MORREY–SOBOLEV INEQUALITIES WITH POWER WEIGHTS ON THE HALF-SPACE

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Abstract. Morrey–Sobolev inequalities are established for functions in weighted Sobolev spaces on the n-dimensional half-space, where the weight is a power of the distance to the boundary, as well as for Sobolev spaces on the n-dimensional hyperbolic space. All the estimates are optimal up to a multiplicative constant.

1. Introduction

The classical *Morrey–Sobolev inequality*, due to Morrey [15, 16], states that, if $p > n$, there exists a constant $C > 0$ such that every Sobolev function

$$u \in \dot{W}^{1,p}(\mathbb{R}^n) := \left\{ u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n) \;\middle|\; \int_{\mathbb{R}^n} |\mathrm{D}u|^p < +\infty \right\} \tag{1}$$

satisfies, for almost every $x, y \in \mathbb{R}^n$,

$$|u(x) - u(y)| \le C|x-y|^{1-\frac{n}{p}} \left(\int_{\mathbb{R}^n} |\mathrm{D}u|^p \right)^{\frac{1}{p}} \tag{2}$$

(see also [4, Thm. 9.12; 13, Thm. 1.4.5; 21, Thm. 6.4.4]). In other words, the Sobolev space $\dot{W}^{1,p}(\mathbb{R}^n)$ is continuously embedded in the space $C^{0,1-\frac{n}{p}}(\mathbb{R}^n)$ of Hölder-continuous functions of order $1 - \frac{n}{p}$.

We consider, for $\gamma \in \mathbb{R}$, the *weighted Sobolev space*

$$u \in \dot{W}^{1,p}_{\gamma}(\mathbb{R}^n_+) := \left\{ u \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n_+) \;\middle|\; \int_{\mathbb{R}^n_+} |\mathrm{D}u(z)|^p {z_n}^{\gamma} \,\mathrm{d}z < +\infty \right\}.$$

The fundamental properties of weighted Sobolev spaces have been studied for wide classe of weights [10, 19]. For $p < n$, Maz'ya has proved scale invariant Sobolev embeddings for compactly supported functions with power weights [13, Cor. 2.1.7/2].

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In the context of $\dot{W}^{1,p}_{\gamma}(\mathbb{R}^n_+)$, a local version of (2) shows that, for almost every x, $y \in \mathbb{R}^n_+ := \mathbb{R}^n \times (0, +\infty)$ satisfying $x_n \le y_n \le 2x_n$ and $|x - y| \le 2x_n$,

$$|u(x) - u(y)| \le C \frac{|x-y|^{1-\frac{n}{p}}}{{x_n}^{\frac{\gamma}{p}}} \left(\int_{\mathbb{R}^n_+} |\mathrm{D}u(z)|^p {z_n}^{\gamma} \, \mathrm{d}z \right)^{\frac{1}{p}}. \tag{3}$$

When $-1 < \gamma < p - 1$, setting $s := 1 - \frac{\gamma+1}{p}$ so that $\gamma = (1 - s)p - 1$, the classical trace theory, due to Gagliardo [8] and Uspenskiĭ [20] (see also [11, Thm. 9.4; 14]), states that any function $u \in \dot{W}^{1,p}_{\gamma}(\mathbb{R}^n_+)$ has a well-defined trace $\operatorname{tr}_{\mathbb{R}^{n-1}} u \in \dot{W}^{s,p}(\mathbb{R}^{n-1})$ on $\mathbb{R}^{n-1} \cong \mathbb{R}^{n-1} \times \{0\} = \partial\mathbb{R}^n_+$, which satisfies the estimate

$$\iint\limits_{\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}} \frac{|\operatorname{tr}_{\mathbb{R}^{n-1}} u(x) - \operatorname{tr}_{\mathbb{R}^{n-1}} u(y)|^p}{|x-y|^{n-1+sp}} \, \mathrm{d}x \, \mathrm{d}y \le C \int_{\mathbb{R}^n_+} |\mathrm{D}u(z)|^p {z_n}^{\gamma} \, \mathrm{d}z. \tag{4}$$

On the other hand, the Morrey–Sobolev inequality in fractional Sobolev spaces [7, §8; 11, Thm. 7.13; 18, Lem. 11] states that, if $v \in \dot{W}^{s,p}(\mathbb{R}^{n-1})$ and $sp > n - 1$, then, for almost every $x, y \in \mathbb{R}^{n-1}$,

$$|v(x) - v(y)| \le C|x-y|^{s-\frac{n-1}{p}} \left(\iint\limits_{\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}} \frac{|v(z) - v(w)|^p}{|z-w|^{n-1+sp}} \, \mathrm{d}z \, \mathrm{d}w \right)^{\frac{1}{p}}. \tag{5}$$

If $\gamma < p - n$, combining (4) and (5) with $s = 1 - \frac{\gamma+1}{p}$, we get

$$|\operatorname{tr}_{\mathbb{R}^{n-1}} u(x) - \operatorname{tr}_{\mathbb{R}^{n-1}} u(y)| \le C|x-y|^{1-\frac{n+\gamma}{p}} \left(\int_{\mathbb{R}^n_+} |\mathrm{D}u(z)|^p {z_n}^{\gamma} \, \mathrm{d}z \right)^{\frac{1}{p}}. \tag{6}$$

The starting point of the present work is to understand whether the continuity of $u \in \dot{W}^{s,p}(\mathbb{R}^n_+)$ in the interior of $\mathbb{R}^n_+$ and of its trace on the boundary $\partial\mathbb{R}^n_+ \cong \mathbb{R}^{n-1}$, that is, the estimates (3) and (6), both follow from a single global continuity result, and what the corresponding estimates would be. This question is answered by the following result.

Theorem 1.1. *Let $n \ge 1$, $\gamma \in \mathbb{R}$, and $n < p < +\infty$. There exists constant $C = C(n, \gamma, p) > 0$, depending at most on n, γ, and p, such that every $u \in \dot{W}^{1,p}_{\gamma}(\mathbb{R}^n_+)$ satisfies for almost every x, $y \in \mathbb{R}^n_+$,*

$$|u(x) - u(y)| \le C \, \Theta_{1-\frac{1}{p}, 1-\frac{n}{p}, 1-\frac{n+\gamma}{p}}(x, y) \left(\int_{\mathbb{R}^n_+} |\mathrm{D}u(z)|^p {z_n}^{\gamma} \, \mathrm{d}z \right)^{\frac{1}{p}}.$$

Here and in the sequel, we use the function $\Theta_{\alpha,\beta,\kappa}\colon \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to [0,+\infty)$ defined for given $\alpha > 0$, $\beta \in \mathbb{R}$, and $\gamma \in \mathbb{R}$, by

$$\Theta_{\alpha,\beta,\kappa}(x,y) = \begin{cases} \dfrac{|x-y|^\beta}{\min(x_n,y_n)^{-\kappa}\max(x_n,y_n,|x-y|)^\beta} & \text{if } \kappa < 0,\\ \left(\ln\left(1+\left(\dfrac{|x-y|}{\min(x_n,y_n)}\right)^{\frac{\beta}{\alpha}}\right)\right)^\alpha & \text{if } \kappa = 0,\\ \dfrac{|x-y|^\beta}{\max(x_n,y_n,|x-y|)^{\beta-\kappa}} & \text{if } 0<\kappa<1+\beta-\alpha,\\ \dfrac{\max(x_n,y_n)^\kappa|x-y|^\beta}{\max(x_n,y_n,|x-y|)^\beta} & \text{if } \kappa \geq 1+\beta-\alpha. \end{cases} \tag{7}$$

Equivalently, Theorem 1.1 states that we have

- if $\gamma > p-n$,

$$|u(x)-u(y)| \leq C\frac{|x-y|^{1-\frac{n}{p}}}{\min(x_n,y_n)^{\frac{n+\gamma}{p}-1}\max(x_n,y_n,|x-y|)^{1-\frac{n}{p}}}\left(\int_{\mathbb{R}^n_+}|\mathrm{D}u(z)|^p {z_n}^\gamma \,\mathrm{d}z\right)^{\frac{1}{p}}, \tag{8}$$

- if $\gamma = p-n$,

$$|u(x)-u(y)| \leq C\left(\ln\left(1+\left(\frac{|x-y|}{\min(x_n,y_n)}\right)^{\frac{p-n}{p-1}}\right)\right)^{1-\frac{1}{p}}\left(\int_{\mathbb{R}^n_+}|\mathrm{D}u(z)|^p {z_n}^\gamma \,\mathrm{d}z\right)^{\frac{1}{p}}, \tag{9}$$

- if $-1 \leq \gamma < p-n$,

$$|u(x)-u(y)| \leq C\frac{|x-y|^{1-\frac{n}{p}}}{\max(x_n,y_n,|x-y|)^{\frac{\gamma}{p}}}\left(\int_{\mathbb{R}^n_+}|\mathrm{D}u(z)|^p {z_n}^\gamma \,\mathrm{d}z\right)^{\frac{1}{p}}, \tag{10}$$

- if $\gamma \leq -1$,

$$|u(x)-u(y)| \leq C\frac{\max(x_n,y_n)^{1-\frac{n+\gamma}{p}}|x-y|^{1-\frac{n}{p}}}{\max(x_n,y_n,|x-y|)^{1-\frac{n}{p}}}\left(\int_{\mathbb{R}^n_+}|\mathrm{D}u(z)|^p {z_n}^\gamma \,\mathrm{d}z\right)^{\frac{1}{p}}. \tag{11}$$

For any fixed n, γ, and p, it can be observed that the inequality among (8)-(11) which applies is always stronger than the preceding ones in the list.

