of Δ' and Δ'' is the simplicial complex on $V' \cup V''$:

$$\Delta' * \Delta'' := \{F' \cup F'' \mid F' \in \Delta', F'' \in \Delta''\} .$$

Fact 4.1.4 ([3])**.** Let Δ' and Δ'' be simplicial complexes as in Definition 4.1.3. Then $\Delta' * \Delta''$ is shellable if and only if Δ' and Δ'' are shellable.

Hence, by Fact 4.1.4, if the stable complex $\mathcal{S}(T)$ of an unmixed tree T can be expressed as the join of two simplicial complexes, $\Delta' * \Delta''$, we only need to show that Δ' and Δ'' are shellable to prove that $\mathcal{S}(T)$ is shellable.

Definition 4.1.5. Let $T = (V, E)$ be a balanced tree. We denote the set of *even height* vertices in T by

$$V_{even}(T) := \{v \in V \mid \text{height}(v) \text{ is even}\}.$$

Similarly, we denote the set of *odd height* vertices in T by

$$V_{odd}(T) := \{v \in V \mid \text{height}(v) \text{ is odd}\}.$$

For $D \subseteq V$, we say D is an *odd-TD-set* if D is an V_{odd}-TD-set in T; see Definition 2.1.2.

The following fact is a consequence of Definition 4.1.5 and the properties of the interior graphs.

Fact 4.1.6. Let $T = (V, E)$ be a tree and let $T_{\mathcal{B}}$ and $T_{\mathcal{R}}$ be its interior graphs. Then

$$V_{even}(T_{\mathcal{B}}) = V_{\mathcal{B}}(T_{\mathcal{B}}) \text{ and } V_{odd}(T_{\mathcal{B}}) = V_{\mathcal{R}}(T_{\mathcal{B}}).$$

Similarly, we have

$$V_{even}(T_{\mathcal{R}}) = V_{\mathcal{R}}(T_{\mathcal{R}}) \text{ and } V_{odd}(T_{\mathcal{R}}) = V_{\mathcal{B}}(T_{\mathcal{R}}).$$

Hence odd-TD-sets in $T_{\mathcal{B}}$ and $T_{\mathcal{R}}$ correspond to the RD-sets and BD-sets (from Subsection 3.2) in $T_{\mathcal{R}}$ and $T_{\mathcal{B}}$, respectively.

Definition 4.1.7. Let $T = (V, E)$ be a balanced tree. The *even-stable complex* of T is the simplicial complex on $V_{even}(T)$:

$$\mathcal{S}_{even}(T) := \langle V_{even}(T) \setminus D \mid D \text{ is a minimal odd-TD-set} \rangle.$$

The following lemma is used to prove the main theorem of this subsection.

Lemma 4.1.8. *Let T be a tree with a 2-coloring χ. Then $V(T)$ can be partitioned as*

$$V(T) = V_{even}(T_{\mathcal{B}}) \cup V_{even}(T_{\mathcal{R}}) \cup V_1(T).$$

Proof. We will show that $V(T)$ is partitioned by $V_{even}(T_{\mathcal{B}})$, $V_{even}(T_{\mathcal{R}})$, and $V_1(T)$. The containment $V(T) \supseteq V_{even}(T_{\mathcal{B}}) \cup V_{even}(T_{\mathcal{R}}) \cup V_1(T)$ is clear by definition, so we only need to show the other containment. Let $v \in V(T)$. By symmetry, assume that v is blue. We consider two cases: $v \in V(T_{\mathcal{B}})$ or $v \notin V(T_{\mathcal{B}})$. First, suppose that $v \in V(T_{\mathcal{B}})$. Then by Corollary 3.2.6 and the construction of $V_{\mathcal{B}}$ (every leaf is blue color), $v \in V_{even}(T_{\mathcal{B}})$. Next, suppose that $v \notin V(T_{\mathcal{B}})$. Then by the construction of $T_{\mathcal{B}}$, v must be a blue support vertex in T since the blue vertices in T that are not in $T_{\mathcal{B}}$ are the blue support vertices of T. Thus $v \in V_1(T)$. Thus we get $V(T) \subseteq V_{even}(T_{\mathcal{B}}) \cup V_{even}(T_{\mathcal{R}}) \cup V_1(T)$.

Next, we show that the sets $V_{even}(T_{\mathcal{B}})$, $V_{even}(T_{\mathcal{R}})$, and $V_1(T)$ are pairwise disjoint. By Corollary 3.2.6, vertices in $V_{even}(T_{\mathcal{B}})$ are blue and vertices in $V_{even}(T_{\mathcal{R}})$ are red, giving us $V_{even}(T_{\mathcal{B}}) \cap V_{even}(T_{\mathcal{R}}) = \emptyset$. By symmetry, it suffices to show that $V_1(T) \cap V_{even}(T_{\mathcal{B}}) = \emptyset$. Let $v \in V_1(T)$. If $\chi(v) = \mathcal{B}$, then $v \notin V(T_{\mathcal{B}})$ by the construction of $T_{\mathcal{B}}$, and thus $v \notin V_{even}(T_{\mathcal{B}})$. If $\chi(v) = \mathcal{R}$, then $v \notin V_{even}(T_{\mathcal{B}})$ since vertices in $V_{even}(T_{\mathcal{B}})$ are blue. $\square$

Theorem 4.1.9. *Let $T = (V, E)$ be a tree. Then $\mathcal{S}(T) = \mathcal{S}_{even}(T_{\mathcal{B}}) * \mathcal{S}_{even}(T_{\mathcal{R}})$.*

Proof. We will only show that $\mathcal{F}(\mathcal{S}(T)) \subseteq \mathcal{F}(\mathcal{S}_{even}(T_{\mathcal{B}}) * \mathcal{S}_{even}(T_{\mathcal{R}}))$ (the reverse inclusion is proved similarly). Let $F \in \mathcal{F}(\mathcal{S}(T))$ be a facet. Then there exists a minimal TD-set $D \subseteq V(T)$ such that $F = V(T) \setminus D$. By Fact 3.2.9, there exist an RD-set $D' \subseteq V_{\mathcal{B}}(T)$ and a BD-set $D'' \subseteq V_{\mathcal{R}}(T)$ in T such that $D = D' \cup D''$. Then by Lemma 3.4.5 and Fact 4.1.6, there are minimal odd-TD-sets $D_1 \subseteq V_{even}(T_{\mathcal{B}})$ in $T_{\mathcal{B}}$ and $D_2 \subseteq V_{even}(T_{\mathcal{R}})$ in $T_{\mathcal{R}}$ such that

$$D' = V_{1,\mathcal{B}} \cup D_1 \text{ and } D'' = V_{1,\mathcal{R}} \cup D_2$$

where $V_{1,C} = V_1(T) \cap V_C(T)$ for all $C \in \{\mathcal{R}, \mathcal{B}\}$. Also, we have $V_1(T) = V_{1,\mathcal{B}} \cup V_{1,\mathcal{R}}$. Thus

$$\begin{aligned}
F &= V(T) \setminus D \\
&= V(T) \setminus (D' \cup D'') \\
&= V(T) \setminus ((V_{1,\mathcal{B}} \cup D_1) \cup (V_{1,\mathcal{R}} \cup D_2)) \\
&= (V_{even}(T_{\mathcal{B}}) \cup V_{even}(T_{\mathcal{R}})) \setminus (D_1 \cup D_2) && \text{(Lemma 4.1.8)} \\
&= (V_{even}(T_{\mathcal{B}}) \setminus D_1) \cup (V_{even}(T_{\mathcal{R}}) \setminus D_2).
\end{aligned}$$

By Definition 4.1.7, we have $V_{even}(T_{\mathcal{B}}) \setminus D_1 \in \mathcal{S}_{even}(T_{\mathcal{B}})$ and $V_{even}(T_{\mathcal{R}}) \setminus D_2 \in \mathcal{S}_{even}(T_{\mathcal{R}})$. □

4.2. **Shellability of even stable complexes.** In this subsection, we will prove that the even-stable complexes of unmixed balanced trees are shellable; see Theorem 4.2.12.

Definition 4.2.1 ([4, Definition 5.1.11])**.** Let Δ be a pure simplicial complex. An ordering $F_1, \dots, F_m$ of facets of Δ is a *shelling* if it satisfies one of the following equivalent conditions:

(i) For all i such that $1 < i \leq m$, the simplicial complex $\langle F_1, \dots, F_{i-1} \rangle \cap \langle F_i \rangle$ is pure of dimension $\dim(\Delta) - 1$.

(ii) for all i, j such that $1 \leq i < j \leq m$, there exist a vertex $v \in F_j \setminus F_i$ and an index $k \in [j-1]$ satisfying $F_j \setminus F_k = \{v\}$.

We say Δ is *shellable* if Δ has a shelling order.

The following equivalent definition of shellability can be proven using Definition 4.2.1.

Fact 4.2.2. Let Δ be a pure simplicial complex. Then Δ is shellable if and only if there is an ordering of facets $F_1, \dots, F_m$ such that for all $1 \leq i < j \leq m$, if $|F_i \cap F_j| < \dim(\Delta)$, then there exists $1 \leq k < j$ such that $F_i \cap F_j \subseteq F_k \cap F_j$ and $|F_k \cap F_j| = \dim(\Delta)$.

Throughout, we will use Fact 4.2.2 to show that a simplicial complex is shellable. We also use the following well-known result about shellable simplicial complexes; see [14, 20].

Fact 4.2.3. Let $|V| = n$ and $1 \leq k \leq n$. Then the simplicial complex $\langle S \subseteq V \mid |S| = k \rangle$ is shellable.

We first consider balanced trees of heights 0 and 1.

Lemma 4.2.4. *Let T be a balanced tree with height$(T) \leq 1$. Then $\mathcal{S}_{even}(T)$ is shellable.*

Proof. The proof of Theorem 3.3.12 implies that T is unmixed, and hence $\mathcal{S}_{even}(T)$ is pure. If height$(T) = 0$, then T is a graph with a single vertex, say v. Hence $\emptyset$ is its only TD-set by assumption, so $\mathcal{S}_{even}(T) = \langle \{v\} \rangle$, which is shellable. Suppose that height$(T) = 1$. Then T is a tree with a unique support vertex s and leaves $\{l_1, \dots, l_k\}$ for some $k > 1$. In particular, every minimal odd-TD-set of T is of the form $\{l_i\}$. Thus we get

$$\mathcal{F}(\mathcal{S}_{even}(T)) = \{V_0 \setminus \{l_i\} \mid 1 \leq i \leq k\} = \{V' \subseteq V_0 \mid |V'| = k - 1\}.$$

By the last equality above, $\mathcal{S}_{even}(T)$ is shellable by Fact 4.2.3. □

Next, we consider unmixed balanced trees of height 3. We use the following convention for the labeling of vertices in unmixed balanced trees of height 3.

