On Minimal Achievable Quotas in Multiwinner Voting

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Abstract

Justified representation (JR) and extended justified representation (EJR) are well-established proportionality axioms in approval-based multiwinner voting. Both axioms are always satisfiable, but they rely on a fixed quota (typically Hare or Droop), with the Droop quota being the smallest one that guarantees existence across all instances. With this observation in mind, we take a first step beyond the fixed-quota paradigm and introduce proportionality notions where the quota is instance-dependent. We demonstrate that all commonly studied voting rules can have an additive distance to the optimum of $\frac{k^2}{(k+1)^2}$. Moreover, we look into the computational aspects of our instance-dependent quota and prove that determining the optimal value of α for a given approval profile satisfying α-JR is NP-complete. To address this, we introduce an integer linear programming (ILP) formulation for computing committees that satisfy α-JR, and we provide positive results in the voter interval (VI) and candidate interval (CI) domains.

1 Introduction

In the context of multiwinner approval voting, fairness is typically understood in terms of proportionality: the elected committee should reflect, to an appropriate extent, the fraction of voters who share similar preferences or approval sets. This guarantees that sufficiently large groups of voters receive adequate representation, while preventing smaller groups from being disregarded. Proportionality axioms, like justified representation (JR) and extended justified representation (EJR), formalize this intuition and have become a central concept in the study of fairness in multiwinner elections. Given the size of the voter set n and committee size k, a group of voters S of size at least n/k that agrees on a common candidate is called *cohesive* and should be able to demand some form of representation in the committee. The Hare quota ($= n/k$) is deeply ingrained in the definitions of most proportionality axioms in multiwinner voting — including JR and EJR, but also proportional justified representation (PJR) and full justified representation (FJR).

Nevertheless, there is a variety of situations where these proportionality axioms fail to capture intuitive fairness. JR demands that every cohesive group has at least one voter $v \in S$ who approves a candidate in the committee. Consider, for example, the election instance provided in Figure 1. To be cohesive, the group has to contain at least 5 voters. It should be easy to see that the committee $W = \{c_3, c_4\}$ satisfies JR, and, thereby, disregards the groups $\{1, 2, 3, 4\}$ and $\{7, 8, 9, 10\}$. These groups fall just short of meeting the quota. This instance, however, permits a more balanced representation of all voters by selecting the committee $W' = \{c_1, c_2\}$.

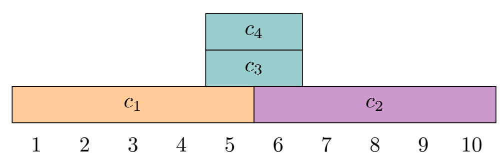

Figure 1: A voting instance with $N = [10]$, $C = \{c_1, c_2, c_3, c_4\}$ and $k = 2$ in which any committee containing c_3 satisfies JR. Voters are denoted by an integer and precisely approve the candidates stacked above them.

This phenomenon extends to all fairness axioms that are defined with respect to these static quota requirements. Even the core is not immune to this issue.[1] Figure 2 illustrates an example in which the committee $\{c_1, c_2, c_3\}$ belongs to the core, even though voters 5–12 remain unrepresented. The three voters supporting c_4 once again fall short of the quota and can therefore not form a valid deviating coalition.

These observations regarding the limitations of proportionality are well established in the literature:

"[P]roperties like the core (and also EJR, PJR, and JR) prevent specific pathological situations, but beyond their definitions, do not provide intuitive justifications

[1] For a formal definition, consider the paper by Peters et al. (2021b).

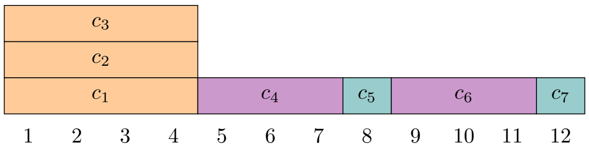

Figure 2: A voting instance with $N = [12]$, $C = \{c_1, \ldots, c_7\}$ and $k = 3$ in which the committee $W = \{c_1, c_2, c_3\}$ lies within the core even though a large group of voters remains unrepresented (example taken from Peters et al. (2021b)).

for why a committee they allow should be selected" — Peters et al. (2021b).

For some proportionality axioms, representation cannot be guaranteed for groups smaller than the Droop quota ($= \frac{n}{k+1}$) (Droop, 1881). Hence, fixed quotas may yield examples like that in Figure 1 or render the axiom unsatisfiable. This raises the question of whether, instead of relying on a uniform quota across all instances, we should make it adaptive with respect to the preference profile. Our experiments show that the quota values for which JR or EJR remains satisfiable tend to be much smaller than $\frac{n}{k+1}$ A similar observation holds for the ordinal case (Bardal et al., 2025).

In this paper, we propose variable quotas. The central idea is to reduce the quota n/k by a factor α, approaching the infimum value at which JR or EJR ceases to be feasible. We refer to this approach as the α-extension of cohesiveness. For the case of JR, finding the optimal α-value corresponds to minimizing the largest group of unrepresented voters approving a common candidate.

To illustrate this concept more concretely, consider the following example: A company has four locations, each comprising five members. The task is to form a representative committee of three members. Within each location, all members approve a location representative aware of the problems at this location. Each location has a manager, salesperson, and engineer, with each role group approving a role representative. The remaining two members per location are working students, who only approve their location representative. Known approval-based multiwinner rules like PAV or CC elect three of the four location representatives as they receive the broadest approval support. However, this outcome fails to represent an entire location. We argue that a more proportionally fair outcome — and the optimal committee with respect to α-JR — would instead include the three group representatives, ensuring that no large and homogeneous segment of the electorate, like an entire location of a company, remains unrepresented.

Our Contributions In this paper, we introduce a class of proportionality axioms, denoted α-JR and α-EJR, which generalize the existing proportionality axioms JR and EJR. There are instances for which existing voting rules perform very badly with respect to these axioms. More specifically, we demonstrate that all commonly studied voting rules can produce committees with an additive distance of $\frac{k^2}{(k+1)^2}$ away from the optimal α-value. Next to that, we show that computing the optimal α-value for a voting instance is NP-hard, even for JR. In order to find the optimal value for JR, we provide an ILP. When considering restricted voting instances, we can show positive results: In the voter- and candidate-interval domain, we can efficiently compute the optimal committee with respect to α-JR. For party-list profiles, we provide an algorithm to construct an optimal α-EJR committee in polynomial time. Lastly, we provide empirical evaluation on the distribution of the optimal α-value for synthetically generated instances and show that existing voting rules can, to some extent, approximate the optimal α-value on average.

2 Related Work

Our work builds on the growing literature on fairness and proportionality in approval-based multiwinner voting (see, e.g., (Lackner and Skowron, 2023; Aziz et al., 2017; Sánchez-Fernández et al., 2017; Skowron et al., 2017a)). The central idea in this line of research is that voters who agree on a set of candidates should receive influence in the elected committee proportional to their size. The notions of JR and EJR, introduced by Aziz et al. (2017), translate this into the setting of multiwinner approval voting and, thereby, mark the starting point of proportionality in approval-based multiwinner voting. Building on these axioms, subsequent work has extended proportionality axioms and analyzed their implications for prominent rules (Brill and Peters, 2023; Brill et al., 2024). These ideas have also been generalized beyond multiwinner elections to participatory budgeting (Peters et al., 2021a), and general voting models with feasibility constraints (Masařík et al., 2024).

One aspect that has received relatively little attention is the role of quotas, i.e., the requirement that a group of like-minded voters must reach a certain threshold to secure representation in the committee. The most common quota that has been used in the literature so far is the Hare quota ($= \frac{n}{k}$). For this, it has been shown that JR and EJR are always satisfiable (Aziz et al., 2017). This also holds for some stronger versions of the axioms, like EJR+ and FJR. More recently, these representation axioms have been considered for the Droop quota ($= \frac{n}{k+1}$) (Casey and Elkind, 2025). In this paper, the authors demonstrate that proportionality axioms based on the Droop quota can be satisfied for any given voting instance. They show that many voting rules and concepts originally developed for the Hare quota can be effectively adapted to the Droop quota setting. Before this work, the Droop quota was primarily studied within the apportionment framework (Pukelsheim, 2017; Brill et al., 2020).

In this work, we go a step further and consider dynamic quotas. Droop (1881) — after whom the quota is named — already observed that elections often result in situations where more voters than the quota support a common candidate, raising the question of how to allocate the "excess" votes that such candidates receive. That being said, we take a quantitative perspective of proportionality, aiming to measure the extent to which a given committee satisfies proportional representation. Similar quantitative approaches have been explored in prior work (Bardal et al., 2025; Lackner and Skowron, 2019; Janson, 2018; Janeczko and Faliszewski, 2022).

Closest related to ours is the extensive work by Janson (2018), who studied and surveyed thresholds of proportionality. More specifically, he asked the question of the smallest proportion of voters that can be guaranteed a good outcome, where a good outcome corresponds to committees satisfying various proportionality notions. While he mostly focused on fixing the proportion of votes achievable by sequential Phragmén and Thiele rules, we consider more voting rules and look into approaches and results around the computational aspects of minimizing the quota.
More recently, Bardal et al. (2025) developed quantitative measures of proportionality in the ordinal setting, thereby strengthening the proportionality guarantees of so-called *solid coalitions*. Their work primarily focused on empirical analyses of real-world data, whereas our study centers around the theoretical aspects of optimizing the α-value in the approval setting and exploring results in the voter- and candidate-interval domain. In parallel to us, they introduced the parameter α, which for values less than 1 makes *proportionality for solid coalitions* more demanding. Skowron et al. (2017b) formalized proportionality in the ranking setting when voters have approval preferences and introduced a measure of group satisfaction within this setting.

Alternative approaches to capturing fairness in multiwinner voting have been explored beyond JR–based axioms (Brill and Peters, 2024; Kehne et al., 2025; Lackner and Skowron, 2020). Brill et al. (2025) introduced the axiom of *individual representation (IR)*, which strengthens the earlier concept of *semi-strong justified representation* proposed by Aziz et al. (2017). While *semi-strong JR* requires that every sufficiently large and cohesive group of voters is represented in the committee at least once, IR refines this principle by assigning each voter a guarantee relative to the largest ℓ-cohesive group she is able to form for a given profile. IR is not always satisfiable: there exist profiles for which no committee provides individual representation, and non-trivial approximation guarantees are unattainable without imposing restrictions on the domain of admissible profiles. Kehne et al. (2025) changed the viewpoint of fairness to be more candidate-centric. Their desideratum of fairness says that candidates with similar sets of supporters should receive similar representation.

Lastly, we mention the *proportionality degree (PD)* as a different way of quantifying proportionality in multiwinner approval voting. Informally, this is a function that specifies the average satisfaction of a voter in an ℓ-cohesive group. This approach strictly enforces the quota while allowing greater flexibility in determining the average satisfaction level of each group. Skowron (2021) used this to measure the proportionality guarantees that different voting rules can achieve. He showed that the sequential variant of the proportional approval voting rule gets higher PD than sequential Phragmén. Janeczko and Faliszewski (2022) looked into computational aspects of the proportionality degree in committee elections. They show that deciding if a committee with a given PD exists is NP-hard and that verifying if a given committee provides a given PD is coNP-complete.

