

Quantifying Transient Dynamics in Heterogeneous Networks under Various Inputs

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Understanding how transient dynamics unfold in response to localized inputs is central to predicting and controlling signal propagation in network systems, including neural processing, epidemic intervention, and power-grid resilience. Existing theoretical frameworks typically assume homogeneous network structures and constant or pulse-like inputs, overlooking how heterogeneity in structure and variety of input shape transient responses, often leading to discrepancies between theory and observation. Here, we develop a general theoretical framework that establishes quantitative relationships between the strength and timing of transient dynamics to various inputs in heterogeneous networks. Using a Neumann series expansion, we disentangle the distinct roles of self-dynamics and network structures beyond the scope of standard spectral theory, yielding intuitive and interpretable formulations. We show that node-to-node propagation can be represented as the cumulative effect of all directed walks, each weighted recursively by the self-dynamics of participating nodes. This framework further quantifies how heterogeneity, such as broad degree distributions or additional motifs, amplifies both response strength and time. Our results advance the understanding of transient dynamics across network structures and input types, extend the existing theory to more general settings, and provide practical guidance for optimizing transient responses.

I. INTRODUCTION

Understanding how transient dynamics are triggered by localized inputs is crucial for elucidating signal propagation in network systems. Propagation characteristics, such as the strength and timing of transient dynamics, play a central role in predicting and controlling signal propagation and are essential for assessing system resilience and stability [1–7]. These dynamics have profound implications for real-world applications, including neural processing [8, 9], epidemic intervention [10–12], communication network design [13, 14], and power-grid resilience [15, 16]. Existing theoretical frameworks have established connections between network structures and transient responses. Typically formulated under idealized conditions such as thermodynamic limits, these frameworks provide analytical tools for identifying propagation patterns across complex systems and for enabling targeted interventions at nodes or links within specific system classes [1, 4, 17, 18].

Despite substantial progress, prevailing frameworks remain constrained by two key limitations: neglecting fine structural details through mean-field approximations, which imply homogeneity, and relying exclusively on deterministic inputs, which overlook the variety of real-world stimuli. These limitations hinder mechanistic understanding and precise prediction of transient dynamics in real finite-size systems that exhibit structural heterogeneity and operate under various input types and

conditions [19–22]. For instance, in theoretical neuroscience, macroscopic models at the neuronal level often assume i.i.d. Gaussian connectivity, justified by the relative insensitivity of order parameters to microscopic details in the large-system limit, consistent with sampling-based experimental approaches [23–25]. However, zooming out to the scale of brain regions, system sizes are way smaller and modern experiments can resolve the full matrix of connectivity elements. At this level, a brain region receives various types of inputs, and structural heterogeneity becomes fundamental to its functional state, rendering homogeneous approximations invalid [20, 21]. This raises pivotal questions: How do structural heterogeneities shape transient dynamics in finite-size networks under different inputs? More fundamentally, do universal, and interpretable principles exist that govern transient dynamics across diverse configurations?

To address these questions, we develop a general framework that explicitly incorporates both structural heterogeneity and input variety. We model dynamical systems as finite-size networks of coupled ordinary differential equations and quantify node-to-node propagation. Our approach advances prior works in two key aspects. First, we extend frameworks restricted to pulse inputs [26, 27] by quantifying responses to a broader range of inputs, including constant, square, and white-noise types. Second, whereas existing studies primarily examine how system-specific nonlinearities interact with network structures under constant inputs [1, 4, 17, 18], and often rely on mean-field approximations that obscure fine structural details, our framework directly isolates the fundamental role of heterogeneous network structures themselves. In

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doing so, we establish a foundational theory for transient dynamics in finite-size networks where structural heterogeneity is explicitly resolved rather than averaged away.

We propose metrics to estimate the strength and timing of local responses, complementing conventional metrics that are limited to system-wide onset or steady-state measures. Our estimated metrics exhibit strong numerical agreement with simulations across multiple network classes, including chains, regular lattices, random networks, small-world networks, scale-free networks, and geometric networks. Crucially, through Neumann series expansion [28], we disentangle the distinct contributions of self-dynamics and network structures across different inputs, which are otherwise opaque to spectral analysis. This framework provides quantitative and interpretable insights into spatiotemporal signal propagation, enabling a comprehensive understanding of response strength and time across various inputs.

This paper is structured as follows. Section II introduces our framework, which incorporates structural heterogeneity and input variety, including the analytically tractable subclass of Negative Strictly Diagonally Dominant (NSDD) systems, which provide the basis for our analysis and subsequent generalizations. Section III establishes metrics derived via matrix inverses and spectral decompositions, quantifying transient response strength (e.g., amplification, peak response) and time (e.g., time constant, response time). After rigorous validation across classical topologies, the Neumann series expansion effectively disentangles distinct contributions from self-dynamics and network structures across inputs. To elucidate general principles governing propagation, we then analyze progressively complex structures: directed chains and sparse random networks (Sec. IV A); homogeneous in-degree networks (Sec. IV B), contrasting path-based simulation and truncation and revealing the role of dominant paths and motifs; heterogeneous in-degree networks (Sec. IV C), quantifying degree distribution effects and local motif effects in the most general cases. Finally, the Discussion (Sec. V) synthesizes the relationships between strength and temporal metrics from deterministic, stochastic, and structural aspects, and clarifies the roles of heterogeneities.

II. MODEL INTRODUCTION

A. General formalism

Consider a general dynamical system comprising N interacting components $\mathbf{y}(t) = (y_1(t), \dots, y_N(t))^T$ governed by

$$\frac{d\mathbf{y}}{dt} = \mathbf{F}(\mathbf{y}),$$

where $\mathbf{F}(\mathbf{y})$ incorporates self-dynamics and pairwise interactions. Assuming the existence of the equilibrium state $\mathbf{y}^*$ that satisfies $\mathbf{F}(\mathbf{y}^*) = \mathbf{0}$, we can characterize

local dynamics around the equilibrium using small perturbations $\mathbf{x}(t) \equiv \mathbf{y}(t) - \mathbf{y}^*$. Linearizing $\mathbf{F}$ around $\mathbf{y}^*$ gives the perturbation dynamics as

$$\frac{d\mathbf{x}}{dt} = \mathbf{H}\mathbf{x},$$

where $\mathbf{H} \equiv \nabla \mathbf{F}|_{\mathbf{y}^*}$ is the Jacobian matrix. The spectral properties of $\mathbf{H}$ provide a foundational framework for analyzing local stability [5, 29–31], transient responses [6, 26, 27, 32, 33], mode decomposition [8, 34, 35], and other critical features of nonlinear systems near equilibrium states [36–38].

Building on the linearized system, we extend our analysis to systems exhibiting heterogeneity and variety: non-uniformity in $\mathbf{F}$, reflected in the spectral properties of the Jacobian $\mathbf{H}$, and spatiotemporally various inputs $\mathbf{I}(t)$. These motivate the generalized driven linear system

$$\frac{d\mathbf{x}}{dt} = \mathbf{H}\mathbf{x} + \mathbf{I}(t), \quad (1)$$

where $\mathbf{I}(t)$ denotes external inputs that may vary across time and components [2, 20, 39, 40]. We focus on systems for which all eigenvalues of $\mathbf{H}$ have negative real parts, ensuring that $\mathbf{H}$ is invertible and the dynamics are asymptotically stable. The interaction structure is represented by a directed weighted graph derived from $\mathbf{H}$, with weakly connected conditions that should be enforced by the irreducibility of $\mathbf{H} + \mathbf{H}^T$, which ensures connectivity in the undirected counterpart. Systems that fail this criterion split into disjoint, connected subgraphs, which are then subjected to respective individual analysis. Four frequently encountered input types are examined in our framework: constant, pulse, square, and white noise, predominantly applied to individual nodes. Extensions to multi-node inputs are also possible.

The generalized model in Eq. (1) is applicable to both linear systems (e.g., linear compartmental models [41], continuous-time Markov chains [42]) and nonlinear systems that are linearized near equilibrium states. Under white noise input, it reduces to a Gaussian linear process [43]. Despite the model's linearity, its transient responses remain analytically intractable due to structural heterogeneity in $\mathbf{H}$ and the variety of input $\mathbf{I}(t)$. These generate transcendental dependencies in weighted term combinations, precluding closed-form results. To address this, we develop a framework that systematically accounts for structural heterogeneity across inputs, focusing on spatial granularity (mesoscale finite-size networks with asymmetric and weighted $\mathbf{H}$, where mean-field approximations break down [1, 4, 17, 18]) and temporal granularity (finite-time dynamics rather than $t \rightarrow 0$ or $t \rightarrow \infty$ asymptotics [6, 32, 44, 45]). These considerations motivate two central questions: (Q1) Can spectral or matrix-based methods quantitatively characterize transient responses ($\mathbf{x}(t)$) in structurally heterogeneous, finite-size networks ($\mathbf{H}$) driven by various inputs ($\mathbf{I}(t)$)? (Q2) Do general, interpretable principles govern transient dynamics across different network configurations and input conditions?

B. Special formalism

A widely studied realization within the general framework is the Negative Strictly Diagonally Dominant (NSDD) structure [46–52], defined as $\mathbf{H} \equiv \mathbf{A} - \mathbf{D} - \text{diag}(\beta_i)$, where $\mathbf{A}$ is the adjacency matrix with weighted directed connections, and $A_{ij} \geq 0$ ($i \neq j$) denoting the connection from node j to node i , and $A_{ii} = 0$; diagonal matrix $\mathbf{D} = \text{diag}(\sum_{j=1}^N A_{ij}) \equiv \text{diag}(D_i)$ represents the nodal in-degree, and $\beta_i > 0$ denotes the self-decay rate. Substituting $\mathbf{H}$ into Eq. (1) yields

$$\begin{aligned} \frac{dx_i}{dt} &= -\beta_i x_i + \sum_{j=1}^N A_{ij} (x_j - x_i) + I_i(t), \\ &= \underbrace{-(\beta_i + D_i)x_i}_{\text{Self-dynamics}} + \underbrace{\sum_{j=1}^N A_{ij} x_j}_{\text{Interactions}} + \underbrace{I_i(t)}_{\text{Inputs}}, \end{aligned} \quad (2)$$

where the NSDD property (strict diagonal dominance with $H_{ii} < 0$) ensures Hurwitz stability [31, 33, 52–56]. Relaxing the sign constraints on β_i and A_{ij} reverts the system to the general formalism, where Hurwitz stability is no longer guaranteed but can be preserved under a moderate amount of negative couplings $A_{ij} < 0$ or negative self-decay rate $\beta_i < 0$.

This structure serves as an analytically tractable foundation for the following reasons. (i) It guarantees stability and well-behaved transient dynamics, enabling rigorous analysis (Appendix A). (ii) This model architecture and its extensions exhibit versatile applicability, spanning disciplines ranging from neuroscience to physiology [8, 57–61]. (iii) The term $\sum_j A_{ij}(x_j - x_i)$ represents a diffusion process on the network governed by the graph Laplacian, analogous to spatial diffusion in continuous media [58]. (iv) The explicit separation of self-dynamics ($-(\beta_i + D_i)x_i$), pairwise interactions from the network structure (A_{ij}), and external inputs ($I_i(t)$) provides a unified template for comparing linearized non-linear systems, and also an extension for theoretical results to more general formalisms [26, 27]. The core idea of NSDD is that each node's self-dynamics must outweigh the total positive input it receives, expressed as $(\beta_i + D_i) > \sum_j A_{ij}$.

III. GENERIC QUANTIFICATION ACROSS INPUTS

Building on the central questions of how networks transform inputs into responses, we establish a general mathematical framework connecting input properties to output characterizations across network structures. This section achieves three interconnected advances: First, we derive exact analytical metrics for typical response characterizations: amplification, peak response, time constant, and response time, which are valid for NSDD struc-

tures and four input classes (Fig. 1). Second, we rigorously validate these metrics across a broad range of structures, demonstrating numerical accuracy of estimations to structural heterogeneity (Fig. 2). Third, we uncover relationships and scaling laws that intrinsically link these metrics through their shared dependence on each other. We also provide simple intuition through Hurwitz stability, see Box 1.

A. Constant input

We start by analyzing the system's response to constant inputs modeled as the Heaviside step functions $\mathbf{I}_0^{\text{const}}(t)$, a widely-used approach for studying signal propagation in large-scale systems [1, 4, 18]. Full-time course of node i receiving the constant input based on the steady-state $x_i^{\text{const}}(0)$ is $\Delta x_i^{\text{const}}(t) \equiv x_i^{\text{const}}(t) - x_i^{\text{const}}(0) = [\mathbf{H}^{-1} (e^{\mathbf{H}t} - \mathbf{I}_N) \mathbf{I}_0^{\text{const}}(t)]_i$, where $\mathbf{I}_N$ is the identity matrix. Under stability, the trajectory $\Delta x_i^{\text{const}}(t)$ converges to its final steady state denoted as $\Delta x_i^{\text{const}}(\infty)$. We characterize the response of node i to a single-node constant input I_0^{const} at node m using four dynamical metrics (Fig. 1): (i) *amplification* Z_{im} , defined as the area between the response curve and its final steady state; (ii) *peak response* R_{im} , the maximum amplitude; (iii) *time constant* τ_{im} , the time to reach $(1-1/e)$ of increasing $\Delta x_i^{\text{const}}(t)$; and (iv) *relative propagation time* t_{im} , the time when $\Delta x_i^{\text{const}}(t_{im})/\Delta x_i^{\text{const}}(\infty) = \eta$ [4, 11, 18, 62–66], where fraction η is the given threshold. Analytical expressions of these metrics are computed and estimated, respectively, as:

$$Z_{im} \triangleq [\mathbf{H}^{-2}]_{im} I_0^{\text{const}} \sim O(1/\lambda_1^2), \quad (3)$$

$$R_{im} \triangleq -[\mathbf{H}^{-1}]_{im} I_0^{\text{const}} \sim O(1/\lambda_1), \quad (4)$$

$$\tau_{im} \triangleq -\frac{[\mathbf{H}^{-2}]_{im}}{[\mathbf{H}^{-1}]_{im}} \sim O(1/\lambda_1), \quad (5)$$

$$t_{im} \triangleq -\tau_{im} \ln(1 - \eta) \sim O(1/\lambda_1). \quad (6)$$

Here, $[\cdot]_{im}$ denotes the (i, m) matrix element and $\lambda_1 \equiv \max_j \text{Re}(\lambda_j)$ is the dominant eigenvalue of $\mathbf{H}$. These scaling relations with λ_1 govern system-level response strength and time. Corresponding spectral decomposition is

$$[\mathbf{H}^{-k}]_{im} = \sum_{j=1}^N \frac{u_{im}^j}{\lambda_j^k}, \quad u_{im}^j \equiv [\mathbf{U}]_{ij} [\mathbf{U}^{-1}]_{jm},$$

where k is an integer; $\mathbf{U}$ is the eigenmatrix diagonalizing $\mathbf{H}$ ($\mathbf{H} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^{-1}$, $\mathbf{\Lambda} = \text{diag}(\lambda_j)$), establishing the metrics via eigenmode projections.

Among these metrics, amplification (Z_{im}) and peak response (R_{im}) admit analytical expressions, while temporal metrics (τ_{im} and t_{im} , labeled by asterisks *'s in Fig. 1) are estimated under the assumption that the residual response ($\Delta x_i^{\text{const}}(\infty) - \Delta x_i^{\text{const}}(t)$) decays exponentially

Box 1. Intuition through Hurwitz stability

A simpler, more intuitive, and more general assumption is *Hurwitz stability* [31, 33]: namely, that dominant eigenvalues (also all eigenvalues) of $\mathbf{H}$ have strictly negative real parts, $\max_j \text{Re}(\lambda_j) < 0$. Under this assumption, the system $\dot{\mathbf{x}} = \mathbf{H}\mathbf{x}$ forgets initial conditions exponentially fast ($e^{\mathbf{H}t} \rightarrow 0$); $\mathbf{H}$ (and hence $-\mathbf{H}$) is invertible; and steady-state quantities, such as the unique solution of the continuous-time Lyapunov equation, are well defined. This assumption also clarifies why the steady-state (step) gain is the resolvent at zero frequency, $-\mathbf{H}^{-1}$. For a constant input $\mathbf{I}$, the equilibrium satisfies $0 = \mathbf{H}\mathbf{x}_{ss} + \mathbf{I}$, hence $\mathbf{x}_{ss} = -\mathbf{H}^{-1}\mathbf{I}$. Equivalently,

$$\int_0^\infty e^{\mathbf{H}s} ds = -\mathbf{H}^{-1},$$

so the long-time effect of a sustained drive is mediated by $-\mathbf{H}^{-1}$.

Intuition in 1d. For $\dot{x} = -\beta x + I(t)$ with $\beta > 0$ (so $\mathbf{H} = -\beta$ is Hurwitz), a step $I(t) = a \mathbf{1}_{t \geq 0}$ yields

$$x(t) = \left(x_0 - \frac{a}{\beta}\right)e^{-\beta t} + \frac{a}{\beta}, \quad x_{ss} = \frac{a}{\beta} = -\mathbf{H}^{-1}a.$$

An impulse-like probe $I(t) = a \delta(t)$ has impulse response $h(t) = a e^{-\beta t}$ for $t \geq 0$, whose *area* equals the step gain:

$$\int_0^\infty h(t) dt = \frac{a}{\beta} = -\mathbf{H}^{-1}a.$$

A square of duration t_s , $I(t) = a \mathbf{1}_{0 \leq t \leq t_s}$, interpolates between impulse- and step-like behavior: as $t_s \rightarrow 0$ the response is impulse-like with peak $\approx at_s$, while as $t_s \rightarrow \infty$ it saturates to a/β . Under white-noise input $u(t) = \sigma \xi(t)$, the stationary variance is $\text{Var}(x) = \sigma^2/(2\beta)$, again set by the same decay rate $1/\beta$.

govern by the time constant τ_{im} . Full derivations are provided in Appendix B. These metrics are well-defined in NSDD systems, where positive constant inputs yield strictly positive, monotonic responses, ensuring temporal solution uniqueness. The metric sign conventions are also rigorously maintained: $[\mathbf{H}^{-1}]_{im} < 0$ and $[\mathbf{H}^{-2}]_{im} > 0$ for all reachable node pairs (i, m) , while $[\mathbf{H}^{-k}]_{im} = 0$ (for all integers $k \geq 1$) when no path exists from m to i (Appendix A). Numerical validation across diverse network topologies in NSDD systems demonstrates high accuracy in metric estimation, with strong Spearman's rank correlations between analytical and simulated values (Fig. 2(a); See Supplementary Material (SM) Sec. I). Extensions to more general forms reveal robustness: estimation accuracy remains above 80% even when 20% of the connections are inhibitory ($A_{ij} < 0$) in large, sparse networks operating near the stability boundary (See SM Fig. S32). Limitations arise primarily in extreme cases, such as near chain-like networks with relatively low average degrees and widespread inhibition, where the loss of monotonicity leads to overshoot (See SM Sec. II). Compared with other inputs analyzed subsequently, constant inputs produce more regular time courses, enabling robust metric estimation and easier theoretical analysis.

The derived metrics exhibit two distinct scaling relations with the input amplitude: strength metrics (Z_{im}, R_{im}) scale linearly with input amplitude ($Z_{im}, R_{im} \propto I_0^{\text{const}}$), while temporal metrics ($\tau_{im} = Z_{im}/R_{im}$) remain invariant. This fundamental distinction enables separate structural interpretations: strength metrics quantify absolute intensities and input amplitude, whereas temporal

metrics characterize relative efficiency. This dichotomy motivates our subsequent analysis of network structure effects in Sec. IV, where we analyze how network structure shapes these metrics and provide intuitive interpretations.

B. Pulse input

Understanding the impulse response of a linear time-invariant (LTI) system is fundamental for characterizing its transient dynamics, as the response to any input can be derived through its convolution with the system's impulse response [67]. Response to the Dirac delta input $\delta \mathbf{I}(t)$ with the total impulse strength $\mathbf{I}_0^{\text{pulse}} \equiv \int_0^\infty \delta \mathbf{I}(t) dt$ is governed by $\Delta x_i^{\text{pulse}}(t) \equiv x_i^{\text{pulse}}(t) - x_i^{\text{pulse}}(0) = \left[e^{\mathbf{H}t} \mathbf{I}_0^{\text{pulse}} \right]_i$. The response to a single-node pulse input I_0^{pulse} applied at node m has already been systematically characterized in prior works [26, 27]. Building on these studies, we adopt the same set of metrics to quantify the temporal and strength properties of the response, as illustrated in the second column of Fig. 1(b). Additionally, we derive an estimation for the decay time constant. The four metrics are given, respectively, by: (i) *amplification* R_{im} , defined as the area under the response curve; (ii) *peak response* $\tilde{P}_{im}$, the maximum amplitude; (iii) *decay time constant* $\tau_{D_{im}}$, the time to reach $1/e$ of the peak during the decay phase; and (iv) *peak response time* $\tilde{\tau}_{im}$, the

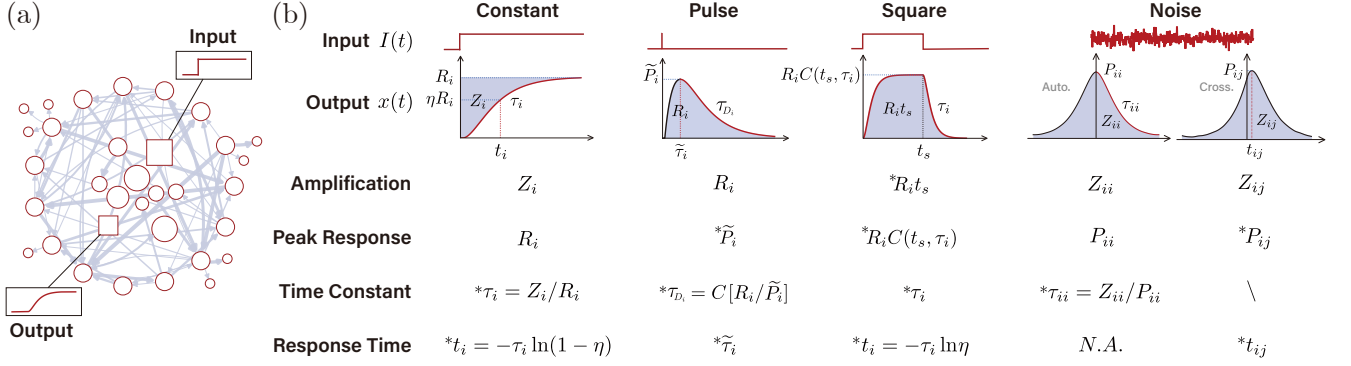


FIG. 1. **Quantification of response strength and time in the general linear network model.** (a) Schematic of node-to-node propagation under constant input. A single node receives a constant input, and the resulting transient responses are quantified for any single node in a heterogeneous network. (b) Summary of metrics quantifying node i 's response across four input types. Response strength is quantified by amplification and peak response, while response time is characterized by time constants and response times. Metrics marked with asterisks ($*$'s) indicate approximations derived from transcendental equations, as detailed in Appendix A. Constant input (first column): Amplification (Z_i , blue area above curve), peak response (R_i , maximum amplitude), time constant (τ_i , rise to $(1 - 1/e)R_i$), and relative propagation time (t_i , time to ηR_i). Pulse input (second column): Amplification (R_i , blue area under curve), peak response ($\tilde{P}_i$, maximum amplitude), decay time constant (D_i , drop to $\tilde{P}_i/e$), and time to peak ($\tilde{\tau}_i$). Square input (third column): Amplification ($R_i t_s$, blue area under curve), peak response ($R_i C(t_s, \tau_i)$), decay time constant (τ_i , drop to $1/e$ of peak), and response time (t_i). Noise input (last two columns): Autocovariance ($i = j$): Amplification (Z_{ii}), peak response (P_{ii} , zero-lag), time constant (τ_{ii} , decay to P_{ii}/e). Crosscovariance ($i \neq j$): Amplification (Z_{ij}), peak response (P_{ij}), and peak response time (t_{ij} , time to peak).

time at which the peak occurs, where, more concretely,

$$R_{im} \triangleq -[\mathbf{H}^{-1}]_{im} I_0^{\text{pulse}} \sim O(1/\lambda_1), \quad (7)$$

$$\tilde{P}_{im} \triangleq C(d) P_{im}, \quad (8)$$

$$= C(d) \frac{([\mathbf{H}^{-1}]_{im})^2 I_0^{\text{pulse}}}{\sqrt{2[\mathbf{H}^{-3}]_{im}[\mathbf{H}^{-1}]_{im} - ([\mathbf{H}^{-2}]_{im})^2}} \sim O(1), \quad (9)$$

$$\tau_{D_{im}} \triangleq \left(1 - \frac{1}{e}\right) \frac{R_{im}}{\tilde{P}_{im}} \sim O(1/\lambda_1), \quad (10)$$

$$\tilde{\tau}_{im} \triangleq \tau_{im} + \frac{1}{\lambda_1} = -\frac{[\mathbf{H}^{-2}]_{im}}{[\mathbf{H}^{-1}]_{im}} + \frac{1}{\lambda_1} \sim O(1/\lambda_1). \quad (11)$$

Here, the bias term $C(d) = (\sqrt{d+1}d^d)/(e^d d!)$ ($\approx 1/\sqrt{2\pi}$ when d is large) depends on the shortest path length d between nodes m and i . For a given network, $C(d)$ typically provides lower bound estimations on the simulated peak response, and setting $C(d) = 1$ gives an upper bound numerically. Similarly, for $\tilde{\tau}_{im}$, the bias term $(1/\lambda_1) < 0$ results in an lower-bound approximation [27], and omitting this term gives practical upper bounds.

Pulse-response metrics build on established methods [26, 27], where normalized responses to the single-node pulse are interpreted as probability distributions. This framework ensures non-negative dynamics under positive pulse inputs in NSDD systems, providing a well-grounded basis for interpretation (Appendix A, [26]). We extend prior works by defining the decay time constant $\tau_{D_{im}}$ through exponential assumptions (Appendix B). Numerical validation across networks demonstrates great

performance (Fig. 2(b); See SM Sec. I), with low relative error and strong rank correlations, particularly in sparse networks with weak network interactions ($\alpha/\lambda_1 \rightarrow 0$ regime where α refers to identical interaction weight [27]).

Pulse-input dynamics inherit properties from constant-input responses through their derivative relationship ($d\Delta x_i^{\text{const}}(t)/dt = \Delta x_i^{\text{pulse}}(t)$) under the same input location and amplitude: pulse amplification equals the peak response under constant input (denoted as R_{im} in Eqs. (4) and (7)). Temporal metrics share complementary interpretations: the constant-input time constant τ_{im} (Eq. (5)) aligns with the pulse-input peak response time $\tilde{\tau}_{im}$ (Eq. (11)). In addition, for LTI systems, this equivalence extends to covariance [68]: single-node pulse responses mirror the crosscovariance function with time-lag s : $\langle \mathbf{x}, \mathbf{I} \rangle = e^{\mathbf{H}s} \mathbf{Q}$ when spectral density matrix $\mathbf{Q} \equiv \mathbf{S}_{\mathbf{I}}(w)$ (Fourier transform of the autocovariance function $\mathbb{E}[\mathbf{I}(t)\mathbf{I}(t+\tau)^\top]$) contains only a single non-zero diagonal element I_0^{pulse} at node m (Appendix B). This existing mathematical equivalence enables direct comparison among inputs while preserving consistent interpretation.

C. Square input

Square inputs combine analytical simplicity with biological relevance, offering precise temporal control for modeling finite-duration stimuli in physiological experiments [69, 70]. The full time course exhibits biphasic dynamics: (i) a rising phase corresponding to a trun-

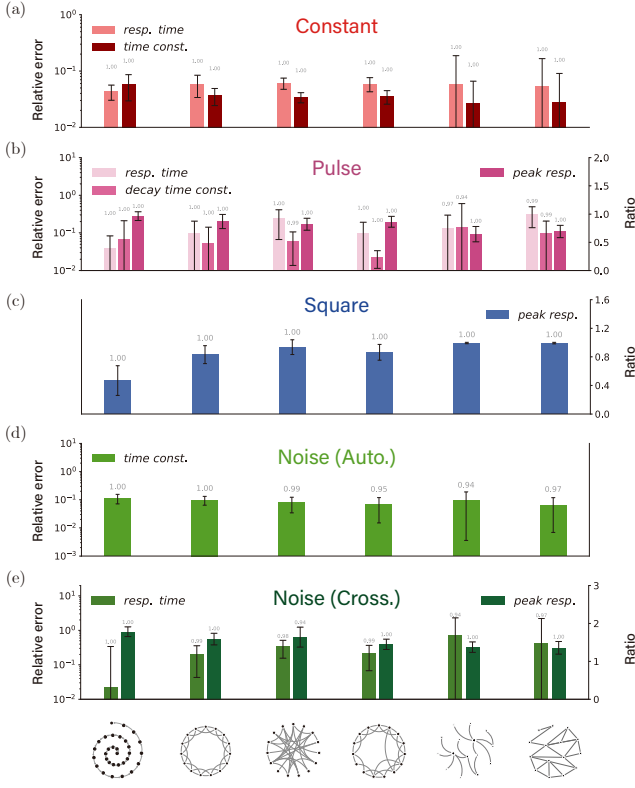


FIG. 2. Accuracy of estimated response metrics across classical network topologies. Network types from left to right: (i) Chain (directed for deterministic inputs; undirected for noise inputs), (ii) Regular lattice (average degree ~ 4), (iii) Erdős-Rényi (ER) random network (edge probability ~ 0.02), (iv) Small-world network (average degree ~ 2 , rewiring probability ~ 0.5), (v) Scale-free network (preferential attachment parameter ~ 1), and (vi) Geometric network (connection radius ~ 0.2). Input types: (a) Constant; (b) Pulse; (c) Square; (d) Noise (for autocovariance); (e) Noise (for crosscovariance). The abscissa (x-axis) is shared across all panels. All networks contain 100 nodes with uniform parameters: self-decay rate $\beta = 1$ and interaction weights set to 1. Inputs are applied as follows: to the first node in the chain, randomly assigned in the lattice, and randomly assigned across 100 independent instances for randomly generated networks (ER, small-world, scale-free, geometric). Time-related metrics (response time, time constant) use relative error $|t_{\text{sim}} - t_{\text{thr}}|/t_{\text{sim}}$ (left bars; t_{sim} : simulated, t_{thr} : theoretical). Strength-related metrics (peak response) use ratio $P_{\text{thr}}/P_{\text{sim}}$ (right bars). Error bars show mean $\pm$ variance across instances. Most time-related errors remain below 10^0 (100%), while strength ratios cluster near ~ 1 (within one order of magnitude), indicating consistent quantitative agreement. Gray labels indicate Spearman's rank correlation for node-wise ordering preservation, with values close to 1 reflecting strong rank consistency. In chain-like or sparse networks, nodes whose shortest path from the input node is ≥ 15 are excluded to avoid numerical artifacts caused by rapid response decay.

cated constant-input response during stimulation period t_s , followed by (ii) a decay phase that mirrors the remaining portion of the constant-input response. We characterize the decay phase by the time constant τ_{im} , which measures the time it takes for the response to drop to $1/e$ of its initial value in response to a single-node square input I_0^{square} at node m . We then quantify two strength metrics: the *amplification* $R_{im}t_s$ (representing total integrated response) and the *peak response* $R_{im}C(t_s, \tau_{im})$ (quantifying maximum amplitude), where $R_{im} = -[\mathbf{H}^{-1}]_{im}I_0^{\text{square}}$ (see Eqs. (4) and (7)).

Precisely,

$$R_{im}t_s \triangleq -t_s[\mathbf{H}^{-1}]_{im}I_0^{\text{square}}, \quad (12)$$

$$R_{im}C(t_s, \tau_{im}) \triangleq -(1 - e^{-t_s/\tau_{im}})[\mathbf{H}^{-1}]_{im}I_0^{\text{square}}. \quad (13)$$

As such, systematic validation across NSDD systems confirms metric robustness (Fig. 1(c)): amplification exhibits negligible error ($< 0.1\%$, omitted for clarity), while peak responses achieve near-unity agreement ratios (> 0.8) under typical topologies (Fig. 2(c); Appendix B). For unit input duration ($t_s = 1$), Eq. (12) establishes the equivalence linking impulse-integrated amplification (Eq. (7)) to constant-input peak response (Eq. (4)). Correspondence for peak response of unit duration (Eq. (13)) extends to impulse-response peaks (Eq. (9); See SM Fig. S17). The asymptotic scaling $C(t_s, \tau_{im}) \sim t_s/\tau_{im}$ (for $t_s \rightarrow 0$) and $C(t_s, \tau_{im}) \rightarrow 1$ (for $t_s \rightarrow \infty$) emerges naturally from τ_{im} -dominated dynamics, confirming time constant (Eq. (5)) as universal regulators of transient dynamics.

D. Noise input

White noise input, characterized by a flat power spectral density, serves as a fundamental tool to probe broadband system dynamics (e.g., in neural processing [20, 71]). To rigorously model its discontinuous and unbounded nature, we reformulate the system dynamics from Eq. (1) as the stochastic differential equation:

$$d\mathbf{x} = \mathbf{H}\mathbf{x}dt + d\beta(t),$$

where $d\beta(t) = \mathbf{I}(t)dt$, and $\beta(t)$ is the Brownian motion process with a zero mean and covariance structure as: $\mathbb{E}[\mathbf{I}(\tau)\mathbf{I}(s)^T] = \mathbf{Q}\delta(\tau - s)$. Here, $\mathbf{Q}$ defines the input spectral density matrix, and also the Fourier transform of the autocovariance function $\mathbb{E}[\mathbf{I}(\tau)\mathbf{I}(s)^T]$. The time-dependent solution, derived via Itô calculus to accommodate the unbounded and discontinuous variation of $\beta(t)$, is [72]:

$$\mathbf{x}^{\text{noise}}(t) = e^{\mathbf{H}(t-t_0)}\mathbf{x}^{\text{noise}}(t_0) + \int_{t_0}^t e^{\mathbf{H}(t-\tau)}d\beta(\tau).$$

In steady state ($t \rightarrow \infty$), the stationary covariance function becomes:

$$\mathbf{C}(\tau) \triangleq \mathbb{E}[\mathbf{x}(t)\mathbf{x}(t-\tau)^\top] = \begin{cases} \mathbf{P}_\infty e^{-\mathbf{H}^\top \tau}, & \tau \leq 0, \\ e^{\mathbf{H}\tau} \mathbf{P}_\infty, & \tau > 0, \end{cases}$$

satisfying $\mathbf{C}(\tau) = \mathbf{C}(-\tau)^\top$ with time-lag τ . The steady-state covariance $\mathbf{P}_\infty$ corresponds to the Lyapunov equation:

$$\mathbf{H}\mathbf{P}_\infty + \mathbf{P}_\infty\mathbf{H}^\top + \mathbf{Q} = 0, \quad (14)$$

and admits equivalent representations:

$$\mathbf{P}_\infty = \int_0^\infty e^{\mathbf{H}\tau} \mathbf{Q} e^{\mathbf{H}^\top \tau} d\tau. \quad (15)$$

For scalar systems ($x \in \mathbb{R}$) with a stable eigenvalue $\lambda < 0$, this reduces to $\mathbf{C}(\tau) = \frac{Q}{2|\lambda|} e^{\lambda|\tau|}$. This covariance function rigorously quantifies steady-state variability and frequency-selective sensitivity under stochastic forcing (Appendix B; [72]).

The covariance function contains two distinct components: diagonal elements (autocovariances, $C_{ii}(\tau)$) quantifying self-evolutions and off-diagonal elements (crosscovariances, $C_{ij}(\tau)$) capturing pairwise relations. Autocovariance functions are even symmetric with maxima at zero lag. NSDD systems exhibit monotonic decay with a large self-decay rate and strictly positive values (Appendix A). When noise input with spectral density I_0^{noise} is applied only to node m , the autocovariance dynamics at node i are characterized by three metrics (fourth column in Fig. 1(b)): (i) *amplification* Z_{ii}^m , total integrated covariance, (ii) *peak response* P_{ii}^m , maximum instantaneous covariance at zero lag, and (iii) *decay time constant* τ_{ii}^m , $1/e$ relaxation time [19, 20, 73], where

$$Z_{ii}^m \triangleq -2 [\mathbf{H}^{-1} \mathbf{P}_\infty]_{ii}^m \sim O(1/\lambda_1^2), \quad (16)$$

$$P_{ii}^m \triangleq [\mathbf{P}_\infty]_{ii}^m \sim O(1/\lambda_1), \quad (17)$$

$$\tau_{ii}^m \triangleq -\frac{[\mathbf{H}^{-1} \mathbf{P}_\infty]_{ii}^m}{[\mathbf{P}_\infty]_{ii}^m} \sim O(1/\lambda_1). \quad (18)$$

The eigenmode decomposition for steady-covariance is

$$[\mathbf{P}_\infty]_{ij}^m = -\sum_{p,q} \frac{u_{im}^p u_{jm}^q}{\lambda_p + \lambda_q} I_0^{\text{noise}}. \quad (19)$$

The metrics of crosscovariance dynamics between input node m and node pair (i, j) (last column in Fig. 1(b)) are: (i) *amplification* Z_{ij}^m , area under the crosscovariance curve; (ii) *peak response* P_{ij}^m , maximal value; and (iii)

peak response time t_{ij}^m , time to maximum, where

$$Z_{ij}^m \triangleq -M_{ij}^{m(1)} \sim O(1/\lambda_1^2), \quad (20)$$

$$P_{ij}^m \triangleq \frac{(M_{ij}^{m(1)})^2}{\sqrt{4M_{ij}^{m(1)}M_{ij}^{m(3)} - 2(M_{ij}^{m(2)})^2}} \sim O(1/\lambda_1), \quad (21)$$

$$t_{ij}^m \triangleq -\frac{M_{ij}^{m(2)}}{M_{ij}^{m(1)}} \sim O(1/\lambda_1), \quad (22)$$

with $M_{ij}^{m(n)} \equiv [\mathbf{H}^{-n} \mathbf{P}_\infty]_{ij}^m + (-1)^{n+1} [\mathbf{H}^{-n} \mathbf{P}_\infty]_{ji}^m$ and $n = 1, 2, 3$. Estimation methods parallel those for pulse inputs, with complete derivations in Appendix B. In NSDD systems, crosscovariance also preserves strict positivity (Appendix A).