Assumptions. Let $T = (V, E)$ be a height-3 unmixed balanced tree. Throughout this subsection, let $p := |V_1|$, $V_1 := \{s_1, \dots, s_p\}$. We set $V_2 := \{u_{1,1}, \dots, u_{p,1}\}$, where $s_i u_{i,1} \in E$ for all $i \in [p]$. (Note that $p = |V_1| = |V_2|$ by the proof of Theorem 3.3.11.) For all $i \in [p]$, set $k_i := |N(s_i)|$ and write $\{u_{i,2}, \dots, u_{i,k_i}\} := N(s_i) \cap V_0$.

Example 4.2.5. Figure 20 demonstrates a vertex labeling based on the assumptions. In

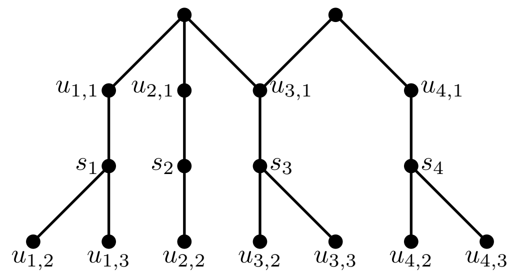


FIGURE 20. Vertex labeling of a height 3 unmixed balanced tree

this example, $p = |V_1| = |V_2| = 4$, $k_1 = |N(s_1)| = 3$, $k_2 = |N(s_2)| = 2$, $k_3 = 3$, and $k_4 = 3$.

To define an ordering on $\mathcal{F}(\mathcal{S}_{even}(T))$, we will enumerate the facets using integer vectors. We begin with a lemma which is essentially proved in Theorem 3.3.11.

Lemma 4.2.6. *Let T be an unmixed balanced tree of height 3, and let D be a minimal odd-TD-set. Let $s \in V$ be a support vertex. Then $|N(s) \cap D| = 1$. Therefore, every minimal odd-TD-set of T is of the form*

$$\left\{u_{1,a_1}, u_{2,a_2}, \ldots, u_{p,a_p}\right\}$$

where $1 \leq a_i \leq k_i$.

The following corollary is not used until Section 5.

Corollary 4.2.7. *Let $T = (V, E)$ be an unmixed balanced tree of height 3. Then every minimal TD-set of T is a union of V_1 and a minimal odd TD-set of T. Therefore by Lemma 4.2.6, every minimal TD-set of T is of the form*

$$V_1 \cup \left\{u_{1,a_1}, u_{2,a_2}, \ldots, u_{p,a_p}\right\}.$$

Proof. Use Lemma 3.4.7 and Fact 4.1.6. □

Definition 4.2.8. Let T be an unmixed balanced tree of height 3. Define

$$P_T := \{(a_1, \ldots, a_p) \mid \text{for all } i \in [p], 1 \leq a_i \leq k_i\}$$

and

$$U_T := \left\{V_{even} \setminus \left\{u_{1,a_1}, \ldots, u_{p,a_p}\right\} \mid 1 \leq a_i \leq k_i\right\}.$$

Define a bijection $\nu_T : U_T \to P_T$ by

$$\nu_T(V_{even} \setminus \left\{u_{1,a_1}, \ldots, u_{p,a_p}\right\}) = (a_1, \ldots, a_p).$$

For $(a_1, \ldots, a_p) \in P_T$, define

$$\ell(a_1, \ldots, a_p) = |\{i \mid a_i = 1, 1 \leq i \leq p\}|.$$

Define a total order $<_P$ on P_T as follows: for $\alpha, \beta \in P_T$, $\alpha <_P \beta$ if and only if

(1) $\ell(\alpha) > \ell(\beta)$, or
(2) $\ell(\alpha) = \ell(\beta)$ and $\alpha <_{lex} \beta$

where $<_{lex}$ is the lexicographic order; $\alpha <_{lex} \beta$ if and only if $a_1 < b_1$ or there is an index $1 < i \leq p$ such that $a_j = b_j$ for all $j \in [i-1]$ and $a_i < b_i$. Define a total order $<_T$ on U_T as follows: for all $F_1, F_2 \in U_T$, $F_1 <_T F_2$ if and only if $\nu_T(F_1) <_P \nu_T(F_2)$.

Example 4.2.9. Consider the sets

$$D_1 = \{u_{1,3}, u_{2,1}, u_{3,2}, u_{4,1}\} \quad \text{and} \quad D_2 = \{u_{1,1}, u_{2,2}, u_{3,1}, u_{4,2}\}$$

in Figure 21. Call this tree T, and set $F_i := V_{even} \setminus D_i$ for $i = 1, 2$. The sets D_1 and D_2 are

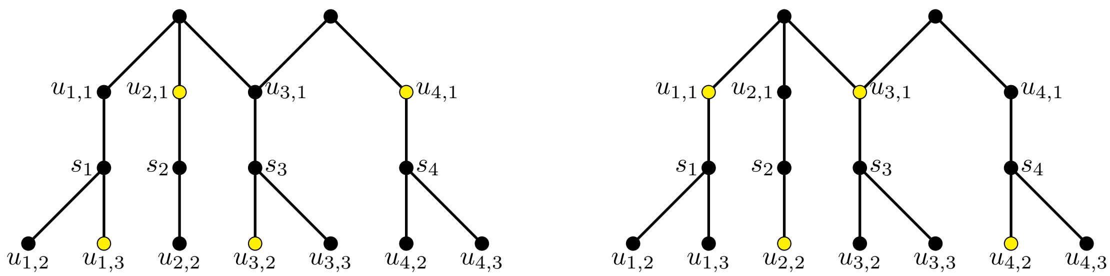


FIGURE 21. Two minimal odd-TD-sets D_1 (left) and D_2 (right)

minimal odd-TD-sets in T, and we have

$$\nu_T(F_1) = (3, 1, 2, 1) \quad \text{and} \quad \nu_T(F_2) = (1, 2, 1, 2).$$

Since $\ell(1, 2, 1, 2) = 2 = \ell(3, 1, 2, 1)$ and $(1, 2, 1, 2) <_{lex} (3, 1, 2, 1)$, we get $F_2 <_T F_1$.

We give some useful lemmas that will be used in the proof of Theorem 4.2.12.

Lemma 4.2.10. *Let T be an unmixed balanced tree of height 3. Let $F_1, F_2 \in \mathcal{F}(\mathcal{S}_{even}(T))$ with $\nu_T(F_1) = (a_1, \dots, a_p)$ and $\nu_T(F_2) = (b_1, \dots, b_p)$. Then*

$$|F_1 \cap F_2| = \dim(\mathcal{S}_{even}(T)) + 1 - |\{i \mid a_i \neq b_i, 1 \leq i \leq p\}|.$$

Proof. Set $I_0 := \{i \mid a_i = b_i, 1 \leq i \leq p\}$ and $I_1 := \{i \mid a_i \neq b_i, 1 \leq i \leq p\}$. Also, write $D_i = V_{even} \setminus F_i$ for $i = 1, 2$. Then by definition of ν_T, we get

$$D_1 = \left\{u_{1,a_1}, \dots, u_{p,a_p}\right\} \quad \text{and } D_2 = \left\{u_{1,b_1}, \dots, u_{p,b_p}\right\}.$$

Thus $|D_1 \cap D_2| = |I_0|$, and $|D_2| - |I_0| = |I_1|$. Hence we get

$$\begin{aligned}
|F_1 \cap F_2| &= |(V_{even} \setminus D_1) \cap (V_{even} \setminus D_2)| \\
&= |V_{even} \setminus (D_1 \cup D_2)| \\
&= |V_{even}| - |D_1 \cup D_2| && (D_1 \cup D_2 \subseteq V_{even}) \\
&= |V_{even}| - (|D_1| + |D_2| - |D_1 \cap D_2|) \\
&= |V_{even}| - |D_1| - |D_2| + |I_0| \\
&= |F_1| - |D_2| + |I_0| && (|F_1| = |V_{even}| - |D_1|) \\
&= |F_1| - |I_1| && (|D_2| - |I_0| = |I_1|) \\
&= \dim(\mathcal{S}_{even}(T)) + 1 - |\{i \mid a_i \neq b_i, 1 \leq i \leq p\}|.
\end{aligned}$$

□

In particular, for two facets $F_1, F_2 \in \mathcal{F}(\mathcal{S}_{even}(T))$, $|F_1 \cap F_2| = \dim(\mathcal{S}_{even}(T))$ if and only if $\nu_T(F_1)$ and $\nu_T(F_2)$ differ by exactly one entry.

Lemma 4.2.11. *Let T be an unmixed balanced tree of height 3, and let*
$$D = \left\{u_{1,a_1}, \dots, u_{p,a_p}\right\} \subseteq V$$
be a minimal odd-TD-set in T. Suppose that there exists some index i such that $a_i \neq 1$. Then for any $c \in [k_i]$, the set $D' = (D \setminus \{u_{i,a_i}\}) \cup \{u_{i,c}\}$ is also a minimal odd-TD-set.

Proof. First, suppose that $c \neq 1$. Then $u_{i,c}$ is a leaf adjacent to s_i. So, D' is obtained by replacing the leaf u_{i,a_i} by another leaf $u_{i,c}$ that is adjacent to the same support vertex s_i. Thus D' is also a minimal odd-TD-set. Next, suppose that $c = 1$. Since D is a minimal odd-TD-set, we get $N(s_i) \cap D = \{u_{i,a_i}\}$ by Lemma 4.2.6. Hence we have $N(D \setminus \{u_{i,a_i}\}) = V_{odd} \setminus \{s_i\}$. Since $N(u_{i,1}) \supseteq \{s_i\}$, we get
$$N(D') = N(D \setminus \{u_{i,a_i}\}) \cup N(u_{i,1}) = V_{odd}.$$
To show that D' is minimal, consider the function $\mathcal{D}' : D' \to N(D')$ given by
$$\mathcal{D}'(v) = \begin{cases} s_j & v = u_{j,a_j} \text{ where } j \neq i \\ s_i & v = u_{i,1} \end{cases}.$$
Then Lemma 4.2.6 implies that $\mathcal{D}'$ is a domination selector, therefore D' is minimal. □

We prove the main result of this subsection.

