

A NOTE ON THE PLEIJEL THEOREM FOR H -TYPE GROUPS

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ABSTRACT. We continue the program initiated by [FH25] and show that the Pleijel theorem holds unconditionally on all but four H -type groups.

1. INTRODUCTION AND MAIN RESULT

In this note, we continue the study of the Pleijel theorem for subriemannian laplacians defined on the product of a Heisenberg group $\mathbb{H}_n$ and $\mathbb{R}^k$ initiated by Frank and Helffer [FH25], and extend their argument to all but four H -type groups. We will follow closely their proof which relates the asymptotic behaviour of the number of nodal domains to the sharp constant of the L^2 -Sobolev inequality and the Weyl constant appearing in the asymptotics of the eigenvalue counting function.

Let $\mathbb{G}$ be a step two stratified Lie group with Lie algebra $\mathfrak{g}$, meaning $\mathfrak{g}$ admits the stratification $\mathfrak{g} = V_1 \oplus [V_1, V_1]$ for some linear subspace $V_1 \subset \mathfrak{g}$. We say $\mathbb{G}$ is a H -type group if $\mathfrak{g}$ is equipped with an inner product $\langle \cdot, \cdot \rangle$ with the property if $\mathfrak{z}$ denotes the centre of $\mathfrak{g}$, then

- (1) $[\mathfrak{g}^\perp, \mathfrak{g}^\perp] = \mathfrak{g}$ and
- (2) for every fixed $z \in \mathfrak{g}$, the map $J_z : \mathfrak{z}^\perp \mapsto \mathfrak{z}^\perp$ defined by

$$\langle J_z(v), w \rangle = \langle z, [v, w] \rangle$$

for all $w \in \mathfrak{z}^\perp$ is orthogonal for all $\langle z, z \rangle = 1$.

If $n = \dim(\mathfrak{z}^\perp)$ and $m = \dim(\mathfrak{z})$, in which case the stratification is $\mathfrak{g} = \mathfrak{z}^\perp \oplus \mathfrak{z}$, then $\mathbb{G}$ can be equipped with a system of coordinates $(x, t) \in \mathbb{R}_x^n \times \mathbb{R}_t^m$ equipped with the group law

$$(x, t) \circ (\xi, \tau) = \left(x + \xi, t_1 + \tau_1 + \frac{1}{2} \langle U^{(1)} x, \xi \rangle, \dots, t_n + \tau_n + \frac{1}{2} \langle U^{(n)} x, \xi \rangle \right)$$

for a collection $U^{(1)}, \dots, U^{(m)}$ of $n \times n$ skew-symmetric, orthogonal, and anticommuting matrices, see for instance [BLU07, Theorem 18.2.1]. These conditions imply the $U^{(1)}, \dots, U^{(m)}$ are linearly independent and that n is even. Consequently in the sequel we shall always write $2n = \dim(\mathfrak{z}^\perp)$ for some $n \in \mathbb{N}$ and $\mathbb{G} \cong \mathbb{R}_x^{2n} \times \mathbb{R}_t^m$. Associated with the group law is a canonical family of vector fields $X_1, \dots, X_{2n}$. The canonical negative sublaplacian on $\mathbb{G}$ is given by

$$(1.1) \quad \Delta^{\mathbb{G}} = - \sum_{i=1}^{2n} X_i^* X_i = \Delta_x^{\mathbb{R}^{2n}} + \frac{1}{4} |x|^2 \Delta_t^{\mathbb{R}^m} + \sum_{j=1}^m \langle U^{(j)} x, \nabla_x \rangle \partial_{t_j}$$

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where X_i^* is the adjoint of X_i with respect to Lebesgue measure $d\xi$ on $\mathbb{G}$.