We validate all metrics in NSDD systems, demonstrating high accuracy and rank correlation (Fig. 2(d, e); Appendix B). Compared with deterministic inputs, noise-driven responses depend critically on the steady-state covariance $\mathbf{P}_\infty$ - computable through the Lyapunov equation (Eqs. (14) and (15)) or its eigenmode (Eq. (19)), though both approaches lack intuitive interpretation of their dependence with $\mathbf{H}$. Through Wiener-Khinchin theorem [72], $\mathbf{P}_\infty = \mathbf{C}(\tau = 0) = \mathcal{F}^{-1}[S_{\mathbf{x}}(\omega)]_{\tau=0}$ admits the representation:

$$\mathbf{P}_\infty = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\mathbf{H} - i\omega \mathbf{I}_N)^{-1} \mathbf{Q} (\mathbf{H} + i\omega \mathbf{I}_N)^{-\top} d\omega, \quad (23)$$

which reduces single-node inputs at m between node pair (i, j) to:

$$[\mathbf{P}_\infty]_{ij}^m = \frac{I_0^{\text{noise}}}{2\pi} \int_{-\infty}^{\infty} [(\mathbf{H} - i\omega \mathbf{I}_N)^{-1}]_{im} [(\mathbf{H} + i\omega \mathbf{I}_N)^{-1}]_{jm} d\omega.$$

While explicitly relating $\mathbf{P}_\infty$ to $\mathbf{H}$, this formulation remains analytically opaque due to its complex-integral nature. This limitation motivates our matrix expansion and complex analysis in subsequent sections, where we unravel how network structure governs transient responses.

Across input classes, we find most metrics share inverse dependencies on the dominant eigenvalue λ_1 , reflecting system-wide coordination between strength and temporal variations. However, critical refinements arise in heterogeneous settings across inputs: (i) heterogeneous connectivity encoded in element-wise inverse terms $[\mathbf{H}^{-n}]_{im}$, $n = 1, 2, 3$, and steady-state covariance $[\mathbf{P}_\infty]_{ij}^m$, (ii) spectral dispersion of λ_j and non-uniform eigenmode participation u_{im}^j that might localize temporal or strength features [19, 74], and (iii) input-specific alignment $([\cdot]_{im})$ governing response profiles. Reconciling these global spectral principles with localized structural and input details motivates the structure-aware framework developed in Sec. IV.

Box 2. How to probe your network and use our framework for Hurwitz systems

Fix a stable network and linearize its dynamics as $\dot{\mathbf{x}} = \mathbf{H}\mathbf{x} + \mathbf{I}(t)$ with $\mathbf{H}$ satisfying Hurwitz stability.

A *constant (step) probe* $\mathbf{I}(t) = a e_m \mathbf{1}_{t \geq 0}$ is the standard baseline: it reveals both the static-gain geometry and the dominant timescale in a single shot. In steady state, the peak response from source m to target i is $-a [\mathbf{H}^{-1}]_{im}$ (Eq. (4)), while the approach to steady state reflects the spectral gap (typical relaxation $\tau \sim 1/|\lambda_1(\mathbf{H})|$). Because the step integrates all effects of directed walks through the network, it is robust and easy to estimate, making it effective for mapping who influences whom and by how much.

A *pulse (impulse-like) probe* is linked to the step by differentiation: the impulse response is the time derivative of the step response. This gives explicit translation rules across inputs. In particular, the pulse *amplification* (area under the target's transient, Eq. (7)) coincides with the step *peak* gain (Eq. (4)), and the pulse *peak response time* (Eq. (9)) tracks the step *time constant* (Eq. (5)). Pulses emphasize latency and sharpness, localizing propagation delays along paths and distinguishing fast feedforward topological routes from slower, loop-mediated ones.

A *square probe* of duration t_s is the difference of two steps separated by t_s , so its response interpolates between impulse- and step-like regimes. For $t_s \ll \tau$ it behaves like a pulse; for $t_s \gg \tau$ it approaches the step. Sweeping t_s is thus a titration of the network's intrinsic time scale: the dependence of peak size and timing on t_s identifies dominant decay rates and reveals when longer feedback walks contribute beyond shortest paths. Because squares are straightforward to implement, t_s acts as a practical control parameter for balancing high temporal resolution (information capacity) and high signal-to-noise level.

White-noise forcing, $\mathbb{E}[\mathbf{I}(\tau)\mathbf{I}(s)^\top] = \mathbf{Q}\delta(\tau - s)$, with $\mathbf{I}(t)$ a zero-mean white noise vector, connects deterministic probes to fluctuations. In steady state the covariance $\mathbf{P}_\infty$ solves the Lyapunov equation $\mathbf{H}\mathbf{P}_\infty + \mathbf{P}_\infty\mathbf{H}^\top + \mathbf{Q} = 0$, and autocovariance and crosscovariance functions follow by propagating $\mathbf{P}_\infty$ through $e^{\mathbf{H}t}$. The same directed walks that determine step and pulse gains set covariance peaks, areas, and correlation times (again scaling with $1/|\lambda_1(\mathbf{H})|$). The picture in the frequency domain is equivalent via the Wiener-Khinchin theorem while its path-based view further exposes how much each sub-walk modulates the ongoing variability for each frequency (Eq. (E12)).

In summary, a constant step provides a direct readout of static gains $-[\mathbf{H}^{-1}]_{im}$ and dominant relaxation times with minimal overhead. Pulses sharpen latency estimates. Squares of duration t_s bridge impulse- and step-like behavior by sweeping t_s . White-noise forcing projects the same transfer properties into second-order statistics, where structural sensitivity appears as covariance patterns determined during steady-state.

Connection between metrics. The derivative $d\Delta x_i^{\text{const}}(t)/dt = \Delta x_i^{\text{pulse}}(t)$ link implies that pulse amplification matches the step peak; short squares behave like pulses while long squares recover steps; and noise covariances are the stochastic counterpart of deterministic gains via the Wiener-Khinchin theorem (Eq. (23)). In practice, one needs to verify Hurwitz stability; if it holds, all formulas apply. If $\mathbf{H}$ is further NSDD, additional qualitative estimates (e.g., monotone, sign-definite responses) are guaranteed. However, the core inferences about response strength and time already hold under the Hurwitz assumption alone.

Taken together, the four input classes furnish a consistent characterization of network structures and dynamics, and can be used to reveal path-dependent bottlenecks.

IV. IMPACT OF NETWORK STRUCTURES ON TRANSIENT DYNAMICS

Our theoretical framework (Fig. 1), employing matrix and spectral formulations, quantifies how structural heterogeneity ($\mathbf{H}$) and input variety ($\mathbf{I}(t)$) shape transient responses (Q1). This framework is validated in NSDD systems (Fig. 2) and further generalized in SM Sec. II. To probe your network and use our framework for Hurwitz systems, see Box. 2. A critical gap remains: How does heterogeneous network structure (A_{ij}) interact with self-

decay rates (β_i) across inputs ($I(t)$) to shape responses (Q2)? While eigen-decompositions and matrix inversions yield general solutions, they obscure interpretable relationships and become computationally prohibitive for large-scale systems. Through systematic expansion and truncation, we analytically disentangle the interactions governing transient responses. This derivation reveals how specific structures shape the relationship between response strength and temporal metrics across different input types under the uniform self-decay rate setting in NSDD systems: $\dot{x}_i = -\beta x_i + \sum_{j=1}^N A_{ij}(x_j - x_i) + I_i(t)$.

A. Directed chain to sparse random networks.

We start analyzing from a simple case: propagation in a directed chain with interaction weights $A_{d \rightarrow d+1} = \alpha$ (Appendix C, Fig. 3(a)). Strength metrics (Z, R, P) basically decay geometrically with the shortest path length d , and scale proportionally with input strength I_0 :

$$\text{Strength} \sim \left(1 + \frac{\beta}{\alpha}\right)^{-d} I_0, \quad (24)$$

while temporal metrics (τ, t) scale linearly:

$$\text{Time} \sim \frac{d}{\alpha + \beta}. \quad (25)$$

These scaling relationships reveal two distinct structural effects: strength attenuation, dominated by the ratio β/α , and temporal accumulation, determined by the inverse $(\alpha + \beta)^{-1}$, which leads to the distinct role of α and β : weak coupling ($\alpha \ll \beta$) results in rapid geometric decay, accompanied by slow temporal growth, governed by d/β . Strong coupling ($\alpha \gg \beta$) yields gradual decay and slow linear time governed by d/α . These scaling laws naturally generalize to sparse ER random networks when considering shortest path lengths (shaded areas in Fig. 3(b)), and temporal metrics are more robust to the variation of interaction weights when self-decay rate dominants ($\beta \geq 1$ or $\beta \geq \alpha$, Appendix C). This simple case thereby disentangles how self-decay rate β and interaction weight α jointly govern input propagation along the chain.

B. Homogeneous in-degree

We generalize directed chains and sparse random networks to homogeneous in-degree NSDD systems, where all nodes share identical in-degree ($D_i \equiv \sum_{j=1}^N A_{ij} = D$). This configuration enables structural diversity through heterogeneous walks while enforcing uniform self-dynamics, a design paradigm characteristic of artificial neural networks and synthetic biological circuits [75–78]. For clarity, we use the term *chains* to denote the acyclic subset of walks, while *walks* refer to the general case that may include revisiting nodes. Fig. 4(a) shows an input propagation example from source m to target i ($m \neq i$) through multiple walk lengths d ($d \geq 1$), with peak response R_{im} for constant input expanding as a weighted sum of $(\beta + D)^{-d}$ terms. Basic elements of metrics can be expanded in terms of walk length p using the Neumann series expansion. Concretely,

$$\begin{aligned} [\mathbf{H}^{-n}]_{im} &= (-1)^n \sum_{p=1}^{\infty} \left(\frac{C_{p+n-1}^{n-1}}{(\beta + D)^{p+n}} \right) [\mathbf{A}^p]_{im}, \\ [\mathbf{P}_{\infty}]_{ij}^m &= \sum_{p,q=1}^{\infty} \left(\frac{C_{p+q}^p}{(2(\beta + D))^{p+q+1}} \right) [\mathbf{A}^p]_{im} [\mathbf{A}^q]_{jm}, \end{aligned}$$

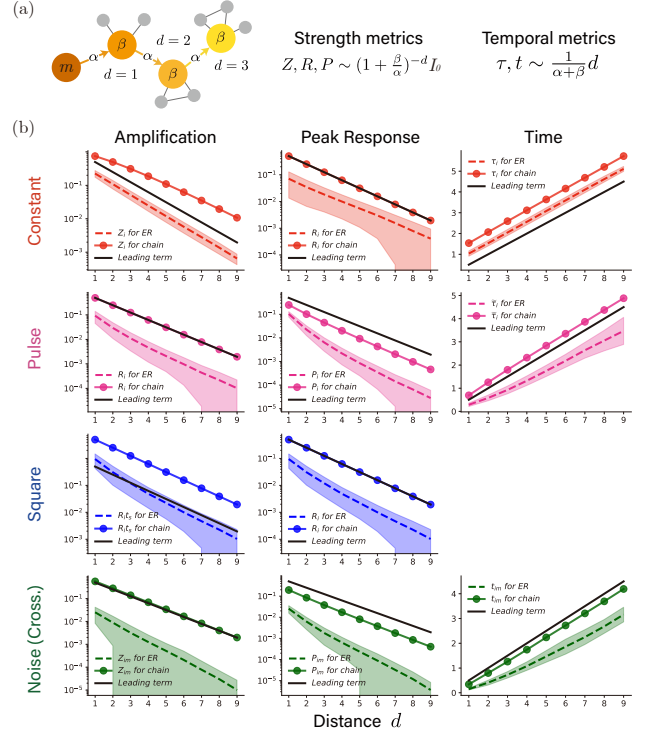


FIG. 3. From directed chains to sparse random networks. (a) Schematic relationships: Directed chain topology with uniform self-decay rate β , uniform interaction weight α , and distinct source-target shortest path length d . For the leading terms in metrics, strength metrics (Z, R, P) exhibit geometric decay with d , while temporal metrics (τ, t) show linear path dependence. (b) Combined validation: Theoretical predictions (solid black: analytic leading terms for the chain; dotted: numerical chain simulations) and sparse ER random networks (dashed: ensemble mean of 100 realizations; shading: ± 1 SD; connection probability of random networks: ~ 0.02). Alignment enables direct structural comparison. Parameters: $\beta = \alpha = 1.0$, stimulus duration $t_s = 10$ for square input, input strength I_0 normalized to unity in simulation.

where $[\mathbf{A}^p]_{im} \equiv \sum_{j_1, \dots, j_{p-1}} A_{ij_1} A_{j_1 j_2} \dots A_{j_{p-1} m}$ quantifies the cumulative influence through all directed walks of length p from source m to target i , with the summation running over all possible intermediate nodes $\{j_1, \dots, j_{p-1}\}$. These expansions reveal how walk diversity ($[\mathbf{A}^p]_{im}$) links the self-dynamics ($\beta + D$) in shaping the responses (Appendix D). The term $1/(\beta + D)$ acts as a weighting factor that modulates the contribution of more distant walks. While $\beta > 2D$ (strong decay dominance) can guarantee convergence for expansion through Gerschgorin's theorem, practical implementations often tolerate weaker decay rates, especially when prioritizing the ranking order (See SM Figs. S36 and S37).

The series expansions naturally motivate truncation rules that identify dominant contributions and simplify metrics while preserving accuracy. We analyze simulated results for node pairs (source m , target i) with walk length d and truncate expansions at order p in random

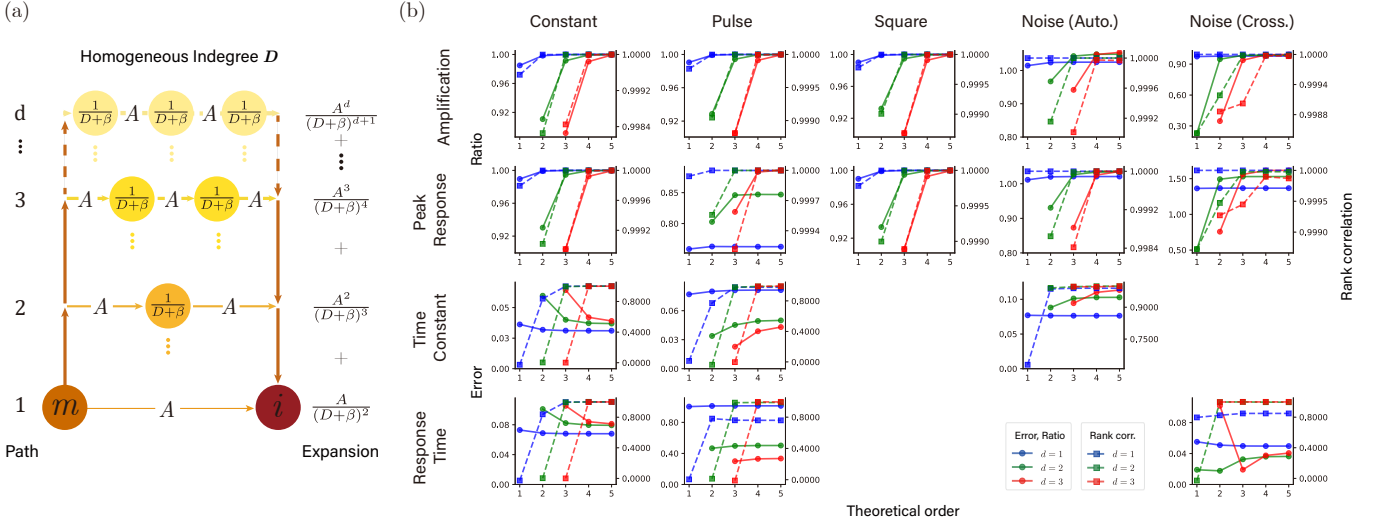


FIG. 4. **Path length vs. truncation order in homogeneous in-degree networks.** (a) Relations for dynamical metric dependence on path length d : Peak response R_{im} (constant input) as weighted sums of $(D + \beta)^{-(d+1)} \times A^d$ terms, where D (homogeneous in-degree) and β (uniform self-decay) combine multiplicatively. The A^d factor accounts for path multiplicity (all length- d paths between nodes), while temporal metrics are derived from ratio relationships. (b) Truncation order effects: Strength metric ratios (circles, left axis; Ratio = $P_{\text{trunc}}/P_{\text{sim}}$) and temporal metric relative errors (circles, left axis; Error = $|t_{\text{trunc}} - t_{\text{sim}}|/t_{\text{sim}}$) vs. rank correlations (squares, right axis). Colors denote shortest path lengths (blue: $d = 1$, green: $d = 2$, red: $d = 3$) and values are averaged from 100 network realizations. Strength metrics require $\geq d$ -order truncation (ratio > 0.9 , error $< 10\%$, rank correlation > 0.9); temporal metrics need $\geq (d+1)$ -order (error $< 10\%$, rank correlation > 0.8). Crosscovariance $C_{im}^m(\tau)$ truncated separately in $\mathbf{H}$ and $\mathbf{P}_\infty$. Parameters: $\beta = 10$ (uniform self-decay rate, satisfying $\beta > 2D$); $\alpha = 0.1$ (identical interaction weight); $N = 100$ (network size); $p = 0.08$ (connection probability).

networks (Fig. 4(b)). This order p determines the maximal walk length ($[\mathbf{A}^p]_{im}$) incorporated in the metrics. Although all expansions converge asymptotically, required truncation depths differ between metric classes: strength metrics (e.g., constant-input peak response R_{im}) need at least $p = d$, while temporal metrics (e.g., constant-input time constant τ_{im}) require at least $p = d+1$. This distinction originates from their mathematical forms. Notice that $R_{im} = \sum_{p=1}^{\infty} ([\mathbf{A}^p]_{im}/(\beta + D)^{p+1})$ is dominated by minimal-length walks ($p = d$), whereas

$$\tau_{im} = \left(\sum_{p=1}^{\infty} \frac{(1+p)[\mathbf{A}^p]_{im}}{(\beta + D)^{p+2}} \right) / \left(\sum_{p=1}^{\infty} \frac{[\mathbf{A}^p]_{im}}{(\beta + D)^{p+1}} \right) \quad (26)$$

demands $p = d + 1$ terms to resolve the $(d+1)$ -vs- d balance between numerator and denominator. Temporal metrics are more sensitive to longer walk lengths than strength metrics, which may limit the generalizability of localized approximations for these metrics (see Discussion Sec. V).

We present three canonical metrics, describing amplification and illustrating how network structure maps onto responses in the case of direct propagation ($d = 1, A_{im} \neq$

0) after truncation ($p = 1$). Concretely,

$$\begin{aligned} Z_{im} &= \sum_{p=1}^{\infty} \left(\frac{1+p}{(\beta + D)^{p+2}} \right) [\mathbf{A}^p]_{im} \approx \frac{2A_{im}}{(\beta + D)^3}, \\ Z_{ii}^m &= 2R_{im} [\mathbf{P}_\infty]_{mi}^m + 2 \sum_{r \neq m} R_{ir} [\mathbf{P}_\infty]_{ri}^m, \\ &\approx \frac{A_{im}^2}{2(\beta + D)^4} + \sum_{r \neq m} \frac{A_{ir} A_{rm} A_{im}}{2(\beta + D)^5}, \\ Z_{im}^m &= R_{mm} [\mathbf{P}_\infty]_{mi}^m + \sum_{r \neq m} R_{mr} [\mathbf{P}_\infty]_{ri}^m \\ &\quad + R_{im} [\mathbf{P}_\infty]_{mm}^m + \sum_{r \neq m} R_{ir} [\mathbf{P}_\infty]_{rm}^m, \\ &\approx \frac{5A_{im}}{4(\beta + D)^3} + \sum_{r \neq m} \frac{A_{mr} A_{rm} A_{im}}{4(\beta + D)^5} + \sum_{r \neq m} \frac{A_{ir} A_{rm}}{4(\beta + D)^4}. \end{aligned}$$

Apart from the direct link A_{im} , feedforward (FF) motifs ($A_{ir} A_{rm} A_{im}$) also govern Z_{ii}^m and indirect pathways ($A_{mr} A_{rm} A_{im}$ and $A_{ir} A_{rm}$) also govern Z_{im}^m (Appendix D).

When dominant walks of length p exist between source-target pairs (i.e., $[\mathbf{A}^p]_{im} \gg [\mathbf{A}^q]_{im}$ for $q \neq p$), strength metric scaling reveals universal alignment patterns across input classes when we omit effects of other walks. By isolating these terms including dominant walks and omitting $(\beta + D)$ -independent coefficients, we obtain:

$$Z_{im} \sim Z_{im}^m \sim \frac{1}{(\beta + D)^{p+2}} [\mathbf{A}^p]_{im}, \quad (27)$$

$$R_{im} \sim P_{im}^m \sim \frac{1}{(\beta + D)^{p+1}} [\mathbf{A}^p]_{im}, \quad (28)$$

$$P_{im} \sim \frac{1}{(\beta + D)^p} [\mathbf{A}^p]_{im}, \quad (29)$$

$$P_{ii}^m \sim \frac{1}{(\beta + D)^{2p+1}} [\mathbf{A}^p]_{im}^2, \quad (30)$$

$$Z_{ii}^m \sim \frac{1}{(\beta + D)^{2p+2}} [\mathbf{A}^p]_{im}^2. \quad (31)$$

Three fundamental relationships emerge under the dominant walks case: (i) Constant-input amplification (Z_{im}) scales with noise-driven crosscovariance (Z_{im}^m), as Eq. (27) shows; (ii) Peak responses under constant (R_{im}), amplification under pulse and unit square inputs all map to noise-driven crosscovariance peaks (P_{im}^m), as Eq. (28) shows; (iii) Impulse-response peaks (P_{im}) serve as natural reference units ($\sim O(1)$), with metric differences arising solely through $1/(\beta + D)$ scaling (Eq. (29)). These relationships reveal universal walk-length-dependent scaling underlying transient dynamics across different inputs.

C. Heterogeneous in-degree

Heterogeneous in-degree distributions are a ubiquitous feature of real-world networks, spanning biological, transportation, and social systems [79–82]. In large-scale settings, global dynamical patterns can be captured by input-specific response profiles shaped by localized structural features [1, 4, 17, 18]. However, such heterogeneity induces asymmetries in signal propagation that mean-field approximations cannot capture accurately, particularly in finite-size networks. To systematically quantify structural heterogeneities, we develop a generalized expansion for arbitrary in-degree distributions. This framework naturally yields walk-length-decomposed solutions:

$$\begin{aligned} [\mathbf{H}^{-k}]_{im} &= (-1)^k \sum_{w \in \mathcal{W}(m \rightarrow i)} h_w^k \mathcal{A}_w, \\ [\mathbf{P}_\infty]_{ij}^m &= I_0 \sum_{\substack{w \in \mathcal{W}(m \rightarrow i) \\ v \in \mathcal{W}(m \rightarrow j)}} p(w, v) \mathcal{A}_w \mathcal{A}_v, \end{aligned} \quad (32)$$

where $\mathcal{W}(m \rightarrow i)$ denotes the set of all walks from node m to node i (with at least one edge for $m \neq i$). The walk term $\mathcal{A}_w \equiv \prod_{t=0}^{n-1} A_{w_{t+1}w_t}$ corresponds to the product of edge weights along the walk $w = (w_0, w_1, \dots, w_n)$ with $w_0 = m$ and $w_n = i$, while the weight terms, for example, $h_w^1 \equiv 1/\prod_{v \in w} (\beta + D_v)$ for $\mathbf{H}^{-1}$ represents the product of $(\beta + D_v)$ terms over all node occurrences in $\mathcal{W}(m \rightarrow i)$ (including multiplicities for revisited nodes; other expressions of weight terms are shown in Appendix E). Although the degree configuration determines the weight

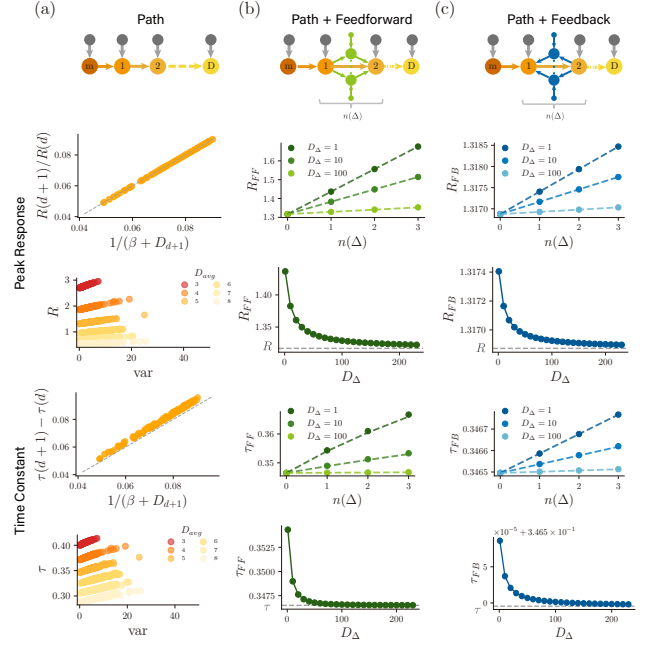


FIG. 5. **Constant input propagation along a single path under heterogeneous degree configurations and triangle motifs.** Constant input at the first node m of the single path produces propagation laws: $R(d+1)/R(d) = A_{d \rightarrow d+1}/(\beta + D_{d+1})$, and $(\tau(d+1) - \tau(d)) = 1/(\beta + D_{d+1})$, where d is the shortest path length. (a) Path-degree means reduce R, τ while variance enhances them. (b, c) Feedforward/feedback triangles amplify $R_{FF/FB}, \tau_{FF/FB} \propto n(\Delta)$ with distinct slopes, where $n(\Delta)$ represents the number of triangular motifs. Dots represent simulations, and dotted lines represent theory. Large triangle-node degrees D_Δ suppress motifs effects, recovering single-path dynamics R, τ (rows 2, 4, and $n(\Delta) = 1$). Note that the y-axis does not start from zero for better visualization of slope differences. Parameters: self-decay rate $\beta = 10$, total path length $D = 5$, input strength $I_0 = 10^6$, and unit weight along the chain $A_{d \rightarrow d+1} = 1$, for all d .

terms h_w^k and $p(w, v)$, the arrangement order of nodes along a walk does not affect weight values. The presence of nodes with strong walk centrality between m and i can substantially alter the weights assigned to different walks [83]. Overall, through Eq. (32), all metrics can be expressed as walk-based decompositions, with their weights determined by self-dynamics of the nodes involved.

Substitution of the expansion into Eqs. (4) and (5) establishes the following relationships under constant input along the setting of one individual directed walk:

$$\frac{R(d+1)}{R(d)} = \frac{A_{d \rightarrow d+1}}{\beta + D_{d+1}}, \quad (33)$$

$$\tau(d+1) - \tau(d) = \frac{1}{\beta + D_{d+1}}, \quad (34)$$

where both the multiplicative ratio between two sequential peak responses (which is always smaller than

1 in the NSDD setup) and the additive latency between two sequential time constant are modulated by the subsequent node's in-degree D_{d+1} . These relationship provide the basic laws of propagation rooted in the minimal structural complexity. Heterogeneous in-degrees thus maintain global scaling structure while enabling local modulation through nodal degrees along the directed walk. We also investigate self-responses ($i = m$; Appendix E), finding that deterministic metrics depend primarily on the self-degree D_m , while noise-driven cases incorporate stronger influences from other nodes, like reciprocal motifs.

Governed by propagation laws, signal propagation unfolds through walk-length iteration, with nodal in-degree configurations modulating its responses. Two key statistical effects emerge: (i) Increased mean in-degree $\langle D \rangle$ suppresses responses via degree-dependent damping, reducing both peak responses ($R \sim 1/(\beta + \langle D \rangle)^d$) and time constants ($\tau \sim 1/(\beta + \langle D \rangle)$); (ii) For fixed $\langle D \rangle$, heterogeneous degree distributions enhance signal propagation – increasing variance σ_D monotonically amplifies R and τ , with both metrics minimized exclusively at homogeneity ($\sigma_D = 0$) as shown in Fig. 5(a) and derived analytically in Appendix E. The universal role of degree heterogeneity across diverse systems is further discussed in Discussion (Sec. V).

To further investigate motif effects, we extend the baseline propagation laws for chains (Eqs. (33) and (34)) by incorporating additional motifs. In principle, the quantifiable influence of motifs of any order (i.e., with arbitrary numbers of edges) can be derived, since their contributions can always be decomposed into walk-based node-to-node propagation (Appendix E, Eq. (32)). For clarity, we highlight two representative cases, feedforward (FF) and feedback (FB) triangular motifs, to illustrate their distinct effects. As such,

$$\begin{aligned} R_{\text{FF}}(d) &= R(d) \left(1 + \frac{n(\Delta)}{\beta + D_\Delta} \right), \\ \tau_{\text{FF}}(d) &= \tau(d) + \frac{n(\Delta)}{(\beta + D_\Delta)(\beta + D_\Delta + n(\Delta))}, \\ R_{\text{FB}}(d) &= \frac{R(d)}{1 - n(\Delta)D_\Delta^\times}, \\ \tau_{\text{FB}}(d) &= \tau(d) + \frac{n(\Delta)D_\Delta^\times D_\Delta^+}{1 - n(\Delta)D_\Delta^\times}, \end{aligned}$$

where $n(\Delta)$ counts the number of triangular motifs, D_Δ denotes the assumed homogeneous in-degree at motif nodes (excluding chain nodes), with $D_\Delta^\times \equiv \prod_{k \in \Delta} (\beta + D_k)^{-1}$ and $D_\Delta^+ \equiv \sum_{k \in \Delta} (\beta + D_k)^{-1}$. Numerical validation (Fig. 5(b, c)) in the strong self-decay regime ($\beta + D_\Delta \gg 1$) confirms that time constants increase approximately linearly with $n(\Delta)$, with a slope scaling as $(\beta + D_\Delta)^{-2}$ for FF motifs, which is larger than the corresponding slope $\sim D_\Delta^\times D_\Delta^+$ for FB motifs under the same parameters. Similarly, peak responses scale proportionally with $n(\Delta)$, modulated by $R(d)(\beta + D_\Delta)^{-1}$

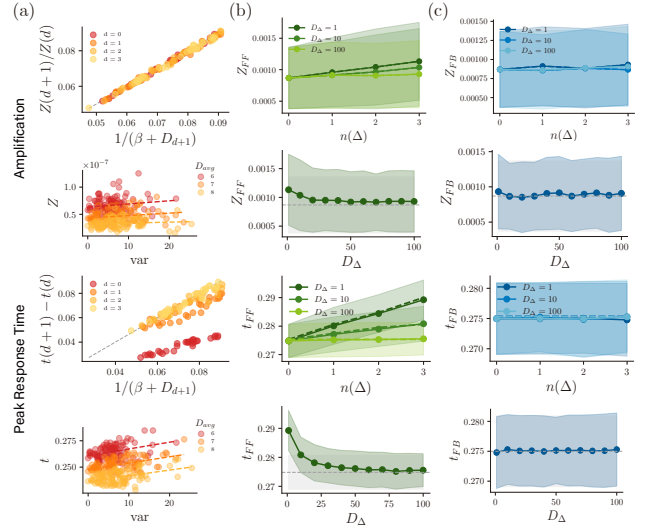


FIG. 6. White-noise input propagation along a single path under heterogeneous degree configurations and triangular motifs. A white-noise input applied to the first node m of the path produces propagation laws analogous to the constant-input case for crosscovariance between source and target pairs: $Z(d+1)/Z(d) \rightarrow A_{d \rightarrow d+1}/(\beta + D_{d+1})$ and $(t(d+1) - t(d)) \rightarrow 1/(\beta + D_{d+1})$ as $d \rightarrow \infty$, where d is the shortest path length. (a) Increasing the mean path degree reduces Z, t , while increasing variance enhances them. (b, c) Feedforward and feedback triangles amplify $Z_{\text{FF/FB}}$ and $t_{\text{FF/FB}}$ proportionally to $n(\Delta)$, the number of triangular motifs, but with distinct slopes (dashed: averages over 1000 realizations; shading: ± 1 SD). Large triangle-node degrees D_Δ suppress motif effects, recovering single-path dynamics Z, t (rows 2 and 4, $n(\Delta) = 3$). Compared with the impact of feedforward triangles ($\sim O(1/\beta^4)$), the impact of feedback triangles ($\sim O(1/\beta^6)$) grows more slowly and is obscured by fluctuations. Parameters: self-decay rate $\beta = 10$, total path length $D = 5$, input strength $I_0 = 100$, and unit chain weight $A_{d \rightarrow d+1} = 1$ for all d .

for FF motifs, which also yields larger values compared to $\sim R(d)D_\Delta^\times$ for FB motifs. The mechanistic divergence emerges through distinct physical walks: FF motifs introduce an additional effect via off-pathway nodes, while FB motifs enable signal amplification through coherent reinforcement along loops (Appendix E). Both effects are suppressed algebraically with increasing D_Δ through degree-dependent damping ($\sim D_\Delta^{-1}$ scaling), restoring baseline chain dynamics shown by $R(d)$ and $\tau(d)$.

The results for crosscovariance between source m and target i are qualitatively similar. Although the covariance computation retains information from preceding degrees, leading to a more intricate iterative form (Appendix E), it converges to Eqs. (33) and (34) for sufficiently long walk length. This convergence likewise reveals the amplifying effect of degree distribution variance to both strength and timing (Fig. 6(a)). Triangles also yield comparable effects, with amplification of both strength and timing as the number of triangles increases,

though the modest contribution of FB triangles is masked by fluctuations. Again, the jamming effect of large nodal in-degrees persists in this setting (Fig. 6(b, c)).

V. DISCUSSION

Summary

In summary, we establish a general framework quantifying transient network dynamics, answering two pivotal problems (Q1 and Q2) in heterogeneous settings, and also answering some of the important questions raised in Timme and Nagler 2019 [3], such as connection between deterministic and stochastic dynamics and full understanding of deterministic local dynamics in the general linear settings. In our work, analytical solutions derived from matrix inverses and spectral analysis directly link nodal response metrics, strength (amplification and peak response) and timing (time constants and response time), across different inputs. Second, systematic matrix expansions reveal structure-dependent governing principles of signal propagation: chain and sparse random networks obey universal scaling laws for strength (Eq. (24)) and temporal metrics (Eq. (25)); homogeneous in-degree networks exhibit distinct sensitivities to path lengths for strength and temporal metrics, with different effects across metrics emerging from two aspects: path-dominated propagation (Eqs. (27)–(31)) and motif-dominated propagation; analysis for heterogeneous in-degree configurations provides the most general case, further helping to map degree statistics ($\langle D \rangle$, σ_D) and motifs to response modulation. This framework enables the quantitative characterization of transient dynamics and provides design principles to optimize networks for signal propagation, highlighting both its predictive and explanatory power.

Relationship between metrics

In real systems, various inputs, ranging from deterministic stimuli to stochastic fluctuations, could act upon the same underlying systems, producing correspondingly diverse responses. This naturally motivates the intuition that unified response metrics and laws might exist across input types due to the same systems. However, such equivalences are mathematically nontrivial and remain uncharacterized for transient dynamics. Establishing cross-input metric relationships is therefore crucial for developing a general framework that quantifies intrinsic system properties. Our framework addresses this by identifying metric relationships across four input classes, organized into three aspects: deterministic properties, stochastic properties, and structural constraints.

(i) *Deterministic input relationships.* Under identical input amplitude (I_0) and location (m), there are mathematical equivalences between metrics for Eq. (1).

First, the constant-input response connects to the pulse-input response through exact temporal differentiation: $d\Delta x_i^{\text{const}}(t)/dt = \Delta x_i^{\text{pulse}}(t)$, directly linking the peak of constant input (Eq. (4)) to the pulse-response amplification (Eq. (7)). Second, square-input responses are truncated versions of constant-input responses, inheriting the same time constants during the relaxation period. The corresponding equivalences in the NSDD system (Eq. (2)) reveal operational correspondences between seemingly distinct metrics: the constant-input time constant (Eq. (5)) becomes operationally equivalent to the pulse-input peak response time (Eq. (11)). A special case emerges for square inputs with unit duration ($t_s = 1$), where dual metric equivalences occur: square and pulse amplifications achieve numerically estimation identity (Eq. (12) $\approx$ Eq. (7)), and peak responses of square and pulse inputs numerically converge through isomorphic temporal evolution (Eq. (13) $\approx$ Eq. (9)).

(ii) *Stochastic input relationships.* Under identical input amplitude ($I_0^{\text{deter}} = I_0^{\text{noise}}$) and location (m), relationships between deterministic inputs and stochastic inputs emerge from two organizing principles. First, homogeneous networks with dominant pathways exhibit direct stochastic-deterministic metric correspondence: cross-covariance amplification quantitatively matches constant-input amplification (Eqs. (27) and (28)), also corresponding to the principle in chain-structure (Eq. (24)). In heterogeneous settings, when the walk length is large enough, the iterative forms along a single walk are similar to the case under the constant input (Fig. 6). Second, first-order motif ($p = 1$) analysis establishes that (a) For direct propagation ($A_{im} \neq 0$), autocovariance peaks for white-noise inputs (Eq. (D6)) and deterministic responses (Eq. (D8)) follow direct pathway $m \rightarrow i$, while all other stochastic metrics (i.e., amplification and time constant for autocovariance $C_{ii}^m(\tau)$ (Eq. (D9))) are additionally governed by feedforward motifs ($m \rightarrow i$, $m \rightarrow r \rightarrow i$; Fig. 12), and metrics for crosscovariance $C_{ij}^m(\tau)$ (Eq. (D11)) are governed by the diverging (1,1) motif $m \rightarrow i, m \rightarrow j$, and also diverging (1,2) motifs, including $m \rightarrow r \rightarrow j, m \rightarrow i$ and $m \rightarrow r \rightarrow i, m \rightarrow j$. (b) For self-node responses ($m = i$), deterministic dynamics depend mainly on its in-degree D_i (Eqs. (E19), (E20)), whereas autocovariance $C_{ii}^i(\tau)$ (Eqs. (E23), (E24)) is additionally more sensitive to reciprocal motifs ($m \rightarrow k \rightarrow m$; Fig. 14).

Based on these relationships between inputs, the constant input makes it ideal for probing intrinsic system dynamics and extending theoretical results to other input types. For more intuitive understanding and practical usage of these types of inputs, please see Box. 2.

(iii) *Structural dependency relationships.* The relationships emerge across three network settings. Chain and sparse random networks exhibit universal scaling laws: strength metrics (amplification, peak response) follow geometric decay with path length (Eq. (24)), while temporal metrics (time constant, response time) scale linearly with path length (Eq. (25)). Homogeneous in-degree

networks operate through two regimes: a dominant-path regime, which unifies strength metrics across input classes (Eqs. (27)-(31)), and a motif-driven regime where deterministic and stochastic responses diverge due to distinct motif dominance (Fig. 12). Heterogeneous in-degree configurations exhibit dual statistical dependence: an increased mean degree $\langle D \rangle$ suppresses both peak response and time constant, while increased degree variance σ_D amplifies these metrics across inputs. Additional motifs on the walks could enhance both strength and timing in different ways (Figs. 5 and 6).

