# On better-quasi-ordering under graph minors

Agelos Georgakopoulos*

Mathematics Institute
University of Warwick
CV4 7AL, UK

October 23, 2025

**Abstract**

In the aftermath of the Robertson–Seymour Graph Minor Theorem, Thomas conjectured that the countable graphs are well-quasi-ordered under the minor relation. We prove that this conjecture, when restricted to graphs with no infinite paths (rays), is equivalent to the statement that the finite graphs are better-quasi-ordered, another well-known open problem. Even more, we prove that the latter implies that the countable rayless graphs are better-quasi-ordered.

We prove several other statements to be equivalent to the above, one of which being that the rayless countable graphs of rank $\alpha$ can be decomposed into exactly $\aleph_0$ minor-twin classes for every ordinal $\alpha < \omega_1$.

By restricting the latter statement to trees, and combining it with Nash-Williams' theorem that the infinite trees are well-quasi-ordered, we deduce as a side result that a minor-closed family of $\mathbb{N}$-labelled rayless forests is Borel —in the Tychonoff product topology— if and only if it does not contain all rayless forests.

As another side-result, we prove Seymour's self-minor conjecture for rayless graphs of any cardinality.

**Keywords:** graph minor, well-quasi-order, better-quasi-order, countable rayless graph, Borel set, self-minor conjecture.

**MSC 2020 Classification:** 05C83, 05C63, 06A07.

## 1 Introduction

The celebrated Robertson–Seymour Graph Minor Theorem [20]–[21] states that the finite graphs are well-quasi-ordered under the minor relation $<$. This paper focuses on two well-known problems that it left open:

**Conjecture 1.1** (Folklore [5, 17])**.** *The finite graphs are better-quasi-ordered (BQO) under the minor relation.*

**Conjecture 1.2** (Thomas' conjecture [25])**.** *The countable graphs are well-quasi-ordered (WQO) under the minor relation.*

*Supported by EPSRC grants EP/V048821/1 and EP/V009044/1.

While WQO is a simple notion, its strengthening BQO is much harder to define but has the benefit of being preserved by natural operations that occur in inductive arguments, making it an indispensable tool even when one is only interested to prove that an order is WQO; this idea has found applications in several areas including combinatorics [15, 26], order theory [13], set theory [14] and topology [3]. See [17] for a survey of these notions. According to Pequignot [17], the poset of finite graphs endowed with the minor relation is

*"the only naturally occurring WQO which is not yet known to be BQO".*

A graph is *rayless*, if it does not contain an (1-way) infinite path. The first main result of this paper provides a strong connection between the above conjectures, and much more:

**Theorem 1.1.** *The following statements are equivalent:*

*(i) The finite graphs are BQO;*

*(ii) The countable rayless graphs are WQO;*

*(iii) The countable rayless graphs are BQO.*

Thus we have reduced the better-quasi-ordering of the finite graphs to a comparatively simple statement about infinite graphs.

Commenting on Thomas' Conjecture 1.2 and similar questions, Robertson, Seymour & Thomas [19] wrote:

> *"There is not much chance of proving these conjectures because they imply that the set of all finite graphs is 'second-level better-quasi-ordered' by minor containment, which in itself seems to be a hopelessly difficult problem".*

The (easier) implication (ii) $\to$ (i) of Theorem 1.1 provides a far reaching strengthening of this claim: it says that the words 'second-level' can be deleted. Moreover, the implication (ii) $\to$ (iii) is a considerable strengthening. Interestingly, the proof of the latter implication passes via the implication (ii) $\to$ (i). Part of this proof applies to arbitrary quasi-orders (Corollary 11.8). Our most difficult result is the implication (i) $\to$ (ii).

While the above quote seems as timely as ever, the results of this paper provide new tools for attacking Conjectures 1.1 and 1.2, and point out interesting special cases. See Section 12 for more.

But let me first try to explain the above quote. What is meant by 'second-level better-quasi-ordered' here is probably the following. Given two sets of graphs $\mathcal{G}, \mathcal{G}'$, we write $\mathcal{G} <_* \mathcal{G}'$ if for every $G \in \mathcal{G}$ there is $H \in \mathcal{G}'$ such that $G < H$. Note that $<_*$ is a quasi-order on the powerset $\mathcal{P}(\mathcal{F})$ of the class $\mathcal{F}$ of finite graphs.

**Problem 1.3.** *Are the sets of finite graphs well-quasi-ordered under* $<_*$*?*

An equivalent formulation is whether the set of minor-closed families of finite graphs is well-quasi-ordered by the inclusion relation.

The connection between Problem 1.3 and Conjecture 1.1 is that one can take the definition of $<_*$ further, and extend it from an ordering of $\mathcal{P}(\mathcal{F})$ into an ordering of $\mathcal{P}(\mathcal{P}(\mathcal{F}))$, and iterate transfinitely in order to define better-quasi-ordering; see Section 2.4 for details. In particular, our Theorem 1.1 implies that if Thomas' Conjecture 1.2 has a positive answer, then so does Problem 1.3.

Motivated by Theorem 1.1, we undertake a thorough study of Thomas' conjecture for rayless graphs, noting that some long-standing open problems for infinite graphs have been settled in the rayless case [1, 2, 18]. As a warm-up, we will prove Seymour's (unpublished) self-minor conjecture for rayless graphs of any cardinality, which to the best of my knowledge was open even for countable graphs:

**Corollary 1.2.** *For every infinite rayless graph $G$, there is a minor model of $G$ in itself which is not the identity.*

(Oporowski [16] has found uncountable counterexamples to Seymour's self-minor conjecture, which of course contain rays.)

A well-known equivalent way to define what it means for a quasi-order $(Q, \leqslant)$ to be well-quasi-ordered is to say that it has no infinite antichain and no infinite descending chain. None of these conditions implies the other in general, but we will show that in our setup they are in fact equivalent. Let $\mathcal{R}$ denote the class of countable rayless graphs. Our second main result can be summarized as follows:

**Theorem 1.3.** *The following statements are equivalent:*

*(a) $\mathcal{R}$ is well-quasi-ordered;*

*(b) $\mathcal{R}$ has no infinite descending chain;*

*(c) $\mathcal{R}$ has no infinite antichain;*

*(d) for every ordinal $\alpha < \omega_1$, the number of minor-twin classes of countable rayless graphs of rank $\alpha$ is $\aleph_0$.*

As a consequence, to prove that $\mathcal{R}$ is well-quasi-ordered, it would suffice to prove that the rayless graphs of rank $\alpha$ are well-quasi-ordered for every ordinal $\alpha < \omega_1$. But the most interesting aspect of Theorem 1.3 is the equivalence of the well-quasi-ordering of $\mathcal{R}$ to the cardinality condition (d), which I will now explain. We say that two graphs $G, H$ are *minor-twins*, if both $G < H$ and $H < G$ hold. Each rayless graph can be assigned an ordinal number, called its *rank*, by recursively decomposing $G$ into graphs of smaller ranks (Section 2.3) similarly to the definition of rank for Borel sets or Hausdorff rank for linear orders. It is known that every ordinal $\alpha$ smaller that the least uncountable ordinal $\omega_1$ is the rank of some countable graph $G$. Assuming $\neg$CH, i.e. that $\aleph_1 < 2^{\aleph_0}$, an alternative way to formulate (d) is to say that $\mathcal{R}$ consists of exactly $\aleph_1$ minor-twin equivalence classes.

We will prove a refinement of Theorem 1.3 (Theorem 9.1) whereby $\mathcal{R}$ is replaced by the subclass of graphs of (up to) a given rank. The rank 0 case of this refinement provides a statement equivalent to the Graph Minor Theorem (Corollary 4.2).

This refinement is also needed for the proof of the implication (i) $\to$ (ii) of Theorem 1.1. Moreover, it adds several further equivalent statements to those of Theorems 1.1 and 1.3 (Theorem 10.1). Theorem 9.1 is proved via an intricate transfinite induction in which we have to show all of these conditions to be equivalent before being able to proceed to the next rank. In fact, we will have to introduce additional equivalent conditions for the induction to work, which involve graphs that have a finite set of their vertices *marked*, endowed with a *marked minor* relation that maps each marked vertex to a branch set containing at least one marked vertex (Section 2.2).

I expect that Theorem 1.3 remains true, with minor modifications of the proof, when replacing $\mathcal{R}$ and $\mathcal{F}$ by a variety of subclasses, e.g. planar graphs. For the case of rayless trees we will explicitly prove the implication (a) → (d). Combining this with Nash-Williams' [15] theorem that the trees are well-quasi-ordered[1], we deduce as above that there are only countably many minor-twin classes of countable rayless trees of rank $\alpha$ for every $\alpha < \omega_1$. Combining this further with ongoing work with J. Grebík [7] connecting minor-closed families with Borel subsets of the space $\mathcal{G}$ of $\mathbb{N}$-labelled graphs[2], we will deduce the following.

**Theorem 1.4.** *Let $\mathcal{T} \subset \mathcal{G}$ be a minor-closed family of $\mathbb{N}$-labelled rayless forests. Then $\mathcal{T}$ is Borel if and only if it is proper, i.e. it does not contain all rayless forests.*

## 1.1 Structure of the paper

After some preliminaries, we prove the implication (ii) → (i) of Theorem 1.1 in Section 3. Much of the rest of the paper is devoted to Theorem 1.3, and we prepare it with a warm-up: Section 4 focuses on graphs of rank 1, introducing some fundamental ideas of the paper, and concluding with Corollary 1.2. Section 5 is devoted to a key tool (Lemma 5.1) implying that, under natural conditions, the minor-twin class of a rayless graph $G$ of rank $\alpha$ is determined by the class of marked-minors of $G$ of ranks $< \alpha$. Lemma 5.1 is needed for the proofs of all three of Theorems 1.1, 1.3 and 1.4. Sections 6 and 7 constitute an Intermezzo devoted to the proof of Theorem 1.4 (this constitutes an early example of a statement about unmarked graphs for the proof of which marked graphs are necessary). After this, we return to the proof of Theorem 1.3, introducing some un-marking techniques in Section 8, followed by the main technical part in Section 9, and concluding with Section 10. We then use some of the tools gathered thereby to prove the implication (i) → (iii) of Theorem 1.1 in Section 11. We finish with some open problems in Section 12.

For a reader wishing to gain an impression of the proofs without reading all details, I recommend reading the following selection. Section 2.3 excluding the proof of Observation 2.12; Section 2.4. Section 3 up to (5). Section 4, skimming the proof of Lemma 4.6. The statement of Lemma 5.1; its proof is the most challenging one in the paper, and Lemmas 4.1 and 4.6 illustrate the main ideas avoiding the more *annoying parts*. Sections 6 and 7 are optional; they only contribute to Theorem 1.4. The statement of Theorem 9.1, and implications (A) → (B) and (D) → (G) of its proof. The statement of Theorem 10.1. Section 11 excluding the proof of Lemma 11.6.

[1] In fact, we will need a strengthening of this, proved by Thomas, saying that the graphs of tree-width $k$ are well-quasi-ordered for every $k \in \mathbb{N}$.

[2] $\mathcal{G}$ denotes the space of graphs $G$ with $V(G) = \mathbb{N}$, encoded as functions from $\mathbb{N}^2$ to $\{0, 1\}$ representing the edges, endowed with the (Tychonoff) product topology. The vertex labelling is ignored when considering minors; it is only used to define the topology on $\mathcal{G}$.

# 2 Preliminaries

We will be following the terminology of Diestel [5] for graph-theoretic concepts and Jech [10] for set-theoretic ones.

## 2.1 (Well)-Quasi-Orders

A quasi-order $(Q, \leqslant)$ consists of a set $Q$ and a binary relation $\leqslant$ on $Q$ which is reflexive and transitive (but not necessarily antisymmetric). A quasi-order $(Q, \leqslant)$ is said to be *well-quasi-ordered*, if for every sequence $(G_n)_{n \in \mathbb{N}}$ of its elements there are $i < j$ such that $G_i \leqslant G_j$. If such $i, j$ exist then we say that $(G_n)_{n \in \mathbb{N}}$ is *good*, otherwise it is *bad*.

A well-known consequence of Ramsey's theorem is

**Proposition 2.1** ([5, Proposition 12.1.1]). *$(Q, \leqslant)$ is WQO if and only if it has no infinite antichain and no infinite sequence $(G_n)_{n \in \mathbb{N}}$ such that $G_{n+1} \leqslant G_n$ and $G_n \nleqslant G_{n+1}$ for every $n$.*

Such a sequence $(G_n)_{n \in \mathbb{N}}$ is called a *descending chain*. The following is also well-known:

**Observation 2.2** ([5, Corollary 12.1.2]). *Every sequence $(G_n)_{n \in \mathbb{N}}$ of elements of a WQO $(Q, \leqslant)$ has a subsequence $\{G_{a_n}\}_{n \in \mathbb{N}}$ such that $G_{a_n} \leqslant G_{a_k}$ for every $1 \leqslant n < k$.*

## 2.2 (Finite) graph minors and marked graphs

Let $G, H$ be graphs. A *minor model* of $G$ in $H$ is a collection of disjoint connected subgraphs $B_v, v \in V(G)$ of $H$, called *branch sets*, and edges $E_{uv}, uv \in E(G)$ of $H$, such that each $E_{uv}$ has one end-vertex in $B_u$ and one in $B_v$. We write $G < H$ to express that such a model exists, and say that $G$ is a *minor* of $H$.

A *minor embedding* of $G$ into $H$ is a map $h$ assigning to each $v \in V(G)$ a connected subgraph $B_v$ of $H$, and to each $uv \in E(G)$ an edge $E_{uv}$ of $H$ such that these sets form a minor model of $G$ in $H$. We write $h : G < H$ to denote that $h$ is such a minor embedding.

A *marked graph* is a pair consisting of a graph $G$ and a subset $A$ of $V(G)$, called the *marked vertices*. Given two marked graphs $(G, A), (H, A')$, a *marked minor (model)* of $G$ in $H$ is defined as above, except that for each marked vertex $v$ of $G$, we require that the corresponding branch set $B_v$ contains at least one marked vertex of $H$. We write $(G, A) < (H, A')$, or $G <_\bullet H$ when $A, A'$ are fixed, if this is possible. We also extend the above definition of minor embedding canonically to marked minors.

Given a set $X$ of graphs, we write $\mathrm{Forb}(X)$ for the class of graphs $H$ such that no element of $X$ is a minor of $H$.

We recall the Robertson–Seymour Graph Minor Theorem, which we will only use in the proof Corollary 1.2 (and the warm-up Lemmas 4.1 and 4.6).

**Theorem 2.3** ([22]). *The finite marked graphs are well-quasi-ordered under $<_\bullet$.*[3]

[3] This version of the Graph Minor Theorem is a special case of statement 1.7 of [22], obtained by replacing $\Omega$ by $\{0, 1\}$.

We conclude this subsection with some basic facts relating connectivity and minors. Firstly, note that if $\mathcal{B}$ a minor model of $G$ in $H$, then it maps each component of $G$ into a component of $H$. The following is a straightforward extension of this which is easy to prove:

**Observation 2.4.** *Let $G, H$ be graphs, let $\mathcal{B}$ a minor model of $G$ in $H$, and let $A \subset V(G)$. Then $\mathcal{B}$ maps each component of $G - A$ into a component of $H - \mathcal{B}(A)$.* □

Here, $\mathcal{B}(A)$ stands for the image of $A$ under $\mathcal{B}$, i.e. $\bigcup_{v \in A} B_v$.

A *block* of a graph is a maximal 2-connected subgraph.

**Lemma 2.5.** *Let $G, H$ be graphs, and $\mathcal{B} = \{B_v \mid v \in V(G)\}$ a minor model of $G$ in $H$. Then for each block $D$ of $G$, there is a block $D'$ of $H$, such that each $B_v, v \in V(D)$ intersects $D'$. Moreover, there is a model $\mathcal{B}'$ of $D$ in $D'$, obtained by intersecting each $B_v$ with $D'$.*

*Proof.* For every $e = uv \in E(D)$, let $B_e$ be a $B_u$–$B_v$ edge in $H$, called a *branch edge*. Since $D$ is 2-connected, any two edges $e, f$ of $D$ lie in a common cycle $C$. Moreover, there is a cycle $C'$ in $\bigcup_{v \in V(C)} B_v \subseteq H$ containing $B_e$ and $B_f$. Since there is a unique block containing any edge of a graph, it follows that there is a block $D'$ of $H$ containing $\{B_e \mid e \in E(D)\}$, and this block intersects each branch edge, and hence each branch set of $D$.

For the second sentence, suppose $x \in V(D')$ is a cut-vertex of $H$. Then all branch sets of $\mathcal{B}$ are contained in the component $K$ of $H - x$ containing $D' - x$, except possibly for a unique branch set $B$ containing $x$. If such a $B$ exists, then after replacing it with $B \cap K$, $\mathcal{B}$ is still a model of $G$, because no branch edge of $\mathcal{B}$ can lie in $H - K$. Thus doing so for every cut-vertex $x \in V(D')$ we modify $\mathcal{B}$ into the desired model $\mathcal{B}'$ of $G$ in $D'$. □

### 2.2.1 Suspensions

Given a (marked) graph $G$, we define its *suspension* $S(G)$ by adding an unmarked vertex $s_G$ and joining $s_G$ to each $v \in V(G)$ with an edge. Given a marked graph $G$, we define its *marked suspension* $S^\bullet(G)$ by adding a marked vertex $s_G$ and joining $s_G$ to each $v \in V(G)$ with an edge.

Note that $S(G)$ is always connected even if $G$ is not. Combining this with the following observation will be often convenient, as it will allow us to assume that any bad sequences we consider consist of connected graphs.

**Lemma 2.6.** *Let $G, H$ be marked graphs. Then $G < H$ if and only if $S^\bullet(G) < S^\bullet(H)$.*

*Proof.* The forward implication is trivial. For the backward implication, let $M = \{B_v \mid v \in V(S^\bullet(G))\}$ be a model of $S^\bullet(G)$ in $S^\bullet(H)$. If no branch set $B_v$ contains $s_H$, then $M$ witnesses that $S^\bullet(G)$, and hence $G$, is a minor of $H$ and we are done. If $s_H$ is in the branch set of $s_G$, then by deleting that branch set from $M$ we obtain a model of $G$ in $H$.

Thus it only remains to consider the case where some $B_v$ with $v \in V(G)$ contains $s_H$. In this case $B_{s_G}$ cannot contain $s_H$ too, and so $B_{s_G} \subseteq H$, and $B_{s_G}$ contains a marked vertex since $s_G$ is marked. Then by removing $B_{s_G}$ from $M$, and re-defining $B_v$ to be $B_{s_G}$, we have modified $M$ into a model of $G$ in $H$. □

The unmarked version of Lemma 2.6 is also true, and easier to prove along the same lines:

**Lemma 2.7.** *Let $G, H$ be (unmarked) graphs. Then $G < H$ if and only if $S(G) < S(H)$.* □

### 2.2.2 Minor-twins

We say that $G$ and $H$ are *minor-twins*, if both $G < H$ and $H < G$ hold. Any two finite minor-twins are isomorphic, but in the infinite case the relation is much more interesting.

The class of countable graphs can be decomposed into its *minor-twin classes*, whereby two graphs belong to the same class whenever they are minor-twins. The minor-twin class of a graph $G$ will be denoted by $[G]_<$. These definitions have obvious analogues for marked minors.

Given a graph class $\mathcal{C}$, we let $|\mathcal{C}|_<$ denote the cardinality of the set of minor-twin classes of elements of $\mathcal{C}$. We define $|\mathcal{C}|_{<_\bullet}$ analogously for marked minors.

## 2.3 The Rank of a rayless graph

A graph is *rayless*, if it does not contain a 1-way infinite path. Schmidt [23] assigned to every rayless graph an ordinal number, its *rank*, reminiscent of the notion of rank for Borel sets. This notion often enables us to prove results about rayless graphs by transfinite induction on the rank.

The notion of rank comes from the observation that it is possible to construct all rayless graphs by a recursive, transfinite procedure, starting with the class of finite graphs and then, in each step, glueing graphs constructed in previous steps along a common finite vertex set, to obtain new rayless graphs as follows.

**Definition 2.8.** *For every ordinal $\alpha$, we recursively define a class of graphs $\mathrm{Rank}_\alpha$ by transfinite induction on $\alpha$ as follows:*

- $\mathrm{Rank}_0$ *consists of the finite graphs; and*
- *if $\alpha > 0$, then a graph $G$ is in $\mathrm{Rank}_\alpha$ if there is a finite $S \subset V(G)$ such that each component of $G - S$ lies in $\mathrm{Rank}_\beta$ for some $\beta < \alpha$.*

Schmidt [23, 5, 9] proved that a graph is rayless if and only if it belongs to $\mathrm{Rank}_\alpha$ for some $\alpha$. Easily, if $G$ is countable then so is this $\alpha$, but it may be greater than $\omega$ (Observation 2.12 below). Let $\mathrm{Rank}_{<\alpha} := \bigcup_{\beta<\alpha} \mathrm{Rank}_{<\beta}$.

The *rank* $\mathrm{Rank}(G)$ of a rayless graph $G$ is the least ordinal $\alpha$ such that $G \in \mathrm{Rank}_\alpha$. For a class $\mathcal{C}$ of graphs, we let $\mathrm{Rank}(\mathcal{C})$ be the least ordinal $\alpha$ such that $\mathrm{Rank}(G) < \alpha$ holds for every $G \in \mathcal{C}$.