Theorem 4.2.12. *Let T be an unmixed balanced tree of height 3. Then the total order $<_T$ on U_T restricted to $\mathcal{F}(\mathcal{S}_{even}(T))$ is a shelling of $\mathcal{S}_{even}(T)$.*

Proof. Since $\mathcal{F}(\mathcal{S}_{even}(T)) \subseteq U_T$ by Lemma 4.2.6, $<_T$ is a total order on $\mathcal{F}(\mathcal{S}_{even}(T))$. Now index the facets in $\mathcal{S}_{even}(T)$ under $<_T$ and write
$$\mathcal{F}(\mathcal{S}_{even}(T)) = \{F_1, F_2, \dots, F_m\}.$$
To show that $<_T$ is a shelling, suppose that there exist indicies i, j with $1 \leq i < j \leq m$ such that $|F_i \cap F_j| < \dim(\mathcal{S}_{even}(T))$. We need to find a facet $F \in \mathcal{F}(\mathcal{S}_{even}(T))$ such that $F <_T F_j$, $F_i \cap F_j \subseteq F \cap F_j$, and $|F \cap F_j| = \dim(\mathcal{S}_{even}(T))$. Write $\nu_T(F_i) = (a_1, \dots, a_p)$ and $\nu_T(F_j) = (b_1, \dots, b_p)$. Since $F_i <_T F_j$, $\nu_T(F_i) <_P \nu_T(F_j)$, so there are two cases to consider.

Case 1: $\ell(\nu_T(F_i)) > \ell(\nu_T(F_j))$. Then $\nu_T(F_i)$ has more 1's then $\nu_T(F_j)$. Hence there exists some index k such that $a_k = 1$ and $b_k \neq 1$. Now consider
$$F = V_{even} \setminus \left\{u_{1,b_1}, \dots, u_{k-1,b_{k-1}}, u_{k,1}, u_{k+1,b_{k+1}}, \dots, u_{p,b_p}\right\} \in U_T$$
with
$$\nu_T(F) = (b_1, \dots, b_{k-1}, 1, b_{k+1}, \dots, b_p).$$
Then $F \in \mathcal{F}(\mathcal{S}_{even}(T))$ by Lemma 4.2.11. Since $\ell(\nu_T(F)) = \ell(\nu_T(F_j)) + 1$, we get $F <_T F_j$. Since we have
$$F_i \cap F_j = V_{even} \setminus \left(\left\{u_{1,a_1}, \dots, u_{p,a_p}\right\} \cup \left\{u_{1,b_1}, \dots, u_{p,b_p}\right\}\right)$$
and
$$\begin{aligned} F \cap F_j &= V_{evem} \setminus \left(\left\{u_{1,b_1}, \dots, u_{k-1,b_{k-1}}, u_{k,1}, u_{k+1,b_{k+1}}, \dots, u_{p,b_p}\right\} \cup \left\{u_{1,b_1}, \dots, u_{p,b_p}\right\}\right) \\ &= V_{even} \setminus \left(\{u_{k,1}\} \cup \left\{u_{1,b_1}, \dots, u_{p,b_p}\right\}\right) \end{aligned}$$
the choice of k giving $u_{k,1} = u_{k,a_k} \in \left\{u_{1,a_1}, \dots, u_{p,a_p}\right\}$ implies
$$\left(\{u_{k,1}\} \cup \left\{u_{1,b_1}, \dots, u_{p,b_p}\right\}\right) \subseteq \left(\left\{u_{1,a_1}, \dots, u_{p,a_p}\right\} \cup \left\{u_{1,b_1}, \dots, u_{p,b_p}\right\}\right)$$

hence $F_i \cap F_j \subseteq F \cap F_j$. Finally, we have $|F \cap F_j| = \dim(\mathcal{S}_{even}(T))$ by Lemma 4.2.10 as $\nu_T(F_j)$ and $\nu_T(F)$ only differ by one digit.

Case 2: $\ell(\nu_T(F_i)) = \ell(\nu_T(F_j))$ and $\nu_T(F_i) <_{lex} \nu_T(F_j)$. Since $\nu_T(F_i) <_{lex} \nu_T(F_j)$, there is an index $k \in [p]$ such that $a_k < b_k$ and $a_t = b_t$ for all $t < k$. Let

$$F = V_{even} \setminus \{u_{1,b_1}, \dots, u_{k-1,b_{k-1}}, u_{k,a_k}, u_{k+1,b_{k+1}}, \dots, u_{p,b_p}\} \in U_T$$

with

$$\nu_T(F) = (b_1, \dots, b_{k-1}, a_k, b_{k+1}, \dots, b_p).$$

Then $F \in \mathcal{F}(\mathcal{S}_{even}(T))$ by Lemma 4.2.11. To show that $F <_T F_j$, we consider two cases. If $a_k = 1$, then $\ell(\nu_T(F)) = \ell(\nu_T(F_j)) + 1$, hence $F <_T F_j$. If $a_k > 1$, then $\ell(\nu_T(F)) = \ell(\nu_T(F_j))$ and $\nu_T(F) <_{lex} \nu_T(F_j)$ as $a_k < b_k$, hence $F <_T F_j$. Similar to Case 1, we get $F_i \cap F_j \subseteq F \cap F_j$, and $|F \cap F_j| = \dim(\mathcal{S}_{even}(T))$ by Lemma 4.2.10. □

Example 4.2.13. We demonstrate Case 2 in the proof of Theorem 4.2.12 using the unmixed balanced tree from Example 4.2.5. Consider the sets F_1 and F_2 from Example 4.2.9. Since $F_2 <_T F_1$ and $|F_2 \cap F_1| = 3 < 6 = \dim(\mathcal{S}_{even}(T))$, we construct a facet $F \in \mathcal{F}(\mathcal{S}_{even}(T))$ such that $F_2 \cap F_1 \subseteq F \cap F_1$ and $|F \cap F_1| = 6$. Let $F \in U_T$ such that $\nu_T(F) = (1, 1, 2, 1)$. Then $F \in \mathcal{F}(\mathcal{S}_{even}(T))$ by Lemma 4.2.11, and

$$|F \cap F_1| = \dim(\mathcal{S}_{even}(T)) + 1 - 1 = \dim(\mathcal{S}_{even}(T))$$

by Lemma 4.2.10. Since

$$F_2 \cap F_1 = V_{even} \setminus (\{u_{1,3}, u_{2,1}, u_{3,2}, u_{4,1}\} \cup \{u_{1,1}, u_{2,2}, u_{3,1}, u_{4,2}\})$$

and

$$F \cap F_1 = V_{even} \setminus (\{u_{2,1}\} \cup \{u_{1,1}, u_{2,2}, u_{3,1}, u_{4,2}\})$$

we get $F_2 \cap F_1 \subseteq F \cap F_1$. Since $\ell(F) = 3$ and $\ell(F_2) = 2$, we get $F <_T F_2$.

Corollary 4.2.14. *Let T be an unmixed balanced forest. Then $\mathcal{S}_{even}(T)$ is shellable.*

Proof. Let $T_1, \dots, T_c$ be the connected components of T, which are all unmixed balanced trees. Then Lemma 4.2.4 and Theorem 4.2.12 implies that $\mathcal{S}_{even}(T_i)$ is shellable for all $i \in [c]$. A set $D = D_1 \cup \dots \cup D_c$ is a minimal odd-TD-set of T if and only if D_i is a minimal odd-TD-set of T_i for all i. Thus we get

$$\mathcal{S}_{even}(T) = \mathcal{S}_{even}(T_1) * \dots * \mathcal{S}_{even}(T_c)$$

therefore $\mathcal{S}_{even}(T)$ is shellable by Fact 4.1.4. □

Theorem 4.2.15. *Let T be an unmixed tree. Then $\mathcal{S}(T)$ is shellable.*

Proof. Let $T_{\mathcal{R}}$ and $T_{\mathcal{B}}$ be the interior graphs of T. Then by Corollary 4.2.14, $\mathcal{S}_{even}(T_{\mathcal{R}})$ and $\mathcal{S}_{even}(T_{\mathcal{B}})$ are shellable. By Theorem 4.1.9, we get $\mathcal{S}(T) = \mathcal{S}_{even}(T_{\mathcal{R}}) * \mathcal{S}_{even}(T_{\mathcal{B}})$, therefore $\mathcal{S}(T)$ is shellable. □

As a final remark, we get the following Cohen-Macaulay theorem.

Corollary 4.2.16. *Let T be an unmixed tree and let T' be an unmixed balanced tree. Then $\Bbbk[V(T)]/\mathcal{N}(T)$ and $\Bbbk[V_{even}(T')]/\mathcal{N}_{V_{odd}(T')}(T')$ are Cohen-Macaulay.*

Proof. The fact that the first quotient $\Bbbk[V(T)]/\mathcal{N}(T)$ is Cohen-Macaulay is a consequence of Theorem 4.2.15 and Fact 4.1.2 (see [4, Theorem 5.1.13]). For the second quotient, one can show that $\mathcal{N}_{V_{odd}(T')}(T')$ is the Stanley-Reisner ideal of $\mathcal{S}_{even}(T')$ using Theorem 2.2.2. □

Remark 4.2.17. Let T be a tree. The *suspension* of T is a tree ΣT obtained by adding a leaf to each vertex of T. It is well known that the edge ideal of ΣT, $\mathcal{I}(\Sigma T)$, is Cohen-Macaulay [18]. A *complete edge subdivision* of a graph G is a graph $\mathcal{E}(G)$ obtained by inserting a new vertex on each edge in G. It is not hard to see that $E(\Sigma T)$ is either a P_2 or a height 3 unmixed balanced tree. Moreover, we have

$$\mathcal{I}(\Sigma T) = \mathcal{N}_{V_{odd}}(\mathcal{E}(\Sigma T)).$$

Hence every Cohen-Macaulay edge ideal of trees arises as an odd-open neighborhood ideal of an unmixed balanced tree.

5. Cohen-Macaulay Type of unmixed trees

In this section, we give a graph theoretic way to compute the Cohen-Macaulay type of open neighborhood ideal of unmixed trees.

