Teacher transfers: equalizing deficits across schools*

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Abstract

The Right to Free and Compulsory Education Act (2009) (RTE) of the Government of India prescribes student-teacher ratios for state-run schools. One method advocated by the Act to achieve its goals is the redeployment of teachers from surplus to deficit (in teacher strength) schools. We consider a model where teachers can either remain in their initially assigned schools or be transferred to a deficit school in their acceptable set. The planner's objective is specified in terms of the post-transfer deficit vector that can be achieved. We show that there exists a transfer whose post-transfer deficit vector Lorenz dominates all achievable post-transfer deficit vectors. We provide a two-stage algorithm to derive the Lorenz-dominant post-transfer deficit vector, and show that this algorithm is strategy-proof for teachers.

Keywords: matching, teacher transfer, Lorenz dominance, network flows.

JEL Classification: C78; D47; D71

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1 Introduction

The equitable allocation of resources is a fundamental concern in economics. It is well-known that various equity criteria can be satisfied in models where resources are perfectly divisible. However, several problems are *discrete* in nature and only allow for integral solutions. Our objective in this paper is to show that an existing equitable solution for perfectly divisible resources can be extended to a discrete problem of practical interest.

The Right to Education Act in India, 2009 (henceforth, RTE) was enacted by the Union Government to improve the quality of school education in India.[1] The Act identified student-teacher ratios in schools as a key determinant of the quality of education and clearly mandated the minimum number of teachers to be deployed relative to student strength. For historical reasons, some schools had teachers in excess of the desired student-teacher ratio while other schools had a deficit of teachers. In addition to fresh hiring, the Act mandated the transfer of teachers from surplus to deficit schools. According to Section 22 of the RTE:

> *"...The sanctioned strength of teachers in a school shall be notified by the Central Government, appropriate Government or the local authority...The Central Government, appropriate Government or the local authority, as the case may be, shall .... redeploy teachers of schools having strength in excess of the sanctioned strength...."*

The implementation of the RTE Act was largely left to the various states in the Union of India. The government of the state of Haryana was especially proactive in the matter of teacher transfers and its policies are enunciated in two documents (Government of Haryana (2016) and Government of Haryana (2023)). According to the vision statements of both documents, the aim of its policies are

> *" ... to ensure equitable, demand based distribution of teachers/heads to protect academic interest of students and optimise job satisfaction amongst its employees in a fair and transparent manner".*

Our paper is an attempt to design a theoretical framework to achieve the goals articulated

[1]For an overview of the act, see https://dsel.education.gov.in/rte.

in the vision statement.[2] We formulate the teacher transfer problem as a network flow problem where teachers have trichotomous preferences. We show that fairness and efficiency of outcomes can be defined in a natural and unambiguous way. Finally, we show that these "fair outcomes" can be achieved by a strategy-proof mechanism. Though we consider teacher transfer as our application, we note that our model and results apply more broadly to environments involving transfer of resources. For instance, civil servants assigned to states are transferred periodically; (multi-tasking) workers are transferred across departments in firms; doctors are transferred within states to meet shortages in hospitals.

We consider a set of teachers and a set of schools. Each school is either a surplus school or a deficit school, depending on the number of teachers currently assigned to it and its target number of teachers. Prior to transfers, every surplus school is associated with a strictly positive integer that is its surplus. Similarly, every deficit school is associated with a strictly positive integer that is its deficit. A transfer is a reassignment of teachers from surplus to deficit schools satisfying several requirements. Therefore, only the teachers assigned to surplus schools can be transferred; each such teacher has a set of acceptable deficit schools to which she can be transferred. Each teacher must either be transferred to an acceptable deficit school[3] or remain in the surplus school to which she belongs. This is an individual rationality requirement for teachers. No surplus school can become a deficit school nor can a deficit school become a surplus school post-transfer. The former is an individual rationality requirement for surplus schools and the latter, a natural fairness assumption. Finally, we do not permit fractional transfers of teachers. This is a natural assumption in our context since teachers are always assigned to a single school in practice.

Every transfer leads to a post-transfer deficit vector of deficit schools. We assume (in broad consonance with the Haryana policy documents) that the planner's objectives are expressed solely in terms of the post-transfer deficit vector. One can conceive of the planner having one of several objectives. A utilitarian planner may wish to minimize the aggregate sum of deficits or transfer as many teachers as possible. An egalitarian planner may wish to "minimize the worst deficits". A welfare maximizing planner may wish to minimize total loss for a continuous and convex loss function of the deficits. These solutions will not coincide

[2]Sharan et al. (2022) and Agarwal et al. (2018) contain empirical analyses of teacher transfers in Haryana: see also Section 6.

[3]In Section 5, we consider the case where a teacher from a surplus school can be transferred to an acceptable surplus school. We do not allow the transfer of teachers from deficit schools in our model.

in an arbitrary problem. Our main result, Theorem 1 shows that there exists an achievable post-transfer deficit vector that Lorenz dominates all other achievable post-transfer deficits. Such a transfer simultaneously reconciles the objectives of utilitarian, egalitarian and welfare maximizing planners. A novel feature of our result is that we demonstrate the existence of a Lorenz dominant solution under integrality constraints.

We provide a two-stage algorithm to find a Lorenz dominant vector. We formulate our problem as one of finding an integer flow in a network. In the first stage of the algorithm, we consider a relaxed version of the network flow problem where flows are allowed to be non-negative real numbers. We then use an existing algorithm by Megiddo (1974) and Dutta and Ray (1989) (henceforth, MDR), to find Lorenz dominant flows in this network. These flows are not integral.[4] The second stage of the algorithm uses the MDR solution to find an appropriate rounding of these flows which corresponds to the Lorenz dominant deficit vector of the teacher transfer problem. This requires solving a maximum flow problem of an augmented network constructed using the original network and the solution of the MDR algorithm. We note that a Lorenz dominant transfer can be computed efficiently (see Remark 1 in Section 3.6).

From a methodological viewpoint, our work extends the work of Dutta and Ray (1989). They consider convex cooperative games and their (the MDR) algorithm finds a Lorenz dominant core allocation — core is non-empty in a convex game with elegant extreme point properties (Shapley, 1971). Consider a convex cooperative game whose characteristic function is integral, which is the case for the teacher transfer problem. Our work addresses the following question: does there exist an *integral* core allocation that Lorenz dominates every other integral core allocation. We answer this question in the affirmative for the specific class of convex games induced by the network flow problems in Megiddo (1974). We show that the output of the MDR algorithm can be used to do a careful rounding to get an integral vector. As long as this rounding remains in the core, it is an integral Lorenz dominant core allocation.

We note that transfers could be made more efficient if teachers could be transferred from

[4] Consider a problem in which there is one surplus school with three teachers and two deficit schools. Suppose every teacher finds both the deficit schools acceptable and, the current deficits of both the deficit schools are equal, say 3. Then, the MDR algorithm will generate flows which will transfer 1.5 teachers to each of the deficit schools, leading to the post-transfer deficit vector (1.5,1.5).

a surplus school to another surplus school while maintaining the restrictions that surplus schools do not become deficit schools and deficit schools do not become surplus schools. Consider the case in which there are three schools: two surplus schools s and s', each with a surplus of one; and a deficit school d with a deficit of two. Suppose there is exactly one teacher t currently assigned to s who is willing to be transferred, but this teacher finds only s' acceptable; and suppose teachers t'_1 and t'_2 are currently assigned to s' but are both willing to be transferred to d. In our basic model, the deficit at d can only be reduced to one because exactly one of the teachers assigned to school s' can be transferred to d. However, if transfers between surplus schools is permitted, the deficit at d can be reduced to zero: transfer t from s to s'; and transfer both t'_1 and t'_2 to d. In Section 5, which discusses extensions of our basic model, we show how our results can be extended to cover this case.

We consider another extension where teachers are not homogeneous. They can be qualified to teach different subjects and a teacher may be able to teach more than one subject. We show that such a model can be accomodated in our framework by a suitable adaptation of the underlying network. In Section 5, we show that our main result holds in this setting as well.

We also consider the teacher transfer problem where the acceptable set of every teacher is private information. The current assignment of teachers and the surplus and deficit vectors are, however, publicly observable. We assume that a teacher's preferences are trichotomous. Her top indifference class consists of the deficit schools she finds acceptable; the second indifference class is the surplus school that she currently belongs to; and the third class is the set of non-acceptable deficit schools. A transfer mechanism must therefore rely on the reports of acceptable sets by the teachers. We show that the two-stage algorithm, supplemented by a consistent tie-breaking condition, is strategy-proof for teachers.

Our work is most closely related to the theoretical literature on matching with constraints and redistribution of matched resources. The key distinguishing feature of our work is that we seek an integral post transfer deficit vector that is Lorenz dominant. Bogomolnaia and Moulin (2004) and Bochet et al. (2012) have considered Lorenz dominance as an objective in specific matching models. We outline the relationship between these papers and ours next.

Bogomolnaia and Moulin (2004) consider two-sided matching problems in which both men and women have dichotomous preferences. For every agent (both a man and a woman), a *random* matching generates the probability of an acceptable match. The paper constructs a random matching where this probability vector is Lorenz dominant. Like us, their proposed matching is the output of the Dutta and Ray (1989) algorithm for a convex cooperative game. There are however, several significant differences between their paper and ours. The first is that they allow for random matching while we are interested in (integral) deterministic transfers. As a result, we have to supplement the MDR algorithm with a rounding stage to achieve a Lorenz dominant transfer. The second is that our problem is not a two-sided matching problem. In our model, teachers are employed in surplus schools which have constraints on the number of teachers they can release for transfer. As we shall show in Example 1, this fact plays an important role in our analysis. Consequently, a teacher is not an independent agent since her matching is tied to the surplus in the school where she is currently assigned. This is a departure from Bogomolnaia and Moulin (2004). Finally, our objective is to equalize deficits, different from that of equalizing transfers.

Bochet et al. (2012) analyse a model in which a divisible, non-disposable homogeneous commodity has to be reallocated between agents with single-peaked preferences. Agents are either suppliers or demanders, and transfers between a supplier and demander are feasible only if they are linked. These links form a bipartite graph. The teachers in our model can be regarded as the commodity being transferred from surplus to deficit schools. However, teachers have preferences over deficit schools and cannot be treated as a homogenous commodity. They cannot also be treated as a divisible commodity since fractional appointments are not permitted in our model. Finally as mentioned earlier, the transfer of teachers is constrained by the surplus of the schools in their initial assignment.

The paper is organized as follows. Section 2 presents the model and the main result. Section 3 describes the procedure to find a Lorenz dominant transfer. Section 4 discusses the strategy-proofness issue, while Section 5 considers extensions of the basic model. Section 6 reviews additional related literature. Finally, Appendix A provides a proof of Lorenz dominance of the output of the MDR algorithm.

2 The model

We consider a set of schools partitioned into two subsets: the set of deficit schools D where there are fewer teachers than the number mandated by the RTE act and the set of surplus schools S with more teachers than the mandated number. Each surplus school has an initial assignment of teachers. Let T denote the set of all teachers assigned to surplus schools. Only the teachers in T are candidates for being transferred; teachers employed in non-surplus schools cannot be transferred and so play no further role in our basic model.[5]

We denote generic deficit schools, surplus schools, and teachers by d_k, s_j, and t_i respectively. Each teacher t_i belongs to a unique surplus school $O(t_i) \in S$. This is the initial assignment of t_i.