Schmidt [23, 9] also proved that each rayless graph $G$ has a unique *kernel* $A(G)$, i.e. a minimal set of vertices $S$ such that each component of $G - S$ lies in $\mathrm{Rank}_\gamma$ for some $\gamma < \mathrm{Rank}(G)$. We will use the following simple observation:

**Proposition 2.9.** *For every ordinal $\alpha > 0$ every $G \in \mathrm{Rank}_\alpha$, and every $\beta < \alpha$, there are infinitely many components $C$ of $G - A(G)$ such that $\mathrm{Rank}(C) \geqslant \beta$.*

*Proof.* If the set $\mathcal{C}$ of such components $C$ is finite, then the finite set $A(G) \cup \bigcup_{C \in \mathcal{C}} A(C)$ separates $G$ into components that all have ranks less than $\beta$, yielding the contradiction $\mathrm{Rank}(G) \leqslant \beta$. □

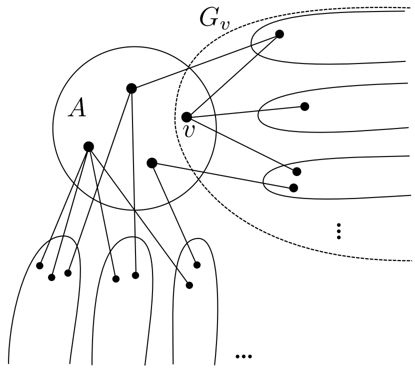


Figure 1: The components of $G - A$ and $G - (A - v)$ in the proof of (1).

#### 2.3.1 Rank and minors

It is well-known, and not hard to prove, that rank is monotone with respect to minors:

**Observation 2.10** ([9, Proposition 4.4.]). *Let $G, H$ be graphs with $G < H$. Then* $\mathrm{Rank}(G) \leqslant \mathrm{Rank}(H)$.

The following may be well-known but I could not find a reference:

**Observation 2.11.** *Let $G, H$ be graphs with* $\mathrm{Rank}(G) = \mathrm{Rank}(H) = \alpha$, *and $\mathcal{B} = \{B_v \mid v \in V(G)\}$ a minor model of $G$ in $H$. Then $B_v$ intersects $A(H)$ for every $v \in A(G)$.*

*Proof.* Let $v \in A := A(G)$, and let $\mathcal{C} = \mathcal{C}_v$ denote the set of components of $G - A$ sending an edge to $v$. We claim that

$$\mathrm{Rank}(\mathcal{C}) = \alpha. \tag{1}$$

Indeed, if $\mathrm{Rank}(\mathcal{C}) < \alpha$, then $G_v := G[\{v\} \cup \bigcup \mathcal{C}]$ has rank less than $\alpha$ too. Moreover, each component of $G - (A - v)$ is either $G_v$ or a component of $G - A$, and therefore it has rank less than $\alpha$ (Figure 1). This contradicts the fact that $A$ is, by definition, a minimal set with this property.

Suppose $B_v$ contains no vertex of $A' := A(H)$ for some $v \in A$. Since $B_v$ is connected, it is contained in a component $C$ of $H - A'$. By (1), we have $\mathrm{Rank}(G_v) = \alpha$. This remains true if we delete from $G_v$ those elements of $\mathcal{C}_v$ containing a branch set intersecting $A'$, since there are at most finitely many such branch sets. Let $G'_v$ be the subgraph of $G_v$ obtained after this deletion. Since $G'_v$ is connected, and all its branch sets avoid $A'$, its image under $\mathcal{B}$ is contained in $C$. But $\mathrm{Rank}(C) < \mathrm{Rank}(H) = \alpha$, while $\mathrm{Rank}(G'_v) = \alpha$ as observed above. This contradicts Observation 2.10, hence $B_v$ must intersect $A'$. ☐

**Observation 2.12.** *For every countable ordinal $\alpha$, there is a countable tree $T_\alpha$ with* $\mathrm{Rank}(T_\alpha) = \alpha$, *such that $T_\alpha < G$ for every non-empty, connected, graph $G$ with* $\mathrm{Rank}(G) \geqslant \alpha$.

*Proof.* We will prove the statement by induction on $\alpha$. For this it will be convenient to think of each $T_\alpha$ as a rooted tree, and denote its root by $r_\alpha$. Instead of $T_\alpha < G$, we will prove the following strengthening, which will be important for our induction:

for every $v \in V(G)$, there is a model of $T_\alpha$ in $G$ such that the branch set of $r_\alpha$ contains $v$. (2)

For $\alpha = 0$, we just let $T_\alpha$ be the tree on one vertex, and note that (2) is trivially satisfied by letting $B_{r_\alpha} = \{v\}$.

For $\alpha > 0$, we construct $T_\alpha$ as follows. We start with the disjoint union of countably infinitely many copies of $T_\beta$ for each $\beta < \alpha$, add a new vertex $r_\alpha$, and join $r_\alpha$ to the root of each such copy of $T_\beta$ with an edge. Note that $A(T_\alpha) = \{r_\alpha\}$ by construction.

This completes the construction of $T_\alpha$ for every ordinal $\alpha$, and it now remains to prove (2), which we do by transfinite induction of $\alpha$. Having checked the start $\alpha = 0$ of the induction above, we may assume that (2) holds for all ordinals $\beta < \alpha$. Given $G$ as above and $v \in V(G)$, we construct the desired model of $T_\alpha$ in $G$ as follows. Pick a $v$–$A(G)$ path $P$ in $G$. Moreover, for every $x, y \in A(G)$, pick a $x$–$y$ path $P_{xy}$ in $G$. Since $A(G)$ is finite, the union of all these paths meets only a finite set $\mathcal{C}$ of components of $G - A(G)$. We let $B_{r_\alpha} = A(G) \cup \bigcup \mathcal{C}$ be the branch set of $r_\alpha$ in our model. All other branch sets will be chosen within $G - \bigcup \mathcal{C}$.

Let $(C_n)_{n \in \mathbb{N}}$ be an enumeration of the components of $T_\alpha - r_\alpha$, and recall that each $C_n$ is isomorphic to $T_\beta$ for some $\beta < \alpha$. For $i = 1, 2, \ldots$, we recursively find a model of $C_i$ in $G - \bigcup \mathcal{C}$ as follows. We pick a component $C'_i$ of $G - \bigcup \mathcal{C}$ with $\mathrm{Rank}(C'_i) \geqslant \beta$ such that $C'_i \notin \mathcal{C}$, and $C'_i \neq C'_j$ for any $j < i$, which $C'_i$ exists by Proposition 2.9. Since $G$ is connected, there is a vertex $v_i \in C'_i$ sending an edge $e_i$ to $A(G)$. Applying the inductive hypothesis (2) with $G$ replaced by $C'_i$, and $v$ replaced by $v_i$, and $\alpha$ replaced by $\beta$, we obtain a minor model of $C_i \cong T_\beta$ in $C'_i$, in which the branch set corresponding to $r_\beta$ contains $v_i$. Adding the edges $e_i$, and the branch set $B_{r_\alpha}$ to these models for all $i \in \mathbb{N}$ we obtain the desired model of $T_\alpha$ in $G$. $\square$

## 2.4 Better-quasi-orders

Rather than repeating the original definition of a better-quasi-order, we will work with an equivalent one. Intuitively, a quasi-order $(Q, \leqslant)$ is better-quasi-ordered if it is well-quasi-ordered, and so are its subsets, sets of subsets, sets of sets of subsets, and so on transfinitely, whereby we recursively extend $\leqslant$ from elements of $Q$ to subsets of $Q$. To make this precise, we need the following terminology.

Let $\mathcal{P}^*(A)$ denote the set of non-empty subsets of a set $A$, i.e. $\mathcal{P}^*(A) := \mathcal{P}(A) \backslash \{\varnothing\}$. Let $Q$ be a quasi-order. For every ordinal $\alpha$ we define, by transfinite induction, the 'iterated power set' $V^*_\alpha(Q)$ as follows. We start our induction by setting $V^*_0(Q) := Q$. Having defined $V^*_\alpha(Q)$, we let $V^*_{\alpha+1}(Q) := \mathcal{P}^*(V^*_\alpha(Q))$. Finally, if $\alpha$ is a limit ordinal, we let $V^*_\alpha(Q) := \bigcup_{\beta<\alpha} V^*_\beta(Q)$. We say that $X \in V^*_\alpha(Q)$ is *hereditarily countable* if it is a countable set of hereditarily countable sets (the latter having been defined recursively, starting by declaring each element of $Q$ to be hereditarily countable).

Having defined $V^*_\alpha(Q)$ for every $\alpha$, we let $V^*(Q) := \bigcup_\alpha V^*_\alpha(Q)$. The *Q-rank* $Q\mathrm{Rank}(X)$ of an element $X \in V^*_\alpha(Q)$ is defined as $\alpha + 1$ where $\alpha$ is the least ordinal such that $X \in V^*_\alpha(Q)$. Thus $Q\mathrm{Rank}(X) = 1$ if and only if $X \in Q$ (the $+1$ may look strange for now, but it will be justified later).

**Definition 2.13.** *Given a quasi-order $(Q, \leqslant)$, recursively define a quasi-order $\leqslant_+$ on $V^*(Q)$ as follows*

*(i) if $X, Y \in Q$, then $X \leqslant_+ Y$ in $V^*(Q)$ if and only if $X \leqslant Y$ in $Q$;*

*(ii) if $X \in Q$ and $Y \notin Q$, then $X \leqslant_+ Y$ if and only if there exists $Y' \in Y$ with $X \leqslant_+ Y'$;*

*(iii) if $X \notin Q$ and $Y \notin Q$, then $X \leqslant_+ Y$ if and only if for every $X' \in X$ there exists $Y' \in Y$ with $X' \leqslant_+ Y'$.*

Let $H^*_{\omega_1}(Q)$ denote the set of hereditarily countable elements of $V^*_{\omega_1}(Q)$ equipped with the above quasi-order induced from $V^*(Q)$.

**Theorem 2.14** ([17, Theorem 3.45][4])**.** *A quasi-order $Q$ is BQO if and only if $H^*_{\omega_1}(Q)$ is WQO.*

We will effectively use Theorem 2.14 as our definition of *better-quasi-ordering*. (We will refrain from repeating Nash-Williams' original definition of better-quasi-ordering, as this will never be used in the paper.)

Let $Seq(Q)$ be the set of all finite or countably infinite sequences with elements in $(Q, \leqslant)$. We endow $Seq(Q)$ by a quasi-ordering $\preceq$ by letting $S \preceq T$ if there exists an embedding from $F$ into $G$, i.e. a strictly increasing map $\phi$ from the index set of $S$ to that of $T$, such that $S(i) \leqslant T(\phi(i))$ for every $i$. We will use the following well-known lemmas about better-quasi-ordering.

**Lemma 2.15** ([15], [12, Lemma 4])**.** *If a quasi-order $Q$ is BQO, then so is $Seq(Q)$.*

**Lemma 2.16** ([15, COROLLARY 22A], [12, Lemma 3])**.** *If two quasi-orders $Q_1, Q_2$ are BQO, then so is $Q_1 \times Q_2$.*

For the proof of the implication (i) $\to$ (iii) of Theorem 1.1 we will use the following closedness property of $H^*_{\omega_1}$.

**Observation 2.17.** *For every quasi-order $Q$, we have*
*$H^*_{\omega_1}(H^*_{\omega_1}(Q)) = H^*_{\omega_1}(Q)$.*

*Proof.* Notice that any countable non-empty subset $X$ of $H^*_{\omega_1}(Q)$ belongs to $H^*_{\omega_1}(Q)$[5]; this is because each $Y \in X$ belongs to some $V^*_{\alpha_Y}(Q)$, and letting $\beta := \sup_{Y \in X} \alpha_Y$ we have $X \in V^*_\beta(Q) \cup V^*_{\beta+1}(Q)$ by the definitions because $\beta < \omega_1$ as $\omega_1$ is a regular ordinal. This means that the hereditarily countable sets in $V^*_1(H^*_{\omega_1}(Q))$ are already in $H^*_{\omega_1}(Q)$. By induction, the same applies to $V^*_\alpha(H^*_{\omega_1}(Q))$ for every $\alpha < \omega_1$, and so $H^*_{\omega_1}(H^*_{\omega_1}(Q)) = H^*_{\omega_1}(Q)$. ☐

[4]Private communication with Yann Pequignot suggests that this theorem was known to experts on BQO, but apparently it first appear in print in this formulation in [17].

[5]This is the crucial observation, and it is used by Pequignot in [17, Theorem 3.46]. It is important here that we are working with $H^*_{\omega_1}$ instead of $V^*$.

# 3 $\mathcal{R}$ is WQO impies $\mathcal{F}$ is BQO

The aim of this section is to prove the implication (ii) → (i) of Theorem 1.1:

**Theorem 3.1.** *If the countable rayless graphs are WQO, then the finite graphs are BQO.*

*Proof.* Let $Q$ be the set of finite *1-connected graphs*, i.e. connected graphs with at least two vertices. By Lemmas 2.15 and 2.16, it suffices to prove that $Q$ is better-quasi-ordered. Indeed, any finite graph $G$ is the disjoint union of a number $n_G$ of isolated vertices and a sequence $s_G$ of 1-connected subgraphs. Thus we can represent $G$ as a pair $(n_G, s_G)$, and apply Lemma 2.16 to such pairs after applying Lemma 2.15 to these sequences, recalling that $\mathbb{N}$ is well-ordered and hence better-quasi-ordered.

It thus remains to prove

$$\text{If the class } \mathcal{R} \text{ of countable rayless graphs is WQO, then } Q \text{ is BQO.} \tag{3}$$

To prove this we will define a map $T$ that assigns to each set $X$ in $H^*_{\omega_1}(Q)$ a rayless graph $T(X)$ in such a way that

$$T(X) < T(Y) \text{ implies } X \leqslant Y \text{ for every } X, Y \in H^*_{\omega_1}(Q), \tag{4}$$

where $\leqslant$ stands for the relation $\leqslant_+$ of Definition 2.13. To see how this implies 3, recall that, by Lemma 2.14, if $H^* := H^*_{\omega_1}(Q)$ is WQO then $Q$ is BQO. Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of elements of $H^*$. If the countable rayless graphs are WQO, then $\{T(X_n)\}_{n\in\mathbb{N}}$ is good, hence so is $(X_n)_{n\in\mathbb{N}}$ by 4, and so $H^*$ is WQO as desired.

It remains to define $T$ and prove that it satisfies 4. We define $T(X), X \in H^*$ by transfinite induction on the $Q$-rank of $X$ as follows. If $Q\text{Rank}(X) = 1$, in which case $X \in Q$ is a finite connected graph (on at least two vertices), then $T(X)$ comprises a countably infinite collection of pairwise disjoint copies of $X$, and an additional vertex $r$, called the *root*, joined by an edge to all other vertices.

If $Q\text{Rank}(X) > 1$, then $T(X)$ comprises a countably infinite collection of pairwise disjoint copies of $T(X')$ for each $X' \in X$, and an additional root vertex $r$ joined by an edge to the root of each such $T(X')$.

It is easy to show, by transfinite induction on $Q\text{Rank}(X)$, that $T(X)$ is a countable graph, and that it is rayless; to see the latter, notice that any ray in $T(X)$ can visit $r$ at most once, hence it would need to have a sub-ray in a copy of $T(X')$ for some $X' \in X$ if $Q\text{Rank}(X) > 1$, or in a copy of $X$ if $Q\text{Rank}(X) = 1$. It is not hard to show, again by transfinite induction on $Q\text{Rank}(X)$, that

$$\text{Rank}(T(X)) = Q\text{Rank}(X) \text{ for every } X \in H^*. \tag{5}$$

(This justifies the $+1$ in the definition of $Q\text{Rank}(X)$; without it this equation would hold only when $Q\text{Rank}(X) \geqslant \omega$.)

We call $r =: r(T(X))$ the *root* of $T(X)$. The *children* of $r$ are the copies of the roots $\{r(T(X')) \mid X' \in X\}$. The *descendant* relation is the transitive closure of the child relation just defined. Given a descendant $r'$ of $r$, we write $\lceil r' \rceil$ for the component of $T(X)$ containing $r'$ formed when deleting the edge from $r'$ to its parent. We set $\lceil r \rceil = T(X)$. Notice that $\lceil r' \rceil$ is isomorphic to $T(Y)$ for some element $Y$ of the transitive closure of $X$.

By construction, we can assign to each vertex $v$ of $T(X), X \in H^*$ an ordinal number called the *level* of $v$, similar to the notion of $Q$Rank, as follows. If $X \in Q$, each vertex of $X$ and its copies has level 0. For $X \in H^*$, we proceed inductively to define the *level* of $r(T(X))$ to be the smallest ordinal that is larger than the level of each vertex $v \neq r(T(X))$ of $T(X)$, which is defined in a previous step of the induction since $v \in V(T(X'))$ for some $X' \in X$.

Notice that for each triangle $\Delta$ of $T(X)$, at least two of the vertices of $\Delta$ have level 0. Recall moreover that each $X \in Q$ contains at least one edge. It thus follows from our construction that, for every $X \in H^*$,

a vertex $v$ of $T(X)$ lies in infinitely many triangles of $T(X)$ if and only if $v$ has level 1, and no vertex of level $> 1$ lies in a triangle. (6)

We now prove that this definition of $T$ satisfies 4, by a nested transfinite induction on $Q\mathrm{Rank}(X)$ and $Q\mathrm{Rank}(Y)$.

**Remark 3.0.1.** More generally, we can let $Q'$ be any non-empty family of 1-connected finite graphs. Then 6 still holds. By restricting (4) to $Q'$, we deduce the following variant of (3): if $T[Q']$, i.e. the image of $H^*_{\omega_1}(Q')$ under $T$, is well-quasi-ordered, then $Q'$ is better-quasi-ordered. The remainder of this proof works verbatim when replacing $Q$ by $Q'$, and $\mathcal{R}$ by $T[Q']$ in (3).

Our inductive proof of (4) starts with $Q\mathrm{Rank}(X)$ being 1: assume that $T(X) < T(Y)$ holds for some $X \in Q$ and $Y \in H^*$. Let $B_r$ denote the branch set corresponding to the root $r = r(T(X))$ in some minor model $\mathcal{B}$ of $T(X)$ in $T(Y)$. We claim that $B_r$ contains at least one vertex of level 1 in $Y$. Indeed, by (6) $r$ lies in infinitely many triangles of $T(X)$, and if $B_r$ avoids level 1 vertices of $T(Y)$, then it lies in at most finitely many triangles of any minor of $T(Y)$.

Let $G$ be one of the copies of $X$ in $T(X)$, let $xy$ be an edge of $G$, and notice that $xyr$ is a triangle of $T(X)$. Then the branch sets $B_x, B_y$ of $\mathcal{B}$ are contained in a component $C$ of level 0 vertices of $T(Y)$. Let $r'$ be the unique level 1 vertex of $T(Y)$ sending edges to $C$. Notice that $r' \in B_r$. Since $G$ is connected, and $r'$ separates $C$ from the rest of $T(Y)$, we deduce that $B_v \subset C$ for every $v \in V(G)$. By our construction of $T(Y)$, there are infinitely many level 0 components $C'$ of $T(Y)$ isomorphic with $C$ and incident with $r'$. It follows that

$$T(X) \leqslant \lceil r' \rceil. \tag{7}$$

If $Q\mathrm{Rank}(Y) = 1$, then $C$ is just a copy of $Y$ and we obtain the desired $X \leqslant Y$ since $G$ was a copy of $X$. If $Q\mathrm{Rank}(Y) > 1$, then let $Y'$ be the element of $Y$ such that some copy of $T(Y')$ in $T(Y)$ contains $r'$, and hence $\lceil r' \rceil$. In this case (7) implies $T(X) \leqslant T(Y')$, and by transfinite induction on $Q\mathrm{Rank}(Y)$, of which the previous case is the initial step, we deduce $X \leqslant Y'$, and hence $X \leqslant Y$ by (ii) of Definition 2.13.

Thus we have completed the initial step $Q\mathrm{Rank}(X) = 1$ of our inductive proof of (4). Assume now that $Q\mathrm{Rank}(X) > 1$, and $T(X) \leqslant T(Y)$ holds for some $Y \in H^*$. We cannot have $Q\mathrm{Rank}(Y) = 1$ by Observation 2.10 and (5). Thus we are in case (iii) of Definition 2.13, and so our task is to find, for each $X' \in X$, some $Y' \in Y$ such that $X' \leqslant Y'$.

To this aim, let again $B_r$ denote the branch set of some minor model $\mathcal{B}$ of $T(X)$ in $T(Y)$ corresponding to the root $r = r(T(X))$. Let $r'$ be the vertex of $B_r$

of maximal level among all vertices of $B_r$; this exists because $B_r$ is connected, and hence if it contains vertices of all levels $\beta < \alpha$ for some ordinal $\alpha$, then it must also contain a vertex of level $\alpha$. By (6), the level of $r'$ is at least 1. Notice that all but at most one of the edges of $r$ are mapped by $\mathcal{B}$ to descendants of $r'$, because $r'$ sends at most one edge to a non-descendant. Call this edge $e$ if it exists. Since $T(X)$ contains infinitely many pairwise disjoint copies of $T(X')$ incident with $r$, at least one (in fact almost all) of these copies $G$ is mapped by $\mathcal{B}$ to a subgraph of $T(Y)$ avoiding $e$. Therefore, $G$ is mapped by $\mathcal{B}$ into $\lceil r'' \rceil$ for some child $r''$ of $r'$, because $G$ is connected, $r'$ separates its children, and $r' \in B_r$ cannot lie in $B_v$ for any $v \in V(G)$. Let $Y'$ be the element of $Y$ for which some copy of $T(Y')$ contains $r''$, which exists since $r'' \neq r(T(Y))$ as $r''$ is a child of another vertex. Then $T(Y')$ contains $\lceil r'' \rceil$, and hence $T(X') < T(Y')$ because $T(X') < \lceil r'' \rceil$. Since $Q\mathrm{Rank}(X') < Q\mathrm{Rank}(X)$, our inductive hypothesis yields $X' \leqslant Y'$ as desired, completing the inductive step. □

By Remark 3.0.1, we immediately deduce

**Corollary 3.2.** *Let $F$ be a set of finite graphs, and let $Q$ be the set of 1-connected elements of $F$. If $T[Q']$ is well-quasi-ordered, then $F$ is better-quasi-ordered.*

It is not hard to prove the converse of (4), by induction on the $Q$-rank by recursively preserving the property that the branch set of the root contains the root of the target graph:

$$X \leqslant Y \text{ implies } T(X) < T(Y) \text{ for every } X, Y \in H^*_{\omega_1}(Q). \tag{8}$$

Thus combining (4), (8) and Theorem 2.14, and recalling that $T[Q]$ denotes the image of $H^*_{\omega_1}(Q)$ under $T$, we deduce

The finite graphs are better-quasi-ordered if and only if $T[Q]$ is well-quasi-ordered. (9)

This can be thought as an additional equivalent condition that could be added to Theorem 1.1.