3 Preliminaries

We consider the standard approval-based multiwinner voting framework. Let $N = \{v_1, \dots, v_n\}$ be the set of voters, $C = \{c_1, \dots, c_m\}$ be the set of candidates, and $k \in \mathbb{N}$ be the committee size with $k \leq m$. For every voter $v \in N$, we denote the set of candidates she approves of as $A_v \subseteq C$, and we write $N_c = \{v \in N : c \in A_v\}$ for the set of supporters of a candidate $c \in C$. A voting instance I is defined by an approval profile $A = (A_v)_{v \in N}$ and the committee size k, i.e., $I = (A, k)$. Finally, a committee W is a subset of the candidates with cardinality k. In order to determine which committee should be selected, different multiwinner voting rules have been proposed. In this paper, we consider commonly used approval-based multiwinner voting rules, like the method of equal shares (MES), sequential Phragmén (seq-Phragmén), Chamberlin-Courant (CC), proportional approval voting (PAV), and greedy justified candidate rule (GJCR). We refer the reader to Section A for the formal definitions of these rules.

4 Quota-Based Strengthening of Proportionality

As stated in the beginning, most proportionality axioms rely on the principle that a group is entitled to representation only if it satisfies some cohesiveness condition. Cohesiveness in its general form demands that a voter group satisfies the Hare quota (Aziz et al., 2017; Sánchez-Fernández et al., 2017).

Definition 4.1 (ℓ-Cohesiveness (Aziz et al., 2017))**.** For $\ell \in \mathbb{N}_+$, a group $S \subseteq N$ is ℓ-cohesive if (1) $|S| \geq \ell \cdot \frac{n}{k}$ and (2) $|\bigcap_{v \in S} A_v| \geq \ell$.

In this work, we introduce a more general version of this definition, which introduces a parameter α, making the quota requirement dynamic. The parameter α acts as a scaling factor that adjusts the group size required for representation. Smaller α values correspond to a more demanding standard of proportionality, as even smaller groups can claim representation.

Definition 4.2 ((α, ℓ)-Cohesiveness)**.** For $\ell \in \mathbb{N}_+$ and $\alpha \in \mathbb{R}_{\geq 0}$, a group $S \subseteq N$ is (α, ℓ)-cohesive if (1) $|S| \geq \alpha \cdot \ell \cdot \frac{n}{k}$ and (2) $|\bigcap_{v \in S} A_v| \geq \ell$.

Remark 1. *Changing the quota size has, up until now, mostly been done in the other direction, i.e., one relaxes the quota to make a proportionality axiom easier to satisfy (Dong et al., 2025a,b; Jiang et al., 2020).*

With this generalized cohesiveness definition, let us now modify existing proportionality axioms:

Definition 4.3 (α-JR)**.** A rule satisfies α-JR if for every $(\alpha, 1)$-cohesive group S, there exists a voter $v \in S$ with $|A_v \cap W| \geq 1$.

Definition 4.4 (α-EJR)**.** A rule satisfies α-EJR if for every (α, ℓ)-cohesive group S, there exists a voter $v \in S$ with $|A_v \cap W| \geq \ell$.

Definition 4.5 (α-EJR+)**.** A committee W satisfies α-EJR+ if there does not exist a candidate $c \in C \setminus W$, a group of voters $S \subseteq N$, and $\ell \in \mathbb{N}_+$ with $|S| \geq \alpha \cdot \ell \cdot \frac{n}{k}$ such that

$$c \in \bigcap_{v \in S} A_v \text{ and } |A_v \cap W| < \ell \text{ for all } v \in S.$$

Clearly, for the case $\alpha = 1$, the three proportionality axioms correspond to the original notions of JR, EJR, and EJR+. Therefore, several voting rules are known to provide guarantees of satisfiability. More recently, it has been established that for any given instance, there exists an $\alpha < 1$ such that α-EJR+ still can be satisfied.

Theorem 4.6 (Casey and Elkind (2025))**.** *For every instance I, and any α with $\alpha > \frac{k}{k+1}$ there exists a committee W that satisfies α-EJR+ and is polynomial-time computable.*

In this paper, we are interested in the minimal possible quota that is still achievable by a given instance. Therefore, we like to find the infimum α-value that still allows for an instance to satisfy α-Φ where $\Phi \in \{\text{JR}, \text{EJR}, \text{EJR+}\}$. That being said, let

$$\alpha_\Phi(W) := \inf\{\alpha \colon W \text{ satisfies } \alpha\text{-}\Phi\}.$$

We refer to $\alpha_\Phi(W)$ as the α-value of a committee W with respect to the proportionality axiom Φ. The minimum achievable α-value for an instance $I = (A, k)$ is denoted by

$$\alpha^*_\Phi(I) = \min_{W \subseteq C : |W| = k} \alpha_\Phi(W).$$

We also refer to this as the optimal α-value for an instance I with respect to Φ. If the instance I is clear from the context, we omit it and just write the optimal α-value with respect to Φ.

The notion of α_Φ is equivalent to finding the largest α-value for which there is a committee W violating α-Φ. Observe that by definition, the value is greater than or equal to 0. If all voters with a non-empty ballot approve a candidate in the committee $\alpha_{\text{JR}} = 0$, and if $|A_v \cap W| \geq \min(|A_v|, k)$ for all $v \in N$, then $\alpha_{\text{EJR}} = 0$.

4.1 General Properties

This approach generates a whole class of proportionality axioms. We, therefore, start by showing some general relational results: The known implications that EJR+ implies EJR and EJR implies JR only hold for α_1-EJR+, α_2-EJR, and α_3-JR when $\alpha_1 \leq \alpha_2 \leq \alpha_3$. A complete proof, as well as all omitted proofs, can be found in the appendix.

Proposition 4.7. *For $\alpha_1 \leq \alpha_2 \leq \alpha_3$, it holds that α_1-EJR+ $\implies$ α_2-EJR $\implies$ α_3-JR.*

Proof Sketch. Let $\alpha_1 \leq \alpha_2$ and $\Phi \in \{\text{JR, EJR, EJR+}\}$. First of all, α_1-Φ implies α_2-Φ since all cohesive groups for α_2 are also considered for α_1. Showing the implications between α-EJR+, α-EJR, and α-JR follows directly from their definitions. ☐

Proposition 4.8. *For $\alpha_1 < \alpha_2 < \alpha_3 \leq 1$, it holds that α_3-EJR+, α_2-EJR, and α_1-JR are incomparable.*

Remark 2. *For $\alpha = 0$, every voter group qualifies as cohesive, so no committee can satisfy the proportionality axioms; hence, they are all equivalent in this degenerate case.*

We next demonstrate that there can be a substantial gap between the optimal α-values for JR and EJR. This highlights that our framework provides greater flexibility in how representation is understood. Under the classical definitions, it is natural to focus on EJR, as it always implies JR. However, introducing the parameter α allows us to adjust the strength of representation: we can choose whether to prioritize smaller, cohesive groups or to emphasize the representation of larger ones.

Proposition 4.9. *The additive gap between JR and EJR for the optimal α-value can be $k/(k+1)$.*

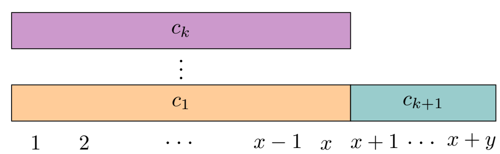

Figure 3: A voting instance $N = [x + y]$, $C = \{c_1, \dots c_{k+1}\}$ in which the optimal α-value for JR is 0, whereas the optimal α-value for EJR is $\min(\frac{x}{x+y}, \frac{k \cdot y}{x+y})$.

Proof Sketch. The committee $W = \{c_1, \dots, c_{k-1}, c_{k+1}\}$ achieves $\alpha_{\text{JR}}(W) = 0$, while for α_{EJR} we need to decide whether to select candidates $c_1, \dots, c_k$ or to replace on of them with c_{k+1}. Either we get a violation by a 1-cohesive group or by a k-cohesive group. When we set $x = ky$ in Figure 3, we can show that $\alpha_{\text{EJR}}(W') = k/(k+1)$ for all W'. ☐

This construction shows that even when a committee perfectly satisfies α-JR ($\alpha = 0$), its α-EJR value can be as large as $k/(k+1)$, demonstrating a maximal additive gap. Since α can be any value in $\mathbb{R}_{\geq 0}$, we prove that it is sufficient to only consider $k \cdot n$ many α-values to determine the optimum for JR and EJR.

Lemma 4.10. *For a given instance I, at most $\lceil \frac{n}{k} \rceil - 1 \leq n$ α-values need to be checked to determine α^*_{JR}. Further, it holds that $\alpha^*_{JR} \in \mathbb{Q}$.*

Proof. For α-JR, it is only relevant what size the smallest 1-cohesive group has. This group size is between 1, meaning that a single voter on its own already forms a 1-cohesive group (together with a candidate she approves) and $\frac{n}{k}$ the size of the Hare quota. The relevant α-values come from the set $X = \{0, \frac{k}{n}, \frac{2k}{n}, \ldots, \frac{rk}{n}\}$ with $r = \lceil \frac{n}{k} \rceil - 1$. Observe that if there does not even exist a violation for $\alpha = \frac{k}{n}$ then α^*_{JR} by definition is 0.

Now let α^*_{JR} be the optimal value. Thus, there exists some α^*_{JR}-JR violation witnessed by some (N', c). Therefore, it holds that $|N'| \geq \alpha^*_{\text{JR}} \cdot \frac{n}{k}$. It even has to be the case that this inequality is tight, since otherwise α^*_{JR} is clearly not optimal with (N', c) also witnessing a violation for a larger α. Thus, $\alpha^*_{\text{JR}} = |N'| \cdot \frac{k}{n} \in X$ which proves the claim. □

Lemma 4.11. *For a given instance I, at most $n \cdot k$ many α-values to determine the α^*_{EJR}. Further,it holds that $\alpha^*_{EJR} \in \mathbb{Q}$.*

The last two lemmas allow us to assume that $\alpha \in \mathbb{Q}$ since for any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ we can consider the next larger α' from Lemma 4.10 or Lemma 4.11 respectively because if there exist a violation for α, then there also exist one for α'.

4.2 Worst-Case Analysis of Existing Rules

Many known rules can always guarantee the Droop quota, meaning that $\alpha^*_{\text{JR}}(I) \leq \frac{k}{k+1}$ for every instance I. Nevertheless, there exist instances where the optimal α-value is smaller by a factor of $\frac{n}{k+1}$. Intuitively, this follows from the fact that many rules greedily select candidates that are approved by a lot of voters and do not consider whether the voters who are underrepresented approve common candidates. Figure 4 visualizes such an instance for $k = 5$.

Example 4.12. Consider the following family of instances. Let $N = [(k+1)(k+1)]$. Furthermore, let $C = B \cup D$, with $B = \{b_1, \ldots, b_{k+1}\}$ such that $N_{b_i} = \{(i-1) \cdot (k+1) + 1, \ldots, i \cdot (k+1)\}$ and $D = \{d_1, \ldots, d_k\}$ such that $N_{d_i} = \{j \in N \colon (j \mod k+1) = i\}$ for all $i \in \{1, \ldots, k+1\}$. Then each candidate in D and each candidate in B is approved by $k+1$ voters (see Figure 4 for a visualization of the case $k = 5$).

Many existing voting rules achieve an α-value of $\frac{k}{k+1}$ for this example since they can return the committee, which is visualized in violet, and leave the large group of voters N_{b_6} unrepresented. Note that in this instance, rules like GJCR or MES would actually return not a single candidate since the instance has to contain a JR violation in order for them to select any candidate. We therefore consider a natural way to adapt these rules to α by lowering the quota and giving more budget to the voters, respectively. They are formally defined in Section 4.2. In the following, we show that all these rules have a comparably bad α-value on the instances defined in *Example* 4.12.