It can be seen that the interior estimate (3) and the boundary estimate (6) are both consequences of Theorem 1.1 with $\gamma < p-n$ (see (10)). Conversely, the classical extension theory for weighted Sobolev spaces (4) can be combined with (6) to recover the fractional Morrey–Sobolev embedding (5).

Remark 1.2. Theorem 1.1 implies that, if $|x-y| \leq \frac{1}{2}\min(x_n,y_n)$, then

$$|u(x)-u(y)| \leq C\frac{|x-y|^{1-\frac{n}{p}}}{\min({x_n}^{\frac{\gamma}{p}},{y_n}^{\frac{\gamma}{p}})}\left(\int_{\mathbb{R}^n_+}|\mathrm{D}u(z)|^p {z_n}^\gamma \,\mathrm{d}z\right)^{\frac{1}{p}}. \tag{12}$$

Indeed, one has then

$$\min(x_n,y_n) \leq \max(x_n,y_n) \leq \min(x_n,y_n) + |x-y| \leq \tfrac{3}{2}\min(x_n,y_n),$$

so that

$$\min(x_n, y_n) \le \max(x_n, y_n, |x-y|) \le \tfrac{3}{2}\min(x_n, y_n). \qquad \triangle$$

Remark 1.3. When $\gamma \le 0$, the estimate (12) holds for almost every $x, y \in \mathbb{R}^{n-1} \cong \partial\mathbb{R}^n_+$ and for almost every $x \in \mathbb{R}^{n-1}$ and almost every $y \in \mathbb{R}^n_+$, provided u is replaced by its trace on the boundary. △

Remark 1.4. As for the classical Morrey–Sobolev embedding, Theorem 1.1 implies that any function $u \in \dot{W}^{1,p}_\gamma(\mathbb{R}^n_+)$ is equal almost everywhere to a unique continuous function $u_* \in C(\mathbb{R}^n_+)$. When $\gamma < p - n$, this function u_* can be taken to be *continuous up to the boundary* $\partial\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times \{0\}$, and when $\gamma \le -1$, to be *constant on the boundary*. △

Remark 1.5. When $0 \le \gamma < p - n$, Theorem 1.1 implies through (10) that, for almost every $x, y \in \mathbb{R}^n_+$,

$$|u(x) - u(y)| \le C|x-y|^{1-\frac{n+\gamma}{p}} \left(\int_{\mathbb{R}^n_+} |Du(z)|^p {z_n}^\gamma \,\mathrm{d}z \right)^{\frac{1}{p}}, \tag{13}$$

since one has then $|x-y|^{\frac{\gamma}{p}} \le \max(x_n, y_n, |x-y|)^{\frac{\gamma}{p}}$. The global Hölder exponent is thus smaller than the one appearing in the classical Morrey–Sobolev inequality (2). The estimate (13) is a particular case of Cabré and Ros-Oton's estimate for monomial weights [5]. △

Remark 1.6. When $-1 < \gamma \le 0$, Theorem 1.1 and (10) can be rewritten as

$$|u(x) - u(y)| \le C\big(|x-y|^{1-\frac{n+\gamma}{p}} + \min(x_n, y_n)^{-\frac{\gamma}{p}}|x-y|^{1-\frac{n}{p}}\big) \left(\int_{\mathbb{R}^n_+} |Du(z)|^p {z_n}^\gamma \,\mathrm{d}z \right)^{\frac{1}{p}},$$

and can be seen as an improvement close to the boundary of the interior Hölder-continuity exponent $1 - \frac{n}{p}$ towards the boundary exporent $1 - \frac{n+\gamma}{p} > 1 - \frac{n}{p}$. △

Remark 1.7. When $\gamma > p - n$ and if for every $R > 0$, $\operatorname{ess\,inf}_{\mathbb{R}^n_+ \setminus \mathbb{B}^n_R} |u| = 0$, then, for almost every $x \in \mathbb{R}^n_+$, letting $y_n \to +\infty$ in (8), we get the estimate

$$|u(x)| \le \frac{C}{{x_n}^{\frac{n+\gamma}{p}-1}} \left(\int_{\mathbb{R}^n_+} |Du(z)|^p {z_n}^\gamma \,\mathrm{d}z \right)^{\frac{1}{p}}. \tag{14}$$

When $n = 1$, the latter inequality (14) corresponds to Berestycki and Lions' radial lemma [1, Lemma A.III; 12, Corollaire II.1]. △

Remark 1.8. When $n = 1$, one has $|x-y| \le \max(x, y)$ on $\mathbb{R}_+ = (0, +\infty)$. Therefore, $\max(x, y, |x-y|) = \max(x, y)$ and $\min(x, y) + |x-y| = \max(x, y)$. It follows then from Theorem 1.1 that

- if $\gamma > p - 1$,

$$|u(x) - u(y)| \le C \frac{|x-y|^{1-\frac{1}{p}}}{\min(x, y)^{\frac{1+\gamma}{p}-1} \max(x, y)^{1-\frac{1}{p}}} \left(\int_{\mathbb{R}_+} |u'(z)|^p z^\gamma \,\mathrm{d}z \right)^{\frac{1}{p}}, \tag{15}$$

- if $\gamma = p - 1$,

$$|u(x) - u(y)| \le C \left|\ln \frac{x}{y}\right|^{1-\frac{1}{p}} \left(\int_{\mathbb{R}_+} |u'(z)|^p z^\gamma \,\mathrm{d}z\right)^{\frac{1}{p}}, \tag{16}$$

- if $\gamma < p - 1$,

$$|u(x) - u(y)| \le C \frac{|x - y|^{1-\frac{1}{p}}}{\max(x, y)^{\frac{\gamma}{p}}} \left(\int_{\mathbb{R}_+} |u'(z)|^p z^\gamma \,\mathrm{d}z\right)^{\frac{1}{p}}. \tag{17}$$

Notably, there is no difference between the cases $\gamma \le -1$ and $-1 \le \gamma < p - 1$ that are merged into a single case (17) from the two cases (10) and (11) in Theorem 1.1.

The estimates (15), (16) and (17) for $n = 1$ can all be obtained by a direct application of the fundamental theorem of calculus and Hölder's inequality:

$$|u(x) - u(y)| \le \int_{[x,y]} |u'(z)| \,\mathrm{d}z \le \left(\int_{\mathbb{R}_+} |u'(z)|^p z^\gamma \,\mathrm{d}z\right)^{\frac{1}{p}} \left(\int_{[x,y]} \frac{1}{z^{\frac{\gamma}{p-1}}} \,\mathrm{d}z\right)^{1-\frac{1}{p}}. \qquad \triangle$$

The proof of (8), (9), and (10) in Theorem 1.1 is based on the Sobolev representation formula, whereas the proof of (11) is obtained as an enhancement of (10) through the observation that traces of Sobolev functions in $\dot{W}^{1,p}_\gamma(\mathbb{R}^n_+)$ with $\gamma \le -1$ are constant.

We also show that the result of Theorem 1.1 is *optimal up to a multiplicative constant*.

Theorem 1.9. *Let $n \ge 1$, $\gamma \in \mathbb{R}$, and $n < p < +\infty$. There exists a constant $C > 0$ such that given $\omega \in C(\mathbb{R}^n_+ \times \mathbb{R}^n_+, [0, +\infty))$ for which every $u \in \dot{W}^{1,p}_\gamma(\mathbb{R}^n_+)$ satisfies for almost every x, $y \in \mathbb{R}^n_+$, the estimate*

$$|u(x) - u(y)| \le \omega(x, y) \left(\int_{\mathbb{R}^n_+} |\mathrm{D}u(z)|^p {z_n}^\gamma \,\mathrm{d}z\right)^{\frac{1}{p}}, \tag{18}$$

then necessarily, for all x, $y \in \mathbb{R}^n_+$,

$$\omega(x, y) \ge C\, \Theta_{1-\frac{1}{p}, 1-\frac{n}{p}, 1-\frac{n+\gamma}{p}}(x, y).$$

The determination of the optimal ω is a delicate question. We show that such an ω is a distance (Proposition 3.1). In general, the optimal constant in the unweighted Morrey–Sobolev inequality (2) is not known [9].

When $n = 1$, the estimate turns out to be optimal for compactly supported functions.

Theorem 1.10. *Let $\gamma \in \mathbb{R}$, and $1 < p < +\infty$. There exists a constant $C > 0$ such that given $\omega \in C(\mathbb{R}_+ \times \mathbb{R}_+, [0, +\infty))$ for which every $u \in C^\infty_c(\mathbb{R}_+)$ satisfies for almost every x, $y \in \mathbb{R}_+$, the estimate*

$$|u(x) - u(y)| \le \omega(x, y) \left(\int_{\mathbb{R}_+} |u'(z)|^p z^\gamma \,\mathrm{d}z\right)^{\frac{1}{p}}, \tag{19}$$

then necessarily, for all x, $y \in \mathbb{R}_+$,

$$\omega(x, y) \ge C\, \Theta_{1-\frac{1}{p}, 1-\frac{1}{p}, 1-\frac{1+\gamma}{p}}(x, y).$$

In higher dimensions $n \ge 2$, the estimate can be improved for functions that are compactly supported in $\mathbb{R}^n_+$. We get the following variant of Theorem 1.1.

