PLANARITY AND DIMENSION I

HEATHER SMITH BLAKE, JĘDRZEJ HODOR, PIOTR MICEK, MICHAŁ T. SEWERYN, AND WILLIAM T. TROTTER

Abstract. The dimension of a partially ordered set P (poset for short) is the least positive integer d such that P is isomorphic to a subposet of $\mathbb{R}^d$ with the natural product order. Dimension is arguably the most widely studied measure of complexity for posets, and standard examples in posets are the canonical structure forcing dimension to be large. In many ways, dimension for posets is analogous to chromatic number for graphs with standard examples in posets playing the role of cliques in graphs. However, planar graphs have chromatic number at most four, while posets with planar diagrams may have arbitrarily large dimension. The key feature of all known constructions of such posets is that large dimension is forced by a large standard example. The question of whether every poset of large dimension and with a planar cover graph contains a large standard example has been a critical challenge in posets theory since the early 1980s, with very little progress over the years. We answer the question in the affirmative. Namely, we show that every poset P with a planar cover graph has dimension $\mathcal{O}(s^8)$, where s is the maximum order of a standard example in P.

(Blake) Mathematics & Computer Science Department, Davidson College, Davidson, North Carolina 28035

(Hodor) Theoretical Computer Science Department, Faculty of Mathematics and Computer Science and Doctoral School of Exact and Natural Sciences, Jagiellonian University, Kraków, Poland

(Micek) Theoretical Computer Science Department, Faculty of Mathematics and Computer Science, Jagiellonian University, Kraków, Poland

(Seweryn) Computer Science Institute, Charles University, Prague, Czech Republic

(Trotter) School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332

E-mail addresses: hsblake@davidson.edu, jedrzej.hodor@gmail.com, piotr.micek@uj.edu.pl, seweryn@iuuk.mff.cuni.cz, trotter@math.gatech.edu.

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Contents

1. Introduction

In this paper, we study finite partially ordered sets, called *posets* for short. The *dimension* of a poset P, denoted $\dim(P)$, is the least positive integer d such that P is isomorphic to a subposet of $\mathbb{R}^d$ equipped with the product order.[1] Dimension is arguably the most widely studied measure of complexity for posets. It captures important concepts in graph theory such as planarity [23] and nowhere denseness [17]. The computational complexity of testing whether a poset has dimension at most d appeared on the famous Garey-Johnson list of problems [11]. Nowadays, we know that dimension is NP-hard to compute [31, 6] and hard to approximate in a strong sense [2].

Dimension was introduced in a foundational paper by Dushnik and Miller [5] in 1941. This paper also includes the canonical structure in posets forcing dimension to be large, namely, the family of standard examples. For each integer n with $n \geqslant 2$, the *standard example* of order n, denoted by S_n, is a poset on $2n$ elements $a_1, \dots, a_n, b_1, \dots, b_n$ such that $a_1, \dots, a_n$ are pairwise incomparable, $b_1, \dots, b_n$ are pairwise incomparable, and for all $i, j \in \{1, \dots, n\}$, we have $a_i < b_j$ in S_n if and only if $i \neq j$. See Figure 1. It is one of the first exercises in dimension theory to show that $\dim(S_n) = n$. Since dimension is a monotone parameter, $\dim(P) \geqslant n$ whenever P contains a subposet isomorphic to S_n.

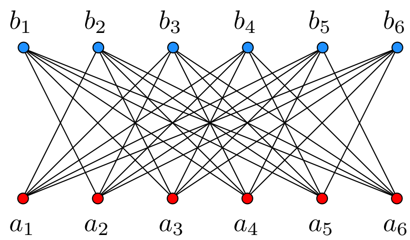


Figure 1. The standard example of order 6.

However, large standard examples are not the only way to drive dimension up. There are families of posets with arbitrarily large dimension such that for some integer n with $n \geqslant 2$, no poset in the family contains S_n, e.g., incidence posets of complete graphs (as proved by Dushnik and Miller [5]), interval orders (see a tight asymptotic bound on their dimension by Füredi, Hajnal, Rödl, and Trotter [10]), adjacency posets of triangle-free graphs with large chromatic number (as shown by Felsner and Trotter [8]). These results motivate the following definitions. The *standard example number* of a poset P, denoted $\mathrm{se}(P)$, is set to be 1 if P does not contain a subposet isomorphic to a standard example; otherwise, $\mathrm{se}(P)$ is the maximum integer n such that P contains a subposet isomorphic to S_n. Clearly, for every poset P, we have $\mathrm{se}(P) \leqslant \dim(P)$. A class of posets $\mathcal{C}$ is dim-*bounded* if there is a function f such that $\dim(P) \leqslant f(\mathrm{se}(P))$ for every P in $\mathcal{C}$. As we discussed, the class of all posets is not dim-bounded.

The dimension of a poset P can be defined equivalently as the chromatic number of the hypergraph on the set of all incomparable pairs of P with the edge set given by the set of all alternating cycles in P (see Subsection 3.3 for the definition of an alternating cycle and Proposition 3 for the equivalence). This links the notion of dimension of posets with graph colorings. The inequality $\mathrm{se}(P) \leqslant \dim(P)$ for all posets P parallels the inequality $\omega(G) \leqslant \chi(G)$ for all graphs G.[2] Both inequalities are far from tight. A class of graphs $\mathcal{C}$ is

[1]In the product order of $\mathbb{R}^d$, for $(x_1, \dots, x_d), (y_1, \dots, y_d) \in \mathbb{R}^d$, we have $(x_1, \dots, x_d) \leqslant (y_1, \dots, y_d)$ if and only if $x_i \leqslant y_i$ for every $i \in \{1, \dots, d\}$.

[2]Here, for a graph G, $\chi(G)$ is its chromatic number and $\omega(G)$ is its clique number.

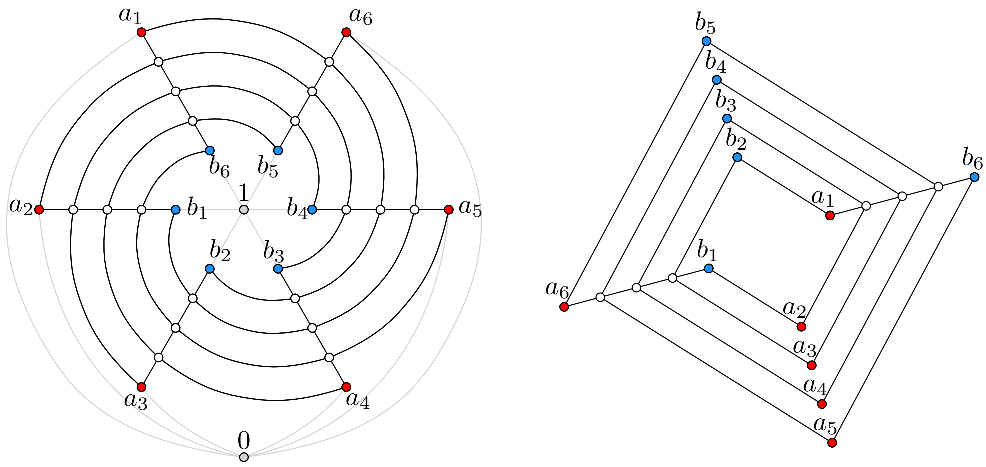


FIGURE 2. Left: The wheel of order 6: it has a planar cover graph (the order relation goes inwards) and it contains a subposet isomorphic to the Kelly poset of order 6. Right: The Kelly poset of order 6: it has a planar diagram and contains a subposet isomorphic to S_6.

χ-*bounded* if there is a function f such that for every G in $\mathcal{C}$, we have $\chi(G) \leqslant f(\omega(G))$. We refer readers to the recent survey by Scott and Seymour [24] on the extensive body of research done on this topic. The analogy breaks down with the celebrated Four Color Theorem, which states that planar graphs have chromatic number at most four, while "planar" posets may have arbitrarily large dimension as we now explain.

An element y in a poset P *covers* an element x in P if $x < y$ in P and there is no z in P with $x < z < y$ in P. The *cover graph* of a poset P is the graph whose vertices are the elements of P and two elements are adjacent if one covers the other. Somewhat unexpectedly, posets with planar cover graphs can have arbitrarily large dimension, as we learned[3] in 1978 [27], see Figure 2. A *diagram* of P is a drawing of the cover graph of P in the plane such that whenever xy is an edge in the cover graph and $x < y$ in P, the relation is represented by a curve from x to y going upwards. In 1981, Kelly [20] published a seminal construction of posets with planar diagrams and arbitrarily large dimension, see Figure 2 again. A key remark is that all known constructions of planar posets with large dimension contain large standard examples. Thus, since the early 1980's, it remained a challenge and perhaps the most important problem in poset theory to settle the following.

Conjecture.

(i) *The class of posets with planar diagrams is* dim-*bounded.*
(ii) *The class of posets with planar cover graphs is* dim-*bounded.*

We believe that the first published reference to Conjecture (i) is an informal comment on page 119 in [28] published in 1992. However, the conjecture was circulating among researchers soon after the constructions illustrated in Figure 2 appeared. Accordingly, Conjecture (i) is more than 40 years old and obviously Conjecture (ii) is a stronger statement. In this paper, we show that both statements are true and prove the following theorem.

Theorem 1. *For every poset P with a planar cover graph,* $\dim(P) \leqslant 64s^6(s+3)^2+12$ *where* $s = \mathrm{se}(P)$.

[3] *We learned* is a bit of an exaggeration as only one author of the manuscript in hand was alive in 1978.

The theorem has an important algorithmic aspect. Following our proof, one can design a polynomial time algorithm that given a poset P with a planar cover graph on the input, returns an embedding of P into $\mathbb{R}^d$, where d is in $\mathcal{O}(\text{se}(P)^8)$. Since $\text{se}(P) \leqslant \dim(P)$, this constitutes an approximation algorithm for poset dimension in the planar setting. We believe that it is not known whether computing dimension of posets with planar cover graphs is NP-hard.

We remark that if P has a planar diagram we can prove that $\dim(P)$ is bounded by a linear function of $\text{se}(P)$. Since this requires separate proof techniques crafted for the planar diagram setup, we leave this line of research for a separate manuscript: Planarity and dimension II. Rephrasing Theorem 1, a large dimensional poset with a planar cover graph contains a large standard example. In another manuscript in preparation, Planarity and dimension III, we show that a poset with a planar cover graph containing a large standard example must contain a large Kelly poset. Thus, large dimensional posets with planar cover graphs contain large Kelly posets.

It seems that for many years, there were no tools to force large standard examples in highly dimensional posets. Hence, there are very little results on dim-boundedness. Blake, Micek, and Trotter [1], proved that each family of posets with planar cover graphs and a unique minimal element (called a zero) is dim-bounded. Note that posets with planar cover graphs and a zero may have arbitrarily large dimension as they contain all the wheels, recall Figure 2. Recently, Joret, Micek, Pilipczuk, and Walczak [18] proved that posets with cover graphs of bounded treewidth (or even posets of bounded cliquewidth) are dim-bounded. Note that posets with cover graphs of treewidth at most 3 may have arbitrarily large dimension as they contain all the Kelly posets, again recall Figure 2. The proof of the main result in [18] builds on Colcombet's deterministic version of Simon's factorization theorem, which is a fundamental tool in formal language and automata theory – therefore, it is very different from the combinatorics in [1] and this paper.

Since planar graphs exclude K_5 as a minor and graphs of treewidth less than t exclude K_{t+1} as a minor, the following statement generalizes both Theorem 1 and the main result of [18].

Conjecture. *For every positive integer t, the class of posets with cover graphs excluding K_t as a minor is* dim*-bounded.*

We remark that the class of posets with cover graphs excluding K_t as a topological minor is not dim-bounded [7, Section 6].

In contrast to dim-boundedness, it is well understood for which minor-closed classes of graphs, posets with cover graphs in this class have dimension bounded by an absolute constant. The very first result in this line is by Trotter and Moore [29] who showed that posets whose cover graphs are forests (so exactly those excluding K_3 as a minor) have dimension at most 3. Felsner, Trotter, and Wiechert [9] proved that posets with outerplanar cover graphs have dimension at most 4. Seweryn [25] proved that posets with cover graphs excluding K_4 as a minor have dimension at most 12. Note that posets with cover graphs excluding K_5 as a minor contain all Kelly posets, so they may have arbitrarily large dimension. Huynh, Joret, Micek, Seweryn, and Wollan [14] proved that posets with cover graphs excluding a $2 \times t$ grid as a minor have dimension bounded by a function of t. All the above is qualitatively generalized by the following statement [18]: for a minor-closed class of graphs $\mathcal{C}$ there exists a constant such that every poset with a cover graph in $\mathcal{C}$ has dimension bounded by this constant if and only if $\mathcal{C}$ excludes the cover graph of some Kelly poset.

Poset dimension was also studied with respect to other poset invariants. Already in 1950, Dilworth [4] proved that dimension of a poset is always bounded by its width (i.e. the largest size of an antichain). This is a consequence of his famous chain decomposition result: every poset of width w can be partitioned into w chains. On the other hand, there are posets of height (i.e. the largest size of a chain) equal to 2 and arbitrarily large dimension, e.g. standard examples. This leads to a line of research initiated by Streib and Trotter [26] who proved that posets with planar cover graphs have dimension bounded by a function of their height. In other words, large dimensional posets with planar cover graphs are not only wide but also tall. This result was later improved and generalized in many directions [15, 30, 22, 19, 16, 21, 12].

Finally, let us mention that in fact we prove a stronger result, which implies Theorem 1 (see Theorem 131 and the necessary definitions in Subsection 3.3). In particular, the stronger result implies that the class of posets that are subposets of posets with planar cover graphs is dim-bounded (see Corollary 132).

2. Outline of the proof

Our proof of Theorem 1 is self-contained and does not rely on any external material beyond the fundamentals of graph theory and poset theory. In this section, we discuss the main ideas and sketch the structure of the proof. The readers unfamiliar with the basics of poset dimension may want to skip this high-level overview and and proceed directly to Section 3. All the notation introduced in the outline is later introduced again.

We begin with an alternative perspective on the dimension of posets, presented in detail in Subsection 3.3, which is more amenable to combinatorial arguments. Let P be a poset. We denote by $\operatorname{Inc}(P)$ all incomparable pairs of elements of P. For an integer k with $k \geqslant 2$, a sequence $((a_1, b_1), \ldots, (a_k, b_k))$ of pairs in $\operatorname{Inc}(P)$ is a *strict alternating cycle* of size k in P if $a_i \leqslant b_j$ in P for all $i, j \in [k]$ if and only if $j = i+1$, cyclically (that is, $(a_{k+1}, b_{k+1}) = (a_1, b_1)$). Let $I \subseteq \operatorname{Inc}(P)$. We say that such a strict alternating cycle is *contained* in I if $(a_i, b_i) \in I$ for all $i \in [k]$. The subset I is *reversible* if I does not contain a strict alternating cycle. A family $\mathcal{S}$ of subsets of $\operatorname{Inc}(P)$ *covers* I if $I \subseteq \bigcup \mathcal{S}$. We define $\dim_P(I)$ as the minimum positive integer d such that I can be covered by d reversible sets. One can show that $\dim(P) = \dim_P(\operatorname{Inc}(P))$, see Proposition 3.

Let P be a poset with a planar cover graph. The proof of the main theorem starts with an application of an unfolding (explained below) to P in order to pinpoint a fragment of P responsible for its dimension. The idea of an “unfolding of a poset” was introduced by Streib and Trotter [26] in 2014, and it is inspired by the following classical application of layerings to graph colorings. A *layering* of a graph G is a family $(Z_i : i \in \mathbb{N})$ of pairwise disjoint subsets of $V(G)$ such that for every edge uv of G, there exists $i \in \mathbb{N}$ with $\{u, v\} \subseteq Z_i \cup Z_{i+1}$. It is well-known that given a graph G and a layering of G, if every layer induces a k-colorable graph, then G is $2k$-colorable (one can use two disjoint palettes of k colors each, one for even layers, and one for odd layers). The counterpart of the above for posets is formulated in terms of the unfolding. An *unfolding* of a poset P is a layering of the comparability graph of P. If the union of any two consecutive sets in the unfolding induces a subposet of dimension at most d, then the dimension of P is at most $2d$, see Proposition 11. We use a specific unfolding of P that emerges from a BFS-layering of the comparability graph of P that starts in a minimal element of P. Given such an unfolding, we localize two consecutive layers whose union maximizes dimension, and we contract all the elements of P that lie in layers before, see Lemma 12. In this way, paying a multiplicative factor of 2, we reduce the problem of bounding $\dim(P)$ to a simpler setting. Namely, we have

$$\dim(P) \leqslant 2 \cdot \dim_Q(I),$$

where Q is a poset and $I \subseteq \mathrm{Inc}(Q)$ such that: Q has a planar cover graph; Q has a distinguished minimal element x_0; the cover graph G of Q has a fixed planar drawing with x_0 in the exterior face; for every $(a,b) \in I$, we have $x_0 < b$ in Q; and $\mathrm{se}(Q) \leqslant \mathrm{se}(P)$. This is encapsulated in the notion of an *instance*, see (I1)–(I5) in Section 4 and Corollary 13. An instance is a tuple $(Q, x_0, G, e_{-\infty}, I)$, where Q, x_0, G, and I are as above, and additionally, $e_{-\infty}$ is a simple curve in the plane that is contained in the exterior face of the drawing and has one endpoint in x_0. This will be an "anchor" determining "directions" in the drawing, as we explain below.

Once an instance $(P, x_0, G, e_{-\infty}, I)$ is fixed, we discuss the topology of the drawing, see Section 4. This part of the material draws from concepts introduced by Blake, Micek, and Trotter [1]. Given an element u in P and an edge (or the anchor) e incident to u, all the edges incident to u are ordered from left to right (i.e. clockwise) around u starting from e (see more details and the notation in Subsection 3.5). A *witnessing path* in P is a path in the cover graph of P of ascending elements in P. Let B be the set of all elements b in P with $x_0 \leqslant b$ in P. Given $b \in B$, we consider all the witnessing paths from x_0 to b in P. Using the ordering of edges described above, we order these paths. Namely, given two such witnessing paths, we consider their longest common prefix starting from x_0. Next, we look at the last edge of this prefix (or $e_{-\infty}$ if the prefix is trivial), and according to this edge, we compare the next edges of the paths to decide which one is *left* and which one is *right* of the other. This ordering is in fact a linear ordering of all witnessing paths from x_0 to b, see Observation 10. This way, we obtain the *leftmost* witnessing path from x_0 to b, denoted by $W_L(b)$, and the *rightmost* witnessing path from x_0 to b, denoted by $W_R(b)$, see Subsection 4.2 and Figure 7.

Given $b \in B$, the paths $W_L(b)$ and $W_R(b)$ start from x_0 and then they may split and rejoin multiple times before they both end in b. Let $z_0, \ldots, z_n$ be all the common elements of $W_L(b)$ and $W_R(b)$ in the natural order. Thus, $z_0 = x_0$ and $z_n = b$. For $i \in [n]$, consider a region enclosed by $W_L(b)$ and $W_R(b)$ between z_{i-1} and z_i. Note that, in case $i \neq n$ both paths may continue outside this region or inside this region. In the latter case, z_i is called a *reversing* element of b. Let $x_1, \ldots, x_d$ be all the reversing elements of b and let $x_{d+1} = b$ (recall that x_0 is already defined). The region enclosed by $W_L(b)$ and $W_R(b)$ from x_i up to x_{i+1} is denoted by $\mathrm{shad}_i(b)$ for each $i \in \{0, \ldots, d\}$. We additionally set $\mathrm{shad}_{d+1}(b) = \emptyset$. We say that x_i is the *initial* element of $\mathrm{shad}_i(b)$. See Subsection 4.3 and Figure 11. In the next step of the proof, we reduce the problem to an instance in which any two pairs (a,b) and (a',b') belonging to one strict alternating cycle in I satisfy $b \notin \mathrm{shad}_0(b')$, and $b' \notin \mathrm{shad}_0(b)$. Therefore, we use the leftmost and rightmost witnessing paths and shadows to topologically order the elements in B, see Subsection 4.4. Namely, for $b, b' \in B$, we say that b is *left* of b' if $W_L(b)$ is left of $W_L(b')$, $W_R(b)$ is left of $W_R(b')$, $b \notin \mathrm{shad}_0(b')$, and $b' \notin \mathrm{shad}_0(b)$.

Within the next step of the argument, we remove twice from I a subset of pairs of dimension at most 2 (non-risky pairs – see Subsection 5.1, and non-dangerous pairs – see Subsection 5.3). Let us skip discussing this detail here and just mention that this is where the additive terms "+2" in the inequality displayed below come from. To each pair $(a,b) \in I$, we assign an *address* (j,x), where j is the minimum nonnegative integer such that $a \notin \mathrm{shad}_j(b)$ and x is the initial element of $\mathrm{shad}_j(b)$. We split the pairs in I into I_0 and I_1 depending on the parity of the first coordinate of their address: $I_\theta = \{(a,b) \in I : (j,x) \text{ is the address of } (a,b) \text{ and } j \equiv \theta \bmod 2\}$. Note that $\dim_P(I) \leqslant \dim_P(I_0) + \dim_P(I_1)$. Let $\theta \in \{0,1\}$. We prove that all pairs in a strict alternating cycle contained in I_θ have the same address, see Lemma 40. This property, together with an elementary fact from poset dimension theory, see Proposition 5, gives $\dim_P(I_\theta) = \max \dim_P(I(j,x))$, where the maximum goes over every possible addresses (j,x) with $j \equiv \theta \bmod 2$ and $I(j,x)$ is the set of all pairs in I with the address (j,x). We fix an address (j,x) such that $\dim_P(I_\theta) = \dim_P(I(j,x))$. We define P' as the poset obtained from P by removing all elements strictly less than x in P. Clearly, all pairs of $I(j,x)$ survive in P'.