5.1. **Cohen-Macaulay type of balanced trees.**

In this subsection, we compute the type of $\mathcal{N}_{V_{\mathrm{odd}}}(T)$ where T is an unmixed balanced tree.

Assumptions. In this subsection, we assume that $T = (V, E)$ is an unmixed balanced tree, and set $R = \Bbbk[V_{\mathrm{even}}]$ and $(V_{\mathrm{even}})R =: \mathfrak{X}$, unless otherwise stated. Also, we write $\mathcal{N}_{\mathrm{odd}}(T) := \mathcal{N}_{V_{\mathrm{odd}}}(T)$.

To compute type$(R/\mathcal{N}_{\mathrm{odd}}(T))$, we first identify a maximal $(R/\mathcal{N}_{\mathrm{odd}}(T))$-regular sequence. First, we consider unmixed balanced trees of height 0 and 1.

Lemma 5.1.1. *Suppose that height$(T) \leq 1$. Let $Z \subseteq R$ be defined as followings:*

(1) If height$(T) = 0$, set $V = \{v\}$ and $Z := \{v\}$;

(2) If height$(T) = 1$, set $V_1 = \{s\}$, $V_0 = \{\ell_1, \ldots, \ell_{n_0}\}$ with $n_0 \geq 2$, and

$$Z := \{\ell_1 - \ell_j \mid 2 \leq j \leq n_0\}.$$

Then Z is a maximal $R/\mathcal{N}_{odd}(T)$-regular sequence in $\mathfrak{X}$.

Proof. (1) If height$(T) = 0$, then T is a graph with a single vertex, and we have $R = \Bbbk[v]$ and $\mathcal{N}_{\mathrm{odd}}(T) = 0$. Thus $R/\mathcal{N}_{\mathrm{odd}}(T) \cong R = \Bbbk[v]$ and $\{v\} = Z$ is a maximum regular sequence for R.

(2) Now suppose that height$(T) = 1$. We have $R = \Bbbk[V_{\mathrm{even}}] = \Bbbk[\ell_1, \ldots, \ell_{n_0}]$ and $\mathcal{N}_{\mathrm{odd}}(T) = \left\langle X_{N(s)} \right\rangle = \langle \ell_1 \cdots \ell_{n_0} \rangle$. Since the m-irreducible decomposition of $\mathcal{N}_{\mathrm{odd}}(T)$ is

$$\mathcal{N}_{\mathrm{odd}}(T) = \langle \ell_1 \cdots \ell_{n_0} \rangle = \bigcap_{i=1}^{n_0} \langle \ell_i \rangle,$$

we have $\dim(R/\mathcal{N}_{\mathrm{odd}}(T)) = n_0 - 1$. Then we have

$$\frac{R}{\mathcal{N}_{\mathrm{odd}}(T) + \langle Z \rangle} = \frac{\Bbbk[\ell_1, \ldots, \ell_{n_0}]}{\langle \ell_1 \cdots \ell_{n_0} \rangle + \langle \{\ell_1 - \ell_j \mid 2 \leq j \leq n_0\} \rangle} \cong \frac{\Bbbk[\ell_1]}{\langle \ell_1^{n_0} \rangle}.$$

Since the ideal $\langle \ell_1^{n_0} \rangle$ is an m-irreducible decomposition, we get that $\dim(\Bbbk[\ell_1]/\langle \ell_1^{n_0} \rangle) = 1 - 1 = 0$. Since $R/\mathcal{N}_{\mathrm{odd}}(T)$ is Cohen-Macaulay and $|Z| = n_0 - 1$, the elements in Z form a maximal $(R/\mathcal{N}_{\mathrm{odd}}(T))$-regular sequence. $\square$

Corollary 5.1.2. *Suppose that height*$(T) \leq 1$*. Then*

$$\mathit{depth}(R/\mathcal{N}_{\mathit{odd}}(T)) = \begin{cases} 1 & \text{if } \mathit{height}(T) = 0 \\ |V_0| - 1 & \text{if } \mathit{height}(T) = 1 \end{cases}.$$

Now we consider unmixed balanced trees of height 3. We begin with a vertex labeling that will be used throughout the proofs in this section for simplicity.

Notation 5.1.3. Assume T has height 3. For $\ell = 0, 1, 2, 3$, set $n_\ell := |V_\ell|$. Theorem 3.3.12 allows us to denote the vertices of T as follows:

(1) write $V_1 := \{s_1, \ \ldots \ , s_{n_1}\}$ and $V_3 := \{r_1, \ \ldots \ , r_{n_3}\}$;
(2) for $i \in [k]$ write u_i for the unique height 2 vertex adjacent to s_i; and
(3) write $\ell_{i,1}, \ \ldots \ , \ell_{i,m_i}$ for the leaves adjacent to s_i $(m_i \geq 1)$.

Since T is unmixed of height 3, we have $n_1 = n_2$.

Example 5.1.4. Using Notation 5.1.3, we can label the vertices of an unmixed balanced tree T of height 3 as in Figure 22. Here, $n_1 = |V_1| = 3$, $n_3 = |V_3| = 2$, $m_1 = 3$, $m_2 = 1$, and

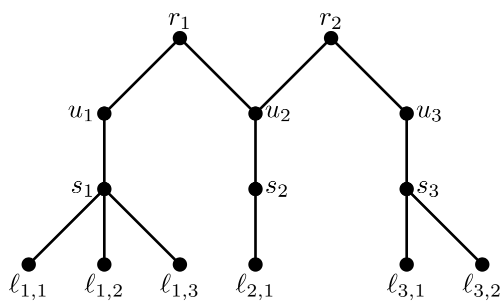


Figure 22. Example of a vertex labeling on a balanced tree using Notation 5.1.3.

$m_3 = 2$. The labeling rule (2) in Notation 5.1.3 is well defined by Theorem 3.3.12. Also, we have $|V_1| = |V_2| = n_2$ (for height 3 unmixed balanced trees), $|V_3| = n_3$, and for $i \in [n_1]$, there are m_i leaves adjacent to s_i. Thus, we have

$$|V_{\text{even}}| = n_2 + \sum_{i=1}^{n_2} m_i = n_2 + n_0 \tag{1}$$

e.g., for the tree in Figure 22, we have

$$9 = |V_{\text{even}}| = 3 + (3 + 1 + 2) = 3 + 6.$$

We begin with a short lemma about the depth of $R/\mathcal{N}_{\text{odd}}(T)$ when height$(T) = 3$.

Lemma 5.1.5. *Let height*$(T) = 3$*. Using Notation 5.1.3, we have*

$$\mathit{depth}(R/\mathcal{N}_{\mathit{odd}}(T)) = n_0.$$

Proof. Recall that $R = \Bbbk[V_{\text{even}}]$. Since T is unmixed, every minimal odd-TD-set of T has the same size; moreover, they all have size n_2 since the set V_2 is a minimal odd-TD-set of T. Thus we get

$$\begin{aligned} \text{depth}(R/\mathcal{N}_{\text{odd}}(T)) &= \dim(R/\mathcal{N}_{\text{odd}}(T)) \\ &= |V_{\text{even}}| - n_2 \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n_2} m_i \\
&= \sum_{i=1}^{n_1} m_i && (n_1 = n_2) \\
&= n_0.
\end{aligned}
$$

□

Now we exhibit a maximal $R/\mathcal{N}_{\text{odd}}(T)$-regular sequence when height$(T) = 3$.

Lemma 5.1.6. *Let height$(T) = 3$. Set*

$$Z := \{u_i - \ell_{i,j} \mid 1 \leq i \leq n_1, 1 \leq j \leq m_i\} \subseteq R$$

Then Z is a maximal $R/\mathcal{N}_{odd}(T)$-regular sequence in $\mathfrak{X}$.

Proof. Set $R' := \Bbbk[u_1, \ldots, u_{n_2}]$. Then we have

$$\frac{R}{\mathcal{N}_{\text{odd}}(T) + \langle Z \rangle} \cong \frac{R'}{\left(u_1^{|N(s_1)|}, \ldots, u_{n_2}^{|N(s_{n_1})|}, X_{N(r_1)}, \ldots, X_{N(r_{n_3})}\right) R'} \tag{$*$}$$

because every leaf $\ell_{i,j}$ is identified with its closest height 2 vertex u_i and $n_1 = |V_1| = |V_2| = n_2$ since T is an unmixed balanced tree of height 3. Set

$$\mathfrak{I} := \left(u_1^{|N(s_1)|}, \ldots, u_{n_1}^{|N(s_{n_1})|}, X_{N(r_1)}, \ldots, X_{N(r_{n_3})}\right) R'.$$

Since $\mathfrak{I}$ contains positive powers of all the u_i's, and the $X_{N(r_i)}$ are non-unit monomials, we get m-rad$(\mathfrak{I}) = (u_1, \ldots, u_{n_1})R'$. This implies that $\dim(R'/\mathfrak{I}) = 0$. Since $R/\mathcal{N}_{\text{odd}}(T)$ is Cohen-Macaulay of depth n_0 by Corollary 4.2.16 and Lemma 5.1.5, the condition

$$|Z| = \sum_{i=1}^{n_2} m_i = n_0,$$

implies that Z is a maximal regular sequence. □

Example 5.1.7. Let T be the tree from Example 5.1.4 and let $R = \Bbbk[V_{\text{even}}]$; see Figure 23, some vertices are colored for guidance. Then the maximal $R/\mathcal{N}_{\text{odd}}(T)$-regular sequence from

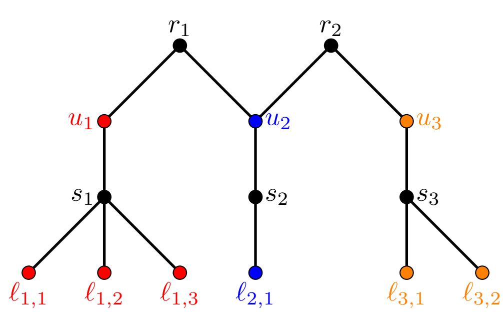


FIGURE 23. Tree T from Example 5.1.4

Lemma 5.1.6 is

$$Z = \{u_1 - \ell_{1,1}, u_1 - \ell_{1,2}, u_1 - \ell_{1,3}, u_2 - \ell_{2,1}, u_3 - \ell_{3,1}, u_3 - \ell_{3,2}\}.$$

We have

$$\mathcal{N}_{\text{odd}}(T) = \langle u_1\ell_{1,1}\ell_{1,2}\ell_{1,3}, u_2\ell_{2,1}, u_3\ell_{3,1}\ell_{3,2}, u_1u_2, u_2u_3\rangle$$

hence

$$\frac{R}{\mathcal{N}_{\text{odd}}(T)+\langle Z\rangle} = \frac{\Bbbk[u_1,u_2,u_3,\ell_{1,1},\ell_{1,2},\ell_{1,3},\ell_{2,1},\ell_{3,1},\ell_{3,2}]}{\langle u_1\ell_{1,1}\ell_{1,2}\ell_{1,3}, u_2\ell_{2,1}, u_3\ell_{3,1}\ell_{3,2}, u_1u_2, u_2u_3\rangle+\langle Z\rangle}$$

$$= \frac{\Bbbk[u_1,u_2,u_3]}{\langle {u_1}^4, {u_2}^2, {u_3}^3, u_1u_2, u_2u_3\rangle}.$$

The ideal $\langle {u_1}^4, {u_2}^2, {u_3}^3, u_1u_2, u_2u_3\rangle$ is the ideal $\mathfrak{J}$ from the proof of Lemma 5.1.6.