We recall the results in the setting of [FH25]. Let $\mathbb{H}_n$ be the $(2n+1)$ -dimensional Heisenberg group, a H -type group according to the previous definition with $m=1$ and $U^{(1)}$ a block diagonal matrix of the form

$$U^{(1)} = \begin{pmatrix} 0 & -\mathbb{I}_n \\ \mathbb{I}_n & 0 \end{pmatrix}$$

Consider the space $\mathbb{H}_n \times \mathbb{R}^k$, where $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$, equipped with its sublaplacian

$$\Delta^{\mathbb{H}_n \times \mathbb{R}^k} = \Delta^{\mathbb{H}_n} \otimes \mathbb{I}^{\mathbb{R}^k} + \mathbb{I}^{\mathbb{H}_n} \otimes \Delta^{\mathbb{R}^k}.$$

If $\Omega \subset \mathbb{H}_n \times \mathbb{R}^k$ is a domain of finite measure, it was shown that $-\Delta_{\Omega}^{\mathbb{H}_n \times \mathbb{R}^k}$, the selfadjoint Dirichlet realisation of $-\Delta^{\mathbb{H}_n \times \mathbb{R}^k}$ on Ω , has discrete spectrum and we denote by $(\lambda_{\ell}(\Omega))_{\ell \geq 1}$ its eigenvalues, which are arranged in nondecreasing order and counted with multiplicity, and by $(\varphi_{\ell}(\Omega))_{\ell \geq 1}$ its corresponding eigenfunctions.

If f is continuous on Ω , then its nodal set is

$$\mathcal{N}(f) = \overline{\{x \in \Omega \mid f(x) = 0\}}$$

and a nodal domain is a connected component of $\Omega \setminus \mathcal{N}(f)$. Since it was also shown the eigenfunctions $\varphi_{\ell}(\Omega)$ are smooth in Ω , the nodal domains are well defined and we denote by $\nu_{\ell}(\Omega)$ the number of nodal domains of $\varphi_{\ell}(\Omega)$. Then [FH25, Theorems 7.1 and 9.1] assert

$$\begin{aligned} \limsup_{\ell \rightarrow \infty} \frac{\nu_{\ell}(\Omega)}{\ell} &\leq \gamma(\mathbb{H}_n \times \mathbb{R}^k) = C^{\text{FK}}(\mathbb{H}_n \times \mathbb{R}^k)^{-Q/2} \mathcal{W}(\mathbb{H}_n \times \mathbb{R}^k)^{-1} \\ (1.2) \qquad \qquad \qquad &\leq C^{\text{Sob}}(\mathbb{H}_n \times \mathbb{R}^k)^{-Q/2} \mathcal{W}(\mathbb{H}_n \times \mathbb{R}^k)^{-1}, \end{aligned}$$

where $C^{\text{FK}}(\mathbb{H}_n \times \mathbb{R}^k)$ and $\mathcal{W}(\mathbb{H}_n \times \mathbb{R}^k)$ are the Faber-Krahn and Weyl constants respectively appearing in the lower bound on the first eigenvalue

$$(1.3) \qquad \qquad \qquad \lambda_1(\Omega) \geq C^{\text{FK}}(\mathbb{H}_n \times \mathbb{R}^k) |\Omega|^{-2/Q}$$

and the Weyl asymptotics for the eigenvalue counting function

$$(1.4) \qquad \lim_{\lambda \rightarrow \infty} N(\lambda, -\Delta_{\Omega}^{\mathbb{H}_n \times \mathbb{R}^k}) = |\{\ell \in \mathbb{N} \mid \lambda_{\ell}(\Omega) < \lambda\}| \sim \mathcal{W}(\mathbb{H}_n \times \mathbb{R}^k) |\Omega| \lambda^{Q/2}$$

and where $Q = Q(n, k) = 2n + 2 + k$ is the homogeneous dimension of $\mathbb{H}_n \times \mathbb{R}^k$, and $C^{\text{Sob}}(\mathbb{H}_n \times \mathbb{R}^k)$ is the sharp constant in the L^2 -Sobolev inequality

$$(1.5) \qquad \int_{\mathbb{H}_n \times \mathbb{R}^k} (-\Delta^{\mathbb{H}_n \times \mathbb{R}^k} u) \bar{u} d\xi \geq C^{\text{Sob}}(\mathbb{H}_n \times \mathbb{R}^k) \left(\int_{\mathbb{H}_n \times \mathbb{R}^k} |u|^{2Q/(Q-2)} d\xi \right)^{(Q-2)/Q}$$

valid for $u \in C_0^{\infty}(\mathbb{H}_n \times \mathbb{R}^k)$.