Moreover, x lies on the exterior face of the inherited drawing of the cover graph of P'. Now, the element x in P' plays the role of x_0 in P. In particular, shad_j in P becomes shad_0 in P', and every pair in $I(j,x)$ has address $(0,x)$ in the new setting.

In other words, again paying a multiplicative factor of 2 (splitting I into I_0 and I_1), we reduce the problem to a simpler setting. Namely, we obtain an instance $(P', x_0', G', e_{-\infty}', I')$ with

$$\dim_P(I) \leqslant 2 \cdot (\dim_{P'}(I') + 2) + 2,$$

and satisfying the following properties. First, $a \notin \operatorname{shad}_0(b)$ for every $(a,b) \in I'$. Next, for every strict alternating cycle $((a_1,b_1),\ldots,(a_k,b_k))$ contained in I', the elements $\{b_1,\ldots,b_k\}$ are linearly ordered by the "left of" relation, see the second assertion in Lemma 40. Finally, all the pairs in I' are dangerous, which is a detail that we omit in the outline. An instance is *good* if it satisfies these additional properties, see (I6)–(I8) in Subsection 5.4 and Corollary 47. For technical reasons, we insist that a good instance that we fix is also *maximal* in a certain sense that we also omit in the outline, see (I9) in Subsection 5.4 and Proposition 48.

Given a maximal good instance $(P, x_0, G, e_{-\infty}, I)$, we split I into reversible sets along the following plan. We devise six auxiliary graphs $H_1,\ldots,H_6$ on the vertex set I such that for every strict alternating cycle $((a_1,b_1),\ldots,(a_k,b_k))$ contained in I, there exist distinct $i,j \in [k]$ such that (a_i,b_i) and (a_j,b_j) are adjacent in one of the graphs, see Lemma 76. Suppose that we are given a proper vertex coloring c_i of H_i, for each $i \in [6]$. Consider the product coloring of I, i.e., define $\kappa((a,b)) = (c_1((a,b)),\ldots,c_6((a,b)))$ for each $(a,b) \in I$. It follows that each color class of κ yields a reversible subset of I. Namely, for each color ξ in the coloring κ, the set $I_\xi = \{(a,b) \in I : \kappa((a,b)) = \xi\}$ is reversible. In particular,

$$\dim_P(I) \leqslant \chi(H_1)\cdots\chi(H_6).$$

The construction of suitable graphs $H_1,\ldots,H_6$ that fit the framework above is an essential part of this paper. All six graphs share some fundamental properties. E.g., they all admit the following orientation. Let $H \in \{H_1,\ldots,H_6\}$. When (a,b) are adjacent (a',b') in H, then either b is left of b' or b' is left of b. Hence, we can orient all edges of H so that when $((a,b),(a',b'))$ is an edge in H, then b is left of b'. In particular, this orientation is acyclic. The natural way of coloring a graph H with a fixed acyclic orientation is to assign to each vertex v of H, the value $\text{max-start-path}(H,v)$, i.e., the maximum order of a directed path in H starting in v. This coloring witnesses $\chi(H) \leqslant \text{max-path}(H)$, where $\text{max-path}(H)$ is the maximum order of a directed path in H.

Four of the six auxiliary graphs admit a suitable upper bound on max-path. Namely, for each $H \in \{H_1,\ldots H_4\}$, we have $\text{max-path}(H) \leqslant \operatorname{se}(P)$, see Propositions 71 to 74 proved in Section 8, and so, $\chi(H) \leqslant \operatorname{se}(P)$. The remaining two graphs must be treated with more care. Let $H = H_5$. When $((a,b),(a',b'))$ is an edge in H, then

$$\text{max-start-path}(H,(a,b)) > \text{max-start-path}(H,(a',b')) > \text{max-start-path}(H,(a,b)) - m,$$

where $m = 2\operatorname{se}(P)\cdot(2\operatorname{se}(P)+6)$. The first inequality is trivial, while proving the second one is a substantial portion of the whole proof. It follows that

$$\text{max-start-path}(H,(a,b)) \not\equiv \text{max-start-path}(H,(a',b')) \bmod m.$$

We color each vertex (a,b) of H with the value $\text{max-start-path}(H,(a,b)) \bmod m$. Therefore, two adjacent vertices in H are assigned distinct colors, and so, the coloring is proper. In conclusion, $\chi(H_5) \leqslant m$. The argument for H_6 is symmetric (roughly, we replace each occurrence of max-start-path by max-end-path). Eventually, we also obtain $\chi(H_6) \leqslant m$. In this outline, we omit a detail that the edges in H_5 and H_6 are weighted with values from $\{0,1\}$, see Lemma 75 for a precise statement. Altogether, we obtain

$$\dim_P(I) \leqslant \chi(H_1)\cdots\chi(H_6) \leqslant \operatorname{se}(P)^4 \cdot (\operatorname{se}(P)\cdot(2\operatorname{se}(P)+6))^2.$$

The definitions of the six auxiliary graphs are based on the topology of the drawing. We associate each strict alternating cycle $((a_1, b_1), (a_2, b_2))$ contained in I with two regions $\mathcal{R}_1$ and $\mathcal{R}_2$ in the plane, see Subsection 6.4. Next, we classify these cycles according to the statements "$a_1 \in \mathcal{R}_2$" and "$a_2 \in \mathcal{R}_1$" being satisfied or not. As a result, each strict alternating cycle of size 2 in I is of one of the four types: In-In, In-Out, Out-In, and Out-Out. This classification inspires the definitions of the six auxiliary graphs: H_{OO}, H_{IIL}, H_{IIR}, H_{IILR}, H_{IO}, and H_{OI}, see Section 7. It also lays the foundations for the key property discussed above: every strict alternating cycle (of arbitrary size) contained in I contains two incomparable pairs adjacent in one of the auxiliary graphs, see Lemma 76.

3. Preliminaries

We denote by $\mathbb{R}$ the set of real numbers and by $\mathbb{N}$ the set of nonnegative integers. For a positive integer k, we write $[k]$ as a compact form of $\{1, \ldots, k\}$.

3.1. **Graphs.** We consider simple and finite graphs. Unless we say otherwise, all graphs are undirected. The vertex set of a graph G is denoted by $V(G)$ and the edge set of G is denoted by $E(G)$. Let G_1, G_2 be two graphs. The *union* of G_1 and G_2, denoted by $G_1 \cup G_2$ is the graph with the vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2)$. The *intersection* of G_1 and G_2, denoted by $G_1 \cap G_2$ is the graph with vertex set $V(G_1) \cap V(G_2)$ and edge set $E(G_1) \cap E(G_2)$.

Let k be a nonnegative integer. A *path* is a graph with the vertex set $\{v_0, \ldots, v_k\}$ and the edges $\{v_{i-1}v_i : i \in [k]\}$, where $v_0, \ldots, v_k$ are pairwise distinct. We often refer to a path by a natural sequence of its vertices, writing, say, $v_0 \cdots v_k$ or equivalently $v_k \cdots v_0$. Writing a path as $v_0 \cdots v_k$ fixes an underlying orientation of the path, where v_0 is the first vertex and v_k is the last vertex. We say that v_0 and v_k are the *endpoints* of $v_0 \cdots v_k$. We also say that $v_0 \cdots v_k$ *starts* in v_0 and *ends* in v_k. A *cycle* is a graph with at least three vertices such that removing each of its edges gives a path. A *forest* is a graph with no cycles. A *tree* is a connected forest. Let T be a tree, and let u, v be two vertices in T. We write $u[T]v$ to denote the unique path in T with endpoints u and v and with an orientation from u to v.

We use the following convenient notation for manipulating paths in graphs. Let G be a graph. Let $v_0, \ldots, v_k$ be vertices of G, and let $T_1, \ldots, T_k$ be trees in G such that each T_i contains the vertices v_{i-1} and v_i. Then we denote by $v_0[T_1]v_1[T_2] \cdots [T_k]v_k$ the union of the paths $v_{i-1}[T_i]v_i$. See Figure 3. When $T_i = v_{i-1}v_i$, then we may omit "$[T_i]$" in this notation. When the resulting graph is a path, we consider it with the orientation from v_0 to v_k. For example, for a path $W = v_0 \cdots v_5$, we have $v_0[W]v_2v_3[W]v_5 = W$.

3.2. **Orders and posets.** Let X be a set. A *partial order* on X is a binary relation R on X that is *reflexive* (for every $x \in X$, $(x, x) \in R$), *antisymmetric* (for all $x, y \in X$, if $(x, y) \in R$ and $(y, x) \in R$, then $x = y$), and *transitive* (for all $x, y, z \in X$, if $(x, y) \in R$ and $(y, z) \in R$, then $(x, z) \in R$). A partial order R is *linear* if for all $x, y \in X$, we have $(x, y) \in R$ or $(y, x) \in R$. When R is a partial order (resp. linear order) on X, we say that R *partially orders* (resp. *linearly orders*) X. When R partially orders (resp. linearly orders) X, we say that $P = (X, R)$ is a *partially ordered set*, or *poset* for short (resp. *linearly ordered set*). We refer to the set X as the *ground set* of P, and to the members of X as the *elements* of P. Note that (X, R^{-1}) is also a poset, we call it the *dual* of P and denote it by P^{-1}.

Let x and y be elements of P. We say that x and y are *comparable* in P (or in R) if $(x, y) \in R$ or $(y, x) \in R$. We use the notation "$x \leqslant y$ in P" whenever $(x, y) \in R$, and "$x < y$ in P"

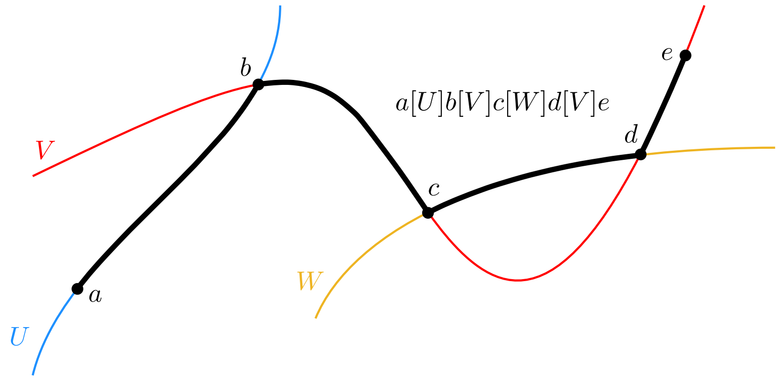


FIGURE 3. A sample application of the notation for path concatenation.

whenever $(x, y) \in R$ and $x \neq y$. Furthermore, we say that x and y are *incomparable* in P (or in R) if they are not comparable in R. In this case, we write "$x \parallel y$ in P".

An element x of P is *minimal* in P if for every element y of P, $y \leqslant x$ in P implies $y = x$. Symmetrically, an element x of P is *maximal* in P if for every element y of P, $x \leqslant y$ in P implies $y = x$. A *chain* in P is a set of elements in P such that every two elements are comparable in P, and an *antichain* in P is a set of elements in P such that every two distinct elements are incomparable in P.

The *comparability graph* of P is a graph on the ground set of P in which two distinct vertices are adjacent if and only if they are comparable in P. An element x of P is *covered* by an element y of P (and y *covers* x) in P if $x < y$ in P and there is no element z of P with $x < z < y$ in P. The *cover graph* of P is a subgraph of the comparability graph in which two vertices are adjacent if either one of them covers the other. Observe that the dual P^{-1} has the same comparability graph and cover graph as P.

Two posets P and Q are *isomorphic* if there exists a bijection f from the ground set of P to the ground set of Q such that for all elements x and y of P, we have $x \leqslant y$ in P if and only if $f(x) \leqslant f(y)$ in Q. For a subset Y of the ground set of P, the *subposet induced* by Y in P is a poset Q with the ground set Y such that for all $x, y \in Y$, we have $x \leqslant y$ in Q if and only if $x \leqslant y$ in P. A poset Q is a *subposet* of a poset P if it is a subposet induced by some subset of the ground set of P in P. A subposet Q of a poset P is *convex* if for all elements x, y, z with $x < y < z$ in P, if x and z are elements of Q, then also y is an element of Q. Note that if Q is a convex subposet of P, then the cover graph of Q is a subgraph of the cover graph of P.

For an element x of P, we denote by $P - x$ the subposet of P induced by the set of all elements of P except x.

A *finite* poset is a poset with a finite ground set. All posets in this paper are finite, unless stated otherwise. A *component* of a poset is a subposet induced by a connected component of its cover graph. A poset is *connected* if it has exactly one component.

3.3. **Dimension of posets.** In the introduction, we presented a concise geometric definition of dimension of posets. However, we (and most other researchers) work with a combinatorial equivalent. In this section, we discuss equivalent definitions of dimension and prove its basic properties.

Let P be a poset. A *linear extension* of P is a linearly ordered set L on the same ground set as P such that $x \leqslant y$ in P implies $x \leqslant y$ in L for all elements x and y of P. Every poset has a linear extension, and furthermore, for all incomparable elements x and y of P there exists a linear extension L with $x < y$ in L (we prove an even stronger statement in Proposition 3). Therefore, if $\mathcal{L}$ is the set of all linear extensions of P then for all elements x and y of P, we have

$$x \leqslant y \text{ in } P \text{ if and only if } x \leqslant y \text{ in } L \text{ for each } L \in \mathcal{L}. \tag{1}$$

Recall that the dimension of a poset P is the least positive integer d such that P is isomorphic to a subposet of $\mathbb{R}^d$ (ordered by the product order). If Q is a finite subposet of $\mathbb{R}^d$, then by slightly perturbing the coordinates of the points we can obtain an isomorphic subposet Q' of $\mathbb{R}^d$ such that for each $i \in [d]$, the ith coordinates of the elements of Q' are pairwise distinct. Then for each $i \in [d]$, the linear order on the ground set of Q' given by the increasing ith coordinate yields a linear extension of Q'. This justifies an equivalent, combinatorial definition of dimension.

The dimension of P is the least positive integer d for which there exist d linear extensions $L_1, \ldots, L_d$ of P such that for all elements x and y of P, we have

$$x \leqslant y \text{ in } P \text{ if and only if } x \leqslant y \text{ in } L_i \text{ for each } i \in [d].$$

Indeed, a linear extension L_i can be seen as the order of elements in the ith coordinate in $\mathbb{R}^d$, and conversely, the order of elements in the ith coordinate in $\mathbb{R}^d$ forms a linear extension. By (1), the dimension is well-defined. In fact, Hiraguchi [13] showed that an n-element poset with $n \geqslant 4$, has dimension at most $n/2$.

For a poset P, let $\operatorname{Inc}(P)$ denote the set of all ordered pairs (x, y) of elements of P with $x \parallel y$ in P. A subset $I \subseteq \operatorname{Inc}(P)$ is *reversible* in P if there exists a linear extension L of P with $y < x$ in L for all $(x, y) \in I$. A family $\mathcal{S}$ of subsets of $\operatorname{Inc}(P)$ *covers* I if $I \subseteq \bigcup \mathcal{S}$. In this case, sometimes we say that $\mathcal{S}$ is a *covering* of I. We state arguably the most useful equivalent definition of dimension of posets.

Observation 2. *For every poset P, the dimension of P is the least positive integer d for which* $\operatorname{Inc}(P)$ *can be covered by d reversible sets.*

Let P be a poset. Motivated by the above, for every $I \subseteq \operatorname{Inc}(P)$, we define the *dimension* of I in P, denoted by $\dim_P(I)$, as the minimum positive integer d such that I can be covered by d reversible sets. Note that by Observation 2, $\dim(P) = \dim_P(\operatorname{Inc}(P))$.

Similarly, we define a restricted version of the standard example number. We say that a set $J \subseteq \operatorname{Inc}(P)$ *induces* a standard example in P if $|J| \geqslant 2$ and for all distinct $(a, b), (a', b') \in J$, we have $a < b'$ and $a' < b$ in P. For every $I \subseteq \operatorname{Inc}(P)$, let $\operatorname{se}_P(I)$ be defined as 1 when there is no subset of I inducing a standard example in P; otherwise, $\operatorname{se}_P(I)$ is the maximum size of a subset of I that induces a standard example in P. Note that $\operatorname{se}(P) = \operatorname{se}_P(\operatorname{Inc}(P))$.

It turns out that there is a relatively simple criterion to verify if a given $I \subseteq \operatorname{Inc}(P)$ is reversible. For an integer k with $k \geqslant 2$, a sequence $((a_1, b_1), \ldots, (a_k, b_k))$ of pairs in $\operatorname{Inc}(P)$ is an *alternating cycle* of size k in P if $a_i \leqslant b_{i+1}$ in P for all $i \in [k]$, cyclically (that is, $b_{k+1} = b_1$). We say that $I \subseteq \operatorname{Inc}(P)$ *contains* an alternating cycle $((a_1, b_1), \ldots, (a_k, b_k))$ if all pairs (a_i, b_i) belong to I. An alternating cycle $((a_1, b_1), \ldots, (a_k, b_k))$ is *strict* if for all $i, j \in [k]$, we have $a_i \leqslant b_j$ in P if and only if $j = i + 1$ (cyclically). Note that in this case, $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_k\}$ are k-element antichains in P. Note also that in alternating cycles, we allow that $a_i = b_{i+1}$ for some or even all values of i. Trotter and Moore [29] made the following elementary observation that has proven over time to be far-reaching in nature.

Proposition 3. *Let P be a poset and let $I \subseteq \operatorname{Inc}(P)$. The following conditions are equivalent:*

(i) *I is reversible,*
(ii) *I does not contain an alternating cycle,*
(iii) *I does not contain a strict alternating cycle.*

Proof. For the implication (i) to (ii), suppose that $I \subseteq \operatorname{Inc}(P)$ is a reversible set which contains an alternating cycle $((a_1, b_1), \dots, (a_k, b_k))$. Since I is reversible, there is a linear extension L of P with $b_i < a_i$ in L and $a_i \leqslant b_{i+1}$ in L (cyclically) for all $i \in [k]$. Therefore,

$$b_1 < a_1 \leqslant b_2 < a_2 \leqslant \cdots < a_k \leqslant b_1 \text{ in } L$$

which is a contradiction.

For the implication (ii) to (i), consider a set $I \subseteq \operatorname{Inc}(P)$ which does not contain an alternating cycle. Let X be the ground set of P. Define an auxiliary binary relation R on X where $(x, y) \in R$ if and only if $x \leqslant y$ in P or there is a sequence $((a_1, b_1), \dots, (a_k, b_k))$ of elements from I where $a_i \leqslant b_{i+1}$ in P for each $i \in [k-1]$, $x \leqslant b_1$ in P, and $a_k \leqslant y$ in P. We show that R is a partial order. Clearly R is reflexive.

Now suppose $(x, y) \in R$ and $(y, z) \in R$ for some elements x, y, z in P. If $x \leqslant y$ in P or $y \leqslant z$ in P, it is quick to verify $(x, z) \in R$. Otherwise, the concatenation of the witnessing sequences for $(x, y) \in R$ and $(y, z) \in R$ is a sequence that shows $(x, z) \in R$. Therefore, R is transitive.

Next, we prove that R is antisymmetric. Consider x, y in P with $(x, y) \in R$ and $(y, x) \in R$. In the case $x < y$ in P, then $y \not\leqslant x$ in P and so there is a sequence $((a_1, b_1), \dots, (a_k, b_k))$ witnessing $(y, x) \in R$. Since $a_k \leqslant x < y \leqslant b_1$ in P, the sequence is actually an alternating cycle with all of its pairs in I, a contradiction. Thus, $x \not< y$ in P and similar reasoning shows that $y \not< x$ in P. As a result, we assume that $x \parallel y$ in P. So each of $(x, y) \in R$ and $(y, x) \in R$ is witnessed by a sequence of pairs in I. The concatenation of these sequences is again an alternating cycle in I, a contradiction. Therefore, $x = y$ and R is antisymmetric.

We have verified that R partially orders X. Let $Q = (X, R)$ and let L be a linear extension of Q. Note that if $x \leqslant y$ in P, then $x \leqslant y$ in Q. Hence, L is also a linear extension of P. Furthermore, for each $(a, b) \in I$, the sequence $((a, b))$ witnesses that $b \leqslant a$ in Q, thus, L reverses all pairs in I. Therefore, I is reversible as desired.

The implication (ii) to (iii) is clear. To prove the implication (iii) to (ii), consider $I \subseteq \operatorname{Inc}(P)$ which contains an alternating cycle. Let $C = ((a_1, b_1), \dots, (a_k, b_k))$ be such a cycle with minimum size. Suppose C is not a strict alternating cycle. So there exist $i, j \in [k]$ such that a_i is comparable to b_j and $j \neq i+1$ (cyclically). Without loss of generality, by rotating the cycle, we can assume that $i = 1$, and so, $j \notin \{1, 2\}$. If $a_1 \leqslant b_j$ in P, then $((a_1, b_1), (a_j, b_j), \dots, (a_k, b_k))$ is an alternating cycle in I shorter than C. If instead $b_j \leqslant a_1$ in P, then $a_{j-1} \leqslant b_2$ in P, and so $2 < j - 1$. Thus $((a_2, b_2), \dots, (a_{j-1}, b_{j-1}))$ is an alternating cycle shorter than C. In each case, we have a contradiction, so C must be a strict alternating cycle in I. □

Next, we develop some easy properties of dimension. For every poset P and for each $I \subseteq \operatorname{Inc}(P)$, recall that $I^{-1} = \{(b, a) : (a, b) \in I\}$. Note that if $I \subseteq \operatorname{Inc}(P)$, then $I^{-1} \subseteq \operatorname{Inc}(P^{-1})$. The following observation is straightforward.