Now we compute the type of $R/\mathcal{N}_{\text{odd}}(T)$ when T is an unmixed balanced tree. We first take care of the cases when $\text{height}(T) < 3$.

Theorem 5.1.8. *If height$(T) < 3$, then type$(R/\mathcal{N}_{odd}(T)) = 1$.*

Proof. First, let $\text{height}(T) = 0$; hence we write $V = \{v\}$. Then $R = \Bbbk[v]$ and $\mathcal{N}_{\text{odd}}(T) = 0$. Hence we have

$$\begin{aligned}\text{type}(R/\mathcal{N}_{\text{odd}}(T)) &= \text{type}(R) && (\mathcal{N}_{\text{odd}}(T)=0)\\ &= \text{type}(R/\langle v\rangle) = 1.\end{aligned}$$

Now suppose that $\text{height}(T) = 1$. Let Z be the maximal $R/\mathcal{N}_{\text{odd}}(T)$-regular sequence from Lemma 5.1.1. Then we have

$$\begin{aligned}\text{type}(R/\mathcal{N}_{\text{odd}}(T)) &= \text{type}\left(\frac{R}{\mathcal{N}_{\text{odd}}(T)+\langle Z\rangle}\right)\\ &= \text{type}(\Bbbk[\ell_{1,1}]/\langle \ell_{1,1}^{m_1}\rangle) && \text{(Lemma 5.1.1 (1))}\\ &= 1\end{aligned}$$

as desired. ☐

Now we consider T with $\text{height}(T) = 3$. The ideal $\mathfrak{J}$ from Lemma 5.1.6 has a parametric decomposition in $\Bbbk[u_1,\ldots,u_{n_2}]$ since $\text{m-rad}(\mathfrak{J}) = \langle u_1,\ldots,u_{n_2}\rangle$. Lemma 5.1.9 gives us the decomposition of $\mathfrak{J}$ explicitly.

Lemma 5.1.9. *Assume that height$(T) = 3$. Let $\mathfrak{J} \leq R' = \Bbbk[u_1,\ldots,u_{n_2}]$ be the ideal from the proof of Lemma 5.1.6. Set*

$$U := \left(u_1^{|N(s_1)|}, \ \ldots \ , u_{n_2}^{|N(s_{n_1})|}\right)R'.$$

The irredundant parametric decomposition of $\mathfrak{J}$ in R' is

$$\mathfrak{J} = \bigcap_{D \ min} (\langle D\rangle + U)$$

where the intersection is taken over all minimal V_3-TD-sets of T.

Proof. Notice that the ideal $\left(X_{N(r_1)}, \dots, X_{N(r_{n_3})}\right) R'$ is the V_3-open neighborhood ideal of T since $V_3 = \{r_1, \dots, r_{n_3}\}$. Thus by Theorem 2.2.2, we get

$$\left(X_{N(r_1)}, \dots, X_{N(r_{n_3})}\right) R' = \bigcap_{D \ min} \langle D \rangle$$

where the intersection is taken over all minimal V_3-TD-set of T. Thus we get

$$\mathfrak{I} = \left(X_{N(r_1)}, \dots, X_{N(r_{n_3})}\right) R' + U = \left(\bigcap_{D \ min} \langle D \rangle\right) + U = \bigcap_{D \ min} \left(\langle D \rangle + U\right)$$

as desired. $\square$

Example 5.1.10. Let $T = (V, E)$ be the tree from Example 5.1.4 and let $\mathfrak{I}$ be the ideal from Example 5.1.7. Setting

$$U = \left\langle u_1^{|N(s_1)|}, u_2^{|N(s_2)|}, u_3^{|N(s_3)|} \right\rangle = \left\langle u_1^4, u_2^2, u_3^3 \right\rangle,$$

we get

$$\begin{aligned} \mathfrak{I} &= \left\langle u_1^4, u_2^2, u_3^3, u_1u_2, u_2u_3 \right\rangle \\ &= \left\langle u_1^4, u_2^2, u_3^3, u_1, u_3 \right\rangle \cap \left\langle u_1^4, u_2^2, u_3^3, u_2 \right\rangle \\ &= \left(\langle u_1, u_3 \rangle + U\right) \cap \left(\langle u_2 \rangle + U\right). \end{aligned}$$

Sets $\{u_1, u_3\}$ and $\{u_2\}$ are minimal V_3-TD-sets of T.

Now we can describe the type$(R/\mathcal{N}_{\text{odd}}(T))$ for an unmixed balanced tree of any height using minimal V_3-TD-sets.

Theorem 5.1.11. *Let $R = \Bbbk[V_{even}]$. Then*

$$\textit{type}(R/\mathcal{N}_{odd}(T)) = \text{“}\textit{number of minimal } V_3\textit{-TD-sets of } T\text{.”}$$

Proof. If height$(T) < 3$, then the empty set is the unique minimal V_3-TD-set, so

$$\begin{aligned} \text{type}(R/\mathcal{N}_{\text{odd}}(T)) &= 1 && \text{(Theorem 5.1.8)} \\ &= |\,\{\emptyset\}\,| \\ &= \text{“number of minimal } V_3\text{-TD-sets of } T\text{.”} \end{aligned}$$

Now suppose that height$(T) = 3$. Let Z, R', and $\mathfrak{I}$ as in Lemma 5.1.6. Then we have

$$\begin{aligned} \text{type}(R/\mathcal{N}_{\text{odd}}(T)) &= \text{type}\left(\frac{R}{\mathcal{N}_{\text{odd}}(T) + \langle Z \rangle}\right) \\ &= \text{type}\left(R'/\mathfrak{I}\right) && \text{((}*\text{) in Lemma 5.1.6)} \\ &= \text{“number of ideals in the parametric decomp. of } \mathfrak{I}\text{”} \\ &= \text{“number of minimal } V_3\text{-TD-sets of } T\text{”} \end{aligned}$$

as desired. $\square$

Example 5.1.12. Let $T = (V, E)$ be the tree from Example 5.1.4. Then by Theorem 5.1.11 with Example 5.1.10, we get

$$\operatorname{type}(\Bbbk[V_{\text{even}}]/\mathcal{N}_{\text{odd}}(T)) = \text{“number of minimal } V_3\text{-TD-set of } T\text{”} = 2.$$

5.2. **Cohen-Macaulay Type of Unmixed Trees.**

Assumptions. Let $T = (V, E)$ be an unmixed tree and let $T_{\mathcal{R}}$ and $T_{\mathcal{B}}$ be the two interior trees (forests) of T derived from a 2-coloring $\chi : V \to \{\mathcal{R}, \mathcal{B}\}$. Set $N_{\mathcal{B}}(v) := N_{T_{\mathcal{B}}}(v)$ and $N_{\mathcal{R}}(v) := N_{T_{\mathcal{R}}}(v)$ for all $v \in V$.

Theorem 5.2.1 gives a connection between open neighborhood ideal of T and the odd open neighborhood ideals of its interior graphs.

Theorem 5.2.1. *We have*

$$\begin{aligned}\mathcal{N}(T) &= \mathcal{N}_{odd}(T_{\mathcal{B}})R + \mathcal{N}_{odd}(T_{\mathcal{R}})R + \langle V_1(T)\rangle \\ &= \mathcal{N}_{odd}(T_{\mathcal{B}})R + \mathcal{N}_{odd}(T_{\mathcal{R}})R + (V_1(T))R.\end{aligned}$$

Proof. ($\subseteq$) It suffices to show that $X_{N_T(v)} \in \mathcal{N}_{\text{odd}}(T_{\mathcal{B}})R + \mathcal{N}_{\text{odd}}(T_{\mathcal{R}})R + \langle V_1(T)\rangle$ for all $v \in V$. Let $v \in V$. By symmetry, assume that $\chi(v) = R$.

Case 1: $v \notin V(T_{\mathcal{B}})$. Then by Definition 3.4.1, there exists some blue support vertex $s \in V(T)$ that is adjacent to v. Suppose that v is a leaf in T. Then $N_T(v) = \{s\}$. Since $s \in \langle V_1(T)\rangle$, we have $X_{N_T(v)} = s \in \langle V_1(T)\rangle$. Now suppose that v is not a leaf. Since s is a support vertex, there exists some leaf $\ell \in V(T)$ that is adjacent to s. Then $X_{N_T(\ell)} = s$ divides $X_{N_T(v)}$ since $s \in N_T(v)$. Hence $X_{N_T(v)}$ is a redundant generator of $\mathcal{N}(T)$.

Case 2: $v \in V(T_{\mathcal{B}})$. Then $N_{T_{\mathcal{B}}}(v) \subseteq N_T(v)$. Thus $X_{N_{\mathcal{B}}(v)} | X_{N_T(v)}$, so $X_{N_T(v)} \in \mathcal{N}(T_{\mathcal{B}})$.

