String graphs are quasi-isometric to planar graphs

James Davies*

Leipzig University, Germany

Abstract

We prove that for every countable string graph S, there is a planar graph G with $V(G) = V(S)$ such that

$$\frac{1}{23660800} d_S(u, v) \leqslant d_G(u, v) \leqslant 162 d_S(u, v)$$

for all $u, v \in V(S)$, where $d_S(u, v)$, $d_G(u, v)$ denotes the distance between u and v in S and G respectively. In other words, string graphs are quasi-isometric to planar graphs.

This theorem lifts a number of theorems from planar graphs to string graphs, we give some examples. String graphs have Assouad-Nagata (and asymptotic dimension) at most 2. Connected, locally finite, quasi-transitive string graphs are accessible. A finitely generated group Γ is virtually a free product of free and surface groups if and only if Γ is quasi-isometric to a string graph.

Two further corollaries are that countable planar metric graphs and complete Riemannian planes are also quasi-isometric to planar graphs, which answers a question of Georgakopoulos and Papasoglu. For finite string graphs and planar metric graphs, our proofs yield polynomial time (for string graphs, this is in terms of the size of a representation given in the input) algorithms for generating such quasi-isometric planar graphs.

1 Introduction

A *string graph* is an intersection graph of a collection $\mathcal{S}$ of path-connected subsets of the plane, where the vertices are $\mathcal{S}$, and $S_1, S_2 \in \mathcal{S}$ are adjacent if they intersect. String graphs were first formally defined by Ehrlich, Even, and Tarjan [28] but were present in earlier works of Benzer [9] on the topology of genetic structures and of Sinden [74] on electrical networks realizable by printed circuit. String graphs are the most general class of intersection graphs in the plane and significantly generalize planar graphs, which by the Koebe-Andreev-Thurston circle packing theorem [50] can be viewed as intersection graphs of discs in the plane with disjoint interiors. String graphs can be thought of as a dense or induced minor analogue of planar graphs.

There are several important structural results for string graphs. For instance, Pach and Tóth [67] proved that recognizing string graphs is decidable. Fox and Pach [31, 32, 34] (see also [62]) proved a separator theorem for string graphs. Every incomparability graph is a string graph and Fox and Pach [33] proved a partial converse of this for dense string graphs.

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This was used by Tomon [75] to prove that string graphs have the Erdős-Hajanl property. Very recently, Abrishami, Briański, Davies, Du, Masaříková, Rzążewski, and Walczak [1] classified string graphs with large chromatic number.

Every planar graph is a string graph, although the structure of string graphs is much more complicated. For instance, while planar graphs can be recognized in linear time [47], recognition of string graphs is NP-hard [52, 73]. There are at most $c^n n!$ planar graphs on n vertices [22, 65], while the number of string graphs on n vertices is $2^{(\frac{3}{4}+o(1))\binom{n}{2}}$ [68]. Every planar graph is 4-colourable [5, 71], however there are triangle-free string graphs with arbitrarily large chromatic number [69].

We prove a partial converse to the fact that planar graphs are string graphs. In a coarse sense, every string graph is equivalent to some planar graph. For vertices u, v of a graph G, we let $d_G(u, v)$ denote the distance between u and v in G.

Theorem 1.1. *Let S be a countable string graph.* [1] *Then there is a planar graph G with $V(G) = V(S)$ such that*

$$\frac{1}{23660800} d_S(u, v) \leqslant d_G(u, v) \leqslant 162 d_S(u, v)$$

for all $u, v \in V(S)$.

In other words, string graphs are quasi-isometric (or even bi-Lipschitz equivalent) to planar graphs. For finite string graphs the proof of Theorem 1.1 yields a polynomial time (with respect to the size of a given representation of S in the input) algorithm for generating such a planar graph G. Unfortunately, are string graphs that require exponentially sized representations [53]. However, natural subclasses such as segment intersection graphs or more generally intersection graphs of convex compact path-connected subsets of $\mathbb{R}^2$, do have polynomially sized representations.

Theorem 1.1 has numerous applications. We first discuss some direct corollaries for string graphs. Afterwards, we shall discuss two further quasi-isometry results on metric planar graphs and complete Riemannian planes.

Asymptotic dimension was introduced by Gromov [43] for metric spaces and groups (via their Cayley graphs). This notion has had significant impact in geometric group theory, for instance Yu [79] proved that finitely generated groups with finite asymptotic dimension satisfy the Novikov conjecture. For a survey of asymptotic dimension from this point of view, see [8]. In coarse graph theory, asymptotic dimension can be thought of as a coarse analogue of graph colouring. There has recently been significant progress on the study of asymptotic dimension in the intersection of structural graph theory and geometric group theory [1, 10, 11, 19, 23, 27, 55]. There are also applications of asymptotic dimension to graph colouring and χ-boundedness, for instance hereditary classes of graphs with bounded asymptotic dimension are either χ-bounded or contain all Burling graphs [1].

One of the most natural classes of graphs to consider is of course planar graphs. Ostrovskii and Rosenthal [66] proved that planar graphs have bounded asymptotic dimension. Fujiwara and Papasoglu [35] improved this bound to 3 and Jørgensen and Lang [49] proved that planar graphs have asymptotic dimension and furthermore Assouad–Nagata dimension at most 2,

[1] We remark that Theorem 1.1 does not hold for uncountable string graphs as can be seen by Wagner's [78] characterization of uncountable planar graphs and the fact that the graph obtained from $|\mathbb{R}|$ 1-ended infinite paths by identifying their ends and then for each vertex adding $|\mathbb{R}|$ new adjacent vertices of degree 1 is a string graph.

which is tight due to the infinite planar grid. Asymptotic dimension and Assouad–Nagata dimension are persevered by quasi-isometries. Thus, by Theorem 1.1, we obtain the following.

Theorem 1.2. *String graphs have Assouad–Nagata dimension at most 2.*

Theorem 1.2 settles the 2-dimensional case of a conjecture of Davies, Georgakopoulos, Hatzel, and McCarty [19] that sphere intersection graphs in $\mathbb{R}^d$ have asymptotic dimension at most d. For $d \geqslant 3$, such graphs have unbounded Assouad–Nagata dimension [19].

Bonamy, Bousquet, Esperet, Groenland, Liu, Pirot, and Scott [11] gave a proof that planar graphs have Assouad–Nagata dimension at most 2, which yields a polynomial time algorithm for computing the corresponding covers. Thus, the corresponding covers in Theorem 1.2 can also be computed in polynomial time (in terms of the size of the provided representation of the string graph given in the input). For some algorithmic applications of such covers, see [7, 12, 13, 30, 48, 54, 56].

A well-known conjecture (see [44]) states that planar graphs admit embeddings into ℓ_1 with bounded distortion. Theorem 1.1 implies that this is equivalent to string graphs having embeddings into ℓ_1 with bounded distortion. Rao [70] showed that n-vertex planar graphs have embeddings into ℓ_1 with distortion at most $O(\sqrt{\log n})$, and Theorem 1.1 extends this to string graphs.

Theorem 1.3. *Every n-vertex string graph has a $O(\sqrt{\log n})$-distortion embedding into ℓ_1.*

For a more extensive survey and discussion of applications of such embeddings, see [44].

Dunwoody [26] showed that every connected, locally finite, quasi-transitive, planar graph is accessible (see also [29, 45, 46]). MacManus [57] recently gave a remarkable extension of this theorem by showing that every connected, locally finite, quasi-transitive graph that is quasi-isometric to a planar graph is accessible. Thus, by Theorem 1.1, Dunwoody's [26] theorem extends from planar graphs to string graphs.

Theorem 1.4. *Every connected, locally finite, quasi-transitive string graphs is accessible.*

There has been significant interest in planar Cayley graphs [6, 25, 24, 26, 38, 40, 41, 39, 61, 76] and Cayley graphs quasi-isometric to planar graphs [14, 15, 37, 57, 58, 59, 63, 77]. Closely related to this, there are several results characterizing certain finitely presented groups Γ in terms of being quasi-isometric to certain Riemannian surfaces. For instance, Γ is quasi-isometric to a complete Riemannian plane if and only if Γ is a virtual surface group [14, 15, 37, 59, 63, 77]. Maillot also showed that Γ is quasi-isometric to a complete, simply connected, planar Riemannian surface with non-empty geodesic boundary if and only if Γ is virtually free. In both these cases, Γ is quasi-isometric to a planar graph since both surface groups and free groups have planar Cayley graphs.

MacManus [57] used his aforementioned theorem on accessibility to extend and unify these two theorems on finitely generated groups. By Theorem 1.1, we obtain the following further extension of MacManus's [57] theorem (our contribution is the equivalence between the first two bullets, the rest is given in [57]).

Theorem 1.5. *Let Γ be a finitely generated group. Then the following are equivalent.*

- *Γ is quasi-isometric to a string graph.*
- *Γ is quasi-isometric to a planar graph.*

- Γ *is quasi-isometric to a planar Cayley graph.*
- *Some finite index subgroup of* Γ *admits a planar Cayley graph.*
- Γ *is virtually a free product of free and surface groups.*

We now turn our attention to some further quasi-isometry results on metric planar graphs and complete Riemannian planes.

A *metric graph* H is a graph along with an assignment of non-negative edge lengths (in this paper, the length of edges in (non-metric) graphs is otherwise always uniformly equal to 1). We can view metric graphs as a length space in which the edges are replaces by intervals of the corresponding lengths. When the edge lengths are within $D \subseteq (0, \infty)$, we say that H is an D-metric graph.

Georgakopoulos and Papasoglu [42] asked whether every planar metric graph is quasi-isometric to a planar graph. As a straightforward application of Theorem 1.1 (see Lemma 2.4), we answer this in positive (note that every planar metric graph can be turned into a $(0, 1]$-metric planar graph by subdividing edges appropriately). Such a planar graph G can be generated in polynomial time.

Theorem 1.6. *Let* H *be a countable* $(0, 1]$*-metric planar graph. Then there is a planar graph* G *with* $V(G) = V(H)$ *such that*

$$\frac{1}{47321600} d_H(u, v) \leqslant d_G(u, v) \leqslant 162 d_H(u, v) + 1$$

for all $u, v \in V(H)$.

Beyond the general interest in metric planar graphs and also in groups acting on planar surfaces [39], one other motivation of Georgakopoulos and Papasoglu [42] was their coarse Kuratowski conjecture that graphs forbidding K_5 and $K_{3,3}$ as K-fat minors are quasi-isometric to planar graphs. Although Theorem 1.6 and Theorem 1.1 provide substantial progress towards this conjecture, we remark that the closely related graph minor analogue that graphs forbidding H as a K-fat minor are quasi-isometric to H-minor-free graphs is false [2, 21], (although it is known to be true for some small graphs H [3, 4, 16, 36, 42, 60, 64]). Such K-fat H-minor-free graphs need not even be quasi-isometric to H'-minor-free graphs for any graph H' [2]. However, with Distel and Hickingbotham [18], we will use the main technical result (see Theorem 5.1) of this paper to prove an approximate graph minor version of Theorem 1.6 that metric H-minor-free graphs are quasi-isometric to H'-minor-free graphs. We remark that the constants of the quasi-isometries necessarily depend on H [20].

It is well known (see [11, 59, 72]) that complete Riemannian planes are quasi-isometric to locally finite[2] metric planar graphs. This was used for instance to show that complete Riemannian planes have Assouad–Nagata dimension at most 2 [11] and in Maillot's [59] aforementioned classification finitely generated groups virtually free groups as being those quasi-isometric to certain planar Riemannian surfaces. By Theorem 1.6, we obtain the following refinement, which significantly generalizes the previously discussed fact that finitely generated groups quasi-isometric to complete Riemannian planes are quasi-isometric to planar graphs.

Theorem 1.7. *Complete Riemannian planes are quasi-isometric to locally finite planar graphs.*

[2] A metric graph is *locally finite* if every bounded radius ball contains at most a finite number of vertices.

In [17] we will use the main technical result (see Theorem 5.1) of the present paper to extend Theorem 1.7 to higher genus Riemannian surfaces and also refine the theorem so that the quasi-isometry from the graph to the Riemannian surface is a continuous embedding. Such a result does not hold for complete Riemannian surfaces of unbounded genus [17], (although, it does if we use metric graphs [11, 59, 72] instead of graphs).

In Section 2 we introduce basic definitions, deduce Theorem 1.6 from Theorem 1.1, and also reduce proving Theorem 1.1 to its finite case (Theorem 2.5). We also introduce the necessary framework for our approach to proving Theorem 1.1, namely through viewing string graphs as region intersection graphs, and the notion of "impressions" which provides a convenient way of finding quasi-isometries from region intersection graphs. The rest of the proof in Sections 3 to 5 is essentially a delicate technical induction. In Section 3 we prove a weakening of Theorem 1.1 for the subclass outerstring graphs (Theorem 3.1). This is used in Section 4 to find a technical structure tailored for use in the inductive argument that we call an "encasing" (see Lemma 4.4). Finally, in Section 5, we prove Theorem 1.1 using Lemma 4.4.

2 Preliminaries

For a set of vertices A of a graph G, we let $G[A]$ denote the *induced subgraph* of G on vertex set A. We say that A is *connected* if the induced subgraph $G[A]$ is connected. We let $N_G(A)$ denoted the *neighbourhood* of the vertex set A in G (the vertices of $V(G)\backslash A$ adjacent to a vertex of A). For a positive integer t, we let $N_G^t[A]$ be the vertices at distance at most t from A. If $A = \{u\}$, then we simply use $N_G(u)$, $N_G^t[u]$. The *weak diameter* of A in G is equal to the maximum distance (in G) between two vertices of A. Given a set F and another collection of sets $\mathcal{H}$, we let $I_{\mathcal{H}}(F)$ denote the subcollection of elements $H \in \mathcal{H}$ that intersect F.

Given a graph G and a collection of connected vertex sets $\mathcal{H}$ of G, we let $\mathbf{RIG}(G, \mathcal{H})$ be the graph with vertex set $\mathcal{H}$ where, $H_1, H_2 \in \mathcal{H}$ are adjacent if they intersect. We call $\mathbf{RIG}(G, \mathcal{H})$ a *region intersection graph*. Region intersection graphs were introduced by Lee [54] as a generalization of string graphs (usually we consider region intersection graphs $\mathbf{RIG}(G, \mathcal{H})$ in which G forbids some graph as a minor, since otherwise the class contains all graphs). Finite string graphs are exactly finite region intersection graphs $\mathbf{RIG}(G, \mathcal{H})$ for which G is planar [54] and this is how we will view string graphs to prove Theorem 1.1 (or its finite version, Theorem 2.5).