Theorem 1.11. *Let $n \geq 2$, $\gamma \in \mathbb{R}$, and $n < p < +\infty$. There exists a constant $C = C(n, \gamma, p) > 0$, depending at most on n, γ, and p, such that, for every $u \in C_c^\infty(\mathbb{R}^n_+)$ and every $x, y \in \mathbb{R}^n_+$,*

$$|u(x) - u(y)| \leq C\, \Theta^0_{1-\frac{n}{p}, 1-\frac{n+\gamma}{p}}(x, y) \left(\int_{\mathbb{R}^n_+} |Du(z)|^p {z_n}^\gamma \, \mathrm{d}z \right)^{\frac{1}{p}}.$$

The function $\Theta^0_{\beta,\kappa} \colon \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to [0, +\infty)$ appearing in Theorem 1.11 is defined for given $\alpha > 0$, $\beta \in \mathbb{R}$, and $\kappa \in \mathbb{R}$, by

$$\Theta^0_{\beta,\kappa}(x, y) = \begin{cases} \dfrac{|x-y|^\beta}{\min(x_n, y_n)^{-\kappa} \max(x_n, y_n, |x-y|)^\beta} & \text{if } \kappa < 0, \\ \dfrac{\max(x_n, y_n)^\kappa |x-y|^\beta}{\max(x_n, y_n, |x-y|)^\beta} & \text{if } \kappa \geq 0. \end{cases} \tag{20}$$

In other words, Theorem 1.11 shows that, if u is compactly supported,

- if $\gamma > p - n$,

$$|u(x) - u(y)| \leq C \frac{|x-y|^{1-\frac{n}{p}}}{\min(x_n, y_n)^{\frac{n+\gamma}{p}-1} \max(x_n, y_n, |x-y|)^{1-\frac{n}{p}}} \left(\int_{\mathbb{R}^n_+} |Du(z)|^p {z_n}^\gamma \, \mathrm{d}z \right)^{\frac{1}{p}}, \tag{21}$$

- if $\gamma \leq p - n$,

$$|u(x) - u(y)| \leq C \frac{\max(x_n, y_n)^{1-\frac{n+\gamma}{p}} |x-y|^{1-\frac{n}{p}}}{\max(x_n, y_n, |x-y|)^{1-\frac{n}{p}}} \left(\int_{\mathbb{R}^n_+} |Du(z)|^p {z_n}^\gamma \, \mathrm{d}z \right)^{\frac{1}{p}}. \tag{22}$$

Remark 1.12. Compared with the definition (7) of the function $\Theta_{\alpha,\beta,\kappa}$ appearing in Theorem 1.1, the only difference in the definition (20) of $\Theta^0_{\beta,\kappa}$ is in the range $0 \leq \kappa \leq 1+\beta-\alpha$. In fact, the two cases $\kappa = 0$ and $0 < \kappa \leq 1 + \beta - \alpha$ merged with the case $\kappa \geq 1 + \beta - \alpha$, which prevails and yields a stronger inequality. We also note that α does not play any role in Theorem 1.11.

In terms of weighted Sobolev spaces $\dot{W}^{1,p}_\gamma(\mathbb{R}^n_+)$, the difference between Theorems 1.1 and 1.11 arises in the range $-1 < \gamma \leq p - n$, for which the estimates merge with those of the range $\gamma \leq -1$. The spaces $\dot{W}^{1,p}_\gamma(\mathbb{R}^n_+)$ in the range $-1 < \gamma \leq p - n$ have traces which are continuous or have vanishing mean oscillations, but are not constant. On the other hand, even though the theory of traces shows that compactly supported functions are not dense in the range $p - n < \gamma < p - 1$, the weighted Morrey–Sobolev-type estimates nevertheless have the same form. △

Remark 1.13. In particular when $\gamma = 0$, Theorem 1.11 implies that, if $u \in C_c^\infty(\mathbb{R}^n_+)$,

$$|u(x) - u(y)| \leq C \min(\max(x_n, y_n), |x-y|)^{1-\frac{n}{p}} \left(\int_{\mathbb{R}^n_+} |Du(z)|^p \, \mathrm{d}z \right)^{\frac{1}{p}}. \tag{23}$$ △

Remark 1.14. If $\gamma < p - n$, it follows from Theorem 1.11 that, for each smooth and compactly supported function $u \in C_c^\infty(\mathbb{R}^n_+)$ and for each $x \in \mathbb{R}^n_+$,

$$|u(x)| \leq C {x_n}^{1-\frac{n+\gamma}{p}} \left(\int_{\mathbb{R}^n_+} |Du(z)|^p {z_n}^\gamma \, \mathrm{d}z \right)^{\frac{1}{p}}$$

(see Lemma 2.3). △

Again, the result of Theorem 1.11 is optimal:

Theorem 1.15. *Let $n \geq 2$, $\gamma \in \mathbb{R}$, and $n < p < +\infty$. There exists a constant $C > 0$ such that, if $\omega \in C(\mathbb{R}^n_+ \times \mathbb{R}^n_+, [0, +\infty))$ satisfies, for every $u \in C^\infty_c(\mathbb{R}^n_+)$ and for almost every x, $y \in \mathbb{R}^n_+$, the estimate*

$$|u(x) - u(y)| \leq \omega(x, y)\left(\int_{\mathbb{R}^n_+} |\mathrm{D}u(z)|^p {z_n}^\gamma \,\mathrm{d}z\right)^{\frac{1}{p}}, \tag{24}$$

then necessarily, for all x, $y \in \mathbb{R}^n_+$,

$$\omega(x, y) \geq C\Theta^0_{1-\frac{n}{p}, 1-\frac{n+\gamma}{p}}(x, y).$$

When $\gamma = p - n$, the weighted Sobolev space $\dot{W}^{1,p}_\gamma(\mathbb{R}^n_+)$ corresponds to the Sobolev space $\dot{W}^{1,p}(\mathbb{H}^n)$ on the n-dimensional hyperbolic space $\mathbb{H}^n$ in the Poincaré half-space model. We obtain, in Section 4, the following estimate and optimality result.

Theorem 1.16. *Let $n \geq 1$ and $n < p < +\infty$. There exists a constant $C = C(n, p) > 0$, depending at most on n and p, such that, for every $u \in \dot{W}^{1,p}(\mathbb{H}^n)$ and for almost every x, $y \in \mathbb{H}^n$,*

$$|u(x) - u(y)| \leq C \max(\mathrm{d}_{\mathbb{H}^n}(x, y)^{1-\frac{n}{p}}, \mathrm{d}_{\mathbb{H}^n}(x, y)^{1-\frac{1}{p}})\left(\int_{\mathbb{H}^n} |\mathrm{D}u|^p\right)^{\frac{1}{p}}.$$

Moreover, if $\theta\colon [0, +\infty) \to [0, +\infty)$ satisfies, for every $u \in \dot{W}^{1,p}(\mathbb{H}^n)$ and for almost every x, $y \in \mathbb{H}^n$,

$$|u(x) - u(y)| \leq \theta(\mathrm{d}_{\mathbb{H}^n}(x, y))\left(\int_{\mathbb{H}^n} |\mathrm{D}u|^p\right)^{\frac{1}{p}},$$

then there exists a constant $C > 0$ such that

$$\theta(t) \geq C \max(t^{1-\frac{n}{p}}, t^{1-\frac{1}{p}}).$$

When $n = 1$, the hyperbolic space $\mathbb{H}^1$ is isometric to the real line $\mathbb{R}$, and we recover the classical one-dimensional Morrey–Sobolev embedding (2).

In the spirit of Theorem 1.11, we have a variant of Theorem 1.16 for compactly supported functions on the hyperbolic space $\mathbb{H}^n$.

Theorem 1.17. *Let $n \geq 2$ and $n < p < +\infty$. There exists a constant $C = C(n, p) > 0$, depending at most on n and p, such that, for every $u \in C^\infty_c(\mathbb{H}^n)$ and for every x, $y \in \mathbb{H}^n$,*

$$|u(x) - u(y)| \leq C \min(\mathrm{d}_{\mathbb{H}^n}(x, y)^{1-\frac{n}{p}}, 1)\left(\int_{\mathbb{H}^n} |\mathrm{D}u|^p\right)^{\frac{1}{p}}.$$

Moreover, if $\theta\colon [0, +\infty) \to [0, +\infty)$ satisfies, for every $u \in \dot{W}^{1,p}(\mathbb{H}^n)$ and for almost every x, $y \in \mathbb{H}^n$,

$$|u(x) - u(y)| \leq \theta(\mathrm{d}_{\mathbb{H}^n}(x, y))\left(\int_{\mathbb{H}^n} |\mathrm{D}u|^p\right)^{\frac{1}{p}},$$

then there exists a constant $C > 0$ such that

$$\theta(t) \geq C \min(t^{1-\frac{n}{p}}, 1).$$

2. Weighted Morrey–Sobolev inequality

We begin with an estimate on averages on a cone based on the Sobolev representation formula.