# 4 Graphs of rank 1, and the self-minor conjecture

This section introduces some of the fundamental techniques used throughout the paper, and serves as a preparation towards the more difficult Section 5. It handles graphs of rank 1. The section culminates with the proof of Theorem 1.2, which is based on the same techniques.

Let $UF$ (to be read "union-finite") denote the class of countable graphs $G$ such that each component of $G$ is finite. Note that $UF \subset \mathrm{Rank}_1$. For $G \in UF$, we let $\mathcal{C}(G)$ denote the class of finite graphs $H$ such that $H < G$. The following is a toy version of Lemma 5.1, a central tool for the proof of Theorem 1.3.

**Lemma 4.1.** *For every $G, G' \in UF$, we have $G < G'$ if and only if $\mathcal{C}(G) \subseteq \mathcal{C}(G')$.*

To see the relevance of this to cardinality of minor-twin classes as in Theorem 1.3 (d), let us prove that Lemma 4.1 implies that $|UF|_< = \aleph_0$. In fact, this statement is equivalent to the Graph Minor Theorem:

**Corollary 4.2.** $|UF|_< = \aleph_0$ *if and only if the finite graphs are well-quasi-ordered.*

*Proof.* For the backward direction, note that Lemma 4.1 says that the minor-twin class $[G]_<$ of $G \in UF$ is determined by $\mathcal{C}(G)$. Using the fact that the finite graphs are well-quasi-ordered, we can express $\mathcal{C}(G)$ as $\mathcal{C}(G) = \mathrm{Forb}(X)$ for a finite set $X$ of finite graphs. Thus there are countably many choices for $\mathcal{C}(G)$, hence for $[G]_<$, since there are countably many finite graphs to choose $X$ from.

For the forward direction, suppose for a contradiction that $(G_n)_{n\in\mathbb{N}}$ is an anti-chain of finite graphs under the minor relation. By Lemma 2.7, we may assume that each $G_n$ is connected. For each $X \subset \mathbb{N}$, let $\mathcal{C}_X$ be the (minor-closed) class of finite graphs $\mathrm{Forb}(X)$. Let $G_X := \bigcup \mathcal{C}_X \in UF$ be the (countably infinite) graph obtained as the disjoint union of all the graphs in $\mathcal{C}_X$. Note that for every finite graph $H$, we have $H < G_X$ if and only if $H \in \mathcal{C}_X$ because each $G_i \in X$ is connected. This implies that $G_X$ is not a minor-twin of $G_Y$ whenever $X \neq Y$, because any graph in the symmetric difference $X \triangle Y$ is a minor of exactly one of $G_X, G_Y$. Since there are continuum many $X \subset \mathbb{N}$, we have obtained continuum many minor-twin classes $[G_X]_<$ (thus we could add $|UF|_< < 2^{\aleph_0}$ as a further equivalent statement). □

Generalising the idea of the proof of Corollary 4.2 to higher ranks will be the key to proving the equivalence (a) $\leftrightarrow$ (d) of Theorem 1.3, the main difficulty being that we do not have an analogue of the Graph Minor Theorem for ranks higher than 0.

We prepare the proof of Lemma 4.1 by recalling a well-known idea:

**Hilbert's Hotel Principle:** Suppose a hotel has infinitely many single rooms, numbered $R_1, R_2, \ldots$, and each $R_i$ is occupied by a guest $G_i$. If a new guest $G$ arrives, they can be accommodated in $R_1$, by moving each $G_i$ to $G_{i+1}$.

*Proof of Lemma 4.1.* The forward implication follows immediately by restricting a minor model of $G$ in $G'$ to any $H \in \mathcal{C}(G)$.

For the backward implication, suppose $\mathcal{C}(G) \subseteq \mathcal{C}(G')$, and let $G_n$ be the union of the first $n$ components of $G$ in a fixed but arbitrary enumeration of its components. Let $G'_n$ be a subgraph of $G'$ such that $G_n < G'_n$, which exists since $\mathcal{C}(G) \subseteq \mathcal{C}(G')$. Easily, we may assume $G'_n$ is finite. Let $h_n : G_n < G'_n$ be a minor embedding (as defined in Section 2.2).

Call a component $C_i$ of $G$ *h-stable*, if $\bigcup_n h_n(C_i)$ is finite; in other words, if $C_i$ is mapped to a finite set of components of $G'$ by the $h_n, n \in \mathbb{N}$. Let $(S_n)_{n\in\mathbb{N}}$ be an enumeration of the $h$-stable components of $G$, and $(U_n)_{n\in\mathbb{N}}$ an enumeration of the other components of $G$. One of these enumerations may be finite, or even empty.

Let $G_S$ denote the (possibly empty) subgraph of $G$ consisting of its $h$-stable components. By a standard compactness argument, there is a minor embedding $h_S : G_S < G'$ such that $h_S(S_i)$ coincides with $h_n(S_i)$ for infinitely many values of $n$. If $G_S = G$ we are done, so suppose from now on $U_1$ exists.

We will now modify $h_S$, recursively in at most $\omega$ steps $i = 1, 2, \ldots$, into a minor embedding of $G$ into $G'$, whereby in step $i$ we handle $U_i$. Importantly, the image of $h_S$ might be all of $G'$, and so we may have to reshuffle $G_S$ inside $G'$ to make space for the $U_i$'s.

We set $h_S^0 := h_S$, and assume recursively that $h_S^j : G_S \cup \{U_1, \ldots, U_{j-1}\}$ has been defined for every $j < i$. Moreover, we assume that a finite number of components of $G$ have been *nailed* into components of $G'$, which means that we promise that $h_S^i$ will coincide with $h_S^{i-1}$ on all components that have been nailed before. We will ensure that every component of $G$—stable or not— will be nailed in some step. No components have been nailed at the beginning of step 1. If $U_i$ does not exist, then we just let $h_S^i = h_S^{i-1}$, and nail $S_i$ —this is the easy case, and we can just terminate the process as $h_S^{i-1}$ is a minor embedding of $G$ into $G'$ in this case.

Otherwise, let $(C_n^i)_{n \in \mathbb{N}}$ be an infinite sequence of distinct components of $G'$ into which $U_i$ is embedded by some $h_n$, which exists since $U_i$ is not $h$-stable. As the finite graphs are well-quasi-ordered by Theorem 2.3, $(C_n^i)_{n \in \mathbb{N}}$ has an infinite subsequence $(Y_n)_{n \in \mathbb{N}}$ such that $Y_r < Y_m$ for every $r < m \in \mathbb{N}$ by Observation 2.2. Pick $k = k(i)$ such that no $Y_m, m \geqslant k$ has been nailed yet.

If $Y_k$ does not intersect the image of $h_S^{i-1}$, we let $h_S^i$ extend $h_S^{i-1}$ by embedding $U_i$ into $Y_k$; this is possible since some $h_n$ embeds $U_i$ into $Y_k$ by the definition of the latter. We nail $U_i$ to $Y_k$ (thereby promising that $h_S^j$ will embed $U_i$ into $Y_k$ for every $j \geqslant i$). Finally, if $S_i$ exists, we nail it to the component containing $h_S^{i-1}(S_i)$, again promising that $h_S^j(S_i)$ is fixed from now on.

It remains to consider the —more difficult— case where $Y_k$ intersects the image of $h_S^{i-1}$. In this case, imitating Hilbert's Hotel principle, we modify $h_S^{i-1}$ into $h_S^i$ by shifting the 'contents' of each $Y_m, m \geqslant k$ to $Y_{m+1}$, and mapping $U_i$ into $Y_k$. To make this precise, fix a minor embedding $g_m : Y_m < Y_{m+1}$ for every $m \geqslant k$, which exists by the definition of $(Y_n)_{n \in \mathbb{N}}$. Then, for every $m \geqslant k$, and every component $C$ of $G$ such that $h_S^{i-1}(C)$ intersects $Y_m$ —and therefore $h_S^{i-1}(C)$ is contained in $Y_m$— we let $h_S^i(C) := g_m \circ h_S^{i-1}(C)$, so that $h_S^i$ embedds $C$ into $Y_{m+1}$. Thus $h_S^i$ now maps the domain $G_S \cup \{U_1, \ldots, U_{i-1}\}$ of $h_S^{i-1}$ to $G' \backslash Y_k$. We extend $h_S^i$ to $U_i$, embedding $U_i$ into $Y_k$ (by imitating some $h_n$), and we nail $U_i$ to $Y_k$.

Again, if $S_i$ exists, we nail it to the component containing $h_S^i(S_i)$ —which may coincide with the component containing $h_S^{i-1}(S_i)$, or have been shifted from some $Y_m$ to $Y_{m+1}$.

This completes the definition of $h_S^i, i \in \mathbb{N}$. Note that each of $U_i, S_i$ that exists has been nailed by step $i$, and its $h_S^\ell$-image is fixed for $\ell \geqslant i$. Thus $h_S^i$ converges, as $i \to \infty$, to a minor embedding $h : G \to G'$, proving our claim $G < G'$. □

Our next result extends Lemma 4.1 from $UF$ to its superclass $\mathrm{Rank}_1$. For this we will need to adapt the above definition of $\mathcal{C}(G)$ (Definition 4.5 below), whereby we will have to consider marked graphs.

**Definition 4.3.** *For $G \in \mathrm{Rank}_\alpha, \alpha \geqslant 1$, a* co-part *of $G$ is a component of $G - A(G)$. Given a co-part $C$ of $G$, we call the subgraph $G[C \cup A(G)]$ induced by $C \cup A(G)$ a* part *of $G$.*

Note that each part of $G$ has lower rank than that of $G$.

**Definition 4.4.** *Let* $\mathrm{Rank}_\alpha^\bullet$ *denote the class of marked graphs* $(G, M)$ *with* $G \in \mathrm{Rank}_\alpha$ *and* $M$ *finite. Define* $\mathrm{Rank}_{<\alpha}^\bullet$ *analogously.*

**Definition 4.5.** *Given a rayless graph* $G \in \mathrm{Rank}_\alpha$*, we let* $\mathcal{C}^\bullet(G)$ *denote the class of marked graphs in* $\mathrm{Rank}_{<\alpha}^\bullet$ *that are marked-minors of* $(G, A(G))$*.*

The following lemma, which extends Lemma 4.1, is again not formally needed for our later proofs; we include it as a warm-up towards the more difficult Lemma 5.1, but the reader will need to be familiar with its proof.

**Lemma 4.6.** *Let* $G, H \in \mathrm{Rank}_1$*, and suppose* $|A(G)| = |A(H)|$*. We have* $G < H$ *if and only if* $\mathcal{C}^\bullet(G) \subseteq \mathcal{C}^\bullet(H)$*.*

*Proof.* As before, the forward implication is straightforward.

For the backward implication, let $(P_n)_{n\in\mathbb{N}}$ be an enumeration of the parts of $G$, and $(P'_n)_{n\in\mathbb{N}}$ an enumeration of the parts of $H$. Let $G_n$ be the graph $\bigcup_{i\leqslant n} P_i$ with $A := A(G)$ marked. Choose a (marked) minor embedding $h_n : G_n < H_n$, where $H_n \subset H$ is finite and has $A' := A(H)$ as its marked vertex set.

We will use $(h_n)_{n\in\mathbb{N}}$ to construct a model of $G$ in $H$ using the ideas of the proof of Lemma 4.1, whereby we need to pay special attention to the vertices in $A$, in particular to how their branch sets intersect $H\backslash A'$.

Since each co-part $P_i\backslash A$ is connected, it is mapped to a co-part of $H$ by each $h_n, n \geqslant i$. Note that $h_n(x) \cap A'$ is a singleton $\{x'\}$ for every $x \in A$ and $n \in \mathbb{N}$ by Observation 2.11. Thus this map $x \mapsto x'$ is a bijection from $A$ to $A'$. By passing to a subsequence of $\{h_n\}$ if necessary, we may assume that this bijection is fixed for every $n \in \mathbb{N}$. Moreover, we can choose subsequences $\{h_n\} \supseteq \{h_n^1\} \supseteq \{h_n^2\} \ldots$ of $\{h_n\}$ such that for every $x \in A$ and every $i \in \mathbb{N}$, the intersection of the branch set $h_n^i(x)$ with the parts $\{P'_1, \ldots, P'_i\}$ is independent of $n$; indeed, having chosen $\{h_n^{i-1}\}$, we observe that since $P'_i$ and $A$ are finite, there is an infinite subsequence $\{h_n^i\}$ along which $h_n^{i-1}(x) \cap P'_i$ is constant for every $x \in A$.

Let $h'_n := h_n^n$. Note that $\{h'_n\}$ is a subsequence of $\{h_n\}$, and that $h'_n(x)$ converges for every $x \in A$, to a connected subgraph $h_A(x)$ of $H$ containing $x'$ and no other vertex of $A'$. This is the beginning of our construction of a minor model of $G$ in $H$. Let us now embed the vertices in $G\backslash A$.

Similarly to the proof of Lemma 4.1, we call a part $P_i$ of $G$ *$h'$-stable*, if $\bigcup_n h'_n(P_i)$ is finite. Let $(S_n)_{n\in\mathbb{N}}$ be an enumeration of the $h'$-stable parts of $G$, and $(U_n)_{n\in\mathbb{N}}$ an enumeration of its other parts. Let $G_S := \bigcup_{n\in\mathbb{N}} S_n$. Again, a standard compactness argument yields a minor embedding $h_S : G_S < H$ such that $h_S(S_i\backslash A)$ coincides with $h'_n(S_i\backslash A)$ for infinitely many values of $n$ whenever $S_i$ exists. Note that $h_S$ extends $h_A$ since the latter is the limit of the restriction of $\{h'_n\}$ to $A$. By construction, $h_S =: h_S^0$ is a minor embedding of $G_S$ into $H$.

We continue by following the lines of the proof of Lemma 4.1: for $i = 1, 2, \ldots$, if $U_i$ exists, we let $\{C_n^i, n \in \mathbb{N}\}$ be an infinite sequence of distinct parts of $H$ such that each $C_n^i\backslash A'$ contains $h'_m(U_i\backslash A)$ for some $m \in \mathbb{N}$, whereby we used the fact that each $h_m$ maps each co-part of $G$ to one of $H$.

Combining the marked-graph version of the Graph Minor Theorem 2.3 with Observation 2.2, we deduce that $(C_n^i)_{n\in\mathbb{N}}$ has an infinite subsequence $(Y_n)_{n\in\mathbb{N}}$ such that $Y_r <_\bullet Y_m$ for every $r < m \in \mathbb{N}$, whereby $A'$ is the set of marked vertices of each $Y_n$.

From now on we will not need to use the assumption that $\mathrm{Rank}(G) = \mathrm{Rank}(H) = 1$; this will be important later, as the rest of this proof is also used for Lemma 5.1.

As before, we pick $k = k(i)$ such that no $Y_m, m \geqslant k$ has been nailed yet, and the interesting case is where $Y_k$ intersects the image of $h_S^{i-1}$. In this case, we want to apply Hilbert's Hotel principle again to 'shift' the $h_S^{i-1}$-image within each $Y_m, m \geqslant k$ to $Y_{m+1}$, but we need to be careful with the image of $A$. For this, we start by picking a marked-minor embedding $g_m : Y_m < Y_{m+1}$ for every $m \geqslant k$. Note that as $A'$ is the set of marked vertices of each $Y_n$, each $g_m$ induces a permutation $\pi_m$ of $A'$. Moreover, by composing consecutive $g_m$'s we obtain marked-minor embeddings $g_{mt} : Y_m < Y_t$ for every $t \geqslant m \geqslant k$, which again induce permutations $\pi_{mt}$ on $A'$. Applying Lemma 4.8 we find a subsequence $(Y'_n)$ of $(Y_n)$ such that each of the corresponding permutations $\pi_{mt}$ is the identity. We may assume without loss of generality that $Y' = Y$. Thus we can now repeat the idea of Lemma 4.1 to shift the $h_S^{i-1}$-image within each $Y_m, m \geqslant k$ to $Y_{m+1}$ and embed, and nail, $U_i$ to $Y_k$ to obtain $h_S^i$. We also nail $S_i$, if it exists, to the part of $H$ containing $h_S^i(S_i \backslash A)$. As before, the $h_S^i$ converge as $i \to \infty$ to a minor embedding of $G$ in $H$. □

## 4.1 Proof of Corollary 1.2

We now use the above techniques to prove Seymour's self-minor conjecture for rayless graphs (Corollary 1.2), which we restate here for convenience:

**Corollary 4.7.** *For every infinite rayless graph $G$, there is a proper minor embedding $g : G < G$.*

We start with a simple lemma about permutations, which we will apply to permutations of $A(G)$ arising from self-minor models of $G$.

**Lemma 4.8.** *For every sequence $(\pi_n)_{n\in\mathbb{N}}$ of permutations of a finite set $A$ there is an infinite index set $Y \subseteq \mathbb{N}$ such that $\pi_{jk} = Id$ for every $j < k \in Y$, where $\pi_{jk} := \pi_{k-1} \circ \pi_{k-2} \circ \ldots \circ \pi_{j+1} \circ \pi_j$.*

*Proof.* For every $k \in \mathbb{N}$ let $S_k := \pi_{k-1} \circ \pi_{k-2} \circ \ldots \circ \pi_0$, let $\pi$ be a permutation of $A$ that coincides with $S_k$ for infinitely many $k$, and let $Y$ be the set of those $k$ except the least one. □

*Proof of Corollary 4.7.* We can find a subgraph $H \subset G$ of rank 1 as follows. If $\text{Rank}(G) = 1$ we just let $H := G =: G_0$. If $\text{Rank}(G) > 1$, we pick a co-part $G_1$ of $G$ with $\text{Rank}(G_1) \geqslant 1$; such a $G_1$ exists, because if all co-parts of $G$ have rank 0 then $G$ has rank 1 by the definitions. We then iterate, with $G$ replaced by $G_1$, to obtain a sequence $G_1, G_2, \ldots$ of subgraphs of $G$, each of rank at least 1. Note that $\text{Rank}(G_i) > \text{Rank}(G_{i+1})$ since $G_{i+1}$ is a co-part of $G_i$. Thus the sequence terminates because the ordinal numbers are well-ordered, and we let $H$ be the final member $G_k$ of this sequence. Clearly, $\text{Rank}(H) = 1$, for the sequence would have continued otherwise.

Let $A'(H) := A(G) \cup A(G_1) \ldots \cup A(G_k)$. (We can think of $A'(H)$ as the union of $A(H)$ with all its 'parent' vertices.) Note that $A'(H)$ is finite. Let $P_i, i \in \mathcal{I}$ be a (possibly transfinite) enumeration of the parts of $H$, and let $P'_i := G[A'(H) \cup P_i]$. Apply Theorem 2.3 to $(P'_n)_{n\in\mathbb{N}}$ as above (with $A'(H)$ always marked) to find an infinite $<$-chain that fixes $A'(H)$, for which we also use Lemma 4.8 below, and then apply the HH principle to form a proper self-minor model of $G[A'(H) \cup \bigcup_{i\in\omega} P_i]$ (note that we are ignoring $P_i$ for $i \geqslant \omega$ so far). Extend this model to $G - H$, and to $P_i, i \geqslant \omega$, by the identity map, to obtain the desired $g : G < G$. □

# 5 Extending Lemma 4.1 to Rank $> 1$

Recall that $\mathcal{C}^\bullet(G)$ denotes the class of marked minors of $(G, A(G))$ of lower rank (Definition 4.5). The following is a key lemma for the proof of Theorem 1.3, and it generalises Lemma 4.6.

**Lemma 5.1.** *Let $G, H$ be countable graphs with* $\mathrm{Rank}(H) = \mathrm{Rank}(G)$ *for some ordinal $\alpha < \omega_1$. Assume $\mathcal{C}^\bullet(H)$ is well-quasi-ordered, and $|\mathrm{Rank}^\bullet_\beta \cap \mathcal{C}^\bullet(G)|_{<_\bullet}$ is countable for every $\beta < \alpha$. Then we have $G < H$ if and only if $\mathcal{C}^\bullet(G) \subseteq \mathcal{C}^\bullet(H)$.*

Compared to Lemmas 4.1 and 4.6, this statement removes the condition $|A(G)| = |A(H)|$, and imposes two additional conditions in order to be able to handle ranks $\alpha > 1$. To appreciate the role of these conditions, recall that when we used Lemma 4.1 to prove Corollary 4.2, we used the Graph Minor Theorem and the fact that there are countably many isomorphism types of finite graphs. We do not have analogous statements for higher ranks, and so our condition that $\mathcal{C}^\bullet(H)$ is well-quasi-ordered replaces the former, and the condition $|\mathrm{Rank}^\bullet_\beta \cap \mathcal{C}^\bullet(G)|_{<_\bullet} = \aleph_0$ replaces the latter. Note that both conditions are about graphs of lower rank than that of $G, H$, which will allow us to apply Lemma 5.1 within inductive arguments.