Role of degree heterogeneity

Unlike homogeneous networks where each node receives similar inputs, heterogeneity in degree configurations, such as those following a power-law distribution, plays a critical role in shaping the dynamical behaviors [1, 4, 79], functionality [13, 22, 84], robustness and resilience [15, 16, 80] of real systems. In such networks, hub nodes, though rare, have a significant influence on spreading processes and can either facilitate or suppress propagation dynamics [1, 4, 10, 17, 18, 82]. Our results demonstrate that in NSDD systems, hub nodes act as avert roles ($\sim D^{-1}$) in both strength domain and time domain, accelerating response decay (Eq. (33)) and suppressing the growth of time constants along propagation pathways (Eq. (34)). Simultaneously, in-degree heterogeneity, described by variance under fixed mean, amplifies signal propagation by enhancing both strength and timing, arising from local structure-dependent iteration.

Extension of framework

To maintain focus on generalizable principles, we strategically leave two aspects for future developments:

First, expanding the framework’s input variety analysis to oscillatory inputs and colored noise inputs is critical for modelling real-world signal processing, especially in neural systems. This framework can also be extended to multi-node input scenarios, particularly for convergent motifs where multiple sources project to a single target, providing insight into signal integration and causal inference [85–88]. Second, incorporating inhibitory connections can induce structure-dependent sign reversals in the system’s response trace; in particular, the positivity of the response trace is no longer guaranteed (see SM Fig. S35). Additionally, inhibition significantly influences the initial phase of the response, potentially leading to non-monotonic behaviors such as overshoots. These effects underscore the need for refined sensitivity metrics, *reactivity* indices, to assess whether impulse responses initially grow before decaying, especially in systems operating near the stability boundary [6, 7, 32, 74, 89].

Furthermore, the numerical accuracy of estimated metrics can still be enhanced for specific topolo-

gies. Chain networks exemplify that higher-order temporal refinements of impulse responses ($\tilde{\tau}_{im}^{(p)} = -[\mathbf{H}^{-(p+1)}]_{im}/[\mathbf{H}^{-p}]_{im}$) reduce estimation errors asymptotically as $p \rightarrow \infty$ (Appendix C). However, such refinements lack natural generalization to arbitrary topologies. Therefore, we retain the first-order estimator ($p = 1$), prioritizing consistent accuracy across tested networks (Fig. 2) and simple spectral interpretability through series expansions.

Although our framework centers on linearized dynamics near equilibrium, it suggests natural insights for extension to nonlinear dynamical behaviors. For example, in homogeneous in-degree networks, we observe that temporal metrics exhibit increased sensitivity to longer path lengths compared to strength metrics. This motivates extending degree-based mean-field approaches (DBMF) [4] to explicitly incorporate higher-order motif interactions (at least second-order) beyond first-order degree approximations [18]. Such extensions can establish more precise relationships between local topology features and collective dynamics, especially for temporal response properties.

Application

The analytical metrics derived from our framework for heterogeneous networks under various inputs closely match numerical benchmarks across canonical structures, and can be decomposed into interpretable topological components that can guide real-world network design. This renders the framework highly applicable across diverse domains.

(i) *Structural heterogeneity.* Our framework excels in finite-size networks that exhibit structural heterogeneity (e.g., asymmetric and weighted connectivity, local motifs) as found in coarse-grained multi-regional models of the mouse, primate, and human cortex, among others [8, 20, 22, 90]. Here, anatomical heterogeneity critically shapes functional states, where predicting localized transients is essential for linking structural connectivity to functional connectivity or effective connectivity during spontaneous and evoked activity [8, 20, 91, 92]. This capability proves crucial for networks with empirically observed features like hubs, inter-areal asymmetries, local motifs, and hierarchical gradients that govern directional propagation patterns [1, 20, 21, 93] and cognitive specialization [22, 71, 94]. Our framework enables a quantitative description of all these effects through the decomposition into diverse walks (Eq. (32)).

While traditional artificial network models often assume structural homogeneity or randomness [95], recent studies have demonstrated that even minor topological variations can directly impact deep learning performance [96]. Specifically, structural properties of a network’s line graph (e.g., high modularity, short average path length, and distinct spectral signatures) facilitate efficient parameter interactions and stable gradient flow,

thereby enhancing learning efficiency and generalization. Conversely, overly homogeneous or excessive hub dominant structures degrade performance through inefficient learning dynamics [96]. Our framework can help clarify how fine-grained structural heterogeneity directly influences model performance in artificial network models.

(ii) *Various setups of local inputs.* Our work could provide intuitive understanding and useful metrics across different setups of local inputs for assessing real-world systems. In neuroscience and synthetic biology, square inputs (on or off) or noise inputs probing at different nodes in the network are well suited for mapping connectivity and dynamics [92, 97–100]. In infrastructure and ecological systems, noise analysis provides insight into transient dynamics and system vulnerability to local disturbances [101, 102]. In social or behavioral interventions, impulse-like or short-square nudges on different units are used to assess immediate responses [103, 104]. These types of local inputs are mathematically linked within our framework, which also reveals how input location and amplitude interact with network structure to generate diverse transient responses and reflects the system’s vulnerability or resilience.

The emergence of these diverse and structured transient responses is well exemplified by the heterogeneous cortical networks in mammalian brains. Computational models based on empirically measured connectivity matrix have shown that the hierarchical cortical network exhibits a timescale hierarchy that is consistent with the experimental observations, regardless of input types. In addition, the information flow can be reconfigured with respect to the cortical hierarchy (e.g., sensory cortices are usually of low hierarchical order, while associative cortices’ are higher) depending on which cortical region receives the input. [19–22, 71, 73, 105–107].

(iii) *Strength-timing trade-off.* The response strength-timing relationship provides a basic design principle for network optimization (Eqs. (33) and (34)). Network structure has already shown distinct effects of signal propagation in time and strength profiles, exemplified by *balanced amplification* in neural systems: tuning of feed-forward or feedback excitation against local inhibition for stable, selective signal enhancement in cortical processing [20, 71, 108]. Analogous trade-offs guide recurrent neural network design for temporal tasks (sequential decisions, credit assignment), where connectivity modulation via gain or structured recurrence controls response latency, dynamic range, and noise robustness [109–111]. Strategic tuning of recurrent coupling strength in rate-based network models establishes an optimal balance between memory retention and signal decay, preserving short-term memory while maintaining dynamical stability [98]. Thus, this principle bridges biological computation and synthetic system design for temporal control, where our framework holds the potential to extend these insights to understand neural computations, as well as applications in social [112] and biological systems [113].

Appendix A: PROPERTIES OF NEGATIVE STRICTLY DIAGONALLY DOMINANT (NSDD) MATRICES

In the NSDD system, the linear matrix $\mathbf{H} \equiv \mathbf{A} - \mathbf{D} - \text{diag}(\beta_i)$ is strictly diagonally dominant with negative diagonal entries, ensuring all eigenvalues have negative real parts. Moreover, $-\mathbf{H}$ is an M -matrix, a structure with many desirable properties, as shown below.

Lemma 1 (All nodal dynamics are positive under positive pulse and constant inputs; under constant inputs, they increase monotonically). *The NSDD system exhibits positive activity across all nodes after positive inputs $I_0 > 0$: the time course $\Delta x_i^{\text{const}}(t)$ satisfies $\Delta x_i^{\text{const}}(t) > 0$ and $d\Delta x_i^{\text{const}}(t)/dt = \Delta x_i^{\text{pulse}}(t) > 0$ for all $i \in \{1, \dots, N\}$.*

Proof. Based on the definition, it is equivalently to prove that $\overline{\Delta x_i^{\text{const}}(t)} \equiv \Delta x_i^{\text{const}}(t) - \Delta x_i^{\text{const}}(\infty) = [e^{\mathbf{H}t} \Delta \mathbf{x}^{\text{const}}(0)]_i < 0$ and that $\overline{\Delta x_i^{\text{const}}(t)}$ increases monotonically.

First, we prove that the initial value $\overline{\Delta x_i^{\text{const}}(0)} = [\mathbf{H}^{-1} \mathbf{I}_0^{\text{const}}]_i < 0$, i.e., $[\mathbf{H}^{-1}]_{im} \leq 0$ for all i, m in the case of a single-node input to m with positive scalar I_0^{const} .

In the NSDD system, the matrix $-\mathbf{H}$ is a non-singular M -matrix, which can be expressed as $-\mathbf{H} = \mathbf{S} - \mathbf{B}$, where $\mathbf{B} = (b_{ij})$ with $b_{ij} \geq 0$. The diagonal matrix $\mathbf{S}$ satisfies $S_{ii} \geq \max |\lambda_i(\mathbf{B})|$, meaning each diagonal entry exceeds the largest absolute eigenvalue of $\mathbf{B}$. Since the inverse of a non-singular M -matrix is non-negative, we have $[\mathbf{H}^{-1}]_{im} \leq 0$ for all i, m .

Next, we prove that $\overline{\Delta x_i^{\text{const}}(t)} < 0$. The expression can be expanded as

$$\overline{\Delta x_i^{\text{const}}(t)} = \sum_{j=1}^N [e^{\mathbf{H}t}]_{ij} \overline{\Delta x_j^{\text{const}}(0)}.$$

Define $\mathbf{C} = \mathbf{H} + b\mathbf{I}_N$, where $b > \max\{|H_{ii}|\}$; then $[\mathbf{C}]_{ii} > 0$ for all i , and $[\mathbf{C}]_{ij} \geq 0$ for all $i \neq j$. Hence, all entries of $\mathbf{C}^n$ are non-negative for any $n \geq 0$, and there exists n^* such that for all $n > n^*$, $[\mathbf{C}^n]_{ij} > 0$. Thus,

$$[e^{\mathbf{C}t}]_{ij} = \sum_n \frac{t^n [\mathbf{C}^n]_{ij}}{n!} > 0.$$

Now,

$$\overline{\Delta x_i^{\text{const}}(t)} = \sum_{j=1}^N [e^{\mathbf{H}t}]_{ij} \overline{\Delta x_j^{\text{const}}(0)}, \quad (\text{A1})$$

$$= \sum_{j=1}^N [e^{(\mathbf{C}-b\mathbf{I}_N)t}]_{ij} \overline{\Delta x_j^{\text{const}}(0)}, \quad (\text{A2})$$

$$= \sum_{j=1}^N e^{-bt} [e^{\mathbf{C}t}]_{ij} \overline{\Delta x_j^{\text{const}}(0)} > 0. \quad (\text{A3})$$

Finally, note that $d\Delta x_i^{\text{const}}(t)/dt = \Delta x_i^{\text{pulse}}(t) > 0$ under the same input nodes and amplitudes, with the positivity of $\Delta x_i^{\text{pulse}}(t)$ established in [26]. This monotonic behavior ensures the uniqueness of the solution for the temporal metrics we defined. $\square$

Lemma 2 (Signatures of $\mathbf{H}$ at negative integer powers). *In the NSDD system, the matrix powers of $\mathbf{H}$ satisfy the following: $[\mathbf{H}^{-p}]_{km} \geq 0$ for all positive even integers p , and $[\mathbf{H}^{-p}]_{km} \leq 0$ for all positive odd integers p . Moreover, $[\mathbf{H}^{-p}]_{km} = 0$ indicates that an input at node m cannot reach node k .*

Proof. From Lemma 1, we know that $[\mathbf{H}^{-1}]_{km} \leq 0$ for all k, m . Then, we have:

$$\begin{aligned} [\mathbf{H}^{-2}]_{km} &= \sum_j [\mathbf{H}^{-1}]_{kj} [\mathbf{H}^{-1}]_{jm} \geq 0, \\ [\mathbf{H}^{-3}]_{km} &= \sum_j [\mathbf{H}^{-2}]_{kj} [\mathbf{H}^{-1}]_{jm} \leq 0, \\ &\vdots \end{aligned} \quad (\text{A4})$$

This pattern holds iteratively, establishing the sign structure of $\mathbf{H}^{-p}$ for all positive integers p .

Consider the Neumann series expansion

$$(\mathbf{I}_N - \mathbf{A})^{-1} = \sum_{q=0}^{\infty} \mathbf{A}^q, \quad (\text{A5})$$

which implies that if node k cannot be reached from node m , then $[\mathbf{A}^q]_{km} = 0$ for all q , since each power q represents walks of length q between nodes.

Given that $\mathbf{H} = \mathbf{A} - \mathbf{D} - \text{diag}_{i \in \{1, \dots, N\}}(\beta_i)$, the additional diagonal terms do not affect the connectivity between node pairs. Therefore, $[\mathbf{H}^{-1}]_{km} = 0$ if and only if node m cannot influence node k through the network. This property is preserved for any power of $\mathbf{H}$, and thus $[\mathbf{H}^{-p}]_{km} = 0$ for all p when no path exists from m to k . $\square$

Lemma 3 (Initial decrease of input node under pulse input). *In the NSDD system, the response of the input node initially decreases: the solution $x_m^{\text{pulse}}(t)$ decreases at the onset of a pulse input.*

Proof. We aim to show that $d\Delta x_m^{\text{pulse}}(t)/dt = [\mathbf{H}e^{\mathbf{H}t} \mathbf{I}_0^{\text{pulse}}]_m < 0$.

$$\frac{d}{dt} \Delta x_m^{\text{pulse}}(t) = \sum_j [\mathbf{H}e^{\mathbf{H}t}]_{mj} [\mathbf{I}_0]_j \quad (\text{A6})$$

$$= [\mathbf{H}e^{\mathbf{H}t}]_{mm} I_0^{\text{pulse}} = \sum_j [\mathbf{H}]_{mj} [e^{\mathbf{H}t}]_{jm} I_0^{\text{pulse}}. \quad (\text{A7})$$

Define $\mathbf{C} = \mathbf{H} + b\mathbf{I}_N$, where $b > \max\{|H_{ii}|\}$. Then $[\mathbf{C}]_{ii} > 0$ for all i and $[\mathbf{C}]_{ij} \geq 0$ for all $i \neq j$. Substituting

$\mathbf{H} = \mathbf{C} - b\mathbf{I}_N$, we get:

$$\begin{aligned} & \sum_j [\mathbf{H}]_{mj} [e^{\mathbf{C}t - b\mathbf{I}_N t}]_{jm} I_0^{\text{pulse}} \\ &= \sum_j e^{-bt} [\mathbf{H}]_{mj} [e^{\mathbf{C}t}]_{jm} I_0^{\text{pulse}} \\ &= I_0^{\text{pulse}} e^{-bt} \left([\mathbf{H}]_{mm} [e^{\mathbf{C}t}]_{mm} + \sum_{j \neq m} [\mathbf{H}]_{mj} [e^{\mathbf{C}t}]_{jm} \right). \end{aligned} \quad (\text{A8})$$

Since the first two terms are positive, we focus on the summation terms. Using the diagonal dominance of $\mathbf{H}$, we obtain:

$$\begin{aligned} & [\mathbf{H}]_{mm} [e^{\mathbf{C}t}]_{mm} + \sum_{j \neq m} [\mathbf{H}]_{mj} [e^{\mathbf{C}t}]_{jm} \\ & < \sum_{j \neq m} [\mathbf{H}]_{mj} \left([e^{\mathbf{C}t}]_{jm} - [e^{\mathbf{C}t}]_{mm} \right). \end{aligned} \quad (\text{A9})$$

Because $[\mathbf{H}]_{mj} > 0$, we examine the sign of the difference. Expanding $[e^{\mathbf{C}t}]_{jm}$ and $[e^{\mathbf{C}t}]_{mm}$ in Taylor series, we find that for small t , the leading order term of the difference is dominated by:

$$\frac{t^d [\mathbf{C}^d]_{jm}}{d!} - 1,$$

where d is the shortest path length from node m to node j . Therefore, for sufficiently small t , we have $t^d < \frac{d!}{[\mathbf{C}^d]_{jm}}$, implying the expression is negative. $\square$

Lemma 4 (Autocovariance and crosscovariance are positive). *If $\mathbf{H}$ is diagonalizable and both nodes i and j are reachable from the input source m , then in the NSDD system, both the autocovariance and crosscovariance satisfy $C_{ij}(\tau) = [e^{\mathbf{H}\tau} \mathbf{P}_\infty]_{ij} > 0$, for all i, j .*

Proof. To show that $[e^{\mathbf{H}\tau} \mathbf{P}_\infty]_{ij} > 0$, it suffices to prove that $\sum_k [e^{\mathbf{H}\tau}]_{ik} [\mathbf{P}_\infty]_{kj} > 0$. In the NSDD system, $[e^{\mathbf{H}\tau}]_{ik} \geq 0$, so we only need to prove that $[\mathbf{P}_\infty]_{kj} > 0$.

Since $\mathbf{H}$ is diagonalizable, we have:

$$\begin{aligned} [\mathbf{P}_\infty]_{kj} &= \int_{-\infty}^t [e^{\mathbf{H}(t-\tau)} \mathbf{Q} e^{\mathbf{H}^\top(t-\tau)}]_{kj} d\tau \\ &= \int_{-\infty}^t \sum_m [e^{\mathbf{H}(t-\tau)}]_{km} Q_{mm} [e^{\mathbf{H}^\top(t-\tau)}]_{mj} d\tau \\ &= Q_{mm} \int_{-\infty}^t \sum_m [e^{\mathbf{H}(t-\tau)}]_{km} [e^{\mathbf{H}(t-\tau)}]_{jm} d\tau, \end{aligned} \quad (\text{A10})$$

If $k = j$, this becomes $[\mathbf{P}_\infty]_{kk} = \int_{-\infty}^t [e^{\mathbf{H}(t-\tau)}]_{km}^2 d\tau > 0$. If $k \neq j$, then $[e^{\mathbf{H}(t-\tau)}]_{km} > 0$ for $(t-\tau) > 0$, ensuring positivity of the integral.

Note that the derivative of autocovariance is not always positive, implying non-monotonic decay in certain cases, particularly with strong interactions or feedback loops. The derivative is given by:

$$\begin{aligned} C'_{ii}(\tau) &= [\mathbf{H} e^{\mathbf{H}\tau} \mathbf{P}_\infty]_{ii}, \\ &= \sum_j [\mathbf{H}]_{ij} C_{ji}(\tau), \\ &= [\mathbf{H}]_{ii} C_{ii}(\tau) + \sum_{j \neq i} [\mathbf{H}]_{ij} C_{ji}(\tau), \\ &= \left(-\sum_{j \neq i} A_{ij} - \beta \right) C_{ii}(\tau) + \sum_{j \neq i} A_{ij} C_{ji}(\tau), \\ &= \sum_{j \neq i} A_{ij} (C_{ji}(\tau) - C_{ii}(\tau)) - \beta C_{ii}(\tau). \end{aligned} \quad (\text{A11})$$

This expression indicates that to ensure monotonic decay of $C_{ii}(\tau)$, one can either increase the self-decay rate β , or ensure $C_{ji}(\tau) < C_{ii}(\tau)$, meaning self-dynamics dominate. $\square$

Appendix B: DERIVATION OF METRICS ACROSS INPUTS

Constant input

In this subsection, we analyze the system's response to constant input (i.e., Heaviside step function), a typical and analytically tractable case often used in large-scale complex systems.

Assume the system is initially at steady state at $t = 0$, where $\mathbf{I}(0)$ satisfies $0 = \dot{\mathbf{x}}(0) = \mathbf{H}\mathbf{x}(0) + \mathbf{I}(0)$. Define variations from this steady state as $\Delta\mathbf{x}(t) \equiv \mathbf{x}(t) - \mathbf{x}(0)$ and $\Delta\mathbf{I}(t) \equiv \mathbf{I}(t) - \mathbf{I}(0)$, yielding the dynamics $\Delta\dot{\mathbf{x}}(t) = \mathbf{H}\Delta\mathbf{x}(t) + \Delta\mathbf{I}(t)$.

For constant input $\mathbf{I}_0^{\text{const}} \equiv \Delta\mathbf{I}(t)$, the system experiences a constant perturbation that eventually stabilizes to a final steady state $\Delta\mathbf{x}(\infty)$. By asymptotic stability,

$$0 = \Delta\dot{\mathbf{x}}(\infty) = \mathbf{H}\Delta\mathbf{x}(\infty) + \mathbf{I}_0^{\text{const}},$$

and the *peak response* at node i is

$$R_i \triangleq \Delta x_i(\infty) = -[\mathbf{H}^{-1} \mathbf{I}_0^{\text{const}}]_i. \quad (\text{B1})$$

Define $\overline{\Delta\mathbf{x}(t)} \equiv \Delta\mathbf{x}(t) - \Delta\mathbf{x}(\infty)$ as the deviation from final steady state. Then,

$$\dot{\overline{\Delta\mathbf{x}(t)}} = \mathbf{H}\overline{\Delta\mathbf{x}(t)},$$

with solution

$$\overline{\Delta x_i^{\text{const}}(t)} \equiv [e^{\mathbf{H}t} \overline{\Delta\mathbf{x}(0)}]_i, \quad (\text{B2})$$

where $\overline{\Delta\mathbf{x}(0)} = -\Delta\mathbf{x}(\infty) = \mathbf{H}^{-1} \mathbf{I}_0^{\text{const}}$.

Note: the form of $\overline{\Delta\mathbf{x}(t)}$ can be regarded as the evolution in the impulse response case (see Eq. (B11)), but

with multiple nonzero elements in $\overline{\Delta \mathbf{x}(0)}$, since each column of $\mathbf{H}^{-1}$ typically contains multiple nonzero entries due to the weak connectivity of the graph.

The evolution can also be expressed as $\overline{\Delta x_i^{\text{const}}(t)} = \sum_j (u_{im}^j e^{\lambda_j t} I_0) / \lambda_j$ in response to a scalar input I_0 at node m , provided that $\mathbf{H}$ is diagonalizable. Here, $u_{im}^j = [\mathbf{U}]_{ij} [\mathbf{U}^{-1}]_{jm}$, where $\mathbf{U}$ and $\mathbf{U}^{-1}$ are the eigenmatrix and its inverse, respectively. If $\mathbf{H}$ is a normal or real symmetric matrix, $\mathbf{U}^{-1}$ can be replaced by $\mathbf{U}^*$ or $\mathbf{U}^\top$, respectively.

Substituting $\overline{\Delta \mathbf{x}(0)}$ into Eq. (B2) and adding $\Delta \mathbf{x}(\infty)$ yields the full time-dependent activity at node i :

$$\Delta x_i^{\text{const}}(t) \equiv [\mathbf{H}^{-1}(e^{\mathbf{H}t} - \mathbf{I}_N) \mathbf{I}_0^{\text{const}}]_i. \quad (\text{B3})$$

In Fig. 7(a), we show the time course of a single node receiving constant input (red line), followed by the propagation of a constant input to other nodes along a directed chain, which responds more slowly and weakly (gray lines). To characterize temporal properties of $\Delta x_i^{\text{const}}(t)$ under constant input (Eq. (B3)), we introduce two temporal metrics: the *relative propagation time* t_i , defined by the relative threshold $\eta = \Delta x_i^{\text{const}}(t_i) / \Delta x_i^{\text{const}}(\infty)$ [4, 18], and the *absolute propagation time* $\tilde{t}_i$, defined by the absolute threshold $\tilde{\eta} = \Delta x_i^{\text{const}}(\tilde{t}_i)$.

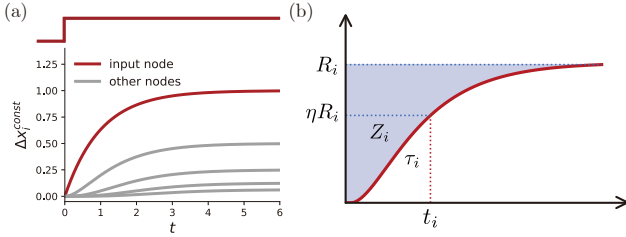


FIG. 7. Quantifying the impact of constant input on network dynamics. (a) Schematic showing the constant input and the resulting responses from the input node and other network nodes. The input node responds immediately with a stronger reaction, while other nodes respond more slowly and weakly. (b) Illustration of response metrics under constant input. Key metrics include peak response (R_i), amplification (Z_i), time constant (τ_i), and relative propagation time (t_i) at threshold ηR_i .

Thresholds η ($0 < \eta < 1$) and $\tilde{\eta}$ ($0 < \tilde{\eta} < R_i$) are chosen based on the timescale of interest: smaller values highlight early responses, while larger values capture slower, sustained dynamics. The half-maximum relative threshold ($\eta \sim \frac{1}{2}$) is commonly used in physics and chemistry, corresponding to the system's half-response time or half-period. The time constant, corresponding to a $(1/e)$ -fraction of the relative response, is also fundamental in physics, engineering, and neuroscience for describing convergence speed to steady state [19]. The absolute propagation time t_i is particularly relevant when the absolute threshold itself is meaningful, such as in the global workspace theory of consciousness [71, 108].

For the NSDD system, we prove that $\overline{\Delta x_i^{\text{const}}(t)}$ monotonically increases and remains negative under positive constant inputs. This confirms the positivity and monotonic growth of $\Delta x_i^{\text{const}}(t)$ (Appendix A). If negative entries A_{ij} are included, the traces may not remain positive. However, the relative response time can still be estimated as $\eta = |\Delta x_i^{\text{const}}(t_i)| / |\Delta x_i^{\text{const}}(\infty)|$, see Supplementary Material (SM) Sec. II.

Based on this, we approximate the activity variation of node i using a single exponential function with *time constant* τ_i :

$$\overline{\Delta x_i^{\text{const}}(t)} \approx [e^{-t/\tau_i} \overline{\Delta \mathbf{x}(0)}]_i. \quad (\text{B4})$$

The time constant is derived by integrating the response over time. The integral, termed the *response amplification*, is

$$Z_i \triangleq - \int_0^{+\infty} \overline{\Delta x_i^{\text{const}}(t)} dt = [\mathbf{H}^{-2} \mathbf{I}_0^{\text{const}}]_i, \quad (\text{B5})$$

and the time constant τ_i is approximated by:

$$\tau_i = - \frac{[\mathbf{H}^{-2} \mathbf{I}_0^{\text{const}}]_i}{[\mathbf{H}^{-1} \mathbf{I}_0^{\text{const}}]_i}. \quad (\text{B6})$$

The time constant τ_i characterizes how quickly the response evolves: for decaying $\overline{\Delta x_i^{\text{const}}(t)}$, it marks the time to reach $1/e \approx 36.8\%$ of the initial value $\overline{\Delta x_i^{\text{const}}(0)}$; for growing $\Delta x_i^{\text{const}}(t)$, it corresponds to reaching $(1 - 1/e)$ of the steady-state value $\Delta x_i^{\text{const}}(\infty)$.

The relative propagation time is:

$$t_i = -\tau_i \ln(1 - \eta), \quad (\text{B7})$$

and the absolute propagation time is:

$$\tilde{t}_i = -\tau_i \ln \left(1 - \frac{\tilde{\eta}}{R_i} \right). \quad (\text{B8})$$

The system's evolution can also be approximated by a sigmoid-shaped curve:

$$\overline{\Delta x_i^{\text{const}}(t)} \approx \left[\frac{2}{1 + e^{t/\tau_i^s}} \overline{\Delta \mathbf{x}(0)} \right]_i. \quad (\text{B9})$$

The sigmoid time constant is given by $\tau_i^s = \tau_i / (2 \ln 2)$. The corresponding relative and absolute propagation times are:

$$t_i^s = -\frac{\tau_i}{2 \ln 2} \ln \left(\frac{1 - \eta}{1 + \eta} \right),$$

$$\tilde{t}_i^s = -\frac{\tau_i}{2 \ln 2} \ln \left(\frac{\Delta x_i^{\text{const}}(\infty) - \tilde{\eta}}{\Delta x_i^{\text{const}}(\infty) + \tilde{\eta}} \right).$$

For a fixed η , both exponential and sigmoid models yield a relative propagation time proportional to the time constant τ_i . The absolute propagation time additionally depends on the peak response. The peak response is determined by the product of the input amplitude and

$\mathbf{H}^{-1}$, while the amplification results from multiplying the input amplitude by $\mathbf{H}^{-2}$. Their ratio gives the time constant. Fig. 7(b) summarizes these metrics for quantifying response dynamics under constant input. Numerical simulations across a broad range of network topologies and interaction weights are shown in SM Sec. II.

All the metrics receiving constant input at source m targeted with i are

$$\begin{aligned} Z_{im} &\triangleq [\mathbf{H}^{-2}]_{im} I_0^{\text{const}} = \sum_{j=1}^n \frac{u_{im}^j}{\lambda_j^2} I_0^{\text{const}} \sim O(1/\lambda_1^2), \\ R_{im} &\triangleq -[\mathbf{H}^{-1}]_{im} I_0^{\text{const}} = -\sum_{j=1}^n \frac{u_{im}^j}{\lambda_j} I_0^{\text{const}} \sim O(1/\lambda_1), \\ \tau_{im} &\triangleq -\frac{[\mathbf{H}^{-2}]_{im}}{[\mathbf{H}^{-1}]_{im}} = -\frac{\sum_{j=1}^n \frac{u_{im}^j}{\lambda_j^2}}{\sum_{j=1}^n \frac{u_{im}^j}{\lambda_j}} \sim O(1/\lambda_1), \\ t_{im} &\triangleq -\tau_{im} \ln(1 - \eta) \sim O(1/\lambda_1). \end{aligned} \quad (\text{B10})$$

Pulse input

Understanding the impulse response of a linear time-invariant system is essential for analyzing its dynamics. The system's output to any input can be constructed from its impulse response in LTI systems.

We first consider the case without external input $\mathbf{I}(t)$. With initial condition $\mathbf{x}_0 \triangleq \Delta \mathbf{x}(0)$ at $t = 0$, we analyze the response defined as $\Delta \mathbf{x}(t) \equiv \mathbf{x}(t) - \mathbf{x}(0)$. The dynamics follow $\Delta \dot{\mathbf{x}}(t) = \mathbf{H} \Delta \mathbf{x}(t)$ with solution $\Delta x_i^{\text{pulse}}(t) \equiv [e^{\mathbf{H}t} \mathbf{x}_0]_i$.

Alternatively, introducing an external input $\delta \mathbf{I}(t)$ modeled as a Dirac delta function, and defining $\mathbf{I}_0^{\text{pulse}} \equiv \int_0^\infty \delta \mathbf{I}(t) dt$, the response becomes:

$$\begin{aligned} \Delta x_i^{\text{pulse}}(t) &\equiv \left[\int_0^t \exp[\mathbf{H}(t - \tau)] \delta \mathbf{I}(\tau) d\tau \right]_i, \\ &= [e^{\mathbf{H}t} \mathbf{I}_0^{\text{pulse}}]_i. \end{aligned} \quad (\text{B11})$$

These two methods produce identical outputs when $\mathbf{x}_0 = \mathbf{I}_0^{\text{pulse}}$. In practice, since a Dirac delta is not implementable in simulation, we approximate it by applying $\mathbf{I}_0^{\text{pulse}}$ at the first time step.

Additionally, if $\mathbf{H}$ is diagonalizable, the response to an input I_0 at node m can also be written as:

$$\Delta x_i^{\text{pulse}}(t) = \sum_j u_{im}^j e^{\lambda_j t} I_0. \quad (\text{B12})$$

The complete time courses are shown in Fig. 8(a). The input node responds to a single pulse with immediate decay (Appendix A), while responses of other nodes increase first and decay slowly in the NSDD system. Although the full temporal evolution can be described analytically, calculating specific temporal metrics is challenging due to the presence of transcendental equations.

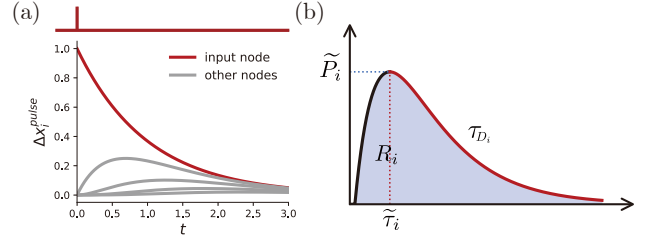


FIG. 8. Quantifying the impact of a pulse input on network dynamics. (a) Schematic showing the pulse input and resulting responses from the input node and other network nodes. The input node exhibits immediate decay from the initial input, whereas other nodes rise first and then decay, reaching lower peaks. (b) Illustration of response metrics for pulse input. Key metrics include the amended peak response ($\tilde{P}_i$), amplification (R_i), amended peak time ($\tilde{\tau}_i$), and decay time constant (τ_{D_i}).

Methods have been developed to characterize system responses using effective probability distributions [26, 27]. This approach normalizes the response into a probability distribution, providing a compact view of transient dynamics, briefly introduced below.

In the NSDD system, the response of each node to a positive input remains positive over time [26]. Normalizing the response trajectory by the total response strength, defined as *amplification*

$$R_i \triangleq \int_0^\infty \Delta x_i^{\text{pulse}}(t) dt = -[\mathbf{H}^{-1} \mathbf{I}_0^{\text{pulse}}]_i, \quad (\text{B13})$$

yields the *probability density*

$$\rho_i(t) \triangleq \frac{\Delta x_i^{\text{pulse}}(t)}{R_i}.$$

Peak response time is then defined as the expected value:

$$\tau_i \triangleq \int_0^\infty t \rho_i(t) dt = -\frac{[\mathbf{H}^{-2} \mathbf{I}_0^{\text{pulse}}]_i}{[\mathbf{H}^{-1} \mathbf{I}_0^{\text{pulse}}]_i}. \quad (\text{B14})$$

Notably, the expressions for Eqs. (B6) and (B14) are similar under the same input amplitude and location. For constant input, the time constant is estimated when the response reaches $(1 - 1/e)$ of the steady state; for pulse input, it estimates the time to peak response.

Additionally, the duration and amplitude of the response are characterized by the *standard deviation* σ_i and the *peak response* P_i . The peak response is defined as the ratio of amplification to standard deviation. In analogy to the normal distribution, where the total area is fixed at 1, a higher peak implies a narrower spread (shorter duration), and a lower peak implies a broader response.

$$\begin{aligned}
\sigma_i &\triangleq \sqrt{\int_0^\infty (t - \tau_i)^2 \rho_i(t) dt} \\
&= \sqrt{\frac{2 \left[\mathbf{H}^{-3} \mathbf{I}_0^{\text{pulse}} \right]_i}{\left[\mathbf{H}^{-1} \mathbf{I}_0^{\text{pulse}} \right]_i} - \left(\frac{\left[\mathbf{H}^{-2} \mathbf{I}_0^{\text{pulse}} \right]_i}{\left[\mathbf{H}^{-1} \mathbf{I}_0^{\text{pulse}} \right]_i} \right)^2}, \\
P_i &\triangleq \frac{R_i}{\sigma_i} \\
&= \frac{\left(\left[\mathbf{H}^{-1} \mathbf{I}_0^{\text{pulse}} \right]_i \right)^2}{\sqrt{2 \left[\mathbf{H}^{-3} \mathbf{I}_0^{\text{pulse}} \right]_i \left[\mathbf{H}^{-1} \mathbf{I}_0^{\text{pulse}} \right]_i - \left(\left[\mathbf{H}^{-2} \mathbf{I}_0^{\text{pulse}} \right]_i \right)^2}}. \tag{B15}
\end{aligned}$$

A refined method [27] improves accuracy by correcting estimation bias:

$$\begin{aligned}
\tilde{\tau}_i &\triangleq \tau_i + \frac{1}{\lambda_1}, \\
\tilde{P}_i &\triangleq \frac{\sqrt{d+1} d^d}{e^d d!} P_i = \frac{P_i}{\sqrt{2\pi}} + O(d^{-1}), \tag{B16}
\end{aligned}$$

where $\tilde{\tau}_i$ is the refined peak time, and $\tilde{P}_i$ is the refined peak response. The correction involves $\lambda_1 \equiv \max_j \text{Re}(\lambda_j) < 0$, which is the dominant eigenvalue of $\mathbf{H}$. The first expression in $\tilde{P}_i$ applies for small shortest path lengths d , while the asymptotic form applies for large d .

These corrections are based on the observation that the response approximates the form $\Delta x_i^{\text{pulse}}(t) = t^d e^{\lambda_1 t}$, where d is the shortest path length from the input node to node i . The rising phase scales with t^d , and the decay phase is governed by λ_1 . This form is similar to the alpha function seen in chain-like structures (Appendix C). The refined estimators are especially suited for weakly coupled systems, where the identical coupling strength α satisfies $\alpha/\lambda_1 \rightarrow 0$ [27].

Notably, the impulse response curve is often asymmetric: rising sharply to its peak and then decaying more gradually, typically following an exponential-form trend. To characterize this decay phase, we estimate a decay rate once the peak response and its timing have been identified. We define a *decay time constant* τ_{D_i} , approximating the descent using a single exponential function:

$$\int_{\tilde{\tau}_i}^{+\infty} [e^{\mathbf{H}t} \mathbf{I}_0^{\text{pulse}}]_i dt = \tilde{P}_i \int_0^{+\infty} e^{-t/\tau_{D_i}} dt,$$

which yields:

$$\tau_{D_i} = -\frac{[e^{\mathbf{H}\tilde{\tau}_i} \mathbf{H}^{-1} \mathbf{I}_0^{\text{pulse}}]_i}{\tilde{P}_i}.$$

This expression involves a matrix exponential and two estimated metrics $\tilde{\tau}_i$ and $\tilde{P}_i$, which is hard for analysis.

Additional insight can be gained by estimating the area under the impulse response from $\tilde{\tau}_i$ to infinity, approximating $(1 - 1/e)R_i$. This leads to a simplified estimate for the decay rate:

$$\tau_{D_i} = \left(1 - \frac{1}{e}\right) \frac{R_i}{\tilde{P}_i}. \tag{B17}$$

Three limitations for estimations might emerge, especially for the temporal metrics: (i) strong interactions may induce multi-peak responses in recurrent loop structures (See SM Fig. S13), (ii) hub-nonhub node pairs in scale-free networks can exhibit obvious asymmetric time course, which leads to biased estimations for peak response time (See SM Figs. S11, S14), and (iii) inhibitory connections may violate non-negativity assumptions (See SM Figs. S15, S16). However, these limitations mainly affect numerical accuracy but do not alter the response order.