Lemma 2.1. *Let $n \geq 1$. If $n < p < +\infty$, then, for all $\gamma \in \mathbb{R}$, there exists a positive constant $C = C(n, \gamma, p) > 0$ such that, if $u \in \dot{W}^{1,p}_{\gamma}(\mathbb{R}^n_+)$, then, for almost all $x = (x', x_n) \in \mathbb{R}^n_+$ and all $R > 0$, one has*

$$\fint_{C_R(x)} |u(x) - u| \leq C\Big(\int_{C_R(x)} |\mathrm{D}u(z)|^p {z_n}^{\gamma}\,\mathrm{d}z\Big)^{\frac{1}{p}} \times \begin{cases} \min(x_n, R)^{1-\frac{n}{p}} {x_n}^{-\frac{\gamma}{p}} & \text{if } \gamma > p - n, \\ \ln\Big(1 + \big(\frac{R}{x_n}\big)^{\frac{p-n}{p-1}}\Big)^{1-\frac{1}{p}} & \text{if } \gamma = p - n, \\ \max(x_n, R)^{-\frac{\gamma}{p}} R^{1-\frac{n}{p}} & \text{if } \gamma < p - n. \end{cases} \tag{25}$$

Here and in the sequel, we have defined, for any $x \in \mathbb{R}^n$ and $R > 0$, the cone

$$C_R(x) = \big\{z \in \mathbb{R}^n \;\big|\; |z' - x'| < z_n - x_n < R\big\}.$$

A straightforward computation using spherical coordinates shows that the volume of this cone depends only on R:

$$\begin{aligned} |C_R(x)| = |C_R(0)| &= \int_{\mathbb{B}^{n-1}_R} \int_{|z'|}^{R} \mathrm{d}z_n\,\mathrm{d}z' \\ &= \frac{|\mathbb{B}^{n-1}|}{n} R^n. \end{aligned} \tag{26}$$

Proof of Lemma 2.1. By the Sobolev representation formula and Hölder's inequality of exponents p and $p' = \frac{p}{p-1}$, we have

$$\begin{aligned} \fint_{C_R(x)} |u(x) - u| &\leq C \int_{C_R(x)} \frac{|\mathrm{D}u(z)|}{|x - z|^{n-1}}\,\mathrm{d}z \\ &\leq C\Big(\int_{\mathbb{R}^n_+} |\mathrm{D}u(z)|^p {z_n}^{\gamma}\,\mathrm{d}z\Big)^{\frac{1}{p}} \Big(\int_{C_R(x)} \frac{{z_n}^{-\frac{\gamma}{p-1}}}{|x - z|^{(n-1)p'}}\,\mathrm{d}z\Big)^{\frac{1}{p'}}. \end{aligned} \tag{27}$$

To estimate the second integral on the right-hand side of (27), we may assume, without loss of generality, that $x = (0, x_n) \in \mathbb{R}^{n-1} \times (0, +\infty)$. By Tonelli's theorem and a change of variable to polar coordinates, we have

$$\begin{aligned} \int_{C_R(x)} \frac{{z_n}^{-\frac{\gamma}{p-1}}}{|x - z|^{(n-1)p'}}\,\mathrm{d}z &= \int_{x_n}^{x_n+R} \int_{\mathbb{B}^{n-1}_{z_n-x_n}} \frac{{z_n}^{-\frac{\gamma}{p-1}}}{(|z'|^2 + (z_n - x_n)^2)^{(n-1)p'/2}}\,\mathrm{d}z'\,\mathrm{d}z_n \\ &\leq (n-1)|\mathbb{B}^{n-1}| \int_{x_n}^{x_n+R} \int_0^{z_n-x_n} \frac{{z_n}^{-\frac{\gamma}{p-1}} r^{n-2}}{(z_n - x_n)^{(n-1)p'}}\,\mathrm{d}r\,\mathrm{d}z_n. \end{aligned} \tag{28}$$

A direct computation yields

$$\int_0^{z_n-x_n} \frac{r^{n-2}}{(z_n - x_n)^{(n-1)p'}}\,\mathrm{d}r = \frac{1}{(n-1)(z_n - x_n)^{\frac{n-1}{p-1}}},$$

which we inject into (28) to reach

$$
\begin{aligned}
\int_{C_R(x)} \frac{{z_n}^{-\frac{\gamma}{p-1}}}{|x-z|^{(n-1)p'}} \,\mathrm{d}z &\le |\mathbb{B}^{n-1}| \int_{x_n}^{x_n+R} \frac{{z_n}^{-\frac{\gamma}{p-1}}}{(z_n - x_n)^{\frac{n-1}{p-1}}} \,\mathrm{d}z_n \\
&= |\mathbb{B}^{n-1}| \int_0^R \frac{(t+x_n)^{-\frac{\gamma}{p-1}}}{t^{\frac{n-1}{p-1}}} \,\mathrm{d}t \qquad (29)\\
&\le \max(1, 2^{-\frac{\gamma}{p-1}}) |\mathbb{B}^{n-1}| \int_0^R \frac{(\max(t,x_n))^{-\frac{\gamma}{p-1}}}{t^{\frac{n-1}{p-1}}} \,\mathrm{d}t,
\end{aligned}
$$

since, for every $t \ge 0$,

$$
\max(t, x_n) \le t + x_n \le 2 \max(t, x).
$$

If $x_n \ge R$, we compute

$$
\begin{aligned}
\int_0^R \frac{(\max(t,x_n))^{-\frac{\gamma}{p-1}}}{t^{\frac{n-1}{p-1}}} \,\mathrm{d}t &= \int_0^R \frac{{x_n}^{-\frac{\gamma}{p-1}}}{t^{\frac{n-1}{p-1}}} \,\mathrm{d}t \\
&= \frac{p-1}{p-n} \Big({x_n}^{-\frac{\gamma}{p}} R^{1-\frac{n}{p}} \Big)^{p'} \qquad (30)\\
&\le \begin{cases} \dfrac{p-1}{p-n} \Big(\min(x_n, R)^{1-\frac{n}{p}} {x_n}^{-\frac{\gamma}{p}} \Big)^{p'} & \text{if } \gamma > p-n, \\ \dfrac{p-1}{(p-n)\ln 2} \ln\Big(1 + \Big(\dfrac{R}{x_n}\Big)^{\frac{p-n}{p-1}}\Big) & \text{if } \gamma = p-n, \\ \dfrac{p-1}{p-n} \Big(\max(x_n, R)^{-\frac{\gamma}{p}} R^{1-\frac{n}{p}} \Big)^{p'} & \text{if } \gamma < p-n, \end{cases}
\end{aligned}
$$

using when $\gamma = p - n$ the fact that, by concavity of the logarithm,

$$
\Big(\frac{R}{x_n}\Big)^{\frac{p-n}{p-1}} \ln 2 \le \ln\Big(1 + \Big(\frac{R}{x_n}\Big)^{\frac{p-n}{p-1}}\Big).
$$

If $x_n \le R$, we have

$$
\begin{aligned}
\int_0^R \frac{(\max(t,x_n))^{-\frac{\gamma}{p-1}}}{t^{\frac{n-1}{p-1}}} \,\mathrm{d}t &= \int_0^{x_n} \frac{{x_n}^{-\frac{\gamma}{p-1}}}{t^{\frac{n-1}{p-1}}} \,\mathrm{d}t + \int_{x_n}^R \frac{t^{-\frac{\gamma}{p-1}}}{t^{\frac{n-1}{p-1}}} \,\mathrm{d}t \\
&= \frac{p-1}{p-n} {x_n}^{(1-\frac{n+\gamma}{p})p'} + \begin{cases} \dfrac{p-1}{\gamma+n-p} {x_n}^{(1-\frac{n+\gamma}{p})p'} & \text{if } \gamma > p-n, \\ \ln \dfrac{R}{x_n} & \text{if } \gamma = p-n, \\ \dfrac{p-1}{p-n-\gamma} R^{(1-\frac{n+\gamma}{p})p'} & \text{if } \gamma < p-n \end{cases}
\end{aligned}
$$

and then

$$\int_0^R \frac{(\max(t,x_n))^{-\frac{\gamma}{p-1}}}{t^{\frac{n-1}{p-1}}}\,\mathrm{d}t \le \begin{cases} \frac{(p-1)\gamma}{(p-n)(\gamma+n-p)}\Big(\min(x_n,R)^{1-\frac{n}{p}}{x_n}^{-\frac{\gamma}{p}}\Big)^{p'} & \text{if } \gamma > p-n,\\ \frac{p-1}{p-n}\Big(\frac{1}{\ln 2}+1\Big)\ln\Big(1+\Big(\frac{R}{x_n}\Big)^{\frac{p-n}{p-1}}\Big) & \text{if } \gamma = p-n,\\ \frac{(p-1)(2p-2n-\gamma)}{(p-n)(p-n-\gamma)}\Big(\max(x_n,R)^{-\frac{\gamma}{p}}R^{1-\frac{n}{p}}\Big)^{p'} & \text{if } \gamma < p-n,\end{cases} \tag{31}$$

since

$$1+\ln\Big(\Big(\frac{R}{x_n}\Big)^{\frac{p-n}{p-1}}\Big) \le \Big(\frac{1}{\ln 2}+1\Big)\ln\Big(1+\Big(\frac{R}{x_n}\Big)^{\frac{p-n}{p-1}}\Big).$$

Combining the estimates (29), (30), and (31), and injecting them into (27), we obtain the conclusion (25). □

Proof of Theorem 1.1 when $\gamma > -1$ (inequalities (8), (9), and (10)). Let $x, y \in \mathbb{R}^n_+$ and fix $R = 2|x-y|$. Then $|C_R(x)\cap C_R(y)| > 0$ and

$$\begin{aligned}|u(x)-u(y)| &\le \fint_{C_R(x)\cap C_R(y)} |u(x)-u| + \fint_{C_R(x)\cap C_R(y)} |u(y)-u|\\ &\le \frac{|C_R(0)|}{|C_R(x)\cap C_R(y)|}\Big(\fint_{C_R(x)}|u(x)-u| + \fint_{C_R(y)}|u(y)-u|\Big).\end{aligned} \tag{32}$$