On $\mathbb{R}^k$, the Courant theorem [Cou23] asserts $\nu_{\ell}(\Omega) \leq \ell$, that is φ_{ℓ} has at most ℓ nodal domains, while the Pleijel theorem [Ple56, §5] asserts the existence of a constant $\gamma(\mathbb{R}^k)$ independent of Ω such that

$$\limsup_{\ell \rightarrow \infty} \frac{\nu_{\ell}(\Omega)}{\ell} \leq \gamma(\mathbb{R}^k)$$

and $\gamma(\mathbb{R}^k) < 1$ for $k \geq 2$, that is $\nu_{\ell}(\Omega) = \ell$ for finitely many ℓ .

It was shown in [FH25, Theorem 7.2] that Pleijel's theorem $\gamma(\mathbb{H}_n \times \mathbb{R}^k) < 1$ holds unconditionally for all but four pairs of $(n, k) \in \mathbb{N} \times \mathbb{N}_0$ as a consequence of (1.2) and the sharp constant in the L^2 -Sobolev inequality for Heisenberg groups [JL88, Corollary C], see also [FL12, Theorem 2.1], and otherwise holds for all pairs of (n, k) assuming the validity of the Pansu conjecture [Pan83] concerning the isoperimetric problem on the Heisenberg group and which gives a better bound than (1.2) on $C^{\text{FK}}(\mathbb{H}_n \times \mathbb{R}^k)$, see [FH25, Proposition 11.1]. The goal of this note is to modestly extend the former result to H -type groups using the recent result of Yang for the sharp constant in the L^2 -Sobolev inequality [Yan24].

Theorem 1. *Let $\mathbb{G} \cong \mathbb{R}_x^{2n} \times \mathbb{R}_t^m$ be a H -type group. Then Pleijel's theorem $\gamma(\mathbb{G}) < 1$ holds for all but $(n, m) \in \{(1, 1), (2, 1), (3, 1), (2, 2)\}$.*

Remark 1. If $m = 1$ then $\mathbb{G}$ is isomorphic to a Heisenberg group. Otherwise, see [BLU07, Example 18.1.3] for an example of a H -type group with $(n, m) = (2, 2)$.

2. PROOF OF MAIN RESULT

We consider the analogue of (1.2) on a H -type group which reads

$$\limsup_{\ell \rightarrow \infty} \frac{\nu_\ell(\Omega)}{\ell} \leq C^{\text{Sob}}(\mathbb{G})^{-Q/2} \mathcal{W}(\mathbb{G})^{-1} =: \tilde{\gamma}(\mathbb{G})$$

for $Q = Q(n, m) = 2n + 2m$ the homogeneous dimension of $\mathbb{G}$. The technical details, for instance concerning the well-definedness of the nodal domain count $\nu_\ell(\Omega)$, the existence of the Weyl constant $\mathcal{W}(\mathbb{G})$, or the validity of (1.2), follow mutatis mutandis from the discussion given in [FH25]. For brevity and since the results depend only on the dimensions $2n$ and m , in the sequel we shall always indicate this dependence in the subscript.