Each teacher t_i has a set of *acceptable* deficit schools $A(t_i) \subseteq D$. As discussed earlier, the set $A(t_i)$ can be interpreted in two different ways. The first is in terms of the preferences of t_i: she is indifferent between schools in $A(t_i)$ and strictly prefers any school in $A(t_i)$ to her initial assignment $O(t_i)$, which in turn is strictly preferred to any other deficit school. The set $A(t_i)$ can also be interpreted as a compatibility requirement. For instance, t_i could be a primary school teacher and $A(t_i)$ is the set of deficit schools with vacancies for primary school teachers. Similarly, one can think of reassignment of doctors across public hospitals where the compatibility is determined by a doctor's specialization. To avoid trivialities, we assume $A(t_i)$ is non-empty for each $t_i \in T$.

Every school $s_j \in S$ has a surplus α_j which is a positive integer. Every school $d_k \in D$ has a deficit β_k which is a positive integer.[6]

We assume that there are L deficit schools, i.e., $|D| = L$. The initial deficit vector is denoted by $\beta \equiv (\beta_1, \ldots, \beta_L)$. Throughout, we will deal with L-dimensional (deficit) vectors of the form $x \equiv (x_1, \ldots, x_L)$. Also for any subset $B \subseteq D$, define $x(B) := \sum_{d_k \in B} x_k$.

A **transfer** is a map $\sigma : T \to D \cup S$ satisfying the following properties:

[5]The RTE explicitly requires teachers to be transferred from surplus to deficit schools. See the quotation from the Act in the Introduction. The Haryana government also followed this practice - see Points (ix) and (x) on Page 7 in Government of Haryana (2016).

[6]In our basic model, we do not consider surplus schools with zero surpluses or deficit schools with zero deficits since they play no role in the analysis. In the extension in Section 5, we will include surplus schools with zero surpluses.

1. **No transfer to an unacceptable deficit school:**

$$\sigma(t_i) \in A(t_i) \cup \{O(t_i)\} \qquad \forall\ t_i \in T,$$

2. **No surplus school can become a deficit school:**

$$|\{t_i : O(t_i) = s_j, \sigma(t_i) \neq s_j\}| \leq \alpha_j \qquad \forall\ s_j \in S,$$

3. **No deficit school can become a surplus school:**

$$|\{t_i \in T : \sigma(t_i) = d_k\}| \leq \beta_k \qquad \forall\ d_k \in D.$$

The first requirement is an individual rationality constraint for teachers. The second can be interpreted as an individual rationality constraint for surplus schools. The third is a natural fairness requirement.

In Section 5, we consider extensions in which some of these restrictions are relaxed, and explain why our main result continues to hold.

2.1 Objectives of the planner

Every transfer σ generates a post-transfer deficit vector β^σ where

$$\beta_k^\sigma := \beta_k - |\{t_i \in T : \sigma(t_i) = d_k\}| \qquad \forall\ d_k \in D.$$

We assume that the planner chooses a transfer by comparing post-transfer deficit vectors. In this respect, our model differs from Combe et al. (2022) where the primary concern is to improve the welfare of teachers. We believe that our assumption is consistent with the provisions of the RTE act — the planner's goal is to improve deficits while maintaining individual rationality for teachers and surplus schools.

A planner may have one of several objectives. A *utilitarian* planner wishes to minimize the aggregate sum of deficits or transfer as many teachers as possible. An *egalitarian* planner wishes to minimize the worst deficits lexicographically. To formalize this objective, it is helpful to introduce some notation and additional notions, which we turn to next.

Let γ be an arbitrary vector in $\Re^L$. Let $[\gamma]$ denote the vector where the components of γ are ordered from highest to lowest, i.e. $[\gamma] = (\gamma_{[1]}, \ldots, \gamma_{[L]})$ and $\gamma_{[1]} \geq \gamma_{[2]} \ldots \geq \gamma_{[L]}$. Consider $\gamma, \gamma' \in \Re^L$. We say γ lexicographically dominates γ' if there exists an integer $r \in \{1, \ldots, L\}$ such that $\gamma_{[1]} = \gamma'_{[1]}, \ldots, \gamma_{[r]} = \gamma'_{[r]}$ and $\gamma_{[r+1]} < \gamma'_{[r+1]}$ or $\gamma_{[r]} = \gamma'_{[r]}$ for all $r \in \{1, \ldots, L\}$. An egalitarian planner would like to choose a transfer σ such that β^σ lexicographically dominates $\beta^{\sigma'}$ for all other $\sigma' \in \Sigma$. A utilitarian planner would like to choose a transfer that minimizes $\sum_k \beta_k^\sigma$ over all $\sigma \in \Sigma$. The set of all possible post-transfer deficit vectors is finite. It follows that the utilitarian and egalitarian solutions exist. In general, there is no reason to believe that these solutions will coincide.[7] However, these objectives can be reconciled if we can find a transfer that generates a *Lorenz dominant* deficit vector, which is defined next.

Consider $\gamma, \gamma' \in \Re^L$. We say γ Lorenz dominates γ' (denoted by $\gamma \succ_{\text{LD}} \gamma'$) if

$$\sum_{i=1}^{k} \gamma_{[i]} \leq \sum_{i=1}^{k} \gamma'_{[i]}, \quad \text{for all } k \in \{1, \ldots, L\}. \tag{1}$$

In the language of majorization (Marshall et al. (1979); Hardy et al. (1952)), γ is weakly majorized by γ' if (1) holds. If Inequality (1) for the case $k = L$ holds with equality, we say γ is majorized by γ'. Our objective is to find a transfer (if it exists) such that the resulting deficit vector is weakly majorized by every deficit vector that can be generated by a transfer. Such a transfer is a Lorenz dominant transfer.

Definition 1 *A transfer $\sigma^\star$ is* **Lorenz dominant** *if $\beta^{\sigma^\star} \succ_{\text{LD}} \beta^\sigma$ for every other transfer σ.*

There are several equivalent definitions of weak majorization — see Chapters 2, 3, and 4 of Marshall et al. (1979). For example,

Fact 1 (Hardy et al. (1929)) *The vector γ Lorenz dominates γ' if and only if for all continuous, increasing, and convex functions $g : \Re \to \Re$, we have*

$$\sum_{k=1}^{L} g(\gamma_k) \leq \sum_{k=1}^{L} g(\gamma'_k).$$

[7] Consider the following example. There are two deficit schools and the set of post-transfer deficit vectors is the set $\{(\beta_1, \beta_2) \in \mathbb{N}^2 : 2\beta_1 + \beta_2 \geq 30\}$. The utilitarian and egalitarian solutions would pick $(15, 0)$ and $(10, 10)$ respectively.

One can interpret g to be the social planner's "cost" of deficits.

Applying Fact 1, a transfer $\sigma^\star$ is Lorenz dominant if for every other transfer σ and every continuous, increasing, and convex function $g : \Re \to \Re$, we have

$$\sum_{d_k \in D} g(\beta_k^{\sigma^\star}) \leq \sum_{d_k \in D} g(\beta_k^{\sigma})$$

The order $\succ_{\mathrm{LD}}$ is a partial order, which implies that a Lorenz dominant transfer need not exist (as in the example in Footnote 7). If a Lorenz dominant transfer exists, it simultaneously reconciles the objectives of utilitarian, egalitarian and welfare maximizing planners — reflected in an appropriate choice of the g function above. Bogomolnaia and Moulin (2004) and Bochet et al. (2012) prove the existence of a Lorenz dominant solution in some matching problems that allow continuous or fractional solutions. However, ours is a discrete problem: we seek a transfer that generates an integer deficit vector which Lorenz dominates every other feasible integer deficit vector. Our main result is that such a transfer exists.

Theorem 1 *A Lorenz dominant transfer exists.*

The proof of Theorem 1 constructs a Lorenz dominant transfer using a two-stage algorithm based on a network flow formulation of the problem, which we describe next.

3 A procedure to find the Lorenz dominant transfer

In this section, we describe a procedure to find a Lorenz dominant transfer to complete the proof of Theorem 1. This is done in three steps, described in next three subsections. In the first step, we formulate the teacher transfer problem as a network flow problem. In the second, we use an existing result of Megiddo (1974), and connect the *reduced-form* version of the network flow problem to the core constraints of a convex cooperative game. We then apply an algorithm due to Megiddo (1974) and Dutta and Ray (1989) (MDR algorithm) to find a Lorenz dominant deficit vector and corresponding transfers. The output of the MDR algorithm typically generates fractional transfers. The final step of our procedure consists of a careful rounding of the output of the MDR algorithm to obtain a Lorenz dominant transfer. This step involves the use of a standard algorithm (like Ford-Fulkerson) to find a maximum flow in an augmented network that is obtained by modifying the network of the first step using the output of the MDR algorithm.

3.1 A network flow formulation

The teacher transfer problem is formulated as a *single source multiple sink network flow problem*.[8] We consider two versions of the problem, one where flows are restricted to be non-negative integers and the other where flows are allowed to be non-negative real numbers.

Both versions use the same network structure that we describe below.

- The set of nodes N in the graph consists of:
  1. a source node N_0
  2. a node for each surplus school in S
  3. a node for each teacher in T
  4. a node for each deficit school in D: we refer to these nodes as *sink* nodes

  Thus, $N \equiv \{N_0\} \cup S \cup T \cup D$.
- The set of edges E and their capacities are as follows:
  1. a directed edge (N_0, s_j) for every surplus school $s_j \in S$, with capacity α_j
  2. a directed edge $(O(t_i), t_i)$ for every teacher $t_i \in T$, with capacity 1
  3. a directed edge (t_i, d_k) for every teacher $t_i \in T$ and every deficit school $d_k \in A(t_i)$, with capacity 1
- Each deficit school (sink) node $d_k \in D$ has a capacity β_k. [9]

We let G denote this network.

A flow is a map $f : E \to \Re_+$ satisfying *flow balancing*:

[8] For a comprehensive survey of network flows see Ahuja et al. (1988).

[9] Note that we can convert node capacities to edge capacities by creating a copy of each sink node and adding a directed edge of capacity β_k from sink node d_k to its copy.

1. $f(N_0, s_j) = \sum\limits_{t_i : s_j = O(t_i)} f(s_j, t_i)$ for each surplus school $s_j \in S$,
2. $f(O(t_i), t_i) = \sum\limits_{d_k \in A(t_i)} f(t_i, d_k)$ for each teacher $t_i \in T$,

and *capacity* constraints:

1. $f(N_0, s_j) \leq \alpha_j$ for all $s_j \in S$,
2. $f(s_j, t_i) \leq 1$ for all edges (s_j, t_i) such that $s_j = O(t_i)$,
3. $f(t_i, d_k) \leq 1$ for all edges (t_i, d_k) such that $d_k \in A(t_i)$,
4. $\sum\limits_{t_i \in T} f(t_i, d_k) \leq \beta_k$ for all $d_k \in D$.

We say a flow is an *integer flow* if the flow on every edge of the network is a non-negative integer.

For every integer flow f in G, we define a corresponding transfer σ as follows:

$$\sigma(t_i) = \begin{cases} d_k, & \text{if } \ f(t_i, d_k) = 1 \\ O(t_i), & \text{if } \ f(t_i, d_k) = 0 \text{ for all } d_k \in A(t_i) \end{cases}$$

It is easy to verify that σ satisfies the feasibility properties in the definition of a transfer. Similarly, given a transfer σ, we define a corresponding **integer** flow f as follows: For the edge (t_i, d_k), let

$$f(t_i, d_k) = \begin{cases} 1, & \text{if } \ \sigma(t_i) = d_k \\ 0, & \text{if } \ \sigma(t_i) = O(t_i) \end{cases}$$

For the edge $(O(t_i), t_i)$, let

$$f(O(t_i), t_i) = \begin{cases} 1, & \text{if } \ \sigma(t_i) = d_k \\ 0, & \text{if } \ \sigma(t_i) = O(t_i) \end{cases}$$

For the edge (N_0, s_j), let

$$f(N_0, s_j) = |\{t_i : s_j = O(t_i), \sigma(t_i) \neq s_j\}|$$

Once again it is easy to verify that f is integer-valued and satisfies flow balancing and capacity constraints. It follows that there is a bijection between transfers and the set of integer flows in G. For a flow f in G, we denote the flow into a sink node (deficit school) d_k by $f(d_k)$.