Lemma 5.1 is an important reason why we are forced to consider marked graphs even though we are mainly interested in unmarked ones:

**Remark 5.0.1.** We cannot replace $\mathcal{C}^\bullet(G)$ in Lemma 5.1 by its unmarked version $\mathcal{C}(G)$, as shown by the following example. Let $G_0$ be the disjoint union of the finite cliques $K_n, n \in \mathbb{N}$. Let $H = S(G_0)$ be its suspension (as defined in Section 2.2.1, and $G := S(H)$. Easily, every finite minor of $G$ is a minor of $H$, and nevertheless $G \nless H$ by Observation 2.11 since $|A(G)| = 2$ and $|A(H)| = 1$. This example also shows that it is important to mark the apex vertices and only those in the definition of $\mathcal{C}^\bullet(G)$.

**Remark 5.0.2.** We can also not replace $\mathcal{C}^\bullet(G)$ by its finitary version $\mathcal{C}^\bullet_{\mathrm{fin}}(G)$, i.e. the class of finite marked graphs that are marked-minors of $(G, A(G))$, as shown by the following example. Let $H = S(G_0)$ be as above, and let $H' = S(\omega \cdot H)$ be the suspension over the disjoint union of $\omega$ copies of $H$ (thus $\mathrm{Rank}(H') = 2$). For every $n \in \mathbb{N}$, let $S_n := S(\omega \cdot K_n)$. Let $G' := S(\bigcup_{n \in \mathbb{N}} S_n)$ (again $\mathrm{Rank}(G') = 2$). Note that $H \nless S_n$ for any $n$, and using this it is not hard to see that $G' \nless H'$. On the other hand, both $\mathcal{C}^\bullet_{\mathrm{fin}}(G'), \mathcal{C}^\bullet_{\mathrm{fin}}(H')$ consist of all finite graphs with at most one marked vertex.

We prepare the proof of Lemma 5.1 with two lemmas:

**Lemma 5.2.** *Suppose* $\mathrm{Rank}(G) = \alpha < \omega_1$, *and* $|\mathrm{Rank}^\bullet_\beta \cap \mathcal{C}^\bullet(G)|_{<_\bullet}$ *is countable for every $\beta < \alpha$. Then there is a sequence $G_1 \subset G_2 \subset \ldots$ of subgraphs of $G$, containing $A := A(G)$, such that*

*(i)* $\mathrm{Rank}(G_i) < \mathrm{Rank}(G)$ *for every $i \in \mathbb{N}$; and*

*(ii) for every $G' \subset G$ containing $A$ with* $\mathrm{Rank}(G') < \mathrm{Rank}(G)$ *there is $n$ such that $(G', A) <_\bullet (G_n, A)$.*

*Proof.* By our assumption, the family of marked graphs

$$\{(G', A) \mid A \subset G' \subset G, \mathrm{Rank}(G') < \mathrm{Rank}(G)\} \subseteq \mathcal{C}^\bullet(G)$$

decomposes into countably many marked-minor-twin classes (whereby we use the fact that there are countably many ordinals $\beta < \alpha$). Enumerate these classes as $(\mathcal{C}_n)_{n\in\mathbb{N}}$, and pick a representative $G'_n \subset G$ from each $\mathcal{C}_n$. Then $G_n := \bigcup_{j\leqslant n} G'_j$ has the desired properties. $\square$

(We can achieve $G = \bigcup_{n\in\mathbb{N}} G_n$ if desired, by adding $P_n$ to $G'_n$, but we will not need this.)

Given two graphs $G, H$ of the same rank satisfying $|A(G)| = |A(H)|$ (for example we could have $G = H$), and a minor embedding $h : G < H$, note that $h$ induces a bijection from $A(G)$ to $A(H)$ by Observation 2.10.

**Definition 5.3.** *We denote this bijection by $h^A$.*

Call a part $P$ (as in Definition 4.3) of $H \in \mathrm{Rank}_\alpha$ *$H$-unstable*, if there is a sequence $(h_n)_{n\in\mathbb{N}}$ of minor embeddings $h_n : H < H$ such that each $h_n$ maps $P$ into a different part of $H$ and $h_n^A$ is the identity. Otherwise, we say that $P$ is *$H$-stable*.

**Lemma 5.4.** *Let $H$ be a rayless graph such that $\mathcal{C}^\bullet(H)$ is well-quasi-ordered. Then at most finitely many parts of $H$ are $H$-stable.*

*Proof.* Suppose not, and let $(P_n)_{n\in\mathbb{N}}$ be an enumeration of the infinitely many $H$-stable parts of $H$. Since $\mathcal{C}^\bullet(H)$ is well-quasi-ordered, $(P_n)_{n\in\mathbb{N}}$ is good, and so by Observation 2.2, there is an infinite chain $P_{a_1} <_\bullet P_{a_2} <_\bullet \ldots$. Let $h_i : P_{a_i} <_\bullet P_{a_{i+1}}$ be corresponding minor embeddings, and note that $h_i$ induces a permutation $\pi_i$ on $A(H)$ by Observation 2.11. By Lemma 4.8 we may assume, by passing to a subsequence if necessary, that each $\pi_i$ is the identity on $A(H)$. Combining this with the Hilbert Hotel Principle as in the proof of Lemma 4.6 we can define $h : H <_\bullet H$ such that $h(P_{a_i}) \subseteq P_{a_{i+1}}$ for every $i$, and $h^A$ is the identity. Let $h_n, n \in \mathbb{N}$ be the composition of $h$ with itself $n$ times. Then $(h_n)_{n\in\mathbb{N}}$ witnesses that $P_{a_1}$ is $H$-unstable, contradicting our assumption. $\square$

**Remark 5.0.3.** Call a part $P$ of $H \in \mathrm{Rank}_\alpha$ *redundant*, if $H < H\backslash P^c$, where $P^c := P\backslash A(H)$ denotes the co-part of $P$. Then Lemma 5.4 remains true if we replace '$H$-stable' by 'irredundant'.

We can now prove the main result of this section.

*Proof of Lemma 5.1.* Again, the forward implication is trivial. For the backward implication we will follow the approach of Lemma 4.6, and it is assumed that the reader is familiar with its proof. The main technical difficulty in comparison to that lemma will be handling the stable parts, and those co-parts using vertices of $A'$ in their branch sets.

Let $(P_n)_{n\in\mathbb{N}}$ be an enumeration of the parts of $G$. Let $(G_n)_{n\in\mathbb{N}}$ be a sequence of subgraphs of $G$ as provided by Lemma 5.2. We may assume that $P_n \subseteq G_n$, because we can add $\bigcup_{i\leqslant n} P_i$ to $G_n$ without increasing its rank. Let $P_i^c := P_i\backslash A$ be the co-part of $P_i$.

Let $A := A(G)$ and $A' := A(H)$. Since $\mathcal{C}^\bullet(G) \subseteq \mathcal{C}^\bullet(H)$, there is a sequence of marked-minor embeddings $h_n : (G_n, A) <_\bullet (H, A')$.

We say that $P_i$ *uses* $x \in A'$ in $h_n$, if $x \in h_n(P_i^c)$, and we say that $P_i$ *uses* $A'$ in $h_n$ if it uses some $x \in A'$. We are going to use a subsequence of $(h_n)$ whose members treat $A'$ in a similar way; to make this precise, given $f : (G_m, A) <_\bullet$

$(H, A')$, we define its *$A'$-signature* $\sigma_f : A' \to X$, where $X := A \cup \{\varnothing\}$, as follows. For each $x \in A'$ used by some $a \in A$ in $f$, we let $\sigma_f(x) = a$. For each other $x \in A'$ we let $\sigma_f(x) = \varnothing$.

Since the target $X$ of $\sigma_{h_n}$ is finite, there is an infinite subsequence of $(h_n)$ along which $\sigma_{h_n}$ is a constant map $z$, and we may assume without loss of generality that this subsequence coincides with $(h_n)$. We call any such sequence $(h_n)_{n \in \mathbb{N}}$ a *$z$-sequence*.

In Lemma 4.6 we had $|A| = |A'|$, which ensured that $z$ was a bijection between $A, A'$, and every co-part $P_i^c$ of $G$ was mapped to a co-part of $H$ by each $h_n$, but neither of this needs to be the case now. Therefore, we will need a more delicate definition of stable and unstable parts, which we prepare with the following notion.

We say that $P_i$ is *$h$-annoying*, if $P_i^c$ uses $A'$ in almost all $h_n$. Let $\mathcal{A} = \mathcal{A}_h$ denote the set of $h$-annoying parts of $G$. Note that

$$|\mathcal{A}_h| \leqslant |A'|, \tag{10}$$

because each $x \in A'$ is used by at most one co-part in each minor embedding. Easily, by passing to a subsequence, we may assume that for every $P_i \notin \mathcal{A}_h$,

$$P_i^c \text{ uses } A' \text{ in at most finitely many } h_n. \tag{11}$$

Indeed, we can construct a subsequence $(h')$ of $h$ with this property recursively as follows. Let $h_n^0 := h_n$, and for $i = 1, 2, \ldots$, let $h_n^i$ be a subsequence of $h_n^{i-1}$ in which $P_i$ never uses $A'$ if $P_i \notin \mathcal{A}_{h^{i-1}}$. If $P_i \in \mathcal{A}_{h^{i-1}}$, just let $h^i = h^{i-1}$. Note that $h^i$ may have annoying parts that were not annoying in $h^{i-1}$, but they are boundedly many by (10).

We may assume, by passing to a co-final subsequence of $h^i$, that each $h^i$-annoying part $P$ uses $A'$ in $h_n^i$ for every $n$. Note that this means that $P$ uses $A'$ for every $n$ in any subsequence of $h_n^i$. Since each $h^i$ is a subsequence of $h^{i-1}$, this implies that $\mathcal{A}_{h^i}$ is increasing with $i$, and therefore its limit $\mathcal{A}$ is a finite set of parts of $G$.

Let $h'_n := h_n^n$. Note that $\mathcal{A}_{h'} = \mathcal{A}$, and therefore (11) is satisfied if we replace $h$ by $h'$ (and $\mathcal{A}_h$ by $\mathcal{A}$). Since $h'$ is a subsequence of $h$, we will from now on just assume $h = h'$ to keep the notation simpler.

We now adapt the definition $h$-stable from Lemma 4.1 to the non-annoying parts of any $z$-sequence $(g_n)_{n \in \mathbb{N}}$. We call a part $P \notin \mathcal{A}_g$ of $G$ *$g$-stable*, if there is a subgraph $H'$ of $H$ consisting of finitely many parts of $H$ with the following property: for all $n$ such that $g_n(P)$ does not use $A'$, we have $g_n(P) \subset H'$. Call $P \notin \mathcal{A}_g$ *$g$-unstable* otherwise. Note that if $P$ is $g$-unstable, then

there is an infinite sequence $(C_n)_{n \in \mathbb{N}}$ of distinct parts of $H$ and a strictly increasing sequence $(\tau_n)_{n \in \mathbb{N}}$ such that $g_{\tau(n)}(P^c) \subseteq C_n^c$ for every $n$. (12)

This is because $g_n(P^c)$ is contained in a co-part of $H$ by Lemma 2.4 whenever $P$ does not use $A'$.

We claim that there is $z$-sequence $(g_n)_{n \in \mathbb{N}}$ maximizing the set of unstable parts in the following sense:

There is a $z$-sequence $(g_n)_{n \in \mathbb{N}}$, $g_n : G_n \to H$, such that every part $P$ of $G$ that is $f^P$-unstable with respect to some $z$-sequence $(f_n^P)_{n \in \mathbb{N}}$ is also $g$-unstable. (13)

To see this, enumerate those parts $P$ as $(U_i)_{i\in\mathbb{N}}$, and form $g_n$ by picking infinitely many members from each sequence $(f^{U_i})$, assuming without loss of generality that $f_n^{U_i}(U_i^c)$ lie in distinct co-parts of $H$ for different values of $n$, which we can by (12), and in particular $f_n^{U_i}(U_i^c)$ never uses $A'$. Note that no $U_i$ can be $g$-annoying, since infinitely members of $(f^{U_i})$ are also members of $g$. This proves (13).

Fix $(g_n)$ as in (13), and let $\mathcal{S}$ be the set of $g$-stable parts of $G$, and $\mathcal{U}$ the set of all other parts of $G$. Thus $\{P_n \mid n \in \mathbb{N}\} = \mathcal{A}_g \,\dot\cup\, \mathcal{S} \,\dot\cup\, \mathcal{U}$.

Let $\mathcal{S}'$ be the set of $H$-stable parts of $H$ —as defined before Lemma 5.4— and $\mathcal{U}'$ the set of all other parts of $H$. Let $H_S := \bigcup \mathcal{S}' \subset H$. By Lemma 5.4,

$$\mathcal{S}' \text{ is finite, and therefore } \mathrm{Rank}(H_S) < \mathrm{Rank}(H). \tag{14}$$

Next, we claim that

$$\text{for every } P \in \mathcal{S}, \text{ and almost every } n \text{ such that } g_n(P) \text{ does not use } A', \text{ we have } g_n(P) \subset H_S. \tag{15}$$

Suppose this fails, which means that there is some $P \in \mathcal{S}$, and an infinite strictly increasing sequence $(\tau_n)_{n\in\mathbb{N}}$, such that $g_{\tau(n)}(P)$ does not use $A'$ and $g_{\tau(n)}(P^c)$ is contained in an element $U_n$ of $\mathcal{U}'$ for every $n$. Since $P$ is $g$-stable, $\bigcup_n g_{\tau(n)}(P)$ meets only finitely many parts of $H$, and so we may assume that these $U_n$ coincide with a fixed $U \in U'$. Let $(f_n)_{n\in\mathbb{N}}, f_n : H <_\bullet H$ be a sequence of minor embeddings witnessing that $U$ is $H$-unstable, i.e. the $f_n$ embed $U$ into infinitely many distinct parts of $H$ and $f_n^A$ is the identity on $A'$ for every $n$. Then the sequence of compositions $f_n \circ g_{\tau(n)}$ embed $P$ into infinitely many distinct parts of $H$, and so $P$ is $(f \circ g \circ \tau)$-unstable. But this contradicts our choice of $g$ because of (13), since $P$ is $g$-stable and $(h \circ g \circ \tau)$ is a $z$-sequence. This contradiction proves (15).

Let $G_S := \bigcup(\mathcal{S} \cup \mathcal{A}_g)$. We now use (15) to prove that

$$\mathrm{Rank}(G_S) < \mathrm{Rank}(H) (= \mathrm{Rank}(G)). \tag{16}$$

Suppose to the contrary $\mathrm{Rank}(G_S) = \mathrm{Rank}(G)$, which is larger than $\beta := \mathrm{Rank}(H_S)$ by (14).

Let $A'' := A(H) \cup \bigcup_{Q\in\mathcal{S}'} A(Q)$ (Figure 2), and note that $A''$ is finite by (14), and that

$$\text{each component of } H_S - A'' \text{ has rank smaller than } \beta \tag{17}$$

by the definition of $A(Q)$.

Since we are assuming $\mathrm{Rank}(G_S) > \beta$, there are infinitely many $P \in (\mathcal{S} \cup \mathcal{A}_g)$ with $\mathrm{Rank}(P^c) \geqslant \beta$ by Proposition 2.9. As $\mathcal{A}_g$ is finite by (10), there are infinitely many such $P$ in $\mathcal{S}$. Let us choose $|A''| + 1$ of them, and denote them $S_0, \ldots, S_{|A''|}$. By (15) we can pick $m$ large enough that for every $i \in [0, \ldots, |A''|]$, either $g_m(S_i^c)$ uses $A'$ or $g_m(S_i) \subseteq H_S$. By the pigeonhole principle, there is $k \in [0, \ldots, |A''|]$ such that $g_m(S_k^c)$ does not use $A''$ —and hence $A'$— because the $S_i^c$ are pairwise disjoint, and so at most one of them can use any fixed vertex of $A''$. But then $g_m(S_k^c)$ is contained in $H_S$ by the choice of $m$, and in fact in a component $C$ of $H_S - A''$ since $S_k^c$ is connected and it avoids $A''$. Thus $S_k^c < C$,

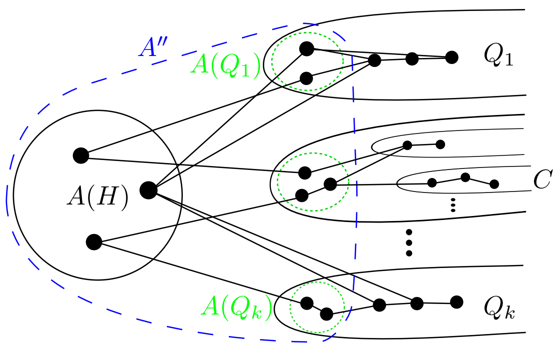


Figure 2: The vertex set $A'' := A(H) \cup \bigcup_{Q \in \mathcal{S}'} A(Q)$ in the proof of (17), enclosed by the dashed curve (blue).

which contradicts Observation 2.10 since $\mathrm{Rank}(S_k^c) \geqslant \beta > \mathrm{Rank}(C)$ by (17). This contradiction proves (16).

From (16) we deduce $(G_S, A) \in \mathcal{C}^\bullet(G)$, and so $(G_S, A) <_\bullet (G_\ell, A)$ for some $\ell$, and therefore $g_n$ induces a marked-minor embedding $(G_S, A) <_\bullet (H, A')$ for every large enough $n$. Pick such an $n$, and let $g_S := g_n$. (We do not claim that $g_S(G_S)$ is contained in $H_S$, or that $G_S$ consists of finitely many parts of $G$.)

From now on we can proceed as in the proof of Lemma 4.6 to extend $g_S$ to $h : G < H$ (and in fact $h : (G, A) <_\bullet (H, A')$, though we will not use this) by recursively applying the Hilbert Hotel Principle to the $i$th $g$-unstable part $U_i$ of $G$. The main difference is that instead of appealing to the Graph Minor Theorem, we now use our assumption that $\mathcal{C}^\bullet(H)$ is well-quasi-ordered—combined with Observation 2.2— in order to find a chain $(Y_n)_{n \in \mathbb{N}}$ within any sequence $(C_n^i)_{n \in \mathbb{N}}$ of parts of $H$ coming from (12) applied to $U_i$. The only other difference is that rather than enumerating the parts in $\mathcal{S}$, and nailing the $i$th one in step $i$, we now enumerate the co-parts of $H$ intersecting $G_S(\mathcal{S} \cup \mathcal{A}_g)$ as $(P'_n)_{n \in \mathbb{N}}$, and ensure that the 'contents' of $P'_i$ are never 'shifted' after step $i$. $\square$

# 6 Countability of minor-twin-types of rayless forests

In this section we prove that for every ordinal $\alpha < \omega_1$ there are countably many minor-twin classes of countable forests of rank $\alpha$ (Theorem 6.3). This is an important step towards the proof of the backward direction of Theorem 1.4, which we will conclude in the next section. Our proof relies on Thomas' theorem that the class $TW(t)$ of countable graphs of tree-width at most $t$ are well-quasi-ordered for every $t \in \mathbb{N}$ [25, (1.7)]. Although we are mainly interested in forests, our proof employs an intricate inductive argument for which it is essential to consider $TW(t)$ for every $t$. For a similar reason, we have to consider marked graphs even though our focus is on unmarked ones. Our final result will apply to $TW(t)$, not just the forests.

We prepare our proof with two lemmas. The first extends Thomas' aforementioned result to marked graphs.

**Lemma 6.1.** *For every $t \in \mathbb{N}$, the class of marked graphs $(G, M)$ with $G \in TW(t)$ is well-quasi-ordered under $<_\bullet$.*

This is easily proved by replacing each marked vertex by a complete graph of the right size:

*Proof.* Let $((G_i, M_i))_{i \in \mathbb{N}}$ be a sequence of marked graphs in $TW(t)$. We need to show that it is good. Easily, we can assume that each $G_i$ has more than $t + 2$ vertices. Let $(G'_i, M'_i) := S^\bullet(S^\bullet((G_i, M_i)))$ —the marked suspension as defined in Section 2.2.1— and note that $G'_i$ is 2-connected, and $TW(G'_i) \leqslant t+2$. Applying Lemma 2.6 twice, we deduce that $((G_i, M_i))_{i \in \mathbb{N}}$ is good if $((G'_i, M'_i))_{i \in \mathbb{N}}$ is, and so it remains to prove the latter.

Next, for each $i$, we modify $(G'_i, M'_i)$ into an unmarked graph $G''_i$ of tree-width $t + 3$ by attaching a copy of $K_{t+4}$ (which has tree-width $t + 3$) to each $v \in M'_i$ by identifying $v$ with an arbitrary vertex of $K_{t+4}$. We claim that for every $i, j \in \mathbb{N}$,

$$G''_i < G''_j \text{ if and only if } (G'_i, M'_i) <_\bullet (G'_j, M'_j). \tag{18}$$

This claim implies our statement, because $(G''_i)_{i \in \mathbb{N}}$ is good by Thomas' aforementioned theorem, implying that $((G'_i, M'_i))_{i \in \mathbb{N}}$ is good too.