If $\mathcal{M}$ is a collection of disjoint connected vertex sets in G, then we let $\mathbf{IM}(G, \mathcal{M})$ be the graph with vertex set $\mathcal{M}$ where $M_1, M_2 \in \mathcal{M}$ are adjacent if there is an edge in G between M_1 and M_2 (which case, we say that they *touch*). This is one way to define *induced minors* of a graph G.

Let G be a graph and $\mathcal{H}$ a collection of connected vertex sets of G. If every edge of G is contained in $G[H]$ for some $H \in \mathcal{H}$, then we say that $\mathcal{H}$ *spans* G. More generally, we say that $\mathcal{H}$ z*-spans* G if for every edge uv of G, there exists $H_u, H_v \in \mathcal{H}$ with $u \in H_u$ and $v \in H_v$, such that $d_{\mathbf{RIG}(G,\mathcal{H})}(H_u, H_v) \leqslant z$. Note that $\mathcal{H}$ spans G if and only if it 0-spans G.

For a graph G and two collections of connected vertex sets $\mathcal{H}, \mathcal{M}$ of G with the sets in $\mathcal{M}$ being disjoint, we say that $(G, \mathcal{M})$ is a (x, y)*-impression* of $(G, \mathcal{H})$ if

- for every $M \in \mathcal{M}$, $I_{\mathcal{H}}(M)$ has weak diameter at most x in $\mathbf{RIG}(G, \mathcal{H})$,
- for every $H \in \mathcal{H}$, $I_{\mathcal{M}}(H)$ has weak diameter at most y in $\mathbf{IM}(G, \mathcal{M})$, and
- $\bigcup_{M \in \mathcal{M}} M = V(G)$.

Impressions provide a convenient way of obtaining a quasi-isometry from a region intersection graph $\mathbf{RIG}(G, \mathcal{H})$ to a minor of G.

Proposition 2.1. *Let $(G, \mathcal{M})$ be a (x, y)-impression of $(G, \mathcal{H})$ where $\mathcal{H}$ z-spans G. Then there is a function $f : \mathcal{H} \to \mathcal{M}$ such that for all $H_1, H_2 \in \mathcal{H}$, we have*

$$\frac{1}{x+z} d_{\mathbf{RIG}(G,\mathcal{H})}(H_1, H_2) - \frac{x}{x+z} \leqslant d_{\mathbf{IM}(G,\mathcal{M})}(f(H_1), f(H_2)) \leqslant 2y d_{\mathbf{RIG}(G,\mathcal{H})}(H_1, H_2),$$

and for every $M \in \mathcal{M}$ there exists some $H \in \mathcal{H}$ such that $d_{\mathbf{IM}(G,\mathcal{M})}(f(H), M) \leqslant y$.

Proof. Let $f : \mathcal{H} \to \mathcal{M}$ be chosen so that $f(H) \in I_{\mathcal{M}}(H)$ for all $H \in \mathcal{H}$.

If H_1 and H_2 are adjacent in $\mathbf{RIG}(G, \mathcal{H})$, then they intersect. So then $I_{\mathcal{M}}(H_1 \cup H_2)$ is a connected set in $\mathbf{IM}(G, \mathcal{M})$ and furthermore has weak diameter at most $2y$. It follows for (possibly non-adjacent $H_1, H_2 \in \mathcal{H}$) that $d_{\mathbf{IM}(G,\mathcal{M})}(f(H_1), f(H_2)) \leqslant 2y d_{\mathbf{RIG}(G,\mathcal{H})}(H_1, H_2)$.

Suppose that $d_{\mathbf{IM}(G,\mathcal{M})}(f(H_0), f(H_t)) = t$ and let $M_0, \dots, M_t$ be a path in $\mathbf{IM}(G, \mathcal{M})$ between $M_0 = f(H_0)$ and $M_t = f(H_t)$. For each $1 \leqslant i < t$, there exists some $u_i v_i \in E(G)$ with $u_i \in M_{i-1}$ and $v_i \in M_i$. For each $1 \leqslant i < t$, let $H_i', H_i \in \mathcal{H}$ be such that $u_i \in V(H_i')$, $v_i \in V(H_i)$, and the distance between H_i' and H_i in $\mathbf{RIG}(G, \mathcal{H})$ is at most z. For each $1 \leqslant i \leqslant t$, we have that $H_{i-1}, H_i' \in I_{\mathcal{H}}(M_{i-1})$. Thus, there is a path in $\mathbf{RIG}(G, \mathcal{H})$ between H_0 and H_t of length at most $(t+1)x + tz$. It now follows that for all $H_1, H_2 \in \mathcal{H}$, we have

$$\frac{1}{x+z} d_{\mathbf{RIG}(G,\mathcal{H})}(H_1, H_2) - \frac{x}{x+z} \leqslant d_{\mathbf{IM}(G,\mathcal{M})}(f(H_1), f(H_2)) \leqslant 2y d_{\mathbf{RIG}(G,\mathcal{H})}(H_1, H_2).$$

Lastly, consider some $M \in \mathcal{M}$. Choose some $H \in I_{\mathcal{H}}(M)$. Then $M, f(H) \in I_{\mathcal{M}}(H)$, and thus M is at distance at most y from $f(M)$ in $\mathbf{IM}(G, \mathcal{M})$. $\square$

In most but not all cases we consider, we will actually have that $\mathcal{H}$ spans G, which means that we simply have $z = 0$ in Proposition 2.1. Given $(G, \mathcal{H})$, we could simply reduce to the case the $\mathcal{H}$ spans G by removing edges of G not contained in $G[H]$ for any $H \in \mathcal{H}$ (which is something we often do). However, adding edges to G can also make finding impressions easier. In fact, for G planar, it is convenient for inductive arguments that we can also add new connected sets to $\mathcal{H}$ in a controlled way so that our final impression can be obtained from a related impression.

Lemma 2.2. *Let $(G, \mathcal{M})$ be a (x, y)-impression of $(G, \mathcal{H}')$. If $\mathcal{H} \subseteq \mathcal{H}'$ has the property that for every $H \in \mathcal{H}' \backslash \mathcal{H}$ we have that $I_{\mathcal{H}}(H)$ has weak diameter at most k in **RIG**$(G, \mathcal{H})$ and $V(G) = \bigcup_{H \in \mathcal{H}} V(H)$, then $(G, \mathcal{M})$ is a (kx, y)-impression of **RIG**$(G, \mathcal{H})$.*

Proof. For each $H \in \mathcal{H} \subseteq \mathcal{H}'$, we have that $I_{\mathcal{M}}(H)$ has weak diameter at most y in $\mathbf{IM}(G, \mathcal{M})$ since $(G, \mathcal{M})$ is a (x, y)-impression of $\mathbf{RIG}(G, \mathcal{H}')$. For $M \in \mathcal{M}$, we have that $I_{\mathcal{H}'}(M)$ has weak diameter at most x in $\mathbf{RIG}(G, \mathcal{H}')$. Since $V(G) = \bigcup_{H \in \mathcal{H}} V(H)$ and $I_{\mathcal{H}}(H)$ has weak diameter at most k in $\mathbf{RIG}(G, \mathcal{H})$ for each $H \in \mathcal{H}' \backslash \mathcal{H}$, it further follows that $I_{\mathcal{H}}(M)$ has weak diameter at most kx in $\mathbf{RIG}(G, \mathcal{H})$. Hence $(G, \mathcal{M})$ is a (kx, y)-impression of $\mathbf{RIG}(G, \mathcal{H})$, as desired. $\square$

It is straight forward to make the quasi-isometry f in Proposition 2.1 into a bijective map by taking a minor and possibly adding some pendent edges. Note in particular that for countable graphs this preserves planarity.

Lemma 2.3. *Let G, H be graphs and let $f : V(G) \to V(H)$ be a function such that for all $u, v \in V(G)$, we have*

$$\frac{1}{x_1} d_G(u,v) - \frac{x_2}{x_1} \leqslant d_H(f(u), f(v)) \leqslant x_3 d_G(u,v),$$

and for every $v \in V(H)$ there exists some $u \in V(G)$ such that $d_H(f(u), v) \leqslant x_4$.

Then there exists some graph H' obtained from H by adding vertices adjacent to individual vertices of $V(H)$ and then taking a minor, and some bijection $f' : V(G) \to V(H')$ such that for all $u, v \in V(G)$, we have

$$\frac{1}{2x_4(x_1+x_2)} d_G(u,v) \leqslant d_{H'}(f'(u), f'(v)) \leqslant (x_3+2) d_G(u,v).$$

Proof. Let $A = f(V(G))$. Since for every $v \in V(H)$ there exists some $u \in V(G)$ such that $d_H(f(u), v) \leqslant x_4$, we can choose a forest F contained in H as a subgraph with $V(F) = V(H)$ such that every connected component of F contains exactly one vertex of A and every vertex of F is at distance at most x_4 from A in F. Let H^* be obtained from H by for each $a \in A$, adding $|f^{-1}(a)| - 1$ vertices to H that are adjacent to a only. Choose $f' : V(G) \to V(H^*)$ so that for every $a \in A$, $f^{-1}(a)$ is mapped bijectivly to a and the newly added $|f^{-1}(a)| - 1$ vertices adjacent to a. Let H' be obtained from H^* by contracting the edges of F onto A. Then f' is a bijection between G and H'.

For every $u, v \in V(G)$, we have

$$\frac{1}{2x_4} d_H(f(u), f(v)) \leqslant d_{H'}(f'(u), f'(v)) \leqslant d_H(f(u), f(v)) + 2.$$

For distinct $u, v \in V(G)$, we clearly have $d_H(f(u), f(v)) + 2 \leqslant 3d_H(f(u), f(v)) \leqslant x_3 d_G(u,v) + 2 \leqslant (x_3+2) d_G(u,v)$. For $t \geqslant 0$, we have that $\frac{1}{x_1+x_2} t \leqslant \max\{\frac{1}{x_1} t - \frac{x_2}{x_1}, 1\}$. Since f' is bijective, it therefore follows that $\frac{1}{2x_4(x_1+x_2)} d_G(u,v) \leqslant d_{H'}(f'(u), f'(v))$. □

Theorem 1.1 for finite string graphs will be quickly deduced in Section 5 from our main technical result (Theorem 5.1) by applying Lemma 2.2, Proposition 2.1, and Lemma 2.3 in order.

Next we show that Theorem 1.1 implies Theorem 1.6. This is immediate from the following lemma.

Lemma 2.4. *Let H be a $(0,1]$-metric planar graph. Then there is a string graph S with $V(S) = V(H)$ such that*

$$\frac{1}{2} d_H(u,v) \leqslant d_S(u,v) \leqslant d_H(u,v) + 1$$

for all $u, v \in V(H)$.

Proof. Consider H embedded in the plane. For each $u \in V(H)$, let S_u be the subset of the plane consisting of all points of H at distance at most 1 in H from u. Let S be the string graph with vertex set $V(H)$ in which each $u \in V(S) = V(H)$ is represented by S_u. Consider $u, v \in V(H)$ and let P be a geodesic path from u to v in H. Choose vertices $x_0, \ldots, x_t$ of H along P so that $x_0 = u$, $x_t = v$ and for each $0 \leqslant i < t$, the distance along P between x_i and x_{i+1} is in $(1, 2]$ (or if $d_H(u,v) \leqslant 1$, we just take $x_0 = u$, $x_1 = v$). So, $\lceil \frac{1}{2} d_H(u,v) \rceil \leqslant t \leqslant \lfloor d_H(u,v) \rfloor$. The lemma now follows since x_i is adjacent to x_{i+1} for each $0 \leqslant i < t$. □

We remark that with a very similar proof to Lemma 2.4, it is easy to extend Theorem 1.1 to $(0,1]$-metric string graphs (again with slightly worse bounds).

For our inductive argument, it is more convenient to prove Theorem 1.1 for finite string graphs. Section 3, Section 4, and Section 5 are dedicated to proving Theorem 1.1 for finite string graphs (Theorem 2.5 below), which in particular, will be quickly deduced in Section 5 from Theorem 5.1 (which is a more technical result tailored for an inductive argument) by applying Lemma 2.2, Proposition 2.1, and Lemma 2.3 in order.

Theorem 2.5. *Let S be a finite string graph. Then there is a planar graph G with $V(G) = V(S)$ such that*

$$\frac{1}{23660800} d_S(u,v) \leqslant d_G(u,v) \leqslant 162 d_S(u,v)$$

for all $u, v \in V(S)$.

We finish this section by showing that Theorem 2.5 implies Theorem 1.1.

Proof of Theorem 1.1 assuming Theorem 2.5. Clearly it is enough to consider the case that S is connected. Let $V(S) = \{x_k : k \geqslant 1\}$ and for each $i \leqslant 1$, let $X_i = \{x_k : 1 \leqslant k \leqslant i\}$. For each $i \geqslant 1$, let $\mathcal{G}_i$ be the collection of embedded planar graphs G' that are topologically inequivelant after possibly relabeling vertices of $V(G') \backslash X_i$, such that $X_i \subseteq V(G') \subseteq V(S)$, $|V(G')| \leqslant i + \sum_{x,y \in X_i} 162 d_S(x,y)$, and for every $x, y \in X_i$, we have that

$$\frac{1}{23660800} d_S(x,y) \leqslant d_{G'}(x,y) \leqslant 162 d_S(x,y).$$

Note that each $\mathcal{G}_i$ is non-empty by Theorem 2.5 (applied to some $S[Y]$ for some Y with $X_i \subseteq Y$, $|Y| \leqslant |X_i| + \sum_{x,y \in X_i} d_S(x,y)$ such that $d_{S[Y]}(x,y) = d_S(x,y)$ for all $x, y \in X_i$). Also, each $|\mathcal{G}_i|$ is finite. Let T be the graph with vertex set $\bigcup_{i=1}^{\infty} \mathcal{G}_i$ where each $G'' \in \mathcal{G}_{i+1}$ is adjacent to some $G' \in \mathcal{G}_i$ such that G' is topologically equivalent to a restriction of G'' after possibly relabeling vertices of $V(G'') \backslash X_i$. Observe that by considering shortest paths between vertices of X_i, such a G' always exists.

By Kőnig's infinity lemma [51], T has an infinite path $G_1, G_2, \ldots$ in which $G_i \in \mathcal{G}_i$ for each $i \geqslant 1$. Note that for each $i \geqslant 1$, there exists some R so that $V(G_i) \subseteq V(G_r)$ for all $r \geqslant R$. We obtain the desired countable planar graph G by iteratively extending an embedded planar graph topically equivalent to G_i to one topologically equivalent to G_{i+1} for each $i \geqslant 1$ in order. □

3 Outerstring graphs

Outerstring graphs are equivalent to region intersection graphs $\mathbf{RIG}(G, \mathcal{H})$ for which G is planar and every $H \in \mathcal{H}$ contains a vertex of the outer face of G. They can be thought of as an "outerplanar" analogue of string graphs, and are thus much simpler. The main aim of this section is to prove the following theorem on finding certain restricted impressions of such region intersection graphs. This will be crucial to our arguments on string graphs (or strictly speaking, Lemma 3.4 and Lemma 3.2 which easilly imply the following theorem will be).