First, by [Yan24, Theorem 1.2], we have

$$(2.1) \quad C_{n,m}^{\text{Sob}} = 4^{n/(n+m)} n(n+m-1) \pi^{(2n+m)/(2n+2m)} \left(\frac{\Gamma(n+m/2)}{\Gamma(2n+m)} \right)^{1/(n+m)}.$$

Note a H -type group in [Yan24] was defined with $m = \dim(\mathfrak{z}^\perp)$ and $n = \dim(\mathfrak{z})$, so (2.1) comes from replacing m and n with $2n$ and m respectively in their Sobolev constant. Second, to compute the Weyl constant, we follow the diagonalisation procedure in [HL08; FH25].

Proposition 1. *Let $\mathbb{G} \cong \mathbb{R}_x^{2n} \times \mathbb{R}_t^m$ be a H -type group. Let*

$$(2.2) \quad c_{n,m} = \sum_{k \geq 0} \binom{k+n-1}{k} \frac{1}{(2k+n)^{n+m}}.$$

Then

$$(2.3) \quad \mathcal{W}_{n,m} = \frac{\omega_{m-1}}{(2\pi)^{n+m}} \frac{1}{n+m} c_{n,m}$$

where ω_{m-1} is the surface volume of the $(m-1)$ -dimensional sphere $\mathbb{S}^{m-1}$. In particular,

$$(2.4) \quad \limsup_{\ell \rightarrow \infty} \frac{\nu_\ell(\Omega)}{\ell} \leq \tilde{\gamma}_{n,m} := \frac{1}{2^{n-m+1}} \frac{n+m}{n^{n+m}(n+m-1)^{n+m}} \frac{\Gamma(m/2)\Gamma(2n+m)}{\Gamma(n+m/2)} \frac{1}{c_{n,m}}.$$

Proof. Reasoning as in [FH25, Equation 8.2], we want to show

$$\mathbb{1}(-\Delta^{\mathbb{G}} < \lambda)((x, t), (x, t)) = \mathcal{W}_{n,m} \lambda^{n+m}$$

for some $\mathcal{W}_{n,m} > 0$. By taking the partial Fourier transform in the central variables $t \in \mathbb{R}_t^m$ with τ the Fourier variable dual to t , we see that while $\mathcal{F}_t(-\Delta^{\mathbb{G}})\mathcal{F}_t^* := -\Delta_{|\tau|}^{\mathbb{G}}$ is *not* a sum of n decoupled Landau hamiltonians

$$-\Delta_{|\tau|}^{\mathbb{H}_n} = -\sum_{j=1}^n \left(\partial_{x_j} + \frac{i}{2} x_{j+n} |\tau| \right)^2 + \left(\partial_{x_{j+n}} - \frac{i}{2} x_j |\tau| \right)^2$$

for $\tau \neq 0$, as in the case of the Heisenberg group, they *are* unitarily equivalent, that is there exists an orthogonal matrix $T_\tau \in O(\mathbb{R}_x^{2n})$ such that $-\Delta_{|\tau|}^{\mathbb{G}} = T_\tau \circ (-\Delta_{|\tau|}^{\mathbb{H}_n}) \circ T_\tau^{-1}$, see for instance [Nie24, Lemma 2.1]. Consequently their spectra agree, and since the densities of the eigenprojections of a single Landau hamiltonian are constant on the diagonal, the densities of the T_τ -conjugated eigenprojections are equal on the diagonal to that same constant and hence each τ -fibre once again contributes $(2\pi)^{-n} |\tau|^n$ to the on-diagonal spectral density. Then taking the inverse partial Fourier transform, we obtain

$$\begin{aligned} \mathbb{1}(-\Delta^{\mathbb{G}} < \lambda)((x, t), (x, t)) &= \int_{\mathbb{R}^m} \frac{d\tau}{(2\pi)^m} \frac{|\tau|^n}{(2\pi)^n} \sum_{k \in \mathbb{N}_0^n} \mathbb{1}(|\tau| (2|k| + n) < \lambda) \\ &= \frac{\omega_{m-1}}{(2\pi)^{n+m}} \int_0^\infty \tau^{n+m-1} \sum_{k \in \mathbb{N}_0^n} \mathbb{1}(\tau(2|k| + n) < \lambda) d\tau \\ &= \frac{\omega_{m-1}}{(2\pi)^{n+m}} \frac{1}{n+m} \sum_{k \in \mathbb{N}_0^n} \left(\frac{\lambda}{2(|k| + n)} \right)^{n+m} \\ &= \frac{\omega_{m-1}}{(2\pi)^{n+m}} \frac{\lambda^{n+m}}{n+m} \sum_{k \geq 0} \binom{k+n-1}{k} \frac{1}{(2k+n)^{n+m}}. \end{aligned}$$