The backward implication of (18) is trivial (and not needed for our proof). For the forward implication, suppose $\mathcal{B}$ is a minor model of $G''_i$ in $G''_j$. Since $G'_i$ is a block of $G''_i$, Lemma 2.5 says that $\mathcal{B}$ can be restricted into a minor model $\mathcal{B}'$ of $G'_i$ within a block of $G''_j$. The latter block can only be $G'_j$, because $|G'_i| > t+4$ by our assumption, and all other blocks of $G''_j$ are copies of $K_{t+4}$. Moreover, applying Lemma 2.5 to each other block $B$ of $G''_i$, which is a $K_{t+4}$, we deduce that for each $v \in V(B)$, the branch set $B_v$ contains a vertex of a single block $B'$ of $G''_j$. This $B'$ must be a $K_{t+4}$ because $TW(B) = t + 3 > TW(G'_j)$. Thus $\mathcal{B}$ induces a 1-1 correspondence between the $t+4$ vertices in $B$ and the $t+4$ vertices in $B'$. It follows that the branch set of the unique vertex in $V(B) \cap M'_i$ contains the unique vertex in $V(B') \cap M''_j$. This means that $\mathcal{B}'$ maps each marked vertex of $G'_i$ to a branch set containing a marked vertex of $G'_j$, which proves (18). $\square$

For our proof of Theorem 1.4 in the next section we will need to prove that there are only countably many minor-twin types of countable forests of any given rank $\alpha < \omega_1$. The ideas involved are similar to the proof of Corollary 4.2, except that instead of $\mathcal{C}(G)$ we will be working with $\mathcal{C}^\bullet(G)$, and therefore with marked graph. Moreover, instead of the Graph Minor Theorem, we now need to use Nash-Williams' theorem [15] that the countable trees are well-quasi-ordered. We have made one step towards adapting our tools to marked graphs with Lemma 6.1, but we will need more: using the same unmarking idea as in the previous proof, we will next upper-bound the number of minor-twin classes of *marked* forests, and more generally graphs in $TW(t)$, by the number of minor-twin classes of *unmarked* graphs in $TW(t + 1)$:

**Lemma 6.2.** *For every ordinal $\alpha < \omega_1$, and every $t \in \mathbb{N}_{>0}$, we have*
$|\mathrm{Rank}^\bullet_\alpha \cap TW(t)|_{<_\bullet} \leqslant |\mathrm{Rank}_\alpha \cap TW(t + 1)|_{<}$.

This lemma is the reason why it does not suffice to use Nash-Williams' theorem about forests, and instead we need Thomas' extension to $TW(t)$.

*Proof.* Pick a representative $(G_X, M_X)$ from each marked-minor-twin class $X$ of $\mathrm{Rank}^{\bullet}_{\alpha} \cap TW(t)$, and use it to define the unmarked graph $G'_X \in \mathrm{Rank}_{\alpha} \cap TW(t+1)$ similarly to the proof of Lemma 6.1, by attaching a copy of $K_{t+2}$ to each marked vertex and unmarking it.

Suppose $G_X, G_Y$ have the same number of marked vertices, say $n$. Similarly to (18), we claim that

$$G'_X < G'_Y \text{ if and only if } (G_X, M_X) <_{\bullet} (G_Y, M_Y). \tag{19}$$

From this we immediately deduce that $G'_X, G'_Y$ are not minor-twins unless $X = Y$. This implies that each $n \in \mathbb{N}$ contributes at most $|\mathrm{Rank}_{\alpha} \cap TW(t+1)|_{<}$ to the count of classes in $\mathrm{Rank}^{\bullet}_{\alpha} \cap TW(t)$, and since the former is infinite, we obtain the desired inequality.

Thus it only remains to check (19). The proof is similar to that of (18), except that we now only apply Lemma 2.5 to the $K_{t+2}$'s of $G'_X$: if $\mathcal{B}$ is a minor model of $G'_X$ in $G'_Y$, then it maps each copy of $K_{t+2}$ in $G'_X$ to one in $G'_Y$, whereby we use the fact that $G_X$ has no $K_{t+2}$ minor as its tree-width is less than that of $K_{t+2}$. It follows easily from this that for each $v \in M_X$ the branch set $\mathcal{B}(v)$ contains a vertex of $M_Y$. Moreover, if $w \in V(G_X)\backslash M_X$, then $\mathcal{B}(w)$ cannot intersect $G'_Y \backslash G_Y$, because all vertices in the latter subgraph are needed to accommodate the copies of $K_{t+2}$ in $G'_X$. Thus by restricting $\mathcal{B}$ to $G_X$ we obtain a marked-minor model of $(G_X, M_X)$ in $(G_Y, M_Y)$, proving (19). □

We are now ready to prove the main result of this section:

**Theorem 6.3.** *For every ordinal $\alpha < \omega_1$, and every $t \in \mathbb{N}_{>0}$, we have $|\mathrm{Rank}^{\bullet}_{\alpha} \cap TW(t)|_{<_{\bullet}} = \aleph_0$.*

*Proof.* We will prove the unmarked version

$$|\mathrm{Rank}_{\alpha} \cap TW(t)|_{<} = \aleph_0 \tag{20}$$

by a simultaneous (transfinite) induction on $\alpha$ and $t$. From this the statement follows immediately from Lemma 6.2.

For $\alpha = 0$, our claim (20) is trivially true for every $t$ as there are countably many (marked) finite graphs; here we do not need the restriction on the tree-width.

For the inductive step $\alpha > 0$, we proceed by induction on $t$. Let $\mathrm{Rank}^{n}_{\alpha}$ be the set of those $G \in \mathrm{Rank}_{\alpha}$ with $|A(G)| = n$. To prove that $|\mathrm{Rank}_{\alpha} \cap TW(t)|_{<}$ is countable, it suffices to prove that $|\mathrm{Rank}^{n}_{\alpha} \cap TW(t)|_{<}$ is countable for every $n \in \mathbb{N}$. Let $G \in \mathrm{Rank}^{n}_{\alpha}(TW(t))$, and apply Lemma 5.1 to deduce that the minor-twin class $[G]_{<_{\bullet}}$ is determined by $n = |A(G)|$ and the class $\mathcal{C}^{\bullet}(G)$; here, to be able to apply Lemma 5.1, we use the fact that $\mathcal{C}^{\bullet}(G) \subseteq \mathrm{Rank}^{n\bullet}_{<\alpha} \cap TW(t)$ is well-quasi-ordered by Lemma 6.1, and our inductive hypothesis that $|\mathrm{Rank}^{\bullet}_{\beta} \cap TW(t)|_{<_{\bullet}}$ is countable for every $\beta < \alpha$.

Since $\mathcal{C}^{\bullet}(G)$ is well-quasi-ordered, we can write $\mathcal{C}^{\bullet}(G)$ as $\mathrm{Forb}(X) \cap \mathrm{Rank}^{n\bullet}_{<\alpha} \cap TW(t)$ for some finite $X \subset \mathrm{Rank}^{n\bullet}_{<\alpha} \cap TW(t)$, whereby we used the fact that $TW(t)$ is minor-closed. We claim that $|\mathrm{Rank}^{n\bullet}_{<\alpha} \cap TW(t)|_{<_{\bullet}}$ is countable. Indeed, for every $\beta < \alpha$, Lemma 6.2 yields

$|\mathrm{Rank}^\bullet_\beta \cap TW(t)|_{<_\bullet} \leqslant |\mathrm{Rank}_\beta \cap TW(t+1)|_<$, which is countable by our inductive hypothesis on $\alpha$. Taking the union over all $\beta < \alpha$, which are countably many as $\alpha < \omega_1$, establishes our claim that $|\mathrm{Rank}^{n\bullet}_{<\alpha} \cap TW(t)|_{<_\bullet}$ is countable. Thus there are countably many ways to choose its finite subset $X$ from above, hence countably many ways to choose $\mathcal{C}^\bullet(G)$, and therefore $[G]_<$. This proves (20). ☐

# 7 Proper minor-closed classes of rayless forests are Borel

In this section we use Theorem 6.3 to prove Theorem 1.4, which we restate for convenience:

**Theorem 7.1.** *Let $\mathcal{T} \subset \mathcal{G}$ be a minor-closed family of $\mathbb{N}$-labelled rayless forests. Then $\mathcal{T}$ is Borel if and only if it does not contain all rayless forests.*

Recall that $\mathcal{G}$ denotes the space of graphs $G$ with $V(G) = \mathbb{N}$, which we call $\mathbb{N}$-*labelled* graphs, encoded as functions from $\mathbb{N}^2$ to $\{0,1\}$ representing the edges, endowed with the product topology. Our condition that $\mathcal{T} \subset \mathcal{G}$ be *minor-closed* here means that if $G \in \mathcal{T}$ and $H \in \mathcal{G}$ is a minor of $G$ (according to the standard definition for unlabelled graphs as in Section 2.2), then $H \in \mathcal{T}$.

The reader will not need to know much about the topology of $\mathcal{G}$ in order to understand this section; the connection between minor-closed families and Borel sets is established by the following result that we will use to prove Theorem 7.1:

**Theorem 7.2** ([7])**.** *Let $G$ be a countable graph, and let $\mathcal{C} \subset \mathcal{G}$ denote the set of countable $\mathbb{N}$-labelled graphs that are isomorphic to a minor of $G$. Suppose $\mathcal{C}(G) = \mathrm{Forb}(\mathcal{X}) \cap \mathcal{G}$ for a countable set $\mathcal{X}$ of (unlabelled) graphs. Then $\mathcal{C}$ is a Borel subspace of $\mathcal{G}$.*

Apart from this, the only topological statement that we will need in our proof is the fact that any countable union of Borel sets is itself Borel.

We will also need the following lemma, which is perhaps well-known when restricted to trees, but we include a proof for completeness.

**Lemma 7.3.** *For every countable rayless tree $T$ there is an ordinal $\alpha(T) < \omega_1$ such that $T < G$ holds for every graph $G$ with* $\mathrm{Rank}(G) > \alpha(T)$*.*

*Proof.* We will state a modified statement that will help us apply transfinite induction on $\mathrm{Rank}(T)$. A *rooted tree* $(T, r)$ is a tree $T$ with one of its vertices $r$ designated as the root. The *tree-order* $\leqslant_r$ on $V(T)$ is the partial order defined by setting $x \leqslant_r y$ for any two vertices $x, y$ such that $x$ lies on the unique path in $T$ from $r$ to $y$. Given rooted trees $(T, r), (T', r')$, we write $(T, r) \preceq (T', r')$ if there is a subgraph embedding $h$ of $T$ into $T'$ that respects the tree-order, i.e. $x \leqslant_r y$ implies $h(x) \leqslant_{r'} h(y)$ for every $x, y \in V(T)$. We will prove:

For every countable rayless rooted tree $(T, r)$ there is an ordinal $\alpha = \alpha(T) < \omega_1$ such that $(T, r) \preceq (T_\alpha, r_\alpha)$ holds, (21)

where $T_\alpha$ denotes the minimal tree of Rank $\alpha$, as provided by Observation 2.12, with rooted $r_\alpha$ provided in its construction.

For finite $T$ it is not hard to see, by induction on the size of $T$, that $\alpha(T) = \omega$ suffices.

For $\mathrm{Rank}(T) \geqslant 1$, the inductive hypothesis is easier to apply when $A(T) = \{r\}$, but this need not be the case. Therefore, we introduce the *spread* $S(T, r)$ of a rooted tree $(T, r)$, defined as $S(T, r) := \max_{x \in A(T)} d(x, r)$, where $d$ denotes the graph distance. Fixing $\mathrm{Rank}(T)$, we prove (21) by induction on $S(T, r)$ as follows.

Let $C_1, C_2, \ldots$ be a (possibly finite) enumeration of the components of $T - r$, and root each $C_i$ at the unique neighbour $r_i$ of $r$ in $C_i$. We claim that

$$\text{for every } i, \text{ either } \mathrm{Rank}(C_i) < \mathrm{Rank}(T), \text{ or } S(C_i, r_i) < S(T, r) \text{ (or both).} \quad (22)$$

To see this, suppose first that $C_i \cap A(T) = \varnothing$. Then $\mathrm{Rank}(C_i) < \mathrm{Rank}(T)$ because $C_i$ is contained in a component of $T - A(T)$ in this case. Otherwise, let $A' := C_i \cap A(T) \neq \varnothing$. Then $\mathrm{Rank}(C_i) = \mathrm{Rank}(T)$, and it is not hard to check that $A(C_i) = A'$. Let $x \in A'$ be a vertex realising $S(C_i, r_i)$. Then $d(x, r) = 1 + d(x, r_i)$, implying the desired $S(T, r) > S(C_i, r_i)$.

Using (22), we can now define $\alpha := 1 + \sup_{i \in \mathbb{N}} \alpha(C_i)$, noting that $\alpha(C_i)$ is well-defined by induction on $S(T, r)$, nested inside our induction on $\mathrm{Rank}(T)$. To start the induction on $S(T, r)$, we note that $S(T, r) = 0$ if and only if $A(T) = \{r\}$, in which case the first possibility always applies in (22), and therefore $\alpha(C_i)$ is well-defined by induction on $\mathrm{Rank}(T)$.

We claim that $(T, r) \preceq (T_\alpha, r_\alpha)$. To prove this, we map $r$ to $r_\alpha$, and use our inductive hypothesis to embed each $C_i$ into an appropriate component of $T_\alpha - r_\alpha$, rooted at the neighbour of $r_\alpha$, preserving the tree order. The latter is possible because, by the construction of $T_\alpha$, there are infinitely many components of $T_\alpha - r_\alpha$ isomorphic to $T_\alpha(C_i)$ for each $i$, and so we can pick a distinct such component to embed each $C_i$ using our inductive hypotheses.

This proves (21). Our statement now follows by forgetting the root, and noting that any graph $G$ with $\mathrm{Rank}(G) > \alpha(T)$ has a component $G'$ with $\mathrm{Rank}(G') \geqslant \alpha(T)$ by the definition of rank, and $G'$ contains $T_\alpha$ as a minor by Observation 2.12. $\square$

We can now prove the main result of this section:

*Proof of Theorem 7.1.* If $\mathcal{T}$ is the family $\mathcal{FR}$ of all $\mathbb{N}$-labelled rayless forests, then it is well-known that it is not Borel (in fact it is co-analytic complete) [11, 7].

So suppose $\mathcal{T}$ excludes some rayless forest $T$ as a minor. Then $\mathcal{T}$ excludes a rayless tree, obtained from $T$ by adding a vertex and joining it to each component, and so Lemma 7.3 implies that $\mathcal{T} \subseteq \mathrm{Rank}_\alpha \cap \mathcal{FR}$ for some ordinal $\alpha < \omega_1$.

We would like to apply Theorem 6.3, but there is a subtlety we need to address: the former result is about unlabelled graphs, while $\mathcal{T} \subset \mathcal{G}$. Therefore, we let $\mathcal{T}'$ denote the class of (unlabelled) graphs $G$ such that $G$ is isomorphic to a minor of some graph in $\mathcal{T}$. Note that $\mathcal{T}'$ is minor-closed in the standard sense, unlike $\mathcal{T}$ which does not contain any graphs with finite vertex set. We still have $\mathcal{T}' \subseteq \mathrm{Rank}_\alpha$, and combining this with Theorem 6.3, we deduce that $\mathcal{T}'$ consists of countably many minor-twin classes because there are countably many ordinals $\beta \leqslant \alpha$. Let $(\mathcal{T}_n)_{n \in \mathbb{N}}$ be an enumeration of these classes, and pick a representative $F_n$ from each $\mathcal{T}_n$. For each infinite $F_n$, let $F'_n$ be an element of

$\mathcal{T}$ isomorphic to $F_n$. For each finite $F_n$, let $F_n'$ be an element of $\mathcal{T}$ consisting of infinitely many isolated vertices and a finite graph isomorphic to $F_n$. Let $\mathcal{C}_n \subset \mathcal{G}$ denote the set of countable $\mathbb{N}$-labelled graphs that are isomorphic to a minor of $F_n'$. Easily,

$$\mathcal{T} = \bigcup \mathcal{C}_n, \tag{23}$$

because $\mathcal{T}$ is minor-closed, and $\bigcup \mathcal{C}_n$ contains a minor-twin of each element of $\mathcal{T}$. Note that, since the countable forests are well-quasi-ordered (Lemma 6.1), $\mathcal{C}_n = \mathrm{Forb}(X_n)$ for a finite set $X_n$ (consisting of $K_3$ and a finite set of forests). Thus each $\mathcal{C}_n$ is a Borel subset of $\mathcal{G}$ by Theorem 7.2, and therefore $\mathcal{T}$ is Borel by (23). □

**Remark 7.0.1.** If Thomas' conjecture, or its restriction to rayless graphs, is true, then following the lines of the proof of Theorem 7.1, but using Theorem 10.1 (a) $\leftrightarrow$ (j) instead of Theorem 6.3, we would deduce that if $\mathcal{T} \subset \mathcal{G}$ is a minor-closed family of $\mathbb{N}$-labelled rayless graphs that does not contain all rayless forests then $\mathcal{T}$ is Borel.

# 8 From marked to unmarked graphs

Marked graphs play an important role in the proof of Theorem 1.3, mainly via Lemma 5.1. This section provides two important tools for the former, which essentially allow us to ignore the marking in certain cases:

**Lemma 8.1.** *For every ordinal $\alpha$, if $\mathrm{Rank}^\bullet_\alpha$ has a (infinite) descending chain, then so does $\mathrm{Rank}_\alpha$.*

**Corollary 8.2.** *For every $0 \leqslant \alpha < \omega_1$, we have $|\mathrm{Rank}_\alpha|_< = |\mathrm{Rank}^\bullet_\alpha|_{<_\bullet}$.*

*Proof of Lemma 8.1.* Suppose there is a $<_\bullet$-descending chain $G_1 \gneqq G_2 \gneqq G_3 \gneqq \ldots$ in $\mathrm{Rank}^\bullet_\alpha$, and let $M_i$ denote the set of marked vertices of $G_i$. By Lemma 2.6 we may assume without loss of generality that each $G_i$ is 2-connected, because we may add a couple of (marked) suspension vertices to each $G_i$ without violating any of the relations $G_i \gneqq G_j$. Moreover, we may assume that $A(G_i)$ is 2-connected for every $i$, since every suspension vertex lies in $A(G_i)$. Here we use the obvious fact that $\mathrm{Rank}(S(G)) = \mathrm{Rank}(G)$ for every graph $G$.

Since $(G', M') <_\bullet (G, M)$ implies $|M| \geqslant |M'|$, we deduce that $|M_i|$ is monotone decreasing. Thus we may assume that it is constant (and at least 1, or there is nothing to prove). Similarly, since $(G', M') <_\bullet (G, M)$ implies $\mathrm{Rank}(G) \geqslant \mathrm{Rank}(G')$ (Observation 2.10), we may assume, by induction on $\alpha$, that $\mathrm{Rank}(G_i) = \alpha$ for every $i$. Letting $A_i := A(G_i)$, it follows from Observation 2.11 that $|A_i|$ is monotone decreasing too, and again we may assume that it is constant. Similarly, we have $|A_i \cap M_i| \geqslant |A_{i+j} \cap M_{i+j}|$ for every $i, j \in \mathbb{N}$, and so we can also assume that $|A_i \cap M_i|$ is constant. Note that this implies that $|M_i \backslash A_i|$ is constant too, and that

any minor model $\mathcal{B}$ of $G_i$ in $G_{i-j}$ maps each vertex in $M_i \backslash A_i$ to a branch set intersecting $M_{i-j} \backslash A_{i-j}$. (24)

We claim that

we may assume that $M_i \subseteq A_i$ for every $i$. (25)

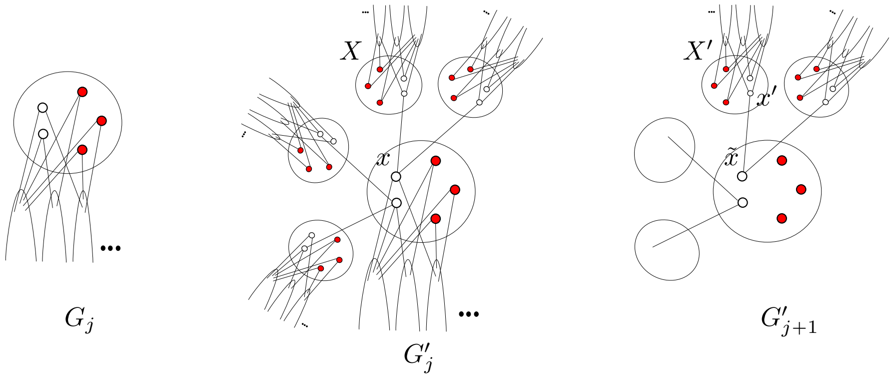


Figure 3: The graph $G'_j$ in the proof of (26) (middle), produced by joining copies of $G_j$ (left), and an attempt to embed it into $G'_{j+1}$ (right).

Indeed, if this is not the case, then let $\tilde{M}_i := M_i \backslash A_i$, and for each $x \in \tilde{M}_i$, attach a copy of the minimal tree $T_\alpha$ of Rank $\alpha$, provided by Observation 2.12, to $x$, by identifying the root of $T_\alpha$ with $x$, to obtain the marked graph $(\tilde{G}_i, M_i \backslash \tilde{M}_i)$ for every $i \in \mathbb{N}$. We will show that $(\tilde{G}_n)_{n \in \mathbb{N}}$ is still a $<_\bullet$-descending chain: firstly, using (24) we can extend $\mathcal{B}$ into a minor model of $\tilde{G}_i$ in $\tilde{G}_{i-j}$. This shows that $\tilde{G}_{i-j} <_\bullet \tilde{G}_i$. To show that $\tilde{G}_i \not<_\bullet \tilde{G}_{i+j}$, suppose $\tilde{\mathcal{B}}$ is such a minor model. Then by Lemma 2.5, since $G_i \subset \tilde{G}_i$ is 2-connected, there is a model $\mathcal{B}'$ of $G_i$ in a block of $\tilde{G}_{i+j}$. Since the only block of $\tilde{G}_{i+j}$ is $G_{i+j}$ by construction, we deduce that $G_i <_\bullet G_{i+j}$, a contradiction that proves (25).