Theorem 3.1. *Let G be a finite planar graph and $\mathcal{H}$ be a spanning collection of connected vertex sets in G, each of which contains a vertex incident to the outer face of G. Then there is a $(770, 9)$-impression $(G, \mathcal{M})$ of $(G, \mathcal{H})$ such that each set in $\mathcal{M}$ contains a vertex incident to the outer face of G.*

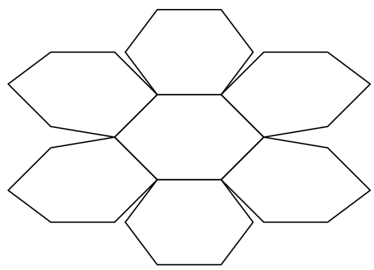

Figure 1: The graph F_6.

Note that if $(G, \mathcal{M})$ is an impression of $(G, \mathcal{H})$ where each graph in $\mathcal{M}$ contains a vertex incident to the outer face of G, then $\mathbf{IM}(G, \mathcal{M})$ is an outerplanar graph.

By Proposition 2.1, Theorem 3.1 implies that outerstring graphs are quasi-isometric to outerplanar graphs. This result is already easily implied by a theorem that graphs forbidding $K_{2,3}$ as a fat minor are quasi-isometric to cacti [16, 36], where a *cacti* is a planar graph in which every edge is on the outer face, or equivalently, a graph in which each maximal 2-connected subgraph is either an edge of a cycle. One might hope that Theorem 3.1 could be improved so that $\mathbf{IM}(G, \mathcal{M})$ is also a cacti. However this is not possible. For every $\ell \geqslant 3$, let F_ℓ be the outerplanar graph consisting of a cycle of length ℓ where for every edge of this cycle, we identify the edge with an edge of another cycle of length ℓ (see Figure 1 for an illustration of the graph F_6). Then it can easily be seen that there is no x, y such that every $(F_\ell, \{\{u, v\} : uv \in E(F_\ell)\})$ has a (x, y)-impression $(F_\ell, \mathcal{M}_\ell)$ in which $\mathbf{IM}(F_\ell, \mathcal{M}_\ell)$ is a cacti. It is however a nice exercise to show that outerplanar graphs are quasi-isometric to cacti. With this, Theorem 3.1 and Proposition 2.1 then yeild another proof that outerstring graphs are quasi-isometric to cacti.

To prove Theorem 3.1, we begin with the easier case of when G is simply an outerplanar graph (so when every vertex of G is incident to the outer face of G). For this, we first define separations. For a connected graph G and connected sets X, Y of G, we say that (X, Y) is a *separation* of G if $X \cup Y = V(G)$, both $X \backslash Y$ and $Y \backslash X$ are non-empty, and $G \backslash (X \cap Y)$ is disconnected. If $|X \cap Y| \leqslant k$, then we say that (X, Y) is a k*-separation* of G.

Lemma 3.2. *Let G be a finite outerplanar graph and $\mathcal{H}$ a spanning collection of connected vertex sets in G. Then there is a $(11, 9)$-impression $(G, \mathcal{M})$ of $(G, \mathcal{H})$.*

Proof. Clearly it is enough to handle the case that G is connected. Choose some $H_0 \in \mathcal{H}$ and for every non-negative integer i, let L_i be the vertices of $\mathbf{RIG}(G, \mathcal{H})$ at distance i from H_0 in $\mathbf{RIG}(G, \mathcal{H})$. For each i we define the following two vertex sets in G. Let $D_i = \bigcup_{H \in L_i} H$, and let $A_i = D_i \backslash D_{i-1}$ (where $D_{-1} = \emptyset$). For each A_i, let $\mathcal{M}_i$ be the maximal connected vertex sets of $G[A_i]$. Let $\mathcal{M} = \bigcup_{i=0}^{\infty} \mathcal{M}_i$. See Figure 2 for an illustration. We must show that $(G, \mathcal{M})$ is a $(11, 9)$-impression of $(G, \mathcal{H})$.

Next, we show for every $M \in \mathcal{M}$ that $I_{\mathcal{H}}(M)$ has weak diameter at most 11 in $\mathbf{RIG}(G, \mathcal{H})$. Let $M \in \mathcal{M}$. Then, for some non-negative integer i, we have that $M \in \mathcal{M}_i$. We may assume that $i \geqslant 2$, since otherwise we clearly have that $I_{\mathcal{H}}(M)$ has weak diameter at most 4 in $\mathbf{RIG}(G, \mathcal{H})$. Since G is outerplanar, observe that there exists a 2-seperation (X, Y) of G with $A_{i-2} \subseteq X$, $X \cap Y \subseteq A_{i-1}$, and $M \subseteq Y$. To see this one can apply Menger's theorem between $\bigcup_{0 \leqslant j \leqslant i-2} A_j$ and the connected component of $\bigcup_{j \geqslant i} A_j$ that contains M. For each $x \in X \cap Y$, let $\mathcal{H}_x = \{H \in L_{i-1} : x \in V(H)\}$. So, $\mathcal{H}_x$ is a clique in $\mathbf{RIG}(G, \mathcal{H})$. Furthermore, each $H \in I_{\mathcal{H}}(M)$ is at distance at most 2 from $\bigcup_{x \in X \cap Y} \mathcal{H}_x$ in $\mathbf{RIG}(G, \mathcal{H})$. Since $|X \cap Y| \leqslant 2$ and $\mathbf{RIG}(G, \mathcal{H})[I_{\mathcal{H}}(M)]$ is connected, it now follows that $I_{\mathcal{H}}(M)$ has weak diameter at most $(2 + 1 + 2) + 1 + (2 + 1 + 2) = 11$ in $\mathbf{RIG}(G, \mathcal{H})$.

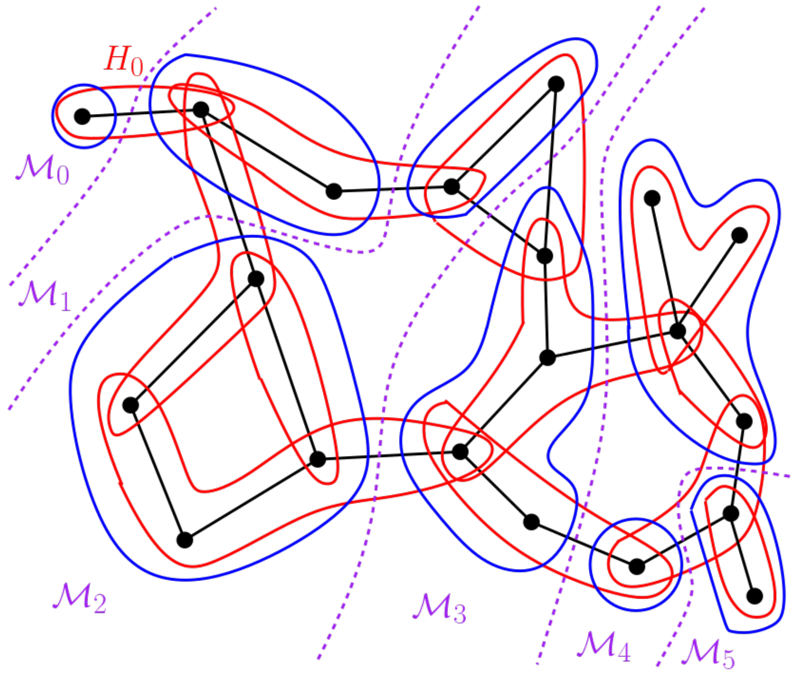

Figure 2: An illustration of the resulting collection of disjoint connected sets $\mathcal{M}$. The connected sets in $\mathcal{H}$ are the red sets and the sets in $\mathcal{M}$ are the blue ones. The dashed purple lines indicate the partition of $\mathcal{M}$ into $\bigcup_{i=0}^{\infty} \mathcal{M}_i$.

Now, we show for every $H \in \mathcal{H}$ that $I_{\mathcal{M}}(H)$ has weak diameter at most 9 in $\mathbf{IM}(G, \mathcal{M})$. This is similar to the previous argument. Let $H \in \mathcal{H}$. Then for some $i \geqslant 0$, we have that $H \in L_i$. We may assume that $i \geqslant 3$, since otherwise we clearly have that $I_{\mathcal{M}}(H)$ has weak diameter at most 6 in $\mathbf{IM}(G, \mathcal{M})$. Similarly to before, by applying Menger's theorem between $\bigcup_{0 \leqslant j \leqslant i-3} A_j$ and the connected component of $\bigcup_{j \geqslant i-1} A_j$ that contains H, there exists a 2-separation (X, Y) of G with $A_{i-3} \subseteq X$, $X \cap Y \subseteq A_{i-2}$, and $H \subseteq Y$. For each $x \in X \cap Y$, choose some $M_x \in \mathcal{M}_{i-2}$ with $x \in M_x$. Then in $\mathbf{IM}(G, \mathcal{M})$, each vertex of $I_{\mathcal{M}}(H)$ is at distance at most 2 from some vertex of $\bigcup_{x \in X \cap Y} M_x$. Since $|X \cap Y| \leqslant 2$ and $\mathbf{IM}(G, \mathcal{M})[I_{\mathcal{M}}(H)]$ is connected, it now follows that $I_{\mathcal{M}}(H)$ has weak diameter at most $(2+2)+1+(2+2) = 9$ in $\mathbf{IM}(G, \mathcal{M})$.

Hence $(G, \mathcal{M})$ is indeed a $(11, 9)$-impression of $(G, \mathcal{H})$, as desired. □

Let G be a finite planar graph and $\mathcal{H}$ a spanning collection of connected vertex sets in G, each of which contains a vertex incident to the outer face of G. For a collection disjoint connected sets $\mathcal{B}$ of G, we say that $(G, \mathcal{B})$ is a *partial* (x, ∞)*-impression* of $(G, \mathcal{H})$ if for every $B \in \mathcal{B}$, $I_{\mathcal{H}}(B)$ has weak diameter at most x in $\mathbf{RIG}(G, \mathcal{H})$.

We aim to find a (x, y)-impression $(G, \mathcal{M})$ of $(G, \mathcal{H})$ in which each $M \in \mathcal{M}$ contains a vertex incident to the outer face of G. In light of Lemma 3.2, we can find such a (x, y)-impression by first finding such a (x', ∞)-impression and then refining this impression. The following theorem comes close to giving us such a (x', ∞)-impression (not every $M \in \mathcal{M}$ contains a vertex incident to the outer face of G, but they are at distance at most 1 from such a $M' \in \mathcal{M}$ in $\mathbf{IM}(G, \mathcal{M})$). The exact technical statement of this theorem is tailored to allow for an inductive argument on extending partial (x', ∞)-impressions.

Theorem 3.3. *Let G be a finite planar graph and $\mathcal{H}$ be a spanning collection of connected vertex sets in G, each of which contains a vertex incident to the outer face of G. Let $(G, \mathcal{B})$ be a partial $(30, \infty)$-impression of $(G, \mathcal{H})$ in which for every $B \in \mathcal{B}$, either B contains a vertex incident to the outer face of G and $I_{\mathcal{H}}(B)$ has weak diameter at most 10 in $\mathbf{RIG}(G, \mathcal{H})$, or B touches some $B' \in \mathcal{B}$*

where B' contains a vertex incident to the outer face of G, and for every maximal connected vertex set K of $G\backslash(\bigcup_{B\in\mathcal{B}} B)$, either

1. *$I_{\mathcal{H}}(K)$ has weak diameter at most 10 in **RIG**$(G,\mathcal{H})$ and contains a vertex incident to the outer face of G, or $I_{\mathcal{H}}(K)$ has weak diameter at most 30 in **RIG**$(G,\mathcal{H})$ and touches some $B\in\mathcal{B}$ that does contain a vertex incident incident to the outer face of G, or*
2. *K contains a vertex incident to the outer face of G and touches no $B\in\mathcal{B}$, or*
3. *K contains a vertex incident to the outer face of G and touches a unique $B\in\mathcal{B}$, or*
4. *K contains a vertex incident to the outer face of G and there exists distinct $B_1,B_2\in\mathcal{B}$ satisfying the following*
   - *K touches B_1 and B_2 and no other $B\in\mathcal{B}\backslash\{B_1,B_2\}$,*
   - *B_1 and B_2 both contain a vertex incident to the outer face of G, and*
   - *B_1 and B_2 both contain a vertex incident to a common face F distinct from the outer-face.*

*Then there is a $(30,\infty)$-impression $(G,\mathcal{M})$ of $(G,\mathcal{H})$ such that for every $M\in\mathcal{M}$, either M contains a vertex incident to the outer face of G and $I_{\mathcal{H}}(B)$ has weak diameter at most 10 in **RIG**$(G,\mathcal{H})$, or M touches some $M'\in\mathcal{M}$ where M' contains a vertex incident to the outer face of G.*

Proof. We shall argue inductively on $|V(G)\backslash(\bigcup_{B\in\mathcal{B}} B)|$. Clearly the theorem holds if we have $V(G)\backslash(\bigcup_{B\in\mathcal{B}} B)=\emptyset$. So, we may assume that there is some connected component of $G\backslash(\bigcup_{B\in\mathcal{B}} B)$, and we let K be its vertex set.

Suppose first that K satisfies Item 1. Let $B^*=K$ and $\mathcal{B}^*=\mathcal{B}\cup\{B^*\}$. Clearly, $(G,\mathcal{B}^*)$ satisfies the inductive hypothesis and $|V(G)\backslash(\bigcup_{B\in\mathcal{B}^*} B)|<|V(G)\backslash(\bigcup_{B\in\mathcal{B}} B)|$, so the theorem follows.

Now, suppose that K satisfies Item 2. Then simply choose some $b\in K$ incident to the outer face of G, let $B^*=\{b\}$ and $\mathcal{B}^*=\mathcal{B}\cup\{B^*\}$. Clearly, $(G,\mathcal{B}^*)$ satisfies the inductive hypothesis and $|V(G)\backslash(\bigcup_{B\in\mathcal{B}^*} B)|<|V(G)\backslash(\bigcup_{B\in\mathcal{B}} B)|$, so the theorem follows.