A flow f is a **maximum flow** if for every other flow f' we have

$$\sum_{d_k \in D} f(d_k) \geq \sum_{d_k \in D} f'(d_k)$$

It is well-known that a maximum integer flow exists whenever the capacities of nodes and edges are integral. Consequently, there is a maximum flow that corresponds to a transfer. By Lorenz dominance condition (1), the following observation is immediate.

Observation 1 *A Lorenz dominant transfer (if it exists) corresponds to an integral maximum flow.*

3.2 An illustrative example

We present an example to highlight some important features of our model.

There are two surplus schools $\{s_1, s_2\}$ each with a surplus of one teacher. Teacher t_1 is initially assigned to surplus school s_1; teachers t_2 and t_3 are initially assigned to surplus school s_2. Surplus school s_2 can transfer at most transfer one teacher. There are three deficit schools $\{d_1, d_2, d_3\}$ with deficits $\beta_1 = 1, \beta_2 = 3, \beta_3 = 2$. Teachers t_1 and t_2 find d_1 and d_2 acceptable, while teacher t_3 finds all three deficit schools acceptable. The corresponding network is shown in Figure 1.

As we will see later, the *fractional* Lorenz dominant solution is the deficit vector $(1, \frac{3}{2}, \frac{3}{2})$. This corresponds to a transfer of no teachers to school d_1, $\frac{3}{2}$ teachers to school d_2 and $\frac{1}{2}$ teacher to school d_3. This transfer equalizes the deficits of schools d_2 and d_3 without changing the deficit of school d_1.

Our procedure will result in an integer deficit vector $(1, 1, 2)$ with d_1 and d_2 having a (post-transfer) deficit of 1, while the deficit of d_3 remains unchanged at 2. This can be achieved by transferring teachers t_1, and either t_2 or t_3 (but not both), to school d_2; the remaining teacher is not transferred out of school s_2 in order to respect the surplus constraint of s_2.

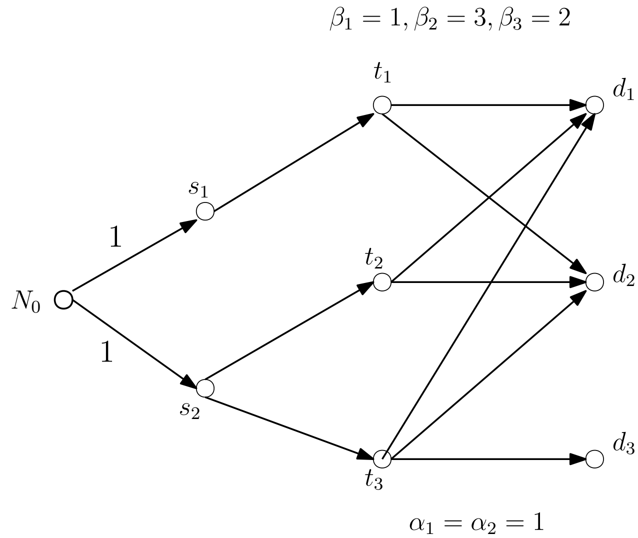

Figure 1: Example 1

The vector $(1, 1, 2)$ Lorenz-dominates all other possible integer post-transfer deficit vectors.

We wish to emphasize some features of our model:

- *Surplus constraints are important.* If we ignored surplus constraints in Example 1, all the three teachers could be transferred. This would reduce deficits further, but would also turn a surplus school into a deficit school, violating a natural participation constraint for the surplus school. In our model the set of teachers is partitioned according to the schools they are currently assigned to. There is an upper bound on the number of teachers that can be transferred from each element of the partition. This constraint distinguishes our model from standard matching problems.

- *Role of existing deficits.* In Example 1, the Lorenz dominant solution does not transfer any teacher to d_1 since its existing deficit is the lowest amongst the three deficit schools. On the other hand, if the deficit in d_1 were high, say 5, then two teachers would be transferred to d_1 to equalize deficits.

- *Many transfer-maximizing solutions.* There are many transfers that maximize the total transfer to deficit schools. For instance, transferring t_1 to d_1 and transferring t_2 to d_2 makes the maximal (2) number of transfers resulting in the deficit vector $(0, 2, 2)$ but that is clearly Lorenz dominated by our solution.

3.3 The reduced form and a cooperative game

In this subsection, we present a result in Megiddo (1974) that allows us to restrict attention *solely* to flows into deficit schools. According to the result, a flow to deficit schools can arise from a feasible flow in network G if and only if it satisfies some conditions. These conditions appear in the form of the core constraints of a suitably constructed cooperative game. This reformulation of the problem in *reduced-form* simplifies our analysis.

In order to describe the reduced-form, we introduce some more concepts. For every $B \subseteq D$, a flow f is a B-**maximum flow** if for every other flow f' we have

$$\sum_{d_k \in B} f(d_k) \geq \sum_{d_k \in B} f'(d_k)$$

Let $v(B) := \sum_{d_k \in B} f(d_k)$, i.e., the maximum flow into sinks in B. An integer B-maximum flow exists, and hence, $v(B)$ is a non-negative integer for every $B \subseteq D$.

Define $w : 2^D \to \Re$ as follows:

$$w(B) := \beta(B) - v(B) \qquad \forall\ B \subseteq D.$$

The pair $\langle D, w \rangle$ is a cooperative game, where D is the set of players and w is an integer-valued characteristic function. Since D will be held fixed throughout the analysis, we will refer to this game simply by w.

Let $h = (h_1, \ldots, h_L)$ be an L-dimensional vector of real numbers. The vector h is **achievable** if there is a flow f in G such that $h_k = \beta_k - f(d_k)$ for all $d_k \in D$. In other words, h is the post-transfer vector generated by some feasible flow. If h is an integral vector, it must be generated by an integral flow vector, i.e., a transfer.

Proposition 1 (Megiddo (1974)) *A vector $h \equiv (h_1, \ldots, h_L)$ is achievable if and only if*

$$h(B) \geq w(B) \qquad \forall\ B \subseteq D \tag{2}$$

Further, w is convex (supermodular) and integer-valued, i.e., for any $A \subsetneq B \subseteq D$ and $d_k \notin B$,

$$w(B \cup \{d_k\}) - w(B) \geq w(A \cup \{d_k\}) - w(A) \tag{3}$$

Proof: The proof follows from a straightforward application of results in Megiddo (1974). Let $x \equiv (x_1, \ldots, x_L)$ be any non-negative vector. Lemma 4.1 in Megiddo (1974) shows that x can be generated from a flow if and only if $x(B) \leq v(B)$ for all $B \subseteq D$. This immediately implies (2).

Lemma 3.2 in Megiddo (1974) shows that v is concave. Since β is linear and $-v$ is convex, it follows that w is convex. ∎

As a consequence of Proposition 1, we can restrict attention to a reduced form problem of analyzing post-transfer deficit vectors. Moreover, achievability consists of verifying a family of subset inequalities. Since w is convex, these inequalities closely resemble the core constraints of a convex game, with one important difference: equality for $B = D$ in (2) is not imposed.

We refer to the constraints in (2) as **relaxed core** constraints.[10]

We can summarize the results thus far, as follows. The teacher transfer problem can be reformulated as a network flow problem where integer flows correspond to transfers. A convex cooperative game w is induced by the network flow problem, where deficit schools are the players. For every coalition $B \subseteq D$, $w(B)$ is an integer. In addition, a real deficit vector is achievable in the network flow problem (where fractional teacher flows are allowed) if and only if it belongs to the relaxed core of w.

3.4 The Megiddo-Dutta-Ray algorithm

The first stage of our algorithm consists of applying a result from cooperative game theory to find a *fractional* Lorenz dominant transfer. Dutta and Ray (1989) show that the core of a convex game always contains a Lorenz dominant allocation and provide an algorithm to identify it.[11] Since our achievability constraints are *relaxed* rather than standard core constraints, we cannot directly apply their result. However, as we show in Appendix A, their algorithm also produces a fractional Lorenz dominant allocation with relaxed core constraints.

[10] In discrete optimization, these constraints are variants of the polymatroid constraints.

[11] Megiddo (1974) proposed the same algorithm to construct a lexicographically optimal flow. Note that a lexicographically optimal solution of a problem always exists but a Lorenz dominant solution may not.

As noted earlier, a fractional solution does not correspond to a transfer. In the second stage of our algorithm we *round* the output of the MDR algorithm. This produces an integral deficit vector that corresponds to a transfer and one that Lorenz dominates all other deficit vectors arising from transfers. Details of the second stage of the algorithm are in the next section.

The MDR algorithm inductively partitions the set of deficit schools D into sets $(D_1, \ldots, D_{k^\star})$. For any $i \in \{1, \ldots, k^\star\}$, let $\overline{D}_i = \cup_{j=1}^{i} D_j$. Define D_1 (and $\overline{D}_1$) as the set having the highest average worth:

$$D_1 \in \arg\max_{B \subseteq D} \frac{w(B)}{|B|}$$

Given $D_1, \ldots, D_k$ for some k, define D_{k+1} as the set having the highest average marginal contribution to $\overline{D}_k$:

$$D_{k+1} \in \arg\max_{B \subseteq D \setminus \overline{D}_k} \frac{w(B \cup \overline{D}_k) - w(\overline{D}_k)}{|B|}$$

Ties in the determination of D_{k+1} are broken arbitrarily.

Since D is finite, the algorithm generates a sequence of disjoint subsets of deficit schools: $(D_1, \ldots, D_{k^\star})$. Using this sequence, we construct a deficit vector $h^\star$ as follows. For every $D_j \in \{D_1, \ldots, D_{k^\star}\}$,

$$h_i^\star := \frac{w(\overline{D}_j) - w(\overline{D}_{j-1})}{|D_j|} \qquad \forall\ d_i \in D_j, \tag{4}$$

where $\overline{D}_0 \equiv \emptyset$ with $w(\emptyset) = 0$.

Theorem 2 in Appendix A shows that $h^\star$ is Lorenz dominant deficit vector. Its proof relies on some key claims. Claim 3 shows that $h^\star$ is achievable. Claim 2 establishes an ordering property of h^*: $h_i^\star \geq h_j^\star$ if $d_i \in D_k$ and $d_j \in D_{k+1}$. Another important property of h^* that follows from (4) is

$$\sum_{d_i \in \overline{D}_j} h_i^\star = w(\overline{D}_j) \qquad \forall\ j \in \{1, \ldots, k^\star\} \tag{5}$$

The sets $\overline{D}_j$, $j = 1, \ldots k^*$ are the subsets of deficit schools where the core constraints are binding for $h^\star$.

According to (4), $h_i^\star$ is a ratio of two integers. This immediately suggests that $h^\star$ is not integral and that rounding is required in order to obtain a meaningful result on teacher transfers. The next two subsections address this issue.

3.5 Arbitrary roundings do not work: An example

This subsection provides an example (Example 2) to show that the rounding issue is a delicate one. The example also illustrates the working of the MDR algorithm.

The sets of deficit schools, surplus schools and teachers are $S = \{s_1, s_2, s_3\}$, $D = \{d_1, d_2, d_3, d_4, d_5, d_5, d_7\}$ and $T = \{t_1, t_2, t_3, t_4, t_5\}$ respectively. The initial assignment of teachers is given by: $O(t_1) = s_1$, $O(t_2) = O(t_3) = s_2$ and $O(t_4) = O(t_5) = s_3$. The acceptable sets are as follows: $A(t_1) = \{d_1, d_2\}$, $A(t_2) = A(t_3) = \{d_1, d_2, d_3, d_4, d_5\}$, $A(t_4) = \{d_6\}$ and $A(t_5) = \{d_6, d_7\}$. The deficit of every school is 5. The surplus of schools s_1, s_2 and s_3 are $\alpha_1 = 1$ and $\alpha_2 = \alpha_3 = 2$ respectively. The example and the associated network flow formulation of the problem is illustrated in Figure 2.