Next, we claim that

we may assume that $M_i = A_i$ for every $i$. (26)

To prove this, we extend each $G_i$ into a supergraph $G'_i$ as follows. For each unmarked $x \in A_i$, add two disjoint copies of $G_i$ to $G_i$, and join $x$ to each of its two copies by an edge (Figure 3, middle). Having done so for each $x \in A_i \backslash M_i$ we have obtained $G'_i$. Note that each block of $G'_i$ is isomorphic to $G_i$ via an isomorphism that preserves the marking. To prove (26) it suffices to check that $G'_{j-1} \gneq G'_j$ still holds for every $j > 1$.

Let us first check $G'_j <_\bullet G'_{j-1}$. Pick a minor embedding $f : G_j < G_{j-1}$. Using (24) we can extend $f$ into a minor embedding of $G'_j$ in $G'_{j-1}$ by embedding each copy of $G_j$ attached to $x$ to the copies of $G'_{j-1}$ attached to the (unique) vertex $x' \in A_{j-1} \backslash M_{j-1}$ contained in $f(x)$, by imitating $f$ inside these copies. This proves $G'_j <_\bullet G'_{j-1}$.

To check $G'_j \not<_\bullet G'_{j+1}$, suppose to the contrary there is a minor embedding $g : G'_j <_\bullet G'_{j+1}$. Recall that, by Observation 2.11, each $g(x), x \in A(G'_j)$ contains a distinct $x' \in A(G'_{j+1})$. Since $|A_i|$ and $|A_i \cap M_i|$ are constant, it follows that $|A(G'_i)|$ is constant too, hence $|A(G'_j)| = |A(G'_{j+1})|$. Thus there is a bijection $x \mapsto x'$ from $A(G'_j)$ to $A(G'_{j+1})$.

Recall we are assuming that $A_j$ is 2-connected, and so $g$ maps $A_j \subset A(G'_j)$ so that each branch set intersects a fixed block $B$ of $G'_{j+1}$ by Lemma 2.5. We claim that this block $B$ must be $G_{j+1}$ (rather than one of its copies in the construction of $G'_{j+1}$). Suppose to the contrary, there is $x \in A_j$ such that $x' \notin A_{j+1}$. Thus $x'$ is a copy of an unmarked vertex of $A_{j+1}$ (Figure 3, right). Note that the branch set $g(x)$ of $x$ cannot contain the unique neighbour $\tilde{x}$ of $x'$ in $A_{j+1}$, because $x \mapsto x'$ and so $\tilde{x}$ is in the branch set of some other vertex of $A(G'_j)$. Thus $g(x)$ avoids $A_{j+1}$. Since at most one of the two copies of $G_j$ adjacent to $x$ can contain the edge $x'\tilde{x}$, it follows that $g$ maps at least one of these copies $X$ inside the copy $X'$ of $G_{j+1}$ containing $x'$. But this copy avoids $x'$ which is already used by $g(x)$, and we therefore reach a contradiction as we do not have enough vertices in $X' \cap A(G_{j+1})$ to accommodate $X \cap A(G_j)$.

Thus $g$ maps $A_j$ so that each branch set intersects $G_{j+1}$. Since $G_j \supset A_j$ is 2-connected, each branch set of a vertex of $G_j$ intersects $G_{j+1}$ by the first sentence of Lemma 2.5. By the second sentence, there is a model of $G_j$ in $G_{j+1}$ respecting the marking, a contradiction that proves (26).

Recall that, by Observation 2.11, any unmarked minor model $f : G_i < G_j$ is a marked minor model of $(G_i, A(G_i))$ in $(G_j, A(G_j))$. Thus (26) implies that $G_1 \gneq G_2 \gneq G_3 \gneq \dots$ is also an unmarked descending chain in $\mathrm{Rank}_\alpha$. □

**Remark 8.0.1.** In this proof we only used the assumption that the family $(G_n)_{n\in\mathbb{N}}$ is a descending chain in order to make each of $|A(G_n)|, |M_n|$ and $\mathrm{Rank}(G_n)$ independent of $n$. Using this, we then produced a modified family $(G'_n)_{n\in\mathbb{N}}$ of unmarked graphs such that $G_i <_\bullet G_j$ if and only if $G'_i < G'_j$ for every $i, j$. This has Corollary 8.2 as an important consequence:

*Proof of Corollary 8.2.* Easily, $|\mathrm{Rank}_0|_< = |\mathrm{Rank}^\bullet_0|_{<_\bullet} = \aleph_0$ since there are countably many (marked) finite graphs, so assume $\alpha \geqslant 1$ from now on.

The inequality $|\mathrm{Rank}_\alpha|_< \leqslant |\mathrm{Rank}^\bullet_\alpha|_{<_\bullet}$ is trivial since each $<$-equivalence class of $\mathrm{Rank}_\alpha$ is contained in a distinct $<_\bullet$-equivalence class of $\mathrm{Rank}^\bullet_\alpha$.

For the converse inequality, let $((G_i, M_i))_{i\in\mathcal{I}}$ be a family of marked graphs, one from each $<_\bullet$-equivalence class of $\mathrm{Rank}^\bullet_\alpha$. If $\mathcal{I}$ is countable then we are done since $|\mathrm{Rank}_\alpha|_<$ is infinite. If it is uncountable, then there is an uncountable subfamily $((G_i, M_i))_{i\in\mathcal{J}\subseteq\mathcal{I}}$ within which each of $|A(G_i)|, |M_i|$ and $\mathrm{Rank}(G_i)$ is constant, because there are countably many choices for each of these numbers. Using Remark 8.0.1 we can transform this subfamily into a family $(G'_i)_{i\in\mathcal{J}}$ of unmarked graphs of the same rank no two of which are minor twins. This completes the proof since $|\mathcal{J}| = |\mathcal{I}| = |\mathrm{Rank}^\bullet_\alpha|_{<_\bullet}$. □

**Remark 8.0.2.** We cannot adapt Lemma 8.1 to antichains, because we cannot keep $|A_i(G)|$ constant.

# 9 Equivalences for a fixed rank

The following is the main technical ingredient of the proof of Theorem 1.3; it strengthens it by providing additional equivalent conditions, and refines it by layering graphs with respect to their rank. Let $\mathrm{Rank}^{n\bullet}_{<\alpha} := \{(G, M) \in \mathrm{Rank}^\bullet_{<\alpha} \mid |M| \leqslant n\}$. (We do not require $M \subseteq A(G)$ here.)

**Theorem 9.1.** *The following are equivalent for every ordinal $1 \leqslant \alpha < \omega_1$:*

*(A)* $\mathrm{Rank}^{n\bullet}_{<\alpha}$ *is well-quasi-ordered for every $n \in \mathbb{N}$;*

*(B)* $|\mathrm{Rank}_\alpha|_<$ *is countable;*

*(C)* $|\mathrm{Rank}^\bullet_\alpha|_{<_\bullet}$ *is countable;*

*(D)* $|\mathrm{Rank}_\alpha|_< < 2^{\aleph_0}$*;*

(E) $\mathrm{Rank}^{\bullet}_{\alpha}$ *has no descending chain;*

(F) $\mathrm{Rank}_{\alpha}$ *has no descending chain;*

(G) $\mathrm{Rank}^{n\bullet}_{<\alpha}$ *has no antichain for every* $n \in \mathbb{N}$*;*

The proof of Theorem 9.1 is involved, and it is not easy to break its statement up into individual equivalences: not only we prove some of the equivalences by cycling through several items, but also to prove some of the implications we perform an induction on $\alpha$ that requires other implications.

Before proving Theorem 9.1, we introduce a way to represent a subclass of $\mathrm{Rank}^{n\bullet}_{<\alpha}$ by a single, unmarked, graph in $\mathrm{Rank}_{\alpha}$:

**Definition 9.2.** *Given a marked-minor-closed class* $\mathcal{C} \subseteq \mathrm{Rank}^{n\bullet}_{<\alpha}$*, we define a (unmarked) graph* $G_{\mathcal{C}} = G^n_{\mathcal{C}}$ *as follows.*

*(i) for every marked-minor-twin class* $\mathcal{H}$ *of elements of* $\mathcal{C}$*, we pick a representative* $(H, M) \in \mathcal{H}$*, and put* $\omega$ *pairwise disjoint copies of* $H$ *into* $G_{\mathcal{C}}$*;*

*(ii) we add a set* $A_{\mathcal{C}}$ *of* $n$ *isolated vertices to* $G_{\mathcal{C}}$*; and*

*(iii) for each copy* $H_i$ *of* $(H, M)$ *as in (i), and each of the (at most* $n$*) marked vertices* $v$ *of* $H_i$*, we add an edge from* $v$ *to a distinct element of* $A_{\mathcal{C}}$*.*

Note that $E(H_i, A_{\mathcal{C}})$ is a complete matching of the set $M(H_i)$ of marked vertices of $H_i$ into $A_{\mathcal{C}}$. But $M(H_i)$ is empty for some $H$, and so $G_{\mathcal{C}}$ is disconnected. We perform step (iii) in such a way that

*(iv) for each possible matching* $m$ *of* $M(H)$ *into* $A_{\mathcal{C}}$*, there are infinitely many indices* $i$ *such that* $E(H_i, A_{\mathcal{C}})$ *coincides with* $m$*.*

Importantly, we have $\mathrm{Rank}(G_{\mathcal{C}}) \geqslant \mathrm{Rank}(\mathcal{C})$ and

$$A(G_{\mathcal{C}}) = A_{\mathcal{C}} \tag{27}$$

by construction. Moreover,

$$\mathrm{Rank}(G_{\mathcal{C}}) \leqslant \alpha, \tag{28}$$

because each component of $G_{\mathcal{C}} \backslash A_{\mathcal{C}}$ belongs to $\mathcal{C}$ and hence to $\mathrm{Rank}^{\bullet}_{<\alpha}$.

The following lemma shows that, under natural conditions, $G_{\mathcal{C}}$ is 'monotone' in $\mathcal{C}$.

**Definition 9.3.** *We say that* $\mathcal{C}$ *is* addable up to rank $\alpha$*, if whenever* $(G_n)_{n \in \mathbb{N}}$ *is a sequence of graphs in* $\mathcal{C}$*, their disjoint union* $G := \dot{\bigcup}_n G_n$ *is also in* $\mathcal{C}$ *unless* $\mathrm{Rank}(G) \geqslant \alpha$*.*

For example, suppose $\mathcal{C} = \mathrm{Rank}_{<\alpha} \cap \mathrm{Forb}(H)$ for some connected graph $H \in \mathrm{Rank}_{<\alpha}$. Then it is easy to see that $\mathcal{C}$ is addable up to rank $\alpha$, because $H < \dot{\bigcup}_n G_n$ only if $H < G_i$ for some $i$.

**Lemma 9.4.** *Let* $\mathcal{C} \subseteq \mathrm{Rank}^{n\bullet}_{<\alpha}$ *and* $\mathcal{C}' \subseteq \mathrm{Rank}^{m\bullet}_{<\alpha}$ *be marked-minor-closed classes for some* $n \leqslant m \in \mathbb{N}$*, and suppose both* $\mathcal{C}, \mathcal{C}'$ *are addable up to rank* $\alpha$*. Suppose* $\mathrm{Rank}(\mathcal{C}) = \mathrm{Rank}(\mathcal{C}') =: \beta \leqslant \alpha$*. Then* $G_{\mathcal{C}} < G_{\mathcal{C}'}$ *if and only if* $\mathcal{C} \subseteq \mathcal{C}'$*.*

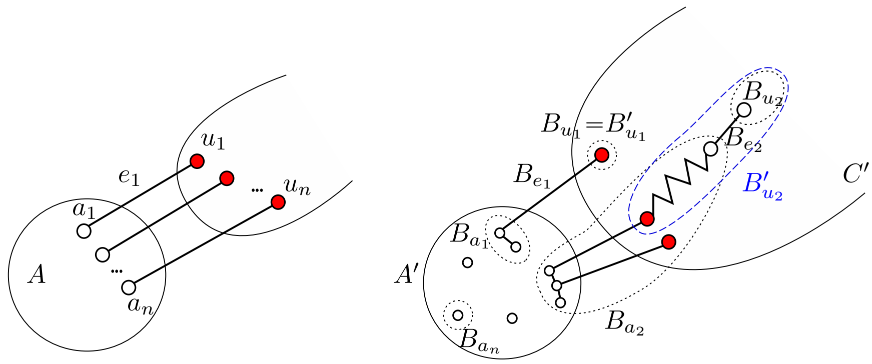


Figure 4: Defining $B'_{u_i}$ in the proof of (29). The left picture depicts a portion of $G$, while the right picture depicts its minor model in $G'$. The dotted curves enclose the original branch sets, while the dashed curve (in blue) encloses $B'_{u_2}$.

We emphasize that the graphs $G_{\mathcal{C}}, G_{\mathcal{C}'}$ are unmarked, even though the classes $\mathcal{C}, \mathcal{C}'$ consist of marked graphs.

*Proof.* For the forward implication, suppose there is a minor model $\mathcal{B} = \{B_v \mid v \in V(G)\}$ of $G \ := G_{\mathcal{C}}$ in $G' := G_{\mathcal{C}}$. By (27) we have $A := A(G) = A_{\mathcal{C}}$ and $A' := A(G') = A_{\mathcal{C}'}$.

Pick $H \in \mathcal{C}$. We will prove $H \in \mathcal{C}'$, thus establishing the forward implication. We may assume that $H$ is connected, because if each component of $H$ lies in $\mathcal{C}'$ then so does $H$ by the addability of $\mathcal{C}'$; indeed, if $H$ has infinitely many components, then the supremum of their ranks is less than $\alpha$ since $H \in \mathcal{C}$. Let $C$ be a component of $G \backslash A$ which is a marked-minor-twin of $H$, which exists by the construction of $G$ and the connectedness of $H$. Recall that $G$ contains infinitely many pairwise disjoint copies of $C$. Only finitely many of those can have a vertex the branch set of which intersects the finite set $A'$, and so we may assume that $\mathcal{B}(C)$ avoids $A'$. Thus by Observation 2.4, $\mathcal{B}(C)$ —i.e. the submodel of $\mathcal{B}$ induced by $C$— is contained in a component $C'$ of $G' \backslash A'$. To prove $H \in \mathcal{C}'$ it suffices to prove

$$C <_{\bullet} C', \tag{29}$$

since $H < C$ and $\mathcal{C}'$ is $<_{\bullet}$-closed. We will prove (29) by slightly modifying $\mathcal{B}$ and restricting it to $C$. This modification is needed to ensure that each marked vertex of $C$ is mapped to a branch set containing a marked vertex of $C'$.

Let $\{u_1, \ldots, u_n\}$ denote the marked vertices of $C$, and recall that each $u_i$ sends a unique edge $e_i$ to $A$; let $a_i \in A$ denote the other end-vertex of $e_i$ (Figure 4).

We consider the following two cases for every $u_i$:

If $B_{e_i}$ has an end-vertex $a' \in A'$, then $B_{u_i} \cap B_{e_i}$ must be the unique neighbour of $a'$ in $C'$ because $\mathcal{B}(C)$ avoids $A'$. That neighbour is a marked vertex of $C'$ by the construction of $G'$. In this case we let $B'_{u_i} := B_{u_i}$.

If not, then $B_{e_i}$ lies in $C'$. In this case, let $P_i$ be a path in $G'$ from the vertex $B_{e_i} \cap B_{a_i}$ to $A'$, which exists since $B_{a_i}$ is connected and intersects $A'$ by

Observation 2.11. Note that the last edge of $P_i$ joins a vertex $u_i' \in C'$ to a vertex of $A'$, and therefore $u_i'$ is a marked vertex of $C'$. Let $B'_{u_i} := B_{u_i} \cup B_{e_i} \cup (P_i \backslash A')$, and note that $B'_{u_i}$ is connected and contains $u_i'$.

In both cases, $B'_{u_i}$ contains a marked vertex. Easily, $B_{u_i} \cap B_{u_j} = \varnothing$ whenever $i \neq j$ as $P_i \subset B_{a_i}$ and $B_{a_i} \cap B_{a_j} = \varnothing$. Therefore, replacing each $B_{u_i}$ by $B'_{u_i}$, and leaving $B'_x := B_x$ for every other vertex $x \in V(C)$, we obtain a minor model $\mathcal{B}'$ of $C$ in $C'$. This proves (29).

For the backward implication, we assume $\mathcal{C} \subseteq \mathcal{C}'$, and construct an embedding of $G$ into $G'$ as follows. We begin by letting $B_a := a'$ for every $a \in A$, where $a \mapsto a'$ is an arbitrary but fixed bijection from $A$ to a subset of $A'$, which is possible since $|A| = n \leqslant m = |A'|$. For each 'part' $H_i$ of $G$, find a part $H'$ of $G'$ such that $H_i <_\bullet H'$ and a model of $H_i$ in $H'$ such that the marked vertices of $H_i$ are joined to $A$ the same way as the marked vertices of $H'$ are joined to $A'$ with respect to $a \mapsto a'$. Since there are infinitely many such $H'$ by item (iv) of Definition 9.2, we can choose them disjointly over the $H_i$'s. □

We are now ready to prove the main result of this section.

*Proof of Theorem 9.1.* The equivalence of (B) and (C) follows from Corollary 8.2. The equivalence of (E) and (F) is Lemma 8.1. The implications (B) → (D) and (A) → (G) are trivial.

For the other implications, we apply induction on $\alpha$. For $\alpha = 0$, we define $\mathrm{Rank}_{<0}$ to be empty, and notice that all items are obviously true.

(A) → (B): The proof of this implication is very similar to that of Theorem 6.3.

To prove that $|\mathrm{Rank}_\alpha|_<$ is countable, it suffices to prove that $|\mathrm{Rank}^n_\alpha|_<$ is countable for every $n \in \mathbb{N}$, where $\mathrm{Rank}^n_\alpha := \{G \in \mathrm{Rank}_\alpha \mid |A(G)| = n\}$. Let $G \in \mathrm{Rank}^n_\alpha$, and apply Lemma 5.1 to deduce that the minor-twin class $[G]_<$ is determined by $n = |A(G)|$ and the class $\mathcal{C}^\bullet(G)$. To be able to apply Lemma 5.1, we use the fact that $\mathcal{C}^\bullet(G) \subseteq \mathrm{Rank}^{n\bullet}_{<\alpha}$ is well-quasi-ordered by our assumption, and that $|\mathrm{Rank}^\bullet_\beta|_{<_\bullet}$ is countable for every $\beta < \alpha$ by our inductive hypothesis (A) → (C)—whereby we use the fact that $\mathrm{Rank}^{n\bullet}_\beta$ is well-quasi-ordered since its superset $\mathrm{Rank}^{n\bullet}_{<\alpha}$ is.

As $\mathcal{C}^\bullet(G)$ is well-quasi-ordered, we can express it as $\mathrm{Forb}(X) \cap \mathrm{Rank}^{n\bullet}_{<\alpha}$ for some finite $X \subset \mathrm{Rank}^{n\bullet}_{<\alpha}$. Since $|\mathrm{Rank}^{n\bullet}_\beta|_<$ is countable for every $\beta < \alpha$ as noted above, $|\mathrm{Rank}^{n\bullet}_{<\alpha}|_<$ is countable too, being the sum of $|\mathrm{Rank}^{n\bullet}_\beta|_<$ over the $\beta < \alpha$, which are countably many as $\alpha < \omega_1$. As $|\mathrm{Rank}^{n\bullet}_{<\alpha}|_<$ is countable, there are countably many ways to choose its finite subset $X$ from above, hence countably many ways to choose $\mathcal{C}^\bullet(G)$, and therefore $[G]_<$. This proves that $|\mathrm{Rank}_\alpha|_<$ is countable.

(G) → (A): It suffices to show that $\mathrm{Rank}^{n\bullet}_{<\alpha}$ has no $<_\bullet$-descending chain by Proposition 2.1, so suppose to the contrary $G_1 \gneq G_2 \gneq \ldots$ is one. Then letting $\mathrm{Rank}(G_1) =: \beta < \alpha$, and noting that $\mathrm{Rank}(G_i) \leqslant \mathrm{Rank}(G_1)$ by Observation 2.10 and $|A(G_i)| \leqslant |A(G_1)|$ by Observation 2.11, we deduce that this is a descending chain in $\mathrm{Rank}^{n\bullet}_\beta$. This contradicts the inductive hypothesis (G) → (E) for $\beta$.

(A) → (F): Suppose, for a contradiction, that $\mathrm{Rank}_\alpha$ has a descending chain $G_1 \gneq G_2 \gneq G_3 \gneq \ldots$. By Observation 2.11 we may assume that $G_i \in \mathrm{Rank}^n_\alpha$

for a fixed $n \in \mathbb{N}$. Then, by Lemma 5.1, $\mathcal{C}^\bullet(G_1) \supsetneq \mathcal{C}^\bullet(G_2) \supsetneq \mathcal{C}^\bullet(G_3) \supsetneq \dots$ is a descending chain of sub-classes of $\mathrm{Rank}^{n\bullet}_{<\alpha}$ with respect to containment. For each $k \in \mathbb{N}$, choose a marked graph $H_k \in \mathcal{C}^\bullet(G_k) \backslash \mathcal{C}^\bullet(G_{k+1})$, which is possible since $\mathcal{C}^\bullet(G_k) \supsetneq \mathcal{C}^\bullet(G_{k+1})$. Then $(H_k)$ is a bad sequence in $\mathrm{Rank}^{n\bullet}_{<\alpha}$[6]. Indeed, if $H_k < H_{k+j}$ for some $k, j > 0$, then $H_k \in \mathcal{C}^\bullet(G_{k+j}) \subseteq \mathcal{C}^\bullet(G_{k+1})$ since $\mathcal{C}^\bullet(G_{k+j})$ is marked-minor closed. This contradicts that $\mathrm{Rank}^{n\bullet}_{<\alpha}$ is well-quasi-ordered.