Now, suppose that K satisfies Item 3. Let B be the unique set in $\mathcal{B}$ that K touches. Then $I_{\mathcal{H}}(B)$ has weak diameter at most 10 in **RIG**$(G,\mathcal{H})$. As K contains a vertex incident to the outer face of G, there exists intersecting (but possibly non-distinct) $H_1,H_2\in\mathcal{H}$ such that $K\cap(H_1\cup H_2)$ contains a vertex b incident to the outer face of G, and $K\cap(H_1\cup H_2)$ touches B. Let B^* be the vertex set of a shortest path between b and $N_G(B)$ in $G[K\cap(V(H_1)\cup V(H_2))]$, and let $\mathcal{B}^*=\mathcal{B}\cup\{B^*\}$. Then $I_{\mathcal{H}}(B^*)$ has weak diameter at most 3 in **RIG**$(G,\mathcal{H})$, and clearly B^* touches B. Let K' be the vertex set of a connected component of $G[K]\backslash B^*$. If K' contains a vertex incident to the outer face of G, then K' satisfies either Item 3 or Item 4. Otherwise, K' contains no vertex incident to the outer face of G. As K' touches none of $\mathcal{B}^*\backslash\{B,B^*\}$ (but at least one of B,B^*) and for every $H\in\mathcal{H}$ that intersects K', we have that H also intersects at least one of B or B^*, we have that $I_{\mathcal{H}}(K')$ has weak diameter at most $10+3=13$ in **RIG**$(G,\mathcal{H})$. Thus, K' then satisfies Item 1. So, $(G,\mathcal{B}^*)$ satisfies the inductive hypothesis and $|V(G)\backslash(\bigcup_{B\in\mathcal{B}^*} B)|<|V(G)\backslash(\bigcup_{B\in\mathcal{B}} B)|$, so the theorem follows.

Lastly, we can now assume that K satisfies Item 4. Let B_1,B_2,F be as in Item 4. Note that the intersection of K with the vertices of the outer face of G induces a connected graph since B_1 and B_2 both contain a vertex on the outer face of G and the face F (which is distinct from the outer face). Let Q be the vertex set of this connected induced subgraph. Let $p_1p_2\dots p_t$ be

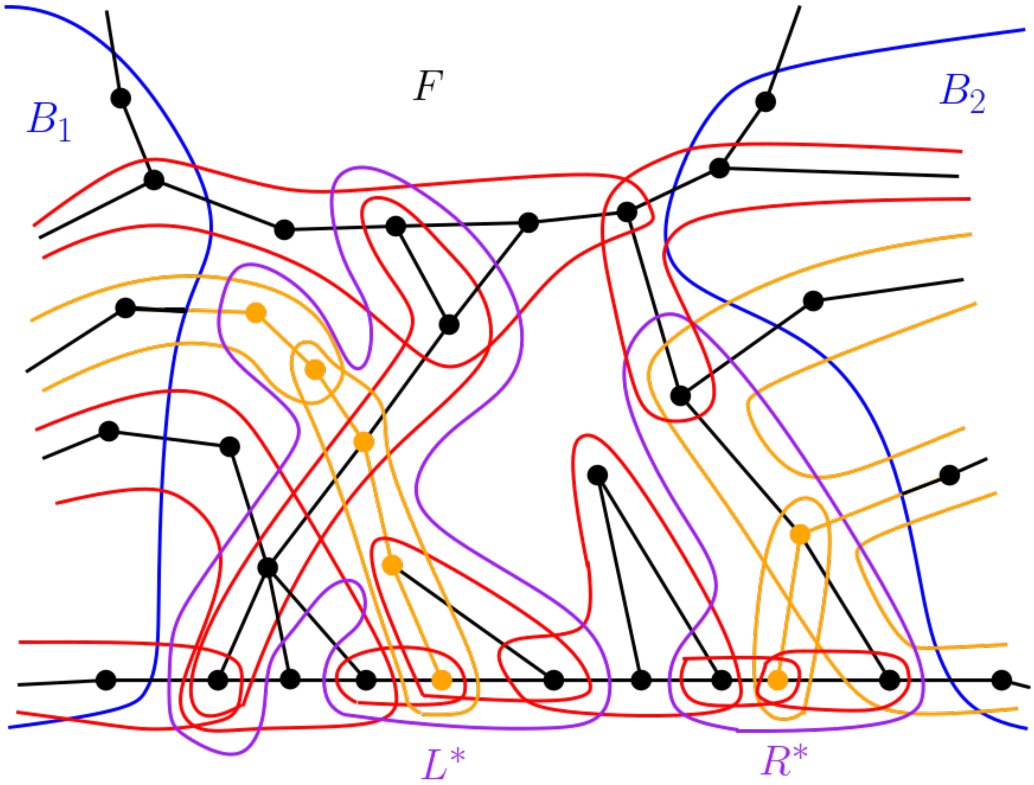

Figure 3: An illustration of the constructed connected sets L^* and Rj (coloured purple) in a case where they do not intersect. Both B_1 and B_2 are coloured blue. The connected sets in $\mathcal{H}$ are coloured red or orange, with the orange ones being $H_\ell, H'_\ell, H_r, H'_r$. The two orange paths indicate L and R.

the walk in $G[Q]$ along the outer face of G (and containing all of) Q, where p_1 touches B_1 and p_t touches B_2. Choose $1 \leqslant \ell \leqslant t$ maximum such that that there exists a path with vertex set L between p_ℓ and $N_G(B_1)$ such that L is contained in $K \cap (H_\ell \cup H'_\ell)$ for some intersecting (but not necessarily distinct) $H_\ell, H'_\ell \in \mathcal{H}$. Similarly, choose $1 \leqslant r \leqslant t$ minimum such that that there exists a path with vertex set R between p_r and $N_G(B_2)$ such that R is contained in $K \cap (H_r \cup H'_r)$ for some intersecting (but not necessarily distinct) $H_r, H'_r \in \mathcal{H}$. Note that such ℓ, r exist since K touches exactly B_1, B_2 from $\mathcal{B}$, and K (and every $H \in \mathcal{H}$) contains a vertex incident to the outer face of G. Let L^* be the vertex set of the connected component of $G\left[K \cap \left(\bigcup_{H \in I_{\mathcal{H}}(L)} H\right)\right]$ containing L, and similarly, let R^* be the vertex set of the connected component of $G\left[K \cap \left(\bigcup_{H \in I_{\mathcal{H}}(R)} H\right)\right]$ containing R. Clearly $I_{\mathcal{H}}(L^*)$ and $I_{\mathcal{H}}(R^*)$ both have weak diameter at most 5 in $\mathbf{RIG}(G, \mathcal{H})$. See Figure 3 for an illustration (in a case where L^* and R^* do not intersect). We split into two cases depending on whether or not L^* and R^* intersect.

Suppose first that L^* and R^* do intersect. Then let $B^* = L^* \cup R^*$, and let $\mathcal{B}^* = \mathcal{B} \cup \{B^*\}$. Clearly $I_{\mathcal{H}}(B^*)$ has weak diameter at most $5 + 5 = 10$ in $\mathbf{RIG}(G, \mathcal{H})$. Note that B^* touches both B_1 and B_2. Consider the vertex set K' of a connected component of $G[K] \backslash B^*$. If K' contains a vertex incident to the outer face of G, then K' clearly satisfies Item 3 or Item 4. So suppose that K' contains no vertex incident to the outer face of G. Then K' touches at least one of B_1, B_2, B^* and for every $H \in I_{\mathcal{H}}(K')$, we have that H intersects at least one of B_1, B_2, B^*. Since B^* touches both B_1 and B_2, it follows that $I_{\mathcal{H}}(B^*)$ has weak diameter at most $10 + 10 + 10 = 30$ in $\mathbf{RIG}(G, \mathcal{H})$. Thus, K' satisfies Item 1. So, $(G, \mathcal{B}^*)$ satisfies the inductive hypothesis and $|V(G) \backslash (\bigcup_{B \in \mathcal{B}^*} B)| < |V(G) \backslash (\bigcup_{B \in \mathcal{B}} B)|$, so the theorem follows.

We may now assume that L^* and R^* do not intersect. This in particular implies that $\ell < r$. Let $\mathcal{B}^* = \mathcal{B} \cup \{L^*, R^*\}$. Consider the vertex set K' of a connected component of $G[K] \backslash (L^* \cup R^*)$. Suppose that K' does not contain a vertex incident to the outer face of G. Then K touches at least one of B_1, B_2, L^*, R^*, and for every $H \in I_{\mathcal{H}}(K')$, we have that H intersects at least one of B_1, B_2, L^*, R^*. Since $I_{\mathcal{H}}(B_1), I_{\mathcal{H}}(B_2)$ both have weak diameter at most 10 in $\mathbf{RIG}(G, \mathcal{H})$ and $I_{\mathcal{H}}(L^*), I_{\mathcal{H}}(R^*)$ both have weak diameter at most 5 in $\mathbf{RIG}(G, \mathcal{H})$, it follows that $I_{\mathcal{H}}(K')$ has weak diameter at most 30 in $\mathbf{RIG}(G, \mathcal{H})$. Thus in this case, K' satisfies Item 1. So, we may now assume that K' does contain a vertex incident to the outer face of G. If K' does not touch both L^* and R^*, then as before, K' satisfies either Item 3 or Item 4. So we assume now that K' touches both L^* and R^*. Showing that K' satisfies Item 4 will complete the proof.

Let Q' be the vertices p_i of the walk Q with $\ell < i < r$. Note that Q' is non-empty since L^* and R^* do not intersect (and L^* contains both p_i and p_{i+1}). Let $\mathcal{H}_{Q'} = I_{\mathcal{H}}(Q')$. Observe that by our choice of L and R, there is no intersecting pair $H_1 \in \mathcal{H}_{Q'}$, $H_2 \in \mathcal{H}$ such that there exists a path in $G[K \cap (H_1 \cup H_2)]$ between Q' and $N_G(B_1 \cup B_2)$. By our choice of L^* and R^*, we therefore have that $I_{\mathcal{H}}(K') \subseteq \mathcal{H}_{Q'}$. In particular, K' does not touch B_1 or B_2. It now follows that there is some face F of G distinct from the outer face containing a vertex of L^* and a vertex of R^*. Thus K' satisfies Item 4.

So, $(G, \mathcal{B}^*)$ satisfies the inductive hypothesis and $|V(G) \backslash (\bigcup_{B \in \mathcal{B}^*} B)| < |V(G) \backslash (\bigcup_{B \in \mathcal{B}} B)|$, so the theorem follows. □

Now we quickly refine the previous theorem so that every graph in $\mathcal{M}$ contains a vertex incident to the outer face of G.

Lemma 3.4. *Let G be a finite planar graph and $\mathcal{H}$ a spanning collection of connected vertex sets of G, each of which contains a vertex incident to the outer face of G. Then there is a $(70, \infty)$-impression $(G, \mathcal{M})$ of $(G, \mathcal{H})$ such that for every $M \in \mathcal{M}$, M contains a vertex incident to the outer face of G.*

Proof. By Theorem 3.3 (applied with $\mathcal{B} = \emptyset$), there is a $(30, \infty)$-impression $(G, \mathcal{M}')$ of $(G, \mathcal{H})$ such that for every $M \in \mathcal{M}'$, either M contains a vertex incident to the outer-face of G and $I_{\mathcal{H}}(B)$ has weak diameter at most 10 in $\mathbf{RIG}(G, \mathcal{H})$, or M touches some $M' \in \mathcal{M}'$ where M' contains a vertex incident to the outer face of G.

Let $\mathcal{M}_1$ be the elements M of $\mathcal{M}'$ that contain a vertex incident to the outer face of G, and let $\mathcal{M}_2$ be the elements M of $\mathcal{M}'$ that do not contain a vertex incident to the outer face of G. Then $\mathcal{M}' = \mathcal{M}_1 \cup \mathcal{M}_2$. Choose some function $f : \mathcal{M}_2 \to \mathcal{M}_1$ such that M touches $f(M)$ for every $M \in \mathcal{M}_2$. Let $\mathcal{M} = \{M \cup f^{-1}(M) : M \in \mathcal{M}_1)\}$. Then for every $M \in \mathcal{M}$, we have that $I_{\mathcal{H}}(M)$ has weak diameter at most $30 + 10 + 30 = 70$ in G. Then $(G, \mathcal{M})$ is a $(70, \infty)$-impression of $(G, \mathcal{H})$ such that for every $M \in \mathcal{M}$, M contains a vertex incident to the outer face of G, as desired. □

We can now quickly prove Theorem 3.1 using Lemma 3.4 and Lemma 3.2. We remark that in our proof of Theorem 2.5, it actually turns out to be more convenient to use Lemma 3.4 and Lemma 3.2 separately as below instead of Theorem 3.1. This is essentially because we end up needing to apply Lemma 3.2 another time later anyway. However for the sake of completeness for outerstring graphs, we still complete the proof of Theorem 3.1 here.

Proof of Theorem 3.1. By Lemma 3.4, there is a $(70, \infty)$-impression $(G, \mathcal{M}')$ of $(G, \mathcal{H})$ such that for every $M \in \mathcal{M}'$, M contains a vertex incident to the outer face of G. Let $G' = \mathbf{IM}(G, \mathcal{M}')$. Then G' is outerplanar. For each $H \in \mathcal{H}$, let H' be the connected vertex set of G' that is exactly the union of all $M' \in \mathcal{M}'$ such that M' intersects H. Let $\mathcal{H}' = \{H' : H \in \mathcal{H}\}$.

By Lemma 3.2, there is a $(11,9)$-impression $(G',\mathcal{M}^*)$ of $(G',\mathcal{H}')$. Let $\mathcal{M}=\{\bigcup_{M'\in M^*} M' : M^*\in\mathcal{M}\}$. Clearly for every $H\in\mathcal{H}$, $I_{\mathcal{M}}(H)$ has weak diameter at most 9 in $\mathbf{IM}(G,\mathcal{M})$. Also, for every $M\in\mathcal{M}$, we have that $I_{\mathcal{H}}(M)$ has weak diameter at most $11\cdot 70=770$ in $\mathbf{RIG}(G,\mathcal{H})$.

Hence, $(G,\mathcal{M})$ is a $(770,9)$-impression of $(G,\mathcal{H})$, as desired. □

4 Encasings

The key to carrying out our inductive proof of Theorem 2.5 in Section 5 will be that we can find what we shall call a (a,b,c,d)-encasing. The aim of this section is to find these encasings. We show that we can find (a,b,c,d)-encasing by first finding a (a,∞,∞,∞)-encasing, then refining it to a (a,∞,c,∞)-encasing, then further refining it to a (a,∞,c,d)-encasing, and then finally refining it to a (a,b,c,d)-encasing. To ease into things, we delay the full definition of an (a,b,c,d)-encasing for now, and start simply with (a,∞,∞,∞)-encasings.