All the metrics receiving pulse input at source m targeted with i are

$$R_{im} \triangleq -[\mathbf{H}^{-1}]_{im} I_0^{\text{pulse}} = -\sum_{j=1} \frac{u_{im}^j}{\lambda_j} I_0^{\text{pulse}} \sim O(1/\lambda_1),$$

$$\begin{aligned}
P_{im} &\triangleq C(d) \frac{([\mathbf{H}^{-1}]_{im})^2 I_0^{\text{pulse}}}{\sqrt{2[\mathbf{H}^{-3}]_{im} [\mathbf{H}^{-1}]_{im} - ([\mathbf{H}^{-2}]_{im})^2}} \\
&= C(d) \frac{\left(\sum_p \frac{u_{im}^p}{\lambda_p}\right)^2 I_0^{\text{pulse}}}{\sqrt{\sum_{p,q} \frac{u_{im}^p u_{im}^q}{\lambda_p^2 \lambda_q (2\lambda_p - \lambda_q)}}} \sim O(1),
\end{aligned}$$

$$\tau_{D_{im}} \triangleq \left(1 - \frac{1}{e}\right) \frac{R_{im}}{\tilde{P}_{im}} \sim O(1/\lambda_1),$$

$$\tilde{\tau}_{im} \triangleq -\frac{[\mathbf{H}^{-2}]_{im}}{[\mathbf{H}^{-1}]_{im}} + \frac{1}{\lambda_1} = -\frac{\sum_j \frac{u_{im}^j}{\lambda_j^2}}{\sum_j \frac{u_{im}^j}{\lambda_j}} + \frac{1}{\lambda_1} \sim O(1/\lambda_1). \tag{B18}$$

Square input

Square inputs are commonly used for their simplicity and analytical tractability, especially in neuroscience and image processing. Conceptually, a square input is a truncated constant input. Assuming exponential evolution with the corresponding time constant (Eq. (B6)), the *peak response* is estimated as:

$$R_i C(t_s, \tau_i) \triangleq -(1 - e^{-t_s/\tau_i}) [\mathbf{H}^{-1} \mathbf{I}_0^{\text{square}}]_i. \tag{B19}$$

As $t_s \rightarrow \infty$, $C(t_s, \tau_i) \rightarrow 1$.

Amplification combines two parts: the truncated constant-input phase and the decay phase, yielding:

$$R_i t_s \triangleq -t_s [\mathbf{H}^{-1} \mathbf{I}_0^{\text{square}}]_i. \tag{B20}$$

All the metrics are shown in Fig. 9(b).

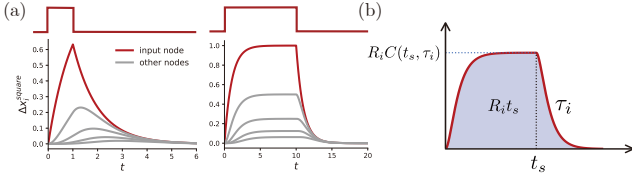


FIG. 9. Quantifying the impact of a square input on network dynamics. (a) Schematic showing the square input and corresponding responses from the input node and other nodes for both short and long durations. For short durations, the response shape resembles the impulse response in downstream nodes. (b) Illustration of response metrics for a square input with duration time t_s . Key metrics include the peak response ($R_i C(t_s, \tau_i)$), amplification ($R_i t_s$), and time constant (τ_i).

Under unit input amplitude and duration conditions ($I_0 = 1$, $t_s = 1$), amplification equals both the pulse-integrated amplification and the constant-input peak response R_i (Eqs. (B1), (B13)). Amplification increases linearly with t_s at rate R_i and remains robust across parameter regimes (See SM Fig. S17). As shown in Fig. 9(a), for short durations, the response closely resembles the impulse response. For unit-duration square input, peak responses match those of impulse input, excluding the input node (See SM Fig. S17).

Peak response accuracy may degrade under: weak interactions and long paths ($\alpha \ll \beta$; See SM Fig. S17(k), S18(d)); or short duration input ($t_s \ll \tau_{im}$), where non-exponential transients dominate [27].

Noise input

We consider white noise input in this subsection with zero mean and spectral density $\mathbf{Q}$. As white noise is discontinuous and unbounded, we adopt the Itô interpretation and rewrite the equation as $d\mathbf{x} = \mathbf{H}\mathbf{x}dt + d\boldsymbol{\beta}$, where $\mathbf{I} = d\boldsymbol{\beta}/dt$ and $\boldsymbol{\beta}$ is Brownian motion. The complete solution is given by:

$$\mathbf{x}^{\text{noise}}(t) = e^{\mathbf{H}t} \mathbf{x}^{\text{noise}}(0) + \int_0^t e^{\mathbf{H}(t-\tau)} d\boldsymbol{\beta}(\tau). \quad (\text{B21})$$

Taking expectations and covariances yields:

$$\begin{aligned} \mathbb{E}[\mathbf{x}(t)] &= e^{\mathbf{H}t} \mathbb{E}[\mathbf{x}(0)] \triangleq \mathbf{m}(t), \\ \mathbb{E}[(\mathbf{x}(t) - \mathbf{m}(t))(\mathbf{x}(t) - \mathbf{m}(t))^{\top}] &= e^{\mathbf{H}t} \mathbb{E}[(\mathbf{x}(0) - \mathbf{m}(0))(\mathbf{x}(0) - \mathbf{m}(0))^{\top}] e^{\mathbf{H}^{\top}t} \quad (\text{B22}) \\ &+ \int_0^t e^{\mathbf{H}(t-\tau)} \mathbf{Q} e^{\mathbf{H}^{\top}(t-\tau)} d\tau \triangleq \mathbf{P}(t). \end{aligned}$$

At steady state, we obtain:

$$\begin{aligned} \frac{d\mathbf{m}(t)}{dt} &= \mathbf{H}\mathbf{m}(t) = 0, \\ \frac{d\mathbf{P}(t)}{dt} &= \mathbf{H}\mathbf{P}(t) + \mathbf{P}(t)\mathbf{H}^{\top} + \mathbf{Q} = 0, \end{aligned} \quad (\text{B23})$$

yielding $\mathbf{m}_{\infty} = 0$ and the Lyapunov equation $\mathbf{H}\mathbf{P}_{\infty} + \mathbf{P}_{\infty}\mathbf{H}^{\top} + \mathbf{Q} = 0$, where $\mathbf{P}_{\infty}$ can be solved numerically.

Taking the limit $t \rightarrow \infty$, the stationary covariance becomes:

$$\mathbf{C}(\tau) \triangleq \mathbb{E}[\mathbf{x}(t)\mathbf{x}(t-\tau)] = \begin{cases} \mathbf{P}_{\infty} \exp(-\mathbf{H}\tau)^{\top}, & \tau \leq 0 \\ \exp(\mathbf{H}\tau) \mathbf{P}_{\infty}, & \tau > 0 \end{cases} \quad (\text{B24})$$

with $\mathbf{C}(\tau) = \mathbf{C}(-\tau)^{\top}$. The steady-state covariance $\mathbf{P}_{\infty}$ can also be expressed as $\int_{-\infty}^t e^{\mathbf{H}(t-\tau)} \mathbf{Q} e^{\mathbf{H}^{\top}(t-\tau)} d\tau$, or $\mathcal{L}^{-1}((s\mathbf{I}_N - \mathbf{H})^{-1} \mathbf{Q} (s\mathbf{I}_N - \mathbf{H}^{\top})^{-1})$ via inverse Laplace transform. The complete derivation of the stationary covariance can be found in [72].

If $\mathbf{Q} = \sigma^2 \mathbf{I}_N$ (i.e., uncorrelated white noise for all nodes) and $\mathbf{H}$ is normal ($\mathbf{H}\mathbf{H}^{\top} = \mathbf{H}^{\top}\mathbf{H}$), then $\mathbf{P}_{\infty} = -\sigma^2(\mathbf{H} + \mathbf{H}^{\top})^{-1}$. For a 1D system, the variance simplifies to $\mathbf{C}(\tau) = -Qe^{\lambda|\tau|}/(2\lambda)$.

Crosscovariance between input and activity can also be derived.

$$\tilde{\mathbf{C}}(s) \triangleq \mathbb{E}[\mathbf{x}(t)\mathbf{I}^{\top}(t-s)] = \begin{cases} e^{\mathbf{H}s} \mathbf{Q}, & s \geq 0, \\ 0, & s < 0. \end{cases} \quad (\text{B25})$$

When the input is applied only at node m , with strength I_0^{noise} , and $\mathbf{H}$ is diagonalizable, the (p, q) -th element of the covariance matrix in the eigendecomposition is given by:

$$\begin{aligned} C_{pq}^m(\tau) &\triangleq \mathbb{E}[x_p(t)x_q(t-\tau)] \\ &= \begin{cases} -\sum_j \sum_k \frac{u_{pm}^j u_{qm}^k}{\lambda_j + \lambda_k} e^{-\lambda_k \tau} I_0^{\text{noise}}, & \tau \leq 0, \\ -\sum_j \sum_k \frac{u_{pm}^j u_{qm}^k}{\lambda_j + \lambda_k} e^{\lambda_j \tau} I_0^{\text{noise}}, & \tau > 0. \end{cases} \end{aligned} \quad (\text{B26})$$

The corresponding element of the crosscovariance matrix between input and activity is [68]:

$$\tilde{C}_{im}(s) \triangleq \mathbb{E}[x_i(t)I_m(t-s)] = \begin{cases} \sum_j u_{im}^j e^{\lambda_j s} I_0^{\text{noise}}, & s \geq 0, \\ 0, & s < 0. \end{cases} \quad (\text{B27})$$

For $s \geq 0$, the crosscovariance between the input and the activity as a function of lag s (Eq. (B27)) resembles the impulse response (Eq. (B12)) when input strength I_0 and location m are identical. This correspondence holds for LTI systems, allowing impulse-response-based metrics to be directly applied to crosscovariance analysis.

The diagonal entries of the covariance matrix (Eq. (B26)) represent autocovariance, which are even functions, attaining their maximum at zero lag by the Cauchy-Schwarz inequality. Assuming an exponential form with *peak response* $P_{ii} \triangleq [\mathbf{P}_{\infty}]_{ii}$ and decay governed by a *time constant* τ_{ii} , we write:

$$C_{ii}(\tau) \approx [P_{\infty}]_{ii} e^{-\tau/\tau_{ii}}. \quad (\text{B28})$$

This form characterizes the time constant of the autocovariance and, upon normalization, the autocorrelation. *Amplification* is defined as: $Z_{ii} \triangleq 2 \int_0^{\infty} C_{ii}(\tau) d\tau =$

$-2[\mathbf{H}^{-1}\mathbf{P}_\infty]_{ii}$, and the time constant is expressed as:

$$\tau_{ii} = -\frac{[\mathbf{H}^{-1}\mathbf{P}_\infty]_{ii}}{[\mathbf{P}_\infty]_{ii}}. \quad (\text{B29})$$

Relative response time for autocovariance is then:

$$\bar{t}_{ii} = -\tau_{ii} \ln \bar{\eta}, \quad (\text{B30})$$

where $\bar{\eta} = \frac{C_{ii}(\bar{t}_i)}{C_{ii}(0)}$. While the structure of $\mathbf{P}_\infty$ is implicit, its spectral form under input I_0^{noise} at node m yields more clear form:

$$Z_{ii}^m = 2 \int_0^\infty C_{ii}^m(\tau) d\tau = 2 \sum_j \sum_k \frac{u_{im}^j u_{im}^k}{\lambda_j + \lambda_k} \frac{1}{\lambda_j} I_0^{\text{noise}},$$

and the corresponding time constant:

$$\tau_{ii}^m = -\frac{\sum_j \sum_k \frac{u_{im}^j u_{im}^k}{\lambda_j + \lambda_k} \frac{1}{\lambda_j}}{\sum_j \sum_k \frac{u_{im}^j u_{im}^k}{\lambda_j + \lambda_k}}.$$

For off-diagonal elements of the covariance matrix (Eq. (B26)), $C_{ij}(\tau)$ is asymmetric. To characterize the temporal properties of crosscovariance, we apply the effective probability distribution framework by treating the response as a probability distribution. *Amplification* is defined as:

$$Z_{ij} \triangleq \int_{-\infty}^\infty C_{ij}(\tau) d\tau = -[\mathbf{P}_\infty \mathbf{H}^{-\top} + \mathbf{H}^{-1} \mathbf{P}_\infty]_{ij}. \quad (\text{B31})$$

We define the normalized *probability density* $\rho_{ij}(\tau) = C_{ij}(\tau)/Z_{ij}$, and estimate the *peak response time* via the expected value:

$$t_{ij} \triangleq \frac{[\mathbf{P}_\infty (\mathbf{H}^{-\top})^2]_{ij} - [\mathbf{H}^{-2} \mathbf{P}_\infty]_{ij}}{[\mathbf{P}_\infty \mathbf{H}^{-\top}]_{ij} + [\mathbf{H}^{-1} \mathbf{P}_\infty]_{ij}}. \quad (\text{B32})$$

Peak response is given by:

$$P_{ij} \triangleq \frac{\left([\mathbf{P}_\infty \mathbf{H}^{-\top}]_{ij} + [\mathbf{H}^{-1} \mathbf{P}_\infty]_{ij}\right)^2}{\sqrt{4 \left([\mathbf{P}_\infty (\mathbf{H}^{-\top})^3]_{ij} + [\mathbf{H}^{-3} \mathbf{P}_\infty]_{ij}\right) \left([\mathbf{P}_\infty \mathbf{H}^{-\top}]_{ij} + [\mathbf{H}^{-1} \mathbf{P}_\infty]_{ij}\right) - 2 \left([\mathbf{P}_\infty (\mathbf{H}^{-\top})^2]_{ij} - [\mathbf{H}^{-2} \mathbf{P}_\infty]_{ij}\right)^2}}. \quad (\text{B33})$$

For diagonalizable $\mathbf{H}$ with input applied at node m

with strength I_0^{noise} , the metrics simplify as:

$$Z_{ij}^m = \sum_{p,q} \frac{u_{ij}^m(p,q)}{\lambda_p \lambda_q} I_0^{\text{noise}}, \quad (\text{B34})$$

$$t_{ij}^m = -\frac{\sum_{p,q} u_{ij}^m(p,q) \left(\frac{1}{\lambda_p \lambda_q} \left(\frac{1}{\lambda_p} - \frac{1}{\lambda_q}\right)\right)}{\sum_{p,q} u_{ij}^m(p,q) \frac{1}{\lambda_p \lambda_q}}, \quad (\text{B35})$$

$$P_{ij}^m = \frac{\left(\sum_{p,q} u_{ij}^m(p,q) \frac{1}{\lambda_p \lambda_q}\right)^2 I_0^{\text{noise}}}{\sqrt{4 \sum_{p,q} u_{ij}^m(p,q) \left(\frac{1}{\lambda_p^3} + \frac{1}{\lambda_q^3}\right) \frac{1}{\lambda_p + \lambda_q} \sum_{p,q} u_{ij}^m(p,q) \frac{1}{\lambda_p \lambda_q} - 2 \left(\sum_{p,q} u_{ij}^m(p,q) \frac{1}{\lambda_p \lambda_q} \left(\frac{1}{\lambda_p} - \frac{1}{\lambda_q}\right)\right)^2}}, \quad (\text{B36})$$

where $u_{ij}^m(p,q) \equiv u_{im}^p u_{jm}^q = [\mathbf{U}]_{ip} [\mathbf{U}^{-1}]_{pm} [\mathbf{U}]_{jq} [\mathbf{U}^{-1}]_{qm}$. Compared to the explicit variance formula for one-dimensional systems, we include a correction term of

$1/\sqrt{2}$ for peak response estimation.

All the metrics receiving noise input at source m for describing autocovariance are

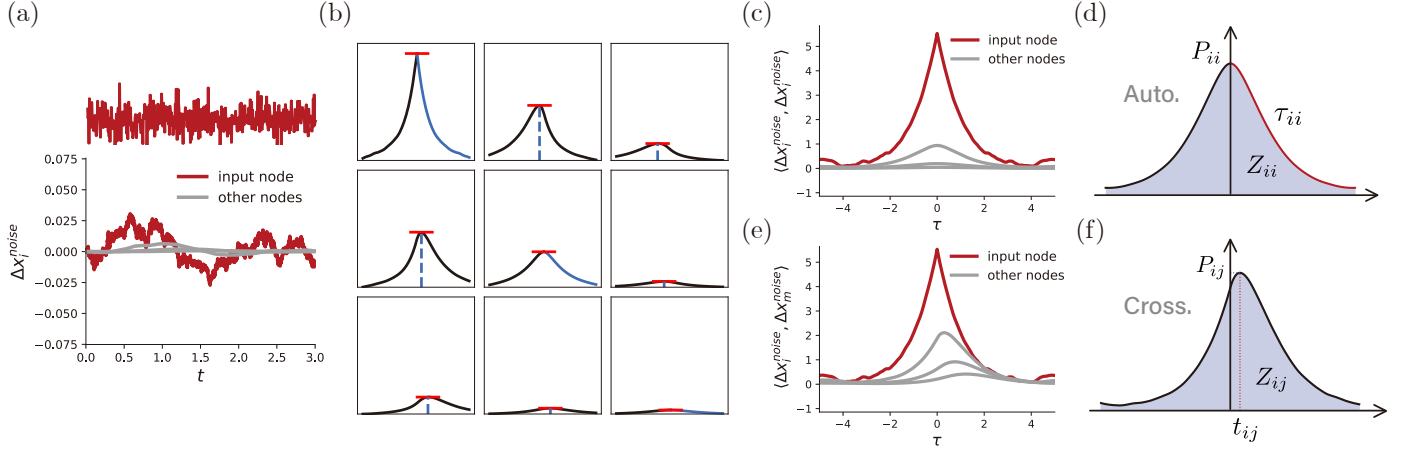


FIG. 10. Quantifying the impact of white noise input on network dynamics. (a) Noise input and resulting responses from the input node and other nodes, showing stochastic fluctuations. (b) Covariance matrix (3×3) with diagonal elements representing autocovariance and off-diagonal elements representing crosscovariance. (c,d) Autocovariance corresponds to diagonal entries in (b). Key metrics include peak response (P_{ii}), amplification (Z_{ii}), and time constant (τ_{ii}). (e,f) Crosscovariance corresponds to off-diagonal entries in (b). Key metrics include peak response (P_{ij}), amplification (Z_{ij}), and peak response time (t_{ij}).

$$\begin{aligned}
 Z_{ii}^m &\triangleq -2 [\mathbf{H}^{-1} \mathbf{P}_\infty]_{ii}^m = 2 \sum_{j,k} \frac{u_{im}^j u_{im}^k}{(\lambda_j + \lambda_k) \lambda_j} I_0^{\text{noise}} \sim O(1/\lambda_1^2), \\
 P_{ii}^m &\triangleq [\mathbf{P}_\infty]_{ii}^m = - \sum_{j,k} \frac{u_{im}^j u_{im}^k}{\lambda_j + \lambda_k} I_0^{\text{noise}} \sim O(1/\lambda_1), \\
 \tau_{ii}^m &\triangleq - \frac{[\mathbf{H}^{-1} \mathbf{P}_\infty]_{ii}^m}{[\mathbf{P}_\infty]_{ii}^m} = \frac{\sum_{j,k} \frac{u_{im}^j u_{im}^k}{(\lambda_j + \lambda_k) \lambda_j}}{\sum_{j,k} \frac{u_{im}^j u_{im}^k}{\lambda_j + \lambda_k}} \sim O(1/\lambda_1).
 \end{aligned} \tag{B37}$$

For crosscovariance, metrics are

$$\begin{aligned}
 Z_{ij}^m &\triangleq - [\mathbf{P}_\infty \mathbf{H}^{-\top} + \mathbf{H}^{-1} \mathbf{P}_\infty]_{ij}^m = \sum_{p,q} \frac{u_{ij}^m(p,q)}{\lambda_p \lambda_q} I_0^{\text{noise}} \sim O(1/\lambda_1^2), \\
 P_{ij}^m &\triangleq \frac{\left([\mathbf{P}_\infty \mathbf{H}^{-\top}]_{ij}^m + [\mathbf{H}^{-1} \mathbf{P}_\infty]_{ij}^m \right)^2}{\sqrt{4 \left([\mathbf{P}_\infty (\mathbf{H}^{-\top})^3]_{ij}^m + [\mathbf{H}^{-3} \mathbf{P}_\infty]_{ij}^m \right) \left([\mathbf{P}_\infty \mathbf{H}^{-\top}]_{ij}^m + [\mathbf{H}^{-1} \mathbf{P}_\infty]_{ij}^m \right) - 2 \left([\mathbf{P}_\infty (\mathbf{H}^{-\top})^2]_{ij}^m - [\mathbf{H}^{-2} \mathbf{P}_\infty]_{ij}^m \right)^2}}, \\
 &= \frac{\left(\sum_{p,q} u_{ij}^m(p,q) \frac{1}{\lambda_p \lambda_q} \right)^2 I_0^{\text{noise}}}{\sqrt{4 \sum_{p,q} u_{ij}^m(p,q) \left(\frac{1}{\lambda_p^3} + \frac{1}{\lambda_q^3} \right) \frac{1}{\lambda_p + \lambda_q} \sum_{p,q} u_{ij}^m(p,q) \frac{1}{\lambda_p \lambda_q} - 2 \left(\sum_{p,q} u_{ij}^m(p,q) \frac{1}{\lambda_p \lambda_q} \left(\frac{1}{\lambda_p} - \frac{1}{\lambda_q} \right) \right)^2}} \sim O(1/\lambda_1), \\
 t_{ij}^m &\triangleq \frac{[\mathbf{P}_\infty (\mathbf{H}^{-\top})^2]_{ij}^m - [\mathbf{H}^{-2} \mathbf{P}_\infty]_{ij}^m}{[\mathbf{P}_\infty \mathbf{H}^{-\top}]_{ij}^m + [\mathbf{H}^{-1} \mathbf{P}_\infty]_{ij}^m} = - \frac{\sum_{p,q} u_{ij}^m(p,q) \left(\frac{1}{\lambda_p \lambda_q} \left(\frac{1}{\lambda_p} - \frac{1}{\lambda_q} \right) \right)}{\sum_{p,q} u_{ij}^m(p,q) \frac{1}{\lambda_p \lambda_q}} \sim O(1/\lambda_1).
 \end{aligned} \tag{B38}$$

Appendix C: CHAIN STRUCTURE

Homogeneous directed chain

For the homogeneous directed chain, the matrix $\mathbf{H}$ takes the form:

$$\begin{pmatrix} -\beta & & & & \\ \alpha & -(\beta + \alpha) & & & \\ & \ddots & \ddots & & \\ & & \alpha & -(\beta + \alpha) & \end{pmatrix}, \quad (\text{C1})$$

where α denotes interaction weight and β the self-decay rate. The eigenvalues of $\mathbf{H}$ are $-\beta$ (with eigenvector $(1, 1, \dots, 1)$) and $-(\beta + \alpha)$ (with multiplicity $N - 1$ and eigenvector $(0, \dots, 0, 1)$).

The analytical time courses for unit pulse input is given by:

$$x_i(t) = \begin{cases} e^{-\beta t}, & d = 1, \\ e^{-\beta t} \left(1 - e^{-\alpha t} \sum_{j=0}^{d-2} \frac{(\alpha t)^j}{j!} \right), & d \geq 2. \end{cases} \quad (\text{C2})$$

The corresponding metric expressions are:

$$\begin{aligned} -[\mathbf{H}^{-1}]_{dm} &= \frac{\alpha^d}{(\alpha + \beta)^d \beta}, \\ [\mathbf{H}^{-2}]_{dm} &= \frac{\alpha^d (\alpha + (d+1)\beta)}{(\alpha + \beta)^{d+1} \beta^2}, \\ -\frac{[\mathbf{H}^{-2}]_{dm}}{[\mathbf{H}^{-1}]_{dm}} &= \frac{\alpha + (d+1)\beta}{\alpha \beta + \beta^2}, \\ &\frac{([\mathbf{H}^{-1}]_{dm})^2}{\sqrt{2[\mathbf{H}^{-3}]_{dm} [\mathbf{H}^{-1}]_{dm} - ([\mathbf{H}^{-2}]_{dm})^2}} \\ &= \frac{(\alpha + \beta)}{\sqrt{\alpha^2 + 2\alpha\beta + (d+1)\beta^2}} \left(\frac{\alpha}{\alpha + \beta} \right)^d, \\ [\mathbf{P}_\infty]_{dm}^m &= \frac{1}{2\beta} \left(\frac{\alpha}{2\beta + \alpha} \right)^d. \end{aligned} \quad (\text{C3})$$

Here, $m = 0$ denotes the first node of the chain, and $d = 1, 2, 3, \dots, N$ refers to the d -th node on the chain.

The directed chain serves as a minimal structure to reveal how metrics scale with path length d . The main

metrics are:

$$\begin{aligned} \ln R_{dm} &= -d \ln \left(1 + \frac{\beta}{\alpha} \right) - \ln \beta, \\ \ln P_{dm} &= -d \ln \left(1 + \frac{\beta}{\alpha} \right) - \frac{1}{2} \ln ((\alpha + \beta)^2 + d\beta^2) \\ &\quad + \ln(\alpha + \beta), \\ \ln Z_{dm} &= -d \ln \left(1 + \frac{\beta}{\alpha} \right) + \ln(\alpha + (d+1)\beta) \\ &\quad - \ln(\alpha + \beta) - 2 \ln \beta, \\ Z_{dm}^m &= \frac{1}{\beta^2} \left(1 + \frac{\beta}{\alpha} \right)^{-d} - \frac{1}{2\beta^2} \left(1 + \frac{2\beta}{\alpha} \right)^{-d}, \\ \ln [\mathbf{P}_\infty]_{dm}^m &= -\ln \left(1 + 2\frac{\beta}{\alpha} \right) d - \ln(2\beta), \\ \tau_{dm} &= \frac{1}{\alpha + \beta} d + \frac{1}{\beta}, \\ t_{dm}^m &= \frac{\frac{1}{2\beta^2} \frac{1}{\beta + \alpha} \left(\frac{\alpha}{\beta + \alpha} \right)^d \left(\left(2 + \frac{\alpha}{\beta} \right) d + \left(1 + \frac{\alpha}{\beta} \right) \left(\frac{\beta + \alpha}{2\beta + \alpha} \right)^d \right)}{\frac{1}{\beta^2} \left(\frac{\alpha}{\beta + \alpha} \right)^d - \frac{1}{2\beta^2} \left(\frac{\alpha}{2\beta + \alpha} \right)^d}. \end{aligned} \quad (\text{C4})$$

Through the metrics, we notice that the scaling of response strength (amplification and peak response) with path length follows approximately $(1 + \beta/\alpha)^{-d}$, and temporal metrics (time constant) basically scale as $d/(\alpha + \beta)$, especially when β dominants.

We define $S \equiv -\ln(1 + \frac{\beta}{\alpha})d$ and $T \equiv \frac{1}{\alpha + \beta}d$. We then perturb the identical weight α by $\Delta\alpha$ to become $\alpha + \Delta\alpha$, and find

$$\begin{aligned} |\Delta S| &= |S(\alpha + \Delta\alpha) - S(\alpha)| = S \left(\frac{\alpha}{\delta\alpha} \right), \\ |\Delta T| &= |T(\alpha + \Delta\alpha) - T(\alpha)| = T(\alpha)\delta\alpha, \end{aligned} \quad (\text{C5})$$

where $0 < \delta\alpha \equiv \left(1 + \frac{\alpha + \beta}{\Delta\alpha} \right)^{-1} < 1$. From these expressions, we conclude that:

1. As d increases, both $|\Delta S|$ and $|\Delta T|$ increase.
2. For $\alpha > \beta$ and $\beta > 1$, $|\Delta S| > |\Delta T|$ and $|\Delta S|/S \approx |\Delta T|/T = \delta\alpha$.
3. For $\beta > \alpha$, $|\Delta S| > |\Delta T|$ and $|\Delta S|/S > |\Delta T|/T = \delta\alpha$.

Alpha function

We notice that the excitatory postsynaptic potentials (EPSPs) modelling, both the amplitude (efficacy) and the temporal dynamics (time constant) of postsynaptic conductance changes, are linked with the chain structure. Interestingly, the evolution of EPSPs strongly resembles the impulse response, reflecting a fundamental aspect of

synaptic transmission. The conductance dynamics governed by the synaptic time constant τ are [114]:

$$\ddot{g} + \frac{2}{\tau}\dot{g} + \frac{1}{\tau^2}g = G_{\text{norm}}u(t), \quad (\text{C6})$$

which is equivalent to the two-dimensional system:

$$\begin{cases} \frac{dz}{dt} = -\frac{z}{\tau} + G_{\text{norm}}u(t), \\ \frac{dg}{dt} = -\frac{g}{\tau} + z(t), \end{cases} \quad (\text{C7})$$

with $z \equiv \frac{g}{\tau} + \dot{g}$. For a pulse input $u(t)$, the solution (alpha function) becomes:

$$g(t) = G_{\text{norm}} t e^{-t/\tau}, \quad \dot{g}(t) = G_{\text{norm}} \left(e^{-t/\tau} - \frac{t}{\tau} e^{-t/\tau} \right). \quad (\text{C8})$$

The peak of $g(t)$ occurs at $t = \tau$. If we set $G_{\text{norm}} = \frac{g_{\text{peak}}}{\tau/e}$, the peak response is $g(\tau) = G_{\text{norm}} \tau e^{-1} = g_{\text{peak}}$.

This system can be rewritten in compact form:

$$\begin{pmatrix} \dot{f} \\ \dot{g} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\tau} & 0 \\ 1 & -\frac{1}{\tau} \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}, \quad \begin{cases} f = \frac{c}{\tau} e^{-t/\tau}, \\ g = \frac{ct}{\tau} e^{-t/\tau}, \end{cases} \quad (\text{C9})$$

assuming a single pulse. Generalizing to an N -dimensional directed chain yields:

$$\dot{\mathbf{x}} = \begin{pmatrix} -(\beta + \alpha) & & & \\ & \alpha & \ddots & \\ & & \ddots & \ddots \\ & & & \alpha & -(\beta + \alpha) \end{pmatrix} \mathbf{x}, \quad (\text{C10})$$

with a solution for node d :

$$x_d(t) = \frac{\alpha^{d-1} t^{d-1} e^{-(\beta+\alpha)t}}{(d-1)!}, \quad (\text{C11})$$

where α denotes the interaction weight. This form closely resembles the response $\Delta x_i^{\text{pulse}}(t) = t^d e^{\lambda_1 t}$ described in [27], which is employed to amend the metrics for impulse responses.

High-order estimations

The primary difference in the solution for the homogeneous directed chain and the alpha function arises from the difference at the first node. Even in a simple two-node model, the resulting dynamics can differ markedly. In the first scenario, inspired by the alpha function, the matrix has repeated eigenvalues $\lambda_1 = \lambda_2 = -(\beta + \alpha)$ and is non-diagonalizable. Applying a pulse input to the first node leads to a peak response at the second node occurring at time $1/(\beta + \alpha)$. While the estimated metrics of time constant at any integer order k can be computed as:

$$-\frac{[\mathbf{H}^{-(k+1)}]_{im}}{[\mathbf{H}^{-k}]_{im}} = \left(\frac{k+1}{k} \right) \frac{1}{\alpha + \beta}. \quad (\text{C12})$$

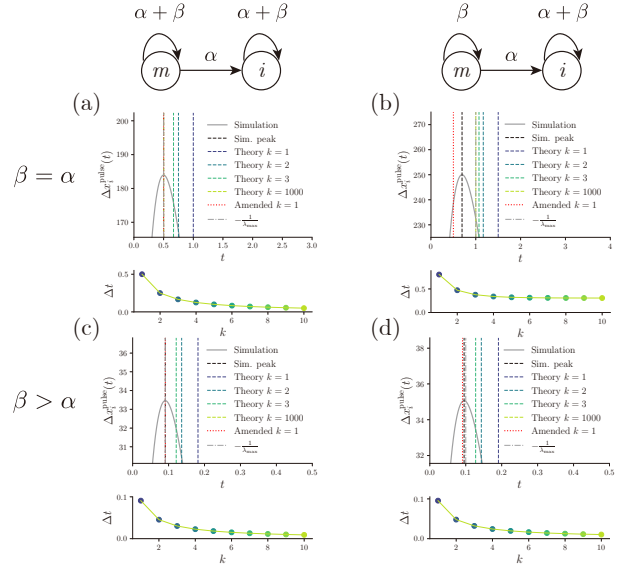


FIG. 11. Order dependence of estimated peak response time in two-node models. Panels (a) and (c) correspond to the alpha-function-inspired model (Eq. (C11)), while (b) and (d) depict the model based on Eq. (C2). The estimation follows the form $-\frac{[\mathbf{H}^{-(k+1)}]_{im}}{[\mathbf{H}^{-k}]_{im}}$, with the order k varied to assess accuracy. The refined estimator $\tilde{\tau}_i$ (Eq. (B16)) is also included. In (a) and (b), the self-decay $\beta = 1$ matches the interaction $\alpha = 1$, while in (c) and (d), $\beta = 10$ exceeds $\alpha = 1$. A pulse input is applied to the first node, and the response of the second node is observed. In (a), (c), and (d), higher-order k yields accurate estimates (Eqs. (C12) and (C13)), with both the refined estimate and $-1/\max_j \text{Re}(\lambda_j)$ aligning well with simulations. In (b), although increasing k improves accuracy, a notable bias persists relative to the simulated peak time.

This expression indicates that estimation improves with increasing order k (Fig. 11(a) and (c)).

In the second scenario, considering the homogeneous directed chain, the second node responds most strongly at time $-\ln(\beta/(\alpha + \beta))/\alpha$, which approximates $1/(\alpha + \beta)$ for large β . The corresponding estimate is:

$$-\frac{[\mathbf{H}^{-(k+1)}]_{im}}{[\mathbf{H}^{-k}]_{im}} = \frac{1}{\alpha + \beta} + \frac{\alpha}{\beta} \cdot \frac{1}{(\alpha + \beta) - \beta \left(\frac{\beta}{\alpha + \beta} \right)^{k-1}}. \quad (\text{C13})$$

As $k \rightarrow \infty$, the expression converges to $(1 + \alpha/\beta)/(\alpha + \beta) \approx 1/(\alpha + \beta)$ for large β (Fig. 11(d)). However, for strong interactions (large α), the theoretical estimations exhibit a consistent bias relative to simulation results (Fig. 11(b)), even when high orders are considered. This suggests that additional bias correction terms are needed to improve metric accuracy. It is worth noting that convergence to high accuracy with increasing order k holds in this simplified case, but does not necessarily extend to general network topologies. This is the main reason why we choose order $p = 1$, as we adhere to the principle of focusing on the generalizability of the metrics, and this choice naturally ensures the easy interpretability of the

metrics.

Appendix D: HOMOGENEOUS IN-DEGREE NETWORKS

Expansion for homogeneous in-degree

We firstly analyze the $\mathbf{H}^{-1}$ expansion under homogeneous in-degree conditions ($\mathbf{D} = D\mathbf{I}$) in NSDD systems. The course converges when

$$\max_{\lambda \in \sigma(\frac{\mathbf{A}-\mathbf{D}}{\beta})} |\operatorname{Re}(\lambda)| < 1, \quad (\text{D1})$$

where $\sigma(\cdot)$ denotes matrix spectrum. This requires all eigenvalues λ of $\mathbf{A} - \mathbf{D}$ to satisfy $|\operatorname{Re}(\lambda)| < \beta$. By Gershgorin's theorem, the spectral bound

$$\max_{\lambda \in \sigma(\mathbf{A}-\mathbf{D})} |\operatorname{Re}(\lambda)| \leq 2D_{\max} \quad (\text{D2})$$

holds for any adjacency matrix $\mathbf{A}$, where $D_{\max}$ is the maximum node degree. Thus, $\beta > 2D_{\max}$ provides a sufficient (non-necessary) convergence criterion. In the homogeneous setting, $\beta > 2D$ can guarantee the convergence.

The expansion is

$$\begin{aligned} \mathbf{H}^{-1} &= (\mathbf{A} - (\beta + D)\mathbf{I}_N)^{-1}, \\ &= -\sum_{p=0}^{\infty} \frac{\mathbf{A}^p}{(\beta + D)^{p+1}}. \end{aligned} \quad (\text{D3})$$

Cases for $\mathbf{H}^{-2}$ and $\mathbf{H}^{-3}$ are similar.

$$\begin{aligned} \mathbf{H}^{-2} &= \frac{1}{\beta^2} \sum_{p=0}^{\infty} \frac{p(\mathbf{A} - D\mathbf{I}_N)^{p-1}}{\beta^{p-1}}, \\ &= \sum_{p=0}^{\infty} \frac{(p+1)\mathbf{A}^p}{(\beta + D)^{p+2}}, \end{aligned} \quad (\text{D4})$$

and

$$\begin{aligned} \mathbf{H}^{-3} &= -\frac{1}{2\beta^3} \sum_{p=0}^{\infty} \frac{(p+1)(p+2)(\mathbf{A} - D\mathbf{I}_N)^p}{\beta^p}, \\ &= -\sum_{p=0}^{\infty} \frac{(p+1)(p+2)\mathbf{A}^p}{2(\beta + D)^{p+3}}. \end{aligned} \quad (\text{D5})$$

The expansions for $\mathbf{H}^{-1}$, $\mathbf{H}^{-2}$ and $\mathbf{H}^{-3}$ cover all metrics for deterministic inputs. Here, we do not consider the bias terms, as they are system-wide parameters primarily introduced to compensate for numerical inaccuracies.

To investigate metrics under noise inputs, we derive the expansion for the steady-state covariance:

$$\begin{aligned} [\mathbf{P}_{\infty}]_{ij}^m &= \frac{I_0}{2\pi} \int_{-\infty}^{\infty} [(\mathbf{H} - i\omega\mathbf{I}_N)^{-1}]_{im} [(\mathbf{H} + i\omega\mathbf{I}_N)^{-1}]_{jm} d\omega \\ &= I_0 \sum_{p,q=0}^{\infty} \frac{(p+q)!}{p!q!} \frac{[\mathbf{A}^p]_{im} [\mathbf{A}^q]_{jm}}{(2(\beta + D))^{p+q+1}} = [\mathbf{P}_{\infty}]_{ji}^m. \end{aligned} \quad (\text{D6})$$

And thus,

$$[\mathbf{P}_{\infty}]_{im}^m = [\mathbf{P}_{\infty}]_{mi}^m = I_0 \sum_{p=0}^{\infty} \frac{[\mathbf{A}^p]_{im}}{(2(\beta + D))^{p+1}}, \quad (\text{D7})$$

which is similar with Eq. (D3).

Next, we present expansions of metrics to identify the dominant terms for direct propagation ($d = 1, [\mathbf{A}]_{im} \neq 0$), including the first term ($p = 1$) for strength metrics and the first two terms ($p = 2$) for temporal metrics, under decay-dominant conditions ($\beta > 2D$), assuming unit input amplitude ($I_0 = 1$) for simplicity. We start from the metrics for deterministic inputs.

$$\begin{aligned} Z_{im} &\approx \frac{2[\mathbf{A}]_{im}}{(\beta + D)^3}, \quad R_{im} \approx \frac{[\mathbf{A}]_{im}}{(\beta + D)^2}, \\ \tau_{im} &\approx \frac{1}{\beta + D} \left(2 + \frac{1}{\frac{[\mathbf{A}]_{im}}{[\mathbf{A}^2]_{im}}(\beta + D) + 1} \right), \quad (\text{D8}) \\ P_{im} &\approx \frac{[\mathbf{A}]_{im}}{\sqrt{2}(\beta + D)}. \end{aligned}$$

From this expansion, we find that to increase the strength metrics (Z, R, P) while decreasing the temporal metric (τ), one can increase the direct link $[\mathbf{A}]_{im}$ and reduce $[\mathbf{A}^2]_{im}$ properly, without altering the in-degree D .