Note that if $m = 1$ then (2.3) recovers [FH25, Equation 8.3] up to a factor of 4, which is due to the choice of normalisation used in their definition of the Heisenberg group, and [FH25, Equation 9.2], which is independent of the normalisation. $\square$

We now show $\tilde{\gamma}_{n,m} < 1$ for all but finitely many pairs of $(n, m) \in \mathbb{N} \times \mathbb{N}$. In fact it suffices to consider a strictly smaller subset since it turns out that there are constraints on n and m , characterised by [Kap80, Corollary 1], in the sense there are no H -type groups for some (n, m) . In particular, if $n \in \{1, 3\}$ then necessarily $m = 1$, and if $n = 2$ then $m \leq 3$. These are precisely the four exceptions given in the statement of the theorem, excluding the specific case of $(n, m) = (2, 3)$ which we treat separately. Note if n is odd then the centre is always 1-dimensional and by [BLU07, Remark 18.2.6] it follows $\mathbb{G} \cong \mathbb{H}_n$.

We first prove $\tilde{\gamma}_{n,m}$ decreases with $n \geq 1$ for all $m \geq 1$. Consider the quotient for $n \geq 2$

$$(2.5) \quad \varphi_{n,m} := \frac{\tilde{\gamma}_{n,m}}{\tilde{\gamma}_{n-1,m}} = \frac{(n-1)^{n+m-1} (n+m-2)^{n+m-1} (n+m) (2n+m-1) c_{n-1,m}}{n^{n+m} (n+m-1)^{n+m+1} c_{n,m}}.$$

In order to prove a lower bound for $c_{n,m}/c_{n-1,m}$, we study the quotient

$$(2.6) \quad \frac{\binom{k+n-1}{k}(2k+n)^{-n-m}}{\binom{k+n-2}{k}(2k+n-1)^{-n-m+1}} = \frac{1}{n-1} \frac{k+n-1}{2k+n} \left(1 - \frac{1}{2k+n}\right)^{n+m-1}$$

of the k -th term in the numerator to the k -th term in the denominator. The derivative with respect to k of the right hand side is $1/(n-1)$ multiplied by

$$\frac{d}{dk} \left(\frac{k+n-1}{2k+n} \left(1 - \frac{1}{2k+n}\right)^{n+m-1} \right) = \frac{2k(m+1) + (n-1)(n+2m)}{(2k+n-1)^2(2k+n)} \left(1 - \frac{1}{2k+n}\right)^{n+m}$$

which is positive. The right hand side of (2.6) therefore minimises at $k=0$ and hence

$$\frac{c_{n,m}}{c_{n-1,m}} \geq \frac{1}{n} \left(1 - \frac{1}{n}\right)^{n+m-1}.$$

Inserting this into (2.7) reduces the problem to showing

$$\varphi_{n,m} \leq \frac{(n+m-2)^{n+m-1}(n+m)(2n+m-1)}{(n+m-1)^{n+m+1}} \leq \frac{1}{e} \frac{(n+m)(2n+m-1)}{(n+m-1)^2} < 1.$$

It suffices to bound the fraction from above by $\frac{5}{2}$ since

$$\begin{aligned} (n+m)(2n+m-1) - \frac{5}{2}(n+m-1)^2 &= -\frac{3}{2}m^2 - 2(n-2)m - \frac{1}{2}(n^2 - 8n + 5) \\ &\leq -\frac{3}{2}m^2 - 4m + \frac{11}{2} \end{aligned}$$

is nonpositive for $m \geq 1$ so $\varphi_{n,m} \leq \frac{5}{2e} < 1$ and hence $\tilde{\gamma}_{n,m}$ is monotonic in n for each m .