($\supseteq$) Every support vertex of T is a generator of $\mathcal{N}(T)$, being the open neighborhood of each leaf it is adjacent to, so we get $\langle V_1(T)\rangle \subseteq \mathcal{N}(T)$. To complete the proof, by symmetry, we show that $\mathcal{N}(T) \supseteq \mathcal{N}_{\text{odd}}(T_{\mathcal{B}})$. Note that all we need to show is that $X_{N_{\mathcal{B}}(v)} \in \mathcal{N}(T)$ for all $v \in V_{\text{odd}}(T_{\mathcal{B}})$ as this shows that the generators of $\mathcal{N}_{\text{odd}}(T_{\mathcal{B}})$ are in $\mathcal{N}(T)$. So, let $v \in V_{\text{odd}}(T_{\mathcal{B}})$. We show that $N_{\mathcal{B}}(v) = N_T(v)$ which will imply that $X_{N_{\mathcal{B}}(v)} = X_{N_T(v)} \in \mathcal{N}(T)$. To this end, assume by way of contradiction that $N_{\mathcal{B}}(v) \subsetneq N_T(v)$. Since $\operatorname{height}(v) \in \{1, 3\}$ by the definition of $V_{\text{odd}}(T_{\mathcal{B}})$ along with Theorem 3.3.12 and 3.4.9, we get $\chi(v) = \mathcal{R}$ as $v \in V(T_{\mathcal{B}})$. Since $N_{\mathcal{B}}(v) \subsetneq N_T(v)$, there exists a vertex $u \in V$ such that $u \in N_T(v) \setminus V(T_{\mathcal{B}})$ and $uv \in E(T)$. Since $\chi(v) = R$, we have $\chi(u) = B$. Since $u \notin V(T_{\mathcal{B}})$, u must be a vertex that is deleted from T while constructing $T_{\mathcal{B}}$. By definition of $T_{\mathcal{B}}$, u must be a support vertex since $\chi(u) = B$. But, this implies that $v \notin V(T_{\mathcal{B}})$ since v must be deleted from T while constructing $T_{\mathcal{B}}$ as $v \in N_T(u)$, a contradiction. □

Example 5.2.2. Let T, $T_{\mathcal{B}}$, and $T_{\mathcal{R}}$ be the graphs from Example 3.4.2 and set $R = \Bbbk[V]$. Then we have

$$\begin{aligned}\mathcal{N}_{\text{odd}}(T_{\mathcal{B}}) &= \langle \ell_3u_3, \ell_4u_4, \ell_5u_5, u_3u_4, u_4u_5\rangle\,, \\ \mathcal{N}_{\text{odd}}(T_{\mathcal{R}}) &= \langle \ell_1u_1, \ell_2u_2, u_1u_2\rangle\,, \\ (V_1(T))R &= \langle s_1, s_2, s_3, s_4, s_5\rangle\,.\end{aligned}$$

The variables are colored using the same coloring of T from Example 3.4.2. The open neighborhood ideal of T is given by

$$\mathcal{N}(T) = \langle s_1, s_2, s_3, s_4, s_5, \ell_1u_1, \ell_2u_2, \ell_3u_3, \ell_4u_4, \ell_5u_5, s_1r_1, s_2u_3r_1, s_3u_2r_2, s_4r_2r_3,$$

$$s_5r_3, u_1u_2, u_3u_4, u_4u_5\rangle$$
$$= \langle s_1, s_2, s_3, s_4, s_5, \ell_1u_1, \ell_2u_2, \ell_3u_3, \ell_4u_4, \ell_5u_5, u_1u_2, u_3u_4, u_4u_5\rangle\ .$$

Notice that the minimal set of generators of $\mathcal{N}(T)$ is exactly the union of the generating sets for $\mathcal{N}_{\text{odd}}(T_{\mathcal{B}})$, $\mathcal{N}_{\text{odd}}(T_{\mathcal{R}})$, and $(V_1(T))R$. Thus we get

$$\mathcal{N}(T) = \mathcal{N}_{\text{odd}}(T_{\mathcal{B}}) + \mathcal{N}_{\text{odd}}(T_{\mathcal{R}}) + (V_1(T))R.$$

By the construction of the interior graphs $T_{\mathcal{B}}$ and $T_{\mathcal{R}}$, we have $\mathcal{N}_{\text{odd}}(T_{\mathcal{B}}) \subseteq (\chi^{-1}(B) \setminus V_1(T))R$ and $\mathcal{N}_{\text{odd}}(T_{\mathcal{R}}) \subseteq (\chi^{-1}(R) \setminus V_1(T))R$; that is, the variables of the generators of $\mathcal{N}_{\text{odd}}(T_{\mathcal{B}})$ and $\mathcal{N}_{\text{odd}}(T_{\mathcal{R}})$ are blue and red, respectively. Hence the ideals $\mathcal{N}_{\text{odd}}(T_{\mathcal{B}})$, $\mathcal{N}_{\text{odd}}(T_{\mathcal{R}})$, and $(V_1(T))R$ use pairwise disjoint sets of variables for their generators.

The following fact is used for Corollary 5.2.4 which shows that the odd open neighborhood ideal of an unmixed balanced forest is also Cohen-Macaulay. The depth computation is used in Theorem 5.2.5.

Fact 5.2.3. Let T_1 and T_2 be unmixed balanced trees with $V(T_1) \cap V(T_2) = \emptyset$. Let $T = (V(T_1) \cup V(T_2), E(T_1) \cup E(T_2))$. Then $\Bbbk[V_{\text{even}}(T)]/\mathcal{N}_{\text{odd}}(T)$ is Cohen-Macaulay over any field, and

$$\operatorname{depth}\left(\frac{\Bbbk[V_{\text{even}}(T)]}{\mathcal{N}_{\text{odd}}(T)}\right) = \operatorname{depth}\left(\frac{\Bbbk[V_{\text{even}}(T_1)]}{\mathcal{N}_{\text{odd}}(T_1)}\right) + \operatorname{depth}\left(\frac{\Bbbk[V_{\text{even}}(T_2)]}{\mathcal{N}_{\text{odd}}(T_2)}\right).$$

Corollary 5.2.4. *Let T be an unmixed balanced forest with connected components $T_1,\dots,T_c$. Then $\Bbbk[V_{even}(T)]/\mathcal{N}_{odd}(T)$ is Cohen-Macaulay and*

$$\operatorname{depth}\left(\frac{\Bbbk[V_{even}(T)]}{\mathcal{N}_{odd}(T)}\right) = \sum_{i=1}^{c} \operatorname{depth}\left(\frac{\Bbbk[V_{even}(T_i)]}{\mathcal{N}_{odd}(T_i)}\right).$$

Proof. Induct on c, and apply Lemma 5.2.3. $\square$

Next, we state a result on a maximal $\Bbbk[V_{\text{even}}]/\mathcal{N}_{\text{odd}}(T)$-regular sequence of an unmixed balanced forest T that is used in Theorem 5.2.7.

Theorem 5.2.5. *Let T be an unmixed balanced forest with connected components $T_1,\dots,T_c$. Let $R = \Bbbk[V_{even}]$ and $R_i = \Bbbk[V_{even}(T_i)]$ for all $i \in [c]$. For each $i \in [c]$, using Notation 5.1.3 for each T_i, let Z_i be the maximal regular sequence constructed in Lemmas 5.1.1 and 5.1.6. Set $Z = \bigcup_{i=1}^{c} Z_i$. Then*

(1) Z is a maximal $R/\mathcal{N}_{odd}(T)$-regular sequence in $\langle V_{even}\rangle$.
(2) type$(R/\mathcal{N}_{odd}(T))$ = "number of minimal V_3-TD-sets in T."

Proof. Since T is a forest with connected components $T_1,\dots,T_c$, we have $V_{\text{odd}} = \bigcup_{i=1}^{c} V_{\text{odd}}(T_i)$ and $V_{\text{even}} = \bigcup_{i=1}^{c} V_{\text{even}}(T_i)$. So, we get

$$\begin{aligned}
\mathcal{N}_{\text{odd}}(T) &= \left\langle \left\{ X_{N_T(v)} \mid v \in V_{\text{odd}}(T) \right\} \right\rangle \\
&= \left\langle \bigcup_{i=1}^{c} \left\{ X_{N_{T_i}(v)} \mid v \in V_{\text{odd}}(T_i) \right\} \right\rangle \\
&= \sum_{i=1}^{c} \left\langle \left\{ X_{N_{T_i}(v)} \mid v \in V_{\text{odd}}(T_i) \right\} \right\rangle \\
&= \sum_{i=1}^{c} \mathcal{N}_{\text{odd}}(T_i).
\end{aligned}$$

Hence we have

$$\frac{R}{\mathcal{N}_{\text{odd}}(T) + \langle Z \rangle} = \frac{R}{\sum_{i=1}^{c} \mathcal{N}_{\text{odd}}(T_i) + \sum_{j=1}^{c} \langle Z_j \rangle}$$

$$= \frac{R}{\sum_{i=1}^{c} \left(\mathcal{N}_{\text{odd}}(T_i) + \langle Z_i \rangle\right)}$$

$$= \bigotimes_{i=1}^{c} \frac{R_i}{\mathcal{N}_{\text{odd}}(T_i) + \langle Z_i \rangle}$$

where the tensor product is taken over $\Bbbk$. Since $\dim\left(R_i/(\mathcal{N}_{\text{odd}}(T_i) + \langle Z_i \rangle)\right) = 0$ for all i, we get $\dim(R/(\mathcal{N}_{\text{odd}}(T) + \langle Z \rangle)) = 0$. Note that we have

$$\operatorname{depth}\left(\frac{\Bbbk[V_{\text{even}}(T)]}{\mathcal{N}_{\text{odd}}(T)}\right) = \sum_{i=1}^{c} \operatorname{depth}\left(\frac{\Bbbk[V_{\text{even}}(T_i)]}{\mathcal{N}_{\text{odd}}(T_i)}\right) \qquad \text{(Corollary 5.2.4)}$$

$$= \sum_{i=1}^{c} |Z_i| \qquad \text{(definition of } Z_i\text{)}$$

$$= |Z|.$$

So, $\Bbbk[V_{\text{even}}(T)]/\mathcal{N}_{\text{odd}}(T)$ is Cohen-Macaulay (Corollary 5.2.4) of depth $|Z|$, hence Z is a maximal regular sequence.