Let G be a finite planar graph and let $\mathcal{H}$ be a spanning collection of connected vertex sets in G. Let $\mathcal{B}$ be a collection of disjoint connected vertex sets in G. We say that $\mathcal{B}$ *encircles* $(G,\mathcal{H})$ if every $B\in\mathcal{B}$ contains a vertex incident to the outer face of G, and every $H\in\mathcal{H}$ that contains a vertex incident to the outer face of G is contained in $\bigcup_{B\in\mathcal{B}} B$. We say that $\mathcal{B}$ is a (a,∞,∞,∞)*-encasing* of $(G,\mathcal{H})$ if $\mathcal{B}$ encircles $(G,\mathcal{H})$, and

- for every $B\in\mathcal{B}$, we have that $I_{\mathcal{H}}(B)$ has weak diameter at most a in $\mathbf{RIG}(G,\mathcal{H})$.

We begin by finding a (a,∞,∞,∞)-encasing. The lemma has an extra condition on the vertices contained in $\bigcup_{B\in\mathcal{B}} B$ that we will need later.

Lemma 4.1. *Let G be a finite planar graph and $\mathcal{H}$ a spanning collection of connected vertex sets of G. Then there is a $(210,\infty,\infty,\infty)$-encasing $\mathcal{B}$ of $(G,\mathcal{H})$ with $\bigcup_{B\in\mathcal{B}} B=\bigcup_{H\in\mathcal{H}_0\cup\mathcal{H}_1} H$, where $\mathcal{H}_0$ are the elements of $\mathcal{H}$ that contain a vertex incident to the outer face of G and $\mathcal{H}_1$ are the elements of $\mathcal{H}\backslash\mathcal{H}_0$ that intersect a element of $\mathcal{H}_0$.*

Proof. Let G^* be the subgraph of G with edges $\bigcup_{H\in\mathcal{H}_0\cup\mathcal{H}_1} E(G[H])$ and no isolated vertices. Note that G^* has the same outer face as G. For each $H\in\mathcal{H}_1$, choose some $H'\in\mathcal{H}_0$ that intersects H and let $H^*=H\cup H'$. Let $\mathcal{H}^*=\mathcal{H}_0\cup\{H^*: H\in\mathcal{H}_1\}$. Then every vertex on the outer face of G is contained in some element of $\mathcal{H}^*$. By Lemma 3.4, there is a $(70,\infty)$-impression $(G^*,\mathcal{B})$ of $(G^*,\mathcal{H}^*)$ such that every vertex of G^* incident to the outer face of G^* is contained in some $B\in\mathcal{B}$. In particular, every vertex of G incident to the outer face of G is contained in some $B\in\mathcal{B}$. Furthermore, observe that as $I_{\mathcal{H}^*}(B)$ has weak diameter at most 70 in $\mathbf{RIG}(G^*,\mathcal{H}^*)$ for each $B\in\mathcal{B}$, we also have that $I_{\mathcal{H}^*}(B)$ has weak diameter at most $3\cdot 70=210$ in $\mathbf{RIG}(G,\mathcal{H})$ for each $B\in\mathcal{B}$. □

Let G be a finite planar graph, $\mathcal{H}$ a spanning collection of connected vertex sets of G, and let $\mathcal{B}$ be a collection of disjoint connected sets of G that encircles $(G,\mathcal{H})$. Let $\mathcal{F}$ be the collection of non-empty vertex sets of G that are maximal with respect to being contained in the interior of a face of $G[\bigcup_{B\in\mathcal{B}} B]$. We call these *facial* vertex sets. We remark that these facial vertex sets are not necessarily connected sets of G. Note that $\mathcal{B}\cup\mathcal{F}$ is a partition of $V(G)$. We denote by $G(\mathcal{B})$ the graph obtained from $\mathbf{IM}(G,\mathcal{B})$ by for every facial vertex set $\mathcal{F}$ of $G[\bigcup_{B\in\mathcal{B}} B]$, we add F as a vertex and make it adjacent to exactly the elements $B\in\mathcal{B}$ that contain a vertex incident to the face of $G[\bigcup_{B\in\mathcal{B}} B]$ that contains F (we remark that B and F need not touch).

We say that some $H\in\mathcal{H}$ is d-caged by $\mathcal{B}$ if for every $u,v\in H$, there exists some sequence of vertex sets $A_0,\dots,A_t$ with $t\leqslant d$ such that

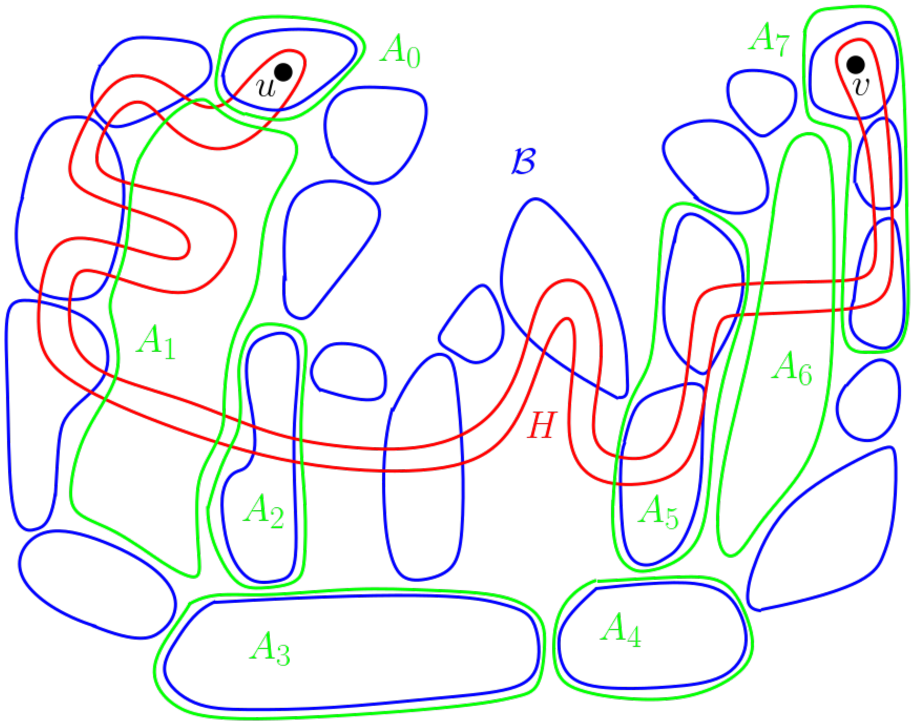

Figure 4: An illustration of some $H \in \mathcal{H}$ (coloured red) that is 7-caged by $\mathcal{B}$ (coloured blue). For $u, v \in H$, the vertex sets $A_0, \ldots, A_7$ (coloured green) are as in the definition of being 7-caged. As an example illustrating the end of the proof of Lemma 4.2, we would have that $A_0 = A_u$, $A_1 \in \mathcal{F}_u$, $A_2, A_5 \in \mathcal{A}_{u,v}$, $\{A_3, A_4\} = \mathcal{V}_{(X,Y)}$, $A_6 \in \mathcal{F}_v$, and $A_7 = A_v$.

- each A_i is either a single element of $\mathcal{B}$ or a single element of $\mathcal{F}$ or the union of a subcollection $\mathcal{B}' \subseteq \mathcal{B}$ such that some connected set $H^* \subseteq H$ of G intersects each member of $\mathcal{B}'$ and is contained in $G[\bigcup_{B \in \mathcal{B}'} B]$,
- $u \in A_0$,
- $v \in A_t$, and
- for every $1 \leqslant i < t$, we have that either H intersects both A_{i-1} and A_i, or neither A_{i-1} or A_i is an element of $\mathcal{F}$ and they touch.

See Figure 4 for an illustration of some 7-caged H. We say that $\mathcal{B}$ is a d-cage of $(G, \mathcal{H})$ if every $H \in \mathcal{H}$ is d-caged by $\mathcal{B}$.

The motivation behind the notion of a d-cage is that it will allow us to "cut up" connected sets H in such a way that each part is in some sense still close via the encircling $\mathcal{B}$. Being able to do this is crucial to the inductive argument of Theorem 2.5.

We say that $\mathcal{B}$ is a (a, ∞, ∞, d)-*encasing* of $(G, \mathcal{H})$ if $\mathcal{B}$ encircles $(G, \mathcal{H})$, and

- for every $B \in \mathcal{B}$, we have that $I_{\mathcal{H}}(B)$ has weak diameter at most a in $\mathbf{RIG}(G, \mathcal{H})$, and
- $\mathcal{B}$ is a d-cage of $(G, \mathcal{H})$.

In other words, $\mathcal{B}$ is a (a, ∞, ∞, d)-encasing of $(G, \mathcal{H})$ if it is both a $(a, \infty, \infty, \infty)$-encasing and a d-cage of $(G, \mathcal{H})$.

We will use our $(a, \infty, \infty, \infty)$-encasing to get a (a, ∞, ∞, d)-encasing. However, we again need this (a, ∞, ∞, d)-encasing to have some stronger structure on the vertices of G that

$\bigcup_{B\in\mathcal{B}} B$ contains. Let $\mathcal{H}_0$ be the elements of $\mathcal{H}$ that contain a vertex incident to the outer face of G, let $\mathcal{H}_1$ be the elements of $\mathcal{H}\backslash\mathcal{H}_0$ that intersect a element of $\mathcal{H}_0$, and similarly, let $\mathcal{H}_2$ be the elements of $\mathcal{H}\backslash(\mathcal{H}_0 \cup \mathcal{H}_1)$ that intersect a element of $\mathcal{H}_1$. We say that a (a, ∞, ∞, d)-encasing $\mathcal{B}$ of $(G, \mathcal{H})$ is *enforced* if $\bigcup_{H\in\mathcal{H}_0\cup\mathcal{H}_1} H \subseteq \bigcup_{B\in\mathcal{B}} B \subseteq \bigcup_{H\in\mathcal{H}_0\cup\mathcal{H}_1\cup\mathcal{H}_2} H$, and for every $v \in \left(\bigcup_{B\in\mathcal{B}} B\right) \setminus \left(\bigcup_{H\in\mathcal{H}_0\cup\mathcal{H}_1} H\right)$, there is some connected set $H' \subseteq H \in \mathcal{H}_2$ of G that contains v, is contained in $\bigcup_{B\in\mathcal{B}} B$, and intersects $\bigcup_{H\in\mathcal{H}_0\cup\mathcal{H}_1} H$.

Lemma 4.2. *Let G be a finite planar graph and $\mathcal{H}$ a spanning collection of connected vertex sets of G. Then there is an enforced $(840, \infty, \infty, 7)$-encasing $\mathcal{B}$ of $(G, \mathcal{H})$.*

Proof. We define $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2$ as above in the definition of an enforced (a, ∞, ∞, d)-encasing of $(G, \mathcal{H})$. By Lemma 4.1, there is a $(210, \infty, \infty, \infty)$-encasing $\mathcal{B}$ of $(G, \mathcal{H})$ with $\bigcup_{B\in\mathcal{B}} B = \bigcup_{H\in\mathcal{H}_0\cup\mathcal{H}_1} H$. We will adjust this encasing to obtain the desired enforced $(840, \infty, \infty, 7)$-encasing.

Let $G^* = G(\mathcal{B})$ as above, where $\mathcal{F}$ are the additional facial vertex sets. Choose some $B_0 \in \mathcal{B}$. We say that a separation (X, Y) of G^* is *1-clean* if $B_0 \in X$, $X \cup Y = V(G^*)$, $|X \cap Y| = 1$, and subject to this, there is no other such (X', Y') with $X' \cap Y' = X \cap Y$ and $|Y'| > |Y|$. A separation (X, Y) of G^* is *2-clean* if $B_0 \in X$, $X \cup Y = V(G^*)$, $X \cap Y$ has two elements and they are adjacent in G^*, and there is no 1-clean separation (X^*, Y^*) with $X^* \cap Y^* \subset X \cap Y$ and $Y^* \subseteq Y$, and there is no other such separation (X', Y') of G^* with $X' \cap Y' = X \cap Y$ and $|Y'| > |Y|$. By abuse of notation, we shall also call $(\{B_0\}, V(G^*))$ a 1-clean separation. A separation (X, Y) of G^* is *clean* if it is 1-clean or 2-clean. Observe that if (X, Y) is a clean separation, then $X \cap Y \subseteq \mathcal{B}$.

For a clean separation (X, Y) of G^*, we let $R_{(X,Y)}$ be the vertices R of G^* in $Y\backslash\mathcal{F}$ (so R is one of the vertex sets $Y\backslash\mathcal{F} = Y \cap \mathcal{B}$) such that there exists some $H \in \mathcal{H}$ that intersects both R and $\bigcup_{A\in X\cap Y} A$. We say that a clean separation (X', Y') of G^* is a *child* of another clean separation (X, Y) if $Y' \subseteq Y$, $R_{(X,Y)} \cap Y' = \emptyset$, and subject to this, there is no other such clean separation (X^*, Y^*) with Y^* containing Y' as a proper subset. We say that a clean separation (X', Y') is a *descendant* of another clean separation (X, Y) if it there is a sequence of clean separations $(X_1, Y_1), \ldots (X_t, Y_t)$ with $(X_1, Y_1) = (X, Y)$, $(X_t, Y_t) = (X', Y')$ and for each $1 \leqslant i < t$, (X_{i+1}, Y_{i+1}) is a child of (X_i, Y_i). Let $\mathcal{T}$ be the collection of clean separations consisting of $(\{B_0\}, V(G^*))$ and all of its descendants. We let $\mathcal{T}_1$ be the 1-clean separations of $\mathcal{T}$, and $\mathcal{T}_2$ be the 2-clean separations of $\mathcal{T}$.