Observation 4. *For every poset P and $I \subseteq \operatorname{Inc}(P)$, we have $\dim_P(I) = \dim_{P^{-1}}(I^{-1})$.*

Let P be a poset, $I \subseteq \text{Inc}(P)$, and let $\{I_i : i \in [s]\}$ be a covering of I. The following inequalities are trivial but still very useful

$$\dim_P(I) \leqslant \dim_P\left(\bigcup_{i\in[s]} I_i\right) \leqslant \sum_{i\in[s]} \dim_P(I_i).$$

The next proposition describes a situation when we get a stronger bound, i.e. $\dim_P(I) \leqslant \max\{\dim_P(I_i) : i \in [s]\}$ (so somehow the problem becomes local with respect to the covering).

Proposition 5. *Let P be a poset, let $I \subseteq \text{Inc}(P)$, and let $I_1, \ldots, I_s \subseteq I$ be pairwise disjoint. Assume that for every strict alternating cycle in P contained in I, there is $i \in [s]$ such that all the pairs in the strict alternating cycle are in I_i. Then,* $\dim_P(I) = \max\{\dim_P(I_i) : i \in [s]\}$.

Proof. Let $d = \max\{\dim_P(I_i) : i \in [s]\}$. Clearly, $\dim_P(I) \geqslant d$, thus, we focus on the other inequality. Since for each $i \in [s]$, $\dim_P(I_i) \leqslant d$, there exists a covering $\{I_{i,j} : j \in [d]\}$ of I_i consisting of reversible sets in P. For each $j \in [d]$, let $J_j = \bigcup_{i=1}^{s} I_{i,j}$. Fix $j \in [d]$. We claim that J_j is reversible. Suppose to the contrary that there is a strict alternating cycle in P contained in J_j. Since $I_1, \ldots, I_s$ are pairwise disjoint, for every $i \in [s]$, we have $J_j \cap I_i \subseteq I_{i,j}$. Now, by the assumption, there is $i \in [s]$ such that all the pairs from the cycle are in I_i. Therefore, all pairs of the cycle are in $I_{i,j}$. This contradicts the fact that $I_{i,j}$ is reversible and completes the proof. □

Back in the 1950s, Hiraguchi [13] proved that if a poset P is the union of disjoint chains, then $\dim(P) \leqslant 2$, and otherwise $\dim(P)$ equals the maximum dimension of a component of P. We will use a slightly refined version of this statement devised for subsets of incomparable pairs.

Proposition 6. *Let P be a poset, and let $I \subseteq \text{Inc}(P)$ with $\dim_P(I) \geqslant 3$. Then there exists a component C of P such that $\dim_P(I) = \dim_C(I_C)$, where $I_C = I \cap \text{Inc}(C)$.*

Proof. Let $C_1, \ldots, C_s$ be components of P, let $I_{C_i} = I \cap \text{Inc}(C_i)$, and $d_i = \dim_{C_i}(I_{C_i})$, for each $i \in [s]$. Let $d = \max(\{d_i : i \in [s]\} \cup \{2\})$. Since for each $i \in [s]$, $\dim_P(I_{C_i}) \leqslant d$, there exists a covering $\{I_{i,j} : j \in [d]\}$ of I_{C_i} consisting of d reversible sets in C_i. Next, we define $J_1 = \{(a, b) \in I : a \text{ in } C_i,\ b \text{ in } C_j, \text{ and } i < j\}$ and $J_2 = \{(a, b) \in I : a \text{ in } C_i, b \text{ in } C_j, \text{ and } i > j\}$. Note that $\{J_1, J_2\} \cup \{I_{i,j} : i \in [s], j \in [d]\}$ is a covering of I.

We need the following abstract observation. Consider $K \subseteq \text{Inc}(P)$ such that

(i) if $(a, b) \in K$ with a in C_i, b in C_j, then $i \leqslant j$;
(ii) $K \cap \text{Inc}(C_i)$ is reversible in P, for each $i \in [s]$.

We claim that K is reversible in P. Let $((a_1, b_1), \ldots, (a_k, b_k))$ be an alternating cycle in P contained in K. Let $i_\alpha \in [s]$ be such that a_α is in C_{i_α}, for each $\alpha \in [k]$. Fix $\alpha \in [k]$. Since $a_\alpha \leqslant b_{\alpha+1}$ in P, it follows that $b_{\alpha+1}$ is in C_{i_α}, and therefore, $i_{\alpha+1} \leqslant i_\alpha$ (cyclically). Thus, $i_1 = \cdots = i_k$. However, this is a contradiction with the fact that $K \cap \text{Inc}(C_{i_\alpha})$ is reversible. This completes the proof that K is reversible in P.

For each $j \in [d]$, let

$$I_j = \begin{cases} \bigcup_{i=1}^{s} I_{i,j} \cup J_j & \text{if } j \in \{1, 2\}, \\ \bigcup_{i=1}^{s} I_{i,j} & \text{otherwise.} \end{cases}$$

Note that $\{I_j : j \in [d]\}$ is a covering of I. By the observation above, I_j is reversible in P for each $j \in [d]$, and so $\dim_P(I) \leqslant d$. It follows that $3 \leqslant \dim_P(I) \leqslant d$, and so, $d \geqslant 3$. In particular, $d = d_i$ for some $i \in [s]$. Since $\dim_P(I) \geqslant d_i$ for every $i \in [s]$ and $\dim_P(I) \geqslant 2$, we have $\dim_P(I) \geqslant d$. Altogether, $\dim_P(I) = d$, and thus, $\dim_P(I) = d_i = \dim_{C_i}(I_{C_i})$ for some $i \in [s]$. □

3.4. **Topology and planarity.** For a set S of points in the plane, we write ∂S for the topological boundary of S and $\operatorname{int} S$ for the topological interior of S. Additionally, we define the *exterior* of S as the interior of the complement of S in the plane.

A *simple curve* γ in the plane is the image of an injective continuous map of a closed segment into the plane. In this case, the *endpoints* of γ are the images of the endpoints of the segment, while the *interior* of γ is the set of non-endpoint points in γ. We say that γ *connects* its endpoints. A *simple closed curve* in the plane is the image of an injective continuous map of a circle into the plane. Since all combinatorial objects considered in this paper are finite, we always assume that curves are finite unions of segments in the plane.

Let γ be a simple closed curve. The Jordan Curve Theorem[4] states that the complement of γ in the plane consists of two arc-connected components, one bounded B and one unbounded U. Moreover, the boundary of each of the components is equal to γ. We define the *region* of γ as the union of γ and B. In particular, if $\mathcal{R}$ is the region of γ, then $\operatorname{int} \mathcal{R} = B$, $\partial\mathcal{R} = \gamma$, and the exterior of $\mathcal{R}$ is U.

Let us state and prove one simple topological observation.

Proposition 7. *Let γ_1 and γ_2 be simple closed curves and let Γ_1 and Γ_2 be the regions of γ_1 and γ_2 respectively. If $\partial\Gamma_1 \subseteq \Gamma_2$, then $\Gamma_1 \subseteq \Gamma_2$.*

Proof. Since $\partial\Gamma_1 \subseteq \Gamma_2$, the exterior of Γ_2 is disjoint from $\partial\Gamma_1$. Since the exterior of Γ_2 is connected, it must be contained in some component of the complement of $\partial\Gamma_1$ in the plane. Since the exterior of Γ_2 is unbounded, it must be contained in the exterior of Γ_1. Thus, $\Gamma_1 \subseteq \Gamma_2$. □

A *drawing* of a graph G is a function f assigning a subset of the plane to each vertex and each edge of G such that $f(u)$ is a point in the plane for every vertex u in G and no two vertices are assigned to the same point; for every edge uv in G, $f(uv)$ is a curve in the plane connecting $f(u)$ and $f(v)$ disjoint from $f(w)$ for every vertex w of G distinct from u and v. We say that a drawing f of a graph G is *planar* if the interiors of the images of edges of G under f are pairwise disjoint. A graph G is *planar* if it admits a planar drawing. A *plane graph* is a planar graph with a fixed planar drawing. In plane graphs, we usually identify vertices with the assigned points in the plane and edges with the assigned curves in the plane. More generally, we identify subgraphs of plane graphs with unions of respective vertices and edges in the plane. For example, every path in a plane graph is a simple curve, and every cycle in a plane graph is a simple closed curve. The complement of a plane graph G in the plane is a union of arc-connected components. The topological closure of such a component is a *face* of G. The only unbounded face is called the *exterior face* of G.

3.5. **Ordering edges in a plane graph.** Let G be a plane graph. Let u be a vertex of G. There is a natural clockwise cyclic ordering of the edges of G incident to u. When $e_0, e_1, \ldots, e_\ell$ are edges (not necessarily distinct) incident to u, we will write $e_0 \preccurlyeq e_1 \preccurlyeq \cdots \preccurlyeq e_\ell$ *in the u-ordering* if starting with e_0 and proceeding in the clockwise manner around u, stopping at e_ℓ (i.e. at the first occurrence of e_ℓ or at the first return to e_0 when $e_\ell = e_0$), we have visited the edges $e_1, \ldots, e_{\ell-1}$ in this order. We can replace some of the $\preccurlyeq$ symbols with $\prec$ when the corresponding edges are distinct.

Sometimes, it is handy to cut the u-ordering. That is, for a given edge e_0 incident to u, there is a clockwise linear order on the edges incident to u with e_0 the least element. When e

[4]Since in this paper we only consider curves that are finite unions of segments, we only need the Jordan Curve Theorem for Polygons [3, Chapter 4].

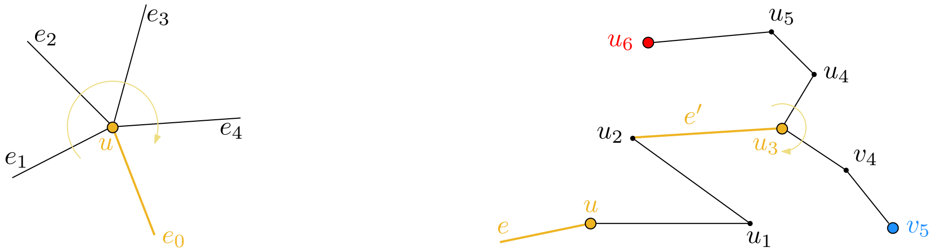


FIGURE 4. Left: We have $e_0 \prec e_1 \prec e_2 \prec e_3 \prec e_4$ in the u-ordering as well as in the (u, e_0)-ordering. On the other hand, $e_4 \prec e_0 \prec e_1$ in the u-ordering, which is not the case in the (u, e_0)-ordering. Right: Let $U = uu_1u_2u_3u_4u_5u_6$ and let $V = uu_1u_2u_3v_4v_5$. Both paths start in u, and neither is a prefix of the other. We have, $u[U]u_3 = u[V]u_3$. Let $e' = u_2u_3$. Then, $u_3u_4 \prec u_3v_4$ in the (u_3, e')-ordering, hence, $U \prec V$ in the (u, e)-ordering. Moreover, the sets $\{u_4, u_5, u_6\}$ and $\{v_4, v_5\}$ are disjoint, thus, U and V are u-consistent.

and e' are edges incident to u , we say that $e \preccurlyeq e'$ *in the* (u, e_0)*-ordering* if either $e_0 = e$ or $e_0 \prec e \preccurlyeq e' \prec e_0$ in the u-ordering. We can replace $\preccurlyeq$ with $\prec$ when the edges e, e' are distinct. See an example on the left side of Figure 4.

Let u be a vertex of G, and let e be an edge incident to u. The (u, e)-ordering can be extended to the family $\mathcal{P}$ of paths in G starting in u in the following way. Let $U, V \in \mathcal{P}$ with $U = u_0 \cdots u_\ell$, $V = v_0 \cdots v_m$. Let i be the maximum nonnegative integer such that $u_0[U]u_i = v_0[V]v_i$, and let $e_0 = e$ if $i = 0$ and $e_0 = u_{i-1}u_i$ otherwise. We say that $U \preccurlyeq V$ *in the* (u, e)*-ordering* if

$$U = V \quad \text{or} \quad i < \ell,\ i < m, \text{ and } u_iu_{i+1} \prec u_iv_{i+1} \text{ in the } (u_i, e_0)\text{-ordering.}$$

In the latter case, we write $U \prec V$ in the (u, e)-ordering. Note that if one of U and V is a proper prefix of the other, then neither of $U \prec V$ and $V \prec U$ holds in the (u, e)-ordering. Moreover, if $\{u_{i+1}, \ldots, u_\ell\}$ is disjoint from $\{v_{i+1}, \ldots, v_m\}$, then we say that U and V are *u-consistent*. See an illustration of these notions on the right side of Figure 4.

Proposition 8. *Let G be a plane graph, let u be a vertex of G, let e be an edge incident to u. Let U, V, W be paths in G starting in u such that $U \prec V$ and $V \prec W$ in the (u, e)-ordering. If v is a common vertex of U and W with $u[U]v = u[W]v$, then v is a vertex of V and $u[U]v = u[V]v = u[W]v$.*

Proof. Let $U = u_0 \cdots u_\ell$, $V = v_0 \cdots v_m$, and let i be such that $u_i = v$. Let j be the maximal nonnegative integer such that $u[U]u_j = u[V]u_j$. If $i \leqslant j$, the assertion holds. Thus, suppose that $j < i$. In this case u_ju_{j+1} lies in both U and W. Let $e_0 = e$ if $j = 0$ and $e_0 = u_{j-1}u_j$ otherwise. Since $U \prec V$ in the (u, e)-ordering, $j < m$ and $u_ju_{j+1} \prec u_jv_{j+1}$ in the (u_j, e_0)-ordering. Since $V \prec W$ in the (u, e)-ordering, $u_jv_{j+1} \prec u_ju_{j+1}$ in the (u_j, e_0)-ordering. This contradiction completes the proof. □

Proposition 9. *Let G be a plane graph, let u be a vertex of G, let e be an edge incident to u, and let $\mathcal{P}$ be a family of paths in G starting in u. Then, the (u, e)-ordering partially orders $\mathcal{P}$.*

Proof. Let $U, V, W \in \mathcal{P}$. The (u, e)-ordering is clearly reflexive and antisymmetric on $\mathcal{P}$. It suffices to show that it is transitive, namely, if $U \prec V$ and $V \prec W$ in the (u, e)-ordering, then $U \prec W$ in the (u, e)-ordering. Let $U = u_0 \cdots u_\ell, V = v_0 \cdots v_m, W = w_0 \cdots w_n$, and let k be the maximum nonnegative integer such that $u_0[U]u_k = w_0[W]w_k$. By Proposition 8, u_k is a vertex of V and $u_0[U]u_k = v_0[V]v_k = w_0[W]w_k$. Since $U \prec V$ and $V \prec W$ in the (u,e)-ordering, $k < \ell$, $k < m$, and $k < n$. Let $e_0 = e$ if $k = 0$ and $e_0 = u_{k-1}u_k$ otherwise. We have $u_ku_{k+1} \preccurlyeq v_kv_{k+1}$ and $v_kv_{k+1} \preccurlyeq w_kw_{k+1}$ in the (u_k,e_0)-ordering. Therefore, $u_ku_{k+1} \preccurlyeq w_kw_{k+1}$ in the (u_k,e_0)-ordering, and so, by the definition of k, $u_ku_{k+1} \prec w_kw_{k+1}$ in the (u_k,e_0)-ordering. This yields $U \prec W$ in the (u,e)-ordering as desired. □

For emphasis, we state the following straightforward observation.

Observation 10. *Let G be a plane graph, let u be a vertex of G, and let e be an edge incident to u. Every pair of paths in G starting in u is comparable in the (u,e)-ordering unless one is a subpath of the other. In particular, if $\mathcal{P}$ is a family of paths in G starting in u such that no path is a subpath of the other, then the (u,e)-ordering linearly orders $\mathcal{P}$.*

3.6. **Unfolding.** Given a poset Q and an element x of Q, we say that a set $J \subseteq \mathrm{Inc}(Q)$ is *singly constrained* by x in Q if for every $(a,b) \in J$ we have $x \leqslant b$ in Q. In 2014, Streib and Trotter [26] introduced a concept that is now known as "poset unfolding", which allows reducing the problem of bounding the dimension of a poset to bounding the dimension of a singly constrained set of pairs in a poset whose cover graph is a minor of the cover graph of the original poset. In particular, they showed that for every poset P with a planar cover graph, there exist a poset Q with a planar cover graph, an element x in Q such that $Q - x$ is a subposet of P or P^{-1}, and a set $J \subseteq \mathrm{Inc}(Q)$ that is singly constrained by x in Q with

$$\dim(P) \leqslant 2\dim_Q(J).$$

In this subsection, we introduce all the necessary tools, and we reprove (with a slightly adjusted setup) the reduction of Streib and Trotter.

Let P be a poset. An *upset* in P is a subset U of elements of P such that for all elements x, y with $x \leqslant y$ in P, if $x \in U$, then $y \in U$. For each element x of P, we denote by $U_P[x]$ the upset in P consisting of all elements y such that $x \leqslant y$ in P. Dually, a *downset* in P is a subset D of elements of P such that for all elements x, y with $x \leqslant y$ in P, if $y \in D$, then $x \in D$. For each element y of P, we denote by $D_P[y]$ the downset in P consisting of all elements x such that $x \leqslant y$ in P. For a set of elements Z in P, we denote by $D_P[Z]$ and $U_P[Z]$ the unions $\bigcup_{z\in Z} D_P[z]$ and $\bigcup_{z\in Z} U_P[z]$, respectively. Note that every upset and downset in a poset induces a convex subposet.

An *unfolding* of a poset P is a family $(Z_i : i \in \mathbb{N})$ of pairwise disjoint subsets of the ground set of P such that for all nonnegative integers i, j and elements $x \in Z_i, y \in Z_j$ with $x \leqslant y$ in P, either $i = j$, or $|i - j| = 1$ and i is even (and thus j is odd). For convenience, for every nonnegative integer k, we write $Z_{\geqslant k} = \bigcup_{i \geqslant k} Z_i$ and $Z_{<k} = \bigcup_{0 \leqslant i < k} Z_i$ where the unions go over integers. Note that for each k, the set Z_k is a downset if k is even, and an upset if k is odd. Moreover, $Z_{\geqslant k}$ and $Z_{<k+1}$ are downsets in P whenever k is even, and they are upsets in P whenever k is odd. In particular, $Z_{\geqslant k}$ and $Z_{<k+1}$ induce convex subposets of P.

An unfolding of a poset is somewhat a parallel of a layering of a graph.[5] Next, we discuss a parallel of BFS-layerings for posets. Let P be a connected poset, and let z_0 be a minimal element of P. Let $(Z_i : i \in \mathbb{N})$ be the BFS-layering of the comparability graph of P from z_0,

[5]A *layering* of a graph G is a family $(Z_i : i \in \mathbb{N})$ of pairwise disjoint subsets of $V(G)$ such that for every edge uv of G, there exists $i \in \mathbb{N}$ such that $\{u,v\} \subseteq Z_i \cup Z_{i+1}$. Given a connected graph G and $u, v \in V(G)$, the *distance* between u and v is the minimum number of edges in a path in G with endpoints u and v. Now, given a connected graph G and a vertex v of G, the BFS-*layering* of G from v is the sequence $(Z_i : i \in \mathbb{N})$, where Z_i is the set of all vertices at distance i from v in G for every nonnegative integer i.

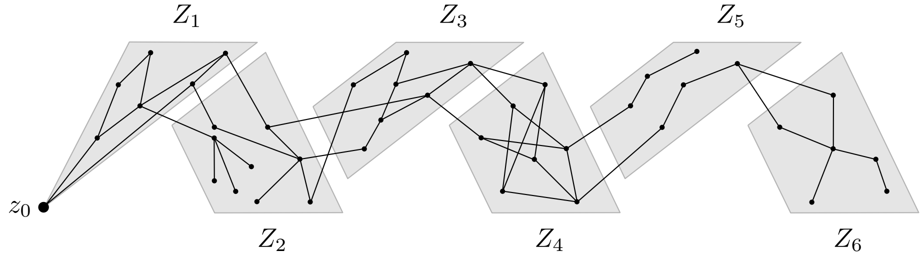


FIGURE 5. The unfolding of a poset from z_0. The drawing is a diagram, that is, for two elements connected by an edge, the lower is less than the higher in the poset. In all the following figures, when a segment has no direction and it is not stated otherwise, the comparability is diagram-like (go upwards).

that is, $Z_0 = \{z_0\}$, and for each positive integer k,

$$Z_k = \begin{cases} U_P[Z_{k-1}] \setminus (Z_0 \cup \cdots \cup Z_{k-1}) & \text{if } k \text{ is odd,} \\ D_P[Z_{k-1}] \setminus (Z_0 \cup \cdots \cup Z_{k-1}) & \text{if } k \text{ is even.} \end{cases}$$

Hence, $(Z_i : i \in \mathbb{N})$ is an unfolding of P, and we call it the unfolding of P *from* z_0. See an example of an unfolding from a minimal element in Figure 5.

Let $(Z_i : i \in \mathbb{N})$ be an unfolding of P. A set $I \subseteq \text{Inc}(P)$ is *supported* by $(Z_i : i \in \mathbb{N})$ if there exists a nonnegative integer k such that

$$\bigcup_{(a,b)\in I} \{a, b\} \subseteq Z_{\geqslant k} \quad \text{and} \quad \begin{cases} \{b : (a,b) \in I\} \subseteq Z_k & \text{if } k \text{ is odd,} \\ \{a : (a,b) \in I\} \subseteq Z_k & \text{if } k \text{ is even.} \end{cases}$$

If k is odd, then we say that I is *supported from below* by $(Z_i : i \in \mathbb{N})$, if k is even, then we say that I is *supported from above* by $(Z_i : i \in \mathbb{N})$. In both cases we say that k *witnesses* that I is supported by $(Z_i : i \in \mathbb{N})$. See the set J' in Figure 6, which is supported from below, witnessed by $k = 3$.