Setting $z = \frac{x+y}{2} + (0,|x-y|)$, we have $C_{\frac{R}{4}}(z) \subset C_R(x)\cap C_R(y)$ and thus, in view of (26),

$$\frac{|C_R(0)|}{|C_R(x)\cap C_R(y)|} \le \frac{|C_R(0)|}{|C_{R/4}(0)|} = 4^n.$$

Using Lemma 2.1 in (32), we have

$$\begin{aligned}|u(x)-u(y)| &\le 4^n C\Big(\int_{\mathbb{R}^n_+}|\mathrm{D}u(z)|^p {z_n}^\gamma\,\mathrm{d}z\Big)^{\frac{1}{p}}\\ &\quad\times\begin{cases}\min(x_n,R)^{1-\frac{n}{p}}{x_n}^{-\frac{\gamma}{p}} + \min(y_n,R)^{1-\frac{n}{p}}{y_n}^{-\frac{\gamma}{p}} & \text{if } \gamma > p-n,\\ \ln\Big(1+\big(\frac{R}{x_n}\big)^{\frac{p-n}{p-1}}\Big)^{\frac{1}{p'}} + \ln\Big(1+\big(\frac{R}{y_n}\big)^{\frac{p-n}{p-1}}\Big)^{\frac{1}{p'}} & \text{if } \gamma = p-n,\\ \max(x_n,R)^{-\frac{\gamma}{p}}R^{1-\frac{n}{p}} + \max(y_n,R)^{-\frac{\gamma}{p}}R^{1-\frac{n}{p}} & \text{if } \gamma < p-n.\end{cases}\end{aligned} \tag{33}$$

Using, in (33), the fact that

$$\begin{aligned}\max(x_n,|x-y|) &\le \max(x_n,2|x-y|) \le 2\max(x_n,|x-y|),\\ \min(x_n,|x-y|) &\le \min(x_n,2|x-y|) \le 2\min(x_n,|x-y|),\\ \max(x_n,|x-y|) &\le \max(x_n,y_n,|x-y|) \le 2\max(x_n,|x-y|),\end{aligned}$$

and

$$\max(y_n,|x-y|) \le \max(x_n,y_n,|x-y|) \le 2\max(y_n,|x-y|),$$

and that, for (8),

$$\begin{aligned}
\max((x_n, |x-y|)^{1-\frac{n}{p}} {x_n}^{-\frac{\gamma}{p}}, \min(y_n, |x-y|)^{1-\frac{n}{p}} {y_n}^{-\frac{\gamma}{p}})\\
&= |x-y|^{1-\frac{n}{p}} \max\Big(\frac{{x_n}^{1-\frac{n+\gamma}{p}}}{\max(x_n, |x-y|)^{1-\frac{n}{p}}}, \frac{{x_n}^{1-\frac{n+\gamma}{p}}}{\max(x_n, |x-y|)^{1-\frac{n}{p}}}\Big)\\
&\le \frac{2^{1-\frac{n}{p}}|x-y|^{1-\frac{n}{p}}}{\max(x_n, y_n, |x-y|)^{1-\frac{n}{p}}} \max({x_n}^{1-\frac{n+\gamma}{p}}, {y_n}^{1-\frac{n+\gamma}{p}})\\
&= \frac{2^{1-\frac{n}{p}}|x-y|^{1-\frac{n}{p}}}{\max(x_n, y_n, |x-y|)^{1-\frac{n}{p}} \min(x_n, y_n)^{\frac{n+\gamma}{p}-1}},
\end{aligned}$$

yields the desired conclusions (8), (9), and (10) and ends the proof. □

We note that we have not used the assumption $\gamma > -1$ in the above proof. It turns out that we will improve the estimate (10) when $\gamma \le -1$ thanks to the observation that Sobolev functions then have constant traces.

Proposition 2.2. *Let $n \ge 1$ and $1 \le p < +\infty$. If $\gamma < p - 1$, then, for any $u \in \dot{W}^{1,p}_{\gamma}(\mathbb{R}^n_+)$, we have $u \in \dot{W}^{1,1}_{\mathrm{loc}}(\overline{\mathbb{R}}^n_+)$ and its trace $v = \mathrm{tr}_{\mathbb{R}^{n-1}} u$ satisfies*

$$\iint_{\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}} \frac{|v(x)-v(y)|^p}{|x-y|^{(n-1)+p-\gamma}} \,\mathrm{d}x\,\mathrm{d}y \le C \int_{\mathbb{R}^n_+} |\mathrm{D}u(z)|^p {z_n}^{\gamma} \,\mathrm{d}z. \tag{34}$$

In particular, if $\gamma \le -1$, then v is constant.

Since our aim is to treat weakly differentiable functions up to the boundary, a density argument would require to approximate Sobolev functions with a quite singular weight — it is not locally integrable near the origin. We rather perform a direct argument from trace theory.

Proof of Proposition 2.2. By Hölder's inequality, we have, if the set $K \subseteq \overline{\mathbb{R}}^n_+$ is compact,

$$\int_K |\mathrm{D}u| \le \Big(\int_{\mathbb{R}^n_+} |\mathrm{D}u(z)|^p {z_n}^{\gamma} \,\mathrm{d}z\Big)^{\frac{1}{p}} \Big(\int_K \frac{1}{{z_n}^{\frac{\gamma}{p-1}}} \,\mathrm{d}z\Big)^{1-\frac{1}{p}} < +\infty,$$

provided $\gamma < p - 1$ so that $u \in W^{1,1}_{\mathrm{loc}}(\overline{\mathbb{R}}^n_+)$.

The estimate (34) follows from the classical proof of the trace estimates in weighted Sobolev spaces [14, Thm. 1.3 & Rem. 3.2].

The last conclusion follows from a direct argument [14, Prop. 5.1] or the Bourgain–Brezis–Mironescu characterisation of constant functions [2,3,6,17]. □

Proof of Theorem 1.1 when $\gamma \le -1$ (inequality (11)). Since $\gamma \le -1 < p - n$ and since the proof of Theorem 1.1 (10) does not use the assumption $\gamma > -1$, we have, for almost every $x, y \in \mathbb{R}^n_+$,

$$|u(x)-u(y)| \le C|x-y|^{1-\frac{n}{p}} \max(x_n, y_n, |x-y|)^{-\frac{\gamma}{p}} \Big(\int_{\mathbb{R}^n_+} |\mathrm{D}u(z)|^p {z_n}^{\gamma} \,\mathrm{d}z\Big)^{\frac{1}{p}}. \tag{35}$$

If $\max(x_n, y_n, |x-y|) = \max(x_n, y_n)$, the inequality (35) is equivalent to

$$|u(x) - u(y)| \le C \frac{\max(x_n, y_n)^{1-\frac{n+\gamma}{p}} |x-y|^{1-\frac{n}{p}}}{\max(x_n, y_n, |x-y|)^{1-\frac{n}{p}}} \left(\int_{\mathbb{R}^n_+} |\mathrm{D}u(z)|^p {z_n}^{\gamma} \,\mathrm{d}z \right)^{\frac{1}{p}}. \tag{36}$$

Assuming that $\operatorname{tr}_{\mathbb{R}^{n-1}} u = u|_{\mathbb{R}^{n-1}}$, this estimate also holds for almost every $x \in \mathbb{R}^{n-1}$ and for almost every $y \in \mathbb{R}^n_+$. By Proposition 2.2, since $\gamma \le -1$, for almost every $x, y \in \mathbb{R}^n_+$, we have $u(x', 0) = u(y', 0)$. Therefore, by the triangle inequality and by (35), we have

$$\begin{aligned}
|u(x) - u(y)| &\le |u(x) - u(x', 0)| + |u(y) - u(y', 0)| \\
&\le C \big({x_n}^{1-\frac{n+\gamma}{p}} + {y_n}^{1-\frac{n+\gamma}{p}} \big) \left(\int_{\mathbb{R}^n_+} |\mathrm{D}u(z)|^p {z_n}^{\gamma} \,\mathrm{d}z \right)^{\frac{1}{p}} \\
&\le 2C \max(x_n, y_n)^{1-\frac{n+\gamma}{p}} \left(\int_{\mathbb{R}^n_+} |\mathrm{D}u(z)|^p {z_n}^{\gamma} \,\mathrm{d}z \right)^{\frac{1}{p}} \\
&= 2C \frac{\max(x_n, y_n)^{1-\frac{n+\gamma}{p}} |x-y|^{1-\frac{n}{p}}}{\max(x_n, y_n, |x-y|)^{1-\frac{n}{p}}} \left(\int_{\mathbb{R}^n_+} |\mathrm{D}u(z)|^p {z_n}^{\gamma} \,\mathrm{d}z \right)^{\frac{1}{p}},
\end{aligned} \tag{37}$$

provided $\max(x_n, y_n, |x-y|) = |x-y|$. The conclusion follows by (36) and (37) when $|x-y| \le \max(x_n, y_n)$ and $|x-y| \ge \max(x_n, y_n)$ respectively. □

In order to prove Theorem 1.11, we first prove an estimate involving the distance to the boundary.