For a clean separation (X, Y) of G^*, we let $H_{(X,Y)}$ be the vertices v of G contained in some vertex set of $\mathcal{F} \cap Y$ such that there exists some $H \in \mathcal{H}_2$ containing both v and intersecting $\bigcup_{A\in X\cap Y} A$. If (X, Y) is 1-clean, then we let $V_{(X,Y)} = H_{(X,Y)} \cup \bigcup_{R\in R_{(X,Y)}} R$. Note that in this case, $V_{(X,Y)}$ is a connected set of G and $I_{\mathcal{H}}(V_{X,Y})$ has weak diameter at most $3 \cdot 210 = 630$ in $\mathbf{RIG}(G, \mathcal{H})$. Furthermore, if $v \in V_{(X,Y)} \setminus \left(\bigcup_{B\in\mathcal{B}} B\right) = V_{(X,Y)} \setminus \left(\bigcup_{H\in\mathcal{H}_0\cup\mathcal{H}_1} H\right)$, then there exists some connected set $H' \subseteq H \in \mathcal{H}_2$ of G that contains v, is contained in $V_{(X,Y)}$, and intersects $\bigcup_{H\in\mathcal{H}_0\cup\mathcal{H}_1} H$ (or more precisely, the unique vertex set of $X \cap Y$). Otherwise, if (X, Y) is 2-clean with $X \cap Y = \{B^1, B^2\}$, then we choose disjoint $V^1_{(X,Y)}, V^2_{(X,Y)}$ such that $B^1 \subseteq V^1_{(X,Y)}$, $B^2 \subseteq V^2_{(X,Y)}$, $V^1_{(X,Y)} \cup V^2_{(X,Y)} = H_{(X,Y)} \cup \bigcup_{R\in R_{(X,Y)}} R$, and both $V^1_{(X,Y)}$ and $V^2_{(X,Y)}$ are connected sets of G (see Figure 5 for an illustration). Then $I_{\mathcal{H}}(V^1_{(X,Y)} \cup V^2_{(X,Y)})$, and in particular both of $I_{\mathcal{H}}(V^1_{(X,Y)})$ and $I_{\mathcal{H}}(V^2_{(X,Y)})$ all have weak diameter at most $4 \cdot 210 = 840$ in $\mathbf{RIG}(G, \mathcal{H})$. Furthermore, if $v \in (V^1_{(X,Y)} \cup V^2_{(X,Y)}) \setminus \left(\bigcup_{B\in\mathcal{B}} B\right) = (V^1_{(X,Y)} \cup V^2_{(X,Y)}) \setminus \left(\bigcup_{H\in\mathcal{H}_0\cup\mathcal{H}_1} H\right)$, then there exists some connected set $H' \subseteq H \in \mathcal{H}_2$ of G that contains v, is contained in $V^1_{(X,Y)} \cup V^2_{(X,Y)}$, and intersects $B^1 \cup B^2 \subseteq \bigcup_{H\in\mathcal{H}_0\cup\mathcal{H}_1} H$.

Let $\mathcal{B}'$ be the elements of $\mathcal{B}$ that do not intersect $V_{(X,Y)}$ for any $(X, Y) \in \mathcal{T}_1$ and do not

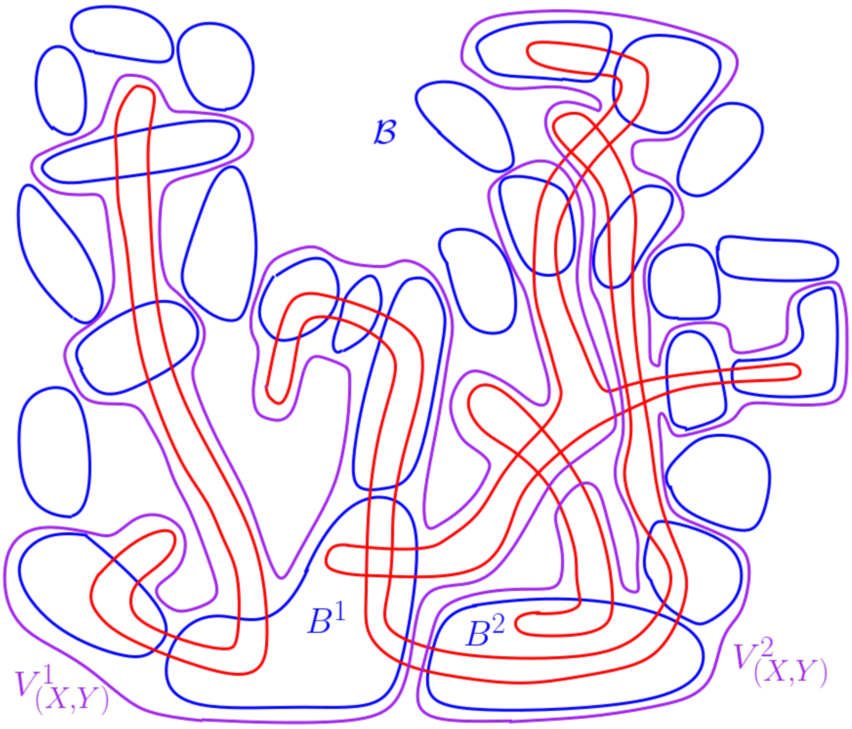

Figure 5: A choice of $V^1_{(X,Y)}, V^2_{(X,Y)}$ (coloured purple) as in the proof of Lemma 4.2. Elements of $\mathcal{B}$ are coloured blue and elements of $\mathcal{H}$ are coloured red.

intersect $V^1_{(X,Y)}$ or $V^2_{(X,Y)}$ for any $(X,Y) \in \mathcal{T}_2$. At last, let $\mathcal{B}^* = \mathcal{B}' \cup \{V_{(X,Y)} : (X,Y) \in \mathcal{T}_1\} \cup \{V^1_{(X,Y)}, V^2_{(X,Y)} : (X,Y) \in \mathcal{T}_2\}$.

Clearly, we have that $\bigcup_{H \in \mathcal{H}_0 \cup \mathcal{H}_1} H \subseteq \bigcup_{B \in \mathcal{B}^*} B \subseteq \bigcup_{H \in \mathcal{H}_0 \cup \mathcal{H}_1 \cup \mathcal{H}_2} H$, and for every $B \in \mathcal{B}^*$, we have that $I_{\mathcal{H}}(B)$ has weak diameter at most 840 in $\mathbf{RIG}(G, \mathcal{H})$. We also have that for every $v \in \left(\bigcup_{B \in \mathcal{B}^*} B\right) \setminus \left(\bigcup_{H \in \mathcal{H}_0 \cup \mathcal{H}_1} H\right)$, there is some connected set $H' \subseteq H \in \mathcal{H}_2$ of G that contains v, is contained in $\bigcup_{B \in \mathcal{B}^*} B$, and intersects $\bigcup_{H \in \mathcal{H}_0 \cup \mathcal{H}_1} H$. To show that $\mathcal{B}^*$ provides the desired $(840, \infty, \infty, 7)$-encasing of $(G, \mathcal{H})$, it now just remains to show that $\mathcal{B}^*$ is a 7-cage of $(G, \mathcal{H})$.

If $H \in \mathcal{H}_0 \cup \mathcal{H}_1$, then H is 0-caged by $\mathcal{B}^*$ since $H \subseteq \bigcup_{B \in \mathcal{B}^*} B$. If $H \in \mathcal{H} \setminus (\mathcal{H}_0 \cup \mathcal{H}_1 \cup \mathcal{H}_2)$, then H lies in some face of $G(\mathcal{B})$. In particular, H intersects at most two elements of $\mathcal{B}^*$ and furthermore if H does intersect two elements of $\mathcal{B}^*$ then they are touching, so it follows that H is 3-caged by $\mathcal{B}^*$.

It remains to consider $H \in \mathcal{H}_2$. To do so, we must first introduce certain subsets $\mathcal{B}^*_{(X,Y)}$ and $\mathcal{F}^*_{(X,Y)}$ of $V(G(\mathcal{B}^*))$ according to clean separations (X,Y) of G^*. Fix some $H \in \mathcal{H}_2$. For each 1-clean separation $(X,Y) \in \mathcal{T}_1$, let $\mathcal{V}_{(X,Y)} = \{V_{(X,Y)}\}$, and for each 2-clean separation $(X,Y) \in \mathcal{T}_2$, let $\mathcal{V}_{(X,Y)} = \{V^1_{(X,Y)}, V^2_{(X,Y)}\}$. Let $\mathcal{F}^*$ be the facial vertex sets of $G(\mathcal{B}^*)$. For each clean separation $(X,Y) \in \mathcal{T}$, let $\mathcal{F}^*_{(X,Y)}$ be the elements F of $\mathcal{F}^*$ adjacent in $G(\mathcal{B}^*)$ to a element of $\mathcal{V}_{(X,Y)}$. Now, let $\mathcal{B}^*_{(X,Y)} = N_{G(\mathcal{B}^*)}(\mathcal{F}^*_{(X,Y)})$.

Consider some $H \in \mathcal{H}_2$. By our choice of $\mathcal{B}^*$, we have that there exists some clean separation $(X,Y) \in \mathcal{T}$ such that H is entirely contained in $\bigcup_{B \in \mathcal{B}^*_{(X,Y)}} B \cup \bigcup_{F \in \mathcal{F}^*_{(X,Y)}} F$. Observe that $G(\mathcal{B}^*)[\mathcal{B}^*_{(X,Y)} \cup \mathcal{F}^*_{(X,Y)}]$ is a planar graph in which the vertices $\mathcal{B}^*_{(X,Y)}$ are incident to the outer face and every $F \in \mathcal{F}^*_{(X,Y)}$ is adjacent to some $V \in \mathcal{V}_{(X,Y)}$, where $|\mathcal{V}_{(X,Y)}| \leqslant 2$ and if $|\mathcal{V}_{(X,Y)}| = 2$, then the two vertices of $\mathcal{V}_{(X,Y)}$ are adjacent.

Consider some $u, v \in H$. Let H_u be a subset of H that contains u and is maximal with respect to being a connected set of G and being contained in $G[\bigcup_{B \in \mathcal{B}} B]$ (or equivalently

$G[\bigcup_{B\in\mathcal{B}^*} B]$). Let A_u be the union of all $B \in \mathcal{B}^*$ that intersect H_u. Let $\mathcal{F}_u$ be elements F of $\mathcal{F}^*$ such that H intersects both F and A_u. We define A_v and $\mathcal{F}_v$ similarly. Crucially, note that if H intersects distinct $F_1, F_2 \in \mathcal{F}^*$, then H intersects some $V \in \mathcal{B}$ incident to the face of $G(\mathcal{B}^*)$ corresponding to F_1 that is also either contained in $\mathcal{V}_{(X,Y)}$ or adjacent to some element of $\mathcal{V}_{(X,Y)}$ in $G(\mathcal{B}^*)$. Let $\mathcal{A}_{u,v}$ be the collection of unions of subcollections $\mathcal{B}' \subseteq \mathcal{B}^*\backslash\mathcal{V}_{(X,Y)}$ such that some connected set $H^* \subseteq H$ of G intersects each member of $\mathcal{B}'$ and is contained in $G[\bigcup_{B\in\mathcal{B}'} B]$, and some $B \in \mathcal{B}'$ touches some $V \in \mathcal{V}_{(X,Y)}$. To see that H is 7-caged by $\mathcal{B}^*$, it is now straight forward to choose the required sequence of vertex sets $A_0, \dots, A_t$ with $t \leqslant 7$ from the collection $\{A_u\} \cup \mathcal{F}_u \cup (\mathcal{A}_{u,v} \cup \mathcal{V}_{(X,Y)}) \cup \mathcal{F}_v \cup \{A_v\}$ (here we take $A_0 = A_u$, $A_t = A_v$, at most one A_i is contained in $\mathcal{F}_u$ (this would be A_1 if so), at most one is contained in $\mathcal{F}_v$ (this would be A_{t-1} if so), at most two are contained in $\mathcal{A}_{u,v}$, and at most two more are contained in $\mathcal{V}_{(X,Y)}$). For an illustrative example, see Figure 4.

So, $\mathcal{B}^*$ is indeed a 7-cage of $(G, \mathcal{H})$, and thus $\mathcal{B}^*$ is an enforced $(840, \infty, \infty, 7)$-encasing of $(G, \mathcal{H})$, as desired. $\square$

Let G be a finite planar graph, $\mathcal{H}$ a spanning collection of connected vertex sets of G, and let $\mathcal{B}$ be a collection of disjoint connected sets of G such that $\mathcal{B}$ encircles $(G, \mathcal{H})$. Let $\mathcal{R}$ be the interiors of the internal faces of $G[\bigcup_{B\in\mathcal{B}} B]$. For $u \in \bigcup_{B\in\mathcal{B}} B$ and $H \in \mathcal{H}$, we say that u is *bounded* by H in $G[\bigcup_{B\in\mathcal{B}} B]$ if either $u \in H$ or u is contained in one of the bounded regions of $\mathbb{R}^2\backslash(G[H] \cup \bigcup_{R\in\mathcal{R}} R)$. We say that $U \subseteq \bigcup_{B\in\mathcal{B}} B$ is *bounded* by H in $G[\bigcup_{B\in\mathcal{B}} B]$ if each $u \in U$ is bounded by H in $G[\bigcup_{B\in\mathcal{B}} B]$. Let F be the vertices of G contained in some face of $G[\bigcup_{B\in\mathcal{B}} B]$. We say that a vertex $v \in F$ is *$\mathcal{B}$-encroached* by some $H \in \mathcal{H}$ with $v \notin H$ if there exists some connected set $U \subseteq \bigcup_{B\in\mathcal{B}} B$ of G and edges $u_1v_1, \dots, u_rv_r$ between F and $\bigcup_{B\in\mathcal{B}} B$ that appear consecutively along the boundary of the face of $G[\bigcup_{B\in\mathcal{B}} B]$ corresponding to F, such that

- U is bounded by H in $G[\bigcup_{B\in\mathcal{B}} B]$,
- for each $1 \leqslant i < r$, there is a facial walk of G between u_i and u_{i+1} that is contained in U,
- $u_1, v_1, u_r, v_r \in H$, and
- $v \in \{v_1, \dots, v_r\}$.

See Figure 6 for an illustrative example. We further say that some $H' \in \mathcal{H}$ is *$\mathcal{B}$-encroached* by some $H \in \mathcal{H}$ if there is some set F of vertices of G contained in some face of $G[\bigcup_{B\in\mathcal{B}} B]$ and some vertex $v \in H' \cap F$ such that v is $\mathcal{B}$-encroached by H.

Let $\mathcal{H}_0$ be the connected sets in $\mathcal{H}$ that contain a vertex incident to the outer face of G. We say that a collection of disjoint connected sets $\mathcal{B}$ of G that encircles $(G, \mathcal{H})$ is a *c-surround* of $(G, \mathcal{H})$ if

- for every pair $H_u, H \in \mathcal{H}$ such that H intersects but is not contained in $\bigcup_{B\in\mathcal{B}} B$, and H_u contains a vertex $u \in \bigcup_{B\in\mathcal{B}} B$ that is bounded by H in $G[\bigcup_{B\in\mathcal{B}} B]$, we have that H_u is at distance at most $c-1$ from H in $\mathbf{RIG}(G, \mathcal{H}\backslash\mathcal{H}_0)$, and
- for every pair $H', H \in \mathcal{H}$ such that H encroaches H', we have that H' is at distance at most c from H in $\mathbf{RIG}(G, \mathcal{H}\backslash\mathcal{H}_0)$.