The following two implications are similar to the forward direction of Corollary 4.2; the reader may want to recall that proof before reading them.

(F) $\to$ (A): Suppose, to the contrary, there is a bad sequence $(H_i)$ in $\mathrm{Rank}^{n\bullet}_{<\alpha}$. We may assume without loss of generality that each $H_i$ is connected by replacing it by $S^\bullet(H_i)$ and applying Lemma 2.6 (and increasing $n$ by 1). Let $\mathcal{C}_i := \mathrm{Rank}^{n\bullet}_{<\alpha} \cap \mathrm{Forb}(H_1, \dots, H_i)$ for each $i \in \mathbb{N}$. Note that $\mathcal{C}_1 \supsetneq \mathcal{C}_2 \supsetneq \dots$ because $\mathcal{C}_i$ contains $H_{i+1}$ but $\mathcal{C}_{i+1}$ does not. Clearly, $\mathcal{C}_i$ is closed under marked minors. Since the $H_i$ are connected, each $\mathcal{C}_i$ is addable up to rank $\alpha$ as remarked after Definition 9.3. Let $G_i := G_{\mathcal{C}_i}$ be as in Definition 9.2. By our inductive hypothesis, (F) $\to$ (C) holds for every $\beta < \alpha$, and therefore $\mathcal{C}_i \subseteq \mathrm{Rank}^{n\bullet}_{<\alpha}$ consists of countably many minor-twin classes. Thus $G_i$ is countable.

We claim that

$$\mathrm{Rank}(\mathcal{C}_i) = \alpha \text{ for every } i \in \mathbb{N}. \tag{30}$$

To see this, note first that if $\mathrm{Rank}(H_n) \leqslant \beta$ holds for some $\beta < \alpha$ and infinitely many $n \in \mathbb{N}$, then this subsequence of $(H_n)_{n\in\mathbb{N}}$ is a bad sequence in $\mathrm{Rank}^{n\bullet}_{\beta}$, contradicting our inductive hypothesis since $\mathrm{Rank}_\beta$ has no descending chain. Thus $\sup_n \mathrm{Rank}(H_n) = \alpha$. Since each $\mathcal{C}_i$ contains every $H_j, j > i$, we deduce that $\mathrm{Rank}(\mathcal{C}_i) \geqslant \alpha$. The converse inequality is obvious since $\mathcal{C}_i \subset \mathrm{Rank}^{n\bullet}_{<\alpha}$.

Lemma 9.4 now implies that $G_n \gneqq G_{n+1}$ for every $i$, i.e. $(G_i)_{i\in\mathbb{N}}$ is an infinite descending $<$-chain in $\mathrm{Rank}_\alpha$, contradicting our assumption that none exists.

(D) $\to$ (G): Suppose $(H_n)_{n\in\mathbb{N}}$ is an anti-chain in $\mathrm{Rank}^{n\bullet}_{<\alpha}$. As above, we may assume that each $H_n$ is connected by Lemma 2.6. Call a subset $X$ of $\mathcal{H} := \{H_n \mid n \in \mathbb{N}\}$ *co-infinite*, if $\mathcal{H}\backslash X$ is infinite. Easily, there are $2^{\aleph_0}$ such $X$, because any subset of the even $H_n$'s is co-infinite. We will follow the lines of the previous implication to produce $2^{\aleph_0}$-many graphs $G_X$ none of which is a twin of another. Let $\mathcal{C}_X := \mathrm{Rank}^{n\bullet}_{<\alpha} \cap \mathrm{Forb}(X)$. Since each $H_n$ is connected, $\mathcal{C}_X$ is addable up to rank $\alpha$.

For each co-infinite $X \subset \mathcal{H}$, let $G_X := G_{\mathcal{C}_X}$ be as in Definition 9.2. Again, $G_X$ is countable by our inductive hypothesis (D) $\to$ (C) and the fact that $\alpha$ is countable. Similarly to (30), we claim

$$\mathrm{Rank}(\mathcal{C}_X) = \alpha \text{ for every co-infinite } X \subseteq \mathcal{H}. \tag{31}$$

Indeed, $\mathrm{Rank}(\mathcal{C}_X) \leqslant \alpha$ is obvious, and to confirm $\mathrm{Rank}(\mathcal{C}_X) \geqslant \alpha$, we observe that each of the infinitely many graphs in $\mathcal{H}\backslash X$ is a minor of $G_X$ by construction. We claim that for every $\beta < \alpha$ there is a graph $G$ in $\mathcal{H}\backslash X$ with $\mathrm{Rank}(G) \geqslant \beta$. For if not, then $\mathcal{H}\backslash X$ is an anti-chain in $\mathrm{Rank}^{n\bullet}_{<\beta}$ contradicting our inductive hypothesis. (This argument is the reason why we are working with co-infinite sets $X$.)

[6]This idea also appears in [4, LEMMA 6].

Thus by Lemma 9.4, $G_X, G_Y$ are never minor-twins for co-infinite $X \neq Y \subset \mathcal{H}$, because $\mathcal{C}_X \neq \mathcal{C}_Y$ as $\mathcal{H}$ is an anti-chain. Thus we have obtained $2^{\aleph_0}$ distinct minor-twin classes, contradicting our assumption (D).

To summarize, we have obtained the implications

$$\begin{aligned}(\text{E}) \leftrightarrow (\text{F}) \leftrightarrow &(\text{A}) \leftrightarrow (\text{G}),\\ &(\text{A}) \rightarrow (\text{B}) \rightarrow (\text{D}) \rightarrow (\text{G}),\\ &(\text{C}) \leftrightarrow (\text{B}),\end{aligned}$$

combining which proves all 7 statements to be equivalent. □

We will use similar ideas to obtain two corollaries that we will also need for our main theorem in the next section:

**Corollary 9.5.** *Let $1 \leqslant \alpha < \omega_1$, and suppose*

$$\mathrm{Rank}_\alpha \textit{ has no antichain of cardinality } 2^{\aleph_0}. \tag{32}$$

*Then $\mathrm{Rank}^{n\bullet}_{<\alpha}$ is well-quasi-ordered for every $n \in \mathbb{N}$.*

*Proof.* We refine the proof of the implication (D) → (G) above to show that if $\mathrm{Rank}^{n\bullet}_{<\alpha}$ has an anti-chain $(H_n)_{n\in\mathbb{N}}$ for some $n \in \mathbb{N}$, then (32) is contradicted. Indeed, given co-infinite sets $X, Y \subset \mathcal{H} := \{H_n \mid n \in \mathbb{N}\}$ note that $G_X < G_Y$ implies $\mathcal{C}_X \subseteq \mathcal{C}_Y$, which in turn implies $Y \subseteq X$. Thus if $\{X_i\}_{i\in\mathcal{I}}$ is a family of co-infinite sets that are pairwise $\subseteq$-incomparable, then $\{G_{X_i}\}_{i\in\mathcal{I}}$ is a $<$-antichain. Easily, there is such a family with $|\mathcal{I}| = 2^{\aleph_0}$, and so (32) implies that $\mathrm{Rank}^{n\bullet}_{<\alpha}$ has no infinite anti-chain. By the equivalence of items (G), (A) of Theorem 9.1, (32) implies the stronger statement that $\mathrm{Rank}^{n\bullet}_{<\alpha}$ is well-quasi-ordered for every $n \in \mathbb{N}$. □

**Remark 9.0.1.** It is not clear whether (32) can be added as a further equivalent condition in Theorem 9.1; we have just seen that it implies (A), but the latter only implies the weaker statement that $\mathrm{Rank}_{<\alpha}$ has no infinite antichain.

Let $\mathcal{R}^\bullet$ denote the class of marked, countable, rayless graphs.

**Corollary 9.6.** *If $\mathrm{Rank}^{n\bullet}_{<\gamma}$ is well-quasi-ordered for every ordinal $1 \leqslant \gamma < \omega_1$ and every $n \in \mathbb{N}$, then $\mathcal{R}^\bullet$ is well-quasi-ordered.*

*Proof.* The proof is similar to the implication (F) → (A) above, but we will have to increase the rank by one.

Suppose, to the contrary, there is a bad sequence $(H_n)_{n\in\mathbb{N}}$ in $\mathcal{R}^\bullet$. As $\omega_1$ is a regular ordinal, and this sequence is countable, we deduce that there is $\alpha < \omega_1$ such that $H_n \in \mathrm{Rank}^\bullet_{<\alpha}$ for every $n$. As usual, we may assume that each $H_i$ is connected by replacing each $H_i$ by $S^\bullet(H_i)$ and applying Lemma 2.6.

Let $\mathcal{C}_i := \mathrm{Rank}^{n_i\bullet}_{<\alpha} \cap \mathrm{Forb}(H_1, \ldots, H_i)$ for each $i \in \mathbb{N}$, where $n_i$ is the maximum number of marked vertices in $\{H_1, \ldots, H_i\}$. We can no longer claim that $\mathcal{C}_1 \supsetneq \mathcal{C}_2 \supsetneq \ldots$, because graphs in $\mathcal{C}_j$ can contain more marked vertices than those in $\mathcal{C}_{j-1}$. To amend this, we introduce $\mathcal{C}^k_i := \mathcal{C}_i \cap \mathrm{Rank}^{k\bullet}_{<\alpha}$ for every $k \in \mathbb{N}$. We now have $\mathcal{C}^k_1 \supseteq \mathcal{C}^k_2 \supseteq \ldots$, and $\mathcal{C}^{i+\ell}_i \supsetneq \mathcal{C}^{i+\ell}_{i+1}$ for every fixed $i$ and large enough $\ell$, because $\mathcal{C}^{i+\ell}_i$ contains $H_{i+1}$ but $\mathcal{C}^{i+\ell}_{i+1}$ does not.

Let $G_{\mathcal{C}_i^k}$ be as in Definition 9.2. By Theorem 9.1 (A) → (C) $\mathcal{C}_i^k \subseteq \mathrm{Rank}^{\bullet}_{<\alpha}$ consists of countably many minor-twin classes, and so $G_{\mathcal{C}_i^k}$ is countable. Similarly to (30), we will prove that $\mathrm{Rank}(\mathcal{C}_i^k) = \alpha$ for every $i, k \in \mathbb{N}$. Indeed, if $\mathrm{Rank}(H_n) \leqslant \beta$ holds for some $\beta < \alpha$ and infinitely many $n \in \mathbb{N}$, then this contradicts our inductive hypothesis. Thus $\sup_n \mathrm{Rank}(H_n) = \alpha$. Since each $\mathcal{C}_i^k, i \in \mathbb{N}$ contains every $H_j, j > i$ as an unmarked graph, we deduce that $\mathrm{Rank}(\mathcal{C}_i^k) \geqslant \alpha$. The converse inequality is obvious since $\mathcal{C}_i^k \subset \mathrm{Rank}_{<\alpha}$.

Let $G_n := \dot{\bigcup}_{k \in \mathbb{N}} G_{\mathcal{C}_n^k}$ for every $n \in \mathbb{N}$, and note that $\mathrm{Rank}(G_n) \leqslant \alpha + 1$. We claim that $(G_n)_{n \in \mathbb{N}}$ is a $<$-descending chain. Easily, we have $G_n > G_{n+1}$ by applying the backward direction of Lemma 9.4 componentwise, since $\mathcal{C}_n^k \supseteq \mathcal{C}_{n+1}^k$. One the other hand, if $G_n < G_{n+1}$ holds for some $n$, then for every $k$ there is $k'$ such that $G_{\mathcal{C}_n^k} < G_{\mathcal{C}_{n+1}^{k'}}$. But there is $k$ large enough that $H_{n+1} \in \mathcal{C}_n^k$, while $H_{n+1} \notin \mathcal{C}_{n+1}^{k'}$ for every $k'$. This contradicts the forward direction of Lemma 9.4. Thus $(G_n)_{n \in \mathbb{N}}$ is a $<$-descending chain in $\mathrm{Rank}_{\alpha+1}$, which contradicts Theorem 9.1 (A) → (F).

□

# 10 Concluding the proof of Theorem 1.3

We can now complete the proof of our main Theorem 1.3 in a more detailed version. Recall that $\mathcal{R}$ denotes the class of countable rayless graphs, and $\mathcal{R}^{\bullet}$ the marked countable rayless graphs. Let $\mathcal{R}^{n\bullet}$ denote the class of countable rayless graphs with at most $n$ marked vertices.

**Theorem 10.1.** *The following statements are equivalent:*

*(a)* $\mathcal{R}$ *is well-quasi-ordered;*

*(b)* $\mathcal{R}^{\bullet}$ *is well-quasi-ordered;*

*(c)* $\mathcal{R}^{n\bullet}$ *is well-quasi-ordered for every* $n \in \mathbb{N}$*;*

*(d)* $\mathrm{Rank}^{n\bullet}_{<\alpha}$ *is well-quasi-ordered for every* $n \in \mathbb{N}$ *and* $\alpha < \omega_1$*;*

*(e)* $\mathcal{R}$ *has no descending chain;*

*(f)* $\mathrm{Rank}_{\alpha}$ *has no descending chain for every* $\alpha < \omega_1$*;*

*(g)* $\mathcal{R}^{\bullet}$ *has no descending chain;*

*(h)* $\mathcal{R}$ *has no infinite antichain;*

*(i)* $\mathcal{R}$ *has no antichain of cardinality* $2^{\aleph_0}$*;*

*(j)* $|\mathrm{Rank}_{\alpha}|_{<}$ *is countable for every* $\alpha < \omega_1$*;*

*(k)* $|\mathrm{Rank}^{\bullet}_{\alpha}|_{<_{\bullet}}$ *is countable for every* $\alpha < \omega_1$*;*

*(l)* $|\mathrm{Rank}_{\alpha}|_{<} < 2^{\aleph_0}$ *for every* $\alpha < \omega_1$*.*

*Proof.* We trivially have (b)→ (c) → (d), and the implication (d) → (b) is Corollary 9.6. Thus items (b), (c), (d) are equivalent.

We trivially have (b)→ (a) → (e). By the implication (F) → (A) of Theorem 9.1, (e) implies (d). Thus (a), (e) are also equivalent to the above items.

We have (f)→ (e) since rank is monotone decreasing in any descending chain by Observation 2.10. Thus (f) is also equivalent to the above items.

The equivalence between (e) and (g) follows easily from Lemma 8.1, by noting again that in any descending chain the rank is monotone decreasing.

We trivially have (a) → (h) → (i). Corollary 9.5 provides the implication (i) → (d), thus adding (h), (i) to be above equivalences.

The equivalences between items (j), (k), (l) and (d) follow by applying Theorem 9.1 to every $\alpha < \omega_1$. □

# 11 $\mathcal{F}$ is WQO impies $\mathcal{R}$ is WQO, and even BQO

Let $\mathcal{F}$ denote the class of finite graphs (ordered by $<$). This section completes the proof of Theorem 1.1 (started with the implication (ii) → (i) in Section 3) by providing the implication (i) → (iii):

**Theorem 11.1.** *If $\mathcal{F}$ is BQO then so is $\mathcal{R}$.*

Given two quasi-orders $(Q, \leqslant)$ and $(R, \preceq)$, a map $f : Q \to R$ is called an *immersion*, if $f(p) \preceq f(q)$ implies $p \leqslant q$ for every $p, q \in Q$. It is easy to see that immersions preserve WQOs:

**Observation 11.2.** *Suppose $f : R \to Q$ is an immersion between quasi-orders. If $Q$ is a WQO, then so is $R$.* □

Given a quasi-order $(Q, \leqslant)$, we obtain a quasi-order $(\mathcal{P}^*(Q), \leqslant_*)$ by restricting Definition 2.13, i.e. by letting $X \leqslant_* Y$ for every $X, Y \in \mathcal{P}^*(Q)$ such that for every $X' \in X$ there exists $Y' \in Y$ with $X' \leqslant Y'$. Any map $m : (R, \preceq) \to (Q, \leqslant)$ between quasi-orders naturally extends to power sets:

**Definition 11.3.** *We define $m^* : (\mathcal{P}^*(R), \preceq_*) \to (\mathcal{P}^*(Q), \leqslant_*)$ by $X \mapsto \{m(X') \mid X' \in X\}$.*

It is straightforward to check from the definitions that this operation preserves immersions too:

**Observation 11.4.** *If $m : (R, \preceq) \to (Q, \leqslant)$ is an immersion between quasi-orders, then so is $m^* : (\mathcal{P}^*(R), \preceq_*) \to (\mathcal{P}^*(Q), \leqslant_*)$.* □

Our approach for the proof of Theorem 11.1 can be summarized as follows. We will immerse $\mathcal{R}$ into $H^*_{\omega_1}(\mathcal{F})$, which is WQO by Theorem 2.14 and our assumption that $\mathcal{F}$ is BQO. This implies that $\mathcal{R}$ is well-quasi-ordered by Observation 11.2.

This immersion starts by defining $f : \mathrm{Rank}_\alpha \to \mathcal{P}^*(\mathrm{Rank}^\bullet_{<\alpha})$ by $G \mapsto \mathcal{C}^\bullet(G)$; we will use Lemma 5.1 to deduce that $f$ is an immersion. Note that $f$ maps rayless graphs to sets of marked rayless graphs of smaller ranks. By recursively composing $f$ with the set operation of Observation 11.4, and noting that compositions of immersions are immersions, we immerse $\mathrm{Rank}_\alpha$ into $H^*_{\omega_1}(\mathcal{F})$. But this proof sketch is incomplete for two reasons.

Firstly, Lemma 5.1 requires $\mathrm{Rank}(H) = \mathrm{Rank}(G)$, and this equality needs to be relaxed in order for $f$ to immerse all of $\mathrm{Rank}_\alpha$; we do this with Lemma 11.6 below. An interesting aspect is that to be able to satisfy the conditions of Lemma 5.1 we will need to apply Theorem 9.1.

Secondly, the image of $f$ is a set of *m*arked graphs, and so we cannot compose it with $f^*$ applied to smaller ranks. We will amend this with an unmarking trick in Section 11.2.

Having constructed this immersion $m : \mathcal{R} \to H^*_{\omega_1}(\mathcal{F})$, it will then be easy to deduce that $\mathcal{R}$ is better-quasi-ordered by combining Observation 11.2 with Observation 11.5 below. To state it, let us extend Definition 11.3 to define a map $m^+ : (H^*_{\omega_1}(Q), \leqslant_+) \to (H^*_{\omega_1}(R), \preceq_+)$, where $\leqslant_+, \preceq_+$ are as in Definition 2.13: we apply $X \mapsto \{m(X') \mid X' \in X\}$ to each $X \in V^*_\alpha(Q)$ by transfinite induction on $\alpha$. Analogously to Observation 11.4, we now have the following, which is straightforward to prove by induction on the $R$-Rank:

**Observation 11.5.** *If $m : (R, \preceq) \to (Q, \leqslant)$ is an immersion between quasi-orders, then so is $m^+ : (H^*_{\omega_1}(R), \preceq_+) \to (H^*_{\omega_1}(Q), \leqslant_+)$.* □

## 11.1 Extending Lemma 5.1

We will need the following variant of the backward direction of Lemma 5.1, obtained be removing the condition $\mathrm{Rank}(H) = \mathrm{Rank}(G)$, and slightly strengthening the other conditions:

**Lemma 11.6.** *Let $G, H$ be countable graphs with $\mathrm{Rank}(H), \mathrm{Rank}(G) \leqslant \alpha < \omega_1$. Assume $\mathrm{Rank}^\bullet_{<\alpha}$ is well-quasi-ordered, and $|\mathrm{Rank}^\bullet_\beta|_{<_\bullet}$ is countable for every $\beta < \alpha$. If $\mathcal{C}^\bullet(G) \subseteq \mathcal{C}^\bullet(H)$, then $G < H$.*

*Proof.* By Proposition 2.9 and Observation 2.10, $\mathcal{C}^\bullet(G) \subseteq \mathcal{C}^\bullet(H)$ easily implies $\mathrm{Rank}(G) \leqslant \mathrm{Rank}(H)$. If $\mathrm{Rank}(G) = \mathrm{Rank}(H)$, then Lemma 5.1 applies.

We may assume that $G, H$ are 2-connected by applying Lemma 2.7.

If $\mathrm{Rank}(G) < \mathrm{Rank}(H) =: \beta$, we modify $G, H$ into supergraphs $G', H'$ with $\mathrm{Rank}(G') = \mathrm{Rank}(H')$ as follows. For each $a \in A(G) \cup A(H)$, attach the root $r_\alpha$ of a copy of the minimal tree $T_\alpha$ of rank $\alpha$ as provided by Observation 2.12. Easily, $\mathrm{Rank}(G') = \mathrm{Rank}(H')$, and $A(G') = A(G)$ and $A(H') = A(H)$. Moreover, we have $\mathcal{C}^\bullet(G') \subseteq \mathcal{C}^\bullet(H')$, because for every $(K, M) \in \mathcal{C}^\bullet(G')$, we can extend any model of $(K \cap G, M)$ in $(H, A(H))$ into a model of $(K, M)$ in $(H', A(H))$ by mapping the $T_\alpha$'s attached to $M \subseteq A(G)$ to those attached to $A(H)$.