Lemma 2.3. *Let $n \ge 1$, $n < p < +\infty$ and $\gamma < p-1$. There exists a constant $C = C(n, \gamma, p) > 0$ depending at most on n, γ, and p, such that, for every $u \in C^\infty_c(\mathbb{R}^n_+)$ and every $x \in \mathbb{R}^n_+$,*

$$|u(x)| \le C \, {x_n}^{1-\frac{n+\gamma}{p}} \left(\int_{\mathbb{R}^n_+} |\mathrm{D}u(z)|^p {z_n}^{\gamma} \,\mathrm{d}z \right)^{\frac{1}{p}}. \tag{38}$$

Even though the estimate keeps the same form for $\gamma < p-1$, it gives a continuity property near the boundary when $\gamma < p-n$, gives a uniform bound when $\gamma = p-n$, and controls the explosion rate near the boundary when $\gamma > p-n$.

Proof of Lemma 2.3. Without loss of generality, we may assume $x = (0, x_n) \in \mathbb{R}^n_+$. Then, since u is compactly supported in $\mathbb{R}^n_+$, we have, for any $h \in \mathbb{B}^{n-1}_{x_n}$,

$$|u(0, x_n)| \le \int_0^1 |\mathrm{D}u((1-t)h, t x_n)| |(h, x_n)| \,\mathrm{d}t. \tag{39}$$

Taking the average over all $h \in \mathbb{B}^{n-1}_{x_n}$ in (39) and applying the change of variable $z = (z', z_n) = ((1-t)h, t x_n)$, we obtain

$$
\begin{aligned}
|u(0, x_n)| &\le \fint_{\mathbb{B}^{n-1}_{x_n}} \int_0^1 |\mathrm{D}u((1-t)h, t x_n)| |(h, x_n)| \,\mathrm{d}t \,\mathrm{d}h \\
&= \frac{1}{|\mathbb{B}^{n-1}|} \int_{U(x)} \frac{|\mathrm{D}u(z)|}{(x_n - z_n)^{n-1}} |(\tfrac{z'}{x_n - z_n}, 1)| \,\mathrm{d}z \\
&\le \frac{\sqrt{2}}{|\mathbb{B}^{n-1}|} \int_{U(x)} \frac{|\mathrm{D}u(z)|}{(x_n - z_n)^{n-1}} \,\mathrm{d}z,
\end{aligned} \tag{40}
$$

where we have defined, for any $x \in \mathbb{R}^n_+$, the cone

$$
U(x) = \{ z \in \mathbb{R}^n \mid |z' - x'| < x_n - z_n < x_n \}.
$$

Applying Hölder's inequality of exponents p and p' to (40), we have

$$
|u(0, x_n)| \le \frac{\sqrt{2}}{|\mathbb{B}^{n-1}|} \Big(\int_{\mathbb{R}^n_+} |\mathrm{D}u(z)|^p {z_n}^{\gamma} \,\mathrm{d}z \Big)^{\frac{1}{p}} \Big(\int_{U(x)} \frac{{z_n}^{-\frac{\gamma}{p-1}}}{(x_n - z_n)^{(n-1)p'}} \,\mathrm{d}z \Big)^{\frac{1}{p'}}. \tag{41}
$$

Using Tonelli's theorem, we obtain, since $n < p$ and $\gamma < p - 1$,

$$
\begin{aligned}
\int_{U(x)} \frac{{z_n}^{-\frac{\gamma}{p-1}}}{(x_n - z_n)^{(n-1)p'}} \,\mathrm{d}z &= \int_0^{x_n} \int_{\mathbb{B}^{n-1}_{x_n - z_n}} \frac{{z_n}^{-\frac{\gamma}{p-1}}}{(x_n - z_n)^{(n-1)p'}} \,\mathrm{d}z' \,\mathrm{d}z_n \\
&= |\mathbb{B}^{n-1}| \int_0^{x_n} \frac{{z_n}^{-\frac{\gamma}{p-1}}}{(x_n - z_n)^{\frac{n-1}{p-1}}} \,\mathrm{d}z_n \\
&\le |\mathbb{B}^{n-1}| \Big(\int_0^{x_n/2} \frac{{z_n}^{-\frac{\gamma}{p-1}}}{(x_n/2)^{\frac{n-1}{p-1}}} \,\mathrm{d}z_n + \int_{x_n/2}^{x_n} \frac{\max(1, 2^{\frac{\gamma}{p-1}}) x_n}{(x_n - y_n)^{\frac{n-1}{p-1}}} \,\mathrm{d}z_n \Big) \\
&= 2^{(1 - \frac{n+\gamma}{p})p'} |\mathbb{B}^{n-1}| \Big(\frac{p-1}{p-1-\gamma} + \frac{p-1}{p-n} \max(1, 2^{\frac{\gamma}{p-1}}) \Big) x_n^{(1 - \frac{n+\gamma}{p})p'}.
\end{aligned} \tag{42}
$$

The conclusion (38) then follows from (41) and (42). ◻

Proof of Theorem 1.11 when $\gamma \le p - n$. We first note that if

$$
|x - y| \le \frac{1}{2} \max(x_n, y_n) \tag{43}
$$

we have

$$
\max(x_n, y_n, |x - y|) = \max(x_n, y_n)
$$

and

$$
\tfrac{1}{2} \min(x_n, y_n) \le \max(x_n, y_n) - |x - y| \le \max(x_n, y_n),
$$

and it follows from Theorem 1.1 that for every $x, y \in \mathbb{R}^n_+$ satisfying (43), we have by monotonicity properties of powers and concavity of the logarithm,

$$
|u(x) - u(y)| \le C \frac{\max(x_n, y_n)^{1 - \frac{n+\gamma}{p}} |x - y|^{1 - \frac{n}{p}}}{\max(x_n, y_n, |x - y|)^{1 - \frac{n}{p}}} \Big(\int_{\mathbb{R}^n_+} |\mathrm{D}u(z)|^p {z_n}^{\gamma} \,\mathrm{d}z \Big)^{\frac{1}{p}}. \tag{44}
$$

On the other hand, if (43) fails, we have $|x-y| > \frac{1}{2}\max(x_n, y_n)$, and thus $|x-y| \geq \frac{1}{2}\max(x_n, y_n, |x-y|)$. Since $\gamma \leq p-n < p-1$, we then have by the triangle inequality and Lemma 2.3 that

$$
\begin{aligned}
|u(x)-u(y)| &\leq |u(x)| + |u(y)| \\
&\leq C \max({x_n}^{1-\frac{n+\gamma}{p}}, {y_n}^{1-\frac{n+\gamma}{p}}) \Big(\int_{\mathbb{R}^n_+} |\mathrm{D}u(z)|^p {z_n}^\gamma \,\mathrm{d}z\Big)^{\frac{1}{p}} \\
&\leq 2^{1-\frac{n}{p}} C \frac{\max(x_n, y_n)^{1-\frac{n+\gamma}{p}} |x-y|^{1-\frac{n}{p}}}{\max(x_n, y_n, |x-y|)^{1-\frac{n}{p}}} \Big(\int_{\mathbb{R}^n_+} |\mathrm{D}u(z)|^p {z_n}^\gamma \,\mathrm{d}z\Big)^{\frac{1}{p}},
\end{aligned}
\tag{45}
$$

which yields the conclusion when $|x-y| \geq \frac{1}{2}\max(x_n, y_n)$. Combining (44) and (45) concludes the proof. ◻

3. Optimality

We discuss in this section the optimality of the estimates obtained in the previous section.

Proof of Theorem 1.15 when $\gamma \neq p-n$. Let us fix a function $\psi \in C^\infty_c(\mathbb{R}^n)$ such that $\psi = 1$ on $\mathbb{B}^n_{1/2}$ and $\operatorname{supp}\psi \subseteq \mathbb{B}^n_1$. Given $R \in (0, +\infty)$ and $y \in \overline{\mathbb{R}}^n_+$, we define $\psi_{y,R} \in C^\infty(\mathbb{R}^n)$ by setting, for $z \in \mathbb{R}^n_+$,

$$
\psi_{y,R}(z) = \psi\Big(\frac{z-y}{R}\Big).
\tag{46}
$$

We compute

$$
\int_{\mathbb{R}^n_+} |\mathrm{D}\psi_{y,R}(z)|^p {z_n}^\gamma \,\mathrm{d}z = \frac{1}{R^p} \int_{\mathbb{B}^n_R(y)\cap\mathbb{R}^n_+} \Big|\mathrm{D}\psi\Big(\frac{z-y}{R}\Big)\Big|^p {z_n}^\gamma \,\mathrm{d}z \leq \frac{C}{R^p} \int_{\mathbb{B}^n_R(y)} {z_n}^\gamma \,\mathrm{d}z.
$$

If $R \leq \frac{y_n}{2}$, then, for all $z \in \mathbb{B}^n_R(y)$, one has $\frac{1}{2}y_n \leq z_n \leq \frac{3}{2}y_n$, and therefore

$$
\int_{\mathbb{R}^n_+} |\mathrm{D}\psi_{y,R}(z)|^p {z_n}^\gamma \,\mathrm{d}z \leq C R^{n-p} {y_n}^\gamma.
\tag{47}
$$

Taking $R = \frac{\min(y_n, |x-y|)}{2}$ ensures that the support of ψ_R is a compact subset of $\mathbb{R}^n_+$, and the assumption (24) on ω together with (47) then implies that

$$
\begin{aligned}
1 = |\psi_{y,R}(x) - \psi_{y,R}(y)| &\leq \omega(x,y) \Big(\int_{\mathbb{R}^n_+} |\mathrm{D}\psi_{y,R}(z)|^p {z_n}^\gamma \,\mathrm{d}z\Big)^{\frac{1}{p}} \\
&\leq C \frac{\omega(x,y)}{{y_n}^{-\frac{\gamma}{p}} \min(y_n, |x-y|)^{1-\frac{n}{p}}}.
\end{aligned}
\tag{48}
$$