It is well-known that given a graph G and a layering of G, if every layer induces a k-colorable graph, then G is $2k$-colorable (one can use two disjoint palettes of k colors each, one for even layers, and one for odd layers). The counterpart of the above for posets is formulated in terms of the unfolding. Given a connected poset P and an unfolding of P, if the union of any two consecutive sets in the unfolding induces a subposet of dimension at most d, then the dimension of P is at most $2d$. We adapt the proof of this fact to reduce the problem of bounding $\dim_P(I)$ for any $I \subseteq \text{Inc}(P)$ to bounding $\dim_P(I')$ for some $I' \subseteq I$ which is supported by the unfolding of P.

Proposition 11. *Let P be a poset, and let $(Z_i : i \in \mathbb{N})$ be an unfolding of P. Then every set $I \subseteq \text{Inc}(P)$ contains subsets $I_0, I_1 \subseteq I$, where I_0 is supported from above by $(Z_i : i \in \mathbb{N})$ in P, I_1 is supported from below by $(Z_i : i \in \mathbb{N})$ in P, and*

$$\dim_P(I) \leqslant \dim_P(I_0) + \dim_P(I_1).$$

Proof. Define

$$I'_0 = \{(a, b) \in I : a \in Z_i, b \in Z_j \text{ with } i < j \text{ or } (i = j \text{ and } i \text{ is even})\}$$
$$I'_1 = \{(a, b) \in I : a \in Z_i, b \in Z_j \text{ with } i > j \text{ or } (i = j \text{ and } i \text{ is odd})\}.$$

Note that $I = I'_0 \cup I'_1$. We will find $I_0 \subseteq I'_0$ supported from above by $(Z_i : i \in \mathbb{N})$ in P and $I_1 \subseteq I'_1$ supported from below by $(Z_i : i \in \mathbb{N})$ in P, such that $\dim_P(I'_0) = \dim_P(I_0)$ and

$\dim_P(I_1') = \dim_P(I_1)$. This way, we will obtain

$$\dim_P(I) \leqslant \dim_P(I_0') + \dim_P(I_1') = \dim_P(I_0) + \dim_P(I_1),$$

as desired.

We start with the construction of $I_1 \subseteq I_1'$. For each odd positive integer j, define $I_{1,j} = \{(a,b) \in I_1' : b \in Z_j\}$. We show that each alternating cycle in P that has all pairs in I_1' is contained in $I_{1,j}$ for some odd positive integer j. Let $((a_1,b_1),\ldots,(a_k,b_k))$ be an alternating cycle in P with all pairs in I_1'. For each $\alpha \in [k]$, let $i_\alpha, j_\alpha \in \mathbb{N}$ be so that $a_\alpha \in Z_{i_\alpha}$ and $b_\alpha \in Z_{j_\alpha}$. We claim that for each $\alpha \in [k]$, we have $j_\alpha \leqslant j_{\alpha+1}$ cyclically. Since $(a_\alpha, b_\alpha) \in I_1'$, either $i_\alpha > j_\alpha$, or $i_\alpha = j_\alpha$ and i_α is odd. Since $a_\alpha \leqslant b_{\alpha+1}$ in P, we have $|i_\alpha - j_{\alpha+1}| \leqslant 1$, so if $i_\alpha > j_\alpha$, then $j_\alpha \leqslant j_{\alpha+1}$. On the other hand, if $i_\alpha = j_\alpha$ and i_α is odd, then Z_{i_α} is an upset in P, so $a_\alpha \leqslant b_{\alpha+1}$ in P implies that $b_{\alpha+1} \in Z_{i_\alpha}$, and thus, $j_\alpha = i_\alpha = j_{\alpha+1}$. In both cases, $j_\alpha \leqslant j_{\alpha+1}$. This holds cyclically for all $\alpha \in [k]$, so $j_1 = \cdots = j_k$. In order to conclude the claim, it suffices to show that j_1 is odd. Suppose to the contrary that j_1 is even. Since Z_{j_1} is a downset in P and $a_k \leqslant b_1$ in P, we have $a_k \in Z_{j_1}$, so $i_k = j_1 = j_k$. As j_1 is even, this contradicts $(a_k, b_k) \in I_1'$. Hence, j_1 is odd and $(a_\alpha, b_\alpha) \in I_{1,j_1}$ for each $\alpha \in [k]$. By Proposition 5, there exists an odd positive integer j such that $\dim_P(I_{1,j}) = \dim_P(I_1')$. Let $I_1 = I_{1,j}$. For each $(a,b) \in I_1$ we have $b \in Z_j$ and $a \in Z_{\geqslant j}$, so j witnesses that I_1 is supported from below by $(Z_i : i \in \mathbb{N})$ in P.

Next, we construct $I_0 \subseteq I_0'$. Note that the construction is symmetric. For each even positive integer i, define $I_{0,i} = \{(a,b) \in I_0 : a \in Z_i\}$. We show that each alternating cycle in P that has all pairs in I_0' is contained in $I_{0,i}$ for some even positive integer i. Let $((a_1,b_1),\ldots,(a_k,b_k))$ be an alternating cycle in P with all pairs in I_0'. For each $\alpha \in [k]$, let $i_\alpha, j_\alpha \in \mathbb{N}$ be so that $a_\alpha \in Z_{i_\alpha}$ and $b_\alpha \in Z_{j_\alpha}$. We claim that for each $\alpha \in [k]$, we have $i_\alpha \leqslant i_{\alpha-1}$ cyclically. Since $(a_\alpha, b_\alpha) \in I_0'$, either $i_\alpha < j_\alpha$, or $i_\alpha = j_\alpha$ and i_α is even. Since $a_{\alpha-1} \leqslant b_\alpha$ in P, we have $|i_{\alpha-1} - j_\alpha| \leqslant 1$, so if $i_\alpha < j_\alpha$, then $i_\alpha \leqslant i_{\alpha-1}$. On the other hand, if $i_\alpha = j_\alpha$ and j_α is even, then Z_{j_α} is a downset in P, so $a_{\alpha-1} \leqslant b_\alpha$ in P implies that $a_{\alpha-1} \in Z_{j_\alpha}$, and thus, $i_\alpha = j_\alpha = i_{\alpha-1}$. In both cases, $i_\alpha \leqslant i_{\alpha-1}$. This holds cyclically for all $\alpha \in [k]$, so $i_1 = \cdots = i_k$. In order to conclude the claim, it suffices to show that i_1 is even. Suppose to the contrary that i_1 is odd. Since Z_{i_1} is an upset in P and $a_1 \leqslant b_2$ in P, we have $b_2 \in Z_{i_1}$, so $j_2 = i_1 = i_2$. As i_1 is odd, this contradicts $(a_2, b_2) \in I_0'$. Hence, i_1 is even and $(a_\alpha, b_\alpha) \in I_{0,i_1}$ for each $\alpha \in [k]$. By Proposition 5, there exists an even positive integer i such that $\dim_P(I_{0,i}) = \dim_P(I_0')$. Let $I_0 = I_{0,i}$. For each $(a,b) \in I_0$ we have $a \in Z_i$ and $b \in Z_{\geqslant i}$, so i witnesses that I_0 is supported from above by $(Z_i : i \in \mathbb{N})$ in P. □

Finally, we apply Proposition 11 to posets with planar cover graphs. Application of the following result is the initial step in the proof of the main result of this paper: Theorem 1.

Lemma 12. *Let P be a poset with a planar cover graph and let $I \subseteq \mathrm{Inc}(P)$. Then there exist a poset Q, a set $J \subseteq \mathrm{Inc}(Q)$, and a minimal element x of Q such that*

(i) *the cover graph of Q is planar,*
(ii) *$Q - x$ is a convex subposet of P and $J \subseteq I$ or $Q - x$ is a convex subposet of P^{-1} and $J \subseteq I^{-1}$,*
(iii) *J is singly constrained by x in Q, and*
(iv) *$\dim_P(I) \leqslant 2\dim_Q(J)$.*

Proof. If $\dim_P(I) \leqslant 2$, then the lemma is satisfied by a poset Q consisting of one element x and $J = \emptyset$. Hence, we assume that $\dim_P(I) \geqslant 3$. By Proposition 6, there is a component C of P with $\dim_P(I) = \dim_C(I_C)$, where $I_C = \mathrm{Inc}(C) \cap I$. Thus, without loss of generality, we assume that P is connected. Let z_0 be a minimal element of P, and let $(Z_i : i \in \mathbb{N})$ be the

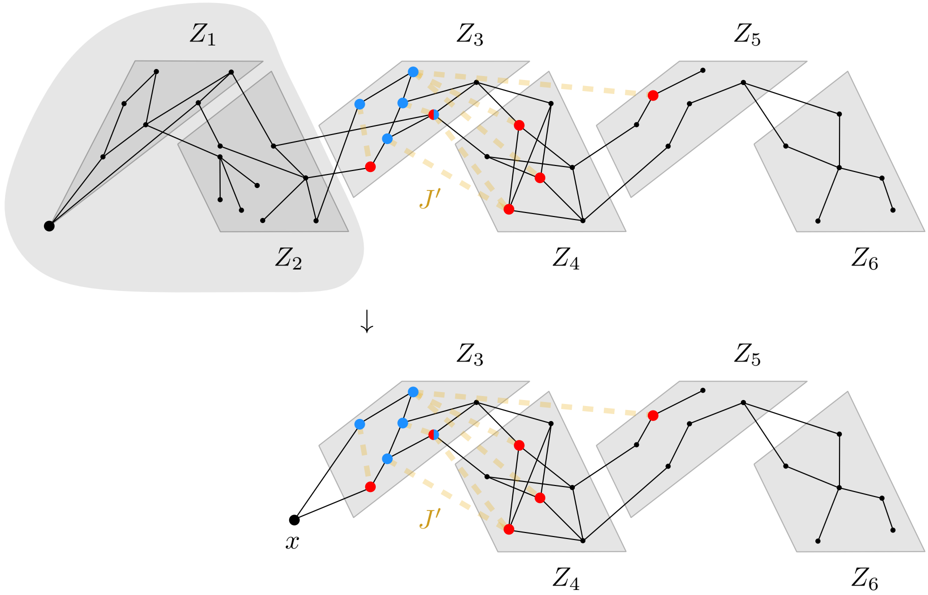


FIGURE 6. With yellow lines, we denote the incomparable pairs in J', which is supported from below by $(Z_i : i \in \mathbb{N})$. We contract $Z_0 \cup Z_1 \cup Z_2$ into a vertex x. The cover graph is still planar, and J' is singly constrained by x.

unfolding of P from z_0. By Proposition 11 applied to I, there exists a set $J' \subseteq I$ supported by $(Z_i : i \in \mathbb{N})$, which is witnessed by the integer k such that

$$\dim_P(I) \leqslant 2 \dim_P(J').$$

If $k = 0$, then for each $(a, b) \in J'$, we have $a = z_0$. Therefore, J' is reversible in P as there is no strict alternating cycle in P containing only pairs in J'. It follows that $\dim_P(J') \leqslant 1$, hence, $\dim_P(I) \leqslant 2 \dim_P(J') \leqslant 2$, which contradicts $\dim_P(I) \geqslant 3$.

Next, we assume that $k > 0$. Let $Y = Z_{<k}$. Since $k > 0$, the subgraph of the cover graph of P induced by Y is nonempty. We claim that it is also connected. Indeed, let P_Y be the subposet of P induced by Y. By the properties of the unfolding from z_0, the comparability graph of P_Y is connected, and P_Y is a convex subposet of P. Since P_Y is a convex subposet of P, the cover graph of P_Y coincides with the subgraph of the cover graph of P induced by Y. Since P_Y is connected, its cover graph is connected as well, and thus, the subgraph of the cover graph of P induced by Y is connected, as claimed.

We obtain Q' from P by "contracting" the set Y into a single element x. Formally, let the ground set of Q' consist of all the elements of $Z_{\geqslant k}$ and a new element x. The order relations within $Z_{\geqslant k}$ are the same as in P. If k is odd, then let $x < z$ in Q' for all $z \in Z_k$ and if k is even, let $z < x$ in Q' for all $z \in Z_k$, while $x \parallel z$ in Q' for all $z \in Z_{\geqslant k+1}$ in both cases. See Figure 6.

Note that the neighborhood of x in the cover graph of Q' is the set of all minimal elements in the subposet of P induced by Z_k if k is odd and the set of all maximal elements in the subposet of P induced by Z_k if k is even. Moreover, since $Z_{\geqslant k}$ induces a convex subposet of P, the cover graph of Q' is a subgraph of the graph H obtained from the cover graph G of P by contracting Y into x (note that it can be a proper subgraph as in Figure 6). Since the subgraph of G induced by Y is nonempty and connected, H is planar. In particular, the cover graph of Q' is planar.

Each $(a,b) \in J'$ remains an incomparable pair in Q', moreover, the subposets induced by $\bigcup_{(a,b)\in J'}\{a,b\}$ are the same in P and Q', and so, $\dim_P(J') = \dim_{Q'}(J')$. Finally, in the case, where k is odd, $x \leqslant b$ in Q' for all $(a,b) \in J'$, and we take $Q = Q'$, $J = J'$. In the case, where k is even, $a \leqslant x$ in Q' for all $(a,b) \in J'$. Let $Q = (Q')^{-1}$ and let $J = (J')^{-1}$. In both cases, J is singly constrained by x in Q and $\dim_P(I) \leqslant 2\dim_{Q'}(J') = 2\dim_Q(J)$. Note that x is a minimal element in Q. □

4. Topology of an instance

We say that $(P, x_0, G, e_{-\infty}, I)$ is an *instance* if

(I1) P is a poset with a planar cover graph,
(I2) x_0 is a minimal element in P,
(I3) G is a plane graph obtained from the cover graph of P by fixing a planar drawing of G with x_0 in the exterior face,
(I4) $e_{-\infty}$ is a curve in the plane contained in the exterior face of G such that the only common point of the curve and G is x_0,
(I5) $I \subseteq \mathrm{Inc}(P)$ and I is singly constrained by x_0 in P.

The notion of unfolding introduced in Subsection 3.6 and Lemma 12 allow us to reduce the main problem to studying instances as captured in the following statement.

Corollary 13. *For every poset P with a planar cover graph and $I \subseteq \mathrm{Inc}(P)$, there exists an instance $(P', x_0, G, e_{-\infty}, I')$ such that*

$$\dim_P(I) \leqslant 2\dim_{P'}(I').$$

Moreover, $P' - x_0$ is a convex subposet of P and $I' \subseteq I$, or $P' - x_0$ is a convex subposet of P^{-1} and $I' \subseteq I^{-1}$.

Proof. Let P be a poset with a planar cover graph and $I \subseteq \mathrm{Inc}(P)$. By Lemma 12, there exists a poset P', a set $I' \subseteq \mathrm{Inc}(P')$, and a minimal element x_0 of P' such that the cover graph of P' is planar, $P' - x_0$ is a subposet of P and $I' \subseteq I$ or $P' - x_0$ is a subposet of P^{-1} and $I' \subseteq I^{-1}$, I' is singly constrained by x_0 in P', and $\dim_P(I) \leqslant 2\dim_{P'}(I')$. Let G be a plane graph obtained from the cover graph of P by fixing a planar drawing of G with x_0 in the exterior face, and let $e_{-\infty}$ be a curve in the plane contained in the exterior face of G such that the only common point of the curve and G is x_0. It follows that $(P', x_0, G, e_{-\infty}, I')$ is a desired instance. □

In this section, we develop a basic understanding of what an instance looks like. We discuss the theory of shadows and formulate many useful properties of instances. To this end, we fix an instance $(P, x_0, G, e_{-\infty}, I)$ for the remainder of this section.

4.1. **Orientations and regions.** Let $G_{-\infty}$ be the plane graph obtained from G by adding a new vertex $x_{-\infty}$ and a new edge $x_{-\infty}x_0$, which is identified with $e_{-\infty}$. We call $G_{-\infty}$ the *rooted graph* of the instance $(P, x_0, G, e_{-\infty}, I)$. When an instance is fixed (like in this section), every (v,e)-ordering of edges or paths (as defined in Subsection 3.5) is considered in the rooted graph of the instance.

Let γ be a cycle in G and let Γ be the region of γ. An *orientation* of γ is a cyclic ordering of the edges of γ such that incident edges are consecutive. In particular, γ has exactly two orientations: *clockwise* and *counterclockwise*. If $e_1, \ldots, e_n$ (cyclically) is an orientation of γ, then we say that e_{i+1} *follows* e_i in this orientation and e_{i-1} *precedes* e_i in this orientation (indices are considered cyclically in $[n]$). The main feature of the counterclockwise orientation is enclosed in the following observation.

Observation 14. *Let γ be a cycle in G and let Γ be the region of γ. Let w be a vertex of γ and let e^- and e^+ be the edges of γ incident to w such that e^+ follows e^- in the counterclockwise orientation in γ. Then, for every edge e incident to w,*

$$e \text{ is contained in } \Gamma \quad \text{if and only if} \quad e^- \preccurlyeq e \preccurlyeq e^+ \text{ in the } w\text{-ordering.}$$

4.2. **Leftmost and rightmost witnessing paths.** For two elements u and v of P, a *witnessing path* from u to v in P is a path $w_0 \cdots w_n$ in G such that $w_0 = u$, $w_n = v$, and w_{i-1} is covered by w_i for every $i \in [n]$. Observe that there exists a witnessing path from u to v in P if and only if $u \leqslant v$ in P.

Let U and V be two paths in G starting in x_0. We say that U is *left* of V if $U \prec V$ in the $(x_0, e_{-\infty})$-ordering. If U is left of V, then we say that V is *right* of U. Moreover, we define $\text{gcpe}(U, V)$ to be the element w in $U \cap V$, where $x_0[U]w = x_0[V]w$ and this prefix is the longest possible. The abbreviation stands for “greatest common prefix-element”.

Let

$$B = U_P[x_0] = \{b \text{ in } P : x_0 \leqslant b \text{ in } P\}.$$

Let $b \in B$ and let $\mathcal{P}$ be the family of all witnessing paths from x_0 to b in P. Note that $\mathcal{P}$ is nonempty. By Observation 10, $\mathcal{P}$ has a unique minimal element and a unique maximal element in the $(x_0, e_{-\infty})$-ordering. We call the minimal element the *leftmost* witnessing path from x_0 to b in P and we call the maximal element the *rightmost* witnessing path from x_0 to b in P. We denote the former $W_L(b)$ and the latter $W_R(b)$. See Figure 7. Note that by definition, for all witnessing paths W from x_0 to b, either $W = W_L(b)$ or $W_L(b)$ is left of W. Similarly, for all witnessing paths W from x_0 to b, either $W = W_R(b)$ or $W_R(b)$ is right of W. Next, we develop some properties of the leftmost and rightmost witnessing paths.

Proposition 15. *Let $b, b' \in B$.*

(L) *The paths $W_L(b)$ and $W_L(b')$ are x_0-consistent.*
(R) *The paths $W_R(b)$ and $W_R(b')$ are x_0-consistent.*

Proof. We prove that $W_L(b)$ and $W_L(b')$ are x_0-consistent. The proof for $W_R(b)$ and $W_R(b')$ is symmetric. Suppose to the contrary that $W_L(b)$ and $W_L(b')$ are not x_0-consistent. Let $u = \text{gcpe}(W_L(b), W_L(b'))$. Since the paths are not x_0-consistent there exists an element v distinct from u in the intersection of $u[W_L(b)]b$ and $u[W_L(b')]b'$. Let $e_0 = e_{-\infty}$ when $u = x_0$ and let e_0 be the last edge in $x_0[W_L(b)]u$ when $u \neq x_0$. Let e be the first edge of $u[W_L(b)]b$ and let e' be the first edge of $u[W_L(b')]b'$. It follows that $e \neq e'$. Without loss of generality, assume that $e \prec e'$ in the (u, e_0)-ordering. Consider the path $W = x_0[W_L(b)]v[W_L(b')]b'$. By definition, W is left of $W_L(b')$, which is a contradiction. □

By Proposition 15, it follows that for all $u, b \in B$ if u lies in $W_L(b)$, then $W_L(u)$ is a prefix of $W_L(b)$. Similarly, if u lies in $W_R(b)$, then $W_R(u)$ is a prefix of $W_R(b)$. We will use this property implicitly many times. The next statement is an immediate corollary of Proposition 8.

Corollary 16. *Let W_1, W_2, W_3 be pairwise x_0-consistent witnessing paths in P such that W_1 is left of W_2 and W_2 is left of W_3. If an element u in P lies in both W_1 and W_3, then u lies in W_2.*

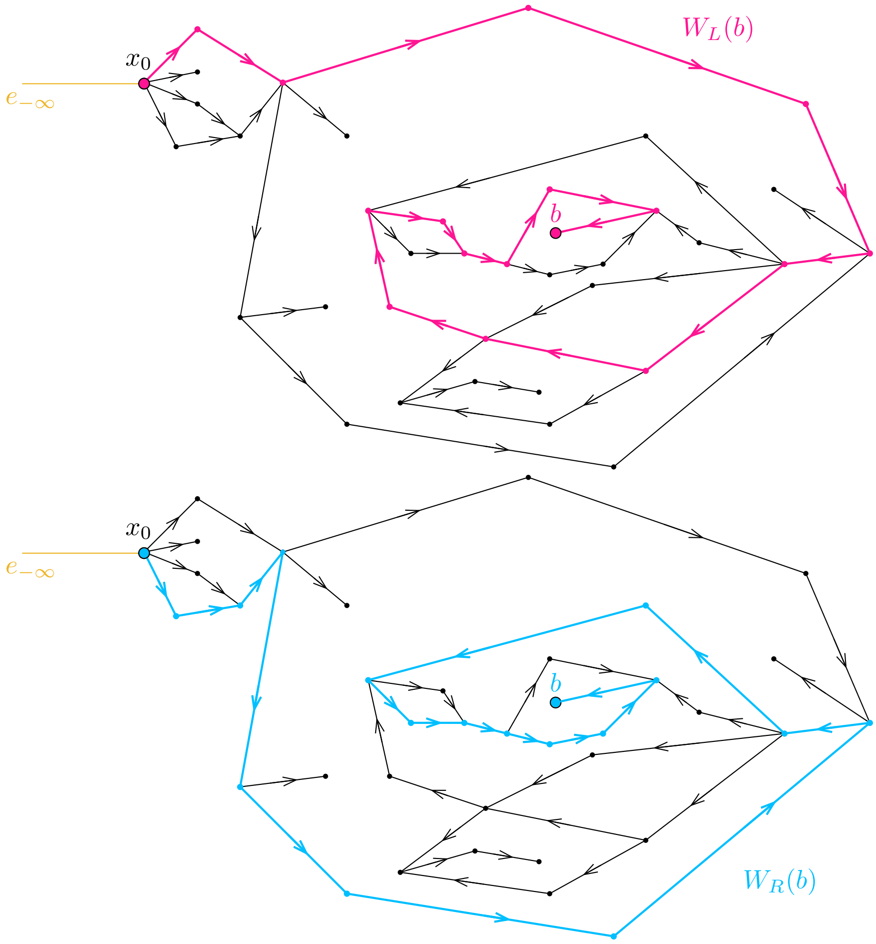


FIGURE 7. Recall that we fixed an instance $(P, x_0, G, e_{-\infty}, I)$. The figure illustrates a part of the drawing of G induced by elements in $B = U_P[x_0]$. In the top part of the figure, we mark the edges and vertices of the leftmost witnessing path from x_0 to b, and in the bottom part, we mark the edges and vertices of the rightmost witnessing path from x_0 to b. We will use the poset depicted in this figure several times to illustrate various notions.