Theorem 4.13. *For CC, seq-CC, seq-Phragmén, PAV, α-MES, and α-GJCR, the distance to the optimal α-value can be of size $\frac{k^2}{(k+1)^2}$.*

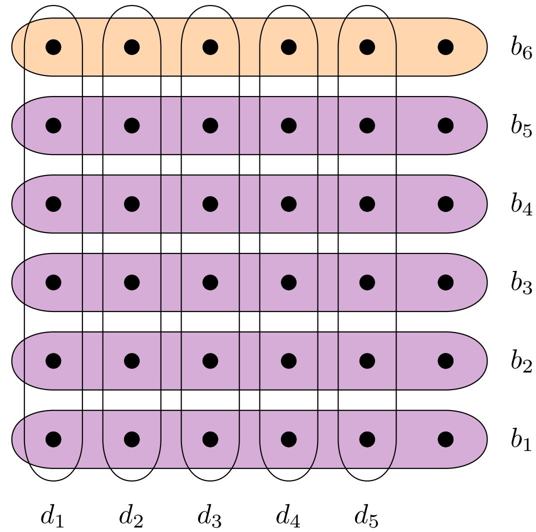

Figure 4: Visualization of Example 4.12 for $k = 5$. Voters are indicated by dots, and hyperedges correspond to the candidates. A committee W returned by most voting rules is indicated in violet, and the largest violation for W is shown in orange. The committee $D = \{d_1, \ldots, d_5\}$ minimizes the α-value in this instance.

Proof. Consider Example 4.12: All of the above rules might choose k from the $k+1$ candidates of the set B, i.e., the violet candidates from Figure 4.

Without loss of generality, let $N_{b_{k+1}}$ be the group of voters that remain unrepresented. For them not to have a legitimate claim for a candidate in the committee, the quota $\alpha \cdot \frac{n}{k}$ must be greater than $|N_{b_{k+1}}|$. Consequently,

$$\alpha > |N_{b_{k+1}}| \cdot \frac{k}{n} = (k+1) \cdot \frac{k}{(k+1)(k+1)} = \frac{k}{k+1}.$$

On the other hand, note that there exists a committee $W = D$, such that for all pairs of non-represented voters v, v' with $v \neq v'$ it holds that $A_v \cap A_{v'} \cap A_{v''} = \emptyset$. Thus, it holds that $\alpha^*_{\text{JR}}(I) \leq \frac{k}{n}$. Note that for this instance, D is an optimal solution since there does not exist any committee that is approved by every voter. It follows that the α-value of these voting rules can be $\frac{k}{k+1} - \frac{k}{n} = \frac{k^2}{(k+1)^2}$. □

Remark 3. *Example 4.12 can be adapted so that it does not rely on tie-breaking. The ratio for this is then $\frac{n}{2(k+1)}$. (See example Example B.1 in Section B.1.)*

For seq-Phragmén, Janson (2018) has already shown that the Droop quota is the smallest achievable quota in every instance. We include the result here for completeness.

Theorem 4.13 complements the results presented by Casey and Elkind (2025). They showed that the proportionality axioms defined with the Droop quota are still achievable by current voting rules. We show that these algorithms cannot provide any axiomatic guarantee in the sense of satisfying the more demanding α-version of proportional representation notions when α is less than or equal to $\frac{k}{k+1}$ — corresponding to the Droop quota.

5 Computing the Optimal α-value

In this section, we analyze how to determine the optimal α-value for a given instance. We begin by showing that, for α-JR, potential violations can be detected efficiently for a given committee W.

Proposition 5.1. *For a given instance $I = (A, k)$ and committee W, we can efficiently compute $\alpha_{JR}(W)$.*

Proof. Given an instance I and a committee W, we want to compute the largest α for which there exists an α-JR violation. For this, we consider every candidate $c \notin W$ and compute the number of voters that support c but no candidate in W. We define the maximal size of such a group as

$$s_{\max} = \max_{c \notin W} |\{v \in N : A_i \cap W = \emptyset \text{ and } c \in A_v\}|.$$

We now claim that the largest α that causes an α-JR violation is equal to $s_{\max} \cdot \frac{k}{n}$. First, observe that by the definition of $s_{\max}$, there exists a group S of size $s_{\max}$ that commonly approves some candidate c but no candidate in W. For $\alpha = s_{\max} \cdot \frac{k}{n}$, S is $(\alpha, 1)$-cohesive since

$$\alpha \cdot \frac{n}{k} = s_{\max} \cdot \frac{k}{n} \cdot \frac{n}{k} = s_{\max},$$

and thus, forms a violation with c. This implies that $\alpha_{\text{JR}} \geq s_{\max} \cdot \frac{k}{n}$.

On the other hand, by definition of α_{JR} there exists some α_{JR}-JR violation with witness (N', c). Thus, it holds that $|N'| \geq \alpha_{\text{JR}} \cdot \frac{n}{k}$. Further, since the voters in N' approve a common candidate c and do not approve any candidate in W, it holds that

$$|N'| \leq |\{v \in N : A_v \cap W = \emptyset \text{ and } c \in A_v\}| \leq s_{\max}.$$

Thus, $s_{\max} \geq \alpha_{\text{JR}} \cdot \frac{n}{k}$ which implies that $\alpha_{\text{JR}} \leq s_{\max} \cdot \frac{k}{n}$. □

Proposition 5.1 proves that, as for JR, we can determine if there exists an α-JR violation for a given committee in polynomial time. Nevertheless, finding $\alpha^*_{\text{JR}}(I)$ for an instance I, is NP-hard for all $0 < \alpha < 1$. The hardness also holds for EJR and EJR+. Remember that for $\alpha = 0$ the proportionality axioms are always violated by definition, and for $\alpha > \frac{k}{k+1}$ they can always be satisfied.

Theorem 5.2. *For all constant $0 < \alpha < 1$, deciding whether there exists a committee that satisfies α-Φ with $\Phi \in \{$JR, EJR, EJR+$\}$ is NP-hard.*

The intuition behind the reduction is as follows: We can ensure that certain candidates are selected if the support they receive is sufficiently large, and that every voter in the support only approves this specific candidate. Further, since $\alpha < 1$, we can choose the support of those candidates small enough that after picking those candidates, there remains a large number of voters unrepresented. Avoiding an α-JR violation for this group is then as hard as solving a vertex cover problem. The formal proof can be found in Section B.2.

Since the problem of determining the optimal α-value for an instance is computationally hard, we provide a method to determine the value by repeatedly solving the following ILP:

$$\sum_{c \in C} x_c = k, \qquad \forall c \in C, \tag{1}$$

$$y_v \leq \sum_{c \in A_v} x_c \qquad \forall v \in N, \tag{2}$$

$$\sum_{v:\, c \in A_v} (1 - y_v) \;\leq\; \lceil \alpha \cdot \frac{n}{k} \rceil - 1 \qquad \forall c \in C, \tag{3}$$

$$y_v, x_c \in \{0, 1\}.$$

Given some α, the ILP finds a committee W satisfying α-JR if one exists. The idea of the ILP is as follows. The binary variables x_c indicate whether the candidate c is included in the committee, and the binary variables y_v can only be 1 if voter v is covered by the output committee, i.e., v has an approved candidate c in the committee (this means $x_c = 1$).

Condition (1) ensures that the committee contains precisely k candidates, and condition (2) guarantees that, if $y_v = 1$, the voter v approves at least one candidate in the committee. The last condition ensures that at most $\lceil \alpha \cdot n/k \rceil - 1$ voters approving some common candidate c are left uncovered, capturing the α-JR requirement. The constraint is equivalent to saying that at least $|N_c| - \lceil \alpha \cdot n/k \rceil + 1$ voters in N_c are covered. Finally, since we are only interested in the feasibility of the constraints, the ILP does not need an objective function. A proof of the correctness of the ILP formulation is established in Section B.2.

Theorem 5.3. *The ILP has a solution if and only if there exists a committee that satisfies α-JR.*

Having addressed the case of JR, we now turn our attention to EJR. In contrast to JR, verifying or optimizing α for EJR is computationally more demanding. In fact, even determining whether a given committee satisfies EJR for $\alpha = 1$ is coNP-hard (Aziz et al., 2017). This hardness directly extends to determining whether, for a fixed α, an α-EJR violation exists.

Theorem 5.4. *For a fixed α, a given instance I and committee W, determining if W satisfies α-EJR is coNP-hard.*

For a given instance I, figuring out the optimal α-value with respect to EJR, i.e. finding $\alpha^*_{\text{EJR}}(I)$, can be solved by considering ℓ-cohesive groups individually with $1 \leq \ell \leq k$:

Namely, we can find for every ℓ the largest group of voters $s_{\max}(W, \ell)$ such that

$$s_{\max}(W, \ell) = \max_{T \subseteq C, |T| = \ell} |\{v \in N : |A_v \cap W| < \ell \text{ and } T \subseteq A_v\}|.$$

Similar to JR, we can now directly determine the largest α_ℓ such that there exists an (α, ℓ)-cohesive group that causes an α-EJR violation. Now for every ℓ, we compute $\alpha_\ell = s_{\max}(W, \ell) \cdot \frac{k}{\ell \cdot n}$ and take the maximum over these values. This gives the largest α-value such that there exists an α-EJR violation. The reason that this does not give an efficient algorithm lies in the fact that determining $s_{\max}(W, \ell)$ is computationally hard for larger ℓ. This computational complexity does not carry over to α-EJR+.

Proposition 5.5. *For a given instance $I = (A, k)$ and committee W, we can efficiently compute $\alpha_{EJR+}(W)$.*

6 Restricted Domains

Since it is computationally hard to compute $\alpha^*_\Phi(I)$ for general instances I and $\Phi \in \{\text{JR, EJR, EJR+}\}$, we look into some restricted domains in which we can efficiently find the optimal α-value. These domains can capture natural situations where preferences are aligned along a single dimension, such as voters and candidates positioned on a political spectrum, geographic proximity in local elections, or a preference for candidates differing along a skill or topic dimension.

6.1 Party-List Profiles

A commonly studied restricted domain is that of *party-list profiles*, where voters' preferences depend solely on the party affiliation of candidates.

Definition 6.1 (Lackner and Skowron (2023))**.** We say that an approval profile A is a *party-list profile* if for every pair of voters $v_i, v_j \in N$ we have either $A_{v_i} = A_{v_j}$ or $A_{v_i} \cap A_{v_j} = \emptyset$. An election instance I is called a *party-list instance* if (1) A is a party-list profile, and (2) for each voter $v \in N$, it holds that $|A_v| \geq k$.

For α-JR, an optimal committee can be obtained by ordering the parties according to their size and iteratively selecting one candidate from each party in this order until $|W| = k$. If every party has received a candidate, the remaining seats may be filled arbitrarily. We can extend this positive result to α-EJR.

Theorem 6.2. *Computing $\alpha^*_{EJR}(I)$ for party-list instances can be done efficiently.*

Proof. Let P be the set of parties. For α-EJR, compute for every party $p \in P$, their support size denoted by s_p. Candidates are added to the committee W sequentially in k rounds. In each round i, add a candidate from party p with

$$\arg\max_{p \in P} \frac{s_p}{w_{p,i} + 1}$$

where $w_{p,i}$ is the number of candidates approved by party p in the committee W at timestep i. The intuition behind the algorithm is the following: Check at every timestep which party causes the largest violation.

For the party p not to get $(\alpha, w_{p,i}+1)$-cohesive, it has to hold that $\frac{n}{k} \cdot \alpha \cdot (w_{p,i}+1) > s_p$. Solving for α, we get that $\alpha > \frac{s_p}{w_{p,i}+1} \cdot \frac{n}{k}$ where $\frac{n}{k}$ is a constant factor for all parties and can thus be ignored.