Note that since $\mathcal{B}$ encircles $(G, \mathcal{H})$, if H encroaches H', then $H \notin \mathcal{H}_0$.

The motivation behind the notion of a c-surround is that it will allow us to add some (not necessarily yet connected until more edges are added to G) sets H' to $\mathcal{H}$ such that $I_{\mathcal{H}}(H')$

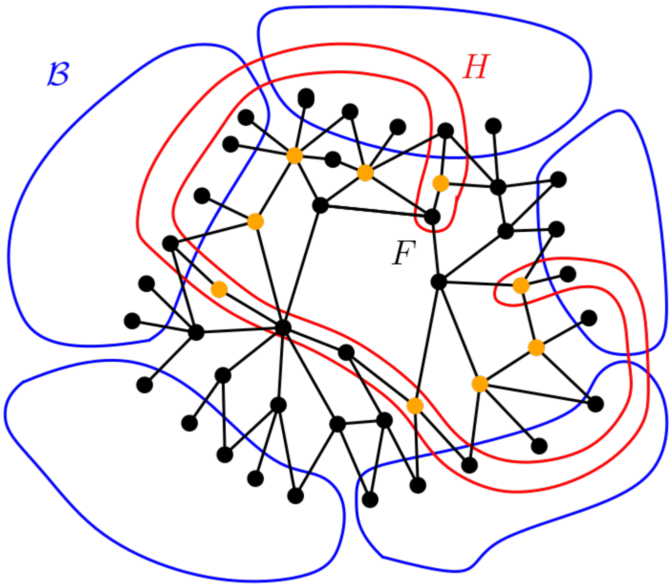

Figure 6: An encircling $\mathcal{B}$ (in blue) and some (red) $H \in \mathcal{H}$. The vertices of the facial vertex set F that are encroached by H are coloured orange.

has bounded weak diameter in $\mathbf{RIG}(G, \mathcal{H} \backslash \mathcal{H}_0)$ and H' is disjoint from $\bigcup_{B \in \mathcal{B}} B$. Such H' will contain for instance the vertices encroached by a given $H \in \mathcal{H}$ and will be crucial for the inductive argument of Theorem 2.5 in which we remove the vertices of $\bigcup_{B \in \mathcal{B}} B$.

Let G be a finite planar graph and $\mathcal{H}$ a spanning collection of connected vertex sets of G. Let $\mathcal{B}$ be a collection of disjoint connected subgraphs of G. We say that $\mathcal{B}$ is a (a, ∞, c, d)-*encasing* of $(G, \mathcal{H})$ if $\mathcal{B}$ encircles $(G, \mathcal{H})$, and

- for every $B \in \mathcal{B}$, we have that $I_{\mathcal{H}}(B)$ has weak diameter at most a in $\mathbf{RIG}(G, \mathcal{H})$,
- $\mathcal{B}$ is a c-surround of $(G, \mathcal{H})$, and
- $\mathcal{B}$ is a d-cage of $(G, \mathcal{H})$.

The enforced encasing found in Lemma 4.2 is actually a 4-surround (which was the purpose of the technical definition of enforced encasings).

Lemma 4.3. *Let G be a finite planar graph and $\mathcal{H}$ a spanning collection of connected vertex sets of G. Then there is a $(840, \infty, 4, 7)$-encasing $\mathcal{B}$ of $(G, \mathcal{H})$.*

Proof. By Lemma 4.2, there is an enforced $(840, \infty, \infty, 7)$-encasing $\mathcal{B}$ of $(G, \mathcal{H})$. Let $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2$ be as in the definition of an enforced encasing, and furthermore, let $\mathcal{H}_3$ be the elements of $\mathcal{H} \backslash (\mathcal{H}_0 \cup \mathcal{H}_1 \cup \mathcal{H}_2)$ that intersect an element of $\mathcal{H}_2$. We will show that $\mathcal{B}$ is in fact a $(840, \infty, 4, 7)$-encasing of $(G, \mathcal{H})$.

First, we show the first bullet in the definition of being a 4-surround. Consider some pair $H_u, H^* \in \mathcal{H}$ such that H^* intersects but is not contained in $\bigcup_{B \in \mathcal{B}} B$, and H_u contains a vertex $u \in \bigcup_{B \in \mathcal{B}} B$ that is bounded by H^* in $G[\bigcup_{B \in \mathcal{B}} B]$. Clearly $H^* \in \mathcal{H}_2 \cup \mathcal{H}_3$, and therefore $u \notin \bigcup_{H \in \mathcal{H}_0} H$. In particular, $H_u, H^* \notin \mathcal{H}_0$.

Suppose first that $u \in \left(\bigcup_{H \in \mathcal{H}_1} H\right) \setminus \left(\bigcup_{H \in \mathcal{H}_0} H\right)$. Then, there is some $H'_u \in \mathcal{H}_1$ that contains u (possibly H'_u is not distinct from H_u). Observe that H'_u must intersect H^* since u is bounded by H^* in $G[\bigcup_{B \in \mathcal{B}} B]$, $H_u \in \mathcal{H}_1$, and $H^* \in \mathcal{H}_2 \cup \mathcal{H}_3$. Therefore, H_u is at distance at most 2 from H^* in $\mathbf{RIG}(G, \mathcal{H} \backslash \mathcal{H}_0)$.

So, we may assume now that $u \in \left(\bigcup_{B\in\mathcal{B}} B\right) \setminus \left(\bigcup_{H\in\mathcal{H}_0\cup\mathcal{H}_1} H\right)$. Then, by the definition of an enforced encasing, there is some connected set $P_u \subseteq H'_u \in \mathcal{H}_2$ of G that contains u, is contained in $\bigcup_{B\in\mathcal{B}} B$, and intersects $\bigcup_{H\in\mathcal{H}_0\cup\mathcal{H}_1} H$. Choose $H''_u \in \mathcal{H}_1$ that intersects P_u (and thus also H'_u). Similarly to before, $H''_u \cup P_u$ must intersect H^* since u is bounded by H^* in $G[\bigcup_{B\in\mathcal{B}} B]$, $u \in H''_u \cup P_u \subseteq \bigcup_{B\in\mathcal{B}} B$, $H''_u \in \mathcal{H}_1$, and $H^* \in \mathcal{H}_2 \cup \mathcal{H}_3$. In particular, H^* intersects at least one of H''_u or H'_u. Therefore, H_u is at distance at most 3 from H^* in $\mathbf{RIG}(G, \mathcal{H}\backslash\mathcal{H}_0)$. Thus, $\mathcal{B}$ satisfies the first bullet of being a 4-surround.

Suppose that some $H' \in \mathcal{H}$ is $\mathcal{B}$-encroached by some $H^* \in \mathcal{H}$. Clearly we have that $H', H^* \notin \mathcal{H}_0$. Note that H^* intersects but is not contained in $\bigcup_{B\in\mathcal{B}} B$. By definition, there is some set F of vertices of G contained in some face of $G[\bigcup_{B\in\mathcal{B}} B]$ and some vertex $v \in H' \cap F$ such that v is $\mathcal{B}$-encroached by H^*. Let u be a neighbour of v in $\bigcup_{B\in\mathcal{B}} B$ that is bounded by H^* in $G[\bigcup_{B\in\mathcal{B}} B]$, and let H_u be some connected set in $\mathcal{H}$ that contains $\{u, v\}$ (possibly H_u is not distinct from H'). Then, H' is at distance at most $1 + 3 = 4$ from H^* in $\mathbf{RIG}(G, \mathcal{H}\backslash\mathcal{H}_0)$.

Thus, $\mathcal{B}$ is a 4-surround of $(G, \mathcal{H})$, and in particular, a $(840, \infty, 4, 7)$-encasing $\mathcal{B}$ of $(G, \mathcal{H})$ as desired. □

We are finally ready to give the entire definition of a (a, b, c, d)-encasing. We say that $\mathcal{B}$ is a (a, b, c, d)-*encasing* of $(G, \mathcal{H})$ if $\mathcal{B}$ encircles $(G, \mathcal{H})$, and

- for every $B \in \mathcal{B}$, we have that $I_{\mathcal{H}}(B)$ has weak diameter at most a in $\mathbf{RIG}(G, \mathcal{H})$,
- for every connected set $H^* \subseteq H \in \mathcal{H}$ of G such that H^* is contained in $\bigcup_{B\in\mathcal{B}} B$, we have that $I_{\mathcal{B}}(H^*)$ has weak diameter at most b in $\mathbf{IM}(G, \mathcal{B})$,
- $\mathcal{B}$ is a c-surround of $(G, \mathcal{H})$, and
- $\mathcal{B}$ is a d-cage of $(G, \mathcal{H})$.

We can at last complete the refinement of the encasing with the use of Lemma 3.2.

Lemma 4.4. *Let G be a finite planar graph and $\mathcal{H}$ a spanning collection of connected vertex sets of G. Then there is a $(9240, 9, 4, 7)$-encasing $\mathcal{B}$ of $(G, \mathcal{H})$.*

Proof. By Lemma 4.3, there is a $(840, \infty, 4, 7)$-encasing $\mathcal{B}'$ of $(G, \mathcal{H})$. Let $G' = \mathbf{IM}(G, \mathcal{B}')$. Then G' is outerplanar. For every connected set $H^* \subseteq H \in \mathcal{H}$ of G such that H^* is contained in $\bigcup_{B\in\mathcal{B}'} B$, let H' be the connected set of G' whose vertex set is exactly the $B \in \mathcal{B}'$ such that H^* intersects B. Let $\mathcal{H}'$ be the collection of all of these connected vertex sets of G'.

By Lemma 3.2, there is a $(11, 9)$-impression $(G', \mathcal{M})$ of $(G', \mathcal{B}')$. Let $\mathcal{B} = \{\bigcup_{B\in M} B : M \in \mathcal{M}\}$. Clearly for every $H \in \mathcal{H}$, $I_{\mathcal{M}}(H)$ has weak diameter at most 9 in $\mathbf{IM}(G, \mathcal{M})$. Also, for every $M \in \mathcal{M}$, we have that $I_{\mathcal{H}}(M)$ has weak diameter at most $11 \cdot 840 = 9240$ in $\mathbf{RIG}(G, \mathcal{H})$.

Hence, $\mathcal{B}$ is a $(9240, 9, 4, 7)$-encasing of $(G, \mathcal{H})$, as desired. □

5 String graphs

Lemma 4.4 is our main technical tool for proving Theorem 2.5. We still need to introduce the notion of fortifications in order to give the precise technical statement that will allow for the inductive proof using Lemma 4.4.

Let G be a planar graph and consider some additional edge e added to a face F. Then e splits F into two faces F_e, F'_e. We call these the *sides* of e in G. We say that a planar graph G' is a *fortification* of a planar graph G if G' can be obtained from G by adding edges to faces in such a way that for every $e \in E(G')\backslash E(G)$ we can pick a side F_e of e in G so that for every

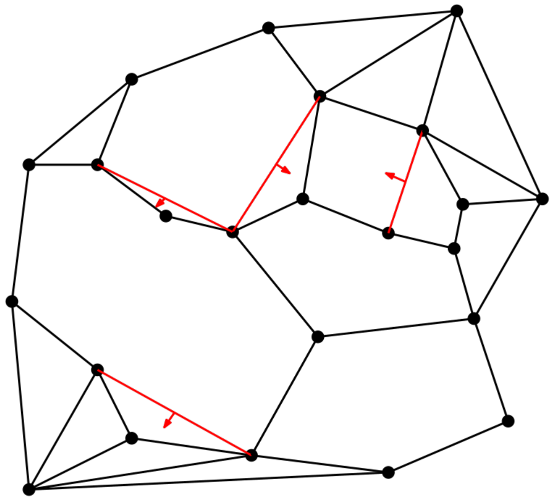

Figure 7: A fortification G' of planar graph G (which is the subgraph on the black edges. The arrows orthogonal to the additional red edges indicate their sides.

distinct $e, e' \in E(G') \backslash E(G)$, the interiors of F_e and $F_{e'}$ do not intersect, and no F_e contains an edge incident to the outer face of G. For an example of a fortification G' of a planar graph G, see Figure 7.

Let G be a planar graph and $\mathcal{H}$ a spanning collection of connected vertex sets $\mathcal{H}$ of G. Let $\mathcal{H}_0$ be the collection of connected sets $H \in \mathcal{H}$ of G that contain a vertex incident to the outer face of G. we say that $(G', \mathcal{H}')$ is a k-*fortification* of $(G, \mathcal{H})$ if

- G' is a fortification of G,
- $\mathcal{H}'$ spans G',
- $\mathcal{H}'$ can be obtained from $\mathcal{H}$ by adding connected sets H' of G such that each $I_{\mathcal{H}}(H')$ has weak diameter at most k in $\mathbf{RIG}(G, \mathcal{H} \backslash \mathcal{H}_0)$, and
- for every $H, H' \in \mathcal{H}$ where H and H' both contain a vertex of a face F_e for some $e \in E(G') \backslash E(G)$ (where F_e is given by the fortification G' of G), we have that the distance between H and H' in $\mathbf{RIG}(G, \mathcal{H} \backslash \mathcal{H}_0)$ is at most k.

We remark that for the purposes of this paper, the last bullet could be weakened so that H and H' just contain an end of e. However, this stronger version does not complicate the proof any further and we need this property for our applications to Riemannian surfaces [17].

We use the following technical statement to facilitate an inductive argument making use of Lemma 4.4.

Theorem 5.1. *Let G be a finite planar graph and $\mathcal{H}$ a spanning collection of connected vertex sets of G. Then there is a $(9242, 80)$-impression $(G', \mathcal{M})$ of a 8-fortification $(G', \mathcal{H}')$ of $(G, \mathcal{H})$ such that for every $M \in \mathcal{M}$ containing a vertex incident to the outer face of G, we have that $I_{\mathcal{H}'}(M)$ has weak diameter at most 9240 in $\mathbf{RIG}(G', \mathcal{H}')$, and for every $H \in \mathcal{H}$ containing a vertex incident to the outer face of G, we have that $I_{\mathcal{M}}(H)$ has weak diameter at most 9 in $\mathbf{IM}(G', \mathcal{M})$.*

Proof. We shall argue inductively on $|V(G)|$ (trivially the theorem holds for small $|V(G)|$). Let $\mathcal{H}_0$ be the collection of connected sets $H \in \mathcal{H}$ that contain a vertex incident to the outer face of G. By Lemma 4.4, there is a $(9240, 9, 4, 7)$-encasing $\mathcal{B}$ of $(G, \mathcal{H})$.