Next, we expand the amplification for noise inputs, and we start from the autocovariance:

$$\begin{aligned} Z_{ii}^m &= -2[\mathbf{H}^{-1}\mathbf{P}_{\infty}]_{ii}^m = 2R_{im}[\mathbf{P}_{\infty}]_{mi}^m + 2\sum_{r \neq m} R_{ir}[\mathbf{P}_{\infty}]_{ri}^m \\ &\approx \frac{[\mathbf{A}]_{im}^2}{2(\beta + D)^4} + \sum_{r \neq m} \frac{[\mathbf{A}]_{ir}[\mathbf{A}]_{rm}[\mathbf{A}]_{im}}{2(\beta + D)^5}. \end{aligned} \quad (\text{D9})$$

Here, $R_{im} \equiv -[\mathbf{H}^{-1}]_{im} = \sum_{p=0}^{\infty} ([\mathbf{A}^p]_{im} / (\beta + D)^{p+1})$ (Eqs. (B1) and (B13)). For the amplification ($i \neq m$), it includes the feedforward triangles: $[\mathbf{A}]_{ir}[\mathbf{A}]_{rm}[\mathbf{A}]_{im}$ (orange part of Z_{ii}^m in Fig. 12), where $[\mathbf{A}]_{im} \neq 0$ modulates the $m \rightarrow r \rightarrow i$ pathway.

Specially, the amplification of autocovariance for the source node m is

$$\begin{aligned} Z_{mm}^m &= -[\mathbf{H}^{-1}\mathbf{P}_{\infty}]_{mm}^m \\ &= R_{mm}[\mathbf{P}_{\infty}]_{mm}^m + \sum_{r \neq m} R_{mr}[\mathbf{P}_{\infty}]_{rm}^m, \quad (\text{D10}) \\ &\approx \frac{1}{2(\beta + D)^2} + \sum_{r \neq m} \frac{[\mathbf{A}]_{mr}[\mathbf{A}]_{rm}}{4(\beta + D)^4}. \end{aligned}$$

When nodes i and m overlap, the motif reduces to a reciprocal motif: $[\mathbf{A}]_{mr}[\mathbf{A}]_{rm}$ (orange part of Z_{mm}^m in Fig. 12).

For crosscovariance ($i \neq j \neq m$), the amplification metric is given by

$$\begin{aligned}
Z_{ij}^m &= -[\mathbf{H}^{-1}\mathbf{P}_\infty]_{ji}^m - [\mathbf{H}^{-1}\mathbf{P}_\infty]_{ij}^m = \sum_r R_{jr} [\mathbf{P}_\infty]_{ri}^m + \sum_r R_{ir} [\mathbf{P}_\infty]_{rj}^m, \\
&= R_{jm} [\mathbf{P}_\infty]_{mi}^m + \sum_{r \neq m} R_{jr} [\mathbf{P}_\infty]_{ri}^m + R_{im} [\mathbf{P}_\infty]_{mj}^m + \sum_{r \neq m} R_{ir} [\mathbf{P}_\infty]_{rj}^m, \\
&\approx \frac{[\mathbf{A}]_{im} [\mathbf{A}]_{jm}}{2(\beta + D)^4} + \sum_{r \neq m} \frac{[\mathbf{A}]_{jr} [\mathbf{A}]_{rm} [\mathbf{A}]_{im}}{4(\beta + D)^5} + \sum_{r \neq m} \frac{[\mathbf{A}]_{ir} [\mathbf{A}]_{rm} [\mathbf{A}]_{jm}}{4(\beta + D)^5}, \\
Z_{im}^m &= R_{mm} [\mathbf{P}_\infty]_{mi}^m + \sum_{r \neq m} R_{mr} [\mathbf{P}_\infty]_{ri}^m + R_{im} [\mathbf{P}_\infty]_{mm}^m + \sum_{r \neq m} R_{ir} [\mathbf{P}_\infty]_{rm}^m, \\
&\approx \frac{5[\mathbf{A}]_{im}}{4(\beta + D)^3} + \sum_{r \neq m} \frac{[\mathbf{A}]_{mr} [\mathbf{A}]_{rm} [\mathbf{A}]_{im}}{4(\beta + D)^5} + \sum_{r \neq m} \frac{[\mathbf{A}]_{ir} [\mathbf{A}]_{rm}}{4(\beta + D)^4}.
\end{aligned} \tag{D11}$$

Besides the diverging (1,1) motif $[\mathbf{A}]_{im} [\mathbf{A}]_{jm}$, Z_{ij}^m also contains diverging (1,2) motifs, including $[\mathbf{A}]_{jr} [\mathbf{A}]_{rm} [\mathbf{A}]_{im}$ and $[\mathbf{A}]_{ir} [\mathbf{A}]_{rm} [\mathbf{A}]_{jm}$ (orange part and yellow part respectively of Z_{ij}^m in Fig. 12). If $i = m$ or $j = m$, which means nodes j and m (or i and m) overlap, the motif simplifies to a direct link $[\mathbf{A}]_{im}$, a second-order chain $[\mathbf{A}]_{ir} [\mathbf{A}]_{rm}$, and a composite motif combining reciprocal and diverging (1,1) motifs: $[\mathbf{A}]_{mr} [\mathbf{A}]_{rm} [\mathbf{A}]_{im}$ (orange part of Z_{im}^m in Fig. 12).

The peak response time is

$$t_{ij}^m = \frac{[\mathbf{H}^{-2}\mathbf{P}_\infty]_{ij}^m - [\mathbf{H}^{-2}\mathbf{P}_\infty]_{ji}^m}{Z_{ij}^m}, \tag{D12}$$

which captures the asymmetry of the temporal profile between nodes i and j . A positive t_{ij}^m indicates that node i responds after node j , whereas a negative t_{ij}^m implies that i leads j . This directionality provides insight into the effective propagation sequence of activity driven by source node m .

The second-order term in the numerator can be written as

$$\begin{aligned}
[\mathbf{H}^{-2}\mathbf{P}_\infty]_{ij}^m &= \sum_k [\mathbf{H}^{-2}]_{ik} [\mathbf{P}_\infty]_{kj}^m, \\
&= [\mathbf{H}^{-2}]_{im} [\mathbf{P}_\infty]_{mj}^m + \sum_{k \neq m} [\mathbf{H}^{-2}]_{ik} [\mathbf{P}_\infty]_{kj}^m, \\
&= I_0 \sum_{p=0}^{\infty} \frac{(p+1) [\mathbf{A}^p]_{im}}{(\beta + D)^{p+2}} \sum_{q=0}^{\infty} \frac{[\mathbf{A}^q]_{jm}}{(2(\beta + D))^{q+1}} \\
&+ I_0 \sum_{k \neq m} \sum_{p=0}^{\infty} \frac{(p+1) [\mathbf{A}^p]_{ik}}{(\beta + D)^{p+2}} \sum_{q=0}^{\infty} \frac{(p+q)!}{p!q!} \frac{[\mathbf{A}^p]_{km} [\mathbf{A}^q]_{jm}}{(2(\beta + D))^{p+q+1}}, \\
&\approx I_0 \left(\frac{[\mathbf{A}]_{im} [\mathbf{A}]_{jm}}{2(\beta + D)^5} + \sum_{k \neq m} \frac{[\mathbf{A}]_{ik} [\mathbf{A}]_{km} [\mathbf{A}]_{jm}}{2(\beta + D)^6} \right).
\end{aligned} \tag{D13}$$

The sign of the peak response time t_{ij}^m is governed by the imbalance between $[\mathbf{H}^{-2}\mathbf{P}_\infty]_{ij}^m$ and $[\mathbf{H}^{-2}\mathbf{P}_\infty]_{ji}^m$ (imbalance between $[\mathbf{A}]_{jm} \sum_{k \neq m} [\mathbf{A}]_{ik}$ and $[\mathbf{A}]_{im} \sum_{k \neq m} [\mathbf{A}]_{jk}$

if we only care about $p = 1$ truncation), revealing the relative temporal ordering of nodal responses.

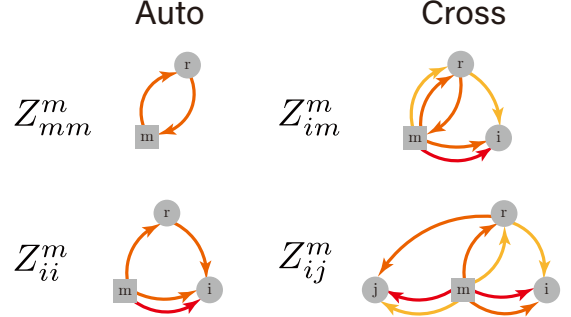


FIG. 12. Dominant motifs ($p = 1$) governing amplification for autocovariance (Z_{ii}^m) and crosscovariance (Z_{ij}^m) under direct propagation ($d = 1$, $[\mathbf{A}]_{im} \neq 0$), with colors representing different multiplicative terms. Nodes are labeled as: source (m), target (i for autocovariance, i and j for crosscovariance), and adjacent node (r).

Appendix E: HETEROGENEOUS IN-DEGREE NETWORKS

Expansion for heterogeneous in-degree

Previous analyses considered homogeneous in-degree configurations, but realistic networks exhibit heterogeneous in-degrees that modifies signal propagation. We now derive expansions for NSDD systems with arbitrary nodal in-degrees while maintaining uniform decay rates ($\beta_i = \beta$). The matrix elements governing deterministic responses admit generalized expansions:

$$\mathbf{H}^{-n} = (-1)^n \sum_{p=1}^{\infty} \left(\frac{C_{p+n-1}^{n-1}}{(\beta + D)^{p+n}} \right) \mathbf{A}^p \tag{E1}$$

We start from the expansion for $\mathbf{H}^{-1}$. For $i \neq m$, the element-wise expansion reveals degree-dependent expansion:

$$\begin{aligned}
[\mathbf{H}^{-1}]_{im} &= -\frac{1}{\beta^2} \left[\left(1 - \frac{D_i + D_m}{\beta} + \frac{D_i^2 + D_m^2 + D_i D_m}{\beta^2} - \dots \right) [\mathbf{A}]_{im} \right. \\
&\quad + \frac{1}{\beta} \left(1 - \frac{D_i + D_m}{\beta} + \dots \right) [\mathbf{A}^2]_{im} + \frac{1}{\beta^2} \left(1 - \frac{D_i + D_m}{\beta} + \dots \right) [\mathbf{A}^3]_{im} \\
&\quad \left. + \left(-\frac{1}{\beta^2} \right) \left(1 - \frac{D_i + D_m}{\beta} + \dots \right) [\mathbf{A} \mathbf{D} \mathbf{A}]_{im} + \frac{1}{\beta^3} \left(1 - \frac{D_i + D_m}{\beta} + \dots \right) [\mathbf{A} \mathbf{D}^2 \mathbf{A}]_{im} + \dots \right], \\
&= -\frac{1}{\beta^2} \sum_{d=0}^{\infty} \sum_{j_1, \dots, j_d \in \{m \rightarrow i\}} \frac{1}{\beta^d} \sum_{k=0}^{\infty} \frac{(-1)^k}{\beta^k} \left(\sum_{p_1 + \dots + p_{d+2} = k} D_{j_1}^{p_1} \dots D_m^{p_{d+2}} \right) A_{m \rightarrow j_1 \rightarrow \dots \rightarrow i}, \\
&= -\frac{1}{\beta^2} \sum_{d=0}^{\infty} \sum_{j_1, \dots, j_d \in \{m \rightarrow i\}} \frac{1}{\beta^d} \frac{A_{m \rightarrow j_1 \rightarrow \dots \rightarrow i}}{\left(1 + \frac{D_{j_1}}{\beta} \right) \dots \left(1 + \frac{D_{j_d}}{\beta} \right) \left(1 + \frac{D_i}{\beta} \right) \left(1 + \frac{D_m}{\beta} \right)}, \\
&= -\sum_{d=0}^{\infty} \sum_{j_1, \dots, j_d \in \{m \rightarrow i\}} \frac{A_{m \rightarrow j_1 \rightarrow \dots \rightarrow i}}{(\beta + D_{j_1}) \dots (\beta + D_{j_d}) (\beta + D_i) (\beta + D_m)}, \\
&\equiv -\sum_{w \in \mathcal{W}(m \rightarrow i)} h_w^1 \mathcal{A}_w,
\end{aligned} \tag{E2}$$

where $h_w^1 \equiv \prod_{v \in w} (\beta + D_i)^{-1}$, and $\mathcal{A}_w = A_{m \rightarrow j_1 \rightarrow \dots \rightarrow i} \equiv \prod_{t=0}^{k-1} A_{w_{t+1} w_t}$ corresponds to the product of edge weights along the walk $w = (w_0, w_1, \dots, w_k)$ with $w_0 = m$ and $w_k = i$.

Under homogeneous conditions, this reduces to:

$$[\mathbf{H}^{-1}]_{im} = -\sum_{d=0}^{\infty} \frac{[\mathbf{A}^{d+1}]_{im}}{(\beta + D)^{d+2}}, \tag{E3}$$

matching Eq. (D3).

The critical identity for these expansions is established via induction:

$$\sum_{k=0}^{\infty} \left(-\frac{1}{\beta} \right)^k \sum_{p_1 + \dots = k} D_1^{p_1} \dots D_n^{p_n} = \frac{1}{\left(1 + \frac{D_1}{\beta} \right) \dots \left(1 + \frac{D_n}{\beta} \right)}$$

Inductive step: Assume validity for n nodes. For $n+1$

nodes:

$$\begin{aligned}
&\sum_{k=0}^{\infty} \left(-\frac{1}{\beta} \right)^k \sum_{p_1 + \dots + p_{n+1} = k} D_1^{p_1} \dots D_n^{p_n} D_{n+1}^{p_{n+1}}, \\
&= \sum_{k=0}^{\infty} \left(-\frac{1}{\beta} \right)^k \sum_{p_{n+1}=0}^k D_{n+1}^{p_{n+1}} \sum_{p_1 + \dots + p_n = k - p_{n+1}} D_1^{p_1} \dots D_n^{p_n}, \\
&= \frac{1}{\left(1 + \frac{D_1}{\beta} \right) \dots \left(1 + \frac{D_n}{\beta} \right)} \left(1 - \frac{D_{n+1}}{\beta} + \frac{D_{n+1}^2}{\beta^2} + \dots \right), \\
&= \frac{1}{\left(1 + \frac{D_1}{\beta} \right) \dots \left(1 + \frac{D_n}{\beta} \right) \left(1 + \frac{D_{n+1}}{\beta} \right)}.
\end{aligned} \tag{E4}$$

The quadratic inverse operator exhibits diverse path weights:

$$\begin{aligned}
[\mathbf{H}^{-2}]_{im} &= \frac{1}{\beta^3} \left[\left(2 - \frac{3(D_i + D_m)}{\beta} + \dots \right) [\mathbf{A}]_{im} + \dots \right], \\
&= \frac{1}{\beta^3} \sum_{d=0}^{\infty} \sum_{j_1, \dots, j_d \in \{m \rightarrow i\}} \frac{1}{\beta^d} \sum_{k=0}^{\infty} \frac{(-1)^k}{\beta^k} (k + d + 2) \left(\sum_{p_1 + \dots + p_{d+2} = k} D_{j_1}^{p_1} \dots D_m^{p_{d+2}} \right) A_{m \rightarrow j_1 \rightarrow \dots \rightarrow i}, \\
&= \frac{1}{\beta^3} \sum_{d=0}^{\infty} \sum_{j_1, \dots, j_d \in \{m \rightarrow i\}} \frac{1}{\beta^d} \frac{(d+2) + (d+1) \left(\frac{D_{j_1}}{\beta} + \dots + \frac{D_{j_d}}{\beta} + \frac{D_i}{\beta} + \frac{D_m}{\beta} \right) + d \left(\frac{D_{j_1} D_{j_2}}{\beta^2} + \dots \right) + \dots}{\left(1 + \frac{D_{j_1}}{\beta} \right)^2 \dots \left(1 + \frac{D_{j_d}}{\beta} \right)^2 \left(1 + \frac{D_i}{\beta} \right)^2 \left(1 + \frac{D_m}{\beta} \right)^2} A_{m \rightarrow j_1 \rightarrow \dots \rightarrow i}, \\
&\equiv \sum_{w \in \mathcal{W}(m \rightarrow i)} h_w^2 \mathcal{A}_w,
\end{aligned} \tag{E5}$$

where

$$h_w^2 \equiv \frac{1}{\beta^{d+3}} \frac{\sum_{r=0}^{d+2} (d+2-r) \sum_{|T|=r}^{T \subset S} \prod_{t \in T} \frac{D_t}{\beta}}{\left(1 + \frac{D_{j_1}}{\beta}\right)^2 \cdots \left(1 + \frac{D_{j_d}}{\beta}\right)^2 \left(1 + \frac{D_i}{\beta}\right)^2 \left(1 + \frac{D_m}{\beta}\right)^2}.$$

Homogeneous reduction confirms consistency, match-

ing Eq. (D4):

$$[\mathbf{H}^{-2}]_{im} = \sum_{d=0}^{\infty} \frac{d+2}{(\beta+D)^{d+3}} [\mathbf{A}^{d+1}]_{im}. \quad (\text{E6})$$

The cubic operator is introduced as:

$$\begin{aligned} [\mathbf{H}^{-3}]_{im} &= -\frac{1}{\beta^4} \left[\left(3 - \frac{6(D_i + D_m)}{\beta} + \cdots \right) [\mathbf{A}]_{im} + \cdots \right], \\ &= -\frac{1}{\beta^4} \sum_{d=0}^{\infty} \sum_{j_1, \dots, j_d \in \{m \rightarrow i\}} \frac{1}{\beta^d} \sum_{k=0}^{\infty} \frac{(-1)^k (k+d+2)(k+d+3)}{\beta^k} \left(\sum_{p_1 + \dots + p_{q+2} = k} D_{j_1}^{p_1} \cdots D_m^{p_{q+2}} \right) A_{m \rightarrow j_1 \rightarrow \dots \rightarrow i}, \\ &\equiv -\frac{1}{\beta^4} \sum_{d=0}^{\infty} \sum_{j_1 \dots j_d \in \{m \rightarrow i\}} \frac{1}{\beta^d} \left(\frac{1}{2} h_{d+2} + \frac{2d+5}{2} g_{d+2} + \frac{(d+2)(d+3)}{2} f_{d+2} \right) A_{m \rightarrow j_1 \rightarrow \dots \rightarrow i}, \\ &\equiv - \sum_{w \in \mathcal{W}(m \rightarrow i)} h_w^3 \mathcal{A}_w, \end{aligned} \quad (\text{E7})$$

where

$$\begin{aligned} f_{d+2} &\triangleq \sum_{k=0}^{\infty} \left(-\frac{1}{\beta} \right)^k \left(\sum_{p_1 + \dots + p_{d+2} = k} D_{j_1}^{p_1} \cdots D_{j_d}^{p_d} D_i^{p_{d+1}} D_m^{p_{d+2}} \right), \\ &= \frac{1}{\left(1 + \frac{D_{j_1}}{\beta}\right) \cdots \left(1 + \frac{D_{j_d}}{\beta}\right) \left(1 + \frac{D_i}{\beta}\right) \left(1 + \frac{D_m}{\beta}\right)}, \\ g_{d+2} &\triangleq \sum_{k=0}^{\infty} \left(-\frac{1}{\beta} \right)^k k \left(\sum_{p_1 + \dots + p_{d+2} = k} D_{j_1}^{p_1} \cdots D_{j_d}^{p_d} D_i^{p_{d+1}} D_m^{p_{d+2}} \right), \\ &= \frac{\sum_{r=0}^{d+2} r \sum_{T \subset S = \{j_1, j_2, \dots, j_d, i, m\}} \prod_{t \in T} \frac{D_t}{\beta}}{\left(1 + \frac{D_{j_1}}{\beta}\right)^2 \cdots \left(1 + \frac{D_{j_d}}{\beta}\right)^2 \left(1 + \frac{D_i}{\beta}\right)^2 \left(1 + \frac{D_m}{\beta}\right)^2}, \\ h_{d+2} &\triangleq \sum_{k=0}^{\infty} \left(-\frac{1}{\beta} \right)^k k^2 \left(\sum_{p_1 + \dots + p_{d+2} = k} D_{j_1}^{p_1} \cdots D_{j_d}^{p_d} D_i^{p_{d+1}} D_m^{p_{d+2}} \right), \\ &= \frac{C_1 \tilde{h}_1 + C_2^2 \tilde{h}_2^2 + C_2^2 \tilde{h}_2^2 + C_3 \tilde{h}_3 + \cdots}{\left(1 + \frac{D_{j_1}}{\beta}\right)^3 \cdots \left(1 + \frac{D_{j_d}}{\beta}\right)^3 \left(1 + \frac{D_i}{\beta}\right)^3 \left(1 + \frac{D_m}{\beta}\right)^3}, \\ h_w^3 &\triangleq \frac{1}{\beta^{d+4}} \left(\frac{1}{2} h_{d+2} + \frac{2d+5}{2} g_{d+2} + \frac{(d+2)(d+3)}{2} f_{d+2} \right). \end{aligned} \quad (\text{E8})$$

Corresponding parameters are

$$\begin{aligned} C_k &= \frac{(d+2)C_{2d+2}^{k-2} - C_{2d+2}^{k-1}}{C_{d+1}^{\frac{k-1}{2}}}, C_k^1 = \left(\frac{k}{2} \right)^2, \\ C_k^2 &= \frac{(d+2)^2 C_{2d+2}^{k-2} - (d+2) C_{2d+2}^{k-1} - \left(\frac{k}{2} \right)^2 C_{d+2}^{\frac{k}{2}}}{C_{d+2}^{\frac{k}{2}-1} C_{d-\frac{k}{2}+3}^2}, \\ \tilde{h}_k &= \sum \frac{D_{j_1}^2 \cdots D_{j_{\frac{k}{2}-1}}^2 D_{j_{\frac{k}{2}+1}}^2}{\beta^k}, \tilde{h}_k^1 = \sum \frac{D_{j_1}^2 \cdots D_{j_{\frac{k}{2}}}^2}{\beta^k}, \\ h_k^2 &= \sum \frac{D_{j_1}^2 \cdots D_{j_{\frac{k}{2}-1}}^2 D_{j_{\frac{k}{2}}}^2 D_{j_{\frac{k}{2}+1}}^2}{\beta^k}. \end{aligned} \quad (\text{E9})$$

Homogeneous reduction confirms consistency, matching Eq. (D5):

$$[\mathbf{H}^{-3}]_{im} = - \sum_{d=0}^{\infty} \frac{(d+1)(d+2)}{2(\beta+D)^{d+3}} [\mathbf{A}^d]_{im}. \quad (\text{E10})$$

For noise input, the main part is to expand the steady-covariance matrix $\mathbf{P}_{\infty}$.

$$\begin{aligned} [\mathbf{P}_{\infty}]_{ij}^m &= [\mathbf{P}_{\infty}]_{ji}^m \\ &= \frac{I_0}{2\pi} \int_{-\infty}^{\infty} [(\mathbf{H} - (i\omega)\mathbf{I}_N)^{-1}]_{im} [(\mathbf{H} + (i\omega)\mathbf{I}_N)^{-1}]_{jm} d\omega. \end{aligned} \quad (\text{E11})$$

The corresponding expansion in the integral (See Eq. (E2)) is

$$\begin{aligned}
[(\mathbf{H} - i\omega)\mathbf{I}_N]^{-1}_{im} &= - \sum_{d_i=0}^{\infty} \sum_{i_1 \dots i_{d_i} \in \{m \rightarrow i\}} \frac{A_{m \rightarrow i_1 \rightarrow \dots \rightarrow i}}{(\beta + D_{i_1} + i\omega) \dots (\beta + D_{i_d} + i\omega) (\beta + D_i + i\omega) (\beta + D_m + i\omega)}, \\
[(\mathbf{H} + i\omega)\mathbf{I}_N]^{-1}_{jm} &= - \sum_{d_j=0}^{\infty} \sum_{j_1 \dots j_{d_j} \in \{m \rightarrow j\}} \frac{A_{m \rightarrow j_1 \rightarrow \dots \rightarrow j}}{(\beta + D_{j_1} - i\omega) \dots (\beta + D_{j_d} - i\omega) (\beta + D_j - i\omega) (\beta + D_m - i\omega)}.
\end{aligned} \tag{E12}$$

After substitution, the expansion will be

$$[\mathbf{P}_{\infty}]_{ij}^m = \frac{I_0}{2\pi} \int_{-\infty}^{\infty} \left[\sum_{d_i} \sum_{i_1, \dots, i_{d_i}} \frac{A_{m \rightarrow i_1 \rightarrow \dots \rightarrow i}}{(\beta + D_{i_1} + i\omega) \dots (\beta + D_m + i\omega)} \right] \left[\sum_{d_j} \sum_{j_1, \dots, j_{d_j}} \frac{A_{m \rightarrow j_1 \rightarrow \dots \rightarrow j}}{(\beta + D_{j_1} - i\omega) \dots (\beta + D_m - i\omega)} \right] d\omega. \tag{E13}$$

We employ Cauchy's residue theorem to evaluate this

integral of rational functions, following the approach outlined in [115].

$$\begin{aligned}
[\mathbf{P}_{\infty}]_{ij}^m &= I_0 \sum_{d_i, d_j} \sum_{i_1 \dots i_{d_i}} \sum_{j_1 \dots j_{d_j}} \left(\prod_{j_q} \frac{1}{2\beta + D_{i_p} + D_{j_q}} \prod_{\substack{i_k \\ k \neq p}} \frac{1}{D_{i_k} - D_{i_p}} \right) A_{m \rightarrow i_1 \rightarrow \dots \rightarrow i} A_{m \rightarrow j_1 \rightarrow \dots \rightarrow j} \\
&\equiv I_0 \sum_{\substack{w \in \mathcal{W}(m \rightarrow i) \\ v \in \mathcal{V}(m \rightarrow j)}} p(w, v) \mathcal{A}_w \mathcal{A}_v,
\end{aligned} \tag{E15}$$

where

$$\begin{aligned}
p(w, v) &\equiv -i \sum_{j_q} \text{Res}_{w \rightarrow -i(\beta + D_{j_q})} f(w), \\
&= \sum_{i_p} \left(\prod_{j_q} \frac{1}{2\beta + D_{i_p} + D_{j_q}} \prod_{\substack{i_k \\ k \neq p}} \frac{1}{D_{i_k} - D_{i_p}} \right), \\
&= \sum_{j_q} \left(\prod_{i_p} \frac{1}{2\beta + D_{i_p} + D_{j_q}} \prod_{\substack{j_k \\ k \neq q}} \frac{1}{D_{j_k} - D_{j_q}} \right), \\
f(w) &\equiv \prod_{i_p} \frac{1}{\beta + D_{i_p} + iw} \prod_{j_k} \frac{1}{\beta + D_{j_k} - iw}.
\end{aligned}$$

Although the expression may at first seem to require distinct degrees, since terms of the form $(D_{i_k} - D_{i_p})$ appear in the denominator, a closer look shows that these differences cancel once the full expression is reduced to a common denominator [115–117]. Intuitively, this means that the formula does not truly depend on the degrees being distinct. When two or more degrees coincide, the cancellation ensures that the expression remains well-

defined, and their effect is captured by the residue theorem through the multiplicity of the pole.

We compare the expansion for steady covariance matrix $\mathbf{P}_{\infty}$ under three-node motifs as an example: chain (denoted as $\mathbf{P}_{\infty}^{\text{chain}}$), feedforward triangles (denoted as $\mathbf{P}_{\infty}^{FF}$) and feedback triangles (denoted as $\mathbf{P}_{\infty}^{FB}$). For the chain structure, only one path goes through node 1 to node 2 from source 0:

$$[\mathbf{P}_{\infty}^{\text{chain}}]_{12}^0 = \frac{x_{01} + x_{12} + x_{02}}{x_{00}x_{01}x_{02}x_{11}x_{12}} [\mathbf{A}]_{10} [\mathbf{A}]_{10} [\mathbf{A}]_{21} I_0, \tag{E16}$$

where $x_{ij} \equiv 2\beta + D_i + D_j$.

When considering the feedforward triangle, the difference is that it additionally considers the other path:

$$[\mathbf{P}_{\infty}^{FF}]_{12}^0 = [\mathbf{P}_{\infty}^{\text{chain}}]_{12}^0 + \frac{x_{01} + x_{02}}{x_{00}x_{01}x_{02}x_{12}} [\mathbf{A}]_{10} [\mathbf{A}]_{21} I_0. \tag{E17}$$

For feedback triangles, we keep the dominant terms for simplicity:

$$[\mathbf{P}_{\infty}^{FB}]_{12}^0 \approx \frac{(x_{01} + x_{12} + x_{02})(D_1 - D_0) [\mathbf{A}]_{10} [\mathbf{A}]_{10} [\mathbf{A}]_{21} I_0}{(A_{\Delta} - x_{00}x_{01}x_{02})(A_{\Delta} - x_{10}x_{11}x_{12})}, \tag{E18}$$

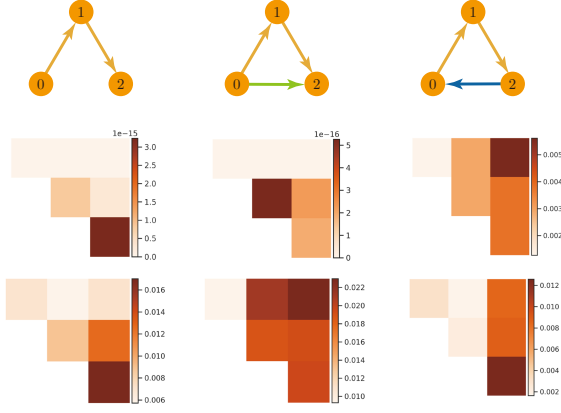


FIG. 13. Steady-state covariance expansions ($\mathbf{P}_\infty$) for three-node motifs (chain, feedforward, and feedback triangles), all with input at node 0: Top row shows motif schematics; middle row displays relative error between Lyapunov equation solutions and theoretical expansions; bottom row presents relative error between time-lag-zero covariance values averaged for 100 rounds and theoretical expansions. Parameters: $\beta = 10$, edge weights $A_{0 \rightarrow 1} = 1$, $A_{1 \rightarrow 2} = 2$, $A_{2 \rightarrow 1} = 4$.

where $A_\Delta \equiv [\mathbf{A}]_{10}[\mathbf{A}]_{21}[\mathbf{A}]_{02}$.

$$\begin{aligned}
 [\mathbf{H}^{-2}]_{ii} &= \frac{1}{(\beta + D_i)^2} + \\
 &\sum_{d=1}^{\infty} \sum_{j_1 \dots j_d \in \{i \rightarrow i\}} \frac{1}{\beta^{d+3}} \frac{(d+2) + (d+1) \left(\frac{D_{j_1}}{\beta} + \dots + \frac{D_{j_d}}{\beta} + \frac{D_i}{\beta} \right) + d \left(\frac{D_{j_1}}{\beta} \frac{D_{j_2}}{\beta} + \dots \right) + \dots}{\left(1 + \frac{D_{j_1}}{\beta} \right)^2 \dots \left(1 + \frac{D_{j_d}}{\beta} \right)^2 \left(1 + \frac{D_i}{\beta} \right)^2} A_{i \rightarrow j_1 \rightarrow \dots \rightarrow i}.
 \end{aligned} \tag{E20}$$

Expansion of the steady-state covariance matrix has

similar forms with Eq. (E15) under unit amplitude ($I_0 = 1$) for simplicity.

$$\begin{aligned}
 [\mathbf{P}_\infty]_{ii} &= \frac{1}{2(\beta + D_i)} + 2 \sum_{d_i} \sum_{j_1 \dots j_{d_j}} \sum_{j_k} \left(\frac{1}{2\beta + D_i + D_{j_k}} \prod_{\substack{j_q \\ q \neq k}} \frac{1}{D_{j_q} - D_{j_k}} \right) A_{i \rightarrow j_1 \rightarrow \dots \rightarrow i} \\
 &+ \sum_{d_i, d_j=1} \sum_{\substack{i_1 \dots i_{d_i} \\ j_1 \dots j_{d_j}}} \sum_{i_p} \left(\prod_{j_q} \frac{1}{2\beta + D_{i_p} + D_{j_q}} \prod_{\substack{i_k \\ k \neq p}} \frac{1}{D_{i_k} - D_{i_p}} \right) A_{i \rightarrow i_1 \rightarrow \dots \rightarrow i} A_{i \rightarrow j_1 \rightarrow \dots \rightarrow i}.
 \end{aligned} \tag{E21}$$

When the decay rate is dominant ($\beta \gg D_{\max}$) or the degree heterogeneity is large ($|D_i - D_j| \gg 0$), the summation terms become negligible compared to the first term. Therefore, retaining only the first term provides a good

approximation in the heterogeneous setting, especially

Fig. 13 presents numerical tests of steady-state covariance expansions ($\mathbf{P}_\infty$) for the motifs. Theoretical predictions closely match both Lyapunov solutions and time-lag-zero simulated covariances, with negligible relative errors, confirming the accuracy of the expansion.

Self-responses

In this subsection, we analyze the self-response case, where the source and target nodes coincide, i.e., $i = m$.

We begin with $\mathbf{H}^{-1}$:

$$\begin{aligned}
 [\mathbf{H}^{-1}]_{ii} &= [(\mathbf{A} - \mathbf{D} - \beta \mathbf{I}_N)^{-1}]_{ii}, \\
 &= -\frac{1}{\beta} \left[\left(\mathbf{I}_N + \frac{1}{\beta} \mathbf{D} \right)^{-1} + \frac{1}{\beta} \left(\mathbf{A} + \frac{1}{\beta} \mathbf{A}^2 + \dots \right) \right]_{ii}, \\
 &= -\left[\frac{1}{\beta + D_i} + \sum_{d=1}^{\infty} \sum_{j_1 \dots j_d \in \{i \rightarrow i\}} \frac{A_{i \rightarrow j_1 \rightarrow \dots \rightarrow i}}{(\beta + D_{j_1}) \dots (\beta + D_{j_d})} \right]_{ii}, \\
 &\equiv -\sum_{d=0}^{\infty} \sum_{j_1 \dots j_d \in \{i \rightarrow i\}} \frac{A_{i \rightarrow j_1 \rightarrow \dots \rightarrow i}}{(\beta + D_{j_1}) \dots (\beta + D_{j_d}) (\beta + D_i)}.
 \end{aligned} \tag{E19}$$

This result generalizes to $\mathbf{H}^{-2}$:

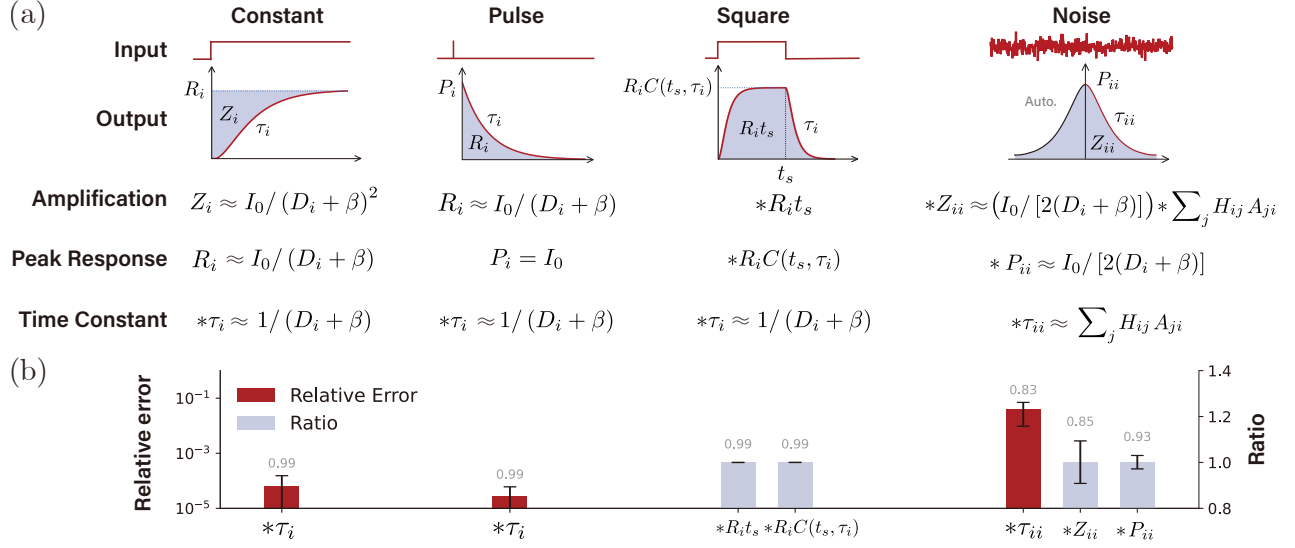


FIG. 14. Summary of self-response metrics for source node i under four input types. Metrics marked with an asterisk (*) are approximations derived from transcendental equations, with relative error $< 10\%$ and estimation ratio ≈ 1 . Gray values indicate rank correlations. All metrics are truncated at leading-order terms ($p = 1$). Parameters: $\beta = 10$, homogeneous weight $\alpha = 0.1$; results averaged over 100 realizations of Erdős-Rényi random networks with connection probability $p = 0.05$. Other numerical validations are presented in SM Sec. IV.

for the source propagating to its adjacency.

$$P_{ii}^i = [\mathbf{P}_\infty]_{ii}^i \approx \frac{I_0}{2(\beta + D_i)}. \quad (\text{E22})$$

The amplification is:

$$\begin{aligned} Z_{ii}^i &= -[\mathbf{H}^{-1} \mathbf{P}_\infty]_{ii} \\ &= R_{ii} [\mathbf{P}_\infty]_{ii}^i + \sum_{j \neq i} R_{ij} [\mathbf{P}_\infty]_{ji}^i \\ &\approx \frac{I_0}{2(\beta + D_i)^2} + \sum_{j \neq i} R_{ij} \sum_{d=0}^{\infty} \sum_{j_1, \dots, j_d \in \{i \rightarrow j\}} \sum_{j_q} \frac{1}{(2\beta + D_i + D_{j_q})} \prod_{\substack{j_h \\ h \neq q}} \frac{1}{D_{j_h} - D_{j_q}} A_{ji} I_0, \end{aligned} \quad (\text{E23})$$

where $R_{ij} \equiv -[\mathbf{H}^{-1}]_{ij}$ (see Eq. (B2)).