We claim that this algorithm returns a committee with optimal α-value with respect to EJR. Assume for the sake of contradiction that there does exist a committee W' with a smaller α'-value. If W and W' had the same number of candidates for each party, their α-values would be equal. Thus, since $|W| = |W'| = k$, there must exist a party p such that $w_p > w'_p$. Now, it holds that $\alpha' \geq \frac{s_p}{w'_p+1} \cdot \frac{k}{n}$ due to party p.

Next, consider the point where p received its last candidate in the algorithm: At this specific timestep i in the algorithm $\frac{s_p}{w_{p,i}+1} \geq \frac{s_t}{w_{t,i}+1}$ for all $t \in P$ and therefore it holds, that

$$\alpha \leq \frac{s_p}{w_{p,i}+1} \cdot \frac{k}{n} \leq \frac{s_p}{w'_p+1} \cdot \frac{k}{n} \leq \alpha'.$$

This contradicts our assumption that $\alpha' < \alpha$. ☐

Remark 4. *α-EJR and α-EJR+ are equivalent for party-list instances since any EJR+ violation (N', c, ℓ) corresponds to an EJR violation (N', A_v, ℓ) for any $v \in N'$ since by definition $|A_v| \geq \ell$.*

6.2 Voter-Interval Domain

Definition 6.3 (Elkind and Lackner (2015))**.** Given an election instance I, we say I has *voter-interval (VI)* preferences if there exists a linear order $\sqsubset$ over N such that for all voters $v_1, v_2, v_3 \in N$ and for each candidate $c \in A_{v_1} \cap A_{v_3}$, we have that $v_1 \sqsubset v_2 \sqsubset v_3 \implies c \in A_{v_2}$.

It has been shown by Aziz et al. (2017) that checking whether W provides EJR for an instance I is coNP-complete. However, to the best of our knowledge, there is no result on efficiently finding violations for ℓ-cohesive groups when we only consider restricted approval profiles. More specifically, we show the following lemma, which lets us efficiently check for an α-EJR violation given an instance I and committee W.

Theorem 6.4. *In the VI domain, verifying whether a committee W satisfies α-EJR can be done efficiently.*

Also, for JR, computationally hard problems become tractable for the VI domain, as the following theorem proves.

Theorem 6.5. *Computing $\alpha^*_{EJR}(I)$ in the VI domain can be done efficiently.*

Proof Sketch. We use the greedy algorithm in Algorithm 1, which constructs a cardinality-minimal subset of candidates that satisfies α-JR for a given I and α. See Lemma B.2 for a proof.
By checking if the size of the returned committee is at most k, we can efficiently verify if such a committee exists for a given α. Due to Lemma 4.11, the number of distinct α-values we need to consider is upper-bounded by n. Thus, we can efficiently construct a committee that satisfies the optimal α-value for I. ☐

In Algorithm 1, we use the observation that in the VI domain, every candidate c has a leftmost and rightmost voter according to the VI order who approves them. We denote these voters by l_c and r_c, respectively. For $i \in [n]$, the algorithm considers the set $\{v_1, \dots v_i\}$ and fixes an α-JR violation, if existent, with a "right-most" candidate according to the VI order. Intuitively, this ensures that the violation to the left of v_i is resolved, while covering the maximum number of voters to the right, thereby postponing the next potential violation as long as possible.

Algorithm 1 Finding Optimal Committees in the VI Domain

Input: Election $I = (A, k)$ in the VI domain, $\alpha \in \mathbb{R}_{\geq 0}$
Output: Committee $W \subseteq C$
$W \leftarrow \emptyset$
for $i = 1, \dots, n$ **do**
    Check whether voters $\{v_1, \dots, v_i\}$ admit an α-JR violation under W
    if a violation is found **then**
        $W \leftarrow W \cup \{c\}$ with $c \in A_i$ and $r_{c'} \sqsubseteq r_c$ for all $c' \in A_i$.
return W

6.3 Candidate-Interval Domain

Definition 6.6 (Elkind and Lackner (2015))**.** Given an election instance I, we say that I has *candidate-interval preferences (CI)* if there exists a linear order $\sqsubset$ over C such that for each voter $v \in N$ and all candidates $a, c \in A_v$, $b \in C$ we have that $a \sqsubset b \sqsubset c \implies b \in A_v$.

Intuitively, each voter approves a consistent interval of candidates. It turns out that the positive results for the VI domain also hold in the CI domain. More specifically, we show that we can find the optimal α-value for a given instance I and committee W with respect to EJR.

Theorem 6.7. *In the CI domain, verifying whether a committee W satisfies α-EJR can be done efficiently.*

Again, as for the VI domain, it is possible to efficiently compute a committee that satisfies α-JR for the optimal value of α.

Theorem 6.8. *Computing $\alpha^*_{EJR}(I)$ in the CI domain can be done efficiently.*

The full proof can be found in Section B.3. The idea is equivalent to the one used in the VI domain. We have at most $n+1$ α-values to check, and for a given α-value, we use a greedy algorithm as presented in *Algorithm* 2 to find a minimal committee satisfying α-JR. The size of the returned committee then indicates the satisfiability of a given α-value. Here, we iterate over the candidates in the linear order $\sqsubset$ given by the CI domain. Intuitively, the algorithm has to keep a candidate if a set of voters with a relatively "early" interval has a violation, and later candidates cannot make up for this violation. This results in the algorithm constructing an optimal committee that pushes the candidates to the right regarding the linear order $\sqsubset$ over the candidates.

Algorithm 2 Finding Optimal Committees in the CI Domain

Input: Election $I = (A, k)$ in the CI domain, parameter $\alpha \in \mathbb{R}_{\geq 0}$
Output: Committee $W \subseteq C$
$W \leftarrow C$
for $i = 1, \dots, m$ **do**
    if $W \setminus \{c_i\}$ satisfies α-JR **then**
        $W \leftarrow W \setminus \{c_i\}$
return W

7 Empirical Evaluation

Finally, we complement our theoretical results with an empirical analysis. We generate synthetic voting instances to evaluate the resulting values of α and to assess the performance of existing multi-winner voting rules. Our primary focus lies on α-EJR; additional plots for α-JR are deferred to Section C. Figure 5 reports the distribution of optimal values, α^*_{EJR}, under the Impartial Culture (IC) and the Euclidean Threshold (EUT) model, two commonly used probabilistic models for generating approval profiles. For each model, we consider instances of varying sizes—ranging from small to moderate—with up to 60 voters and 15 candidates. Both models are well used in approval-based multiwinner voting (Bredereck et al., 2019; Faliszewski et al., 2017; Elkind et al., 2017). Further details on the parameter configurations of them are provided in Section C.

The results demonstrate that in many instances α can be selected to be significantly smaller than $\frac{k}{k+1}$. In fact quite often α^*_{EJR} is closer to 0 than it is to 1. Bardal et al. (2025) show similar results when analyzing the α-value in the ranked setting for *proportionality of solid coalitions*.

In addition to our theoretical analysis in Section 4.2, we evaluate the performance of existing voting rules on randomly generated instances. Figure 6 demonstrates that MES, seq-Phragmén, and PAV typically achieve committees with comparatively low α-values with respect to EJR, indicating that these rules perform well on average. Note that MES is completed with seq-Phragmén. Our experiments, however, also uncover instances in which all three rules yield committees whose α-values exceed the optimum by more than a factor of four, highlighting the presence of substantial worst-case deviations even in average-case data. These cases can be mostly seen among larger instances. In Section C, we provide more fine-grained information on how the voting rules perform under different instance sizes. CC does not perform very well, which is unsurprising because its only goal is to satisfy as many voters as possible and because it does not satisfy EJR for all instances. In Figure 8 we show that, on the contrary, CC performs best out of the considered rules for α^*_{JR}.

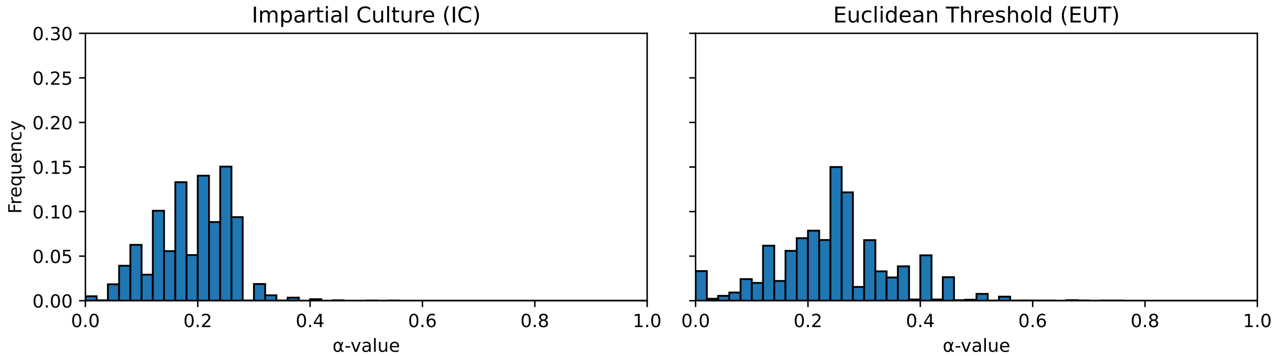

Figure 5: Distribution of optimal α-values, α^*_{EJR}, under the Impartial Culture and the Euclidean Threshold model, based on 6400 generated instances each.

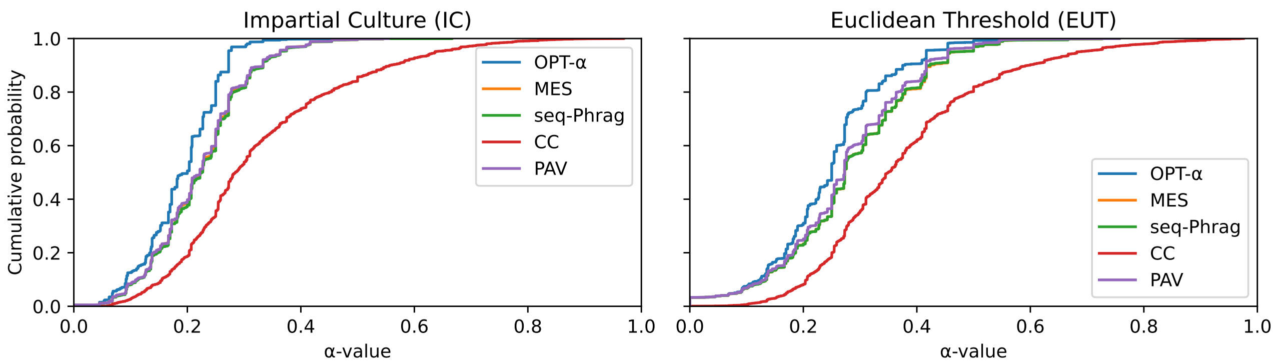

Figure 6: Cumulative distribution of the optimal α-value, α^*_{EJR}, and the α-value achieved by MES, seq-Phragmén, CC, and PAV under the Impartial Culture and Euclidean Threshold model, based on 6400 samples.

8 Conclusion

We demonstrate that our dynamic extensions of JR and EJR can lead to much fairer outcomes for various instances. Our axioms possess a clear fairness appeal, as they seek to minimize the underrepresentation of large aligned groups, which in real-world political settings may otherwise foster resentment. However, as for many other voting rules that try to maximize some objective, like CC or PAV, computing the optimal α-value is NP-hard, even for JR.

While existing voting rules can exhibit poor worst-case behavior theoretically, they perform reasonably well on our artificially generated datasets. A natural direction for future work is to evaluate these rules on real-world data, for instance, using the Polkadot dataset (Boehmer et al., 2024). When restricting the set of voting instances, we obtain several positive results regarding the efficient computation and verification of α-values. Empirical results underline the importance of considering adaptive quotas in multiwinner approval voting and perhaps other settings. Further extensions could include generalizing our framework to weakly cohesive groups or developing broader notions of cohesiveness based on justified representation functions. Such generalizations may also prove useful in continuous settings, such as budget aggregation.