Let $\mathcal{F}$ be the facial vertex sets of $G(\mathcal{B})$. Consider some $F \in \mathcal{F}$ and $H \in \mathcal{H}$ such that H intersects both $\bigcup_{B\in\mathcal{B}} B$ and F. Let $R'_{F,H}$ be the vertices of F that are encroached by H, and let $R_{F,H} = R'_{F,H} \cup (F \cap H)$. Consider some $H' \in \mathcal{H}$ intersecting $R_{F,H}$ but not H. Then, H' intersects $R'_{F,H}$. Since $\mathcal{B}$ is a 4-surround of $(G, \mathcal{H})$, the distance between H' and H in $\mathbf{RIG}(G, \mathcal{H}\backslash\mathcal{H}_0)$ is at most 4. Therefore, $I_{\mathcal{H}}(R_{F,H})$ has weak diameter at most $2 \cdot 4 = 8$ in $\mathbf{RIG}(G, \mathcal{H}\backslash\mathcal{H}_0)$. We wish to add edges to G so that $R_{F,H}$ is a connected set.

Consider some $v \in R_{F,H}$. By definition of being encroached, there exists connected set $U \subseteq \bigcup_{B\in\mathcal{B}} B$ of G and edges $u_1v_1, \ldots, u_rv_r$ between F and $\bigcup_{B\in\mathcal{B}} B$ that appear consecutively along the boundary of the face of $G[\bigcup_{B\in\mathcal{B}} B]$ corresponding to F, such that

- U is bounded by H in $G[\bigcup_{B\in\mathcal{B}} B]$,
- for each $1 \leqslant i < r$, there is a facial walk of G between u_i and u_{i+1} that is contained in U,
- $u_1, v_1, u_r, v_r \in H$, and
- $v \in \{v_1, \ldots, v_r\}$.

Clearly $\{v_1, \ldots, v_r\} \subseteq R_{F,H}$. Let $E_{F,H}$ be the collection of such edges v_iv_{i+1} (across each $v \in R_{F,H}$) that are not already an edge of G. For each such $v_iv_{i+1} \in E_{F,H}$, let $F_{v_iv_{i+1}}$ be the side of v_iv_{i+1} drawn in the corresponding face of $G[\bigcup_{B\in\mathcal{B}} B]$ that contains the facial walk of G between u_i and u_{i+1} that is contained in U (note that the vertices of this facial walk are bounded H in $G[\bigcup_{B\in\mathcal{B}} B]$ and the only two other vertices incident to this face are v_i, v_{i+1}). Since $\mathcal{B}$ is a 4-surround of $(G, \mathcal{H})$, it follows that for every $H' \in \mathcal{H}$ that contains a vertex incident to $F_{v_iv_{i+1}}$, the distance between H and H' in $\mathbf{RIG}(G, \mathcal{H}\backslash\mathcal{H}_0)$ is at most 4. This in particular implies that $F_{v_iv_{i+1}}$ will satisfy the 4th bullet in the definition of a 8-fortification of $(G, \mathcal{H})$. This in particular implies that no vertex incident to $F_{v_iv_{i+1}}$ is incident to the outerface of G. Then observe that $R_{F,H}$ is a connected set in $G \cup E_{F,H}$. For an illustration of such a $R_{F,H}, E_{F,H}$, see Figure 8. Let G^* be the planar graph obtained from adding these edges from the union of all such $E_{F,H}$. Let $\mathcal{R}$ be the collection of all such $R_{F,H}$. Then every element of $\mathcal{R}$ is a connected set of G^*.

Let $\mathcal{H}_C$ be the elements H of $\mathcal{H}$ that are contained in $\bigcup_{B\in\mathcal{B}} B$, and let $\mathcal{H}_I$ be the elements H of $\mathcal{H}$ that intersect but are not contained in $\bigcup_{B\in\mathcal{B}} B$. Let $\mathcal{R}^* = (\mathcal{H}\backslash(\mathcal{H}_C \cup \mathcal{H}_I)) \cup \mathcal{R}$. Let $G_0^* = G^* \backslash \left(\bigcup_{B\in\mathcal{B}} B\right)$, and let $\mathcal{R}_0^*$ be the elements of $\mathcal{R}^*$ that contain a vertex incident to the outer face of G_0. Note that $\mathcal{R}^*$ is a spanning collection of connected vertex sets of G_0 and that $\mathcal{R} \subseteq \mathcal{R}_0^*$.

By the inductive hypothesis, there is a $(9242, 80)$-impression $(G'_0, \mathcal{M}')$ of a 8-fortification $(G'_0, \mathcal{R}')$ of $(G_0, \mathcal{R}^*)$ such that for every $M \in \mathcal{M}'$ containing a vertex incident to the outer face of G'_0, we have that $I_{\mathcal{R}'}(M)$ has weak diameter at most 9240 in $\mathbf{RIG}(G'_0, \mathcal{R}')$, for every $H \in \mathcal{R}_0^*$, we have that $I_{\mathcal{M}'}(H)$ has weak diameter at most 9 in $\mathbf{IM}(G'_0, \mathcal{M}')$. Let G' be obtained from G by adding the edges in $E(G'_0)\backslash E(G)$. Then, G' is a fortification of G. Let $\mathcal{H}' = \mathcal{H} \cup \mathcal{R}'$. Then, $(G', \mathcal{H}')$ is furthermore a 8-fortification of $(G, \mathcal{H})$. Let $\mathcal{M} = \mathcal{B} \cup \mathcal{M}'$. It remains to show that $(G', \mathcal{M})$ provides the desired impression of $(G', \mathcal{H}')$.

If $M \in \mathcal{M}$ contains a vertex incident to the outer face of G', then $M \in \mathcal{B} = \mathcal{M}\backslash\mathcal{M}'$. In particular, $I_{\mathcal{H}}(M) = I_{\mathcal{H}'}(M)$ has weak diameter at most 9240 in $\mathbf{RIG}(G, \mathcal{H})$, and therefore also in $\mathbf{RIG}(G', \mathcal{H}')$. Consider some $H \in \mathcal{H}'$ that is contained in $\mathcal{H}_C$ (which in particular covers the case that H contains a vertex incident to the outer face of G'). Then, $H \subseteq \bigcup_{B\in\mathcal{B}}$, so since $\mathcal{B}$ is a $(9240, 9, 4, 7)$-encasing of G, $I_{\mathcal{B}}(H)$ has weak diameter at most 9 in $\mathbf{IM}(G, \mathcal{B})$. Therefore

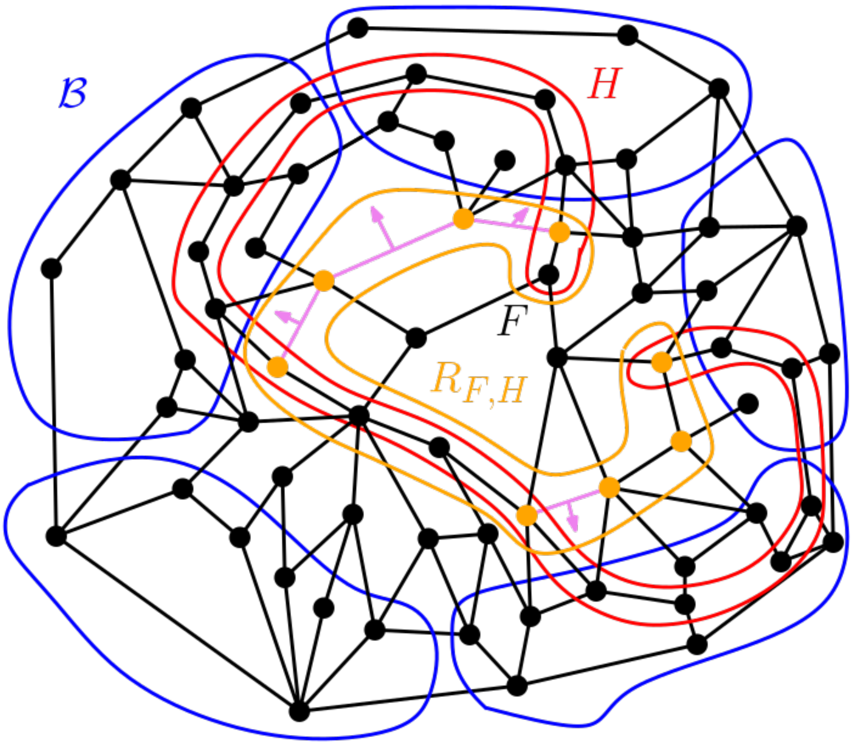

Figure 8: The (orange) vertex set $R_{F,H}$ given the (blue) encasing $\mathcal{B}$, the (red) $H \in \mathcal{H}$, and $F \in \mathcal{F}$. The orange vertices are the vertices of $R'_{F,H}$. The pink edges are $E_{F,H}$, and the arrows orthogonal to them indicate their sides.

$I_{\mathcal{M}'}(H)$ has weak diameter at most 9 in $\mathbf{IM}(G, \mathcal{M})$. It remains to show that $(G', \mathcal{M})$ is a $(9242, 80)$-impression of $(G', \mathcal{H}')$.

If $M \in \mathcal{M}' \subseteq \mathcal{M}$ does not contain a vertex incident to the outer face of G'_0, then $I_{\mathcal{H}'}(M) = I_{\mathcal{R}'}(M)$ has weak diameter at most 9242 in $\mathbf{RIG}(G'_0, \mathcal{R}')$, and therefore also in $\mathbf{RIG}(G', \mathcal{H}')$. If $M \in \mathcal{M}' \subseteq \mathcal{M}$ contains a vertex incident to the outer face of G'_0, then $I_{\mathcal{R}'}(M)$ has weak diameter at most 9240 in $\mathbf{RIG}(G'_0, \mathcal{R}')$. Since $I_{\mathcal{H}'}(M) \subseteq I_{\mathcal{R}'}(M) \cup N_{\mathbf{RIG}(G',\mathcal{H}')}(I_{\mathcal{R}'}(M))$, it follows that $I_{\mathcal{H}'}(M)$ has weak diameter at most $9240 + 2 = 9242$ in $\mathbf{RIG}(G', \mathcal{H}')$. So, $(G', \mathcal{M})$ is a $(9242, \infty)$-impression of $(G', \mathcal{H}')$.

If $H \in \mathcal{R}' \subseteq \mathcal{H}'$, then $I_{\mathcal{M}'}(H) = I_{\mathcal{M}}(H)$ has weak diameter at most 80 in $\mathbf{IM}(G'_0, \mathcal{M}')$ and therefore also in $\mathbf{IM}(G', \mathcal{M})$. To show that $(G', \mathcal{M}')$ is a $(9242, 80)$-impression of $(G', \mathcal{H}')$, it now just remains to show that if $H \in \mathcal{H}_I$, then $I_{\mathcal{M}}(H)$ has weak diameter at most 80 in $\mathbf{IM}(G', \mathcal{M})$.

Fix some $H \in \mathcal{H}_I$. Consider some $u, v \in H$ and let $M_u, M_v \in \mathcal{M}$ be such that $u \in M_u$ and $v \in M_v$. Since $\mathcal{B}$ is a 7-cage of $(G, \mathcal{H})$, there exists some sequence of vertex sets $A_0, \dots, A_t$ with $t \leqslant 7$ such that

- each A_i is either a single element of $\mathcal{B}$ or a single element of $\mathcal{F}$ or the union of a subcollection $\mathcal{B}' \subseteq \mathcal{B}$ such that some connected set $H^* \subseteq H$ of G (or equivalently G') intersects each member of $\mathcal{B}'$ and is contained in $G[\bigcup_{B \in \mathcal{B}'} B] = G'[\bigcup_{B \in \mathcal{B}'} B]$,
- $u \in A_0$,
- $v \in A_t$, and
- for every $1 \leqslant i < t$, we have that either H intersects both A_{i-1} and A_i, or neither A_{i-1} or A_i is an element of $\mathcal{F}$ and they touch.

If $A_i \in \mathcal{B} \subseteq \mathcal{M}$, then clearly $\{A_i\}$ has weak diameter 0 in $\mathbf{IM}(G', \mathcal{M})$. If A_i is the union of a subcollection $\mathcal{B}' \subseteq \mathcal{B}$ such that some connected set $H^* \subseteq H$ of G intersects each member of $\mathcal{B}'$ and is contained in $G[\bigcup_{B \in \mathcal{B}'} B]$, then it follows from $\mathcal{B}$ being a $(9240, 9, 4, 7)$-encasing of $(G, \mathcal{H})$ that A_i has weak diameter at most 9 in $\mathbf{IM}(G, \mathcal{B})$ and therefore in $\mathbf{IM}(G', \mathcal{M})$. If A_i is a single element F of $\mathcal{F}$, then $R_{F,H}$ contains $H \cap F$. As $R_{F,H} \in \mathcal{R}_0^*$, we have that $I_{\mathcal{M}'}(R_{F,H})$ has weak diameter at most 9 in $\mathbf{IM}(G_0, \mathcal{M}')$, and therefore $I_{\mathcal{M}}(R_{F,H})$ has weak diameter at most 9 in $\mathbf{IM}(G', \mathcal{M})$. As $t \leqslant 7$, it now follows that $d_{\mathbf{IM}(G',\mathcal{M})}(M_u, M_v) \leqslant 7 + 8 \cdot 9 = 80$. Therefore, $I_{\mathcal{M}}(H)$ has weak diameter at most 80 in $\mathbf{IM}(G', \mathcal{M})$.

Hence $(G', \mathcal{M})$ is a $(9242, 80)$-impression of $(G', \mathcal{H}')$, as desired. □

By Theorem 5.1 and Lemma 2.2, we obtain the following.

Theorem 5.2. *Let G be a finite planar graph and $\mathcal{H}$ a spanning collection of connected vertex sets of G. Then there is a $(73936, 80)$-impression $(G', \mathcal{M})$ of $(G', \mathcal{H})$, where G' is a fortification of G and $\mathcal{H}$ 8-spans G.*

Since for every finite string graph S there is a finite planar graph G and a spanning collection $\mathcal{H}$ of connected vertex sets of G such that $\mathbf{RIG}(G, \mathcal{H}) = S$ [54], by Theorem 5.2 and Proposition 2.1, we finally get that string graphs are quasi-isometric to planar graphs.

Theorem 5.3. *Let S be a finite string graph. Then there is a planar graph G and a function $f : V(S) \to V(G)$ such that for all $u, v \in V(S)$,*

$$\frac{1}{73944} d_S(u, v) - \frac{73936}{73944} \leqslant d_G(f(u), f(v)) \leqslant 160 d_S(u, v),$$

and for every $v \in V(G)$ there exists some $u \in V(S)$ such that $d_G(f(u), v) \leqslant 80$.

Theorem 2.5 (and thus Theorem 1.1) then finally follows from Theorem 5.3 and Lemma 2.3.

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