Inverting the roles of x and y in (48), we obtain, since $1 - \frac{n}{p} > 0$,

$$
\begin{aligned}
C\omega(x,y) &\geq \max\big({x_n}^{-\frac{\gamma}{p}} \min(x_n, |x-y|)^{1-\frac{n}{p}}, {y_n}^{-\frac{\gamma}{p}} \min(y_n, |x-y|)^{1-\frac{n}{p}}\big) \\
&= \max\Big(\frac{{x_n}^{1-\frac{n+\gamma}{p}}|x-y|^{1-\frac{n}{p}}}{\max(x_n,|x-y|)^{1-\frac{n}{p}}}, \frac{{y_n}^{1-\frac{n+\gamma}{p}}|x-y|^{1-\frac{n}{p}}}{\max(y_n,|x-y|)^{1-\frac{n}{p}}}\Big) \\
&\geq \frac{\max({x_n}^{1-\frac{n+\gamma}{p}}, {y_n}^{1-\frac{n+\gamma}{p}})|x-y|^{1-\frac{n}{p}}}{\max(x_n,y_n,|x-y|)^{1-\frac{n}{p}}}.
\end{aligned}
\tag{49}
$$

Noting that

$$
\max({x_n}^{1-\frac{n+\gamma}{p}}, {y_n}^{1-\frac{n+\gamma}{p}}) = \begin{cases} \min(x_n,y_n)^{1-\frac{n+\gamma}{p}} & \text{if } 1-\frac{n+\gamma}{p} < 0, \\ 1 & \text{if } 1-\frac{n+\gamma}{p} = 0, \\ \max(x_n,y_n)^{1-\frac{n+\gamma}{p}} & \text{if } 1-\frac{n+\gamma}{p} > 0. \end{cases}
$$

we have, by definition (20) of $\Theta^0_{\beta,\kappa}$,

$$
C\omega(x,y) \geq \Theta^0_{1-\frac{n}{p},1-\frac{n+\gamma}{p}}(x,y), \tag{50}
$$

completing the proof. □

Proof of Theorem 1.10. Although the statement Theorem 1.15 requires $n \geq 2$, one can check that this assumption is not used in the proof. This covers the case $\gamma \neq p-1$, and we assume that $\gamma = p - 1$.

We fix a function $\psi \in C^\infty_c(\mathbb{R})$ such that $\psi(0) = 1$ and $\psi = 0$ on $\mathbb{R} \setminus (-1, 1)$, and we define for $x, y \in (0, +\infty)$ the function

$$
\psi_{x,y}(z) = \psi\Big(\frac{\ln(z/y)}{\ln(x/y)}\Big),
$$

so that $\psi_{x,y}(x) = 0$ and $\psi_{x,y}(y) = 1$. Moreover, $\psi_{x,y} = 0$ on $\mathbb{R} \setminus (y^2/x, x)$. We compute then, under the change of variable $z = y(x/y)^t$,

$$
\begin{aligned}
\int_0^\infty |\psi'_{x,y}(z)|^p z^{p-1} \,\mathrm{d}z &= \frac{1}{|\ln(x/y)|^p} \int_0^\infty \Big|\psi'\Big(\frac{\ln(z/y)}{\ln(x/y)}\Big)\Big|^p \frac{1}{z} \,\mathrm{d}z \\
&= \frac{1}{|\ln(x/y)|^{p-1}} \int_0^\infty |\psi'(t)| \,\mathrm{d}t.
\end{aligned}
$$

In view of the hypothesis (19) on ω, we then have

$$
1 = |\psi_{x,y}(x) - \psi_{x,y}(y)| \leq \omega(x,y) \Big(\int_0^\infty |\psi'_{x,y}(z)|^p z^{p-1} \,\mathrm{d}z\Big)^{\frac{1}{p}} \leq C \frac{\omega(x,y)}{|\ln(x/y)|^{\frac{p}{p-1}}},
$$

from which the conclusion follows. □

Proof of Theorem 1.9. In view of Theorems 1.10 and 1.15 and the relationship between $\Theta_{\alpha,\beta,\gamma}$ and $\Theta^0_{\beta,\gamma}$ defined in (7) and (20) respectively, it suffices to consider the case $-1 < \gamma \leq p - n$.

We first assume that $-1 < \gamma < p - n$. We consider $\psi_{y,R} \in C^\infty(\mathbb{R}^n_+)$ defined by (46) in the proof of Theorem 1.15. If $R \geq \frac{y_n}{2}$, then $\mathbb{B}^n_R(y) \subset \mathbb{B}^n_{3R}(y', 0)$. Through an analysis of cases depending on the sign of γ, it follows that

$$\begin{aligned} \int_{\mathbb{R}^n_+} |\mathrm{D}\psi_{y,R}(z)|^p {z_n}^\gamma \,\mathrm{d}z &\leq \frac{C}{R^p} \int_{\mathbb{B}^n_{3R}(y',0)} {z_n}^\gamma \,\mathrm{d}z \\ &\leq C R^{n+\gamma-p}. \end{aligned} \tag{51}$$

Combining (47) and (51), we obtain

$$\int_{\mathbb{R}^n_+} |\mathrm{D}\psi_{y,R}(z)|^p {z_n}^\gamma \,\mathrm{d}z \leq C R^{n-p} \max(y_n, R)^\gamma. \tag{52}$$

By assumption (18) on ω, the inequality (52) then implies that

$$\begin{aligned} 1 = |\psi_{y,|x-y|}(x) - \psi_{y,|x-y|}(y)| &\leq \omega(x,y) \Big(\int_{\mathbb{R}^n_+} |\mathrm{D}\psi_{y,|x-y|}(z)|^p {z_n}^\gamma \,\mathrm{d}z \Big)^{\frac{1}{p}} \\ &\leq C \frac{\max(y_n, |x-y|)^{\frac{\gamma}{p}}}{|x-y|^{1-\frac{n}{p}}} \omega(x,y). \end{aligned} \tag{53}$$

Interchanging the roles of x and y, we thus get

$$\begin{aligned} \omega(x,y) &\geq \mathrm{C} |x-y|^{1-\frac{n}{p}} \max(\max(x_n, |x-y|)^{-\frac{\gamma}{p}}, \max(y_n, |x-y|)^{-\frac{\gamma}{p}}) \\ &\geq \mathrm{C} \min(1, 2^{-\frac{\gamma}{p}}) |x-y|^{1-\frac{n}{p}} \max(x_n, y_n, |x-y|)^{-\frac{\gamma}{p}}, \end{aligned} \tag{54}$$

since,

$$\max(x_n, |x-y|) \leq \max(x_n, y_n, R) \leq 2 \max(x_n, |x-y|),$$

and

$$\max(y_n, |x-y|) \leq \max(x_n, y_n, R) \leq 2 \max(y_n, |x-y|).$$

We deduce from (54) and by definition (7) of $\Theta_{\alpha,\beta,\kappa}$, that, if $-1 < \gamma < p - n$,

$$C\omega(x,y) \geq \Theta_{1-\frac{1}{p}, 1-\frac{n}{p}, 1-\frac{n+\gamma}{p}}(x,y).$$

It remains to treat the case $\gamma = p - n$. In order to do this, we take a smooth function $\theta \in C^\infty(\mathbb{R})$ such that $\theta = 1$ on $(-\infty, 0]$ and $\theta = 0$ on $[1, +\infty)$. We then define, for all $z \in \mathbb{R}^n_+$ and $R > y_n$, the function

$$\theta_{y,R}(z) = \theta\Big(\frac{\ln\big(|z-y|/y_n\big)}{\ln\big(R/y_n\big)} \Big).$$

If $R > y_n$, then $|z - y| \ge y_n$ implies that $z_n \le y_n + |z - y| \le 2|z - y|$. Since $\gamma = p - n > 0$, we compute

$$\begin{aligned}\int_{\mathbb{R}^n_+} |\mathrm{D}\theta_{y,R}(z)|^p {z_n}^\gamma \,\mathrm{d}z &\le \frac{C}{(\ln \frac{R}{y_n})^p} \int_{y_n \le |z-y| \le R} \frac{{z_n}^\gamma}{|z-y|^p} \,\mathrm{d}z \\ &\le \frac{C}{(\ln \frac{R}{y_n})^p} \int_{y_n \le |z-y| \le R} \frac{1}{|z-y|^{p-\gamma}} \,\mathrm{d}z \\ &\le \frac{C}{(\ln \frac{R}{y_n})^{p-1}} \\ &= \frac{C}{\Big(\ln\big((\frac{R}{y_n})^{\frac{p-n}{p-1}}\big)\Big)^{p-1}},\end{aligned} \tag{55}$$

where the last equality follows from the identity $\ln t = \frac{1}{\alpha} \ln(t^\alpha)$. In particular, we obtain from (55) that, if $|x - y| > y_n$,

$$\begin{aligned}1 = |\theta_{y,|x-y|}(x) - \theta_{y,|x-y|}(y)| &\le \omega(x, y) \Big(\int_{\mathbb{R}^n_+} |\mathrm{D}\theta_{y,y_n}(z)|^p {z_n}^\gamma \,\mathrm{d}z \Big)^{\frac{1}{p}} \\ &\le \frac{C}{\Big(\ln\big((\frac{|x-y|}{y_n})^{\frac{p-n}{p-1}}\big)\Big)^{1-\frac{1}{p}}} \omega(x, y).\end{aligned} \tag{56}$$

By definition (7) of $\Theta_{\alpha,\beta,\kappa}$, this implies that

$$C\omega(x, y) \ge \Theta_{1-\frac{1}{p},1-\frac{n}{p},1-\frac{n+\gamma}{p}}(x, y),$$

when $\gamma = p - n$, concluding the proof. □

Proposition 3.1. *Let $n \ge 1$, $\gamma \in \mathbb{R}$, and $n < p < +\infty$. There exists a function $\omega_* \colon \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to [0, +\infty)$ such that, for every $\omega \colon \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to [0, +\infty)$, the following are equivalent*

(i) for every $u \in \dot{W}^{1,p}_\gamma(\mathbb{R}^n_+) \cap C(\mathbb{R}^n_+)$ and every $x, y \in \mathbb{R}^n_+$,

$$|u(x) - u(y)| \le \omega(x, y) \Big(\int_{\mathbb{R}^n_+} |\mathrm{D}u(z)|^p {z_n}^\gamma \,\mathrm{d}z \Big)^{\frac{1}{p}}, \tag{57}$$

(ii) $\omega \ge \omega_*$.