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A Mentioned Voting Rules

In this section, we formally define the voting rules that we use in the main document.

Chamberlin-Courant Rule (CC): Selects all committees that maximize the CC-score, i.e. $\arg\max_{W\subseteq C,|W|=k} \mathrm{CC}(W)$. The CC-score is equal to the number of voters approving at least one candidate in the committee. Formally,

$$\mathrm{CC}(W) = |\{i \in N : A_i \cap W \neq \emptyset\}|.$$

Its sequential variant chooses candidates one after another, maximizing the number of newly covered voters in every step.

Proportional Approval Voting (PAV): Select all committees that maximize the PAV-score, i.e. $\arg\max_{W\subseteq C,|W|=k} \mathrm{PAV}(W)$. The PAV score is defined as

$$\mathrm{PAV}(W) = \sum_{i\in N} H(|A_i \cap W|),$$

where $H(t) = \sum_{j=1}^{t} 1/j$, is the t-th harmonic number, with $H(0) = 0$.

Sequential Phragmén (seq-Phragmén) (Brill et al., 2024): The rule *seq-Phragmén* starts with an empty committee and iteratively adds candidates, always choosing the candidate that minimizes the (new) maximum voter load (under the assumption that previously assigned loads cannot be redistributed). Let $\bar{x}^{(j)}$ denote the voter loads after round j. At first, all voters have a load of 0, i.e., $\bar{x}_i^{(0)} = 0$ for all $i \in N$. In each round, we keep the already assigned loads, but we may further increase them and give the additional load to a new candidate c. In other words, we require

$$\bar{x}_i^{(j)} \geq \bar{x}_i^{(j-1)} \quad \text{for all } i,$$

with equality unless $i \in N_c$. Moreover, the sum of the loads added in the round should be 1. We select the candidate c and the loads $\bar{x}^{(j)}$ that satisfy these conditions and minimize $\max_i \bar{x}_i^{(j)}$.

Method of Equal Shares(MES) (Peters and Skowron, 2020): This rule starts with a budget of k/n per voter and an empty committee. Adding a candidate to the committee incurs a cost of 1. In each step, MES selects a candidate that minimizes the maximum amount of budget any voter has to pay. A detailed description can be found in Algorithm 3. The budget of each voter is $b = k/n$.

Algorithm 3 MES(b)

$W \leftarrow \emptyset$
$b(v) \leftarrow b$ for all $v \in N$
$D \leftarrow \{c \in C \setminus W | \sum_{v\in N_c} b(v) \geq 1\}$
while $D \neq \emptyset$ **do**
  for $c \in D$ **do**
    $q(c) \leftarrow \min\{q \geq 0 | \sum_{v\in N_c} \min\{b(v), q\} \geq 1\}$
  $c^* \leftarrow \arg\min_{c\in D} q(c)$
  $W \leftarrow W \cup \{c^*\}$
  for $v \in N_{c^*}$ **do**
    $b(v) \leftarrow b(v) - \min\{b(v), q(c^*)\}$
  $D \leftarrow \{c \in C \setminus W | \sum_{v\in N_c} b(v) \geq 1\}$
return W

Based on the idea by Casey and Elkind (2025), we define α-MES to be the algorithm as above, but voters obtain the minimum amount of budget b such that at least k candidates are bought, i.e., $b = \min\{b \in \mathbb{R}_+ : |\mathrm{MES}(b)| \geq k\}$. If more than k candidates can be bought, we stop after k candidates have been bought.

Greedy Justified Candidate Rule (GJCR)(Brill and Peters, 2023): This rule iteratively detects the largest current EJR+ violation and adds the responsible candidate to the committee. Under GJCR, the addition of a candidate incurs a cost of 1, which is uniformly split among all voters involved in the violation. For a formal definition, see Algorithm 4.

Algorithm 4 GJCR

$W \leftarrow \emptyset$
$p(v) \leftarrow 0$ for all $v \in N$
for $\ell \in \{k, k-1, ..., 1\}$ **do**
  $N(c) \leftarrow \{v \in N_c : |A_v \cap W| < \ell\}$ for $c \notin W$
  while there is $c \notin W$ s.t. $|N(c)| \geq \ell \cdot \frac{n}{k}$ **do**
    Add candidate c maximizing $|N(c)|$ to W
    $p(v) \leftarrow p(v) + \frac{1}{|N(c)|}$ for all $v \in N(c)$
return W

Similar to Droop-GJCR defined in Casey and Elkind (2025), we define α-GJCR to be the rule that looks for candidates $c \notin W$ such that $|N(c)| \geq \alpha \cdot \ell \cdot \frac{n}{k}$. We minimize the α-value until the algorithm chooses at least k candidates. We again stop after k candidates have been chosen.

B Omitted Proofs

In this appendix, we present the proofs missing from the main body of our paper.

B.1 General Properties

Proposition 4.7. *For $\alpha_1 \leq \alpha_2 \leq \alpha_3$, it holds that α_1-EJR+ $\implies$ α_2-EJR $\implies$ α_3-JR.*

Proof. Let $\alpha_1 \leq \alpha_2$ and $\Phi \in \{\text{JR, EJR, EJR+}\}$. First, it holds that α_1-$\Phi \implies \alpha_2$-Φ since increasing α can only decrease the number of cohesive groups, which therefore can only make the axiom weaker. Therefore, α_1-EJR+ $\implies$ α_2-EJR+. Let α_2-EJR+ be satisfied by committee W. Assume for contradiction that α_2-EJR is violated with witness (N', C', ℓ) where $N' \subseteq N$ and $C' \subseteq C$. For all $v \in N'$, $|A_v \cap W| < \ell$ and due to cohesiveness of N', there is a candidate $c \in C' \setminus W$. Since $|N_c| \geq \alpha_2 \cdot \ell \frac{n}{k}$, this candidate causes an α_2-EJR+ violation; a contradiction. Next, we show that α_2-EJR implies α_3-JR. Again, α_2-EJR implies α_3-EJR, and since α_3-JR is a special case of α_3-EJR, the implication follows. □

Proposition 4.8. *For $\alpha_1 < \alpha_2 < \alpha_3 \leq 1$, it holds that α_3-EJR+, α_2-EJR, and α_1-JR are incomparable.*

Proof. First, we show that α_1-EJR+ does not imply α_3-JR. For this, consider the following instance. Let $\alpha_2 = \frac{a}{b}$, $N = \{1, \dots b\}$ and $C = \{c_1, c_2\}$, $k = 1$, with $A_v = \{c_1\}$ for $v \in \{1, \dots, a\}$. Then, the committee $W = \{c_2\}$ satisfies α_1-EJR+. This follows from the fact that since

$$\alpha_1 \cdot \frac{n}{k} > \alpha_2 \cdot \frac{n}{k} = \frac{a}{b} \cdot b = a,$$

there does not exist any cohesive groups for α_1 in the instance. On the other hand, the voters $\{1, \dots, a\}$ together with candidate c_1 are witnessing an α_2-JR violation.

The same example can also be used to show that α_1-EJR+ does not imply α_2-EJR and that α_2-EJR does not imply α_3-JR.

Next, we show that α_2-EJR does not imply α_1-EJR+. So let $\alpha_1 = \frac{a_1}{b_2}$ and $\alpha_2 = \frac{a_2}{b_2}$ with $b_1 \geq 2$. We create the instance $N = [(2 \cdot b_1 \cdot b_2) \cdot (a_1 \cdot b_2)]$, with $C = \{c_1, \dots, c_{2 \cdot a_1 \cdot b_2}\} \cup \{c\}$ and $k = 2 \cdot a_1 \cdot b_2$ and $A_{v_i} = \{c, c_i\}$ for $i \in [2 \cdot a_1 \cdot b_2]$. Thus, $\alpha_1 \cdot \frac{n}{k} = \frac{a_1}{b_1} \cdot \frac{(2 \cdot b_1 \cdot b_2) \cdot (a_1 \cdot b_2)}{2 \cdot a_1 \cdot b_2} = (a_1 \cdot b_2)$ and $\alpha_2 \cdot \frac{n}{k} = \frac{a_2}{b_2} \cdot \frac{(2 \cdot b_1 \cdot b_2) \cdot (a_1 \cdot b_2)}{2 \cdot a_1 \cdot b_2} = a_2 \cdot b_1$.

We claim that the committee $W = \{c_1, \dots, c_{2 \cdot a_1 \cdot b_2}\}$ satisfies α_2-EJR but violates α_1-EJR+. First, observe that the committee violates α_1-EJR+. This follows, since $|N_c| = 2 \cdot \alpha_1 \cdot \frac{n}{k}$, no voter in N approves 2 candidates in W and $c \notin W$. On the other hand, we claim that there does not exist an $(\alpha, 2)$-cohesive group. Since $|A_v \cap A_{v'}| \leq 1$ for any $v \neq v' \in N$ an $(\alpha, 2)$-cohesive group can contain at most one voter. But as computed above, an $(\alpha, 2)$-cohesive group has to be of size at least $a_2 \cdot b_1 \geq 2$. On the other hand, every voter who approves of any candidate approves of at least one candidate in W. Thus, W satisfies $(\alpha, 2)$-EJR.

The same example can be used to prove that α_3-JR does not imply α_1-EJR+.

Lastly, we show that α_2-JR does not imply α_1-EJR for $\alpha_1 > \alpha_2$. Consider the following instance $N = [4], C = \{c_1, c_2, c_3\}, k = 2$ with $A_i = \{c_1, c_2\}$ for $i \in [4]$. Then, $W = \{c_1, c_3\}$ satisfies α_2-JR for all $\alpha_2 > 0$ since every voter approves at least one candidate in W. On the other hand, W does not satisfy EJR for any $\alpha_1 \leq 1$ since N together with $\{c_1, c_2\}$ witness an $(\alpha_1, 2)$-EJR violation. □

Proposition 4.9. *The additive gap between JR and EJR for the optimal α-value can be $k/(k+1)$.*

Proof. Consider a generalization of the example from Figure 3. We have a set of voters $\{1, \dots, x\}$ approving candidates c_1 to c_k and a set of voters $\{x+1, \dots, x+y\}$ approving candidate c_{k+1}. Without loss of generality, let c_1 to c_{k-1} be in the committee W. If c_{k+1} is also in the committee, then the voters approving candidate c_k are (α_1, k)-cohesive for an α_1-value of $\frac{x}{x+y}$. If c_k instead of c_{k+1} is in the committee, then the voters approving c_{k+1} are (α_2, 1)-cohesive for an α_1-value of $\frac{k \cdot b}{a+b}$. So the optimal α-value with respect to EJR is

$$\min(\frac{x}{x+y}, \frac{k \cdot y}{x+y}).$$

By setting $x = k \cdot y$, both values simplify to $k/(k+1)$. On the other hand, there exists a committee such that every voter approves at least one candidate, for instance, the committee $W = \{c_{k+1}, c_1, \dots, c_{k-1}\}$, the optimal α-value is equal to $1/n$. The optimal committee with respect to α-JR $\{c_1, c_{k-1}\}$, resulting in an optimal α-value of $1/n$ where $n = x + y = k \cdot y$. Therefore, the additive gap between JR and EJR for the optimal α-value tends to $k/(k+1)$. □

Lemma 4.11. *For a given instance I, at most $n \cdot k$ many α-values to determine the α^*_{EJR}. Further,it holds that $\alpha^*_{EJR} \in \mathbb{Q}$.*

Proof. Similar to Lemma 4.11, it also holds for α-EJR that there are only a polynomial number of α-values that need to be considered.