On the other hand, although the numerics (see appendix) appear to support the hypothesis $\tilde{\gamma}_{n,m}$ also decreases with respect to m uniform in n , it is not clear how to generalise the previous proof since the analogue of (2.6), that is the quotient of the k -th term in $c_{n,m}$ and $c_{n,m-1}$ respectively, is $1/(2k+n)$ which is not bounded below.

However, if we first bound $c_{n,m}$ from below by truncating at the first term $n^{-(n+m)}$ in the series so that

$$\tilde{\gamma}_{n,m} \leq \bar{\gamma}_{n,m} := \frac{1}{2^{n-m+1}} \frac{n+m}{(n+m-1)^{n+m}} \frac{\Gamma(m/2)\Gamma(2n+m)}{\Gamma(n+m/2)},$$

we can prove $\bar{\gamma}_{n,m}$ decreases with $m \geq 1$ for all $n \geq 1$. Consider the quotients for $m \geq 2$

$$\begin{aligned} \psi_{n,m} &:= \frac{\bar{\gamma}_{n,m}}{\bar{\gamma}_{n,m-1}} = 4 \frac{(n+m-2)^{n+m-1}(n+m)}{(n+m-1)^{n+m+1}} \frac{\Gamma(m/2)\Gamma(n+m/2+1/2)}{\Gamma(m/2-1/2)\Gamma(n+m/2)} \\ (2.7) \quad &\leq \frac{4}{e} \frac{n+m}{(n+m-1)^2} \frac{\Gamma(m/2)\Gamma(n+m/2+1/2)}{\Gamma(m/2-1/2)\Gamma(n+m/2)}. \end{aligned}$$

We distinguish the particular case $n=1$ where (2.7) reads

$$\psi_{1,m} \leq \frac{4}{e} \frac{m+1}{m^2} \frac{(m-1)(m+1)}{2m} = \frac{2}{e} \frac{(m-1)(m+1)^2}{m^3}.$$

The derivative of the right hand side is zero at $m = 3$ where the second derivative is negative so $\psi_{1,m} \leq \frac{64}{27e} < 1$. Otherwise for $n \geq 2$ and replacing m with $m + 1$ for convenience, by Wendel's inequality

$$\psi_{n,m+1} \leq \frac{4}{e} \frac{n+m+1}{(n+m)^2} \sqrt{m/2} \sqrt{n+m/2+1/2}.$$

Squaring and performing a change of variable $\ell = n + m \geq 2 + m \geq 3$ we find

$$\begin{aligned} \psi_{n,m+1}^2 &\leq \frac{4}{e^2} \frac{m(n+m+1)^2(2n+m+1)}{(n+m)^4} = \frac{4}{e^2} \frac{m(\ell+1)^2(2\ell-m+1)}{\ell^4} \\ &= \frac{4}{e^2} \frac{(\ell+1)^2}{\ell^4} (-m^2 + (2\ell+1)m). \end{aligned}$$

For each fixed ℓ the parabola $-m^2 + (2\ell+1)m$ is increasing on $m \in (0, \ell + \frac{1}{2})$. But since $m \leq \ell - 2$, it maximises at this boundary and we obtain

$$\psi_{n,m+1}^2 \leq \frac{4}{e^2} \frac{(\ell+1)^2(\ell^2 + \ell - 6)}{\ell^4} \leq \frac{4}{e^2} \left(1 + \frac{3}{\ell} - \frac{3}{\ell^2}\right).$$

The derivative of the right hand side is negative for $\ell \geq 3$ so $\psi_{n,m+1}^2 \leq \frac{20}{3e^2} < 1$ and hence $\bar{\gamma}_{n,m}$ is monotonic in m for each n .