Noting that $R \cong \otimes_{i=1}^{c} R_i$ (tensoring over $\Bbbk$), the type of $R/\mathcal{N}_{\text{odd}}(T)$ is given by

$$\operatorname{type}\left(\frac{R}{\mathcal{N}_{\text{odd}}(T)}\right) = \operatorname{type}\left(\bigotimes_{i=1}^{c} \frac{R_i}{\mathcal{N}_{\text{odd}}(T_i)}\right)$$

$$= \dim_{\Bbbk}\left(\operatorname{Ext}^0_{\otimes_{i=1}^{c} R_i}\left(\Bbbk, \bigotimes_{i=1}^{c} \frac{R_i}{\mathcal{N}_{\text{odd}}(T_i)}\right)\right)$$

$$= \dim_{\Bbbk}\left(\bigotimes_{i=1}^{c} \operatorname{Ext}^0_{R_i}\left(\Bbbk, \frac{R_i}{\mathcal{N}_{\text{odd}}(T_i)}\right)\right)$$

$$= \prod_{i=1}^{c} \dim_{\Bbbk}\left(\operatorname{Ext}^0_{R_i}\left(\Bbbk, \frac{R_i}{\mathcal{N}_{\text{odd}}(T_i)}\right)\right)$$

$$= \prod_{i=1}^{c} \operatorname{type}(R_i/\mathcal{N}_{\text{odd}}(T_i)).$$

By Theorem 5.1.11, $\operatorname{type}(R_i/\mathcal{N}_{\mathrm{odd}}(T_i))$ is given by the number of minimal $V_3(T_i)$-TD-sets of T_i for each i. Since T is the union of the connected components $T_1, \ldots, T_c$, the number of minimal V_3-TD-set of T is the product of the number of minimal $V_3(T_i)$-TD-sets of T_i for all i. Thus we get

$$\begin{aligned}\operatorname{type}\left(\frac{R}{\mathcal{N}_{\mathrm{odd}}(T)}\right) &= \prod_{i=1}^{c} \operatorname{type}(R_i/\mathcal{N}_{\mathrm{odd}}(T_i)) \\ &= \text{“number of minimal } V_3\text{-TD-sets in } T\text{”}.\end{aligned}$$

□

Example 5.2.6. We demonstrate the computations in Theorem 5.2.5. Consider $T_{\mathcal{B}}$ from Example 3.4.3 shown below in Figure 24. For each connected component, labeling them as

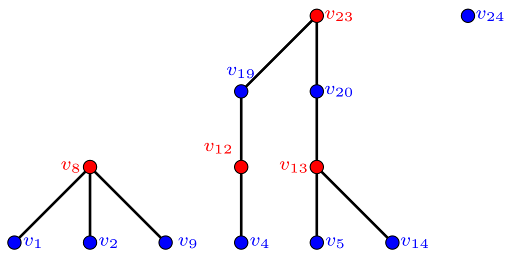


FIGURE 24. $T_{\mathcal{B}}$ from Example 3.4.3

T_1, T_2, and T_3 from left to right, we get

$$\frac{R_1}{\mathcal{N}_{\mathrm{odd}}(T_1)} = \frac{\mathbb{k}[v_1, v_2, v_9]}{\langle v_1 v_2 v_9 \rangle}$$

$$\frac{R_2}{\mathcal{N}_{\mathrm{odd}}(T_2)} = \frac{\mathbb{k}[v_4, v_5, v_{14}, v_{19}, v_{20}]}{\langle v_4 v_{19}, v_5 v_{14} v_{20}, v_{19} v_{20} \rangle}$$

$$\frac{R_3}{\mathcal{N}_{\mathrm{odd}}(T_3)} = \frac{\mathbb{k}[v_{24}]}{\langle v_{24} \rangle}.$$

Also, we have

$$\frac{R}{\mathcal{N}_{\mathrm{odd}}(T_{\mathcal{B}})} = \frac{\mathbb{k}[v_1, v_2, v_9, v_4, v_5, v_{14}, v_{19}, v_{20}, v_{24}]}{\langle v_1 v_2 v_9, v_4 v_{19}, v_5 v_{14} v_{20}, v_{19} v_{20}, v_{24} \rangle}.$$

Thus we get

$$\begin{aligned}\frac{R}{\mathcal{N}_{\mathrm{odd}}(T_{\mathcal{B}})} &= \frac{\mathbb{k}[v_1, v_2, v_9, v_4, v_5, v_{14}, v_{19}, v_{20}, v_{24}]}{\langle v_1 v_2 v_9, v_4 v_{19}, v_5 v_{14} v_{20}, v_{19} v_{20}, v_{24} \rangle} \\ &= \frac{\mathbb{k}[v_1, v_2, v_9]}{\langle v_1 v_2 v_9 \rangle} \otimes_{\mathbb{k}} \frac{\mathbb{k}[v_4, v_5, v_{14}, v_{19}, v_{20}]}{\langle v_4 v_{19}, v_5 v_{14} v_{20}, v_{19} v_{20} \rangle} \otimes_{\mathbb{k}} \frac{\mathbb{k}[v_{24}]}{\langle v_{24} \rangle} \\ &= \frac{R_1}{\mathcal{N}_{\mathrm{odd}}(T_1)} \otimes_{\mathbb{k}} \frac{R_2}{\mathcal{N}_{\mathrm{odd}}(T_2)} \otimes_{\mathbb{k}} \frac{R_3}{\mathcal{N}_{\mathrm{odd}}(T_3)}.\end{aligned}$$

The only minimal V_3-TD-set for T_1 and T_3 is the empty set since $V_3(T_1) = \emptyset = V_3(T_3)$, and the minimal V_3-TD-sets for T_2 are $\{v_{19}\}$ and $\{v_{20}\}$. Thus by Theorem 5.1.11, we get

$$\operatorname{type}(R_1/\mathcal{N}_{\text{odd}}(T_1)) = 1 \qquad \operatorname{type}(R_2/\mathcal{N}_{\text{odd}}(T_2)) = 2 \qquad \operatorname{type}(R_3/\mathcal{N}_{\text{odd}}(T_3)) = 1.$$

Therefore, the type of $R/\mathcal{N}_{\text{odd}}(T_{\mathcal{B}})$ is

$$\operatorname{type}(R/\mathcal{N}_{\text{odd}}(T_{\mathcal{B}})) = \operatorname{type}(R_1/\mathcal{N}_{\text{odd}}(T_1)) \cdot \operatorname{type}(R_2/\mathcal{N}_{\text{odd}}(T_2)) \cdot \operatorname{type}(R_3/\mathcal{N}_{\text{odd}}(T_3)) = 2.$$

Here is the main result of this thesis. It shows how to compute the Cohen-Macaulay type of $R/\mathcal{N}(T)$ for unmixed trees using only graph-theoretic information about T.

Theorem 5.2.7. *Let $R = \Bbbk[V]$, $R_{\mathcal{B}} = \Bbbk[V_{even}(T_{\mathcal{B}})]$, and $R_{\mathcal{R}} = \Bbbk[V_{even}(T_{\mathcal{R}})]$. Let $m_{\mathcal{B}}$ and $m_{\mathcal{R}}$ be the numbers of minimal V_3-TD-sets in $T_{\mathcal{B}}$ and $T_{\mathcal{R}}$, respectively. The Cohen-Macaulay type of $R/\mathcal{N}(T)$ is*

$$\begin{aligned} \mathit{type}(R/\mathcal{N}(T)) &= \mathit{type}(R_{\mathcal{B}}/\mathcal{N}_{odd}(T_{\mathcal{B}})) \cdot \mathit{type}(R_{\mathcal{R}}/\mathcal{N}_{odd}(T_{\mathcal{R}})) \\ &= m_{\mathcal{B}} \cdot m_{\mathcal{R}}. \end{aligned}$$

Proof. Let $Z_{\mathcal{B}}$ and $Z_{\mathcal{R}}$ be the maximal regular sequences for $\Bbbk[V_{\text{even}}(T_{\mathcal{B}})]/\mathcal{N}_{\text{odd}}(T_{\mathcal{B}})$ and $\Bbbk[V_{\text{even}}(T_{\mathcal{R}})]/\mathcal{N}_{\text{odd}}(T_{\mathcal{R}})$ computed in Theorem 5.2.5, respectively, and let $Z = Z_{\mathcal{B}} \cup Z_{\mathcal{R}}$. Then we have

$$\begin{aligned} \frac{R}{\mathcal{N}(T) + \langle Z\rangle} &= \frac{R}{\mathcal{N}_{\text{odd}}(T_{\mathcal{B}}) + \mathcal{N}_{\text{odd}}(T_{\mathcal{R}}) + \langle V_1(T)\rangle + \langle Z\rangle} && \text{(Theorem 5.2.1)} \\ &= \frac{\Bbbk[V_{\text{even}}(T_{\mathcal{B}}) \cup V_{\text{even}}(T_{\mathcal{R}}) \cup V_1(T)]}{\mathcal{N}_{\text{odd}}(T_{\mathcal{B}}) + \mathcal{N}_{\text{odd}}(T_{\mathcal{R}}) + \langle V_1(T)\rangle + \langle Z\rangle} && \text{(Lemma 4.1.8)} \\ &\cong \frac{\Bbbk[V_{\text{even}}(T_{\mathcal{B}}) \cup V_{\text{even}}(T_{\mathcal{R}})]}{\mathcal{N}_{\text{odd}}(T_{\mathcal{B}}) + \mathcal{N}_{\text{odd}}(T_{\mathcal{R}}) + \langle Z\rangle} \\ &= \frac{\Bbbk[V_{\text{even}}(T_{\mathcal{B}})]}{\mathcal{N}_{\text{odd}}(T_{\mathcal{B}}) + \langle Z_{\mathcal{B}}\rangle} \otimes_{\Bbbk} \frac{\Bbbk[V_{\text{even}}(T_{\mathcal{R}})]}{\mathcal{N}_{\text{odd}}(T_{\mathcal{R}}) + \langle Z_{\mathcal{R}}\rangle}. \end{aligned}$$

Note that we have

$$\dim\left(\frac{\Bbbk[V_{\text{even}}(T_{\mathcal{B}})]}{\mathcal{N}_{\text{odd}}(T_{\mathcal{B}}) + \langle Z_{\mathcal{B}}\rangle}\right) = 0 = \dim\left(\frac{\Bbbk[V_{\text{even}}(T_{\mathcal{R}})]}{\mathcal{N}_{\text{odd}}(T_{\mathcal{R}}) + \langle Z_{\mathcal{R}}\rangle}\right)$$

by Theorem 5.2.5. Hence we get

$$\dim\left(\frac{R}{\mathcal{N}(T) + \langle Z\rangle}\right) = 0$$

so

$$\operatorname{depth}\left(\frac{R}{\mathcal{N}(T) + \langle Z\rangle}\right) = 0.$$