For the proof, consider ℓ-cohesive groups separately for $\ell \in \{1, \dots, k\}$.

We claim that for ℓ-cohesive groups it is sufficient to consider the α-values $X_\ell = \{0, \frac{k}{\ell \cdot n}, \frac{2k}{\ell \cdot n}, \dots, \frac{r_\ell \cdot k}{\ell \cdot n}\}$ with $r_\ell = \lceil \frac{n \cdot \ell}{k} \rceil - 1$. Since for each ℓ there are at most n different α-values in total, there are at most $k \cdot n$ many such values. Let $X = \cup_{\ell \in \{1, \dots, k\}} X_\ell$ be the set of all α-values ordered from small to large.

We now claim that $\alpha^*_{\text{EJR}} \in X$. So let $\alpha = \alpha^*_{\text{EJR}}$ be the optimal value. Thus, there exists some α-EJR violation witnessed by some (N', C', ℓ). Let this again be the largest violation, meaning that the ratio of $|N'|$ and ℓ is maximal. Therefore, it holds that $|N'| \geq \alpha \cdot \ell \cdot \frac{n}{k}$. As for JR, it must hold that this inequality is tight, since otherwise α is clearly not optimal. Thus, $\alpha = |N'| \cdot \frac{k}{\ell \cdot n} \in X_\ell \subseteq X$ which proves the claim. □

In the main body of the paper Example 4.12 gave an example which relied on tie breaking to show that many rules can return bad committees. In the following we provide an example which does not rely on tie-breaking. The arguments work the same as for Example 4.12, with the difference that the optimal value in this instance is $2 \cdot \frac{k}{n}$ achieved by the committee $W = D$.

Example B.1. Consider the following family of instances. Let $N = [(k+1)(k+2)]$. Furthermore, let $C = B \cup D$, with $B = \{b_1, \ldots, b_{k+1}\}$ such that $N_{b_i} = \{(i-1)\cdot(k+2)+1, \ldots, i\cdot(k+2)\}$ and $D = \{d_1, \ldots, d_k\}$ such that $N_{d_i} = \{j \in N\colon (j \mod (k+2)) = i\}$ for all $i \in \{1, \ldots, k+1\}$. Then each candidate in D and each candidate in B is approved by $k+1$ voters.

B.2 Complexity Results

Theorem 5.2. *For all constant $0 < \alpha < 1$, deciding whether there exists a committee that satisfies α-Φ with $\Phi \in \{JR, EJR, EJR+\}$ is NP-hard.*

Proof. We first prove the statement for JR. For hardness, we give a reduction from vertex cover, which is one of Karp's original 21 NP-complete problems (Karp, 1972): Given a graph $G = (V, E)$, where $V = \{v_1, \ldots v_n\}$ is the set of vertices and E the set of edges, the question is whether there exists a vertex cover of size at most r.

Let $\alpha = \frac{a}{b}$, where $a \geq 2$. (If $a = 1$ just multiply a, b with a factor of 2.)

We construct the following multiwinner voting instance: Let $N = V \cup X \cup Y \cup Z \cup F$ with

$$X = \{x_1, \ldots, x_{a(|V|+a-2)}\},$$
$$Y = \{y_1, \ldots, y_{ar}\},$$
$$Z = \{z_1, \ldots, z_{(b-a-1)(|V|+a-2)+r\cdot(b-a)}\},$$
$$F = \{f_1, \ldots, f_{a-2}\}.$$

The set of candidates is defined as $C = C_X \cup C_Y \cup C_E$ with

$$C_X = \{c^X_1, \ldots c^X_{|V|+a-2}\},$$
$$C_Y = \{c^v_i\colon i \in [r], v \in V\},$$
$$C_E = \{c^E_e\colon e \in E\},$$

and the committee size $k = |V| + a - 2 + r$. Finally, the approval ballots for all voters are as follows:

$$A_v = \{c^E_e\colon v \in e\} \cup \{c^v_j\colon j \in [r]\} \text{ for all } v \in V,$$
$$A_{x_i} = \{c^X_{\lceil i/a \rceil}\} \text{ for all } x_i \in X,$$
$$A_{y_i} = \{c^v_{\lceil i/a \rceil}\colon v \in V\} \text{ for all } y_i \in Y,$$
$$A_f = C_E \text{ for all } f \in F.$$

In the following, we want to argue that there exists a committee that satisfies α-JR if and only if there exists a vertex cover of size r. For this, we make the following observations about the instance:

1. Observation: The voters in Z do not approve any candidate and therefore cannot cause JR violations.
2. Observation: In total, there are $a(|V|+a-2) + ar + (b-a-1)(|V|+a-2) + r(b-a) + a - 2 + |V| = b(|V|+a-2+r)$ many voters. Thus, in order to be an α-cohesive group, the group has to contain at least $\alpha \cdot \frac{n}{k} = \frac{a}{b} \cdot \frac{b(|V|+a-2+r)}{|V|+a-2+r} = a$ many voters.
3. Observation: Since for every candidate in C_X there exists a voters in X that precisely approve that candidate, this implies that every committee W that satisfies α-JR has to contain all candidates C_X.
4. Observation: Every committee W satisfying α-JR has to contain a candidate c^v_i for every $i \in [r]$, otherwise there exists a group of a voters in Y that does not approve of any candidate in W.

From the latter two observations, we get that every committee that satisfies α-JR only contains candidates in $C_X \cup C_Y$ and that at least $|V| + a - 2 + r$ $(= k)$ many candidates need to be selected such that no α-JR violation can occur.

Put differently, we need to select one candidate c^v_i for every $i \in [r]$, as well as every candidate in C_X. For every such committee, it holds that every voter in X and Y approves at least one candidate and, thus, cannot be part of an α-JR violation.

Lastly, due to our approval ballot construction, every voter in F does not approve of any candidate in such a committee, and at most r voters in V approve a candidate in such committees.

With these observations, we can finally prove the equivalence: First, we show that if there exists a vertex cover for the original instance, then there exists a committee that satisfies α-JR. So let $\{v^{VC}_1, \ldots v^{VC}_r\}$ be a vertex cover. We create the following committee

$$W = C_X \cup \{c^{v^{VC}_i}_i : i \in [r]\}.$$

Observe that every voter of the vertex cover approves at least one candidate in W. We claim that this committee satisfies α-JR. The only violations can be due to a group of voters in $V \cup F$ and a candidate $c \in C_E$. Since every candidate in C_E is approved by precisely $\alpha \cdot \frac{n}{k} = a$ many voters, a violation would imply that the whole support of a candidate is not represented in W. Since the support of every candidate $c \in C_E$ contains two voters $v_1, v_2 \in V$ such that $\{v_1, v_2\} \in E$, at least one of them approves a candidate in W, and thus we have found a committee that satisfies α-JR.

For the reverse direction, assume that there exists a committee W that satisfies α-JR. As argued above, it has to hold that $C_X \subseteq W \subseteq C_X \cup C_Y$. Further, it holds that at most r voters $V' \subseteq V$ approve any candidate in W (at most one for every $c \in W \cap C_y$). We claim that V' forms a vertex cover. Assume for the sake of contradiction that this does not hold. Then, there exists an edge $e = \{u, v\}$ that is not covered by any vertex in V'. Thus, neither u nor v approves any candidate in W but both approve the common candidate c_e^E. Further, every voter in F does not approve any candidate in W and also approves c_e^E. Together they form an α-JR violation contradicting the assumption that W satisfies α-JR. This concludes the proof for α-JR.

In the following, we argue that the same reduction can also be used to show hardness for α-EJR and α-EJR+. For this, we analyze how many voters approve of the different candidates. It holds that $|N_c| = a$ for every $c \in C_X$ since a voters in X approve the candidate. For every candidate in C_E, it holds that all $a-2$ voters in E and two voters in V approve the candidate, so again, a total of a. For every candidate in C_y, a voters in Y, and one voter in V approve the candidate for a total of $a+1$ voters. Every 2-cohesive group must be of size at least

$$\alpha \cdot 2 \cdot \frac{n}{k} = \frac{a}{b} \cdot 2 \cdot \frac{b(|V| + a - 2 + r)}{|V| + a - 2 + r} = 2a.$$

Therefore, it has to hold that $2a \leq a + 1$, which only holds for $a = 1$. By assumption, $a > 1$. Thus, the instance does only contain 1-cohesive groups. Since for 1-cohesive groups EJR+, EJR, and JR are equivalent, the reduction also works for EJR and EJR+. $\square$

Theorem 5.3. *The ILP has a solution if and only if there exists a committee that satisfies α-JR.*

Proof. First, assume that the ILP has a solution. Define $W = \{c \in C \colon x_c = 1\}$ as our committee. We claim that W does not have any α-JR violation. Assume for the sake of contradiction, there exists one with witness (N', c'). This means all voters in N' approve c' and all voters in N' do not approve any candidate in W. Therefore, we can conclude from (1) that $x_c = 0$ for all $c \in A_i$ with $i \in N'$. Therefore, using (2), we get that $y_i = 0$ for all $i \in N'$. Finally, considering (3) for candidate c', we get that $\sum_{i\colon c' \in A_i}(1 - y_i) \leq |N'|$ and that $|N'| \geq \lceil \alpha \cdot \frac{n}{k} \rceil$, which contradicts the assumption that the ILP had a solution for α.

For the other direction, assume there is a committee W that, for a given α, satisfies α-JR. We can define the following solution for the ILP. We set $x_c = 1$ if and only if $c \in W$ and $y_i = 1$ if and only if $W \cap A_i \neq \emptyset$ for all $c \in C, i \in N$. We claim that this is a valid solution for the ILP. First note that since $|W| = k$, condition (1) is satisfied. Consider now condition (2). If $y_i = 0$, then the condition trivially holds, so assume that $y_i = 1$. If $y_i = 1$, then by definition there exists a candidate $c \in A_i \cap W$ and therefore $x_c = 1$. So, finally assume for the sake of contradiction that (3) is violated for some c. Then, there exists a group N' such that $|N'| \geq \lceil \alpha \cdot \frac{n}{k} \rceil$ and for every member $i \in N'$, we have $y_i = 0$ and $c \in A_i$. If $y_i = 0$, this implies that $A_i \cap W = \emptyset$. But, then (N', c) are witnessing an α-JR violation contradicting our assumption that W satisfies α-JR. This concludes the proof. $\square$

Theorem 5.4. *For a fixed α, a given instance I and committee W, determining if W satisfies α-EJR is coNP-hard.*

Proof. We give a reduction which can be used for basically all hardness results and counterexamples that already hold for some larger α (in this case $\alpha = 1$).

We know that for smaller α, all conditions get even harder to satisfy. To be able to use the same arguments as in the original instance ($\alpha = 1$), we can rescale it accordingly.

Formally, let $\alpha = \frac{a}{b}$ with $a, b \in \mathbb{N}$. Given any instance (N, C, k) for the Hare quota, we transform the instance to the following instance (N_α, C, k) such that $N_\alpha = \bigcup_{i=1}^{a} N^{(i)} \cup Z$, where each $N^{(i)}$ is an exact copy of the voters in N meaning that the voters approve precisely the same candidates in C, and $Z = \{z_1, \ldots z_{(b-a)\cdot|N|}\}$ are voters that do not approve any candidate in C.

We now claim that in this adapted instance, there exists an α-EJR violation for a committee W if and only if in the original instance there was an EJR violation.