Combining the two monotonicity arguments yields for $n \geq 4$ and $m \geq 2$:

$$\tilde{\gamma}_{n,m} \leq \tilde{\gamma}_{4,m} \leq \bar{\gamma}_{4,m} \leq \bar{\gamma}_{4,2} = \frac{2268}{3125} \approx 0.7258 < 1$$

while the case $m = 1$ is the content of [FH25, Theorem 7.2]. All that remains is to also verify $\tilde{\gamma}_{2,3} < 1$ and indeed $\tilde{\gamma}_{2,3} \leq \bar{\gamma}_{2,3} = \frac{15}{16} = 0.9375 < 1$.

In light of the previous results of Frank and Helffer for $\mathbb{H}_n \times \mathbb{R}^k$ and the present results for H -type groups $\mathbb{G}_{n,m} \cong \mathbb{R}_x^{2n} \times \mathbb{R}_t^m$, it seems reasonable to expect the Pleijel theorem (at least when deduced via the L^2 -Sobolev constant, not the Faber-Krahn constant) holds also for $\mathbb{G}_{n,m} \times \mathbb{R}^k$ with the exceptions depending only on n , m , and k , or better yet perhaps only on their (homogeneous) sum. Another possible direction may be extension to the larger class of M etivier groups where the sharp L^2 -Sobolev constant is not known and the spectral decomposition of the sublaplacian is more complicated, see for instance [MM14,  2].

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APPENDIX

For the reader's convenience we collect some numeric computations of $\tilde{\gamma}_{n,m}$ and $\bar{\gamma}_{n,m}$. Rows correspond to fixed n , and columns to fixed m . Grey shading indicates inadmissible pairs of (n, m) while red highlighting indicates values greater than 1. Note in the first figure the first column agrees with the numerics found in [FH25, p. 44].

n/m	1	2	3	4	5	6	7	8	9	10
1	3.2423	2.1392	1.5574	1.1666	0.8835	0.6718	0.5115	0.3893	0.2960	0.2248
2	1.8238	1.2325	0.8662	0.6221	0.4530	0.3329	0.2462	0.1828	0.1361	0.1015
3	1.0689	0.7141	0.4892	0.3413	0.2414	0.1726	0.1244	0.0903	0.0659	0.0482
4	0.6249	0.4120	0.2771	0.1893	0.1310	0.0917	0.0647	0.0461	0.0330	0.0237
5	0.3626	0.2365	0.1568	0.1054	0.0718	0.0494	0.0343	0.0240	0.0169	0.0120
6	0.2089	0.1350	0.0885	0.0588	0.0395	0.0268	0.0184	0.0127	0.0088	0.0062
7	0.1196	0.0767	0.0499	0.0328	0.0218	0.0146	0.0099	0.0068	0.0046	0.0032
8	0.0681	0.0434	0.0280	0.0183	0.0120	0.0080	0.0054	0.0036	0.0025	0.0017
9	0.0386	0.0245	0.0157	0.0102	0.0067	0.0044	0.0029	0.0020	0.0013	0.0010
10	0.0218	0.0138	0.0088	0.0057	0.0037	0.0024	0.0016	0.0011	0.0007	0.0005

FIGURE 1. Values of $\tilde{\gamma}_{n,m}$ for $1 \leq n, m \leq 10$.