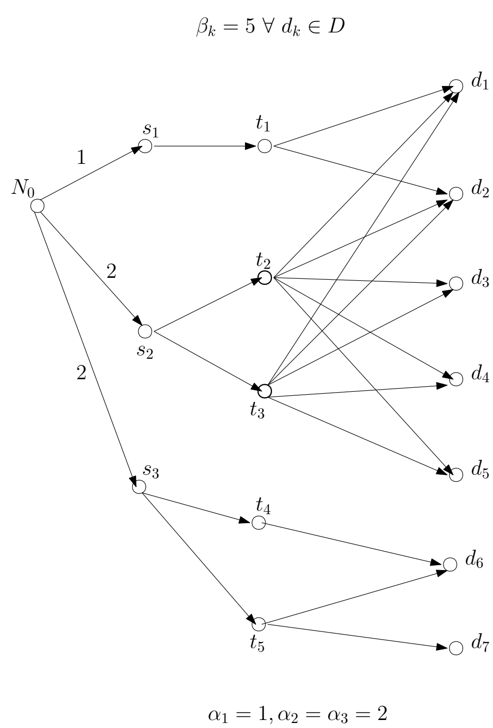

Figure 2: Example 2

We omit the details of the computation of the characteristic function w and simply note that the maximum average worth of a coalition is achieved by coalition $D_1 = \{d_1, d_2, d_3, d_4, d_5\}$.

$$\frac{w(D_1)}{|D_1|} = \frac{22}{5} = \max_{B:B \subseteq D} \frac{w(B)}{|B|}.$$

This generates D_1 of the MDR algorithm. Having identified D_1, the average marginal contributions of subsets of remaining deficit schools are computed.

$$\begin{aligned} w(D_1 \cup \{d_6\}) - w(D_1) &= 3 \\ w(D_1 \cup \{d_7\}) - w(D_1) &= 4 \\ \frac{w(D_1 \cup \{d_6, d_7\}) - w(D_1)}{2} &= 4 \end{aligned}$$

There are two possible choices for D_2: either $D_2 = \{d_7\}$ or $D_2 = \{d_6, d_7\}$. Either choice can be made, so we set $D_2 = \{d_6, d_7\}$. The partition chosen by the MDR algorithm is therefore $\{\{d_1, \ldots, d_5\}, \{d_6, d_7\}\}$. The resultant Lorenz dominant deficit vector is:

$$h_i^\star = \begin{cases} 4.4 & \text{if } d_i \in D_1 \\ 4 & \text{if } d_i \in D_2 \end{cases}$$

It is clear that the vector $h^\star$ cannot be obtained from an integer flow (or transfer).

An important observation here is that $h^\star$ cannot be approximated by an arbitrary rounding. For example, consider the following rounding of $h^\star$:

$$\hat{h}_1 = \hat{h}_2 = 5, \hat{h}_3 = \hat{h}_4 = \hat{h}_5 = 4, \hat{h}_6 = \hat{h}_7 = 4.$$

This is a "valid" rounding of $h^\star$ in the sense that for each $d_i \in D$, we have

$$\hat{h}_i \in \{\lfloor h_i^\star \rfloor, \lceil h_i^\star \rceil\}.$$

Further, $h^\star(D_1) = \hat{h}(D_1)$ and $h^\star(D_2) = \hat{h}(D_2)$.[12]

The difficulty is that $\hat{h}$ is **not achievable**, i.e., there is no feasible transfer that can generate this deficit vector. To see this, consider $B = \{d_3, d_4, d_5\}$ and verify that $w(B) = 13$. But $\hat{h}(B) = 12 < w(B)$ implying non-achievability. This can also be directly verified by noting that only two teachers t_2 and t_3 can be transferred to $\{d_3, d_4, d_5\}$. Consequently, the aggregate deficit of $\{d_3, d_4, d_5\}$ cannot be reduced from 15 to 12.

[12] A rounding of $h^\star$ satisfying these two properties will be referred to as an MDR-consistent rounding. See Section 3.6.

3.6 MDR-consistent rounding

The previous example demonstrates that an arbitrary rounding of the output of the MDR algorithm may not be achievable. The critical step in the proof of Theorem 1 is to show that there exists an achievable rounding where the core binding constraints (5) continue to hold for the new integer deficit vector. The proof is completed by showing that this vector Lorenz dominates every other integral achievable deficit vector.

Definition 2 *A deficit vector h is* **MDR-consistent** *if it is an integral achievable deficit vector such that*

$$h_i \in \{\lfloor h_i^\star \rfloor, \lceil h_i^\star \rceil\} \qquad \forall\ d_i \in D \tag{6}$$

$$h(\overline{D}_j) = w(\overline{D}_j) \qquad \forall\ j \in \{1, \ldots, k^\star\} \tag{7}$$

An MDR-consistent deficit vector corresponds to a transfer as it is integral and achievable.

Proposition 2 (Rounding) *An MDR-consistent deficit vector exists.*

Proof: An auxiliary network G^{AU} is constructed by augmenting the network G using the output of the MDR algorithm $\{D_1, \ldots, D_{k^\star}\}$. Abusing notation slightly, the deficit of any school in D_j generated by the MDR algorithm is denoted by $h_j^\star$. Though $h_j^\star$ is not integral, $h_j^\star |D_j|$ is integral and equals $w(\overline{D}_j) - w(\overline{D}_{j-1})$. The following set of nodes and edges is added to the network G to construct G^{AU}:

1. For each $j \in \{1, \ldots, k^\star\}$, a node is added representing the subset of deficit schools D_j. Each node D_j has a capacity of $\sum_{d_i \in D_j} \left(\beta_i - h_i^\star\right)$ (integral).
2. For each $j \in \{1, \ldots, k^\star\}$ and each deficit school $d_i \in D_j$, an edge (d_i, D_j) is added. The flow in each such edge must lie in $[\lfloor \beta_i - h_i^\star \rfloor, \lceil \beta_i - h_i^\star \rceil]$.

All capacities and bounds on flows in G^{AU} are integers. The construction of G^{AU} for the example in Figure 2 is illustrated in Figure 3.

Since $h^\star$ is achievable, there exists flow $f^\star$ such that $f^\star(d_k) = \beta_k - h_k^\star$ for all $d_k \in D$. Construct the flow $f^{\star\star}$ by augmenting $f^\star$ as follows: for every d_k, the flow on the edge

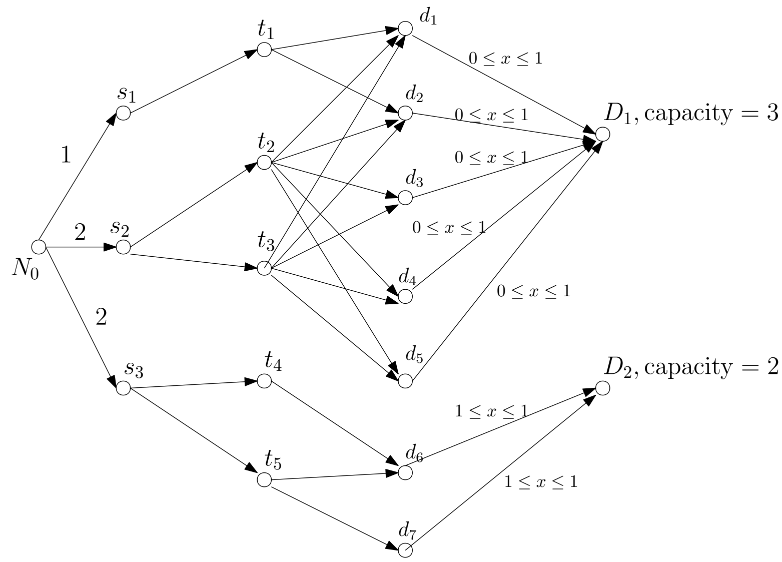

Figure 3: Graph G^{AU} in Example in Figure 2

(d_k, D_j) where $d_k \in D_j$ is $\beta_k - h^\star_k$. Clearly $f^{\star\star}$ satisfies the additional flow balancing and capacity constraints in G^{AU}, and hence, is a feasible flow in the augmented network. The flow received by a (sink) node D_j in $f^{\star\star}$ is,

$$f^{\star\star}(D_j) = \sum_{d_k \in D_j} f^{\star\star}(d_k, D_j) = \sum_{d_k \in D_j} \left(\beta_k - h^\star_k\right) = \beta(D_j) - h^\star(D_j) \tag{8}$$

The RHS of the equation above is the capacity of node D_j. It follows that $f^{\star\star}$ is a maximum flow in G^{AU}.

Since all capacities in G^{AU} are integers, the integrality theorem in network flows (see Page 186 in Ahuja et al. (1988)) implies that there exists a integral maximum flow $\hat{f}$. Therefore $\sum_{j=1}^{k^\star} f^{\star\star}(D_j) = \sum_{j=1}^{k^\star} \hat{f}(D_j)$. In order to satisfy capacity constraints of each node D_j, we must have

$$f^{\star\star}(D_j) = \hat{f}(D_j) \qquad \forall\, j \in \{1, \ldots, k^\star\}. \tag{9}$$

Define a new deficit vector $\hat{h}_k := \beta_k - \hat{f}(d_k)$ for all $d_k \in D$. Hence,

$$\hat{h}(D_j) = \beta(D_j) - \hat{f}(D_j) \qquad \forall\, j \in \{1, \ldots, k^\star\} \tag{10}$$

Note that $\hat{h}$ is the post-transfer deficit vector corresponding to $\hat{f}$. From (8), (9), and (10), we get $\hat{h}(D_j) = h^\star(D_j)$ for all $j \in \{1, \dots, k^\star\}$. This further implies that

$$\hat{h}(\overline{D}_j) = h^\star(\overline{D}_j) = w(\overline{D}_j) \qquad \forall\, j \in \{1, \dots, k^\star\}, \tag{11}$$

where the last equality follows from the property we established for $h^\star$ (see Equation (5)).

We complete the proof by showing that the integrality constraint is satisfied. By the construction of G^{AU}, $\hat{f}$ induces an integer flow in the original network G. It follows that $\hat{h}$ is an integral achievable post-transfer deficit vector in G.