Proposition 17. *Let $b, b' \in B$.*

(L) If $W_L(b)$ is left of $W_L(b')$, then $b \not\leqslant b'$ in P.
(R) If $W_R(b)$ is right of $W_R(b')$, then $b \not\leqslant b'$ in P.

Proof. We prove statement (L). The argument for statement (R) is symmetric. Suppose to the contrary that $W_L(b)$ is left of $W_L(b')$ and $b \leqslant b'$ in P. Let W be a witnessing path from b to b'. Since $W_L(b)$ is left of $W_L(b')$, we have $x_0[W_L(b)]b[W]b'$ left of $W_L(b')$, which is a contradiction. □

We finish the subsection on witnessing paths with a statement which is useful in a definition of some specific regions in G that we study in Subsection 6.2.

Proposition 18. *Let $u, v \in B$ such that $W_L(u)$ is left of $W_L(v)$. Let q be a common element of $W_L(u)$ and $W_R(v)$, let e be the first edge of $q[W_L(u)]u$, and let f be the first edge of $q[W_R(v)]v$. Let e^- be the last edge of $x_0[W_L(u)]q$ if $q \neq x_0$, and $e^- = e_{-\infty}$ if $q = x_0$. Then,*

$$e \preccurlyeq f \text{ in the } (q, e^-)\text{-ordering.}$$

Proof. Suppose to the contrary that $f \prec e$ in the (q, e^-)-ordering. Consequently, the witnessing path $x_0[W_L(u)]q[W_R(v)]v$ is left of the witnessing path $x_0[W_L(u)]q[W_L(u)]u = W_L(u)$. Hence $W_L(v)$, which is the leftmost witnessing path from x_0 to v in P, is also left of $W_L(u)$, a contradiction. □

4.3. **Blocks and shadows.** Let x, y be elements of P with $x < y$ in P. Let U, V be two witnessing paths from x to y in P such that the only common elements of U and V are x and y. Note that there are two possibilities: (a) x is covered by y in P and $U = V = xy$; and (b) each of U and V has some internal elements, and $U \cup V$ is a simple closed curve[6] contained in G. The *block* enclosed by U and V is a subset $\mathcal{B}$ of the plane defined as $\mathcal{B} = U$ if (a) holds; $\mathcal{B}$ is the region of $U \cup V$ if (b) holds. In the case, where (a) holds, we say that $\mathcal{B}$ is *degenerate*, and if (b) holds, we say that $\mathcal{B}$ is *non-degenerate*. We will refer to the element x as $\min \mathcal{B}$, while the element y will be $\max \mathcal{B}$. Since $\partial\mathcal{B} = U \cup V$, for every element $w \in \partial\mathcal{B}$, we have $\min \mathcal{B} \leqslant w \leqslant \max \mathcal{B}$ in P. In particular, the following holds.

Observation 19. *Let $\mathcal{B}$ be a block, let u and v be elements of P with $u \leqslant v$ in P such that one of u and v is in $\mathcal{B}$ and the other is not in $\operatorname{int} \mathcal{B}$. Then $\min \mathcal{B} \leqslant v$ in P and $u \leqslant \max \mathcal{B}$ in P.*

Let $\mathcal{B}$ be a non-degenerate block enclosed by U and V that are witnessing paths from $\min \mathcal{B}$ to $\max \mathcal{B}$. We will designate one of U and V to be the *left side* of $\mathcal{B}$, while the other path will be the *right side* of $\mathcal{B}$. The distinction is made using the following convention. If e and f are the edges of $U \cup V$ incident to $\min \mathcal{B}$ such that f follows e in the counterclockwise orientation of $U \cup V$, then the path (one out of U and V) containing e is the left side and the path containing f is the right side. The elements $\min \mathcal{B}$ and $\max \mathcal{B}$ belong to both sides. An element u on the left side of $\mathcal{B}$, with $\min \mathcal{B} < u < \max \mathcal{B}$ in P is said to be *strictly* on the left side of $\mathcal{B}$. Symmetrically, an element u on the right side of $\mathcal{B}$, with $\min \mathcal{B} < u < \max \mathcal{B}$ in P is said to be *strictly* on the right side of $\mathcal{B}$. When $\mathcal{B}$ is a degenerate block, we consider the whole block to be both the left side and the right side of $\mathcal{B}$, and there are no elements that are strictly on one of the two sides.

For the discussion to follow, we fix an element $b \in B$. We will introduce notation that defines sequences, paths, blocks, and elements of P, all depending on the choice of b. The elements of P in $W_L(b) \cap W_R(b)$ form a chain containing x_0 and b. The elements of this chain can be listed sequentially as $(z_0, \ldots, z_m)$ such that $z_0 < \cdots < z_m$ in P. We will refer to $(z_0, \ldots, z_m)$ as the *sequence of common points* of b. Note that $z_0 = x_0$ and $z_m = b$. For all $i \in [m]$, we define $\mathcal{B}_i$ to be the block enclosed by $z_{i-1}[W_L(b)]z_i$ and $z_{i-1}[W_R(b)]z_i$. We will refer to the blocks obtained in such a way as *shadow blocks* of b. A *shadow block* is a shadow block of an element in B. The sequence $(\mathcal{B}_1, \ldots, \mathcal{B}_m)$ is called the *sequence of blocks* of b. See Figure 8. We prove the following seemingly obvious statement.

Proposition 20. *Let $\mathcal{B}$ be a shadow block, let $x = \min \mathcal{B}$ and $y = \max \mathcal{B}$. Then $x[W_L(y)]y$ is the left side of $\mathcal{B}$ while $x[W_R(y)]y$ is the right side of $\mathcal{B}$.*

[6]Recall that witnessing paths are identified with curves in the drawing.

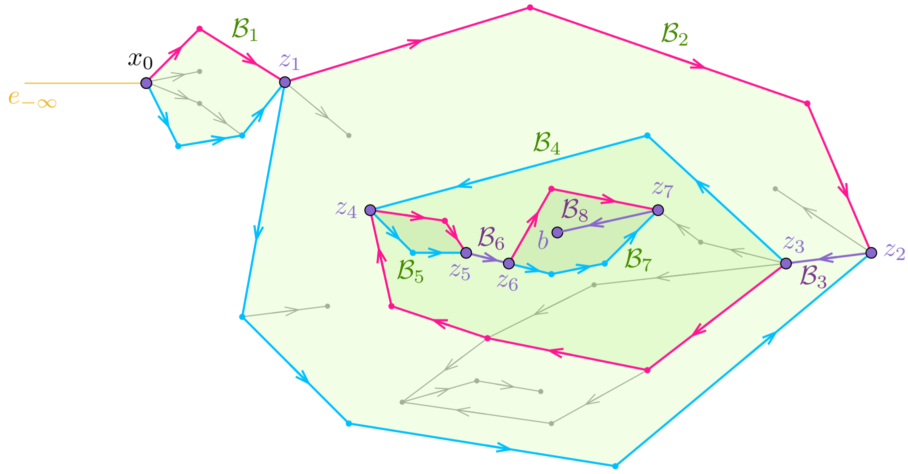


FIGURE 8. We mark in purple the common edges and vertices of the paths $W_L(b)$ and $W_R(b)$, and we mark in red (resp. blue) the remaining edges and vertices of the path $W_L(b)$ (resp. $W_R(b)$). The purple vertices $z_1, \ldots, z_7$ along with $z_0 = x_0$ and $z_8 = b_1$ form the sequence of common points of b. The shadow block $\mathcal{B}_1$ is a non-degenerate block with the initial element x_0 and the terminal element z_1, the shadow block $\mathcal{B}_2$ is a non-degenerate block with the initial element z_1 and the terminal element z_2, the shadow block $\mathcal{B}_3$ is a degenerate block consisting of the edge z_2z_3, and so on. We obtain non-degenerate shadow blocks $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_4, \mathcal{B}_5, \mathcal{B}_7$ marked in green, and degenerate shadow blocks $\mathcal{B}_3, \mathcal{B}_6, \mathcal{B}_8$ marked in purple as common edges of $W_L(b)$ and $W_R(b)$.

Proof. The statement holds when $\mathcal{B}$ is degenerate, thus, assume otherwise. Let e be the edge of $x[W_L(y)]y$ incident to x, and let f be the edge of $x[W_R(y)]y$ incident to x. Since $x \leqslant z$ in P for every element z in $\partial\mathcal{B}$, we get that the only common element of $W_L(x) = x_0[W_L(y)]x$ and $\partial\mathcal{B}$ is x. Let $e^- = e_{-\infty}$ if $x = x_0$ and let e^- be the last edge of $W_L(x)$ otherwise. By (I4), e^- is contained in the exterior of $\mathcal{B}$. Therefore, by Observation 14, $e^- \prec e \prec f$ in the x-ordering. In particular, f follows e in the counterclockwise orientation of $\partial\mathcal{B}$, so $x[W_L(y)]y$ is the left side of $\mathcal{B}$ and $x[W_R(y)]y$ is the right side of $\mathcal{B}$. □

Let $\mathcal{B}$ be a shadow block of b, let $x = \min\mathcal{B}$, and let $y = \max\mathcal{B}$. Let u be an element of P in $\partial\mathcal{B}$. If u is on the left side of $\mathcal{B}$, let e_L^+ and e_L^- be, respectively, the edges (provided they exist) immediately after and immediately before u on the path $W_L(b)$. Also, if u is on the right side of $\mathcal{B}$, let e_R^+ and e_R^- be, respectively, the edges (provided they exist) immediately after and immediately before u on $W_R(b)$. Note that if $u \in \{x, y\}$, then u is on both sides of $\mathcal{B}$. By Observation 14, an edge e incident to u lies in $\mathcal{B}$ if and only if one of the following holds (see Figure 9):

(sb1) $u = x$ and $e_L^+ \preccurlyeq e \preccurlyeq e_R^+$ in the x-ordering;
(sb2) $u = y$ and $e_R^- \preccurlyeq e \preccurlyeq e_L^-$ in the y-ordering;
(sb3) u is strictly on the left side and $e_L^+ \preccurlyeq e \preccurlyeq e_L^-$ in the u-ordering;
(sb4) u is strictly on the right side and $e_R^- \preccurlyeq e \preccurlyeq e_R^+$ in the u-ordering.

Since x_0 is in the exterior face of G, we have $x_0 \notin \operatorname{int}\mathcal{B}_i$ for all $i \in [m]$. Furthermore, let $i, j \in [m]$ with $i < j$. The boundaries of $\mathcal{B}_i$ and $\mathcal{B}_j$ have at most one common point. In fact, their only common point can be z_i, and this is the case if and only if $j = i + 1$. In particular, if $j \neq i + 1$, then $\mathcal{B}_i$ and $\mathcal{B}_j$ are either disjoint or one is contained in the other. Finally, we

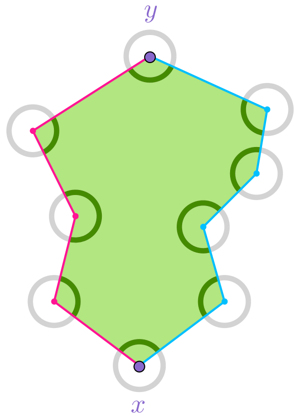


FIGURE 9. For each vertex on the boundary of a non-degenerate block, we mark the angle, where the incident edges are in the interior of the block (dark shade) and in the exterior of the block (light shade). This illustrates, where the items (sb1)–(sb4) come from.

claim that $\mathcal{B}_i$ is not contained in $\mathcal{B}_j$. Suppose to the contrary that $\mathcal{B}_i \subseteq \mathcal{B}_j$. Since $x_0 \notin \operatorname{int} \mathcal{B}_j$ and $z_{i-1} \in \mathcal{B}_i$, by Observation 19, $z_{j-1} = \min \mathcal{B}_j \leqslant z_{i-1}$ in P, which is a contradiction as $z_{i-1} < z_{j-1}$ in P by definition. We conclude that either $\mathcal{B}_i$ and $\mathcal{B}_j$ are disjoint or $\mathcal{B}_j \subseteq \mathcal{B}_i$.

Let $i \in [m-1]$. Let e_L^+ and e_L^- be, respectively, the edges immediately after and immediately before z_i on the path $W_L(b)$ and let e_R^+ and e_R^- be, respectively, the edges immediately after and immediately before z_i on $W_R(b)$. Since $i < m$, we know $z_i \neq b$, and therefore, $z_i < b$ in P. We have two cases (illustrated in Figure 10):

Case 1: b is in the exterior of $\mathcal{B}_i$.

We claim that in this case, the interiors of $\mathcal{B}_i$ and $\mathcal{B}_{i+1}$ are disjoint. Since we know that z_i is the only common point of the boundaries of $\mathcal{B}_i$ and $\mathcal{B}_{i+1}$, in order to prove the claim, we have to exclude the situation where $\mathcal{B}_{i+1} \subseteq \mathcal{B}_i$. If $\mathcal{B}_{i+1} \subseteq \mathcal{B}_i$, then $z_{i+1} \in \operatorname{int} \mathcal{B}_i$. Since $b \notin \mathcal{B}_i$, by Observation 19, $z_{i+1} \leqslant \max \mathcal{B}_i = z_i$ in P, which is a contradiction. Therefore, as claimed, the interiors of $\mathcal{B}_i$ and $\mathcal{B}_{i+1}$ are disjoint. Moreover, by simple induction, we obtain that $\mathcal{B}_i$ and $\mathcal{B}_j$ are disjoint for every j with $i+1 < j \leqslant m$.

Furthermore, we claim that $e_L^- \prec e_L^+ \preccurlyeq e_R^+ \prec e_R^-$ in the z_i-ordering. Indeed, by (sb2) for $\mathcal{B}_i$ and each $e \in \{e_L^+, e_R^+\}$, we obtain $e_L^- \prec e \prec e_R^-$ in the z_i-ordering. Moreover, by (sb1) for $\mathcal{B}_{i+1}$ and e_L^-, we obtain $e_R^+ \prec e_L^- \prec e_L^+$ in the z_i-ordering, in other words, $e_L^- \prec e_L^+ \preccurlyeq e_R^+$ in the z_i-ordering. Combining the above, we obtain $e_L^- \prec e_L^+ \preccurlyeq e_R^+ \prec e_R^-$ in the z_i-ordering as desired.

Case 2: b is in the interior of $\mathcal{B}_i$.

We claim that in this case, the interiors of $\mathcal{B}_i$ and $\mathcal{B}_{i+1}$ are not disjoint, and thus, $\mathcal{B}_{i+1} \subseteq \mathcal{B}_i$. If $\mathcal{B}_i$ and $\mathcal{B}_{i+1}$ have disjoint interiors, then $z_{i+1} \notin \operatorname{int} \mathcal{B}_i$. Since $b \in \mathcal{B}_i$, by Observation 19, $z_{i+1} \leqslant \max \mathcal{B}_i = z_i$ in P, which is a contradiction. Therefore, as claimed, $\mathcal{B}_{i+1} \subseteq \mathcal{B}_i$. Moreover, by simple induction, we obtain that $\mathcal{B}_j \subseteq \mathcal{B}_i$ contains for every j with $i+1 < j \leqslant m$. Furthermore, again by (sb1) and (sb2), $e_R^- \prec e_L^+ \preccurlyeq e_R^+ \prec e_L^-$ in the z_i-ordering.

When the second case holds, i.e., b is in the interior of $\mathcal{B}_i$, we call the element z_i a *reversing element* of b. The number of reversing elements in the sequence of common elements of b is the *shadow depth* of b, denoted $\operatorname{sd}(b)$. Note that b may have no reversing elements and then, the shadow depth of b is 0.

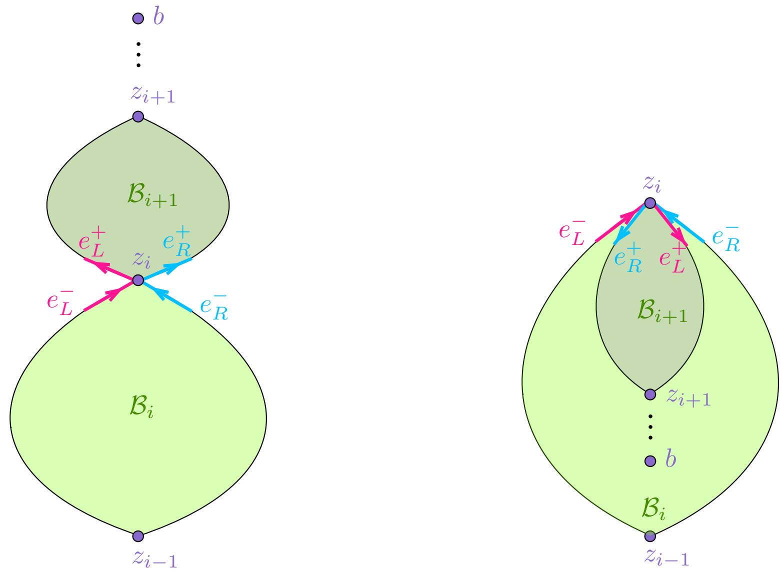


FIGURE 10. In some figures, for simplicity, instead of drawing all elements and edges, we depict witnessing paths as curves and mark only some "important" elements and edges. Here, on the left-hand side, we depict the case when b is in the exterior of $\mathcal{B}_i$, and on the right-hand side, we depict the case when b is in the interior of $\mathcal{B}_i$. In the latter case, z_i is a reversing element.

Next, when $\mathrm{sd}(b) = r$, we define the sequence $(\mathrm{shad}_0(b), \ldots, \mathrm{shad}_r(b))$ of sets called the *shadow sequence* of b. First, let $(i_1, \ldots, i_r)$ be the subsequence of $(1, \ldots, m-1)$ determined by the subscripts of the reversing elements of b. We expand this sequence by adding $i_0 = 0$ at the beginning and adding $i_{r+1} = m$ at the end. For each $j \in \{0, \ldots, r\}$, we then set

$$\mathrm{shad}_j(b) = \bigcup_{i_j < i \leqslant i_{j+1}} \mathcal{B}_i.$$

Also, we refer to $(\mathcal{B}_{i_j+1}, \ldots, \mathcal{B}_{i_{j+1}})$ as the *sequence of blocks* of $\mathrm{shad}_j(b)$. We call $\mathcal{B}_{i_{j+1}}$ the *terminal block* of $\mathrm{shad}_j(b)$. Also, we call the element call z_{i_j} the *initial element* of $\mathrm{shad}_j(b)$, and we call $z_{i_{j+1}}$ the *terminal element* of $\mathrm{shad}_j(b)$. Additionally, we define $\mathrm{shad}_{r+1}(b) = \emptyset$. See an example in Figure 11.

We continue with a series of statements concerning the combinatorics of shadow blocks and shadows. Observation 19 immediately gives the following facts.

Observation 21. *Let $\mathcal{B}$ be a shadow block and let u and v be elements of P with $u \leqslant v$ in P and $u \not\leqslant \max \mathcal{B}$ in P. Then, $u \in \mathcal{B}$ if and only if $v \in \mathcal{B}$. More precisely, either $u, v \in \mathrm{int}\, \mathcal{B}$ or $u, v \notin \mathcal{B}$.*

Observation 22. *Let $b \in B$, let u and v be elements of P, and let j be a nonnegative integer. Assume that $u \not\leqslant b$ and $u \leqslant v$ in P. Then, $u \in \mathrm{shad}_j(b)$ if and only if $v \in \mathrm{shad}_j(b)$. More precisely, either $u, v \in \mathrm{int}\, \mathrm{shad}_j(b)$ or $u, v \notin \mathrm{shad}_j(b)$.*

The next proposition is an abstraction of two corollaries stated right after the proof. Namely, we show that if a witnessing path has both endpoints in a shadow block (or in $\mathrm{shad}_j(b)$ for some $b \in B$ and $j \in \{0, \ldots, \mathrm{sd}(b)\}$), then the path lies entirely in the shadow block (or $\mathrm{shad}_j(b)$, respectively).