Thus Lemma 5.1 applies to $G', H'$, and yields a model of $G'$ in $H'$. By Lemma 2.5, this model yields a model of $G$ in $H$ since the former is 2-connected. □

## 11.2 The unmarking trick

We define a map $u : \mathrm{Rank}^\bullet_\alpha \to \mathcal{P}^*(\mathrm{Rank}_\alpha)$, i.e. mapping each marked graph to a set of unmarked graphs of the same rank.

Given $(G, M) \in \mathcal{R}^\bullet$ with $\mu := |M| \geqslant 2$, and $n \in \mathbb{N}$, we define the *depth-$n$ self-amalgamation $u_n(G)$ of $G$* as follows. We enumerate the elements of $M$ as $m_1, \ldots, m_\mu$. We let $T_n$ be the finite rooted tree of depth $n$ each non-leaf vertex of which (including the root $r$) has degree $\mu$. For each vertex $v$ of $T_n$, we let $G^v$ be a copy of $(G, M)$, disjoint from all other copies. We label each of the $\mu$ edges of $r$ by a distinct number in $[\mu]$. We then label the remaining edges of $T_n$ so that for each non-leaf vertex $x$, the labels of the edges of $x$ are in bijection with $[\mu]$; this is easy to achieve via a breadth-first process.

We let $G^{T_n} = (G, M)^{T_n}$ be the (unmarked) graph obtained from $\dot{\bigcup}_{x \in V(T_n)} G^x$ by identifying, for each edge $xy$ of $T_n$ with label $i \in [\mu]$, the copies of $m_i$ in $G^x$ and $G^y$ into a single vertex. By construction, if $G$ is 2-connected, then

$$\text{Each block of } G^{T_n} \text{ is isomorphic to } G. \tag{33}$$

Finally, let $S_G := S^\bullet(S^\bullet((G,M)))$, and let $u((G,M)) := \{S_G^{T_n} \mid n \in \mathbb{N}\}$, which we consider as an element of $\mathcal{P}^*(\mathrm{Rank}_\alpha)$ where $\alpha := \mathrm{Rank}(G) = \mathrm{Rank}(S_G)$. The reason we applied the self-amalgamation to $S_G$ instead of $G$ is that this ensures our above requirement $\mu \geqslant 2$, so that $u((G,M))$ is well-defined for every $G$. It also ensures that $S_G$ is 2-connected, and so (33) holds.

**Lemma 11.7.** *For every $0 < \alpha < \omega_1$, $u$ is an immersion of $\mathrm{Rank}^\bullet_{<\alpha}$ to $\mathcal{P}^*(\mathrm{Rank}_{<\alpha})$.*

*Proof.* Let $(G,M),(H,M')$ be two marked graphs as in the statement. We have to show that if $u((G,M)) \leqslant u((H,M'))$ then $(G,M) <_\bullet (H,M')$. So suppose the former is the case, which means that

$$\text{for every } n \in \mathbb{N} \text{ there is } k \in \mathbb{N} \text{ such that } S_G^{T_n} < S_H^{T_k}. \tag{34}$$

This implies in particular that $\mathrm{Rank}(G) \leqslant \mathrm{Rank}(H)$ by Observation 2.10, since $\mathrm{Rank}(S_G^{T_n}) = \mathrm{Rank}(S_G) = \mathrm{Rank}(G)$ by construction.

For a graph $R \in \mathcal{R}$, we let $||R|| := |A(R)|$ if $\mathrm{Rank}(R) > 0$, and let $||R|| := |V(R)|$ if $\mathrm{Rank}(R) = 0$ (i.e. $R$ is finite). Note that

$$\text{If } \mathrm{Rank}(R) = \mathrm{Rank}(R') \text{ and } R < R', \text{ then } ||R|| \leqslant ||R'|| \tag{35}$$

by Observation 2.11.

Fix some $n > ||S_H||$, and let $f : S_G^{T_n} < S_H^{T_k}$ be a minor embedding as provided by (34). Let $G' := S_G^{T_n}$ and $H' := S_H^{T_k}$. Recall that $S_G^r$ denotes the copy of $S_G$ corresponding to the root of $T_n$ in the construction of $S_G^{T_n}$. By Lemma 2.5, there is a block $B$ of $S_H^{T_k}$ such that each branch set of $S_G^r$ intersects $B$. By (33), $B$ is isomorphic to $S_H$. For $v \in V(S_G^r)$, let $K_v$ denote the union of components of $G' - v$ other than $S_G^r - v$, and let $\overline{K_v} := S_{G'}[\{v\} \cup V(K_v)]$. Note that $||\overline{K_v}|| \geqslant n$, because $||S_G|| \geqslant 2$ by assumption, and therefore each non-root layer of $T_n$ contributes at least one to each component of $K_v$.

For a marked vertex $v$ of $S_G^r$, let $f'(v) := \bigcup_{w \in V(\overline{K_v})} f(w)$. If $v$ is unmarked we just let $f'(v) := f(v)$. Note that $f'(v)$ spans a connected subgraph of $H'$ because $\overline{K_v}$ is connected. Thus we can think of $f'$ as a minor embedding of $S_G^r$ into $H'$. We claim that

$$\text{for each marked vertex } v \text{ of } S_G^r, \text{ the set } f'(v) \text{ contains a marked vertex of } B. \tag{36}$$

This claim implies $S_G <_\bullet S_H$, because by the second sentence of Lemma 2.5 we can restrict $f' : S_G^r \to H'$ into a marked-minor embedding of $S_G^r$ into $B$.

To prove (36), suppose to the contrary that $f'(v)$ fails to contain a marked vertex of $B$. Since the marked vertices of $B$ separate it from the rest of $H'$, we deduce that $f'(v) = f(\overline{K_v}) \subseteq B$, and therefore $\overline{K_v} < S_H$ since $B \cong S_H$. However, we have $||\overline{K_v}|| \geqslant n > ||H||$, which contradicts (35).

Thus we have proved $S_G <_\bullet S_H$. Applying Lemma 2.6 twice we deduce the desired $(G,M) <_\bullet (H,M')$. □

## 11.3 Completing the proof of Theorem 11.1

We have now gathered all ingredients to complete the proof we sketched at the beginning of this section:

*Proof of Theorem 11.1.* For every ordinal $\alpha < \omega_1$ and every $n \in \mathbb{N}$ we will define immersions $m_\alpha : \mathrm{Rank}_\alpha \to H^*_{\omega_1}(\mathcal{F})$ and $m^\bullet_\alpha : \mathrm{Rank}^\bullet_\alpha \to H^*_{\omega_1}(\mathcal{F})$ by transfinite induction on $\alpha$. It is $m_\alpha$ that we really care about, but we need $m^\bullet_\alpha$ to be able to perform our inductive step. More precisely, our inductive hypothesis is that such immersions exist, and that both families $\{m_\alpha\}_{\alpha<\omega_1}$ and $\{m^\bullet_\alpha\}_{\alpha<\omega_1}$ are *compatible*, i.e. for every $G \in \mathrm{Rank}_\delta$, and ordinals $\beta > \gamma \geqslant \delta$, we have $m_\beta(G) = m_\gamma(G)$ and $m^\bullet_\beta((G, M)) = m^\bullet_\gamma((G, M))$ for every finite $M \subset V(G)$.

To start the induction, for $G \in \mathrm{Rank}_0$ (i.e. $G \in \mathcal{F}$) we just let $m_0(G) = G$. (We will define $m^\bullet_0$ later.)

For the induction step, for any ordinal $0 < \alpha < \omega_1$, we assume that immersions $m_\beta : \mathrm{Rank}_\beta \to H^*_{\omega_1}(\mathcal{F})$ and $m^\bullet_\beta : \mathrm{Rank}^\bullet_\beta \to H^*_{\omega_1}(\mathcal{F})$ have been defined in all previous steps $\beta < \alpha$. We then define $m_\alpha : \mathrm{Rank}_\alpha \to H^*_{\omega_1}(\mathcal{F})$ by a composition of immersions

$$\mathrm{Rank}_\alpha \xrightarrow{\mathcal{C}^\bullet} \mathcal{P}^*(\mathrm{Rank}^\bullet_{<\alpha}) \xrightarrow{u^*} \mathcal{P}^*(\mathcal{P}^*(\mathrm{Rank}_{<\alpha})) \xrightarrow{m^{**}_{<\alpha}} H^*_{\omega_1}(\mathcal{F}) \tag{37}$$

that we will now define. We let $u$ be the immersion of Lemma 11.7, and $u^*$ its extension as in Observation 11.4. Thus $u^*$ is an immersion by combining these two results. Similarly, $m^{**}_{<\alpha}$ is obtained by applying Definition 11.3 twice to $m_{<\alpha}$, defined as $m_{<\alpha} := \bigcup_{\beta<\alpha} m_\beta$. Here the $m_\beta$ have been defined in previous steps of our induction, and their limit $m_{<\alpha}$ is well-defined by our compatibility assumption. Note that $m_{<\alpha}$ is an immersion since each $m_\beta$ is, and being an immersion is a 'local' property. (If $\alpha = \beta + 1$ is a successor ordinal, then we can replace $m_{<\alpha}$ by $m_\beta$.) Applying Observation 11.4 twice to $m_{<\alpha}$ we deduce that $m^{**}_{<\alpha}$ is an immersion too. (The image of $m^{**}_{<\alpha}$ is not all of $H^*_{\omega_1}(\mathcal{F})$, but a subset of $V^*_\gamma(\mathcal{F})$ for some ordinal $\gamma < \omega_1$.) Finally, we let

$$m_\alpha(G) := m^{**}_{<\alpha}(u^*(\mathcal{C}^\bullet(G))). \tag{38}$$

To prove that $m_\alpha$ is an immersion, it suffices to check that each of the three maps it is composed of is an immersion, since compositions of immersions are immersions. We have already checked this for $m^{**}_{<\alpha}$ and $u^*$, and so it remains to check that $\mathcal{C}^\bullet$ is an immersion.

Lemma 11.6 says exactly that $\mathcal{C}^\bullet$ is an immersion of $\mathrm{Rank}_\alpha$ into $\mathcal{P}^*(\mathrm{Rank}^\bullet_{<\alpha})$, assuming its conditions are satisfied, namely a) $\mathrm{Rank}^\bullet_{<\alpha}$ is well-quasi-ordered, and b) $|\mathrm{Rank}^\bullet_\beta|_{<_\bullet}$ is countable for every $\beta < \alpha$. We will prove that they are satisfied using our induction hypothesis. To confirm a), recall that we have a compatible family of immersions $m^\bullet_\beta : \mathrm{Rank}^\bullet_\beta \to H^*_{\omega_1}(\mathcal{F}), \beta < \alpha$. As above, compatibility implies that the limit $m^\bullet_{<\alpha} := \bigcup_{\beta<\alpha} m^\bullet_\beta$ is well-defined, and it is an immersion of $\mathrm{Rank}^\bullet_{<\alpha}$ in $H^*_{\omega_1}(\mathcal{F})$. Thus $\mathrm{Rank}^\bullet_{<\alpha}$ is well-quasi-ordered by Observation 11.2, i.e. a) holds. To confirm b), we plug a) into the implication (A) $\to$ (C) of Theorem 9.1 and obtain the stronger $|\mathrm{Rank}^\bullet_\alpha|_{<_\bullet} = \aleph_0$. This completes the proof that $\mathcal{C}^\bullet$ is an immersion of $\mathrm{Rank}_\alpha$ into $\mathcal{P}^*(\mathrm{Rank}^\bullet_{<\alpha})$, and therefore that $m_\alpha$ is an immersion.

It remains to ensure that the image of $m_\alpha$ is contained in $H^*_{\omega_1}(\mathcal{F})$, i.e. it consists of hereditarily countable sets. Both $m^{**}_{<\alpha}$ and $u^*$ preserve countability by definition, but we need to be careful with $\mathcal{C}^\bullet$. The easiest way to handle this issue is to choose a representative from each marked-minor-twin class of $\mathrm{Rank}^\bullet_{<\alpha}$, and replace $\mathcal{C}^\bullet(G)$ by its subset $\mathcal{C}'(G)$ consisting of the representative of each marked-minor-twin class of $\mathcal{C}^\bullet(G)$. Note that Lemma 11.6 is unaffected by replacing $\mathcal{C}^\bullet(G)$ by $\mathcal{C}'(G)$. Using (b) again, we deduce that $\mathcal{C}'(G)$ is countable, and so $m_\alpha(G) \in H^*_{\omega_1}(\mathcal{F})$ as desired.

Having defined $m_\alpha$, we define $m^\bullet_\alpha$ by

$$(G, M) \mapsto m^*_\alpha(u((G, M))). \tag{39}$$

This is an immersion to $H^*_{\omega_1}(\mathcal{F})$ by Lemma 11.7 and Observation 11.4 applied to $m_\alpha$. This formula can be applied to finite $(G, M)$, and we consider this as the definition of $m^\bullet_0$.

To see that $m_\alpha$ and $m^\bullet_\alpha$ are both compatible, note that their definitions (38) and (39) are independent of the rank of $G$ (as long as the latter is at most $\alpha$), whereby for the former we use the inductive hypothesis of compatibility. This completes the proof of our inductive hypothesis.

Using compatibility as in the definition of $m_{<\alpha}$ above, we can now define an immersion $m : \mathcal{R} \to H^*_{\omega_1}(\mathcal{F})$ by $m := \bigcup_{\alpha<\omega_1} m_\alpha$, and apply Lemma 11.2 to deduce that $\mathcal{R}$ is well-quasi-ordered. (Similarly, we can deduce that $\mathcal{R}^\bullet$ is well-quasi-ordered.)

To prove that $\mathcal{R}$ is better-quasi-ordered, we plug $m$ into Observation 11.5, with $H^*_{\omega_1}(\mathcal{F})$ playing the role of $Q$, and obtain an immersion $m^+ : H^*_{\omega_1}(\mathcal{R}) \to H^*_{\omega_1}(H^*_{\omega_1}(\mathcal{F}))$. But $H^*_{\omega_1}(H^*_{\omega_1}(\mathcal{F})) = H^*_{\omega_1}(\mathcal{F})$ by Observation 2.17, and so $m^+$ immerses $H^*_{\omega_1}(\mathcal{R})$ into the well-quasi-ordered set $H^*_{\omega_1}(\mathcal{F})$. Thus $H^*_{\omega_1}(\mathcal{R})$ is well-quasi-ordered by Observation 11.2, and so $\mathcal{R}$ is better-quasi-ordered by Theorem 2.14. $\square$

The last paragraph of this proof is applicable to arbitrary quasi-orders $\mathcal{R}, \mathcal{F}$, and so we obtain

**Corollary 11.8.** *Let $\mathcal{R}, \mathcal{F}$ be quasi-orders. Suppose $\mathcal{F}$ is better-quasi-ordered, and there is an immersion $m : \mathcal{R} \to H^*_{\omega_1}(\mathcal{F})$. Then $\mathcal{R}$ is better-quasi-ordered.*

# 12 Open problems

Our main open problem, motivated by Thomas' conjecture and Theorem 1.1 is

**Problem 12.1.** *Are the (countable, or of arbitrary cardinality) rayless graphs well-quasi-ordered?*

Thomas [24] provides an example of a bad sequence of graphs of the cardinality of the continuum, which however contain plenty of rays.

I find the following special case of Problem 12.1 particularly interesting:

**Problem 12.2.** *Are the countable, planar, rayless graphs well-quasi-ordered?*

I expect that a positive answer to this would imply a positive answer to the following problem, by following the lines of the proof of Theorem 1.1 (and Corollary 3.2):

**Problem 12.3.** *Are the finite planar graphs better-quasi-ordered?*

Recall that Thomas proved that $TW(k)$ is better-quasi-ordered for every $k \in \mathbb{N}$ [25, (1.7)]. Let $TW_{<\infty} = \bigcup_{k\in\mathbb{N}} TW(k)$ be the class of countable rayless graphs of finite (but not bounded) tree-width. To appreciate the difficulty of the last two problems, the reader may try the following:

**Problem 12.4.** *Is $TW_{<\infty}$ well-quasi-ordered?*

See [6, Conjecture 10.2.] for a problem of similar flavour.

Another interesting special case of Thomas' conjecture is whether the countable Cayley graphs are well-quasi-ordered; see [8, §5] for some progress.

Our next problem is motivated by item (d) of Theorem 1.3.

**Problem 12.5.** *Is there a family $\mathcal{C}$ of countable —not necessarily rayless— graphs with $|\mathcal{C}| = 2^{\aleph_0}$ such that no two elements of $\mathcal{C}$ are minor-twins?*

A similar question can be asked for countable (rayless or not) marked graphs with no restriction on the number of marked vertices.

Our last problems are motivated by Theorem 1.4.

**Problem 12.6.** *Let $\mathcal{C} \subseteq \mathcal{G}$ be a family of $\mathbb{N}$-labelled rayless graphs which is closed under minor-twins. Is it true that $\mathcal{C}$ is well-quasi-ordered if and only if $\mathcal{C} \cap \mathrm{Rank}_\alpha$ is Borel for every $\alpha < \omega_1$?*

(The forward direction is true.)

Similarly, one can ask

**Problem 12.7.** *Let $Q$ be a family of finite graphs. Is it true that $Q$ is better-quasi-ordered if and only if $\mathrm{Rank}^1_\alpha(Q)$ is Borel for every $\alpha < \omega_1$?*

Recall that Robertson, Seymour, & Thomas [19] wrote that there is not much chance of proving Thomas' conjecture, and even our restriction to the rayless case seems out of reach. Theorem 10.1 provides new tools for attacking it, and perhaps there is now a chance of disproving it:

**Problem 12.8.** *Is there a family of rayless $\mathbb{N}$-labelled graphs which is closed under minors, has rank less than $\omega_1$, and is not Borel?*

A positive answer, combined with Theorem 7.2 and the implication (a) $\to$ (j) of Theorem 10.1, would disprove Thomas' conjecture.

# References


[1] A. Bonato, H. Bruhn, R. Diestel, and P. Sprüssel. Twins of rayless graphs. *J. Combin. Theory (Series B)*, 101:60–65, 2011.

[2] H. Bruhn, R. Diestel, A. Georgakopoulos, and P. Sprüssel. Every rayless graph has an unfriendly partition. *Combinatorica*, 30:521–532, 2010.

[3] R. Carroy. A quasi-order on continuous functions. *The Journal of Symbolic Logic*, 78(2):633–648, 2013.

[4] R. Diestel. Relating Subsets of a Poset, and a Partition Theorem for WQOs. *Order*, 18:275–279, 2001.

[5] Reinhard Diestel. *Graph Theory* (6th edition). Springer-Verlag, 2025. Electronic edition available at: `http://www.math.uni-hamburg.de/home/diestel/books/graph.theory`.

[6] A. Georgakopoulos. The excluded minors for embeddability into a compact surface. *Combinatorica*, 45(6), 2025.

[7] A. Georgakopoulos and J. Grebik. A notion of computability for classes of infinite structures. In preparation.

[8] A. Georgakopoulos and M. Hamann. A full Halin grid theorem. *Discrete & Computational Geometry*, 2024.

[9] R. Halin. The Structure of Rayless Graphs. *Abh. Math. Sem. Univ. Hamburg*, 68:225–253, 1998.

[10] Thomas Jech. *Set Theory.* Springer Monographs in Mathematics, 2003.

[11] Alexander S. Kechris. *Classical descriptive set theory. Graduate texts in mathematics, no. 156.* Springer-Verlag, 1995.

[12] D. Kühn. On well-quasi-ordering infinite trees – Nash-Williams's theorem revisited. *Math. Proc. Camb. Phil. Soc.*, 130:401–408, 2001.

[13] R. Laver. On Fraïssé's order type conjecture. *Annals of Mathematics*, 93:89–111, 1971.

[14] C. Martinez-Ranero. Well-quasi-ordering Aronszajn lines. *Fund. Math.*, 213:197–211, 2011.

[15] C.St.J.A. Nash-Williams. On well-quasi-ordering infinite trees. *Math. Proc. Camb. Phil. Soc.*, 61:697–720, 1965.

[16] B. Oporowski. A counterexample to Seymour's self-minor conjecture. *J. Graph Theory*, 14:521–524, 1990.

[17] Y. Pequignot. Towards better: A motivated introduction to better-quasi-orders. *EMS Surveys in Mathematical Sciences*, 4(2):185–218, 2017.

[18] M. Pitz and J. Stegemann. The strong Nash-Williams orientation theorem for rayless graphs. arXiv:2409.10378.

[19] N. Robertson, P. Seymour, and R. Thomas. Excluding infinite clique minors. *Memoirs Amer. Math. Soc.*, 118(566):185–227, 1995.

[20] N. Robertson and P.D. Seymour. Graph Minors. I. Excluding a Forest. *J. Combin. Theory (Series B)*, 35:39–61, 1983.

[21] N. Robertson and P.D. Seymour. Graph Minors. XX. Wagner's Conjecture. *J. Combin. Theory (Series B)*, 92(2):325–357, 2004.

[22] N. Robertson and P.D. Seymour. Graph minors XXIII. Nash-Williams' immersion conjecture. *J. Combin. Theory (Series B)*, 100:181–205, 2010.

[23] R. Schmidt. Ein Ordnungsbegriff für Graphen ohne unendliche Wege mit einer Anwendung auf $n$-fach zusammenhängende Graphen. *Arch. Math.*, 40:283–288, 1983.

[24] R. Thomas. A counter-example to 'Wagner's conjecture' for infinite graphs. *Math. Proc. Camb. Phil. Soc.*, 103:55–57, 1988.

[25] R. Thomas. Well-Quasi-Ordering Infinite Graphs with Forbidden Finite Planar Minor. *Trans. Am. Math. Soc.*, 312(1):279–313, 1989.

[26] S. Thomassé. On Better-Quasi-Ordering Countable Series-Parallel Orders. *Trans. Am. Math. Soc.*, 352(6):2491–2505, 2000.