For any school $d_k \in D$, we have $\hat{f}(d_k) = \beta_k - \hat{h}_k$. By flow balancing at node d_k where $d_k \in D_j$, we have $\hat{f}(d_k) = \hat{f}(d_k, D_j)$. Since the flow $\hat{f}(d_k, D_j)$ must satisfy the capacity constraint of edge (d_k, D_j), this implies

$$\begin{aligned} & \lfloor \beta_k - h_k^\star \rfloor \le \hat{f}(d_k) \le \lceil \beta_k - h_k^\star \rceil \\ \implies & \lfloor \beta_k - h_k^\star \rfloor \le \beta_k - \hat{h}_k \le \lceil \beta_k - h_k^\star \rceil \\ \implies & \beta_k - \lceil h_k^\star \rceil \le \beta_k - \hat{h}_k \le \beta_k - \lfloor h_k^\star \rfloor \\ \implies & \lfloor h_k^\star \rfloor \le \hat{h}_k \le \lceil h_k^\star \rceil \end{aligned}$$

This establishes that $\hat{h}$ is MDR-consistent. ■

Proposition 2 together with Proposition 3 below complete the proof of Theorem 1. An MDR consistent deficit vector in Example 2 is:

$$\hat{h}_1 = 5, \hat{h}_2 = \hat{h}_3 = \hat{h}_4 = 4, \hat{h}_5 = 5, \hat{h}_6 = \hat{h}_7 = 4.$$

Proposition 3 (Lorenz dominance) *An MDR-consistent deficit vector Lorenz dominates every other integral achievable deficit vector.*

Proof: Consider an MDR-consistent deficit vector $\hat{h}$ — by definition, it is achievable and integral. Recall that the output of the MDR algorithm is the achievable deficit vector $h^\star$ (not necessarily integral) and a partition of deficit schools $(D_1, \dots, D_{k^\star})$ satisfying the following: for every $k \in \{1, \dots, k^\star\}$ and every $d_i \in D_k$,

$$\begin{aligned} h_i^\star &= \frac{w(\overline{D}_k) - w(\overline{D}_{k-1})}{|D_k|} \\ h^\star(\overline{D}_k) &= w(\overline{D}_k). \end{aligned}$$

MDR-consistency implies $h^\star(\overline{D}_k) = \hat{h}(\overline{D}_k) = w(\overline{D}_k)$ and $\hat{h}_i \in \{\lfloor h_i^\star \rfloor, \lceil h_i^\star \rceil\}$. There may exist sets D_k, D_{k+1} such that $\{\lfloor h_i^\star \rfloor, \lceil h_i^\star \rceil\} = \{\lfloor h_j^\star \rfloor, \lceil h_j^\star \rceil\}$, where $d_i \in D_k$ and $d_j \in D_{k+1}$. Sets whose integral floor and ceilings are the same in $h^\star$ are merged to form a new partition $(D_1^+, \dots, D_{\ell^\star}^+)$. Clearly, $\ell^\star \leq k^\star$. Different schools in D_k^+ may have different deficits in $h^\star$. However, there are three important facts about the new partition that we note below. The first is that $\{\lfloor h_i^\star \rfloor, \lceil h_i^\star \rceil\} = \{\lfloor h_j^\star \rfloor, \lceil h_j^\star \rceil\}$ for any $d_i, d_j \in D_k^+$, The second is that, for any $k \in \{1, \dots, \ell^\star\}$, we have

$$\hat{h}(\overline{D}_k^+) = h^\star(\overline{D}_k^+) = w(\overline{D}_k^+). \tag{12}$$

Finally, we have $\hat{h}_i \geq \hat{h}_j$ for any $d_i \in D_k^+$ and $d_j \in D_{k+1}^+$.

Assume without loss of generality that $\hat{h}$ is ordered from highest to lowest in each D_k^+. Let h be an arbitrary achievable integer deficit vector. Pick an arbitrary $j \in \{1, \dots, L\}$. The proof is completed by showing $\sum_{i=1}^j \hat{h}_{[i]} \leq \sum_{i=1}^j h_{[i]}$. We consider two cases.

Case A: $j = |\overline{D}_\ell^+|$ for some ℓ. From (12), we have

$$\sum_{i=1}^{j} \hat{h}_{[i]} = \hat{h}(\overline{D}_\ell^+) =_{(a)} w(\overline{D}_\ell^+) \leq_{(b)} h(\overline{D}_\ell^+) \leq_{(c)} \sum_{i=1}^{j} h_{[i]}.$$

Equality (a) follows from MDR-consistency of $\hat{h}$. Inequality (b) follows from achievability of h. Inequality (c) follows because the RHS is the sum of the j highest deficits in h.

Case B: Case A does not hold, i.e. $j \neq |\overline{D}_\ell^+|$ for any ℓ. Suppose the j-th highest deficit school in $\hat{h}$ belongs to D_k^+. Let $|\overline{D}_{k-1}^+| \equiv \ell_{k-1}$ and $|\overline{D}_k^+| \equiv \ell_k$ [13], so that $|D_k^+| = \ell_k - \ell_{k-1}$. From Case A, the Lorenz domination inequality holds for ℓ_{k-1} and ℓ_k:

$$\sum_{i=1}^{\ell_{k-1}} h_{[i]} \geq \sum_{i=1}^{\ell_{k-1}} \hat{h}_{[i]} = \hat{h}(\overline{D}_{k-1}^+) \tag{13}$$

$$\sum_{i=1}^{\ell_k} h_{[i]} \geq \sum_{i=1}^{\ell_k} \hat{h}_{[i]} = \hat{h}(\overline{D}_k^+). \tag{14}$$

Since $|D_k^+| = \ell_k - \ell_{k-1}$, the deficit vector $\hat{h}$ restricted to D_k^+ is an $(\ell_k - \ell_{k-1})$-dimensional vector of integers that are either the floor or the ceiling of deficit vectors in $h^\star$ restricted to

[13] If $k = 1$, let $\overline{D}_{k-1}^+ = \emptyset$ and $\ell_{k-1} = 0$.

D_k^+. Denote the floor and ceiling of $h^\star$ vector in D_k^+ by $\overline{h}^\star$ and $\underline{h}^\star$ respectively. It follows that $\hat{h}$ restricted to D_k^+ is a vector of the following form:

$$(\underbrace{\overline{h}^\star, \ldots, \overline{h}^\star}, \underbrace{\underline{h}^\star, \ldots, \underline{h}^\star}).$$

Figure 4 shows the distribution of the deficit vectors $\hat{h}$ and h for schools in D_k^+ in an illustrative case. Let D_k^+ contain 10 schools. The vectors $\hat{h}$ and h (restricted to D_k^+) are ordered from high to low. Both $\hat{h}$ and h are integral, so that a (deficit, school) pair in each vector is a point on the grid in the figure. The vector $\hat{h}$ either takes the floor $\underline{h}^\star$ or the ceiling $\overline{h}^\star$ value and is represented by RED squares. The seven schools with highest deficits have value $\overline{h}^\star$ while the remaining three schools have value $\underline{h}^\star$. The vector h is represented by BLUE crosses. The five highest deficits in h are at least as high as $\overline{h}^\star$, the next two coincide with $\underline{h}^\star$ and the remaining three are below $\underline{h}^\star$. A crucial observation is the following: since $\hat{h}$ is integral and $\overline{h}^\star$ and $\underline{h}^\star$ are consecutive integers, h values are less than or equal to $\underline{h}^\star$ from the sixth school onwards. In other words, $\hat{h}$ and h satisfy a *single-crossing* property.[14] This allows us to break the proof of Case B into two parts. The first part considers j where the all schools upto the j-th highest school have deficit values in h greater than or equal to $\overline{h}^\star$. In the figure, this corresponds to $j \leq 5$. The second part considers the case where the deficit of the j-th highest school in h is $\underline{h}^\star$ or lower, i.e. $j \geq 6$ in the figure.

Pick any $j \in \{\ell_{k-1}+1, \ell_{k-1}+2, \ldots, \ell_k\}$. If $h_{[j]} \geq \overline{h}^\star$, then $h_{[\ell_{k-1}+1]} \geq \ldots \geq h_{[j]} \geq \overline{h}^\star$ implies

$$\sum_{i=\ell_{k-1}+1}^{j} h_{[i]} \geq \sum_{i=\ell_{k-1}+1}^{j} \overline{h}^\star \geq \sum_{i=\ell_{k-1}+1}^{j} \hat{h}_{[i]},$$

where the second inequality uses the fact that $\hat{h}_i \leq \overline{h}^\star$ for all $i \in \{\ell_{k-1}+1, \ldots, \ell_k\}$. Now, using inequality (13) and adding $\sum_{i=1}^{\ell_{k-1}} h_{[i]}$ on the LHS and $\hat{h}(\overline{D}_{k-1}^+)$ on the RHS, we get

$$\sum_{i=1}^{j} h_{[i]} \geq \sum_{i=1}^{j} \hat{h}_{[i]}.$$

[14]The single-crossing property is trivially achieved in the continuous version, and exploited in Claim 1 due to Hardy et al. (1929).

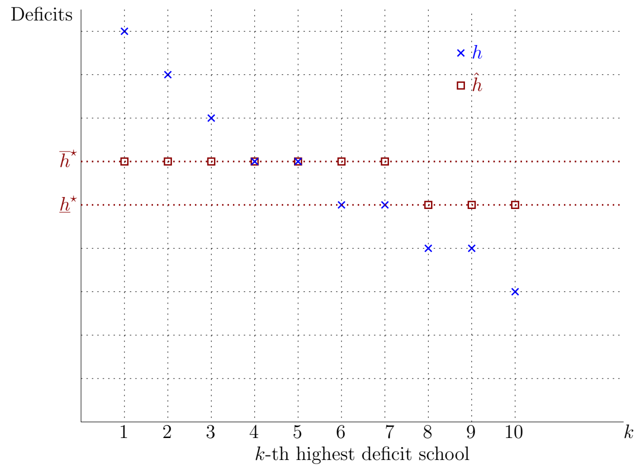

Figure 4: Single crossing of ordered integral vectors $\hat{h}$ and h.

Suppose $h_{[j]} < \overline{h}^\star$. Since $h_{[j]}$ is an integer and $\overline{h}^\star = \underline{h}^\star + 1$, we have $h_{[j]} \leq \overline{h}^\star - 1 = \underline{h}^\star$. But then, $\underline{h}^\star \geq h_{[j]} \geq h_{[j+1]} \geq \ldots \geq h_{[\ell_k]}$. Hence,

$$\sum_{i=1}^{j} h_{[i]} = \sum_{i=1}^{\ell_k} h_{[i]} - \sum_{i=j+1}^{\ell_k} h_{[i]} \geq_{(a)} \hat{h}(\overline{D}_k^+) - \sum_{i=j+1}^{\ell_k} h_{[i]} \geq_{(b)} \hat{h}(\overline{D}_k^+) - \sum_{i=j+1}^{\ell_k} \underline{h}^\star$$
$$\geq_{(c)} \hat{h}(\overline{D}_k^+) - \sum_{i=j+1}^{\ell_k} \hat{h}_i =_{(d)} \sum_{i=1}^{j} \hat{h}_i.$$

Inequality (a) is due to (14). Inequality (b) follows because $\underline{h}^\star \geq h_{[j]}$. Inequality (c) follows because $\hat{h}_i \geq \underline{h}^\star$ for all $i \in [1, L]$. Equality (d) uses the definition of $\hat{h}$.

This completes the proof of Proposition 3 and Theorem 1. ■

Remark 1 We briefly comment on the computational complexity of finding the Lorenz optimal transfer. This transfer can be computed efficiently (i.e., polynomial time in the size of the input) using well-known combinatorial algorithms to compute a maximum-flow in a network. Recall that our two-stage procedure relies on (i) finding the Lorenz optimal fractional assignment in the initial network; and on (ii) finding a maximum-flow in an auxiliary network that is constructed using the knowledge of the Lorenz optimal fractional assignment. Megiddo (1977) describes an efficient algorithm for finding a lexicographically optimal flow in a network; this algorithm, in fact, computes a Lorenz optimal flow because a Lorenz optimal flow *is* lexicographically optimal.[15]

[15]The problem of finding a lex-optimal flow in a network is fundamental and has received a lot of attention

The auxiliary network can be constructed efficiently once a Lorenz optimal fractional assignment is known. Moreover, an integer Lorenz optimal assignment can be found by finding an integer max-flow in the auxiliary network that has upper and lower bounds on arc-flows, and this can be computed using any efficient combinatorial algorithm for finding a max-flow such as the Edmonds-Karp algorithm (Edmonds and Karp, 1972).

4 Strategy-proofness

In this section, we consider the problem of a planner who can observe the surpluses and deficits of schools but not the acceptable schools of teachers. We show that the planner can nevertheless achieve the Lorenz dominant post-transfer deficit vector by incentivizing teachers to report their acceptable schools truthfully.