So, assume first that W violates EJR. Thus, there exists a witness set (N', C', ℓ) with $|N'| \geq \ell \cdot \frac{n}{k}$, $|C'| \geq \ell$ such that for all voters in $v_i \in N'$ it holds that $|A_{v_i} \cap W| < \ell$. We now consider the set $N'_\alpha = \bigcup_i^a N'^{(i)}$ in the transformed instance. First, since $|N_\alpha| = n_\alpha = b \cdot |N|$, it holds that

$$|N'_\alpha| = a \cdot |N'| \geq a \cdot \ell \cdot \frac{n}{k} = a \cdot \ell \cdot \frac{n_\alpha}{b \cdot k} = \alpha \cdot \ell \cdot \frac{n_\alpha}{k}.$$

By definition, every voter in N'_α approves all candidates in C' but at most $\ell - 1$ candidates in W. Therefore, it holds that (N'_α, C, ℓ) is a witness for an α-EJR violation.

For the reverse direction, let (N'_α, C', ℓ) be an EJR violation. Thus, it holds that $|N'_\alpha| \geq \alpha \cdot \ell \cdot \frac{n_\alpha}{k} = \ell \cdot \frac{n \cdot a \cdot b}{k \cdot b} = a \cdot \ell \cdot \frac{n}{k}$. We now define the set

$$N' = \{v \in N \colon \text{there exists an } i \text{ in } 1 \leq i \leq a \text{ with } v^{(i)} \in N'_\alpha\}.$$

First, observe that $|N'| \geq \frac{|N'_\alpha|}{a} = \ell \cdot \frac{n}{k}$. Furthermore, no voter in N' approves more than $\ell - 1$ voters in W. Thus, (N', C, ℓ) witnesses an EJR violation for the original instance. This concludes the proof. $\square$

Proposition 5.5. *For a given instance $I = (A, k)$ and committee W, we can efficiently compute $\alpha_{EJR+}(W)$.*

Proof. The proof resembles the one for Proposition 5.1. We compute for every candidate $c \notin W$ and $\ell \in \{1, \ldots, k\}$ the set of all voters that approve the candidate c but approve at most $\ell - 1$ candidates in W. Formally,

$$N'_{c,\ell} := \{i \in N_c \colon |A_i \cap W| < \ell\}.$$

Observe that this group of voters $N'_{c,\ell}$ forms an α-EJR violation for $\alpha = |N'_{c,\ell}| \cdot \frac{k}{\ell n}$ since $|N'_{c,\ell}| \geq |N'_{c,\ell}| \cdot \frac{k}{\ell n} \cdot \ell \cdot \frac{n}{k} = \alpha \cdot \ell \cdot \frac{n}{k}$.

Finally, let $\alpha = \max_{c \notin W, \ell \in \{1,\ldots,k\}} |N'_{c,\ell}| \cdot \frac{k}{\ell n}$. Therefore, $\alpha \leq \alpha_{\text{EJR}+}$.

It remains to show that $\alpha \geq \alpha_{\text{EJR}+}$. By definition, there exists some $\alpha_{\text{EJR}+}$-EJR+ violation (N', c, ℓ) for

$N' \subseteq N$ and $c \in C$. Thus, it holds that $|N'| \geq \alpha_{\text{EJR+}} \cdot \ell \cdot \frac{n}{k}$. Further, since the voters in N' all approve c but at most $\ell - 1$ candidates in W, we get that $|N'| \leq |\{i \in N_c \colon |A_i \cap W| < \ell\}| \leq |N'_{c,\ell}|$. Consequently, $|N'_{c,\ell}| \geq \alpha_{\text{EJR+}} \cdot \ell \cdot \frac{n}{k}$ which implies that $\alpha_{\text{EJR+}} \leq |N'_{c,\ell}| \cdot \frac{k}{\ell n} \leq \alpha$. This concludes the proof. □

B.3 Restricted Domains

Theorem 6.4. *In the VI domain, verifying whether a committee W satisfies α-EJR can be done efficiently.*

Proof. Let W be the committee, $N = \{v_1, \dots, v_n\}$ be the voters ordered according to the VI order, and A be the approval profile. Furthermore, let $T(i, j)$ be the set of candidates that are approved by voters v_i, v_j and all voters between them according to the linear order $\sqsubset$. Since each candidate is approved by a voter interval, we can conclude that every voter v such that $v_i \sqsubset v \sqsubset v_j$ approves every candidate in $T(i, j)$. Now, for every interval, we compute the following value, which can be thought of as similar to the h index. We want to find the largest integer ℓ such that in the interval i, j there exist at least $\lceil \frac{n}{k} \rceil \cdot \alpha \cdot \ell$ many voters $S \subseteq \{v_i, \dots, v_j\}$ that approve at most $\ell - 1$ many candidates in W. We can compute this value by doing a binary search over ℓ to find the largest ℓ for each interval (i, j).

Now, to find the largest violation, it is sufficient to consider every possible interval of which $O(n^2)$ many exist.

Trivially, every violation we find with this procedure is an α-EJR violation. In the following, we also prove that if there exists an α-EJR violation (N', C', ℓ) then the algorithm above finds an α-EJR violation (N'', C'', ℓ') with $\ell' \geq \ell$, and $N' \subseteq N''$.

So let N' be the set of voters witnessing the violation, and let v_i be the first voter, as well as v_j be the last voter in N', according to the VI order. When the algorithm considered the interval (i, j), there were at least $|N'| \geq \frac{n}{k} \cdot \ell \cdot \alpha$ many voters approving at most $\ell - 1$ many candidates in W. Further, each of them approves at least ℓ common candidates C' with $C' \subseteq T(i, j)$. Thus, it holds that the algorithm finds a violation of size at least ℓ for this interval (i, j). This concludes the proof for EJR.

For JR, the same arguments work if we only consider 1-cohesive groups. □

Theorem 6.5. *Computing $\alpha^*_{EJR}(I)$ in the VI domain can be done efficiently.*

Proof. To prove the theorem, we use Algorithm 1. Observe that the algorithm itself does not necessarily return a subset of size k. The algorithm returns the smallest subset W that does not contain any α-JR violation. Lemma B.2 proves this statement. Thus, we can execute Algorithm 1 and a committee satisfying α-JR exists if and only if the algorithm returns a subset with at most k candidates. □

Lemma B.2. *Let W be the set of candidates returned by Algorithm 1. Then, there does not exist any set $W' \subseteq C$ with $|W'| < |W|$ that satisfies α-JR.*

Proof. We prove this statement by induction over the voters according to the VI order. We say a subset of candidates is feasible if it satisfies α-JR and optimal if it is a feasible subset with minimal cardinality. Observe that the algorithm returns a feasible subset of candidates, so there always exists at least one optimal feasible subset of candidates. Let C_j be the set of candidates selected up to timestep j from our algorithm.

Claim. For every $j \in \{1, \dots, n\}$, there exists an optimal subset of candidates $W^{(j)}$ such that every candidate that the greedy algorithm selected at some index $\leq j$ belongs to $W^{(j)}$, i.e., $C_i \subseteq W^{(j)}$.

Base: $j = 1$. If $C_i = \emptyset$, we are done, so assume that the set is non-empty. This means voter v_1 causes an α-JR violation by herself, and every feasible subset of candidates must select a candidate that is approved by voter v_1. Observe that the candidate c selected by Algorithm 1 is defined such that $r_{c'} \sqsubseteq r_c$ for all $c' \in A_{v_1}$. Since we are in the VI domain, this implies that $N_{c'} \subseteq N_c$. Therefore, if an optimal subset does not contain c it has to contain such a c', and we can swap them while upholding α-JR. This proves the induction base.

Step: Let the claim be satisfied for some $j - 1 \geq 0$. If no candidate is selected in iteration j, we have $C_{j-1} = C_j$ and the claim follows from the induction hypothesis since $C_{j-1} \subseteq W^{(j-1)}$. So assume a candidate c_j is selected in iteration j. Since there is an α-JR violation caused by a set of voters N'' with $v \sqsubseteq v_j$ for every $v \in N''$ and containing voter v_j, $W^{(j-1)} \setminus C_{j-1}$ must contain a candidate c'' that is approved by some voter $v \in N''$ and thus $v \sqsubseteq v_j$.

We now claim that removing c'' from $W^{(j-1)}$ and adding c_j ends in a committee $W^{(j)}$ that also satisfies α-JR.

So, assume for the sake of contradiction that this is not the case. Therefore, there exists a witness of a violation (N', c). Further, since $W^{(j-1)}$ satisfies α-JR there exists a voter $v'' \in N'$ that approves a candidate in $W^{(j-1)}$, but no candidate in $W^{(j)}$ which implies that $c'' \in A_{v''}$.

First, consider the case that $v_j \sqsubseteq v''$. Then, since $v \sqsubseteq v_j \sqsubseteq v''$ and $c'' \in A_{v''} \cap A_v$ it holds that $c'' \in A_{v_j}$. Therefore, by definition of c_j we can conclude that $r_{c''} \sqsubseteq r_{c_j}$ which implies that every voter v^* with $v_j \sqsubseteq v^*$ that approves c'' also approves c_j. Thus, voter v'' approves c_j which contradicts $v'' \in N'$.

It remains to consider the case that $v'' \sqsubset v_j$. Now, since by definition C_{j-1} does not contain any α-JR violation for $\{v_1, \dots, v_{i-1}\}$, and $C_{j-1} \subseteq W^{(j)}$ there has to exist a voter $v^* \in N'$ such that $v_j \sqsubseteq v^*$. But then $v'' \sqsubset v_j \sqsubseteq v^*$ which implies that $c \in A_{v_j}$. Similar to the first case, this implies that $r_c \sqsubseteq r_{c_j}$ and thus it holds that every voter v^* with $v_j \sqsubseteq v^*$ that approves c also approves c_j. But this is a contradiction to $v^* \in N'$.

Consequently, $W^{(n)}$ is minimal and feasible and corresponds to the output of our algorithm. Therefore, there cannot be any feasible committee W' with $|W'| < |W|$. □

Theorem 6.7. *In the CI domain, verifying whether a committee W satisfies α-EJR can be done efficiently.*

Proof. We can use a similar proof as for the VI domain, but instead of considering every interval of voters, we consider every interval of candidates.

More precisely, denote by $N_\ell^{i,j} = \{v \in N : \{c_i, \dots, c_j\} \subseteq A_v, |A_v \cap W| < \ell\}$ for $1 \leq \ell \leq k$ the voters that approve every candidate in the interval but at most $\ell - 1$ many candidates in W.

Then, denote by $\ell_{i,j} = \max_\ell(\min((j - i + 1, \lfloor |N_\ell^{i,j}| \cdot \frac{k}{n} \cdot \alpha \rfloor) \geq \ell)$. Observe that if there does not exist any cohesive group, the value is equal to 0. We claim that with this procedure, we can find an EJR violation if one exists.

To prove this, let (N', C', ℓ) be any EJR violation for the committee W. Further, let c_i and c_j be the first and last candidates in C' according to the CI order. By definition, every voter in N' approves at most $\ell-1$ many candidates in W and $|C'| \geq \ell$. Thus, also $(N', \{c_i, \dots, c_j\}, \ell)$ is a witness of an EJR violation. When the algorithm considers the interval i, j, it therefore finds a violation $(N'', \{c_i, \dots, c_j\}, \ell)$ with $N' \subseteq N''$. This concludes the proof. □

Theorem 6.8. *Computing $\alpha^*_{EJR}(I)$ in the CI domain can be done efficiently.*

Proof. We claim that Algorithm 2 returns a subset of size at most k if and only if there exists a committee of size k that satisfies α-JR. First, observe that Algorithm 2 always outputs a subset of candidates that satisfies α-JR by construction. So if the output committee is of size at most k, there trivially also exists a committee satisfying α-JR. It remains to show that if there exists a committee of size k, then the algorithm finds one as well.