The corresponding time constant is:

$$\begin{aligned} \tau_{ii}^i &= \frac{Z_{ii}^i}{P_{ii}^i} \\ &\approx \frac{1}{\beta + D_i} + \sum_{j \neq i} R_{ij} \sum_{d=0}^{\infty} \sum_{j_1, \dots, j_d \in \{i \rightarrow j\}} \sum_{j_q \neq i} \frac{1}{(2\beta + D_i + D_{j_q})} \prod_{\substack{j_h \\ h \neq q}} \frac{1}{D_{j_h} - D_{j_q}} A_{ji}. \end{aligned} \quad (\text{E24})$$

If we restrict attention to the adjacency-level effects

($p = 1$ truncation), the metrics simplify to:

$$\begin{aligned} Z_{ii}^i &\approx \frac{I_0}{2(\beta + D_i)} \left(\frac{1}{\beta + D_i} + \sum_{j \neq i} \frac{R_{ij} [\mathbf{A}]_{ji}}{2\beta + D_i + D_j} \right) \\ &\equiv \frac{I_0}{2(\beta + D_i)} \sum_j A_{ji} H_{ij}, \end{aligned} \quad (\text{E25})$$

where $H_{ii} \equiv 1/(\beta + D_i)$ and $H_{ij} \equiv R_{ij}/(2\beta + D_i + D_j)$; $A_{ii} = 1$ and $A_{ji} = [\mathbf{A}]_{ji}$.

The time constant then becomes:

$$\begin{aligned} \tau_{ii}^i &\approx \frac{1}{\beta + D_i} + \sum_{j \neq i} \frac{R_{ij} [\mathbf{A}]_{ji}}{2\beta + D_i + D_j} \\ &= \sum_j A_{ji} H_{ij}, \end{aligned} \quad (\text{E26})$$

which highlights the importance of reciprocal motifs $[\mathbf{A}]_{ij} [\mathbf{A}]_{ji}$ in shaping the response. Fig. 14 summarizes self-response metrics for a source node under four input types. The results show that leading-order expansions ($p = 1$) already yield accurate estimates, with approximation errors below 10% and estimation ratios close to unity. Consistent rank correlations across metrics further support the reliability of the framework, with additional validations provided in SM Sec. IV. Notably, the first-order effect vanishes in simple graphs without self-loops, i.e., $[\mathbf{A}]_{ii} = 0$. This structure explains why temporal information seems to be well captured under the $p = 1$ truncation (as the contribution for $p = 2$ is zero), particularly in heterogeneous degree settings.

Iterative characterization under heterogeneous in-degree configurations

We analyze the constant-input propagation behavior along a single path under heterogeneous in-degree conditions and regard it as a baseline model due to its simplicity. The transient response to constant input along the path can be characterized by the following metrics:

$$\begin{aligned} Z(d) &= \frac{1}{\beta^{d+3}} \frac{A_{m \rightarrow \dots \rightarrow i} \sum_{r=0}^{d+2} (d+2-r) \sum_{|T|=r}^{T \subset S} \prod_{t \in T} \frac{D_t}{\beta}}{\left(1 + \frac{D_{j_1}}{\beta}\right)^2 \cdots \left(1 + \frac{D_i}{\beta}\right)^2 \left(1 + \frac{D_m}{\beta}\right)^2}, \\ R(d) &= \frac{1}{\beta^{d+2}} \frac{A_{m \rightarrow \dots \rightarrow i}}{\left(1 + \frac{D_{j_1}}{\beta}\right) \cdots \left(1 + \frac{D_i}{\beta}\right) \left(1 + \frac{D_m}{\beta}\right)}, \\ \tau(d) &= \frac{Z(d)}{R(d)}. \end{aligned} \quad (\text{E27})$$

By comparing the response metrics at path length d and $d+1$, we obtain the following iterative relations:

$$\begin{aligned} \frac{Z(d+1)}{Z(d)} &= \left(\frac{1}{\beta + D_{d+1}} + \frac{1}{(\beta + D_{d+1})^2} \frac{R(d)}{Z(d)} \right) A_{d \rightarrow d+1}, \\ &\rightarrow \frac{A_{d \rightarrow d+1}}{\beta + D_{d+1}}, d \rightarrow \infty, \\ \frac{R(d+1)}{R(d)} &= \frac{A_{d \rightarrow d+1}}{\beta + D_{d+1}}, \\ \tau(d+1) - \tau(d) &= \frac{1}{\beta + D_{d+1}}. \end{aligned} \quad (\text{E29})$$

This behavior which appears “memoryless” can be interpreted as an iterative process, depending only on the current layer d and the next layer $d+1$, with no regard for the preceding layers. The main modulation is governed by local variables D_{d+1} . In this view, the response metrics for a path of length d take the form without loss of generality assuming unit linkage between chain $A_{i \rightarrow i+1} = 1$ for all i :

$$\begin{aligned} R(D_1, \dots, D_d) &= f(D_1) \cdots f(D_d), \\ \tau(D_1, \dots, D_d) &= f(D_1) + \cdots + f(D_d), \end{aligned} \quad (\text{E30})$$

where $f(D) = 1/(\beta + D)$. Eq. (E30) suggests that a higher average degree $\langle D \rangle$ leads to lower response values R and τ .

In contrast, the case of crosscovariance under white-noise input differs slightly, as the covariance computation necessarily retains contributions from earlier degree information. However, these additional effects diminish as the path length grows, so that for sufficiently long paths the behavior converges to the same “memoryless” form.

$$\begin{aligned} Z_{\text{noise}}(d+1) &= Z_{\text{noise}}(d) \frac{A_{d \rightarrow d+1}}{\beta + D_{d+1}} + \frac{1}{2(\beta + D_m)(\beta + D_{d+1})} \prod_{v=0}^d \frac{A_{v \rightarrow v+1}}{2\beta + D_m + D_{v+1}}, \\ t_{\text{noise}}(d+1) - t_{\text{noise}}(d) &= \frac{1}{\beta + D_{d+1}} \frac{1}{\left(1 - \frac{p_{0 \rightarrow d+1}}{\sum_j^d p_{0 \rightarrow j} h_{j \rightarrow d}^1}\right)} + \frac{1}{\beta + D_{d+1}} \frac{1}{\left(1 - \frac{\sum_j^d p_{0 \rightarrow j} h_{j \rightarrow d}^1}{p_{0 \rightarrow d+1}}\right)} \\ &\quad + \frac{1}{\frac{\sum_j^d p_{0 \rightarrow j} h_{j \rightarrow d}^1}{p_{0 \rightarrow d+1}} - 1} \frac{\sum_j^d p_{0 \rightarrow j} h_{j \rightarrow d}^2}{\sum_j^d p_{0 \rightarrow j} h_{j \rightarrow d}^1}, \end{aligned} \quad (\text{E31})$$

where

$$\begin{aligned} p_{k \rightarrow j} &\equiv \prod_{v \in \{k \rightarrow j\}} (2\beta + D_m + D_v)^{-1}, \\ h_{k \rightarrow j}^1 &\equiv \prod_{v \in \{k \rightarrow j\}} (\beta + D_v)^{-1}, \\ h_{k \rightarrow j}^2 &\equiv \frac{1}{\beta^{d+3}} \frac{\sum_{r=0}^{j-k+1} (j-k+1-r) \sum_{|T|=r}^{T \subset S} \prod_{t \in T} \frac{D_t}{\beta}}{\left(1 + \frac{D_k}{\beta}\right)^2 \cdots \left(1 + \frac{D_j}{\beta}\right)^2}. \end{aligned} \quad (\text{E32})$$

Since $p_{0 \rightarrow d+1} / \sum_j^d p_{0 \rightarrow j} h_{j \rightarrow d}^1 \rightarrow 0$ as $d \rightarrow \infty$, $Z_{\text{noise}}(d+1) / Z_{\text{noise}}(d) \rightarrow A_{d \rightarrow d+1} / (\beta + D_{d+1})$ and $(t_{\text{noise}}(d+1) - t_{\text{noise}}(d)) \rightarrow (\beta + D_{d+1})^{-1}$ for sufficiently large d (see Fig. 6).

To isolate the effect of degree variance, we consider the case where the average degree is fixed:

$$D_1 + \cdots + D_d = d\bar{D}_d.$$

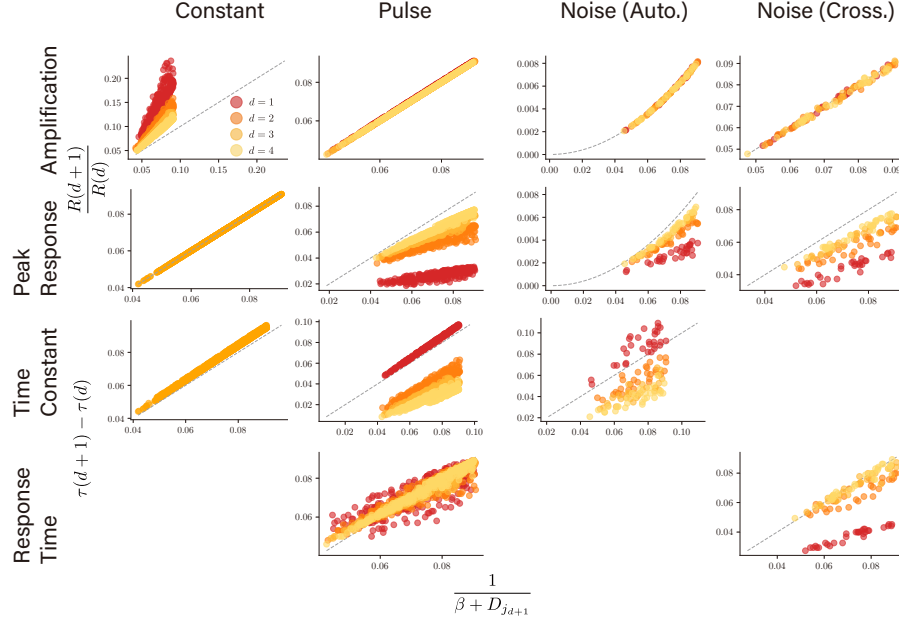


FIG. 15. Propagation laws for heterogeneous degree configurations along a single chain. Strength metrics (Z, R, P) follow the scaling relation $R(d+1)/R(d) \approx 1/(\beta + D_{d+1})$, particularly when d is large, apart from autocovariance, which obeys a squared form, $1/(\beta + D_{d+1})^2$. Temporal metrics satisfy $\tau(d+1) - \tau(d) \approx 1/(\beta + D_{d+1})$. Parameters: $\beta = 10$, chain length = 5, uniform edge weights $A_{d \rightarrow d+1} = 1$, with input applied to the first node.

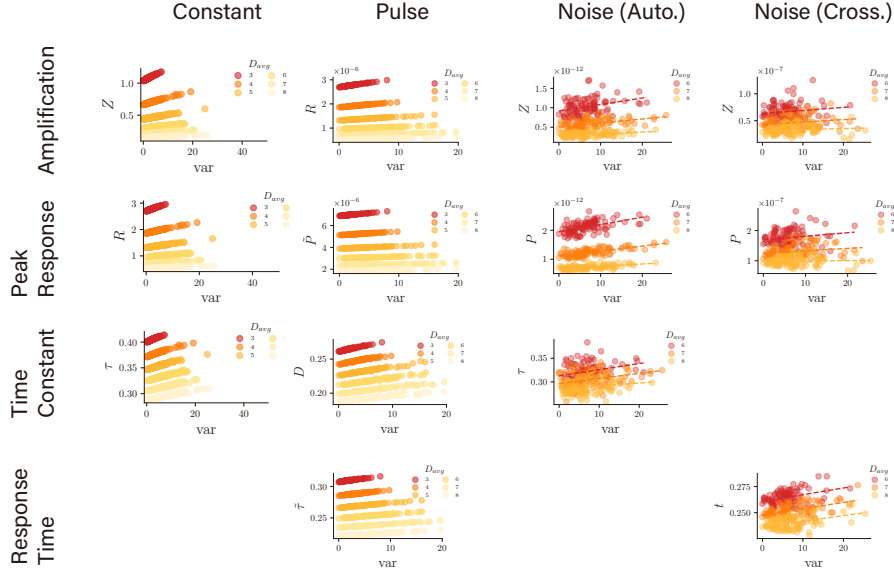


FIG. 16. Impact of mean and variance of degree distributions on a single chain. All metrics indicate that larger mean degrees suppress the metrics, whereas larger variance enhances them. Parameters: $\beta = 10$, chain length = 5, uniform edge weights $A_{d \rightarrow d+1} = 1$ along the chain, with input applied to the first node.

Using the method of Lagrange multipliers with

$$L = \tau - \lambda \left(\sum_i D_i - \bar{D}_d \right),$$

and solving $\frac{\partial L}{\partial D_i} = f'(D_i) - \lambda = 0$, we obtain the condi-

tion

$$\lambda = f'(D_i) = -(\beta + D_i)^{-2}, \quad \forall i.$$

Thus, the minimum value of τ is

$$\tau = \frac{d}{\beta + \bar{D}_d}$$

achieved when all degrees are equal: $D_1 = \dots = D_d = \bar{D}_d$, consistent with Jensen's inequality. The fixed estimation bias for the time constant does not alter the conclusion.

A similar analysis applies to the peak response R . Defining the Lagrangian

$$L = R - \eta \left(\sum_i D_i - \bar{D}_d \right),$$

and solving $\frac{\partial L}{\partial D_i} = f(D_1) \dots f'(D_i) \dots f(D_d) - \eta = 0$, we find

$$\eta = -(\beta + D_i)^{-1} R, \quad \forall i.$$

This yields the minimum peak response

$$R = \left(\frac{1}{\beta + \bar{D}_d} \right)^d$$

for the homogeneous in-degree configuration. Since $\ln R$ is a convex function of D , this homogeneous configuration indeed minimizes the peak response.

Eq. (E30) can alternatively be expressed as

$$R(d) = \left[\beta^d \left(d \sum_{n=1}^d \sum_{j=1}^n \frac{(-1)^{j-1}}{n} s^{(n-j)} \mu^{(j)} + 1 \right) \right]^{-1},$$

$$\tau(d) = \frac{d \left[\sum_{n=1}^d \sum_{j=1}^n \frac{d-n}{n} (-1)^{j-1} s^{(n-j)} \mu^{(j)} + 1 \right]}{\beta \left[d \sum_{n=1}^d \sum_{j=1}^n \frac{(-1)^{j-1}}{n} s^{(n-j)} \mu^{(j)} + 1 \right]}, \quad (\text{E33})$$

where $s^{(k)} \equiv \sum_{1 \leq i_1 < \dots < i_k \leq n} (D_{i_1}/\beta) \dots (D_{i_k}/\beta)$ denote elementary symmetric polynomials and $\mu^{(k)} \equiv n^{-1} \sum_{i=1}^n (D_i/\beta)^k$ represent raw moments. For propagation only restricted to source m and target i , the metrics simplify to

$$R(D_m, D_i) = \frac{1}{\beta^2} \frac{1}{(\mu + 1)^2 - \sigma^2},$$

$$\tau(D_m, D_i) = \frac{1}{\beta} \frac{2(\mu + 1)}{(\mu + 1)^2 - \sigma^2}, \quad (\text{E34})$$

with $\mu \equiv \frac{1}{2}(D_m/\beta + D_i/\beta)$ and $\sigma^2 \equiv \frac{1}{2}[(D_m/\beta - \mu)^2 + (D_i/\beta - \mu)^2]$ being the mean and variance of $\{D_m, D_i\}$. These dependencies align with the general cases: increasing μ reduces both P and τ , while increasing σ^2 elevates both.

Effects of motifs

The single chain is the baseline model, to which we can add motifs to observe their effects. In this subsection, we mainly focus on the effects of triangular motifs (feedforward and feedback triangles).

Effects of Feedforward (FF) Motifs

We examine how the presence of feedforward (FF) triangles alters signal propagation. The propagation metrics in the presence of FF motifs are given by:

$$R_{\text{FF}}(d) = R(d) \frac{x}{y},$$

$$\tau_{\text{FF}}(d) = \tau(d) + x - y, \quad (\text{E35})$$

where $x = 1/(\beta + D_\Delta)$, $y = 1/(\beta + D_\Delta + n(\Delta))$, and hence $y < x$. Here, D_Δ denotes the uniform in-degree assumed of the node participating in the triangle motif but not on the main chain path, and $n(\Delta)$ represents the number of FF motifs.

According to Eq. (E35), increasing the number of feedforward triangles $n(\Delta)$ enhances both the peak response $R_{\text{FF}} \rightarrow \infty$ and the time constant $\tau_{\text{FF}} \rightarrow \tau(d) + x$, indicating that FF motifs facilitate stronger and more sustained signal propagation. On the other hand, increasing the uniform in-degree D_Δ reduces this effect, as it leads to

$$R_{\text{FF}}(d) \rightarrow R(d), \quad \tau_{\text{FF}}(d) \rightarrow \tau(d),$$

effectively attenuating the influence of the FF motifs. This illustrates the jamming effect of nodal in-degree: as D_Δ grows, the additional flow introduced by the FF triangles converges back toward the main path, reducing their impact.

For the white-noise input, we illustrate the effect of FF triangles using a simple toy model: node 0 is the perturbed source, node 2 is the target, and node 1 represents the set of identical FF nodes forming the triangles. All edge weights are set to 1.

$$Z_{\text{FF}}^{\text{noise}} = Z^{\text{noise}} + n(\Delta) \sum_{j=0,1,2} h_{j \rightarrow 2}^1 p_{0 \rightarrow j}, \quad (\text{E36})$$

where Z^{noise} corresponds to the case without triangular effects, i.e., $h_{0 \rightarrow 2}^1 p_{2 \rightarrow 2} + h_{2 \rightarrow 2}^1 p_{0 \rightarrow 2}$. We find that amplification grows with the number of triangles, but the contribution from triangles, $n(\Delta) \sum_{j=0,1,2} h_{j \rightarrow 2}^1 p_{0 \rightarrow j} \sim O(1/\beta^4)$, vanishes as the degree of node 1 increases, verifying the jamming effect of nodal in-degree again.

Effects of Feedback (FB) Motifs

The analysis of feedback motifs is more intricate due to the presence of recurrent loops within their structure. These loops effectively split the signal into multiple paths with varying increases geometrically in path length. The resulting response metrics can be expressed as:

$$\begin{aligned}
R_{\text{FB}}(d) &= R(d) \sum_{k=0}^{\infty} (n(\Delta) D_{\Delta}^{\times})^k, \\
&= R(d) \frac{1}{1 - n(\Delta) D_{\Delta}^{\times}}, \\
\tau_{\text{FB}}(d) &= \frac{Z_{\text{FB}}(d)}{R_{\text{FB}}(d)}, \\
&= \tau(d) + \frac{n(\Delta) D_{\Delta}^{\times} D_{\Delta}^{+}}{1 - n(\Delta) D_{\Delta}^{\times}},
\end{aligned} \tag{E37}$$

where $n(\Delta)$ denotes the number of triangular motifs, $D_{\Delta}^{\times} \equiv \prod_{k \in \Delta} (\beta + D_k)^{-1}$, and $D_{\Delta}^{+} \equiv \sum_{k \in \Delta} (\beta + D_k)^{-1}$.

For the white-noise input, we consider a similar toy model with FB triangles, obtained by reversing the directions of the edges between 0 and 1 and between 1 and 2. All edge weights are set to 1.

$$\begin{aligned}
Z_{\text{FB}}^{\text{noise}} &= \frac{h_{0 \rightarrow 2}^1 p_{0 \rightarrow 0} + h_{2 \rightarrow 2}^1 p_{0 \rightarrow 2} + n(\Delta) h_{\text{FB}} p_{\text{FB}}}{(1 - n(\Delta) h_{\text{FB}}) (1 - n(\Delta) p_{\text{FB}})}, \\
&\approx Z^{\text{noise}} + n(\Delta) h_{\text{FB}} p_{\text{FB}},
\end{aligned} \tag{E38}$$

where $h_{\text{FB}} \equiv h_{0 \rightarrow 1 \rightarrow 2}^1$ and $p_{\text{FB}} \equiv p_{0 \rightarrow 1 \rightarrow 2}$. Similarly, amplification grows with the number of motifs, but here the slope is much smaller, with $h_{\text{FB}} p_{\text{FB}} \sim O(1/\beta^6)$. Also, this contribution vanishes when the in-degree of node 1 increases.

For higher-order feedback motifs $\mathcal{M}_k$ with k edges, appended to the main propagation chain, the expressions naturally extend to:

$$\begin{aligned}
R_{\mathcal{M}_k}(d) &= R(d) \frac{1}{1 - n(\mathcal{M}_k) D_{\mathcal{M}_k}^{\times}}, \\
\tau_{\mathcal{M}_k}(d) &= \tau(d) + \frac{n(\mathcal{M}_k) D_{\mathcal{M}_k}^{\times} D_{\mathcal{M}_k}^{+}}{1 - n(\mathcal{M}_k) D_{\mathcal{M}_k}^{\times}},
\end{aligned} \tag{E39}$$

where $n(\mathcal{M}_k)$ represents the number of such motifs, and the products and sums in $D_{\mathcal{M}_k}^{\times}$ and $D_{\mathcal{M}_k}^{+}$ are taken over all nodes within the motif.

These results highlight how recurrent motifs can enhance both the amplitude and duration of the response by effectively increasing the number of signal propagation routes and their persistence. The jamming effects of nodal in-degree still hold, as $R_{\text{FB}}(d) \rightarrow R(d)$, $\tau_{\text{FB}}(d) \rightarrow \tau(d)$ when nodal in-degree of the motif increases.

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Supplementary Material

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1 Numerical accuracy test in NSDD system

1.1 Constant input

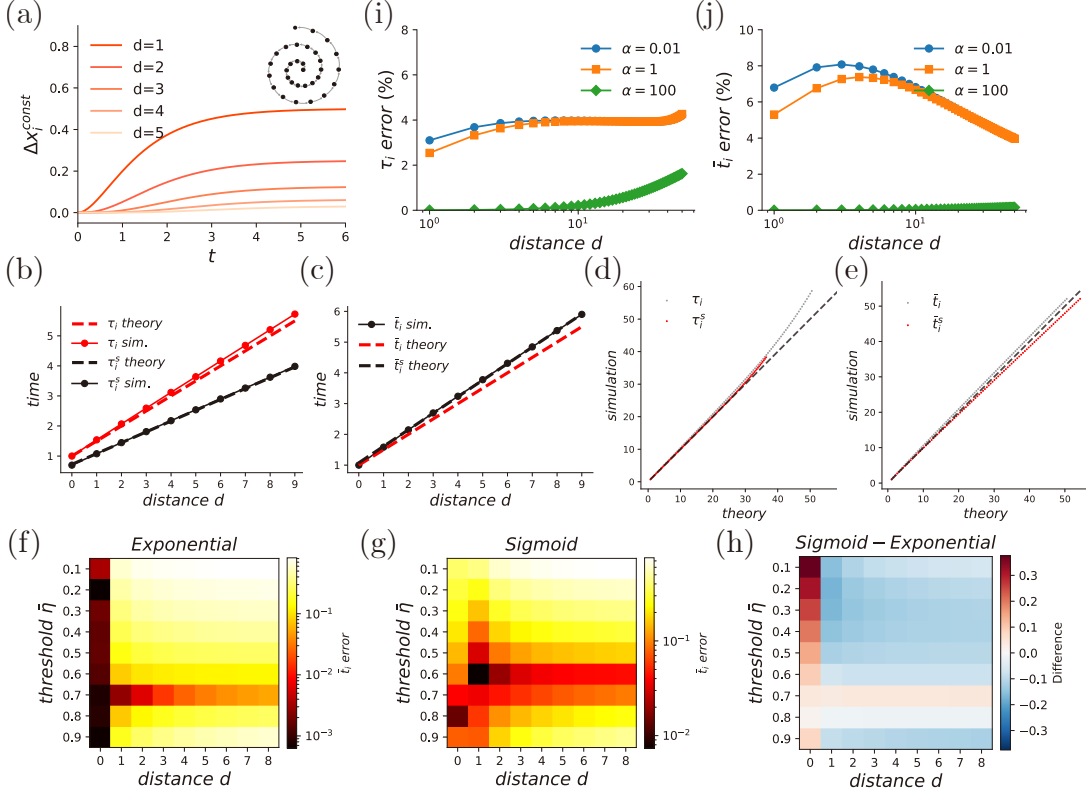


Figure S1: Constant perturbation spreading through a directed chain of length 100. All panels (a) to (h) use NSDD system parameters: topology matrix entries $\alpha = 1$ and self-decay parameter $\beta = 1$. The threshold $\bar{\eta} = 1 - 1/e$ is used in panels (c), (e), and (i). (a) Time courses for nodes at different shortest path lengths from input. (b) Simulated vs. theoretical time constants τ_i and τ_i^s (exponential and sigmoidal functions; dashed lines show simulations). (c) Simulated vs. theoretical relative propagation times $\bar{t}_i$ and $\bar{t}_i^s$ (exponential and sigmoidal; sigmoid shows better fit). (d) Simulated vs. theoretical time constants for exponential (τ_i) and sigmoidal (τ_i^s) functions (dotted lines indicate perfect match). (e) Simulated vs. theoretical relative propagation times for exponential ($\bar{t}_i$) and sigmoidal ($\bar{t}_i^s$) estimations. (f) Relative error in exponential fitting (minimum error near $\bar{\eta} \approx 0.7$). (g) Relative error in sigmoid fitting (minimum error near $\bar{\eta} \approx 0.6$). (h) Sigmoid vs. exponential relative error comparison (blue indicates superior sigmoid performance). (i) Time constant relative error vs. topological weights α (error decreases with larger α , increases with distance). (j) Propagation time relative error vs. α (optimal distance for minimal error). Relative error for panels (f)-(j) is defined as $|\text{sim. } t - \text{thr. } t|/\text{sim. } t$.

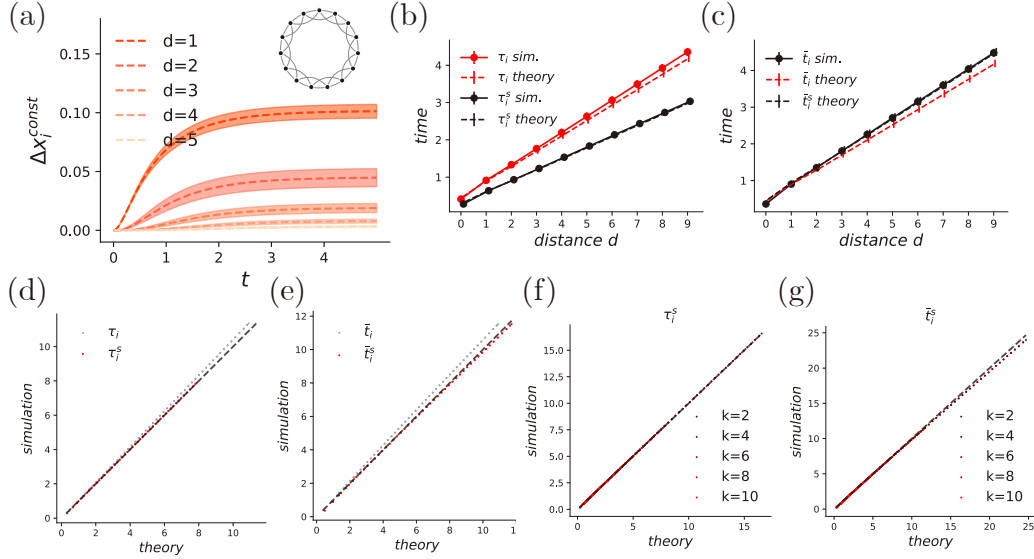


Figure S2: Constant perturbation spreads through a ring lattice of length 100, with one input projecting to any node. The NSDD system uses topology matrix entries $\alpha = 1$ and self-decay parameter $\beta = 1$ for all panels (a) to (g), with the threshold $\bar{\eta} = 1 - 1/e$ in panels (c), (e), and (g). Nodes have degree 4 in panels (a) to (e). (a) Time courses vs. shortest path lengths from input. (b) Simulated versus theoretical time constants τ_i and τ_i^s for exponential and sigmoidal functions (dashed lines: theoretical results). (c) Simulated versus theoretical relative propagation times $\bar{t}_i$ and $\bar{t}_i^s$ for exponential and sigmoidal functions (sigmoid shows better fit). (d) Simulated vs. theoretical time constants for exponential (τ_i) and sigmoidal (τ_i^s) functions (dotted lines: perfect matches). (e) Simulated versus theoretical relative propagation times for exponential ($\bar{t}_i$) and sigmoidal ($\bar{t}_i^s$) estimations. (f) Simulated vs. theoretical time constants for sigmoidal functions (τ_i^s) across different node degrees. (g) Simulated versus theoretical relative propagation times for sigmoidal functions ($\bar{t}_i^s$) across different node degrees.

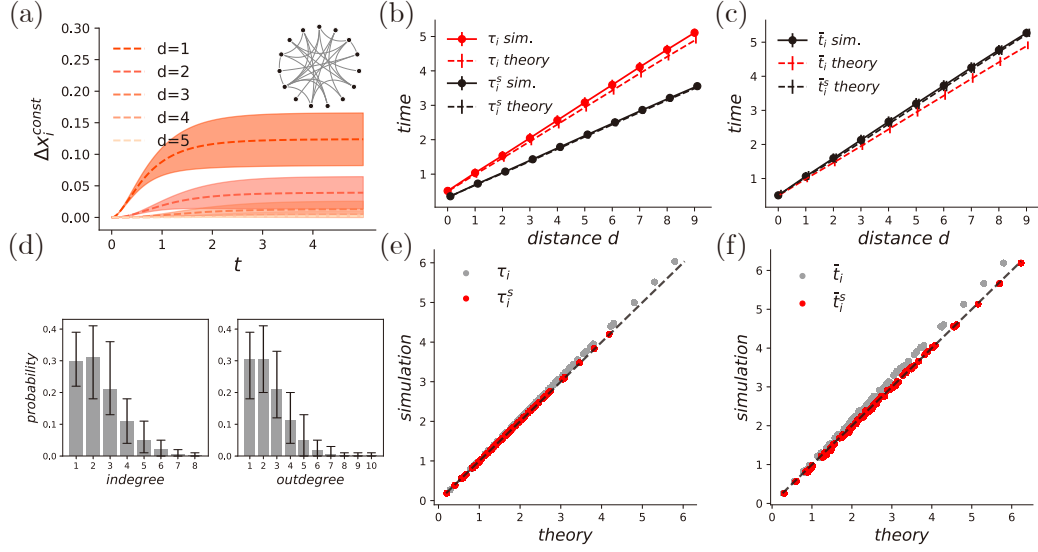


Figure S3: Constant perturbation spreading through an Erdős-Rényi random network ($N = 100$ nodes) over 100 simulation rounds, with one input projecting to random nodes. NSDD system parameters: $\alpha = 1$ (edge weights), $\beta = 1$ (self-decay), $\bar{\eta} = 1 - 1/e$ (decay threshold; c,f). (a) Time courses vs. shortest path lengths from input (shaded area: std. for same-path-length nodes). (b) Simulated vs. theoretical time constants τ_i and τ_i^s for exponential and sigmoidal functions (vertical lines: error ranges). (c) Simulated versus theoretical relative propagation times $\bar{t}_i$ and $\bar{t}_i^s$ for exponential and sigmoidal functions (sigmoid shows better fit; small errors). (d) In-degree and out-degree distributions. (e) Scatter plot: simulated versus theoretical time constants for exponential (τ_i) and sigmoidal (τ_i^s) functions across all rounds. (f) Scatter plot: simulated versus theoretical relative propagation times for exponential ($\bar{t}_i$) and sigmoidal ($\bar{t}_i^s$) functions across all rounds.

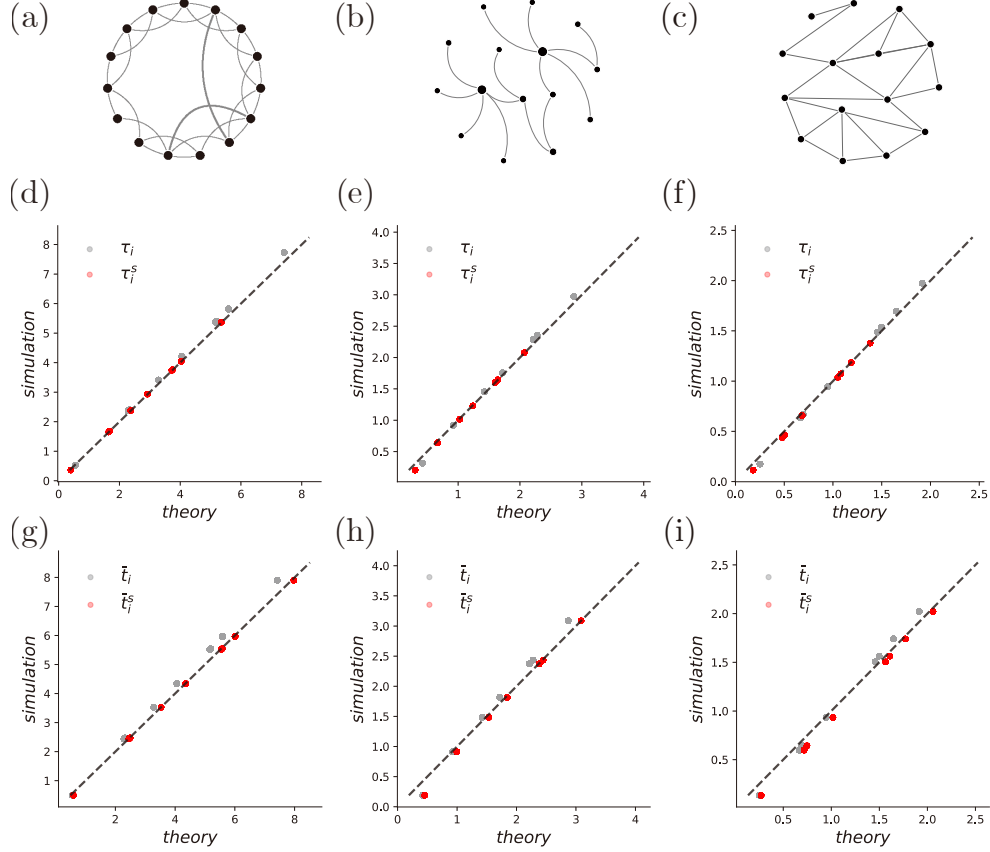


Figure S4: Constant perturbation spreading through small-world, scale-free, and geometric networks ($N = 100$ nodes). Each network type simulated for 100 rounds with random node perturbation. NSDD parameters: $\alpha = 1$ (edge weights), $\beta = 1$ (self-decay). (a) Small-world network sketch: initial ring with connections to 2 nearest neighbors, edge rewiring probability 0.5. (b) Scale-free network sketch: each new node attaches via 1 edge to existing nodes. (c) Geometric network sketch: edges created between nodes within distance threshold 0.2. (d-f) Simulated vs. theoretical time constants for exponential (τ_i) and sigmoidal (τ_i^s) functions across all rounds (small-world: d; scale-free: e; geometric: f). (g-i) Simulated vs. theoretical relative propagation times for exponential ($\bar{\tau}_i$) and sigmoidal ($\bar{\tau}_i^s$) functions across all rounds (small-world: g; scale-free: h; geometric: i).

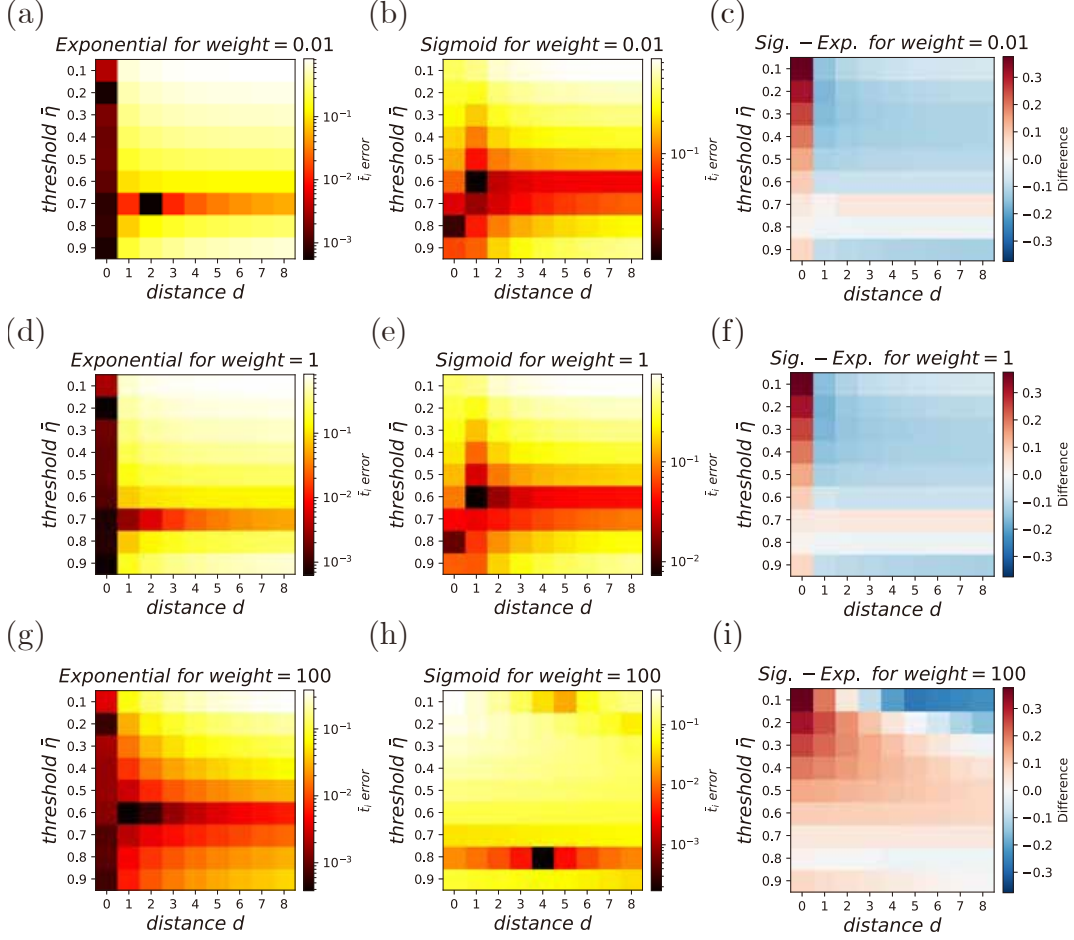


Figure S5: Relative error for thresholds $\bar{\eta}$ and distances d at edge weights $\alpha = 0.01, 1, 100$ in a directed chain with constant inputs. The self-decay parameter $\beta = 1$. The sigmoid model yields better fits for smaller α values (c, f), while the exponential model is better for larger α (i).