We assume that every teacher has "trichotomous" preferences: the top indifference class consists of her acceptable deficit schools, followed by the surplus school she is assigned to, and lastly, the set of non-acceptable deficit schools. In particular, each teacher t_i is indifferent between acceptable deficit schools in $A(t_i)$ and strictly prefers being transferred to a school in $A(t_i)$ than remaining in her initial assignment. Her initial assignment is strictly preferred to any school in $D \setminus A(t_i)$. As mentioned in the literature review, variants of such preferences have been used in Manjunath and Westkamp (2021) and Andersson et al. (2021).

Let $A \equiv (A_1, \dots, A_{|T|})$ denote a profile of acceptable sets (or schools). Let $\Sigma(A)$ denote the set of all transfers that result in the Lorenz dominant deficit vector. Although $\Sigma(A)$ is non-empty (Theorem 1), it can contain multiple transfers. For instance, in the example in Figure 1, the Lorenz dominant transfer $(1, 1, 2)$ can be achieved by three transfers. In all of them no teacher is transferred to school d_1. In the first, teachers t_1 and t_2 are transferred to d_2; in the second, teachers t_1 and t_3 are transferred to d_2; in the third, teacher t_1 is transferred to d_2 and teacher t_3 is transferred to d_3. Suppose that for every profile A, a transfer is selected from $\Sigma(A)$ according to a *fixed tie-breaking rule*. We show in Proposition 4 that this mechanism is strategy-proof.

in the network flow literature. Since Megiddo (1977), several authors have identified faster algorithms for solving this problem, culminating in the work of Gallo et al. (1989), who describe how to pipeline various max-flow computations to find a lex-optimal flow in time that is asymptotically the same as finding a single max-flow.

A *matching* $\mu : T \to D$ is a mapping from set of teachers to deficit schools. A matching does not take into account acceptable schools of teachers or surplus constraints of surplus schools. A matching therefore differs from a transfer. Let $\mathcal{M}$ be the set of all possible matchings and let $\rhd$ be an arbitrary strict ordering of the elements of $\mathcal{M}$. Let $\max_{\rhd}(\Sigma(A))$ denote the maximal transfer in $\Sigma(A)$ according to $\rhd$. A **Lorenz dominant transfer mechanism (supplemented by tie-breaking $\rhd$)**, $\text{LDT}_{\rhd}$, selects the transfer $\text{LDT}_{\rhd}(A) := \max_{\rhd}(\Sigma(A))$ at every profile A. We denote the deficit school assigned to teacher $t_j \in T$ at profile A by $\text{LDT}_{\rhd}(t_j, A)$. Note that $\text{LDT}_{\rhd}(t_j, A) \in A(t_j) \cup \{O(t_j)\}$ since our algorithm does not transfer any teacher to a non-acceptable school (according to reported preferences).

The Lorenz dominant mechanism $\text{LDT}_{\rhd}(A)$ is **strategy-proof** if for every teacher $t_j \in T$, every $A(t_j) \subseteq D$ and every $A(t_{-j})$ the following holds:

$$\Big[\text{LDT}_{\rhd}(t_j, A(t_j), A(t_{-j})) = O(t_j)\Big] \implies \Big[\text{LDT}_{\rhd}(t_j, B, A(t_{-j})) \notin A(t_j) \qquad \forall\ B \subseteq D\Big].$$

Strategy-proofness requires the following: if a teacher is not transferred (remains in her initial surplus school) by reporting her true preference, then she *cannot* be transferred to an acceptable school by reporting any other set of acceptable schools. It must be emphasized that a change in a teacher's report of acceptable schools affects the set of feasible transfers, which in turn has a bearing on achievable deficit vectors.

Proposition 4 *The Lorenz dominant transfer mechanism* $\text{LDT}_{\rhd}$ *is strategy-proof.*

Proof: Pick $t_j \in T$ and a profile A such that $\text{LDT}_{\rhd}(t_j, A) = O(t_j)$. Let $B \subseteq D$ be any non-empty subset of deficit schools. Assume for contradiction that $\text{LDT}_{\rhd}(t_j, (B, A(t_{-j}))) \in A(t_j)$. Since the acceptable sets of other teachers do not change from profile A to profile $(B, A(t_{-j}))$ and $\text{LDT}_{\rhd}(t_j, B, A(t_{-j})) \in A(t_j)$, it must be the case that $\text{LDT}_{\rhd}(B, A(t_{-j}))$ is a feasible transfer in profile A. Similarly, since $\text{LDT}_{\rhd}(t_j, A) = O(t_j)$, transfer $\text{LDT}_{\rhd}(t_j, A)$ is a feasible transfer in profile $(B, A(t_{-j}))$. Hence, the deficit vectors generated by $\text{LDT}_{\rhd}(t_j, (B, A(t_{-j})))$ and $\text{LDT}_{\rhd}(t_j, A)$ Lorenz dominate each other, implying that both are Lorenz dominant transfers at both profiles. But, according to the definition of $\text{LDT}_{\rhd}$ we cannot pick $\text{LDT}_{\rhd}(t_j, B, A(t_{-j}))$ at $(B, A(t_{-j}))$ when $\text{LDT}_{\rhd}(t_j, A)$ is available. We have arrived at a contradiction. ■

Remark 2 There is a more general way of making strategy-proof selections from the set $\Sigma(A)$. One could use a *choice function* that picks a transfer from every feasible subset of transfers. If the choice function satisfies the property of *contraction consistency*, (see Osborne and Rubinstein (1994)) an argument similar to that of the proof of Proposition 4 can be used to show that the mechanism defined by $\Sigma(A)$ and the choice function, is strategy-proof. [16]

5 Extensions

In this section, we explore two extensions of our basic model.

5.1 Allowing for transfers from surplus schools to surplus schools

In our analysis so far, a teacher can only move from a surplus school (her initial assignment) to a deficit school. In this section, we relax this assumption. We consider an extension of our earlier model where a teacher can move from a surplus school (her initial assignment) to either a deficit school or a surplus school. Each teacher t_i has a set of acceptable schools $A(t_i) \subseteq D \cup S \setminus \{O(t_i)\}$. This set consists of all deficit and surplus schools that she would like to be transferred to. The other assumptions remain unchanged, i.e. no surplus school can become a deficit school post-transfer and no deficit school can become a surplus school post-transfer. Our goal is to show that our earlier analysis can be easily modified to cover the extended model.

We begin by describing the network flow formulation for the extended model. This requires making some minor modifications to the original network described in Section 3. In particular, the set of nodes and the edges in the original network remain as before. However, some "backward edges" are added to the original network.

Recall that the set of nodes N in the graph consists of source N_0, the set of surplus schools S, the set of teachers T and the set of deficit schools D, where D is the set of sinks. A directed capacity graph G with edges E is constructed as follows. There is an edge between the source node N_0 and each surplus school $s_j \in S$. There is an edge between each surplus school s_j and each teacher t_i such that $O(t_i) = s_j$. There is an edge from teacher t_i to every

[16]The use of choice function to make strategy-proof selections appears in Nicolò et al. (2022). The greater generality of this approach over the fixed tie-breaking one, leads to a characterization result in their model (in conjunction with other axioms).

acceptable school $A(t_i)$. The capacity of an edge (N_0, s_j) is α_j. The capacity of an edge (s_j, t_i), where $O(t_i) = s_j$, is 1. Similarly the capacity of an edge (t_i, d_k) where $d_k \in A(t_i)$, is 1. The capacity of an edge (t_i, s_j) where $s_j \in A(t_i)$, is 1. Each deficit school d_k has a node capacity β_k. We let G' denote this network. Notice the modified network is exactly like the earlier network G except that the definition of acceptable schools has changed.

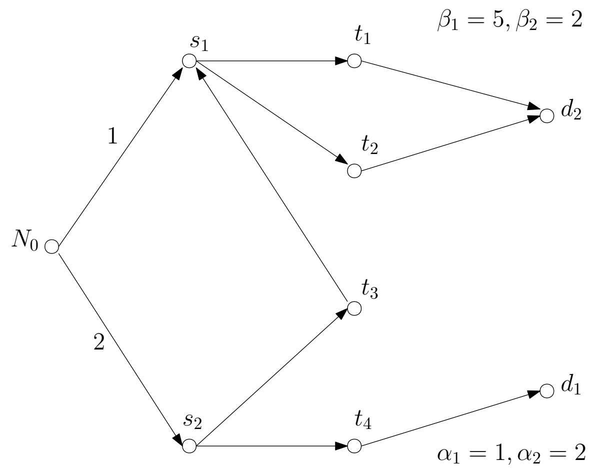

Figure 5: An example where teachers can have surplus schools acceptable

We give an example to illustrate the modified network. In this example, the sets of surplus schools, deficit schools and teachers are $S = \{s_1, s_2\}$, $D = \{d_1, d_2\}$ and $T = \{t_1, t_2, t_3, t_4\}$ respectively. The initial assignment of teachers is given by $O(t_1) = O(t_2) = s_1$ and $O(t_3) = O(t_4) = s_2$. The acceptable sets are as follows: $A(t_1) = A(t_2) = \{d_2\}$, $A(t_3) = \{s_1\}$ and $A(t_4) = \{d_1\}$. The current deficit of schools d_1 and d_2 are $\beta_1 = 5$ and $\beta_2 = 2$. The surplus of schools s_1 and s_2 are $\alpha_1 = 1$ and $\alpha_2 = 2$. The modified network is shown in Figure 5.

In the original model where teachers can only be transferred to deficit schools, teacher t_3 is *not* included in the network as she does not find any deficit school acceptable. The Lorenz dominant deficit vector is $(4, 1)$ achieved by transferring one teacher, say t_1 from s_1 to d_2 and teacher t_4 from s_2 to d_1.

In the extended model, teachers can be transferred either to deficit schools or to surplus schools provided they are acceptable. Teacher t_3 is included in the network with a forward edge (s_2, t_3) and a backward edge (t_3, s_1) respectively (see Figure 5). The Lorenz dominant deficit vector is $(3, 1)$. This can be achieved by the following transfer: transfer t_4 to deficit school d_1, transfer t_3 to surplus school s_1 and transfer t_1, t_2 to deficit school d_2. It is now possible to transfer both teachers t_1, t_2 from surplus school s_1 since teacher t_3 is transferred to surplus school s_1. This ensures that surplus school s_1 does not become a deficit school post-transfer.

A crucial observation is that the achievability result in Megiddo (1974) continues to hold for the new network G'. As a result, we can define an integer- valued, convex cooperative game w for the new network as well. The "reduced form" of the extended problem is identical to that of the original problem. Consequently, Theorem 1 and Proposition 4 continue to hold for the extended model and the proofs remain unchanged.

5.2 Teacher specializations

In the model so far, we have assumed that all teachers are homogeneous in their characteristics. In this extension, we consider a model where teachers have types depending on the subjects they can teach. If every teacher is qualified to teach only a single subject, a separate network can be constructed for each subject and our earlier analysis applies to each subject. In this subsection, we consider the case where a teacher may be qualified to teach several subjects but teaches only one subject in her assigned school. The goal of the planner is to transfer teachers respecting subject-wise surpluses and deficits, to equalize deficits across (school, subject) tuples. We show how our earlier analysis can be modified to cover this case.

To simplify notation, we assume that there are only two subjects: Chemistry (C) and Physics (P). The set of teachers is T and each $t_i \in T$ has a type $\theta(t_i)$ that indicates which subjects she is qualified to teach. Here $\theta(t_i) \in \{C, P, CP\}$ where C (or P) signifies that t_i is qualified to teach only Chemistry (or only Physics) while CP signifies that t_i is qualified to teach both subjects.