We prove the statement inductively over the set of candidates $\{c_1, \dots, c_m\}$, going over them according to the linear order $\sqsubset$. W is the committee returned by our algorithm. Let $C_j := W \cap \{c_1, \dots, c_j\}$ be the subset of candidates selected by the algorithm out of the first j candidates according to the CI order.

Claim. For a given j exists a minimal committee $W^{(j)}$ that satisfies α-JR and $C_j \subseteq W^{(j)}$.

Base: $j = 1$. As discussed above, the subset of candidates returned by the algorithm satisfies α-EJR. Therefore, there exists a subset of candidates that satisfies α-JR and therefore also one of minimal cardinality. Now, if c_1 is selected by our algorithm, then c_1 must also be in every optimal committee since the set of candidates $\{c_2, \dots, c_m\}$ and therefore all possible subsets of size k of this set would result in an α-JR violation. Therefore, $C_1 \subseteq W^{(1)}$.

Step. Let the claim hold for $j - 1 \geq 0$. We show that it also holds for j. If c_j is not selected by the algorithm, then $C_j = C_{j-1} \subseteq W^{(j-1)}$ by the induction hypothesis. Therefore, assume our algorithm selects c_j. Since c_j is selected by the algorithm, there is an α-JR violation when considering $C_{j-1} \cup \{c_{j+1}, \dots, c_m\}$ by a group of voters N' approving a common candidate c with $c \sqsubseteq c_j$ and no voter $v \in N'$ approves a candidate $c' \in \{c_{j+1}, \dots, c_m\}$. By assumption, $W^{(j-1)}$ does not have any violations, which means there is some candidate $c'' \in W^{(j-1)}$ which is approved by at least one voter $v \in N'$ and $c'' \sqsubseteq c_j$. If $c'' = c$, the statement trivially holds, so assume in the following that $c \neq c''$. Now, we denote as $W^{(j)}$ the committee after replacing c'' in $W^{(j-1)}$ with candidate c_j and claim that $W^{(j)}$ does not violate α-JR. For this, assume for the sake of contradiction that there exists a cohesive group (N', c) that causes an α-JR violation for $W^{(j)}$. This implies that no voter in N' approves a candidate in $W^{(j)}$. First, consider the case $c_j \sqsubset c$. Since $W^{(j-1)}$ satisfies α-JR there exists a $v \in N'$ with $c^* \in A_v \cap W^{(j-1)}$ for some c^*. $c^* \in W^{(j)}$ would imply that i approves a candidate in $W^{(j)}$. Thus, $c^* = c''$. But since v approves c and c^* and $c^* \sqsubset c_j \sqsubset c$ this implies that i approves c_j; a contradiction.

Thus, in the following, we can assume that $c \sqsubset c_j$. By definition of the algorithm, we know that the subset of candidates $C_{j-1} \cup \{c_j, \dots, c_m\}$ satisfies α-JR. Therefore, there exists a voter $v \in N'$ that approves a candidate $c^* \in C_{j-1} \cup \{c_j, \dots, c_m\}$, but no candidate in $W^{(j)}$. Since $C_{j-1} \cup \{c_j\} \subseteq W^{(j)}$ it has to hold that $c_j \sqsubset c^*$. But then voter v approves both c^* and c, which together with $c \sqsubset c_j \sqsubset c^*$ implies that $c_j \in A_v$. Since we have a CI instance, this implies that v approves c_j, which contradicts our assumption that $v \in N'$. This concludes the proof. □

C Further Empirical Insights

C.1 Model Configuration

For the experiments, we use the Python voting library by Lackner et al. (2023). We generate approval ballots under two commonly used preference models: *Impartial culture (IC)* and the *Euclidean Threshold model*. For both models, we vary the number of voters $n \in \{11, 12, 29, 59\}$, the number of candidates $m \in \{5, 9, 15\}$, and the committee size $k \in \{3, 5, 8, 11\}$. Under IC, each voter independently approves candidates with probability $p \in \{0.3, 0.5\}$. Under the Euclidean model, candidates and voters are embedded in the plane with voters sampled around the origin $(0, 0)$ using a Gaussian distribution with $\sigma = 0.5$, and approvals are determined by a distance threshold $t \in \{1.7, 2.3\}$ (see (Lackner et al., 2023) for more details). For each of the considered voting rules, we compute at most five

Table 1: Additive distance between α-value achieved by the four different voting rules and the optimal achievable α-value for α-EJR. This is averaged over 400 instances each and contains the IC and EUT model.

k	m	n	**MES**	**seq-Phrag**	**CC**	**PAV**
3	5	11	0.010133	0.010473	0.055392	0.007784
3	5	12	0.008507	0.007760	0.059616	0.008455
3	5	29	0.017004	0.017047	0.034153	0.015237
3	5	59	0.013083	0.013210	0.027110	0.012722
3	9	11	0.019667	0.019884	0.101460	0.018074
3	9	12	0.032172	0.033167	0.102177	0.028019
3	9	29	0.028139	0.029149	0.049896	0.026893
3	9	59	0.028475	0.028294	0.035160	0.027219
3	15	11	0.043034	0.043642	0.159519	0.039949
3	15	12	0.047342	0.046887	0.146009	0.045783
3	15	29	0.035782	0.036655	0.069678	0.036067
3	15	59	0.036084	0.036719	0.048757	0.035427
5	9	11	0.038204	0.038030	0.180881	0.023542
5	9	12	0.037530	0.034058	0.166367	0.027283
5	9	29	0.041997	0.041559	0.081327	0.031221
5	9	59	0.036419	0.037163	0.058316	0.034893
5	15	11	0.044592	0.044854	0.275377	0.036544
5	15	12	0.039661	0.039713	0.253620	0.034005
5	15	29	0.055968	0.055875	0.118074	0.048463
5	15	59	0.051038	0.050536	0.085691	0.045847
8	9	11	0.011152	0.014509	0.158543	0.011192
8	9	12	0.013481	0.016815	0.139576	0.009226
8	9	29	0.017217	0.020142	0.073613	0.013810
8	9	59	0.018103	0.018436	0.044746	0.014925
8	15	11	0.046080	0.047793	0.321579	0.036526
8	15	12	0.055936	0.054837	0.320223	0.036548
8	15	29	0.050834	0.053788	0.170933	0.039221
8	15	59	0.058181	0.059281	0.107438	0.053653
11	15	11	0.039556	0.042946	0.298409	0.027728
11	15	12	0.033503	0.038808	0.301895	0.027706
11	15	29	0.054533	0.058232	0.175163	0.039851
11	15	59	0.050336	0.053688	0.109126	0.043724

committees. For the computation of the performance of each voting rule, we compare the optimal α-value with the mean α-value of the returned committees.

C.2 Further Results on α-EJR

We complement the results on α-EJR presented in Section 7 with Table 1. We separate the performance of the voting rules according to the parameters k, m, and n. Every entry is averaged over 400 randomly generated instances. This comes from the fact that for every (k, m, n)-tuple, we have two sampling configurations for both sampling models.

PAV performs the best out of the considered methods. MES and sequential Phragmén still perform reasonably well. For all combinations of parameters, the average additive distance between the α-value achieved by the voting rules and the optimal α-value is between 0.01 and 0.2, showing that these three rules are actually not as bad as the worst-case would suggest. This is especially surprising for sequential Phragmén, which does not satisfy 1-EJR. Unsurprisingly, CC performs badly relative to the other rules as its only concern is to maximize the number of voters that approve a candidate. Nevertheless, CC often contains committees that also satisfy EJR, resulting in an overall moderate performance in approximating the optimal α-value. (Bredereck et al., 2019). Therefore, we need to note that if one were not to consider the worst of the committees returned by CC, it would have performed better in our analysis.

C.3 Plots for JR

We present the same plots that we have considered for EJR, for JR. The distribution of the α-values is significantly closer towards 0 than for EJR. The fact that the optimal value is 0 in almost 80 per cent of the instances in the IC model shows that there are many instances in which committees satisfying every voter with at least one candidate exist. Figure 8 again shows the performance of the four voting rules we considered in this analysis. CC is very close to the optimal α-value for both the IC and EUT model. This is not surprising, since CC maximizes the coverage, and therefore has an optimal α-value of 0 whenever α^*_{JR} is 0.

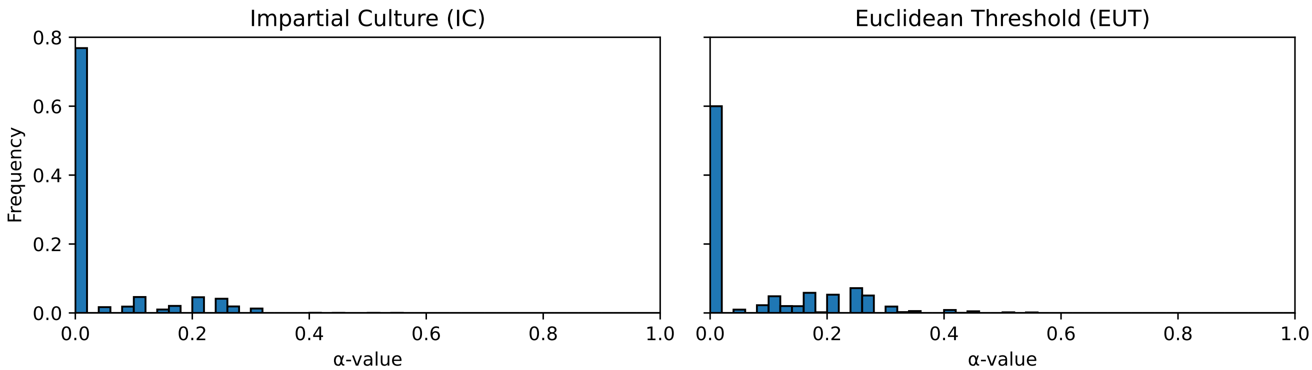

Figure 7: Distribution of optimal α-values, α^*_{JR}, under the Impartial Culture and the Euclidean Threshold model, based on 6400 generated instances each.

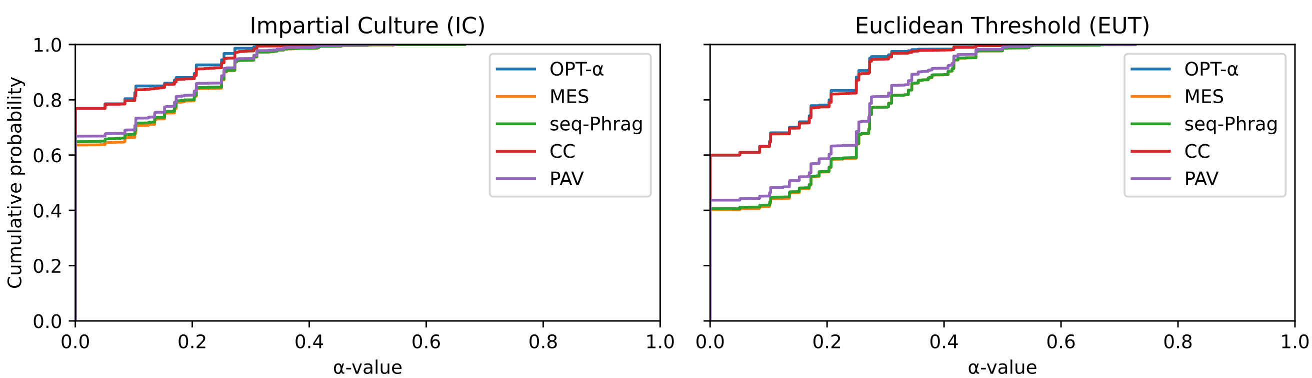

Figure 8: Cumulative distribution of the optimal α-value, α^*_{JR}, and the α-value achieved by MES, seq-Phragmén, CC, and PAV under the Impartial Culture and Euclidean Threshold model, based on 6400 samples each.