n/m	1	2	3	4	5	6	7	8	9	10
1	4.0000	2.2500	1.5803	1.1719	0.8847	0.6722	0.5116	0.3893	0.2960	0.2248
2	3.0000	1.4815	0.9375	0.6451	0.4609	0.3357	0.2472	0.1832	0.1362	0.1016
3	2.3704	1.0254	0.5898	0.3781	0.2558	0.1784	0.1269	0.0913	0.0663	0.0484
4	1.8750	0.7258	0.3841	0.2308	0.1483	0.0992	0.0681	0.0476	0.0337	0.0241
5	1.4746	0.5199	0.2558	0.1450	0.0888	0.0571	0.0379	0.0257	0.0178	0.0124
6	1.1523	0.3751	0.1730	0.0930	0.0545	0.0337	0.0217	0.0143	0.0096	0.0066
7	0.8953	0.2718	0.1184	0.0606	0.0341	0.0204	0.0127	0.0081	0.0054	0.0036
8	0.6921	0.1977	0.0817	0.0401	0.0217	0.0125	0.0076	0.0047	0.0030	0.0020
9	0.5329	0.1440	0.0568	0.0267	0.0140	0.0078	0.0046	0.0028	0.0018	0.0011
10	0.4087	0.1051	0.0397	0.0180	0.0091	0.0049	0.0028	0.0017	0.0010	0.0006

FIGURE 2. Values of $\bar{\gamma}_{n,m}$ for $1 \leq n, m \leq 10$.

REFERENCES

- [BLU07] Bonfiglioli, A., Lanconelli, E., and Uguzzoni, F. (2007). *Stratified Lie groups and potential theory for their sub-Laplacians*. Springer (cit. on pp. 1, 3, 4).
- [Cou23] Courant, R. (1923). “Ein allgemeiner Satz zur theorie der Eigenfunktionen selbstadjungierter Differentialausdrücke”. In: *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse* 1923, pp. 81–84 (cit. on p. 2).
- [FH25] Frank, R. L. and Helffer, B. (2025). “On Courant and Pleijel theorems for sub-Riemannian Laplacians”. en. In: *Journal de l’École polytechnique — Mathématiques* 12, pp. 1083–1160 (cit. on pp. 1–4, 6).
- [FL12] Frank, R. L. and Lieb, E. H. (2012). “Sharp constants in several inequalities on the Heisenberg group”. In: *Annals of Mathematics*, pp. 349–381 (cit. on p. 3).
- [HL08] Hansson, A. M. and Laptev, A. (2008). “Sharp spectral inequalities for the Heisenberg Laplacian”. In: *Groups and Analysis* 354.1, pp. 100–115 (cit. on p. 3).

- [JL88] Jerison, D. and Lee, J. M. (1988). “Extremals for the Sobolev inequality on the Heisenberg group and the CR Yamabe problem”. In: *Journal of the American Mathematical Society* 1.1, pp. 1–13 (cit. on p. 3).
- [Kap80] Kaplan, A. (1980). “Fundamental solutions for a class of hypoelliptic PDE generated by composition of quadratic forms”. In: *Transactions of the American Mathematical Society* 258.1, pp. 147–153 (cit. on p. 4).
- [MM14] Martini, A. and Müller, D. (2014). “Spectral multiplier theorems of Euclidean type on new classes of two-step stratified groups”. In: *Proceedings of the London Mathematical Society* 109.5, pp. 1229–1263 (cit. on p. 6).
- [Nie24] Nierdorf, L. (2024). “An L^p -Spectral Multiplier Theorem with Sharp p -Specific Regularity Bound on Heisenberg Type Groups”. In: *Journal of Fourier Analysis and Applications* 30.2, p. 22 (cit. on p. 4).
- [Pan83] Pansu, P. (1983). “An isoperimetric inequality on the Heisenberg group”. In: *Conference on differential geometry on homogeneous spaces (Turin, 1983)*. *Rend. Sem. Mat. Univ. Politec. Torino, Special*. 1983, pp. 159–174 (cit. on p. 3).
- [Ple56] Pleijel, A. (1956). “Remarks on Courant’s nodal line theorem”. In: *Communications on Pure and Applied Mathematics* 9.3, pp. 543–550 (cit. on p. 2).
- [Yan24] Yang, Q. (2024). “The optimal constant in the L^2 Folland-Stein inequality on the H-type group”. In: *Journal of Functional Analysis* 286.2, p. 110209 (cit. on p. 3).

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