There are three types of schools: *pure surplus schools* S that have surplus teachers in both subjects, *pure deficit schools* D that have deficit of teachers in both subjects and *mixed schools* M that have a surplus in one subject and a deficit in the other. For each $s_j \in S$, α_j^C and α_j^P denote the surpluses of s_j in C and P respectively. For each $d_k \in D$, β_k^C and β_k^P denotes the deficits of d_k in C and P by respectively. For all $m_\ell \in M$, α_ℓ^x denotes the surplus of m_ℓ in subject x and β_ℓ^y is the deficit of m_ℓ in subject y. We assume all surpluses and deficits are strictly positive integers.

Each $t_i \in T$ belongs to a unique school and teaches exactly one of the subjects she is qualified to teach in the school. Let $O(t_i)$ and $a(t_i)$ denote the school t_i belongs to and the subject she teaches there, respectively. We make some natural assumptions regarding the deployment of teacher t_i of type CP. If $O(t_i) = d_k \in D$, then t_i teaches the subject with the higher deficit in d_k; if $O(t_i) = m_\ell \in M$, then t_i teaches the subject for which m_ℓ has a deficit. This assumption ensures that deficits cannot be equalized by transfers *within* the same school.

In keeping with our earlier model, we assume that all teachers in S can be transferred while no teacher in D can be transferred. In school $m_\ell \in M$, the only teachers who can be transferred are those of type x where $\alpha_\ell^x > 0$.[17] We assume (i) no school with a surplus in a subject can have a deficit in that subject post transfer (ii) no school with a deficit in a subject can have a surplus in that subject post transfer and (iii) no surplus can grow larger, and no deficit can grow larger post transfer.

Let T' be the set of teachers who can be transferred. Each $t_i \in T'$ has a set of alternative schools that are acceptable to them: $A(t_i) \subseteq D \cup M$. A network G'' is defined as follows.

- The set of nodes in the graph consists of:
  1. A source node N_0.
  2. Two nodes for each $s_j \in S$: (s_j, P) and (s_j, C).
  3. Two nodes for each $d_k \in D$: (d_k, P) and (d_k, C).
  4. Two nodes for each $m_\ell \in M$ that has a surplus in subject x and deficit in subject y (where x, y are distinct and $x, y \in \{C, P\}$): (m_ℓ, x) and (m_ℓ, y).
  5. A node for every teacher in T'.

  The nodes for pure deficit schools and the deficit "subject" nodes for mixed schools are *sink* nodes.

- The set of edges E and their capacities are as follows:
  1. Two directed edges $(N_0, (s_j, P))$ and $(N_0, (s_j, C))$ for every pure surplus school $s_j \in S$ with capacities α_j^C and α_j^P respectively.

[17] Recall that all teachers of type CP in m_ℓ are teaching subject y where $\beta_\ell^y > 0$.

2. A directed edge $(N_0, (m_\ell, x))$ (where $x \in \{C, P\}$ is the subject in which school m_ℓ has a surplus) for every $m_\ell \in S$ with capacity α_ℓ^x.

3. For every $t_i \in T'$ with $O(t_i) = s_j \in S$, one of the following holds: a directed edge $((s_j, P), t_i))$ with capacity 1 if $\theta(t_i) = P$, or a directed edge $((s_j, C), t_i))$ with capacity 1 if $\theta(t_i) = C$, or a directed edge $((s_j, a(t_i)), t_i)$ with capacity 1 if $\theta(t_i) = CP$ and t_i teachers $a(t_i)$ in school s_j.

4. For every $t_i \in T'$ with $O(t_i) = m_\ell \in M$ (where m_ℓ has a surplus in subject x), we have a directed edge $((m_\ell, x), t_i)$ with capacity 1.

5. For every $t_i \in T'$ with $O(t_i) \in S$, we have
   - for every pure deficit school $d_k \in A(t_i) \cap D$, one of the following holds: a directed edge $(t_i, (d_k, C))$ with capacity 1 if $\theta(t_i) = C$, or a directed edge $(t_i, (d_k, P))$ with capacity 1 if $\theta(t_i) = P$, or two directed edges $(t_i, (d_k, C))$ and $(t_i, (d_k, P))$ with capacities 1 if $\theta(t_i) = CP$.
   - for every mixed school $m_\ell \in A(t_i) \cap D$ with deficit in subject y (sink node (m_ℓ, y)), a directed edge $(t_i, (m_\ell, y))$ with capacity 1 if either $\theta(t_i) = y$ or $\theta(t_i) = CP$.

6. For every $t_i \in T'$ with $O(t_i) = m_\ell \in M$ and mixed school m_ℓ has a surplus in subject x, we have a directed edge $(t_i, (\gamma, x))$ with capacity 1 for every $\gamma \in A(t_i)$.[18]

- For every pure deficit school $d_k \in D$, the two associated sink nodes (d_k, C) and (d_k, P) have capacities β_k^C and β_k^P respectively.

- For every mixed school $m_\ell \in M$ with a deficit in subject $y \in \{C, P\}$, the associated sink node (m_ℓ, y) has a capacity β_ℓ^y.

Like our earlier analysis, we are interested in teacher reassignments that would result in a Lorenz dominant post-transfer deficit vector.

It is routine to check that the achievability result in Megiddo (1974) holds for the network G''. As before, we can define an integer-valued convex cooperative game for this new network

[18]Since teacher t_i belongs to a mixed school with surplus in subject x and is included in the transfer process, we know by definition that type of teacher t_i is x. Thus there is a directed edge between t_i and sink node (γ, x) for every school γ in her acceptable set.

as well. Theorem 1 and Proposition 4 applied to this network establishes the existence of a Lorenz dominant integer transfer, which can be found using the MDR algorithm discussed earlier.

6 Other related literature

Our paper is partly inspired by empirical work on teacher transfers in the state of Haryana, especially Agarwal et al. (2018) and Sharan et al. (2022). These papers look at data in Haryana and show the improvement of deficits in deficit schools as a result of transfers. Agarwal et al. (2018) show that even local transfers (transfers up to a certain distance from the currently assigned school) leads to significant improvements in deficits. Our work is an attempt to address the issue of deficit reduction from a theoretical perspective.

There is a recent theoretical literature on teacher transfers that is motivated by concerns different from ours. We comment briefly on some of these papers.

Combe et al. (2022) study a model of teacher reassignment in the French school system. They consider a two-sided model where teachers and schools have preferences and characterize reassignments that are "maximally efficient and fair". Unlike our model where teachers have to be transferred from surplus to deficit schools, a teacher in their model can be transferred from a school only if another teacher is transferred to that school. The algorithm they propose is a variant of the top-trading cycle that does not change the initial distribution of teachers. Indeed, improvements in the distribution of teachers is not a concern in the paper.

Combe et al. (2025) study an extension of Combe et al. (2022) where each teacher has a type (from an ordered set of types) and each school is initially assigned a set of teachers. This creates a distribution of types at each school. They propose a family of indices to measure inequality of "teacher quality" across schools. They define a generalization of the top-trading cycle mechanism which is strategy-proof, "constrained" Pareto efficient, and improves inequality from the status-quo in large markets. As in Combe et al. (2022), the initial distribution of teachers across schools remains unchanged post transfers although the distribution of teacher types in each schools can change.

Hafalir et al. (2025b) study a similar model in the two-sided matching market and characterize different choice rules used by one side of the market that results in different distributional outcomes. In a related paper, Hafalir et al. (2025a) show that improving distributional objectives (for a family of such objectives) is incompatible with strategy-proofness, individual rationality and constrained efficiency. They introduce a new distributional objective called, "pseudo $M^\natural$-concavity" (generalizing a well-known notion of discrete concavity called $M^\natural$-concavity), that is compatible with these axioms. They also propose a mechanism that satisfies these properties. Echenique et al. (2025) compare diversity in a school choice model using majorization technqiues. They axiomatically characterize choice rules that are consistent with a modified majorization notion which they introduce for school choice problems. In the standard allocation model with unit demand, Pycia and Unver (2014) show that a particular variant of the top trading cycle Lorenz dominates the traditional top trading cycle, in terms of agent utilities.

Dur and Ünver (2019) consider a model where students have already been admitted to a set of colleges. Students have to be transferred across colleges with constraints on "imports" and "exports" . In particular, they require that these exchanges are *balanced*, implying that exports equal imports at each college. Students have strict preferences over colleges and colleges have priorities over students. They introduce the "two-sided top-trading-cycles" mechanism and show that it is the unique mechanism that is priority respecting, balanced-efficient, student-strategy-proof, acceptable and individually rational. It is clear that our model is different from theirs. Importantly, there are also no balancing constraints in our model.

There is a recent literature on matching with constraints: see Kamada and Kojima (2015, 2024) and references therein. This literature considers the two-sided matching (school choice) problem, where teachers have preferences (strict orders) over schools and schools have priorities over students. In addition to these, there are distributional constraints, which are modelled as restrictions on subsets of students that a school can admit. The objectives in these papers vary: it is *fairness* (no justified envy) in Kamada and Kojima (2024); while it is efficiency in Kamada and Kojima (2015). In contrast, our objective is to achieve a Lorenz dominant post-transfer deficit vector assuming teachers preferences are restricted. We note that our restrictions on transfers from surplus to deficit schools cannot be modelled as constraints on matching as in these papers.

Afacan et al. (2024) considers the school choice problem with the option of redistributing additional seats. In their model, schools can redistribute seats between themselves to improve efficiency based on the reported preferences of the students and given the priorities of schools. They introduce a simple class of algorithms that characterizes the set of efficient matchings in their model.

We model teacher preferences as being either dichotomous or trichotomous. Preferences of this nature appear frequently in the matching literature. We have already noted their use in Bogomolnaia and Moulin (2004). Manjunath and Westkamp (2021) analyze an "object reallocation" model where agents are endowed with bundles of objects and their preferences over bundles of objects satisfy a trichotomous structure. They define a class of individually rational, Pareto-efficient, and strategy-proof mechanisms for this model. Andersson et al. (2021) study a model of exchange of goods, where each agent can receive goods from other agents. The preference of an agent over the bundles of goods received are dichotomous in a particular way: each agent partitions the set of agents as acceptable and unacceptable, and bundles of goods from acceptable agents (satisfying some upper bound on quantity) are preferred over other bundles. They define a class of mechanisms called priority mechanisms and show them to be strategy-proof. Nicolo et al. (2019) study a model where agents need to be matched in pairs to complete a project. Each agent has preferences over partner, project pairs that are separable. Marginal preferences over partners and projects are dichotomous. In other words, each agent has an acceptable set of partners and an acceptable set of projects. They propose a mechanism in this setting and show that it is weakly stable and strategy-proof. We note that none of these papers are concerned with the issues that we are.