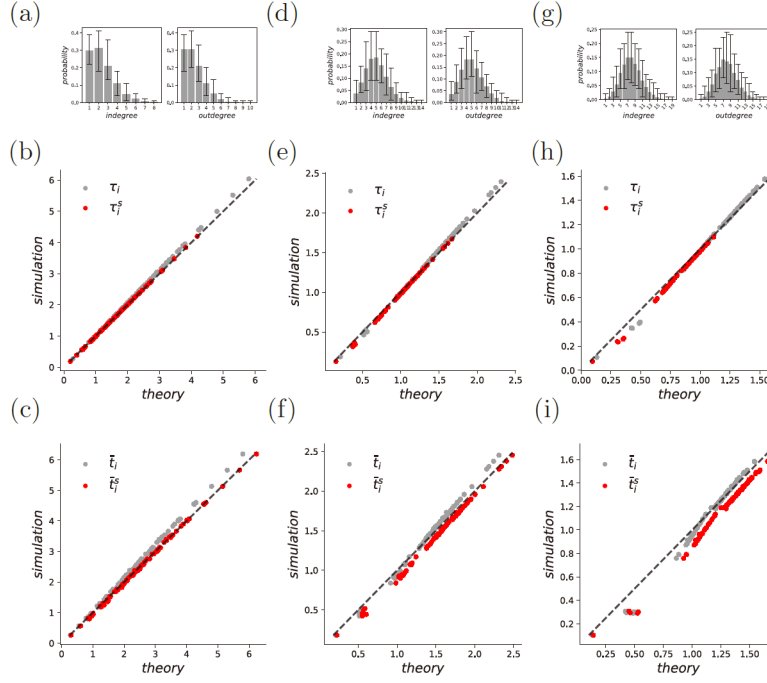


Figure S6: Constant perturbation spreads through in the ER random networks with different edge connecting probability. Results for probability 0.02 (a-c), 0.05 (d-f), and 0.08 (g-i). (a,d,g) Degree distribution for in-degree and out-degree. (b,e,h) Comparison between simulated and theoretical results for time constants for both exponential (τ_i) and sigmoidal (τ_i^s) functions for all rounds. (c,f,i) Comparison between simulated and theoretical results for relative time for both exponential (t_i) and sigmoidal (t_i^s) functions for all rounds. All theoretical results fit well with the simulated results.

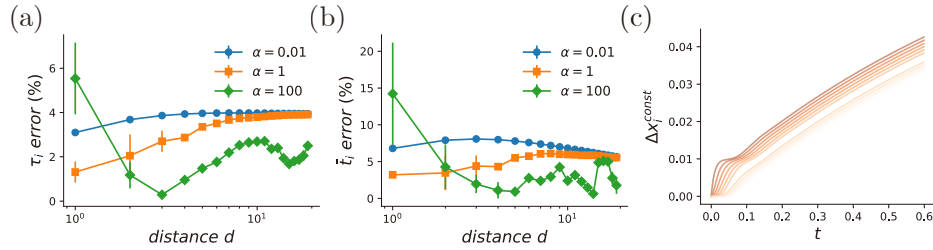


Figure S7: Relative error analysis for constant perturbation in Erdős-Rényi random networks ($p = 0.02$) with different interaction weights. (a) Time constant (τ) relative error at $\alpha = 0.01, 1, 100$. (b) Propagation time relative error at the threshold $\bar{\eta} = 1 - 1/e$ for $\alpha = 0.01, 1, 100$. (c) Time courses at large α (100): flat curve segment corresponds to multi-peak impulse response (Fig. S13); estimations maintain high accuracy despite this feature.

1.2 Pulse input

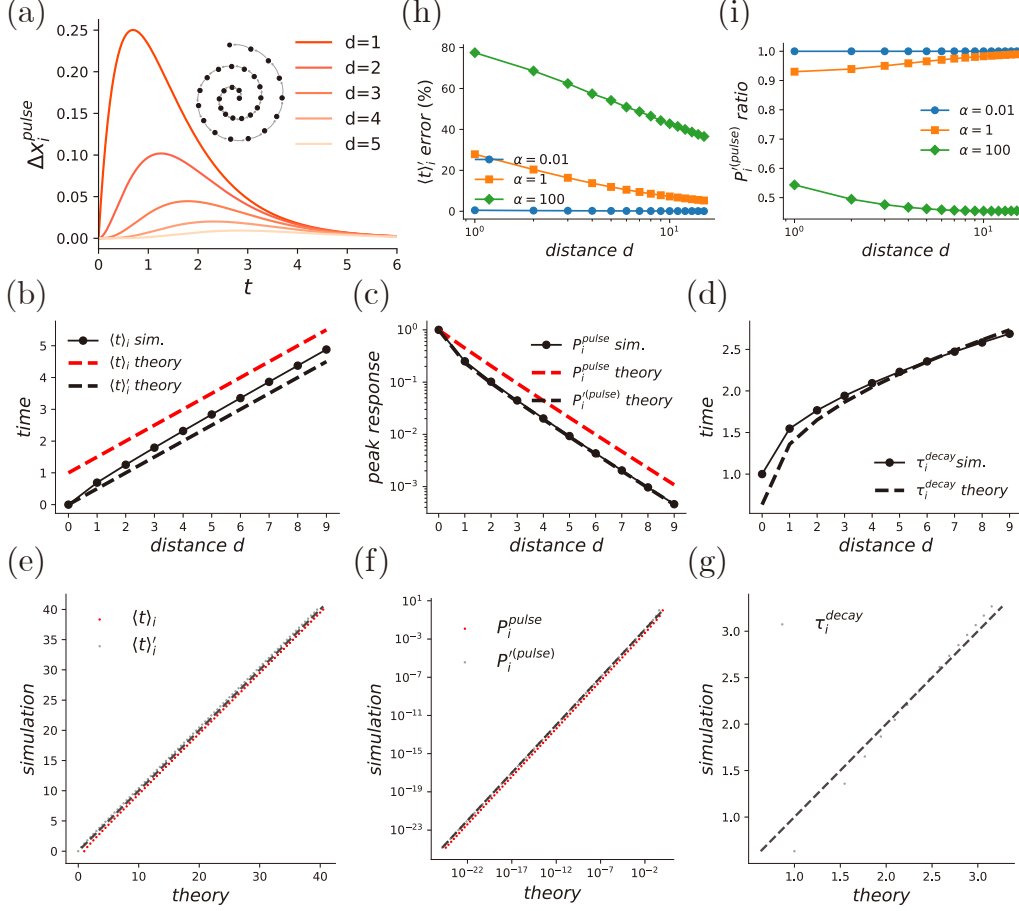


Figure S8: Impulse response in a directed chain ($N = 80$). NSDD parameters: $\alpha = 1$, $\beta = 1$ (panels a-g). (a) Time course vs. shortest path length from input. (b) Peak response time vs. path length d : simulated results vs. theoretical predictions vs. amended predictions. (c) Peak response vs. path length d : simulated results vs. theoretical predictions vs. amended predictions. (d) Simulated vs. theoretical decay time constants. (e) Simulated vs. theoretical and amended peak response time. (f) Simulated vs. theoretical and amended peak response. (g) Simulated decay time constants vs. theoretical predictions for $d < 15$ (differences increase with distance due to sharp decay; dashed lines: perfect matches). (h) Time constant relative error ($|\text{sim. } t - \text{thr. } t|/\text{sim. } t$) for $\alpha = 0.01, 1, 100$: error decreases with distance, minimal at smallest α . (i) Peak response ratio ($\text{thr. } P/\text{sim. } P$) for $\alpha = 0.01, 1, 100$: ratio tends to 1 at smallest α .

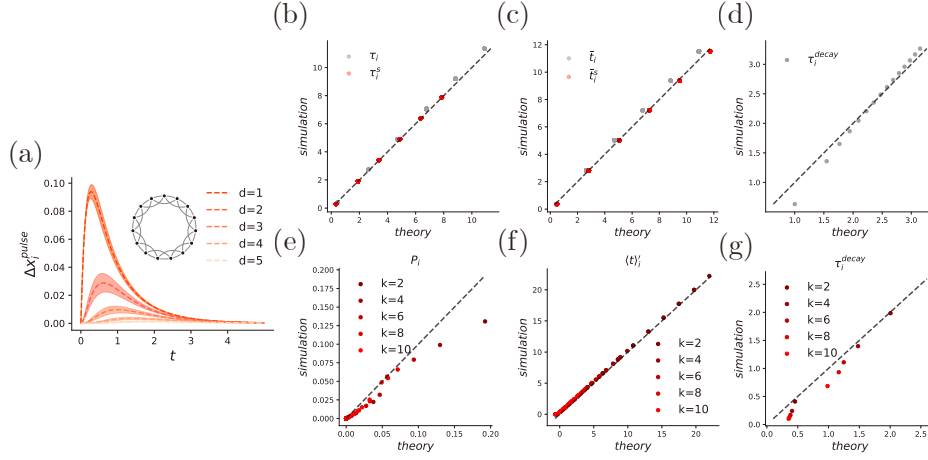


Figure S9: Impulse response in a ring lattice ($N = 80$). NSDD parameters: $\alpha = 1$, $\beta = 1$. Node degree = 4 for panels (b)-(d). (a) Time course vs. shortest path length from input. (b) Simulated vs. theoretical and amended peak response. (c) Simulated vs. theoretical and amended peak response time. (d) Simulated decay time constants vs. theoretical predictions. (e) Simulated vs. theoretical amended peak response across different node degrees (bias increases with degree). (f) Simulated vs. theoretical amended peak response time across different node degrees. (g) Simulated decay time constants vs. theoretical predictions across different node degrees.

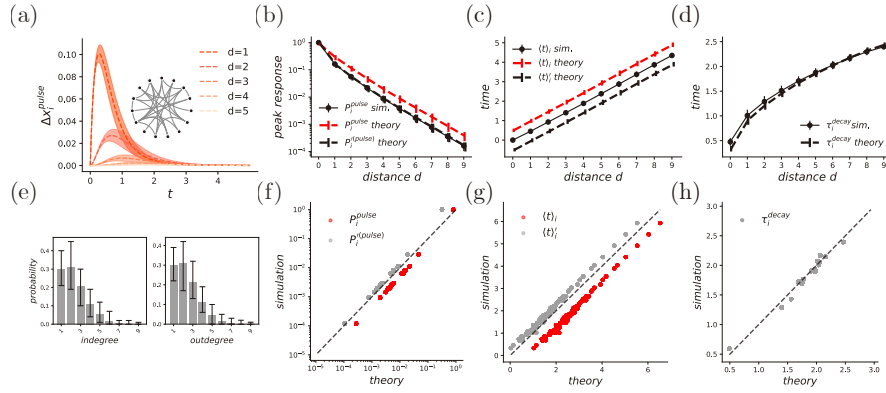


Figure S10: Impulse response in an Erdős-Rényi random network ($N = 80$). Results from 100 simulation rounds with NSDD parameters: $\alpha = 1$, $\beta = 1$. (a) Time course vs. shortest path length from input. (b) Peak response vs. path length d : simulated results vs. theoretical predictions vs. amended predictions. (vertical lines: error ranges). (c) Peak response time vs. path length d : simulated results vs. theoretical predictions vs. amended predictions. (simulations lie between initial and amended predictions). (d) Simulated decay time constants vs. theoretical predictions. (e) In-degree and out-degree distributions. (f) Simulated vs. theoretical and amended peak response across all rounds. (g) Simulated vs. theoretical and amended peak response time across all rounds. (h) Simulated decay time constants vs. theoretical predictions across all rounds.

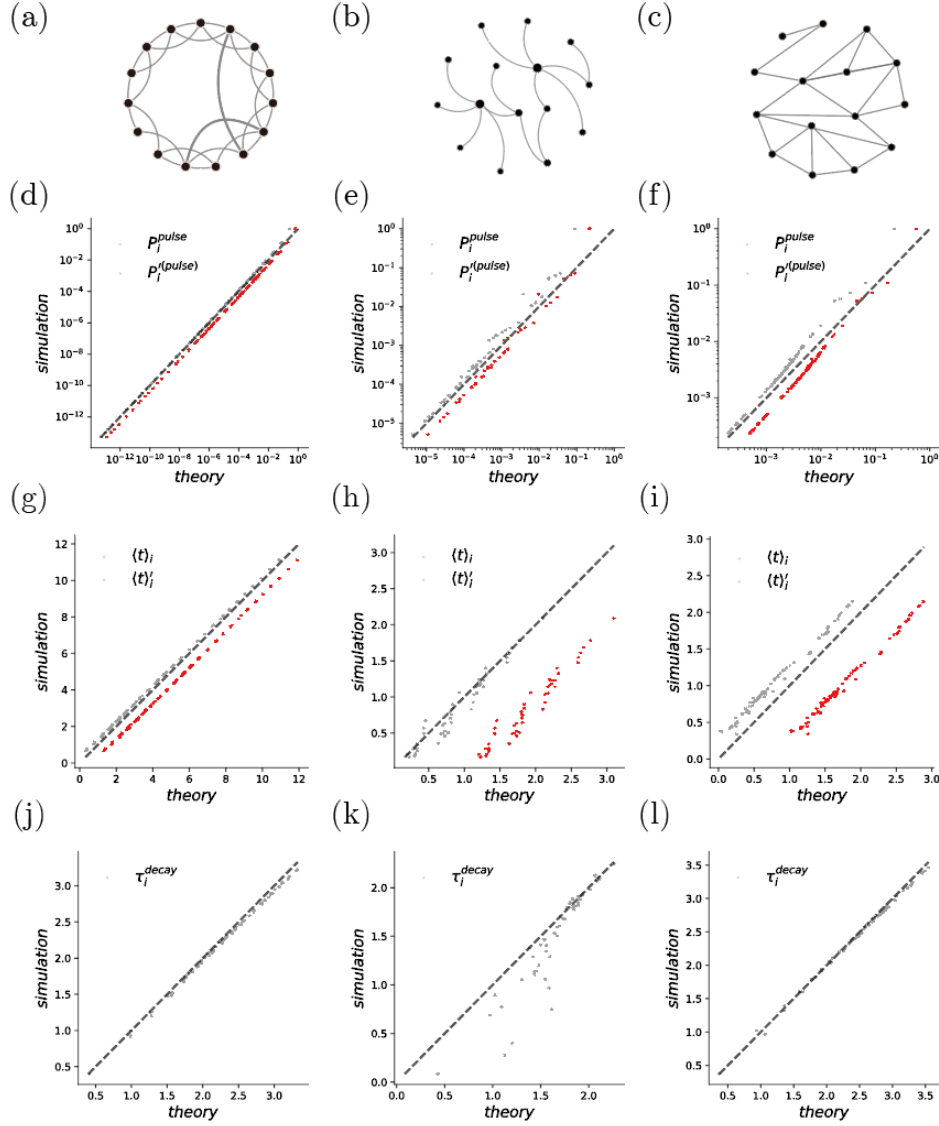


Figure S11: Pulse perturbation in small-world, scale-free, and geometric networks ($N = 100$). Each network simulated for 100 rounds. NSDD parameters: $\alpha = 1$, $\beta = 1$. (a) Small-world sketch: initial ring with connections to 2 nearest neighbors, edge rewiring probability 0.5. (b) Scale-free sketch: each new node attaches via 1 edge to existing nodes. (c) Geometric sketch: edges created between nodes within distance threshold 0.2. (d-f) Simulated vs. theoretical peak response for initial theory (P_i^{pulse}) and amended theory ($P_i'^{(\text{pulse})}$) across all rounds (d: small-world; e: scale-free; f: geometric). (g-i) Simulated vs. theoretical peak response time for initial theory ($\langle t \rangle_i$) and amended theory ($\langle t \rangle_i'$) across all rounds (g: small-world; h: scale-free; i: geometric). (j-l) Simulated vs. theoretical peak response time from decay time constant (τ_i^{decay}) across all rounds (j: small-world; k: scale-free; l: geometric).

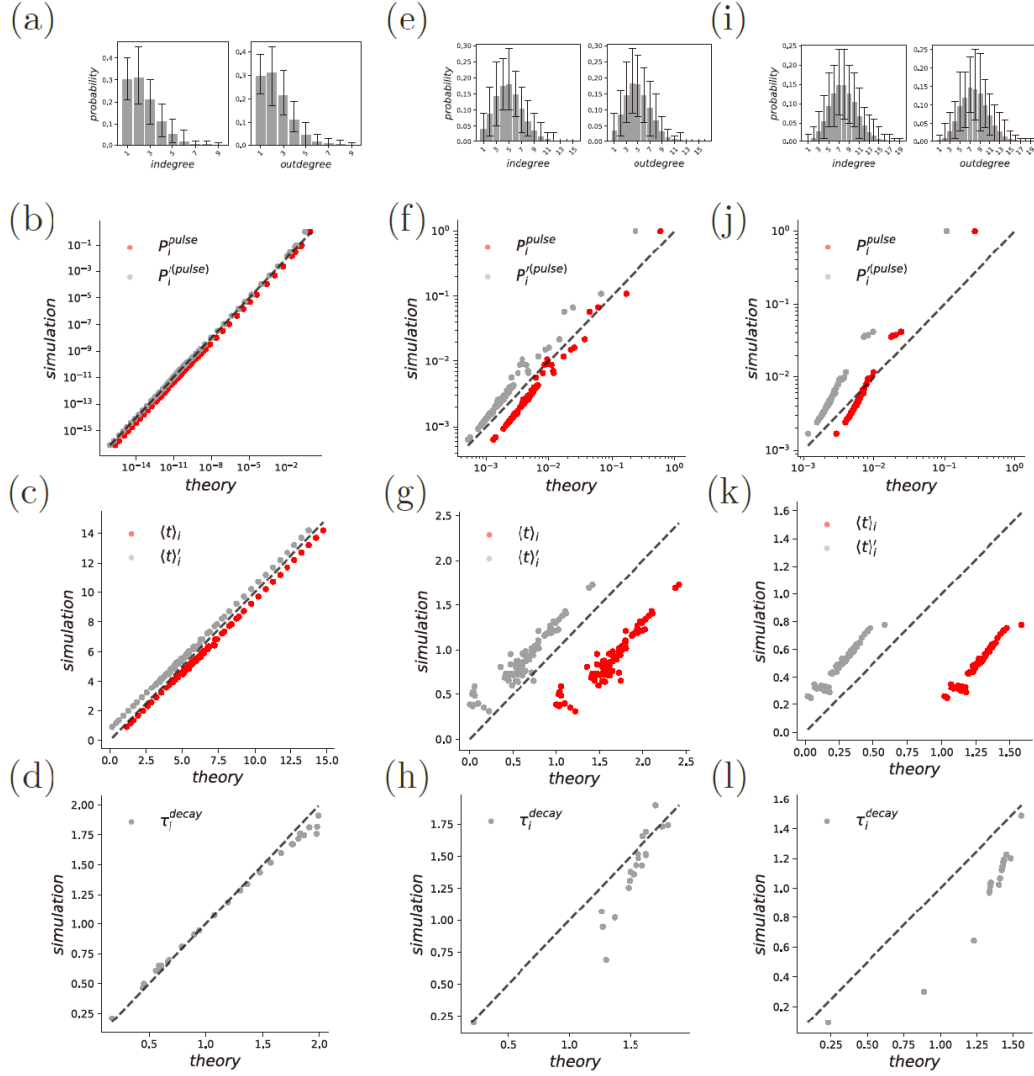


Figure S12: Pulse perturbation in Erdős-Rényi random networks with edge probabilities $p = 0.02$ (a-d), $p = 0.05$ (e-h), and $p = 0.08$ (i-l). (a,e,i) In-degree and out-degree distributions. (b,f,j) Simulated vs. theoretical peak response for initial (P_i^{pulse}) and amended ($P_i'^{\text{(pulse)}}$) theories across all rounds. (c,g,k) Simulated vs. theoretical peak response time for initial ($\langle t \rangle_i$) and amended ($\langle t \rangle'_i$) theories across all rounds. (d,h,l) Simulated decay time constants vs. theoretical predictions (τ_i^{decay}) across all rounds. At higher p values (e.g., 0.08), theoretical predictions show systematic deviations: peak responses exhibit fixed-slope discrepancies and decay time constants are overestimated.

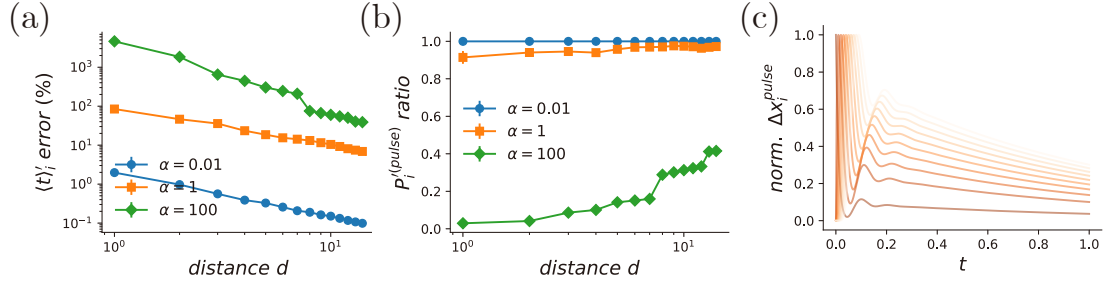


Figure S13: Estimation error for pulse perturbation in Erdős-Rényi random networks ($p = 0.02$) at edge weights $\alpha = 0.01, 1, 100$. (a) Peak response time relative error: increases with α (largest at $\alpha = 100$). (b) Peak response ratio (theoretical/simulated): decreases with α (smallest at $\alpha = 100$). (c) Time course at $\alpha = 100$: recurrent loops generate multi-peak responses where estimations fail.

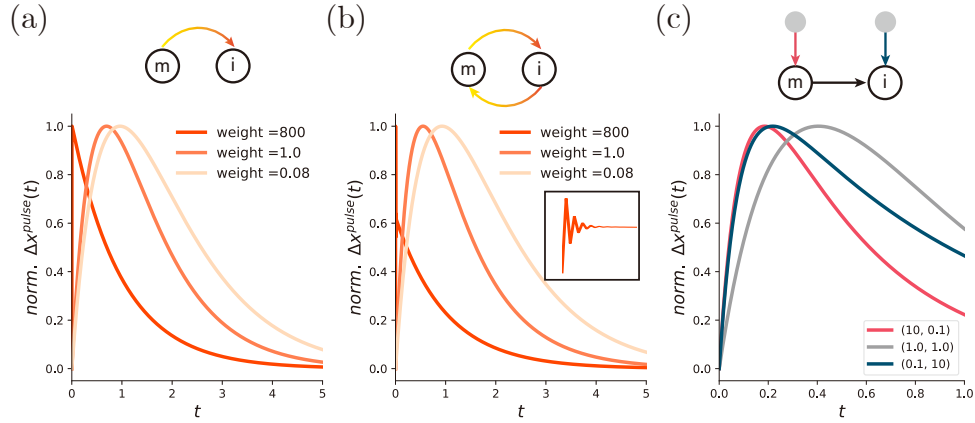


Figure S14: Toy model illustrating combined effects of topology, interaction weight, and degree heterogeneity. (a) Two-node directed chain: large interaction weights (α) create significant asymmetry, explaining peak response time estimation errors at high α due to distribution skewness. (b) Two-node bidirectional chain: asymmetry-induced bias with damped oscillations affects peak response time estimation accuracy. (c) Degree heterogeneity effect: constant in-degree product maintained; greater heterogeneity increases asymmetry, explaining reduced estimation accuracy in scale-free networks.

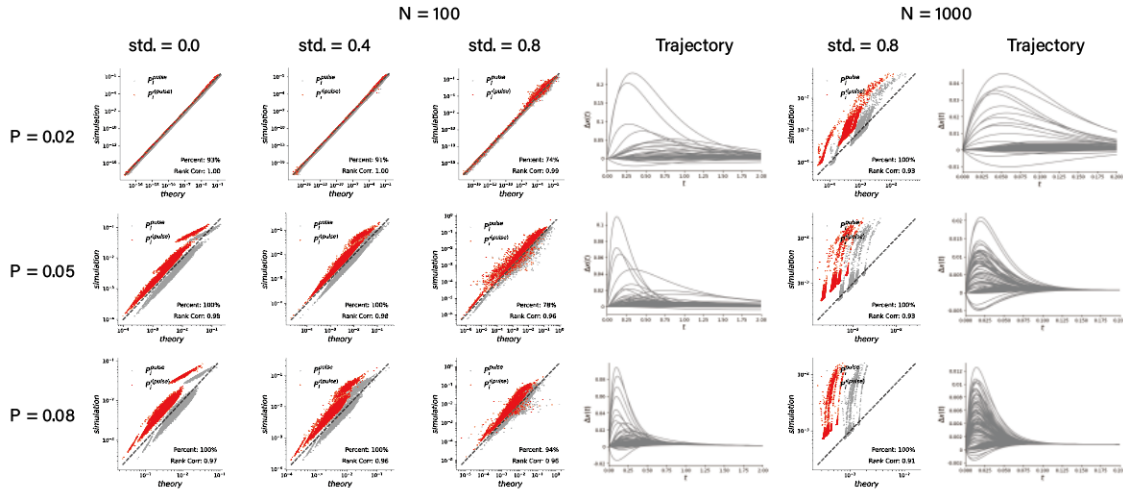


Figure S15: Peak response for pulse perturbation in Erdős-Rényi random networks with Gaussian weight distributions (mean = 1.0) at different standard deviations. Network sizes: 100 and 1000 nodes. Edge probabilities: 0.02 (first row), 0.05 (second row), 0.08 (third row). Simulated vs. theoretical extended relative time comparisons for exponential ($\bar{t}_i$) and sigmoidal ($\bar{t}_i^s$) functions over 100 rounds. Key observations: (1) Theoretical estimates match simulations closely in dense networks (higher average degrees). (2) For larger networks (size 1000), simulated results consistently exceed estimates. (3) At std. = 0.8, negative weights distort nodal traces (especially in sparse networks); negative estimates and negligible responses excluded (inclusion percentages shown in lower-right corners).

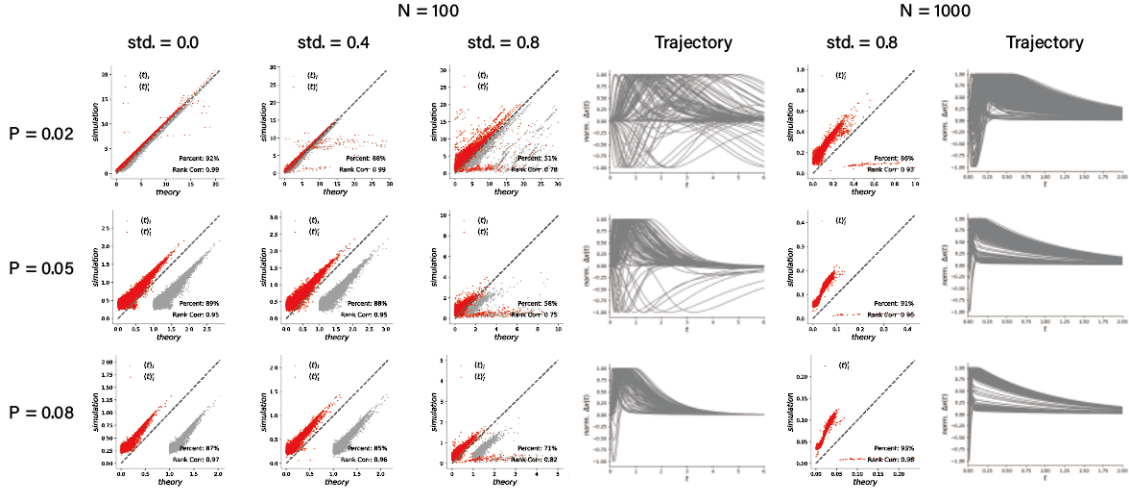


Figure S16: Peak response time for pulse perturbation spreading in ER random networks. Networks of size $N = 100$ and $N = 1000$ with edge probabilities $p = 0.02$ (top row), 0.05 (middle), and 0.08 (bottom). Edge weights follow Gaussian distributions (mean = 1.0) with varied standard deviations. Results show extended peak response times from 100 simulation rounds. Theoretical estimates match simulations closely in sparse networks (low p) without negative links. For $std. = 0.8$, negative weights significantly alter nodal traces, particularly in sparse networks. Negative theoretical values and negligible responses are excluded (inclusion percentages in bottom-right).

1.3 Square input

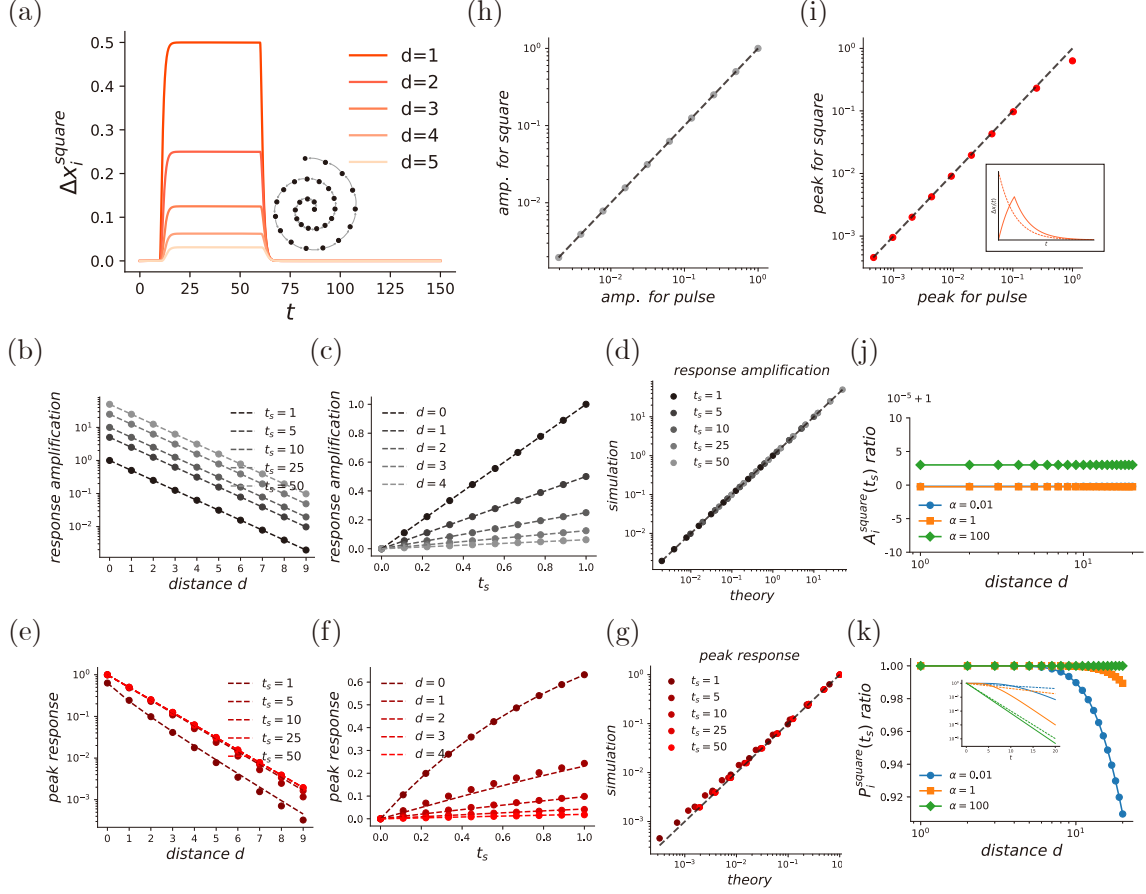


Figure S17: Dynamics of square input propagation in a directed chain ($N = 100$) with NSDD parameters ($\alpha = 1, \beta = 1$). Input is applied to the first node. (a) Time courses at nodes with varying distances from the source. (b) Response amplification vs. distance for different duration lengths (t_s). (c) Response amplification vs. t_s for different distances. (d) Theoretical vs. simulated amplification across t_s . (e) Peak response vs. distance for different t_s . (f) Peak response vs. t_s for different distances. (g) Theoretical vs. simulated peak response (simulations show larger peaks at small t_s). (h) Impulse response amplification matches unit-step ($t_s = 1$) square input (equal input strength). (i) Impulse peak response aligns with unit-step input apart from the first node; inset shows response comparison at input node. (j) Relative error in amplification theory for $\alpha = 0.01, 1, 100$ (accurate predictions). (k) Theoretical/simulated peak ratio for $\alpha = 0.01, 1, 100$; deviations occur for small α at large distances due to early evolution effects (inset: exponential fitting).

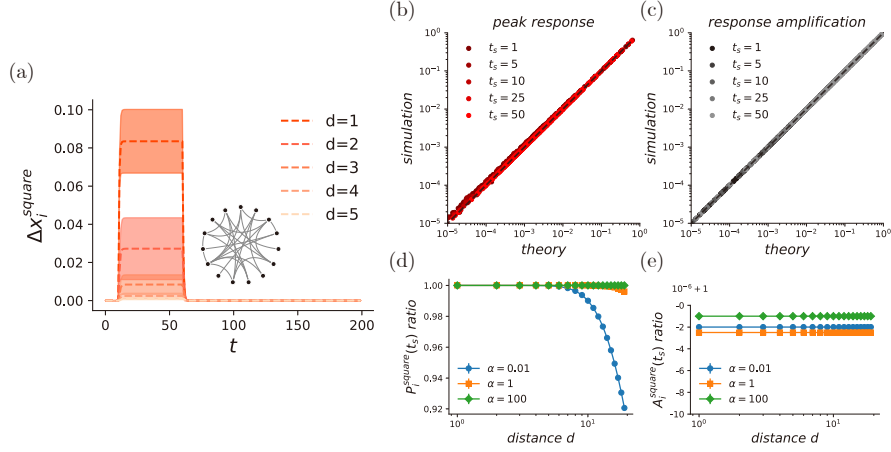


Figure S18: Propagation dynamics of square inputs in random networks ($N = 100$) with NSDD parameters ($\alpha = 1, \beta = 1$). A single input is applied at random nodes. (a) Time courses at varying distances from input node; shading indicates response deviations for several realizations. (b) Theoretical vs. simulated amplification across duration lengths (t_s). (c) Theoretical vs. simulated peak response across t_s . (d) Theoretical/simulated peak ratio (thr. P/sim. P) for $\alpha = 0.01, 1, 100$, showing deviations at small α and large distances. (e) Theoretical/simulated amplification ratio for $\alpha = 0.01, 1, 100$, demonstrating accurate estimation.

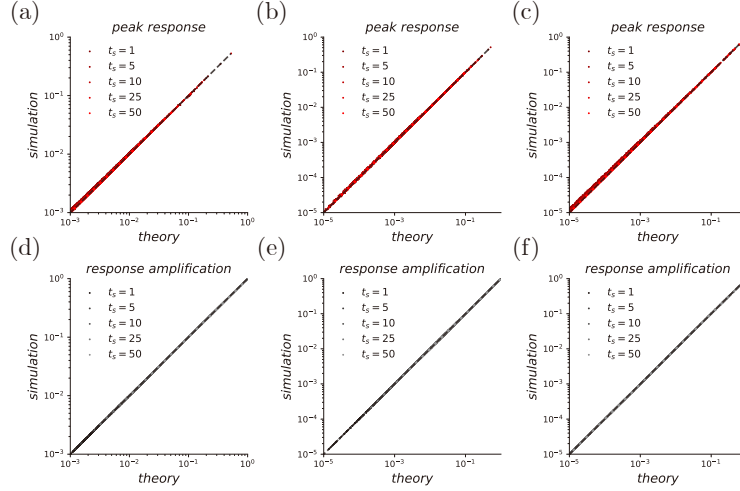


Figure S19: Propagation dynamics of square perturbations in three networks (each $N = 100$): small-world (a,d), scale-free (b,e), and geometric (c,f). All simulations use $\alpha = 1, \beta = 1$ and 100 rounds, with any single-node inputs. Network specifications: (a,d) Small-world: Initial ring with 2 nearest neighbors per node, edge rewiring probability 0.5; (b,e) Scale-free: Preferential attachment (1 edge per new node); (c,f) Geometric: Edge creation threshold radius 0.2. Panels show theoretical estimations of (a-c) peak response and (d-f) response amplification.

1.4 Noise input

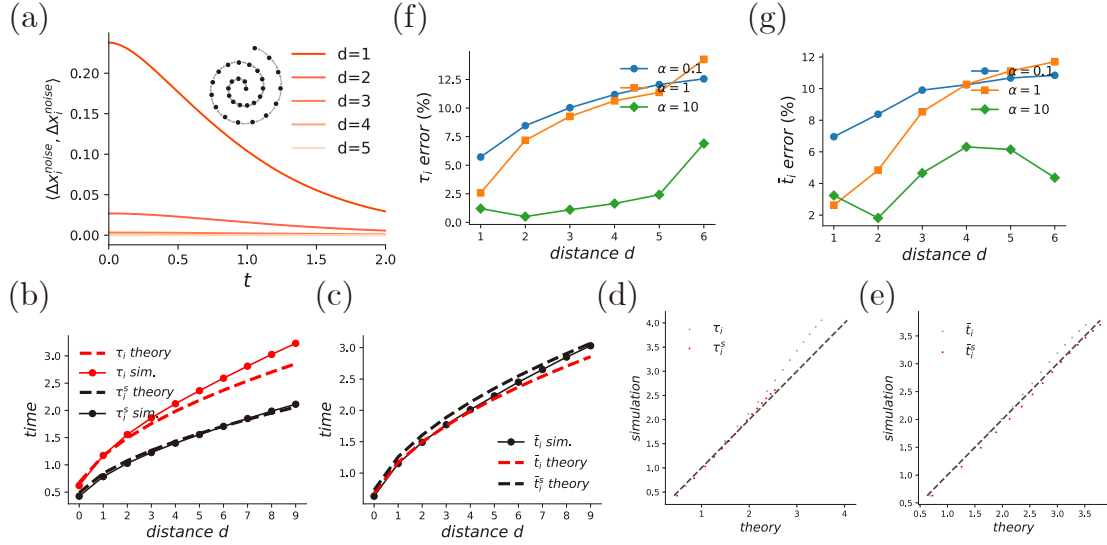


Figure S20: White noise response dynamics in an undirected chain ($N = 100$) with NSDD parameters ($\alpha = 1, \beta = 1$). Input targets the first node. Threshold $\bar{\eta} = 1/e$ applies to panels (c, e, g). (a) Autocovariance dynamics for different shortest path distance d from input node. (b) Simulated and theoretical time constant with exponential form τ_i and sigmoidal form τ_i^s vs. distance d . (c) Simulated and theoretical relative propagation time with exponential form $\bar{t}_i$ and sigmoidal form $\bar{t}_i^s$ vs. distance d . (d) Theoretical vs. simulated time constants (initial/amended). (e) Theoretical vs. simulated relative propagation times (initial/amended). (f) Relative error of τ (|sim. - thr. | / sim.) for $\alpha = 0.1, 1, 10$: error increases with d and minimizes at largest α . (g) Relative error of $\bar{t}_i$ for $\alpha = 0.1, 1, 10$: similar trend as (f).