Moreover,

(iii) ω_* is a distance inducing the natural topology on $\mathbb{R}^n_+$,

(iv) there exist constants $c, C \in (0, +\infty)$ such that for every $x, y \in \mathbb{R}^n_+$

$$c\,\Theta_{1-\frac{1}{p},1-\frac{n}{p},1-\frac{n+\gamma}{p}}(x, y) \le \omega_*(x, y) \le C\,\Theta_{1-\frac{1}{p},1-\frac{n}{p},1-\frac{n+\gamma}{p}}(x, y),$$

(v) for every $t \in (0, +\infty)$ and $x, y \in \mathbb{R}^n_+$,

$$\omega_*(tx, ty) = t^{1-\frac{n+\gamma}{p}} \omega_*(x, y),$$

(vi) for every $h \in \mathbb{R}^{n-1} \times \{0\}$ and $x, y \in \mathbb{R}^n_+$,

$$\omega_*(x + h, y + h) = \omega_*(x, y).$$

Proof. We define ω_* as the pointwise infimum of the functions $\omega\colon \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to [0, +\infty)$ satisfying the condition (i), ensuring the equivalence between (i) and (ii). The assertion (iv) follows from Theorem 1.1 and Theorem 1.9. The minimality property of ω_* ensures that, for every $x, y \in \mathbb{R}^n_+$,

$$\omega_*(x, y) = \omega_*(y, x)$$

and, for every $x, y, z \in \mathbb{R}^n_+$,

$$\omega_*(x, y) \leq \omega_*(x, z) + \omega_*(z, x).$$

In view of Theorem 1.1, (iii) holds.

The assertions (v) and (vi) follow from the minimality of ω_* and applying the corresponding transformations to u in (i). □

Proposition 3.1 combined with standard separability argument shows that any function $\omega\colon \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to [0, +\infty)$ such that, for every $u \in \dot{W}^{1,p}_{\gamma}(\mathbb{R}^n_+) \cap C(\mathbb{R}^n_+)$ and almost every x, $y \in \mathbb{R}^n_+$,

$$|u(x) - u(y)| \leq \omega(x, y)\Big(\int_{\mathbb{R}^n_+} |\mathrm{D}u(z)|^p {z_n}^{\gamma} \,\mathrm{d}z\Big)^{\frac{1}{p}},$$

satisfies $\omega \geq \omega_*$ almost everywhere on $\mathbb{R}^n_+ \times \mathbb{R}^n_+$.

Proposition 3.2. *Let $n \geq 2$, $\gamma \in \mathbb{R}$, and $n < p < +\infty$. There exists a function $\omega_*\colon \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to [0, +\infty)$ such that, for every $\omega\colon \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to [0, +\infty)$, the following are equivalent*

(i) for every $u \in C^\infty_c(\mathbb{R}^n_+)$ and $x, y \in \mathbb{R}^n_+$,

$$|u(x) - u(y)| \leq \omega(x, y)\Big(\int_{\mathbb{R}^n_+} |\mathrm{D}u(z)|^p {z_n}^{\gamma} \,\mathrm{d}z\Big)^{\frac{1}{p}}, \tag{58}$$

(ii) $\omega \geq \omega_*$.

Moreover,

(iii) ω_* is a distance inducing the natural topology on $\mathbb{R}^n_+$,

(iv) there exist constants $c, C \in (0, +\infty)$ such that for every $x, y \in \mathbb{R}^n_+$

$$c\, \Theta^0_{1-\frac{n}{p}, 1-\frac{n+\gamma}{p}}(x, y) \leq \omega_*(x, y) \leq C\, \Theta^0_{1-\frac{n}{p}, 1-\frac{n+\gamma}{p}}(x, y),$$

(v) for every $t \in (0, +\infty)$ and $x, y \in \mathbb{R}^n_+$,

$$\omega_*(tx, ty) = t^{1-\frac{n+\gamma}{p}} \omega_*(x, y),$$

(vi) for every $h \in \mathbb{R}^n$ and $x, y \in \mathbb{R}^n_+$,

$$\omega_*(x + h, y + h) = \omega_*(x, y).$$

Again one can deduce from Proposition 3.2 that any function $\omega\colon \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to [0, +\infty)$ such that for every $u \in C^\infty_c(\mathbb{R}^n_+)$ and almost every $x, y \in \mathbb{R}^n_+$,

$$|u(x) - u(y)| \leq \omega(x, y)\Big(\int_{\mathbb{R}^n_+} |\mathrm{D}u(z)|^p {z_n}^{\gamma} \,\mathrm{d}z\Big)^{\frac{1}{p}},$$

satisfies $\omega \geq \omega_*$ almost everywhere on $\mathbb{R}^n_+ \times \mathbb{R}^n_+$.

4. Hyperbolic Morrey–Sobolev inequality and optimality

The case $\gamma = p-n$ corresponds to the Sobolev–Morrey inequality on the n-dimensional hyperbolic space $\mathbb{H}^n$ in the Poincaré half-space model, whose distance is given by

$$\mathrm{d}_{\mathbb{H}^n}(x,y) = 2\sinh^{-1}\Big(\frac{|x-y|}{2\sqrt{x_n y_n}}\Big). \tag{59}$$

Lemma 4.1. *Let $n \geq 1$. For all $x, y \in \mathbb{H}^n$, one has*

$$\frac{|x-y|}{\sqrt{x_n y_n}} \leq \frac{|x-y|}{\min(x_n, y_n)} \leq \max\Big(\sqrt{2}\frac{|x-y|}{\sqrt{x_n y_n}}, 2\frac{|x-y|^2}{x_n y_n}\Big).$$

Proof. We assume, without loss of generality, that $x_n \leq y_n$. We then have

$$\frac{|x-y|}{\sqrt{x_n y_n}} \leq \frac{|x-y|}{x_n},$$

which is the first inequality.

For the second inequality, we either have $|x-y| \leq x_n$ and then $y_n \leq x_n + |x-y| \leq 2x_n$, so that

$$\frac{|x-y|}{x_n} \leq \sqrt{2}\frac{|x-y|}{\sqrt{x_n y_n}},$$

or we have $|x-y| \geq x_n$, and therefore $y_n \leq x_n + |x-y| \leq 2|x-y|$, which then implies that

$$\frac{|x-y|}{x_n} \leq 2\frac{|x-y|}{x_n}\frac{|x-y|}{y_n} = 2\frac{|x-y|^2}{x_n y_n}. \qquad \square$$

Proof of Theorem 1.16. Thanks to Lemma 4.1 and noting that

$$\ln(1+t) \leq \sinh^{-1}(t) = \ln(t+\sqrt{1+t^2}) \leq \ln(1+2t),$$

we get

$$c\mathrm{d}_{\mathbb{H}^n}(x,y) \leq \ln\Big(1+\frac{|x-y|}{\min(x_n,y_n)}\Big) \leq C\mathrm{d}_{\mathbb{H}^n}(x,y),$$

and thus

$$\begin{aligned} &c\max(\mathrm{d}_{\mathbb{H}^n}(x,y)^{\frac{p-n}{p-1}}, \mathrm{d}_{\mathbb{H}^n}(x,y)) \\ &\qquad\leq \ln\Big(1+\Big(\frac{|x-y|}{\min(x_n,y_n)}\Big)^{\frac{p-n}{p-1}}\Big) \\ &\qquad\qquad\leq C\max(\mathrm{d}_{\mathbb{H}^n}(x,y)^{\frac{p-n}{p-1}}, \mathrm{d}_{\mathbb{H}^n}(x,y)).\end{aligned}$$

The conclusion then follows from Theorems 1.1 and 1.9. $\square$

Proof of Theorem 1.17. We proceed as in the proof of Theorem 1.16 to get

$$c\max(\mathrm{d}_{\mathbb{H}^n}(x,y),1) \leq \frac{|x-y|}{\max(\min(x_n,y_n),|x-y|)} \leq C\max(\mathrm{d}_{\mathbb{H}^n}(x,y),1).$$

Noting that

$$\max(\min(x_n,y_n),|x-y|) \leq \max(x_n,y_n,|x-y|) \leq 2\max(\min(x_n,y_n),|x-y|),$$

the conclusion then follows from Theorem 1.11 and Theorem 1.15. □

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