References

Afacan, M. O., U. Dur, and M. Van der Linden (2024): "Capacity design in school choice," *Games and Economic Behavior*, 146, 277–291. 6

Agarwal, S., A. Kayina, A. Mukhopadhyay, and A. N. Reddy (2018): "Redistributing teachers using local transfers," *World Development*, 110, 333–344. 2, 6

Ahuja, R. K., T. L. Magnanti, and J. B. Orlin (1988): *Network flows*, Prentice Hall. 8, 3.6

Andersson, T., A. Cseh, L. Ehlers, and A. Erlanson (2021): "Organizing time exchanges: Lessons from matching markets," *American Economic Journal: Microeconomics*, 13, 338–373. 4, 6

Bochet, O., R. Ilkiliç, H. Moulin, and J. Sethuraman (2012): "Balancing supply and demand under bilateral constraints," *Theoretical Economics*, 7, 395–423. 1, 2.1

Bogomolnaia, A. and H. Moulin (2004): "Random matching under dichotomous preferences," *Econometrica*, 72, 257–279. 1, 2.1, 6

Combe, J., U. Dur, O. Tercieux, C. Terrier, and U. Unver (2025): "Market design for redistributional objectives in (re)assignment: An application to improve the distribution of teachers in schools," Working Paper. 6

Combe, J., O. Tercieux, and C. Terrier (2022): "The design of teacher assignment: Theory and evidence," *Review of Economic Studies*. 2.1, 6

Dur, U. M. and M. U. Ünver (2019): "Two-sided matching via balanced exchange," *Journal of Political Economy*, 127, 1156–1177. 6

Dutta, B. and D. Ray (1989): "A concept of egalitarianism under participation constraints," *Econometrica*, 615–635. 1, 3, 3.4, 2, 3

Echenique, F., T. Mekonnen, and B. M. Yenmez (2025): "Diversity in Choice as Majorization," Working Paper. 6

Edmonds, J. and R. M. Karp (1972): "Theoretical Improvements in Algorithmic Efficiency for Network Flow Problems," *J. ACM*, 19, 248?264. 1

Gallo, G., M. D. Grigoriadis, and R. E. Tarjan (1989): "A Fast Parametric Maximum Flow Algorithm and Applications," *SIAM J. Comput.*, 18, 30–55. 15

Government of Haryana (2016): "Teacher Transfer Policy 2016," https://cdnbbsr.s3waas.gov.in/s3cf2226ddd41b1a2d0ae51dab54d32c36/uploads/2021/05/2021053128.pdf, accessed: 2024-15-08. 1, 5

——— (2023): "Teacher Transfer Policy 2023," https://cdnbbsr.s3waas.gov.in/s3cf2226ddd41b1a2d0ae51dab54d32c36/uploads/2023/08/20230828327116533.pdf, accessed: 2024-15-08. 1

Hafalir, I., F. Kojima, and B. M. Yenmez (2025a): "Market Design wtih Distributional Objectives: Efficiency, Incentives, and Property Rights," Working Paper. 6

Hafalir, I., F. Kojima, B. M. Yenmez, and K. Yokote (2025b): "Market Design wtih Distributional Objectives," Working Paper. 6

Hardy, G. H., J. E. Littlewood, and G. Poĺya (1929): "Some simple inequalities satisfied by convex functions," *Messenger Math.*, 58, 145–152. 1, 14, 1, A

Hardy, G. H., J. E. Littlewood, and G. Pólya (1952): *Inequalities*, Cambridge university press. 2.1

Kamada, Y. and F. Kojima (2015): "Efficient matching under distributional constraints: Theory and applications," *American Economic Review*, 105, 67–99. 6

——— (2024): "Fair matching under constraints: Theory and applications," *Review of Economic Studies*, 91, 1162–1199. 6

Manjunath, V. and A. Westkamp (2021): "Strategy-proof exchange under trichotomous preferences," *Journal of Economic Theory*, 193, 105197. 4, 6

Marshall, A., I. Olkin, and B. C. Arnold (1979): *Inequalities: Theory of Majorization and Its Applications*, Academic Press, Inc. 2.1, 2.1

Megiddo, N. (1974): "Optimal flows in networks with multiple sources and sinks," *Mathematical Programming*, 7, 97–107. 1, 3, 3.3, 1, 3.3, 11, 5.1, 5.2, 2, 3

——— (1977): "A good algorithm for lexicographically optimal flows in multi-terminal networks," *Bulletin of the American Mathematical Society*, 83, 407 – 409. 1, 15

Nicolò, A., P. Salmaso, A. Sen, and S. Yadav (2022): "Fostering collaborations in matching platforms," Working Paper. 16

Nicolo, A., A. Sen, and S. Yadav (2019): "Matching with partners and projects," *Journal of Economic Theory*, 184, 104942. 6

Osborne, M. J. and A. Rubinstein (1994): *A course in game theory*, MIT Press. 2

Pycia and Unver (2014): "Incentive Compatible Allocation and Exchange of Discrete Resources," http://pycia.bol.ucla.edu/pycia-unver-tcbo.pdf, accessed: 2025-21-10. 6

Shapley, L. S. (1971): "Cores of convex games," *International journal of game theory*, 1, 11–26. 1

Sharan, A., N. Jain, P. Vashistha, and P. Singh (2022): "The Teachers? Transfer Policy, Haryana (2016): A Critical Evaluation," *Indian Journal of Public Administration*, 68, 202–218. 2, 6

A Proof of Lorenz domination of the MDR algorithm

Theorem 2 (Dutta and Ray (1989), Megiddo (1974)) *The MDR algorithm produces a Lorenz dominant transfer.*

Remark 3 Note that Dutta and Ray (1989) show that the MDR algorithm produces a deficit vector that Lorenz dominates every achievable deficit vector corresponding to maximum flow.[19] It is not clear that if a such deficit vector will Lorenz dominate achievable deficit vectors that *do not* correspond to a maximum flow. The current proof provides this minor extension. Megiddo (1974) did not consider Lorenz domination but *lex-optimality*.

The proof of Theorem 2 uses the following claim.

Claim 1 (Hardy et al. (1929)) *Let $a \equiv (a_1, \ldots, a_\ell)$ be a vector and $a^{\text{avg}} := \frac{a_1+\ldots+a_\ell}{\ell}$. Then, $(a^{\text{avg}}, \ldots, a^{\text{avg}})$ Lorenz dominates a.*

We give a proof for completeness here though it is a straightforward consequence of Fact 1, which is due to Hardy et al. (1929).

[19] In their terminology, the provide a Lorenz dominant core point of a convex game. Since core constraints are binding for the grand coalition and the achievability constraints are not necessarily binding, we cannot directly apply their result.

Proof: Without loss of generality, assume $a_1 \geq \ldots \geq a_\ell$. Pick $k \in \{1, \ldots, \ell\}$. Suppose k is such that $a_{k-1} \geq a^{\text{avg}} \geq a_k$. Clearly, for any $j \leq k-1$, $\sum_{i=1}^{j} a_i \geq j a^{\text{avg}}$. Now, for any $j \geq k$, we see that

$$\sum_{i=1}^{j} a_i = \ell a^{\text{avg}} - \sum_{i=j+1}^{\ell} a_i \geq \ell a^{\text{avg}} - (\ell - j) a^{\text{avg}} = j a^{\text{avg}},$$

where the inequality follows because $a^{\text{avg}} \geq a_i$ for all $i \geq k$. This completes the proof. ■

Proof of Theorem 2:

Proof: We prove a series of claims. First, we show how $h^\star$ is ordered. Clearly, if $d_i, d_j \in D_k$ for some k, then $h_i^\star = h_j^\star$.

Claim 2 *Suppose $d_i \in D_k$ and $d_j \in D_{k+1}$. Then $h_i^\star \geq h_j^\star$.*

Proof: Consider Step k of the MDR algorithm where D_k is chosen as the argmax and $D_k \cup D_{k+1}$ is a feasible option. Then by definition,

$$\begin{aligned}
\frac{w(\overline{D}_k) - w(\overline{D}_{k-1})}{|D_k|} &\geq \frac{w(\overline{D}_{k+1}) - w(\overline{D}_{k-1})}{|D_k| + |D_{k+1}|} \\
\implies |D_{k+1}|\Big(w(\overline{D}_k) - w(\overline{D}_{k-1})\Big) &\geq |D_k|\Big(w(\overline{D}_{k+1}) - w(\overline{D}_k)\Big) \\
\implies h_i^\star = \frac{w(\overline{D}_k) - w(\overline{D}_{k-1})}{|D_k|} &\geq \frac{w(\overline{D}_{k+1}) - w(\overline{D}_k)}{|D_{k+1}|} = h_j^\star
\end{aligned}$$

■

The next claim says $h^\star$ is achievable.

Claim 3 *The deficit vector $h^\star$ is achievable.*

Proof: By construction, $\sum_{d_i \in D} h_i^\star = w(D)$. Hence, pick any $B \subsetneq D$. Partition B into $(B_1, \ldots, B_{k^\star})$ such that $B_1 \subseteq D_1, B_2 \subseteq D_2, \ldots, B_{k^\star} \subseteq D_{k^\star}$. Denote $|B_j| \equiv \ell_j$ for each

$j \in \{1, \ldots, k^\star\}$ and note that some ℓ_j can be equal to zero also. Then,

$$\begin{aligned}
\sum_{d_i \in B} h_i^\star = \sum_{j=1}^{k^\star} \sum_{d_i \in B_j} h_i^\star &= \sum_{j=1:\ell_j \neq 0}^{k^\star} \ell_j \frac{w(\overline{D}_j) - w(\overline{D}_{j-1})}{|D_j|} \\
&\geq_{(a)} \sum_{j=1}^{k^\star} \ell_j \frac{w(\overline{D}_{j-1} \cup B_j) - w(\overline{D}_{j-1})}{\ell_j} \\
&\geq_{(b)} \sum_{j=1}^{k^\star} \Big[w(\overline{B}_j) - w(\overline{B}_{j-1}) \Big] \qquad (\overline{B}_j = \cup_{i=1}^{j} B_i \text{ and } \overline{B}_{j-1} = \cup_{i=1}^{j-1} B_i) \\
&= w(B)
\end{aligned}$$

where the first inequality (a) is by construction of D_j:

$$\frac{w(\overline{D}_j) - w(\overline{D}_{j-1})}{|D_j|} \geq \frac{w(\overline{D}_{j-1} \cup B) - w(\overline{D}_{j-1})}{|B|} \qquad \forall\ B \subseteq D \setminus \overline{D}_{j-1}$$

and the second inequality (b) is due to convexity. This shows that $h^\star$ is achievable. ■

Claim 4 *The deficit vector $h^\star$ Lorenz dominates every achievable deficit vector.*

Proof: Let h be an arbitrary achievable deficit vector. Pick $k \in \{1, \ldots, L\}$. Suppose the k-th highest deficit in $h^\star$ lies in D_j. Let k' be the number of components in D_j and rest $(k - k')$ components be in $\overline{D}_{j-1}$. By definition

$$\begin{aligned}
\sum_{i=1}^{k} h_{[i]}^\star &= w(\overline{D}_{j-1}) + k' \frac{w(\overline{D}_j) - w(\overline{D}_{j-1})}{|D_j|} \\
&= \frac{1}{|D_j|} \Big[(|D_j| - k') w(\overline{D}_{j-1}) + k' w(\overline{D}_j) \Big] \\
&\leq \frac{1}{|D_j|} \Big[(|D_j| - k') h(\overline{D}_{j-1}) + k' h(\overline{D}_j) \Big] \\
&= h(\overline{D}_{j-1}) + \frac{k'}{|D_j|} \Big(h(\overline{D}_j) - h(\overline{D}_{j-1}) \Big) \\
&= h(\overline{D}_{j-1}) + k' \frac{h(D_j)}{|D_j|},
\end{aligned} \tag{15}$$

where the first equality is by definition of $h^\star$, the inequality is by achievability of h and the fact that $k' \leq |D_j|$, and the final equality in (15) is due to the fact that h is linear (implying $h(D_j) = h(\overline{D}_j) - h(\overline{D}_{j-1})$).

Suppose there are ℓ deficit schools in D_j. Denote the ordered deficits of h in D_j as $h_{[1]}^{D_j} \geq \ldots \geq h_{[\ell]}^{D_j}$. By Claim 1, we see that for any $k' \in \{1, \ldots, \ell\}$,

$$k' \frac{h(D_j)}{|D_j|} \leq \sum_{i=1}^{k'} h_{[i]}^{D_j}$$

Using this with inequality (15), we get

$$\sum_{i=1}^{k} h_{[i]}^{\star} \leq h(\overline{D}_{j-1}) + \sum_{i=1}^{k'} h_{[i]}^{D_j} \leq \sum_{i=1}^{k} h_{[i]}.$$

This establishes the desired claim. ■

These claims establish the achievability and Lorenz dominance of $h^{\star}$. ■