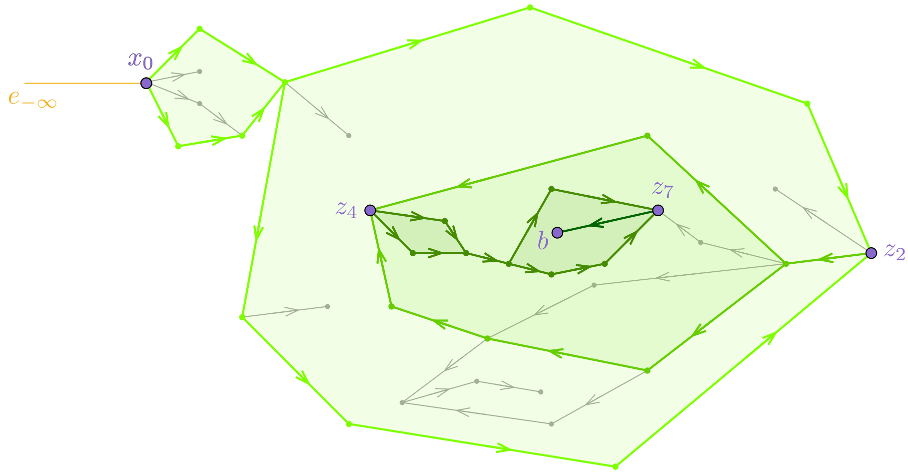


FIGURE 11. There are three reversing elements of b, namely, z_2, z_4, z_7. Hence, $\mathrm{sd}(b) = 3$. $\{\mathrm{shad}_j(b)\}_{j\in[3]}$ are depicted with various shades of green. Recall that in Figure 8, we discussed the shadow blocks of b. In particular, $\mathrm{shad}_0(b) = \mathcal{B}_1 \cup \mathcal{B}_2$, $\mathrm{shad}_1(b) = \mathcal{B}_3 \cup \mathcal{B}_4$, $\mathrm{shad}_2(b) = \mathcal{B}_5 \cup \mathcal{B}_6 \cup \mathcal{B}_7$, and $\mathrm{shad}_3(b) = \mathcal{B}_8$, and finally, $\mathrm{shad}_4(b) = \emptyset$.

Proposition 23. *Let $b \in B$ and let j be a nonnegative integer. Let $(\mathcal{D}_1, \ldots, \mathcal{D}_m)$ be a subsequence of consecutive elements of the sequence of blocks of* $\mathrm{shad}_j(b)$ *and let $\mathcal{D} = \bigcup_{i\in[m]} \mathcal{D}_i$. Let W be a witnessing path in P with both endpoints in $\mathcal{D}$. Then all edges of W lie in $\mathcal{D}$.*

Proof. We argue by contradiction. Let u and v be elements in the intersection of W and $\partial\mathcal{D}$ with $u \leqslant v$ in P such that $u[W]v$ is in the exterior of $\mathcal{D}$ except for the elements u and v. For each $i \in [m]$, let $x_i = \min \mathcal{D}_i$. Additionally, let $x = x_1$ and $y = \max \mathcal{D}_m$. Then $x \leqslant u < v \leqslant y$ in P. In particular, $u \neq y$. Let W' be a witnessing path from v to y in P. For convenience, let $U = u[W]v[W']y$. Let e be the first edge of $u[W]v$. Let $i \in [m]$ be the least integer such that $u \in \mathcal{D}_i$ and let $\mathcal{B} = \mathcal{D}_i$.

By Proposition 20, the left side of $\mathcal{B}$ is contained in $W_L(y)$ and the right side of $\mathcal{B}$ is contained in $W_R(y)$. If u is on the left side of $\mathcal{B}$, let e_L^+ and e_L^- be, respectively (provided they exist), the edges immediately after and immediately before u on the path $W_L(y)$. Of course, in the case $u = x_0$ we set $e_L^- = e_{-\infty}$. Also, if u is on the right side of $\mathcal{B}$, let e_R^+ and e_R^- be, respectively (provided they exist), the edges immediately after and immediately before u on $W_R(y)$. Again, if $u = x_0$ we set $e_R^- = e_{-\infty}$. Note that if $u = x_i$, then u is on both sides of $\mathcal{B}$.

First, assume that

$$u \text{ is on the left side of } \mathcal{B} \text{ and } e_L^- \prec e \prec e_L^+ \text{ in the } u\text{-ordering.} \tag{2}$$

In particular, $e \prec e_L^+$ in the (u, e_L^-)-ordering. It follows that $x_0[W_L(y)]u[U]y$ is left of $W_L(y)$, which is a contradiction.

Next, symmetrically assume that

$$u \text{ is on the right side of } \mathcal{B} \text{ and } e_R^+ \prec e \prec e_R^- \text{ in the } u\text{-ordering.} \tag{3}$$

In particular, $e_R^+ \prec e$ in the (u, e_R^-)-ordering. It follows that $x_0[W_R(y)]u[U]y$ is right of $W_R(y)$, which is a contradiction.

Now, assume that neither (2) nor (3) hold. If u was strictly on the left side of $\mathcal{B}$, then since e is not in $\mathcal{B}$, by (sb3), (2) would hold. If u was strictly on the right side of $\mathcal{B}$, then since e is not in $\mathcal{B}$, by (sb4), (3) would hold. If $i \neq 1$ and $u = x_i$, then (sb1) applied to $\mathcal{D}_i$ and (sb2) applied to $\mathcal{D}_{i-1}$, one of (2) and (3) would hold. If $i = 1$ and $u = x_1 = x$ and x was a reversing element, then $e_R^- \prec e_L^+ \prec e_R^+ \prec e_L^-$ in the x-ordering, and since e is not in $\mathcal{B}$, by (sb1), $e_R^+ \prec e \prec e_L^+$ in the x-ordering; altogether, (2) or (3) or both hold. Therefore, we may assume that $i = 1$, $u = x$, and x is not a reversing element. In this case, $e_L^- \prec e_L^+ \prec e_R^+ \prec e_R^-$ in the x-ordering. Since e is not in $\mathcal{B}$, by (sb1), we obtain that $e_R^+ \prec e \prec e_L^+$. It follows that one of the three holds: (2) or (3) or $e_R^- \prec e \prec e_L^- \prec e_R^-$ in the x-ordering. Thus, we assume the latter. In particular, $\mathcal{B}$ is not the first block in the sequence of blocks of y, and the preceding block $\mathcal{B}'$ is non-degenerate. By (sb2), e is in the interior of $\mathcal{B}'$. Since x is not reversing, the interiors of $\mathcal{B}$ and $\mathcal{B}'$ are disjoint. Therefore, $u[W]v$ must intersect the boundary of $\mathcal{B}'$. This produces a cycle in P, which is a contradiction. □

As indicated before, Proposition 23 yields the following two statements.

Corollary 24. *Let $\mathcal{B}$ be a shadow block, and let W be a witnessing path in P with both endpoints in $\mathcal{B}$. Then all edges of W lie in $\mathcal{B}$.*

Corollary 25. *Let $b \in B$ and let j be a nonnegative integer. Let W be a witnessing path in P with both endpoints in $\operatorname{shad}_j(b)$. Then all edges of W lie in $\operatorname{shad}_j(b)$.*

When $\mathcal{B}$ is a shadow block and $u \in B$, then we somehow control the behavior of $W_L(u)$ and $W_R(u)$ outside $\mathcal{B}$. In Proposition 26, we show that $\min \mathcal{B}$ lies in both of these paths. In particular, by Proposition 15, $W_L(\min \mathcal{B})$ is a prefix of $W_L(u)$ and $W_R(\min \mathcal{B})$ is a prefix of $W_R(u)$. Moreover, in Proposition 27, we show that if we also assume $\max \mathcal{B} < u$ in P, then $\max \mathcal{B}$ lies in both $W_L(u)$ and $W_R(u)$. Proposition 28 is an adjustment of this statement to shadows.

Proposition 26. *Let $\mathcal{B}$ be a shadow block and let $u \in B$. Assume that $u \in \mathcal{B}$. Then $\min \mathcal{B}$ is in both $W_L(u)$ and $W_R(u)$.*

Proof. We set $x = \min \mathcal{B}$ and $y = \max \mathcal{B}$. We show that x is in $W_L(u)$. The proof that x is in $W_R(u)$ is symmetric. Since x_0 is in the exterior face of G and u is in $\mathcal{B}$, the path $W_L(u)$ intersects $\partial\mathcal{B}$. Recall that $\partial\mathcal{B}$ is the union of $x[W_L(y)]y$ and $x[W_R(y)]y$. In particular, every element w in $\partial\mathcal{B}$ satisfies $x \leqslant w \leqslant y$ in P. If $W_L(u)$ intersects $x[W_L(y)]y$ in an element w, then by the fact that $W_L(u)$ and $W_L(y)$ are x_0-consistent (Proposition 15.(*L*)), we obtain $x_0[W_L(u)]w = x_0[W_L(y)]w$, and so, x lies in $W_L(u)$.

Therefore, for the remainder of the proof we assume that $W_L(u)$ does not intersect $x[W_L(y)]y$, and so, let w be an element in the intersection of $W_L(u)$ and $x[W_R(y)]y$ with $w \notin \{x, y\}$. We will show that this leads to a contradiction. Note that in this setting, $W_L(w)$ is a prefix of $W_L(u)$. Since x is not in $W_L(w)$, the path $W_L(x)$ is not a subpath of $W_L(w)$. Therefore, since $x < w$ in P and by Proposition 17.(*L*), $W_L(w)$ is left of $W_L(x)$. Since $W_L(w)$ is left of $W_L(x)$ and $W_L(x)$ is a subpath of $W_L(y)$, we obtain that $W_L(w)$ is left of $W_L(y)$. However, $w < y$ in P, thus for every witnessing path W from w to y in P, we have $x_0[W_L(w)]w[W]y$ left of $W_L(y)$, which is a desired contradiction. □

Proposition 27. *Let $\mathcal{B}$ be a shadow block and let $u \in B$. Assume that $u \in \mathcal{B}$ and $\max \mathcal{B} < u$ in P. Then $\max \mathcal{B}$ belongs to both $W_L(u)$ and $W_R(u)$.*

Proof. We set $x = \min \mathcal{B}$, and $y = \max \mathcal{B}$. By Proposition 26, x lies in $W_L(u)$. Therefore, $x_0[W_L(u)]x = x_0[W_L(y)]x$. Note that u does not lie on $W_L(y)$ as this yields $u \leqslant y$ in P contrary to the assumption. Let $w = \mathrm{gcpe}(W_L(u), W_L(y))$. Thus, $x \leqslant w$ in P and we claim that $y \leqslant w$ in P. Suppose otherwise that $x \leqslant w < y$ in P. By Corollary 24, the whole path $x[W_L(u)]u$ lies in $\mathcal{B}$. Let W be an arbitrary witnessing path from y to u in P. If $x = w$ then by (sb1), $x_0[W_L(y)]y[W]u$ is left of $W_L(u)$ which is a contradiction. If $x < w < y$ in P then w is strictly on the left side, and so, by (sb3), again $x_0[W_L(y)]y[W]u$ is left of $W_L(u)$, which is a contradiction. This shows that y belongs to $W_L(u)$. The proof that y belongs to $W_R(u)$ is symmetric. □

Proposition 28. *Let $u, y \in B$ and let j be a nonnegative integer with $\mathrm{sd}(y) = j$. Assume that $u \in \mathrm{shad}_j(y)$ and $y < u$ in P. Then, u lies in the terminal block of $\mathrm{shad}_j(y)$, and y lies in both $W_L(u)$ and $W_R(u)$.*

Proof. Let $\mathcal{B}$ be the block of $\mathrm{shad}_j(b)$ such that $u \in \mathcal{B}$. Suppose to the contrary that $\mathcal{B}$ is not the terminal block of $\mathrm{shad}_j(y)$. By the construction of $\mathrm{shad}_j(b)$, y lies only in the terminal block of $\mathrm{shad}_j(b)$, and so, $y \notin \mathcal{B}$. Therefore, a witnessing path from y to u in P intersects $\partial\mathcal{B}$. This yields a directed cycle in P, which is a contradiction. Thus, $\mathcal{B}$ is the terminal block of $\mathrm{shad}_j(u)$. This gives the first part of the assertion. To get the second part, it suffices to apply Proposition 27. □

We conclude this subsection with an exhaustive description of what we know about $\mathrm{sd}(u)$ and $\mathrm{shad}_i(u)$ for $i \in \{0, \ldots, \mathrm{sd}(u)\}$ given that u lies in a shadow block of an element $b \in B$.

Proposition 29. *Let $u, b \in B$, let $j \in \{0, \ldots, \mathrm{sd}(b)\}$, let $\mathcal{B}$ be a block of $\mathrm{shad}_j(b)$. If $u \in \mathcal{B}$, then the following statements hold:*

(i) *if u is the initial element of $\mathrm{shad}_j(b)$, then $\mathrm{sd}(u) = j - 1$, unless $j = 0$ and $u = x_0$;*
(ii) *if u is not the initial element of $\mathrm{shad}_j(b)$, then $\mathrm{sd}(u) \geqslant j$;*
(iii) *if u is not the initial element of $\mathrm{shad}_j(b)$ and u lies in $\partial\mathcal{B}$, then $\mathrm{sd}(u) = j$;*
(iv) *$\mathrm{shad}_i(u) = \mathrm{shad}_i(b)$ for all $i \in \{0, \ldots, j-1\}$;*
(v) *$\mathrm{shad}_j(\min \mathcal{B}) \subseteq \mathrm{shad}_j(u) \subseteq \mathrm{shad}_j(\max \mathcal{B}) \subseteq \mathrm{shad}_j(b)$;*
(vi) *$\mathrm{shad}_j(u) = \mathrm{shad}_j(\max \mathcal{B})$ if and only if $\max \mathcal{B} \leqslant u$ in P.*

Proof. Let $x = \min \mathcal{B}$ and $y = \max \mathcal{B}$, let $(z_0, \ldots, z_m)$ be the sequence of common points of b. By definition, x is a common point of b, so let i be an integer such that $z_i = x$. In particular, x is an element of $W_L(b) \cap W_R(b)$, hence, $W_L(x)$ is a prefix of $W_L(b)$ and $W_R(x)$ is a prefix of $W_R(b)$. It follows that $(z_0, \ldots, z_i)$ is the sequence of common points of x. Since $\mathcal{B}$ is a block of $\mathrm{shad}_j(b)$ and $x = \min \mathcal{B}$, exactly j elements of $(z_0, \ldots, z_i)$ are reversing elements of b. For every reversing element z of b that is contained in $(z_0, \ldots, z_i)$, either z is a reversing element of x, or $z = z_i = x$ and this element coincides with the initial element of $\mathrm{shad}_j(b)$. If x is the initial element of $\mathrm{shad}_j(b)$, then either $x = x_0$ (and $j = 0$) or x has $j - 1$ reversing elements, thus, $\mathrm{sd}(x) = j - 1$. If x is not the initial element of $\mathrm{shad}_j(b)$, then x has j reversing elements, so $\mathrm{sd}(x) = j$.

Proposition 26 implies that x is a vertex of $W_L(u) \cap W_R(u)$, thus, x is also a common point of u, and moreover, $(z_0, \ldots, z_i)$ is a prefix of the sequence of common points of u. Therefore, every block of the sequence of shadow blocks of b that precedes $\mathcal{B}$ is also a shadow block of x, u, and y.

If u is the initial element of $\mathrm{shad}_j(b)$, then $u = x$, and x is a reversing element of b, hence, $\mathrm{sd}(x) = \mathrm{sd}(u) = j - 1$ (unless $j = 0$ and $x = u = x_0$), which proves (i). If u is not the initial element of $\mathrm{shad}_j(b)$, then either $u \neq x$ or x is not the initial element of $\mathrm{shad}_j(b)$, hence, $(z_0, \ldots, z_i)$ contains j reversing elements of u, and so, $\mathrm{sd}(u) \geqslant j$, which proves (ii).

Next, we prove (iii). Suppose that u is not the initial element of $\mathrm{shad}_j(b)$, but u lies in $\partial\mathcal{B}$. If $u = x$ or $u = y$, then $(z_0, \ldots, z_i)$ or resp. $(z_0, \ldots, z_i, y)$ is exactly the sequence of common elements of u, hence, $\mathrm{sd}(u) = j$. Now, assume that u is strictly on the left side of $\mathcal{B}$ (the proof in the case, where u is strictly on the right side of $\mathcal{B}$ is symmetric). We know that $\mathrm{sd}(u) \geqslant j$ and we want to prove that $\mathrm{sd}(u) \leqslant j$. To this end, we need to show that there are no reversing elements z of u with $z_i = x < z < u$ in P. Note that $W_L(u)$ is a subpath of $W_L(b)$. In particular, $x[W_L(u)]u$ is a segment of $\partial\mathcal{B}$. Since $W_R(u)$ contains x, by Corollary 24, $x[W_R(u)]u$ is contained in $\mathcal{B}$. Let z be an element of $W_L(u) \cap W_R(u)$ with $x < z < u$ in P. Let e_L^+ and e_L^- be, respectively, the edges immediately after and immediately before z on $W_L(u)$, and let e_R^+ and e_R^- be, respectively, the edges immediately after and immediately before z on $W_R(u)$. Since z lies strictly on the left side of $\mathcal{B}$ and both e_R^- and e_R^+ lie in $\mathcal{B}$, by (sb3), $e_L^+ \preccurlyeq e_R^- \preccurlyeq e_L^-$ and $e_L^+ \preccurlyeq e_R^+ \preccurlyeq e_L^-$ in the z-ordering. Recall that if z was a reversing element of u, then $e_R^- \prec e_L^+ \preccurlyeq e_R^+ \prec e_L^-$ in the z-ordering. These two sets of inequalities can not hold simultaneously, which shows that z is not a reversing element of u, and so, completes the proof of (iii).

Since the sequence of common points of b and the sequence of common points of u agree up to at least x, and thus, up to at least the initial element of $\mathrm{shad}_j(b)$, we obtain (iv).

The union of all shadow blocks of $\mathrm{shad}_j(b)$ that precede $\mathcal{B}$ is equal to $\mathrm{shad}_j(x)$, hence, $\mathrm{shad}_j(x) \subseteq \mathrm{shad}_j(u)$. Moreover, $\mathrm{shad}_j(x) \cup \mathcal{B} = \mathrm{shad}_j(y)$, thus, $\mathrm{shad}_j(y) \subseteq \mathrm{shad}_j(b)$. In order to prove (v), it remains to prove that $\mathrm{shad}_j(u) \subseteq \mathrm{shad}_j(y)$. Consider a block $\mathcal{D}$ of $\mathrm{shad}_j(u)$ such that $x \leqslant \min \mathcal{D}$. We have, $x \leqslant \min \mathcal{D} \leqslant \max \mathcal{D} \leqslant u$ in P. Both sides of $\mathcal{D}$ are subpaths of witnessing paths from x to u. It follows from Proposition 24 that all edges and vertices of the boundary of $\mathcal{D}$ are in $\mathcal{B}$. Thus by Proposition 7, $\mathcal{D} \subseteq \mathcal{B} \subseteq \mathrm{shad}_j(y)$, which yields $\mathrm{shad}_j(u) \subseteq \mathrm{shad}_j(y)$, and completes the proof of (v).

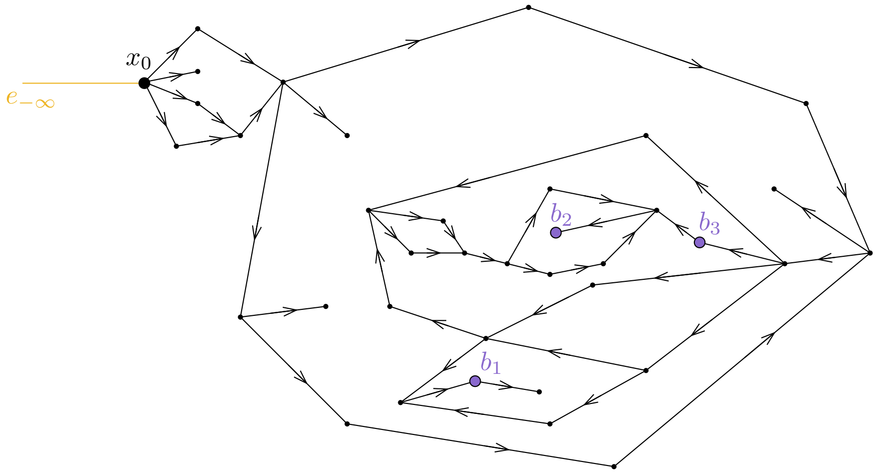
Finally, we prove (vi). If $\mathrm{shad}_j(y) = \mathrm{shad}_j(u)$, then since y is on the boundary of $\mathrm{shad}_j(u)$, we have $y \leqslant u$ in P. In order to prove the reverse implication, suppose that $y \leqslant u$ in P. It suffices to justify that y is a common point of u and that y is a reversing element of u. The former follows from Proposition 27 and the latter follows from the fact that u is in $\mathcal{B}$. □

4.4. **Ordering elements in B.** Given two elements in B, we inspect the interaction of their leftmost and rightmost witnessing paths. This establishes four scenarios, and we relate each case with the comparability status of the elements and their shadows as defined in Subsection 4.3. Let $b_1, b_2 \in B$. We say that (b_1, b_2) is

- an *inside pair* whenever $W_L(b_2)$ is left of $W_L(b_1)$ and $W_R(b_1)$ is left of $W_R(b_2)$,
- an *outside pair* whenever $W_L(b_1)$ is left of $W_L(b_2)$ and $W_R(b_2)$ is left of $W_R(b_1)$,
- a *left pair* whenever $W_L(b_1)$ is left of $W_L(b_2)$ and $W_R(b_1)$ is left of $W_R(b_2)$,
- a *right pair* whenever $W_L(b_2)$ is left of $W_L(b_1)$ and $W_R(b_2)$ is left of $W_R(b_1)$.

See Figure 12 for some examples. Note that a pair (b_1, b_2) is an outside pair whenever (b_2, b_1) is an inside pair and (b_1, b_2) is a right pair whenever (b_2, b_1) is a left pair. The above distinction is in fact a classification of all incomparable pairs of elements in B. That is, if $b_1 \parallel b_2$ in P, the pair (b_1, b_2) is always of one of the four types: inside, outside, left, right. On the other hand, a comparable pair of elements in B is not necessarily of one of the four types. However, all left and right pairs are incomparable by Proposition 17.