Let $\mathcal{D}$ be the set of minimal TD-sets of T, let $\mathcal{D}_{\mathcal{B}}$ and $\mathcal{D}_{\mathcal{R}}$ be the sets of minimal odd TD-sets of $T_{\mathcal{B}}$ and $T_{\mathcal{R}}$, respectively. Then Corollary 3.1.9 and 4.2.7 gives us that $D \in \mathcal{D}$ if and only if $D = D_{\mathcal{B}} \cup D_{\mathcal{R}} \cup V_1(T)$ for some $D_{\mathcal{B}} \in \mathcal{D}_{\mathcal{B}}$ and $D_{\mathcal{R}} \in \mathcal{D}_{\mathcal{R}}$; note that the sets $D_{\mathcal{B}}$, $D_{\mathcal{R}}$, and $V_1(T)$ are pairwise disjoint by Lemma 4.1.8 since $D_{\mathcal{B}} \subseteq V_{\text{even}}(T_{\mathcal{B}})$ and $D_{\mathcal{R}} \subseteq V_{\text{even}}(T_{\mathcal{R}})$. Fix $D \in \mathcal{D}$, $D_{\mathcal{B}} \in \mathcal{D}_{\mathcal{B}}$, and $D_{\mathcal{R}} \in \mathcal{D}_{\mathcal{R}}$ such that $D = D_{\mathcal{B}} \cup D_{\mathcal{R}} \cup V_1(T)$. Then we have

$$\begin{aligned}
\dim\left(\frac{R}{\mathcal{N}(T)}\right) &= |V(T)| - |D| \\
&= |V_{\text{even}}(T_{\mathcal{B}}) \cup V_{\text{even}}(T_{\mathcal{R}}) \cup V_1(T)| - |D| && \text{(Lemma 4.1.8)} \\
&= |V_{\text{even}}(T_{\mathcal{B}}) \cup V_{\text{even}}(T_{\mathcal{R}}) \cup V_1(T)| - |D_{\mathcal{B}} \cup D_{\mathcal{R}} \cup V_1(T)| \\
&= |V_{\text{even}}(T_{\mathcal{B}})| + |V_{\text{even}}(T_{\mathcal{R}})| - |D_{\mathcal{B}}| - |D_{\mathcal{R}}| \\
&= \dim\left(\frac{R_{\mathcal{B}}}{\mathcal{N}_{\text{odd}}(T_{\mathcal{B}})}\right) + \dim\left(\frac{R_{\mathcal{R}}}{\mathcal{N}_{\text{odd}}(T_{\mathcal{R}})}\right) \\
&= \operatorname{depth}\left(\frac{R_{\mathcal{B}}}{\mathcal{N}_{\text{odd}}(T_{\mathcal{B}})}\right) + \operatorname{depth}\left(\frac{R_{\mathcal{R}}}{\mathcal{N}_{\text{odd}}(T_{\mathcal{R}})}\right) && \text{(Corollary 5.2.4)} \\
&= |Z_{\mathcal{B}}| + |Z_{\mathcal{R}}| \\
&= |Z|.
\end{aligned}$$

Thus Z is a maximal $R/\mathcal{N}(T)$-regular sequence. Writing $V_{\mathcal{B}} := V_{\text{even}}(T_{\mathcal{B}})$ and $V_{\mathcal{R}} := V_{\text{even}}(T_{\mathcal{R}})$, we get

$$\begin{aligned}
&\operatorname{type}(R/\mathcal{N}(T)) \\
&= \operatorname{type}\left(\frac{R}{\mathcal{N}(T) + \langle Z\rangle}\right) \\
&= \operatorname{type}\left(\frac{\Bbbk[V_{\mathcal{B}}]}{\mathcal{N}_{\text{odd}}(T_{\mathcal{B}}) + \langle Z_{\mathcal{B}}\rangle} \otimes_{\Bbbk} \frac{\Bbbk[V_{\mathcal{R}}]}{\mathcal{N}_{\text{odd}}(T_{\mathcal{R}}) + \langle Z_{\mathcal{R}}\rangle}\right) \\
&= \dim_{\Bbbk}\left(\operatorname{Ext}^0_{\Bbbk[V_{\mathcal{B}} \cup V_{\mathcal{R}}]}\left(\Bbbk, \frac{\Bbbk[V_{\mathcal{B}}]}{\mathcal{N}_{\text{odd}}(T_{\mathcal{B}})} \otimes_{\Bbbk} \frac{\Bbbk[V_{\mathcal{R}}]}{\mathcal{N}_{\text{odd}}(T_{\mathcal{R}})}\right)\right) \\
&= \dim_{\Bbbk}\left(\operatorname{Ext}^0_{\Bbbk[V_{\mathcal{B}}]}\left(\Bbbk, \frac{\Bbbk[V_{\mathcal{B}}]}{\mathcal{N}_{\text{odd}}(T_{\mathcal{B}}) + \langle Z_{\mathcal{B}}\rangle}\right) \otimes_{\Bbbk} \operatorname{Ext}^0_{\Bbbk[V_{\mathcal{R}}]}\left(\Bbbk, \frac{\Bbbk[V_{\mathcal{R}}]}{\mathcal{N}_{\text{odd}}(T_{\mathcal{R}}) + \langle Z_{\mathcal{R}}\rangle}\right)\right) \\
&= \dim_{\Bbbk}\left(\operatorname{Ext}^0_{\Bbbk[V_{\mathcal{B}}]}\left(\Bbbk, \frac{\Bbbk[V_{\mathcal{B}}]}{\mathcal{N}_{\text{odd}}(T_{\mathcal{B}}) + \langle Z_{\mathcal{B}}\rangle}\right)\right) \cdot \dim_{\Bbbk}\left(\operatorname{Ext}^0_{\Bbbk[V_{\mathcal{R}}]}\left(\Bbbk, \frac{\Bbbk[V_{\mathcal{R}}]}{\mathcal{N}_{\text{odd}}(T_{\mathcal{R}}) + \langle Z_{\mathcal{R}}\rangle}\right)\right)
\end{aligned}$$

$$= \text{type}\left(\frac{\mathbb{k}[V_{\mathcal{B}}]}{\mathcal{N}_{\text{odd}}(T_{\mathcal{B}}) + \langle Z_{\mathcal{B}}\rangle}\right) \cdot \text{type}\left(\frac{\mathbb{k}[V_{\mathcal{R}}]}{\mathcal{N}_{\text{odd}}(T_{\mathcal{R}}) + \langle Z_{\mathcal{R}}\rangle}\right)$$

$$= \text{type}(R/\mathcal{N}_{\text{odd}}(T_{\mathcal{B}})) \cdot \text{type}(R/\mathcal{N}_{\text{odd}}(T_{\mathcal{R}}))$$

$$= m_{\mathcal{B}} \cdot m_{\mathcal{R}} \qquad \text{(Theorem 5.2.5)}$$

as desired. □

We end this chapter with some examples applying Theorem 5.2.7.

Example 5.2.8. Let T be the tree from Example 3.4.2. Consider its blue and red interior graphs $T_{\mathcal{B}}$ and $T_{\mathcal{R}}$ in Figure 25. The minimal V_3-TD-sets of $T_{\mathcal{B}}$ are $\{u_3, u_5\}$ and $\{u_4\}$. The

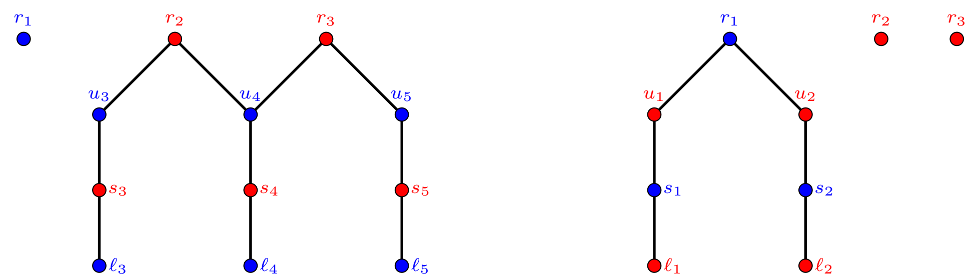


FIGURE 25. $T_{\mathcal{B}}$ and $T_{\mathcal{R}}$ from Example 3.4.2

minimal V_3-TD-sets of $T_{\mathcal{R}}$ are $\{u_1\}$ and $\{u_2\}$. Therefore, we have $\text{type}(R/\mathcal{N}(T)) = 2 \cdot 2 = 4$
.

Example 5.2.9. Let T be the tree in Example 3.4.3 and let $R = \mathbb{k}[V(T)]$. We have

$$\text{type}(\mathbb{k}[V_{\text{even}}(T_{\mathcal{B}})]/\mathcal{N}_{\text{odd}}(T_{\mathcal{B}})) = 2$$

from Example 5.2.6. Since $T_{\mathcal{B}}$ and $T_{\mathcal{R}}$ are isomorphic graphs, we get

$$\mathbb{k}[V_{\text{even}}(T_{\mathcal{B}})]/\mathcal{N}_{\text{odd}}(T_{\mathcal{B}}) \cong \mathbb{k}[V_{\text{even}}(T_{\mathcal{R}})]/\mathcal{N}_{\text{odd}}(T_{\mathcal{R}})$$

which gives

$$\text{type}(\mathbb{k}[V_{\text{even}}(T_{\mathcal{R}})]/\mathcal{N}_{\text{odd}}(T_{\mathcal{R}})) \cong \text{type}(\mathbb{k}[V_{\text{even}}(T_{\mathcal{B}})]/\mathcal{N}_{\text{odd}}(T_{\mathcal{B}})) = 2.$$

Thus we get

$$\begin{aligned}\text{type}(R/\mathcal{N}(T)) &= \text{type}(\mathbb{k}[V_{\text{even}}(T_{\mathcal{B}})/\mathcal{N}_{\text{odd}}(T_{\mathcal{B}})]) \cdot \text{type}(\mathbb{k}[V_{\text{even}}(T_{\mathcal{R}})]/\mathcal{N}_{\text{odd}}(T_{\mathcal{R}}))\\ &= 2 \cdot 2 = 4.\end{aligned}$$

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