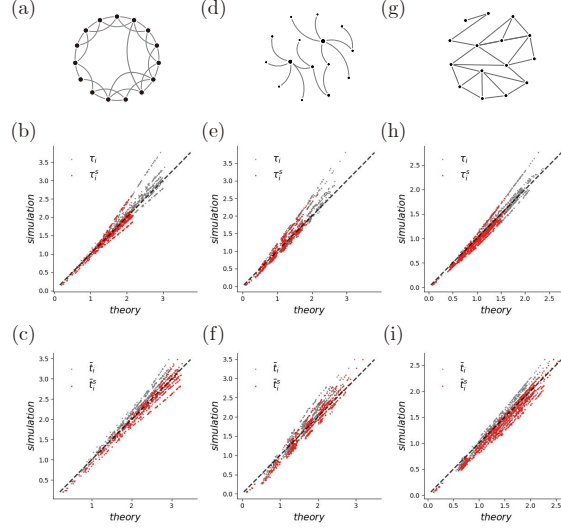


Figure S21: Autocovariance dynamics of white noise perturbations in three networks: small-world (a-c), scale-free (d-f), and geometric (g-i). Simulations use $N = 100$, NSDD parameters ($\alpha = 1$, $\beta = 1$), with 10 random network instances each run for 10 rounds. Single-node perturbations applied randomly. Network schematics: Small-world: initial ring with 2 nearest neighbors, edge rewiring probability 0.5; Scale-free: preferential attachment (1 edge per new node); Geometric: connection radius threshold 0.2; (d-f) Time constant comparison: Theoretical vs. simulated values for exponential (τ_i) and sigmoidal (τ_i^s) models across all networks and rounds; (g-i) Relative propagation time comparison: Theoretical vs. simulated values for exponential ($\bar{t}_i$) and sigmoidal ($\bar{t}_i^s$) models at threshold $\bar{\eta} = 1/e$

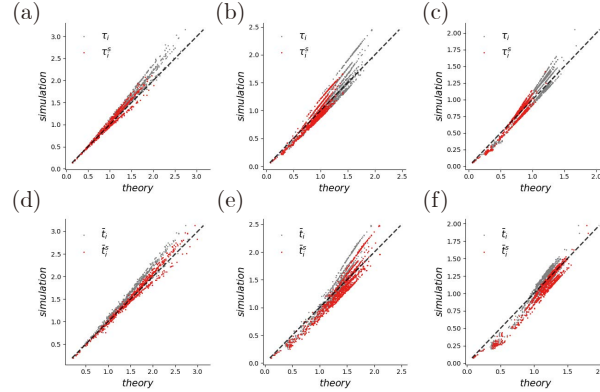


Figure S22: Auto-covariance dynamics of white noise perturbations in ER random networks with varying edge probabilities p . Results compare simulations and theory for: $p = 0.02$ (a, d); $p = 0.05$ (b, e); $p = 0.08$ (c, f). (a-c) Time constants: theoretical vs. simulated for exponential (τ_i) and sigmoidal (τ_i^s) forms; (d-f) Relative propagation times: theoretical vs. simulated for exponential ($\bar{t}_i$) and sigmoidal ($\bar{t}_i^s$) forms. All theoretical predictions show excellent agreement with simulations.

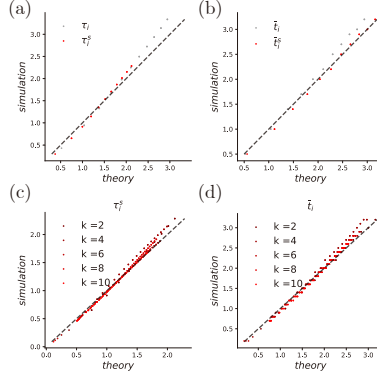


Figure S23: White noise dynamics in a ring lattice network ($N = 100$) with NSDD parameters ($\alpha = 1, \beta = 1$). Perturbations target random nodes. Threshold $\bar{\eta} = 1/e$ applies to (b,d). (a) Time constants (τ): Theoretical vs. simulated for exponential (τ_i) and sigmoidal (τ_i^s) models (degree $k = 4$); dotted lines indicate perfect match. (b) Relative propagation times ($\bar{t}$): Theoretical vs. simulated for exponential ($\bar{t}_i$) and sigmoidal ($\bar{t}_i^s$) forms ($k = 4$); (c) Time constants (τ_i^s): Theoretical vs. simulated across degrees $k = 2, 4, 6, 8, 10$; (d) Relative propagation times ($\bar{t}_i^s$): Theoretical vs. simulated across $k = 2, 4, 6, 8, 10$

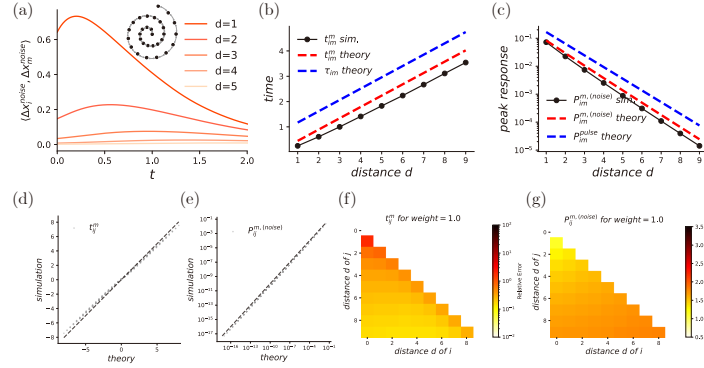


Figure S24: White noise response dynamics in an undirected chain ($N = 100$) with NSDD parameters ($\alpha = 1, \beta = 1$). Input targets the first node. Threshold $\bar{\eta} = 1/e$ applies to panels (c,e,g). (a) Crosscovariance dynamics for different shortest path distance d from input node. (b) Simulated and theoretical peak times between source-target node pairs (m, i), denoted as t_{im}^m , plotted against distance d . The corresponding time constants τ_{im} under constant input are also shown, exhibiting trends that closely align with those of t_{im}^m . (c) Simulated and theoretical peak response ($P_{im}^{m,(noise)}$) vs. d . The corresponding peak response P_{im}^{pulse} under pulse input are also shown, exhibiting trends that closely align with those of $P_{im}^{m,(noise)}$. (d) Theoretical vs. simulated peak times (initial/amended). (e) Theoretical vs. simulated peak response (initial/amended). (f) Relative error of t_{ij}^m ($|\text{sim} - \text{thr}|/\text{sim}$) for node pairs (i, j) when receiving input at node m : error increases with distance from input. (g) Peak amplitude ratio ($\text{thr } P / \text{sim } P$) for node pairs: accuracy higher near input. Only lower half of symmetric estimation matrix shown.

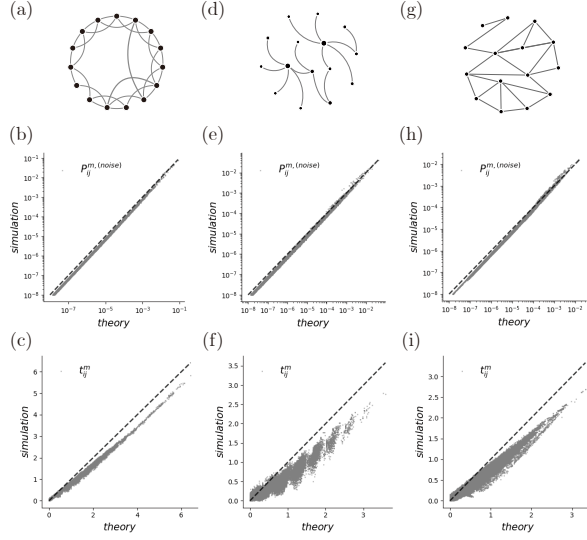


Figure S25: Crosscovariance dynamics following white noise perturbations in three networks. Simulations use $N = 100$, NSDD parameters ($\alpha = 1$, $\beta = 1$), with 10 random instances per network type, each run for 10 rounds. (a-c) Small-world schematic: initial ring with 2 nearest neighbors, edge rewiring probability 0.5; (d-f) Scale-free schematic: preferential attachment (1 edge per new node); (g-i) Geometric schematic: connection radius threshold 0.2; (b,e,h) Peak response ($P_{ij}^{m,(noise)}$): Theoretical vs. simulated values. (c,f,i) Peak response time (t_{ij}^m): Theoretical vs. simulated values. Theoretical peak response time estimates are systematically higher than simulated values.

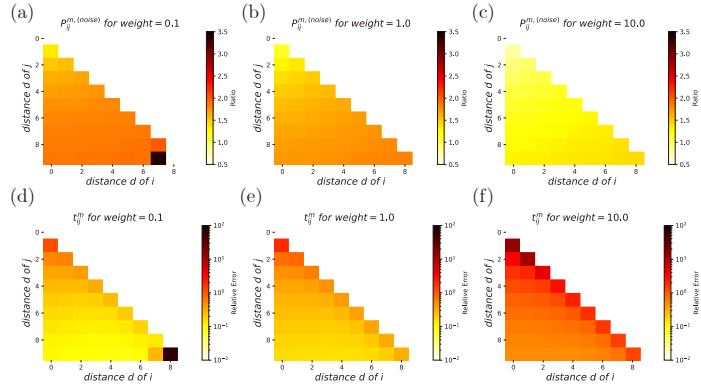


Figure S26: Crosscovariance dynamics in an undirected chain (NSDD system, $\beta = 1$) examining accuracy of peak response metrics for node pairs. Panels (a-c) show peak response ratio (thr P / sim P) for interaction weights $\alpha = 0.1, 1.0, 10.0$ respectively, where accuracy improves near the input node across all weights and generally increases with higher α values (consistent color bar). Panels (d-f) display relative error of peak time ($|\text{sim } t - \text{thr } t|/\text{sim } t$) for $\alpha = 0.1, 1.0, 10.0$ respectively, showing improved error far from the input node, with best overall accuracy at $\alpha = 0.1$ (lightest colors in d).

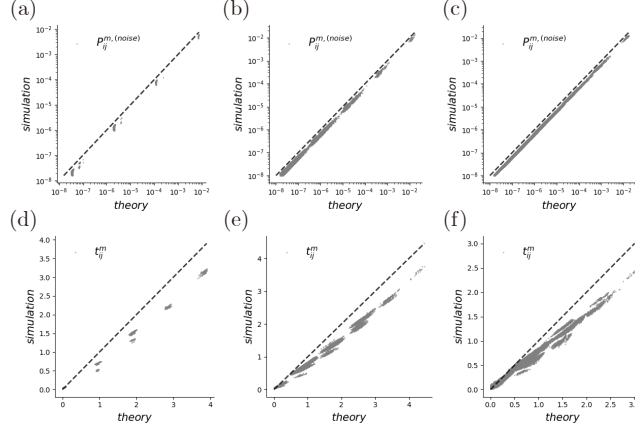


Figure S27: White noise perturbation in ER random networks ($N = 100$) with varying connecting probabilities p . Results for $p = 0.02$ (a,d), 0.05 (b,e), and 0.08 (c,f) show: (a-c) Simulated vs. theoretical peak response; (d-f) Simulated vs. theoretical peak response time (theoretical estimates generally higher than simulations). For each p , 10 random networks were generated and each simulated for 10 rounds. Node pairs with distance ≥ 10 are excluded due to significant errors.

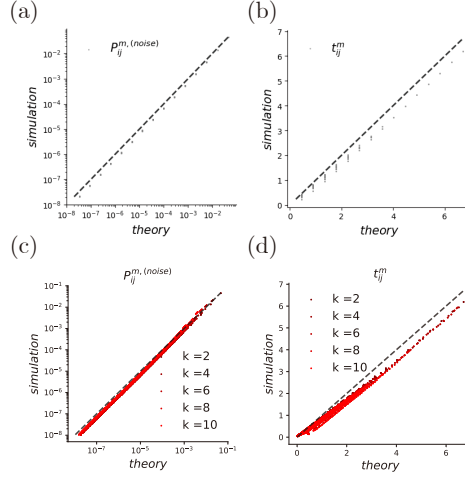


Figure S28: White noise perturbation in a ring lattice network ($N = 100$) with NSDD parameters ($\alpha = 1$, $\beta = 1$), targeting random nodes. Panels (a) and (b) show node degree $k = 4$: (a) Simulated vs. theoretical peak response ($P_{ij}^{m,(noise)}$) for all node pairs (dotted lines indicate perfect agreement); (b) Simulated vs. theoretical peak response time (t_{ij}^m). Panels (c) and (d) show results across degrees $k = 2, 4, 6, 8, 10$: (c) Peak response comparisons; (d) Peak response time comparisons. All panels analyze covariance between node pairs.

2 Extension to general formalism for constant input

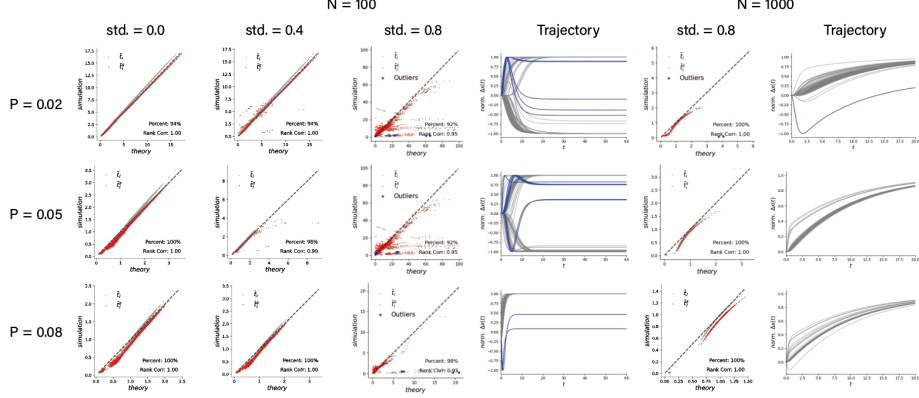


Figure S29: Constant perturbation spreading in ER random networks with Gaussian weight distributions (mean = 1.0). Results compare size $N = 100$ vs. 1000 and edge probabilities $p = 0.02$ (top row), 0.05 (middle), 0.08 (bottom). Theoretical vs. simulated relative times ($\bar{t}_i$, $\bar{t}_i^s$) are shown from 100 rounds. Theoretical estimates match simulations well in large dense networks (high average degree). For $std. = 0.8$, negative weights alter nodal traces in sparse networks, causing non-monotonic traces with prominent peaks and larger theoretical estimates. Negative theoretical values and negligible responses are excluded (inclusion % in bottom-right).

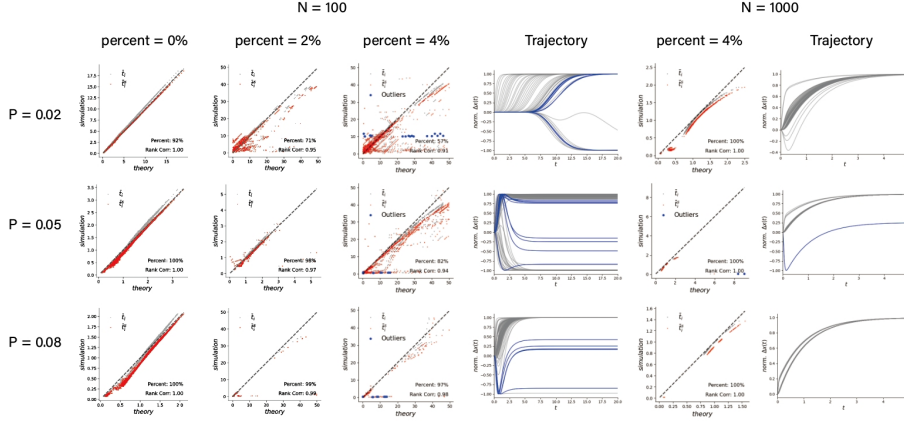


Figure S30: Constant perturbation spreading in ER random networks with varying negative link percentages. Results compare sizes $N = 100$ vs. 1000 and negative links: 0% (left column), 2% (middle), 4% (right). Theoretical vs. simulated relative times ($\bar{t}_i$, $\bar{t}_i^s$) from 100 rounds show that higher negative link percentages significantly alter nodal traces, particularly in low average degree networks.

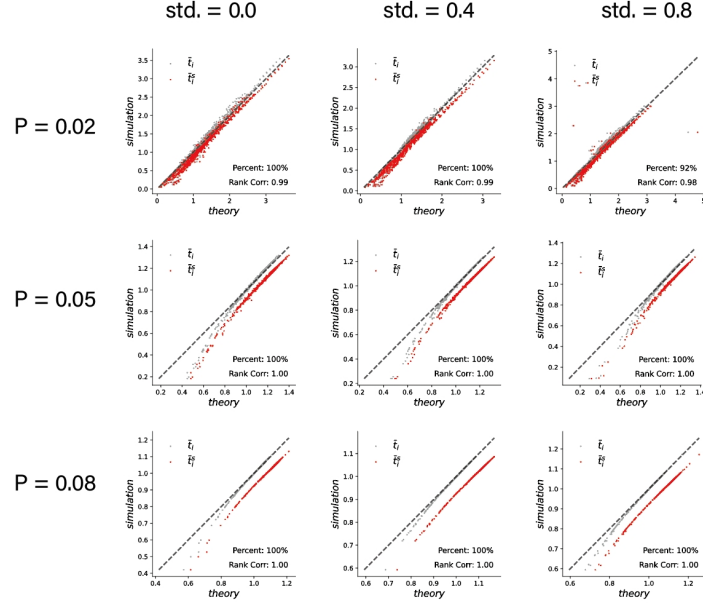


Figure S31: Constant perturbation spreading in ER random networks ($N = 100$) with Gaussian weights (mean = 10.0, varying $std.$). Results for edge probabilities $p = 0.02$ (top row), 0.05 (middle), 0.08 (bottom). Theoretical vs. simulated relative times ($\bar{t}_i$, $\bar{t}_i^s$) from 100 rounds show excellent agreement, particularly with exponential estimates $\bar{t}_i$.

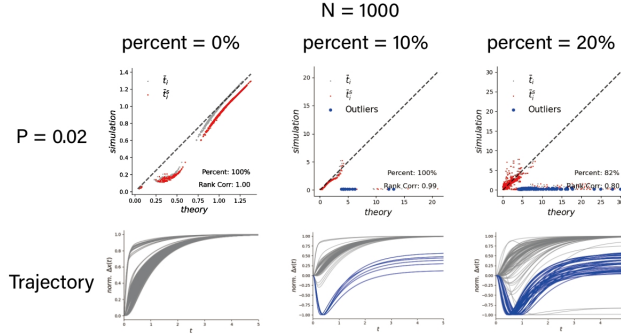


Figure S32: Constant perturbation propagation in ER random networks with connection probability 0.02 and varying proportions of negative links, for a network size of 1000. Simulated and theoretical results are compared over 100 realizations using extended relative time based on both exponential ($\bar{t}_i$) and sigmoidal ($\bar{t}_i^s$) measures. Theoretical estimates maintain high accuracy and strong rank correlations with simulation results for up to 20% negative links. Beyond this threshold, increasing inhibitory links renders the system unstable.

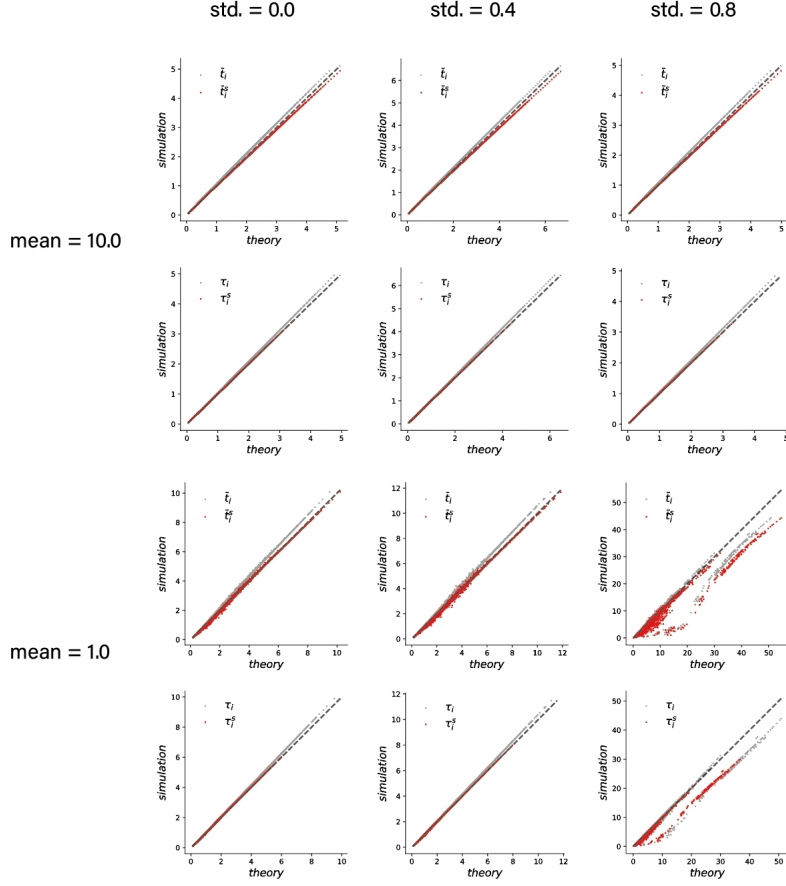


Figure S33: Constant perturbation propagation in ER random networks with connection probability 0.02 and network size 100. Simulated and theoretical results for relative time are compared using both exponential ($\bar{\tau}_i$) and sigmoidal ($\bar{\tau}_i^s$) functions over 100 realizations. The self-decay parameter is drawn from Gaussian distributions with means of 1.0 and 10.0, and varying standard deviations. When the mean is low, larger standard deviations reduce estimation accuracy due to the combined effects of self-decay and network interactions, causing the response traces to deviate from an exponential form, especially during the early, fast-response phase.

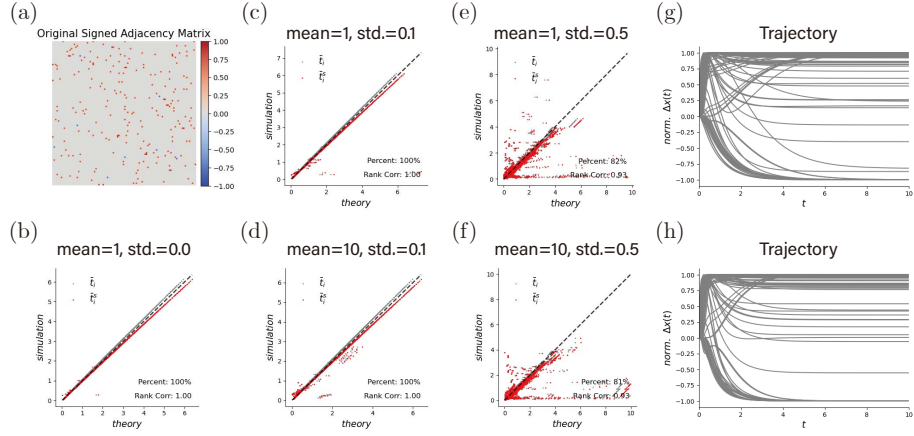


Figure S34: Constant perturbation propagation in ER random networks with connection probability 0.02, a fixed negative link ratio of 4%, and network size 100. Simulated and theoretical results are compared for extended relative time using both exponential ($\bar{t}_i^e$) and sigmoidal ($\bar{t}_i^s$) functions over 100 realizations. The self-decay parameter is set to -11 . Increasing the standard deviation reduces accuracy, and excessively large values can lead to system instability.

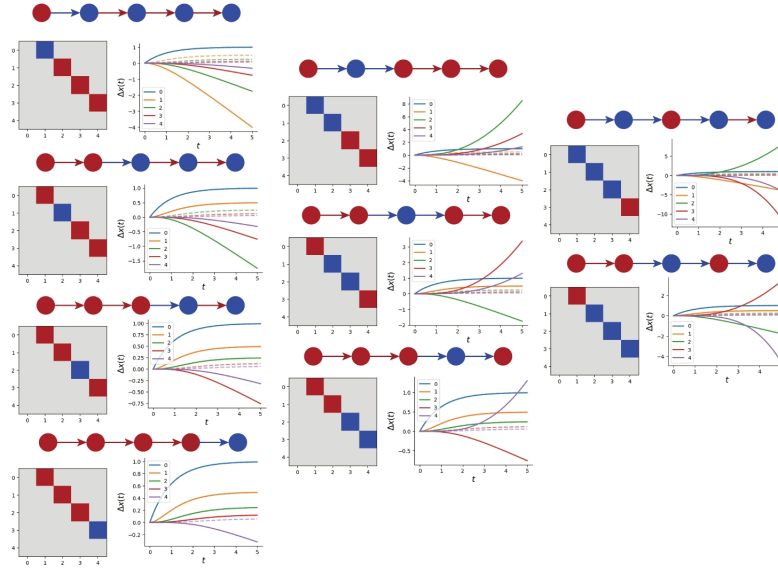


Figure S35: Constant perturbation propagation in a directed chain with substituted negative links, indicated in blue. Node color reflects the sign of the response series, with blue denoting a sign reversal.

3 Expansion for homogeneous and heterogeneous in-degree

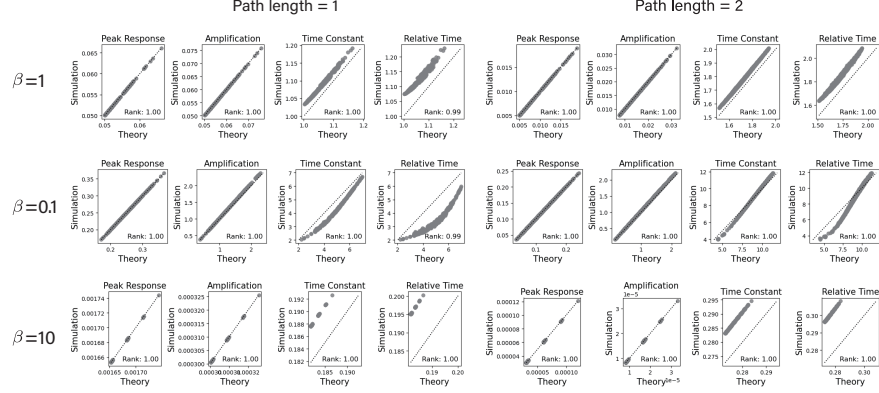


Figure S36: Expansion under homogeneous in-degree for constant input in sparse ER random networks with varying decay rates. Theoretical expansions at order 15 are evaluated on networks with $N = 100$ nodes and connection probability 0.05, resulting in an average unweighted degree of 5 and a weighted degree of 1 (with interaction weight $\alpha = 0.2$). The response time threshold is set to $\eta = 1 - 1/e$. Although temporal estimations show a fixed numerical bias, their rank correlations remain high.

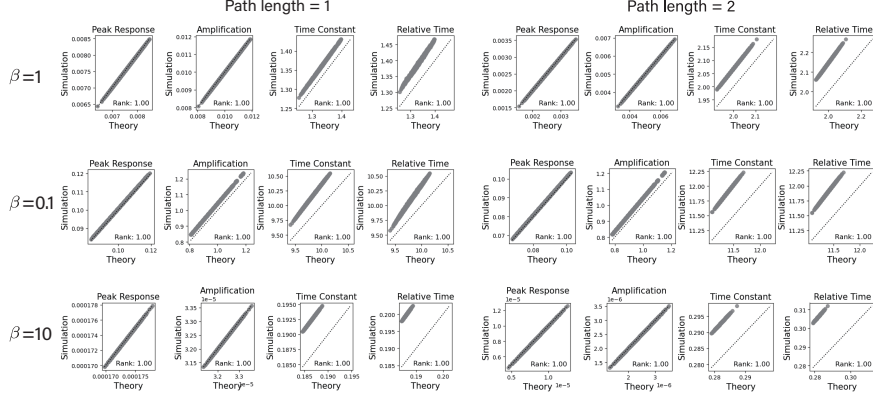


Figure S37: Expansion under homogeneous in-degree for constant input in dense ER random networks with varying decay rates. Theoretical expansions at order 15 are evaluated on networks with $N = 100$ nodes and connection probability 0.5, yielding an average unweighted degree of 50 and a weighted degree of 1 (with interaction weight $\alpha = 0.02$). The response time threshold is set to $\eta = 1 - 1/e$. Although temporal estimations exhibit a fixed numerical bias, their rank correlations remain high and improve with increasing path length.

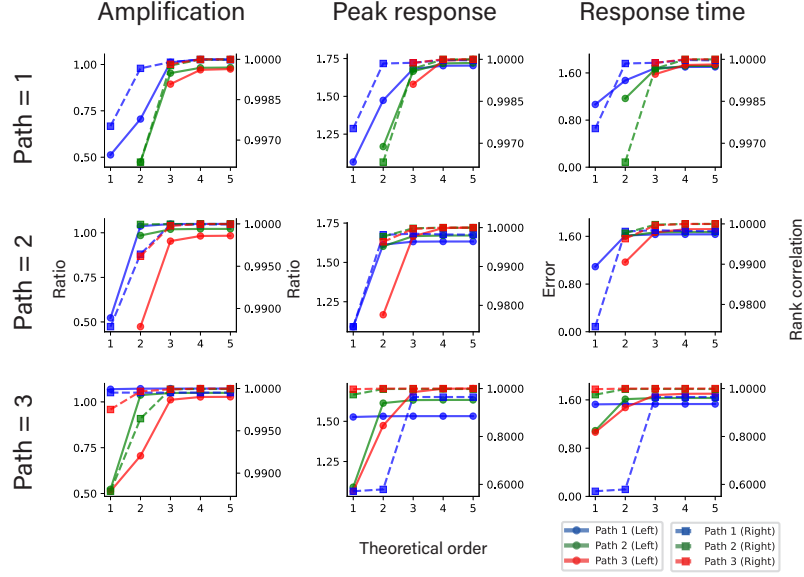


Figure S38: Expansion accuracy for crosscovariance metrics $C_{ij}^m(\tau)$ in homogeneous in-degree ER networks with $i \neq j \neq m$. Results are averaged over 100 realizations of networks with $N = 100$ nodes and connection probability 0.08. The shortest path length from source node m to target node i is denoted as Path, while color coding—blue (Path = 1), green (Path = 2), red (Path = 3)—represents the path length from m to j . The x-axis indicates the theoretical expansion order p , corresponding to the power A^p used in the series. For amplification, higher expansion orders and shorter paths lead to more accurate estimates and stronger rank correlations. For peak response and response time, a fixed estimation bias is present, but rank correlations still improve with increasing expansion order. Overall, deeper expansions enhance correlation performance despite persistent numerical bias.

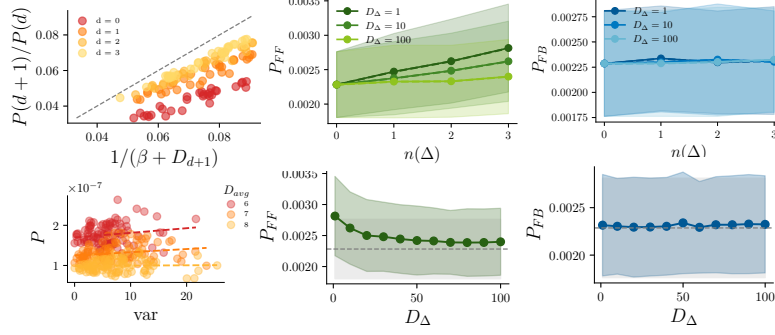


Figure S39: Propagation of white-noise input along a single path under heterogeneous degree configurations and triangular motifs. Left (upper): A white-noise input applied to the first node m produces propagation laws analogous to the constant-input case, with crosscovariance between source and target pairs scaling as $P(d+1)/P(d) \rightarrow A_{d \rightarrow d+1}/(\beta + D_{d+1})$, where d is the shortest path length. Left (lower): Increasing the mean degree decreases P , whereas increasing degree variance enhances P . Middle/Right: Feedforward (FF) and feedback (FB) triangles amplify $P_{\text{FF/FB}}$ proportionally to $n(\Delta)$, the number of triangular motifs, but with distinct slopes. Results are shown as averages over 1000 realizations, with shading indicating ± 1 SD. Large triangle-node degrees D_Δ suppress motif effects, recovering single-path dynamics ($n(\Delta) = 3$). Parameters: nodal decay rate $\beta = 10$; total path length $D = 5$; input strength $I_0 = 100$; unit chain weight $A_{d \rightarrow d+1} = 1$ for all d .

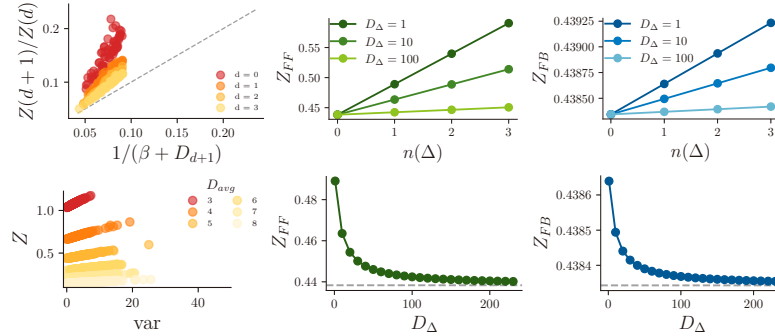


Figure S40: Propagation of constant input along a single path under heterogeneous degree configurations and triangular motifs. Left (upper): A constant input applied to the first node m produces propagation laws, with amplification between source and target pairs scaling as $Z(d+1)/Z(d) \rightarrow A_{d \rightarrow d+1}/(\beta + D_{d+1})$, where d is the shortest path length. Left (lower): Increasing the mean degree decreases Z , whereas increasing degree variance enhances Z . Middle/Right: Feedforward (FF) and feedback (FB) triangles amplify $Z_{\text{FF/FB}}$ proportionally to $n(\Delta)$, the number of triangular motifs, but with distinct slopes. Large triangle-node degrees D_Δ suppress motif effects, recovering single-path dynamics ($n(\Delta) = 1$). Parameters: nodal decay rate $\beta = 10$; total path length $D = 5$; input strength $I_0 = 10^6$; unit chain weight $A_{d \rightarrow d+1} = 1$ for all d .

4 Numerical accuracy test for self-response

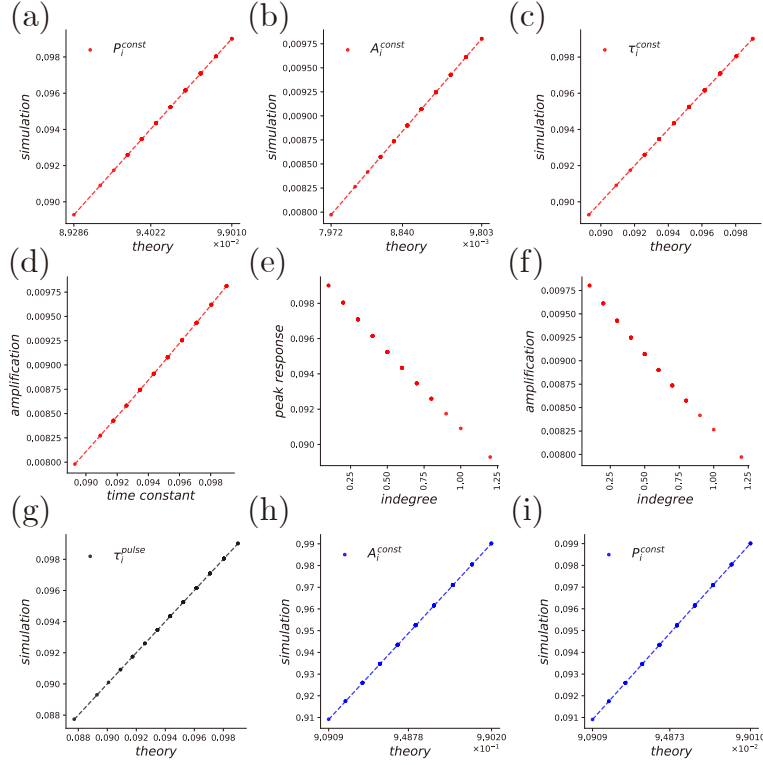


Figure S41: Response metrics for target nodes receiving constant, pulse, and square inputs. Simulations are conducted on an ER network of size 100, where each node receives input individually to observe its response. To satisfy expansion conditions, the interaction weight is set to 0.1 and the self-decay parameter to 10. (a–c) Simulated vs. theoretical results for peak response (a), amplification (b), and time constant (c) under constant input. (d) Peak response and estimated time constant share the same expression, yielding nearly identical simulated results. (e–f) Both peak response (e) and amplification (f) are inversely proportional to the indegree of the perturbed node. (g) Simulated vs. estimated time constant under pulse input. (h–i) Simulated vs. theoretical results for amplification (h) and peak response (i) under square input. Input duration is set to $10ms$.

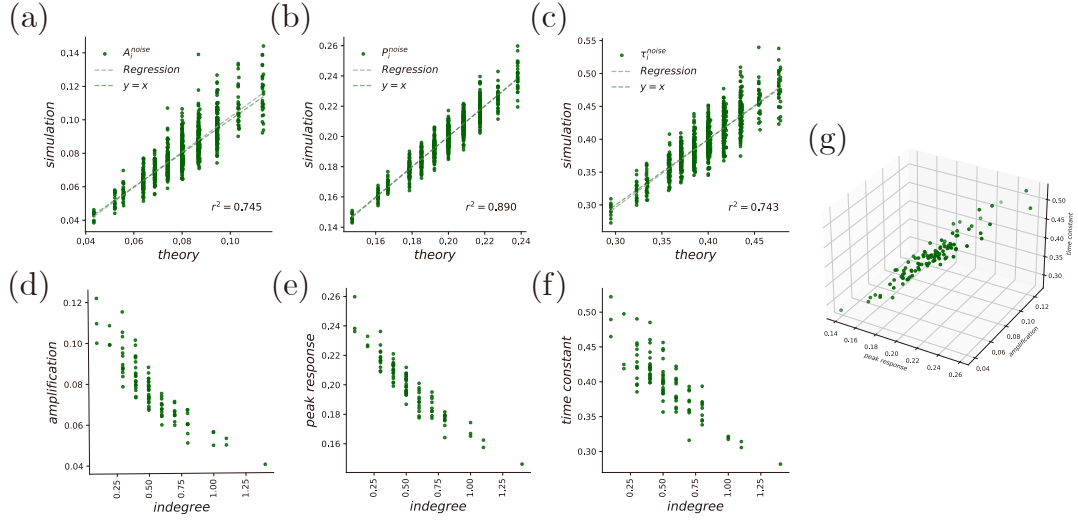


Figure S42: Numerical validation of theoretical estimations for self-response under noise input. Three response metrics are evaluated: amplification (a), peak response (b), and time constant (c), with theoretical predictions based on first-order truncation for simplicity. Simulations are conducted over 100 realizations of ER random networks. All metrics show inverse dependence on nodal indegree (d–f) and exhibit consistent trends, approximately lying along a one-dimensional manifold in the three-dimensional metric space (g).