Within the next two propositions, we show that under some mild assumptions, two elements $b_1, b_2 \in B$ form an inside pair (b_1, b_2) if and only if there is a nonnegative integer j such that



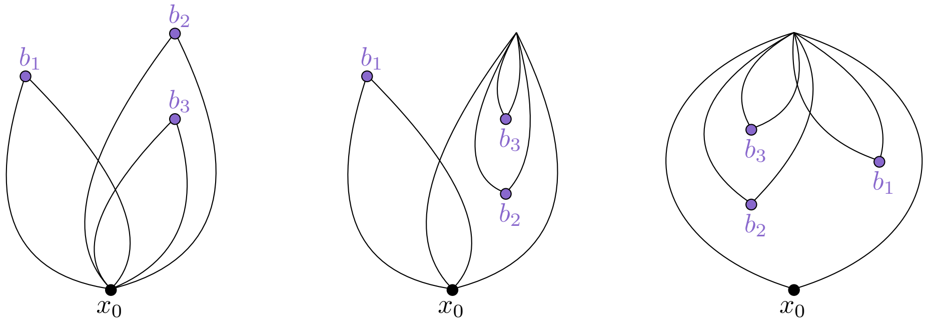


FIGURE 12. In the poset on the top of the figure, the pairs (b_1, b_2) and (b_1, b_3) are left pairs while (b_2, b_3) is an outside pair. It follows that (b_2, b_1) and (b_3, b_1) are right pairs and (b_3, b_2) is an inside pair. Below, we present more schematic drawings in which we draw only the witnessing paths forming boundaries of shadows of respective elements in B. For readability, we omit the arrows in such drawings. We also do not mark intersection of curves, although each such intersection must be an element of the poset. Still, (b_1, b_2) and (b_1, b_3) are left pairs while (b_2, b_3) is an outside pair.

$\text{shad}_j(b_1) \subsetneq \text{shad}_j(b_2)$. Next in Proposition 32, we give some necessary conditions on shadows of elements $b_1, b_2 \in B$ forcing them to form a left or right pair.

Proposition 30. *Let j be a nonnegative integer, let $u, y \in B$ with $\text{sd}(y) = j$ and $u \in \text{shad}_j(y)$.*

(L) $W_L(u)$ is not left of $W_L(y)$.
(R) $W_R(u)$ is not right of $W_R(y)$.

Moreover, if u is in the interior of $\text{shad}_j(y)$, then either $y < u$ in P or (u, y) is an inside pair.

Proof. We prove statement (L). The argument for statement (R) is symmetric. If u lies in $W_L(y)$, then the assertion is clear since $W_L(u)$ is a subpath of $W_L(y)$. If $y < u$ in P, then by Proposition 28, y lies in $W_L(u)$, hence, $W_L(y)$ is a subpath of $W_L(u)$ and the assertion holds as well. Therefore, we can assume that u is not an element of $W_L(y)$ and $y \not< u$ in P. In particular, either $W_L(y)$ is left of $W_L(u)$ or $W_L(u)$ is left of $W_L(y)$. In order to finish the proof we have to prove that $W_L(y)$ is left of $W_L(u)$.

Let $\mathcal{B}'$ be a shadow block of $\text{shad}_j(y)$ containing u, and let x', y' be the initial and terminal elements of $\mathcal{B}'$, respectively. By Proposition 26, x' belongs to $W_L(u)$. First, assume that $y' < u$

in P. Note that this implies that $\mathcal{B}'$ is not the terminal block of $\mathrm{shad}_j(y)$. By Proposition 27, y' lies in $W_L(u)$. In particular, y' lies in both $W_L(y)$ and $W_L(u)$. In this case, $u \in \mathrm{int}\,\mathcal{B}'$ and $y \notin \mathcal{B}'$. Moreover, $y'[W_L(u)]u$ is contained in $\mathcal{B}'$ and $y'[W_L(y)]y$ intersects $\mathcal{B}'$ only in y'. Therefore, by (sb2), $W_L(y)$ is left of $W_L(u)$, as desired. Next, assume that $y' \not< u$ in P. By Corollary 24, $x'[W_L(u)]u$ is contained in $\mathcal{B}'$. Observe that neither of the paths $W_L(y')$ and $W_L(u)$ is a subpath of the other. Let $w = \mathrm{gcpe}(W_L(y'), W_L(u))$. We have $x' \leqslant w < y'$ in P and $w[W_L(u)]u$ lies in $\mathcal{B}'$. If $w = x'$, then by (sb1), $W_L(y')$ is left of $W_L(u)$. Similarly, if w is strictly on the left side, then by (sb3), $W_L(y')$ is left of $W_L(u)$. Since $W_L(y')$ is a subpath of $W_L(y)$, we conclude the proof of (*L*).

Finally, let us prove the "moreover" part. Assume that u lies in the interior of $\mathrm{shad}_j(y)$, and that $y \not< u$ in P. To complete the proof, we will show that (u, y) is an inside pair. Since $y \not< u$ in P, $W_L(y)$ is not a subpath of $W_L(u)$ and $W_R(y)$ is not a subpath of $W_R(u)$. On the other hand, since u lies in the interior of $\mathrm{shad}_j(y)$, $W_L(u)$ is not a subpath of $W_L(y)$ and $W_R(u)$ is not a subpath of $W_R(y)$. It follows that (u, y) is a pair of one of the four types. Furthermore, by (*L*) and (*R*), (u, y) is an inside pair. □

Proposition 31. *Let j be a nonnegative integer, let $u, b \in B$ with $\mathrm{sd}(u), \mathrm{sd}(b) \geqslant j$, and let x be the initial element of $\mathrm{shad}_j(b)$. Assume that $x \leqslant u$ in P. If (u, b) is an inside pair, then $u \in \mathrm{shad}_j(b)$.*

Proof. Suppose to the contrary that (u, b) is an inside pair and $u \notin \mathrm{shad}_j(b)$. Let y be the terminal element of $\mathrm{shad}_j(b)$. Let W be a witnessing path from x to u, and let w be the last element of W that is in $\mathrm{shad}_j(b)$. In particular, w lies either in $x[W_L(b)]b$ or $x[W_R(b)]b$. Note that $w \neq u$. If w lies in $W_L(b)$, then let $W_1 = x_0[W_L(b)]w[W]u$, and if w lies in $W_R(b)$, then let $W_2 = x_0[W_R(b)]w[W]u$. Recall that $W_L(u)$ is either equal or left of any witnessing path from x_0 to u, and $W_R(u)$ is either equal or right of any witnessing path from x_0 to u. Moreover, the relation of being left/right is transitive (by Proposition 9). In particular, since (u, b) is an inside pair, W_1 (if defined) can not be left of $W_L(b)$ and W_2 (if defined) can not be right of $W_R(b)$.

Remember that $w \neq u$, and so, let e be the edge following w in W. By definition, e is in the exterior of $\mathrm{shad}_j(b)$. If w is in $W_L(b)$, let e_L^+ and e_L^- be, respectively (provided they exist), the edges immediately after and immediately before w on the path $W_L(b)$. Of course, in the case $w = x_0$ we set $e_L^- = e_{-\infty}$. Also, if w is in $W_R(b)$, let e_R^+ and e_R^- be, respectively (provided they exist), the edges immediately after and immediately before w on $W_R(b)$. Again, if $w = x_0$ we set $e_R^- = e_{-\infty}$.

We split the reasoning into cases depending on where w lies. If w lies strictly on the left (resp. right) side of some shadow block of $\mathrm{shad}_j(b)$, then by (sb3) (resp. (sb4)), we obtain $e \prec e_L^+$ in the (w, e_L^-)-ordering (resp. $e_R^+ \prec e$ in the (w, e_R^-)-ordering). This yields that W_1 is left of $W_L(b)$ (resp. W_2 is right of $W_R(b)$), which is a contradiction.

Next, assume that w is an element in $W_L(b) \cap W_R(b)$ with $w \notin \{x, y\}$. Let $\mathcal{B}'$ and $\mathcal{B}$ be the blocks of $\mathrm{shad}_j(b)$ such that $w = \max \mathcal{B}'$ and $w = \min \mathcal{B}$. We have $e_L^- \prec e_L^+ \preccurlyeq e_R^+ \prec e_R^-$ in the w-ordering. By (sb1) applied to $\mathcal{B}$ and (sb2) applied to $\mathcal{B}'$, either $e_L^- \prec e \prec e_L^+$ in the w-ordering or $e_R^+ \prec e \prec e_R^-$ in the w-ordering. In other words, either $e \prec e_L^+$ in the (w, e_L^-)-ordering or $e_R^+ \prec e$ in the (w, e_R^-)-ordering. In the former case, we obtain that W_1 is left of $W_L(b)$ and in the latter case, we obtain that W_2 is right of $W_R(b)$. Each of these outcomes leads to a contradiction.

Now, consider the case where $w = x$. If $j = 0$, then $w = x = x_0$. Note that $e_L^- = e_R^- = e_{-\infty}$. By (sb1), either $e \prec e_L^+$ in the (w, e_L^-)-ordering or $e_R^+ \prec e$ in the (w, e_R^-)-ordering. Similarly as in the other cases, this gives a contradiction. Thus, we assume that $j > 0$. In particular, x is a reversing element of b. We have $e_R^+ \prec e_L^- \prec e_R^- \prec e_L^+$ in the w-ordering. By (sb1) applied to the first block of $\mathrm{shad}_j(b)$, we have $e_R^+ \prec e \prec e_L^-$ in the w-ordering. We obtain that either $e_R^+ \prec e \prec e_R^-$ in w-ordering or $e_L^- \prec e \prec e_L^+$ in the w-ordering. In other words, either $e \prec e_L^+$ in the (w, e_L^-)-ordering or $e_R^+ \prec e$ in the (w, e_R^-)-ordering. Again, this is a contradiction.

Finally, assume that $w = y$. Note that $y \neq b$ as otherwise $b \leqslant u$ in P, which contradicts Proposition 17.(L) (since $W_L(b)$ is left of $W_L(u)$). In particular, y is a reversing element of b, and so, $e_R^+ \prec e_L^- \prec e_R^- \prec e_R^+$ in the w-ordering. By (sb2), $e_L^- \prec e \prec e_R^-$ in the w-ordering. We obtain that $e \prec e_L^+$ in the (w, e_L^-)-ordering, and so, W_1 is left of $W_L(b)$, which is a contradiction. This completes the proof. □

Proposition 32. *Let j be a nonnegative integer, let $b_1, b_2 \in B$ with $\mathrm{sd}(b_1), \mathrm{sd}(b_2) \geqslant j$, and assume that $\mathrm{shad}_j(b_1)$ and $\mathrm{shad}_j(b_2)$ have the same initial element. If $b_1 \notin \mathrm{shad}_j(b_2)$ and $b_2 \notin \mathrm{shad}_j(b_1)$, then either (b_1, b_2) is a left pair or (b_1, b_2) is a right pair.*

Proof. If $W_L(b_1)$ was a subpath of $W_L(b_2)$, then $b_1 \in \mathrm{shad}_j(b_2)$, hence, this is not the case. By reversing the roles of b_1 and b_2 and/or replacing W_L with W_R in the above sentence, we obtain that neither of $W_L(b_1)$ and $W_L(b_2)$ is a subpath of the other and neither of $W_R(b_1)$ and $W_R(b_2)$ is a subpath of the other. Therefore, (b_1, b_2) is a pair of one of the four types: inside, outside, left, right. By Proposition 31, if (b_1, b_2) is an inside pair, then $b_1 \in \mathrm{shad}_j(b_2)$, and if (b_1, b_2) is an outside pair, then $b_2 \in \mathrm{shad}_j(b_1)$. It follows that neither of the above holds, thus, (b_1, b_2) is either a left pair or a right pair. □

In Section 5, we give a reduction that allow us to restrict our attention to instances $(P, x_0, G, e_{-\infty}, I)$ such that for every $(a, b) \in I$ we have $a \notin \mathrm{shad}_0(b)$. Therefore, we develop a number of statements just for $\mathrm{shad}_0(b)$ for $b \in B$. Also, to simplify the notation, for every $b \in B$, we write

$$\mathrm{shad}(b) = \mathrm{shad}_0(b).$$

The next proposition is a very intuitive fact about shadows that is quite technical to prove directly but follows nicely from the classification of incomparable pairs.

Proposition 33. *Let $b_1, b_2 \in B$. If $b_1 \leqslant b_2$ in P, then $\mathrm{shad}(b_1) \subseteq \mathrm{shad}(b_2)$.*

Proof. If $W_L(b_1)$ is a subpath of $W_L(b_2)$ or $W_R(b_1)$ is a subpath of $W_R(b_2)$, then $b_1 \in \mathrm{shad}(b_2)$, and the inclusion $\mathrm{shad}(b_1) \subseteq \mathrm{shad}(b_2)$ follows from Proposition 29.(v). Otherwise, (b_1, b_2) is of one of the four types: left, right, inside, outside. By Proposition 17, $W_L(b_1)$ is not left of $W_L(b_2)$ and $W_R(b_1)$ is not right of $W_R(b_2)$, hence, (b_1, b_2) is neither left nor right. Therefore, (b_1, b_2) is either inside or outside. Hence, by Proposition 31, either $b_1 \in \mathrm{shad}(b_2)$ or $b_2 \in \mathrm{shad}(b_1)$. If $b_1 \in \mathrm{shad}(b_2)$ then by Proposition 29.(v) $\mathrm{shad}(b_1) \subseteq \mathrm{shad}(b_2)$ as desired. So we assume that $b_2 \in \mathrm{shad}(b_1)$. Let y be the terminal element of $\mathrm{shad}(b_1)$. In particular, $b_2 \in \mathrm{shad}(y)$ and $y \leqslant b_1 \leqslant b_2$ in P. Thus, by Proposition 28, b_2 lies in the terminal block of $\mathrm{shad}(y) = \mathrm{shad}(b_1)$. Finally, by Proposition 29.(vi), $\mathrm{shad}(b_1) = \mathrm{shad}(y) = \mathrm{shad}(b_2)$, which concludes the proof. □

Let $b_1, b_2 \in B$. We say that b_1 is *left* of b_2 if (b_1, b_2) is a left pair, and $b_1 \notin \mathrm{shad}(b_2)$, $b_2 \notin \mathrm{shad}(b_1)$. We say that b_1 is *right* of b_2 if b_2 is left of b_1. In the bottom part of Figure 12, b_1 is left of b_2 in the examples on the left and in the middle, however, it is not the case in the example on the right.

Next, we prove simple facts about the left of and right of notions. The goal (achieved in Proposition 37) is to show that the relation of being left of (right of) partially orders B. We start with a straightforward corollary of Proposition 32 stated for emphasis.

Corollary 34. *Let $b_1, b_2 \in B$. If $b_1 \notin \text{shad}(b_2)$ and $b_2 \notin \text{shad}(b_1)$, then either b_1 is left b_2 or b_1 is right of b_2.*

Note that the above corollary allows us to employ the following reasoning. Say that we know that $b_1 \notin \text{shad}(b_2)$ and $b_2 \notin \text{shad}(b_1)$ for some $b_1, b_2 \in B$ and we want to prove that b_1 is left of b_2. Then, by Corollary 34, either b_1 is left b_2 or b_1 is right of b_2. This means that (b_1, b_2) is either a left pair or a right pair. At this point, it suffices to prove only one of the conditions required for a pair to be a left pair (or a right pair). That is, the fact that $W_L(b_1)$ is left of $W_L(b_2)$ (or that $W_R(b_1)$ is left of $W_R(b_2)$) implies that b_1 is left of b_2.

Proposition 35. *Let $b_1, b_2 \in B$. Let y_1 be the terminal element of $\text{shad}(b_1)$, and let y_2 be the terminal element of $\text{shad}(b_2)$.*

(L) b_1 is left of b_2 if and only if y_1 is left of b_2.
(R) b_1 is left of b_2 if and only if b_1 is left of y_2.

Note that the above statements imply that b_1 is left of b_2 if and only if y_1 is left of y_2.

Proof. We prove statement (L). The argument for statement (R) is symmetric. Note that $\text{shad}(b_1) = \text{shad}(y_1)$. It follows that $b_2 \notin \text{shad}(b_1)$ if and only if $b_2 \notin \text{shad}(y_1)$. Additionally, we claim that $b_1 \notin \text{shad}(b_2)$ if and only if $y_1 \notin \text{shad}(b_2)$. Indeed, by Proposition 29.(v), if $y_1 \in \text{shad}(b_2)$, then $\text{shad}(y_1) \subseteq \text{shad}(b_2)$ (by Proposition 33), and so, $b_1 \in \text{shad}(b_2)$. On the other hand, if $b_1 \in \text{shad}(b_2)$, then $\text{shad}(b_1) \subseteq \text{shad}(b_2)$, and so, $y_1 \in \text{shad}(b_2)$. Summarizing, $b_1 \notin \text{shad}(b_2)$ and $b_2 \notin \text{shad}(b_1)$ if and only if $y_1 \notin \text{shad}(b_2)$ and $b_2 \notin \text{shad}(y_1)$.

Assume that y_1 is left of b_2. By the above, to prove that b_1 is left of b_2, it suffices to show that (b_1, b_2) is a left pair. Since (y_1, b_2) is a left pair, $W_L(y_1)$ is a subpath of $W_L(b_1)$, and $W_R(y_1)$ is a subpath of $W_R(b_1)$, we obtain that (b_1, b_2) is a left pair.

Finally, assume that b_1 is left of b_2. We will prove that y_1 is left of b_2. Again by the equivalence above, $y_1 \notin \text{shad}(b_2)$ and $b_2 \notin \text{shad}(y_1)$. Hence, by Corollary 34, either y_1 is left b_2 or y_1 is right of b_2. It suffices to prove that $W_L(y_1)$ is left of $W_L(b_2)$. Let $w = \text{gcpe}(W_L(b_1), W_L(b_2))$. We claim that $w < y_1$ in P. Indeed, if $y_1 \leqslant w$ in P, then since y_1 lies in $W_L(b_1)$, we obtain that y_1 lies in $W_L(b_2)$. However, this implies $\text{shad}(y_1) \subseteq \text{shad}(b_2)$ (by Proposition 33). This is false, hence, indeed $w < y_1$ in P. Since $w < y_1$ in P and $W_L(y_1)$ is a subpath of $W_L(b_1)$, we obtain that $W_L(y_1)$ left of $W_L(b_2)$. This completes the proof. ☐

Proposition 36. *Let $b_1, b_2, d \in B$ with b_1 left of b_2.*

(L) If $d \in \text{shad}(b_1)$ and $d \notin \text{shad}(b_2)$, then d is left of b_2.
(R) If $d \in \text{shad}(b_2)$ and $d \notin \text{shad}(b_1)$, then d is right of b_1.

Proof. We prove statement (L) – see Figure 13. The argument for statement (R) is symmetric. Assume that $d \in \text{shad}(b_1)$ and $d \notin \text{shad}(b_2)$. Note that by Proposition 29.(v), $\text{shad}(d) \subseteq \text{shad}(b_1)$. Since b_1 is left of b_2, we have $b_2 \notin \text{shad}(d)$, and so, $b_2 \notin \text{shad}(d)$. By Corollary 34, either d is left b_2 or d is right of b_2.

First, assume that d is on the boundary of $\text{shad}(b_1)$. If d lies in $W_L(b_1)$, then $W_L(d)$ is a subpath of $W_L(b_1)$. Moreover, $\text{gcpe}(W_L(b_1), W_L(b_2))$ is strictly less than d in P, since otherwise $d \in \text{shad}(b_2)$. Since $W_L(b_1)$ is left of $W_L(b_2)$, we obtain $W_L(d)$ left of $W_L(b_2)$. Symmetrically,

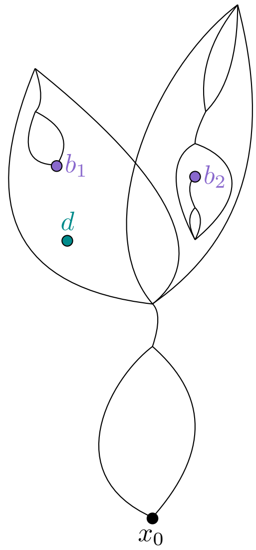


FIGURE 13. An illustration of the statement of Proposition 36.(*L*). We only draw the paths forming boundaries of shadows of b_1 and b_2.

if d lies in $W_R(b_1)$, we obtain $W_R(d)$ left of $W_R(b_2)$. In both cases, we conclude that d is left of b_2.

Next, assume that d is in the interior of $\text{shad}(b_1)$. Let y_1 be the terminal element of $\text{shad}(b_1)$. Since b_1 is left of b_2, by Proposition 35.(*L*), y_1 is left of b_2. By Proposition 30, either $y_1 < d$ in P or (d, y_1) is an inside pair. In the case $y_1 < d$, Proposition 29.(vi) implies $\text{shad}(y_1) = \text{shad}(d)$, and so, y_1 is the terminal element of $\text{shad}(d)$. Then by Proposition 35.(*L*), d is left of b_2 as desired. Otherwise, assume (d, y_1) is an inside pair. Therefore $W_R(d)$ is left of $W_R(y_1)$. And since y_1 is left of b_2, $W_R(y_1)$ is left of $W_R(b_2)$. By transitivity (Proposition 9), we obtain $W_R(d)$ left of $W_R(b_2)$, which completes the proof. □

Next, we argue that the relation of being left (and as follows right as well) is transitive, and henceforth, its reflexive closure partially orders B.

Proposition 37. *For all $b_1, b_2, b_3 \in B$, if b_1 is left of b_2 and b_2 is left of b_3, then b_1 is left of b_3.*

Proof. Assume that b_1 is left of b_2 and b_2 is left of b_3. By Proposition 9, we immediately obtain that (b_1, b_3) is a left pair. It suffices to argue that $b_1 \notin \text{shad}(b_3)$ and $b_3 \notin \text{shad}(b_1)$. Suppose to the contrary that $b_1 \in \text{shad}(b_3)$. Since $b_1 \notin \text{shad}(b_2)$ and b_2 is left of b_3, by Proposition 36.(*R*), we obtain that b_1 is right of b_2, which is a contradiction. Similarly, if $b_3 \in \text{shad}(b_1)$, then we obtain that b_3 is left of b_2, which is again a contradiction and ends the proof. □

5. Reduction to a maximal good instance

In Section 4, we defined instances and developed a small theory describing them. In Corollary 13, we showed that the main difficulty of Theorem 1 is captured by the notion of instances. The goal of this section is a further reduction from instances to good instances, defined in Subsection 5.4. Given an instance $(P, x_0, G, e_{-\infty}, I)$, we focus only on pairs in I that we call "risky" as two reversible sets can cover all the non-risky pairs, see Proposition 38. In Subsection 5.2,

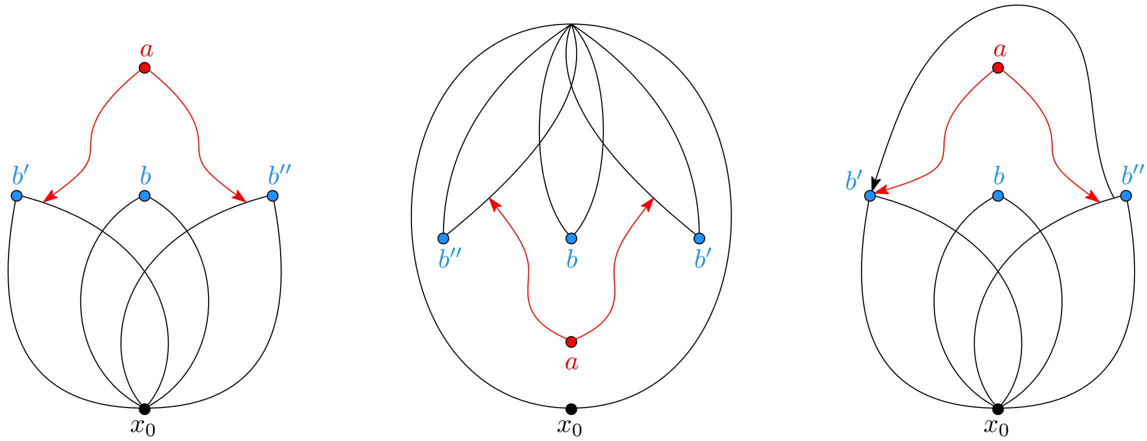


FIGURE 14. In all three drawings the pair (a, b) is a risky pair witnessed by b' and b''. Note that b' is not necessarily left of b'' (the middle drawing), in fact, (b', b) is not even necessarily a left pair (the right drawing). Note that in the right drawing, the element b' also can play a role of b'', that is, $W_R(b)$ is left of $W_R(b'')$. The situation in the left drawing is visibly the "cleanest" one. Further in the paper, we will introduce the notion of "dangerous" pairs, the left drawing is the only one among the three depicting a dangerous pair. Note that a is not necessarily in B. We tend to mark such elements in red and the elements in B in blue.

for a risky pair $(a, b) \in I$, we define its escape number to be the least nonnegative integer such that $a \notin \text{shad}_j(b)$. Next, we split I into I_0 and I_1 depending on the parity of their escape number. The main technical statement of this section is Lemma 40, which describes the structure of strict alternating cycles with all pairs in I_θ for a fixed $\theta \in \{0, 1\}$. E.g., all pairs in such an alternating cycle must have the same escape number. This structure allows us eventually to reduce the problem to instances that carry additional properties (I6)–(I8) and also (I9). The final outcome of this section is Corollary 47.

We fix an instance $(P, x_0, G, e_{-\infty}, I)$ within Subsections 5.1 to 5.3.

5.1. **Risky pairs.** We say that $(a, b) \in I$ is a *risky* pair if

(r1) there exists $b' \in B$ with $a \leqslant b'$ in P such that $W_L(b')$ is left of $W_L(b)$, and
(r2) there exists $b'' \in B$ with $a \leqslant b''$ in P such that $W_R(b)$ is left of $W_R(b'')$.

See Figure 14. In fact, the "interesting" pairs are risky pairs, which is formally proved in the next proposition. Later, in Proposition 39, we establish a key property of risky pairs.

Proposition 38. *Let I' be the set of all non-risky pairs in I. Then* $\dim_P(I') \leqslant 2$.

Proof. Let I'_1 be all the pairs in I that do not satisfy (r1) and let I'_2 be all the pairs in I that do not satisfy (r2). Note that $I' = I'_1 \cup I'_2$. We claim that I'_1 and I'_2 are reversible. We will prove that I'_1 is reversible, the proof for I'_2 is symmetric. Suppose to the contrary that I'_1 is not reversible, and so by Proposition 3, I'_1 contains a strict alternating cycle $((a_1, b_1), \ldots, (a_k, b_k))$. Let $i \in [k]$. Since (a_i, b_i) does not satisfy (r1), $a_i \leqslant b_{i+1}$ in P (we consider the indices cyclically, e.g. $b_{k+1} = b_1$), and $b_i \parallel b_{i+1}$ in P we have $W_L(b_{i+1})$ right of $W_L(b_i)$. However, this cannot hold cyclically for all $i \in [k]$. The contradiction completes the proof. □

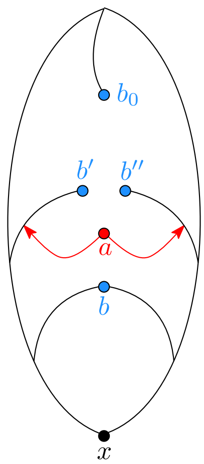


FIGURE 15. Illustration of the statement of Proposition 39. The element x is the initial element of $\text{shad}_j(b_0)$ and $\text{shad}_j(b)$. The brief intuition of why the proposition holds is the following. Since $a \in \text{shad}_j(b_0)$, the elements b' and b'' witnessing that (a, b) is a risky pair are in $\text{shad}_j(b_0)$ and in turn "trap" b in $\text{shad}_j(b_0)$.

Proposition 39. *Let $(a, b) \in I$ be a risky pair, let $b_0 \in B$ with $a \parallel b_0$ in P, and suppose that for some nonnegative integer j, $\text{shad}_j(b_0)$ and $\text{shad}_j(b)$ have the same initial element. If $a \in \text{shad}_j(b_0)$, then $b \in \text{shad}_j(b_0)$.*

Proof. See Figure 15. Since (a, b) is a risky pair, there exist $b', b'' \in B$ as in (r1) and (r2). By Observation 22, we obtain that b' and b'' lie in the interior of $\text{shad}_j(b_0)$.

Let x and y be the initial and terminal elements of $\text{shad}_j(b_0)$, respectively. By Proposition 30 applied to b', either $W_L(y)$ is left of $W_L(b')$ (as (b', y) is an inside pair) or $y < b'$ in P. In the case where $y < b'$ in P, by Proposition 28, $W_L(y)$ is a subpath of $W_L(b')$. Similarly, by Proposition 30 applied to b'', either $W_R(b'')$ is left of $W_R(y)$ or $W_R(y)$ is a subpath of $W_R(b'')$. We claim that

$$\begin{aligned} &\text{no witnessing path from } x_0 \text{ to } b \text{ in } P \text{ is left of } W_L(y), \text{ and} \\ &\text{no witnessing path from } x_0 \text{ to } b \text{ in } P \text{ is right of } W_R(y). \end{aligned} \tag{4}$$

Indeed, if U is a witnessing path from x_0 to b in P and U is left of $W_L(y)$, then U is left of $W_L(b')$. However, this implies that $W_L(b)$ is left of $W_L(b')$, a contradiction. The proof of the statement for rightmost paths goes analogously.

First, suppose that neither of $W_L(b), W_L(y)$ is a subpath of the other and that neither of $W_R(b), W_R(y)$ is a subpath of the other. By (4), it follows that $W_L(y)$ is left of $W_L(b)$ and $W_R(b)$ is left of $W_R(y)$. Therefore, (b, y) is an inside pair, and by Proposition 31, $b \in \text{shad}_j(y) = \text{shad}_j(b_0)$.

If $W_L(b)$ is a subpath of $W_L(y)$, then $b \in \text{shad}_j(b_0)$. Similarly, if $W_R(b)$ is a subpath of $W_R(y)$, then $b \in \text{shad}_j(b_0)$.

Finally, it remains to consider the case, where $W_L(y)$ is a subpath of $W_L(b)$ or $W_R(y)$ is a subpath of $W_R(b)$. Assume that $W_L(y)$ is a subpath of $W_L(b)$, the proof in the other case is symmetric. It follows that $y \leqslant b$ in P. Recall that either $W_L(y)$ is left of $W_L(b')$ or $W_L(y)$ is a subpath of $W_L(b')$. If $W_L(y)$ is left of $W_L(b')$, then $W_L(b)$ is left of $W_L(b')$, which is false as

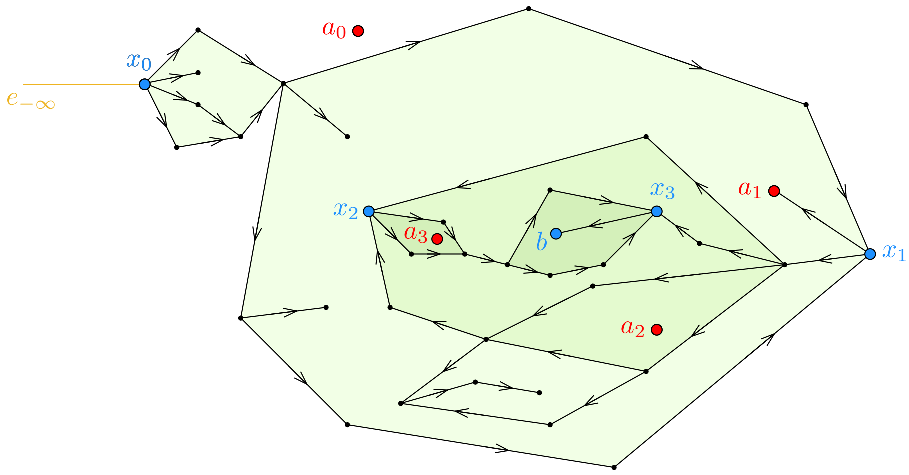


FIGURE 16. For every $j \in \{0,1,2,3\}$, the escape number of (a_j, b) is equal to j, and the address is equal to (j, x_j).

b' witnesses (r1) for (a,b). Hence, $W_L(y)$ is a subpath of $W_L(b')$. Since b' is in the interior of $\text{shad}_j(y)$, by Corollary 25 the path $y[W_L(b')]b'$ lies in $\text{shad}_j(y)$. Suppose to the contrary that $b \notin \text{shad}_j(y)$ and let W be a witnessing path from y to b in P. Note that the only element of W in $\text{shad}_j(y)$ is y. By (sb2), this yields that $x_0[W_L(y)]y[W]b$ is left of $W_L(b')$, which is left of $W_L(b)$ (by (r1)), which is a contradiction that ends the proof. □

5.2. **Escape number and address.** Let $(a,b) \in I$. We define the *escape number* of (a,b) to be the least nonnegative integer j such that $a \notin \text{shad}_j(b)$. Note that this is well-defined as if $j = \text{sd}(b)+1$, then $\text{shad}_j(b) = \emptyset$. For each $\theta \in \{0,1\}$, we let

$$I_\theta = \{(a,b) \in I : \text{ the escape number of } (a,b) \text{ is } j \text{ with } j \equiv \theta \bmod 2\}.$$

Note that I is partitioned into I_0 and I_1.

We define the *address* of (a,b) to be the pair (j,x), where j is the escape number of (a,b) and x is the initial element of $\text{shad}_j(b)$. See Figure 16.

We prove now a comprehensive lemma that provides structural information about strict alternating cycles with all pairs being risky and in I_θ for a fixed $\theta \in \{0,1\}$. See Figure 17.

Lemma 40. *Let $\theta \in \{0,1\}$, and let $((a_1,b_1),\ldots,(a_k,b_k))$ be a strict alternating cycle in P with all the pairs being risky pairs in I_θ. Then there is a nonnegative integer j and an element x of P such that for every $\alpha \in [k]$, the address of (a_α,b_α) is (j,x); and for all distinct $\alpha,\beta \in [k]$, $b_\beta \notin \text{shad}_j(b_\alpha)$.*

Proof. The indices in $[k]$ are considered cyclically, e.g. $a_{k+1} = a_1$. For each $\alpha \in [k]$, let (j_α, x_α) be the address of (a_α,b_α). Set $j = \min\{j_\alpha : \alpha \in [k]\}$. Fix $\gamma \in [k]$ such that $j = j_\gamma$ and let $x = x_\gamma$. Eventually, we will show that (j,x) is the common address of (a_α,b_α) for all $\alpha \in [k]$. The proof is split into three intermediate statements encapsulated by the following claims.

Claim 41. For all $\alpha \in [k]$, if $j > 0$, then x is the terminal element of $\text{shad}_{j-1}(b_\alpha)$.

Proof. If $j = 0$, then the claim holds vacuously, thus, suppose that $j > 0$ and let $\alpha \in [k]$. Since j is at most the escape number of (a_α,b_α), the element a_α lies in $\text{shad}_{j-1}(b_\alpha)$. Let $\mathcal{B}_\alpha$ be the block of $\text{shad}_{j-1}(b_\alpha)$ such that a_α lies in $\mathcal{B}_\alpha$. Since $a_\alpha \leqslant b_{\alpha+1}$ and $a_\alpha \parallel b_\alpha$ in P, by

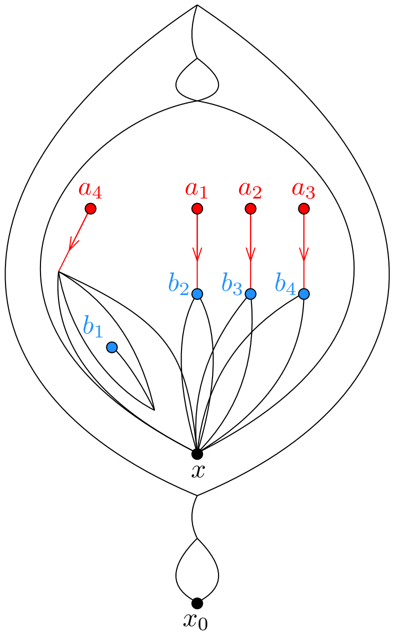


FIGURE 17. An example of a strict alternating cycle with all the pairs in I_0 (the address of (a_i, b_i) for each $i \in [4]$ is equal to $(2, x)$). In particular, in order to study the cycle, it suffices to study the terminal block of $\text{shad}_2(x)$. Lemma 40 shows that every strict alternating cycle with all the pairs in I_θ for some $\theta \in \{0, 1\}$ "looks like this". Note, however, that the shadow depth of each b_i may be much higher than the number in the common address of the pairs in the cycle (as it indeed is for b_1).

Observation 21, $b_{\alpha+1}$ lies in the interior of $\mathcal{B}_\alpha$. By Proposition 29.(v),

$$\text{shad}_{j-1}(b_{\alpha+1}) \subseteq \text{shad}_{j-1}(\max \mathcal{B}_\alpha) \subseteq \text{shad}_{j-1}(b_\alpha).$$

Since these inclusions hold for all $\alpha \in [k]$ cyclically, we conclude that for all $\alpha \in [k]$,

$$\text{shad}_{j-1}(b_{\alpha+1}) = \text{shad}_{j-1}(\max \mathcal{B}_\alpha) = \text{shad}_{j-1}(b_\alpha).$$

Since $\text{shad}_{j-1}(b_\alpha)$ are the same for all $\alpha \in [k]$, their terminal blocks also coincide. Let $\mathcal{B}$ be the common terminal block. Recall that for every $\alpha \in [k]$, $b_{\alpha+1}$ lies in the terminal block of $\text{shad}_{j-1}(b_{\alpha+1})$, thus, $\mathcal{B} = \mathcal{B}_\alpha$. Since $x = x_\gamma$ is the terminal element of $\text{shad}_{j-1}(b_\gamma)$, we conclude that $x = \max \mathcal{B}$ and the assertion holds. $\triangleleft$

Claim 42. Let $\alpha, \delta \in [k]$ with $\delta \neq \alpha - 1$ and such that a_δ lies in the interior of $\text{shad}_j(b_\alpha)$. Then, a_α lies in the interior of $\text{shad}_j(b_\alpha)$.

Proof. Without loss of generality, assume that $\alpha = 1$. Note that $\delta \neq k$. We show that if $\beta \in [k-1]$ and a_β lies in the interior of $\text{shad}_j(b_1)$, then either $\beta = 1$ or $a_{\beta-1}$ is also in the interior of $\text{shad}_j(b_1)$. The statement of the claim immediately follows.

Suppose that $\beta \in \{2, \ldots, k-1\}$ and a_β lies in the interior of $\text{shad}_j(b_1)$. Recall that by Claim 41, $\text{shad}_j(b_1)$ and $\text{shad}_j(b_\beta)$ have a common initial element (namely x). Since the pair (a_β, b_β) is a risky pair, $a_\beta \parallel b_1$ in P, and a_β is in the interior of $\text{shad}_j(b_1)$, by Proposition 39, b_β lies in the interior of $\text{shad}_j(b_1)$. Since $a_{\beta-1} \leqslant b_\beta$ in P and $1 \neq \beta$, by Observation 22, we obtain that $a_{\beta-1}$ is in the interior of $\text{shad}_j(b_1)$, as desired. $\triangleleft$

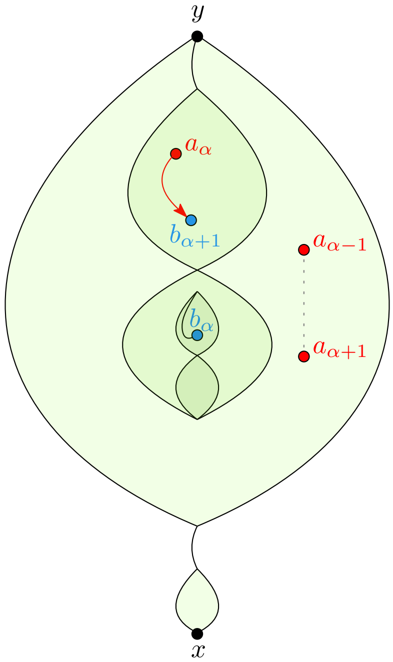


FIGURE 18. An illustration of the proof of Claim 43. We show $\text{shad}_j(b_\alpha)$ (equal to $\text{shad}_j(b_{\alpha+1})$). We arrive at $a_{\alpha+1} \in \text{shad}_j(b_{\alpha+1})$, which is a contradiction.

Claim 43. For all $\alpha \in [k]$, we have $j_\alpha = j$.

Proof. Suppose to the contrary that there is some $\alpha \in [k]$ with $j_\alpha \neq j$. Since we consider only pairs in I_θ and j is the minimum value of the escape number among all the pairs in the strict alternating cycle, we have $j_\alpha \geqslant j + 2$. Without loss of generality, we can assume that $j_{\alpha+1} = j$.

The argument is illustrated in Figure 18. By definition of the escape number, the element a_α lies in the interior of $\text{shad}_{j+1}(b_\alpha)$. Since $a_\alpha \leqslant b_{\alpha+1}$ in P, by Observation 22, $b_{\alpha+1}$ lies in the interior of $\text{shad}_{j+1}(b_\alpha)$. By Proposition 29.(iv), we have $\text{shad}_j(b_\alpha) = \text{shad}_j(b_{\alpha+1})$.

Let y be the terminal element of $\text{shad}_j(b_\alpha) = \text{shad}_j(b_{\alpha+1})$. It follows that $y \leqslant b_\alpha$ and $y \leqslant b_{\alpha+1}$ in P, hence, by definition of strict alternating cycles, for every $\beta \in [k]$, we have $a_\beta \parallel y$ in P. In particular, $a_{\alpha-1} \parallel y$ in P. Since $a_{\alpha-1} \leqslant b_\alpha$ in P, by Observation 22, the element $a_{\alpha-1}$ lies in the interior of $\text{shad}_j(b_\alpha) = \text{shad}_j(b_{\alpha+1})$. Therefore, by Claim 42, $a_{\alpha+1}$ lies in the interior of $\text{shad}_j(b_{\alpha+1})$, which contradicts $j_{\alpha+1} = j$ and ends the proof of the claim. ◁

To complete the proof of the lemma, it suffices to prove that for all distinct $\alpha, \beta \in [k]$, we have $b_\beta \notin \text{shad}_j(b_\alpha)$. Suppose to the contrary that there are distinct $\alpha, \beta \in [k]$ such that $b_\beta \in \text{shad}_j(b_\alpha)$. Since $b_\alpha \parallel b_\beta$ in P, we obtain a stronger statement that b_β lies in the interior of $\text{shad}_j(b_\alpha)$. Since $\alpha \neq \beta$, we have $a_{\beta-1} \parallel b_\alpha$ in P. This combined with the fact that $a_{\beta-1} \leqslant b_\beta$ in P and Observation 21 implies that $a_{\beta-1}$ lies in the interior of $\text{shad}_j(b_\alpha)$. Therefore, by Claim 42, we obtain that a_α lies in the interior of $\text{shad}_j(b_\alpha)$, which yields $j_\alpha > j$, which contradicts Claim 43. This completes the proof of the lemma. □

5.3. **Dangerous pairs.** In this subsection, we refine the notion of risky pairs to the so-called dangerous pairs. We say that $(a, b) \in I$ is a *dangerous* pair if

(d1) there exists $b' \in B$ with $a \leqslant b'$ in P such that b' is left of b, and
(d2) there exists $b'' \in B$ with $a \leqslant b''$ in P such that b is left of b''.

See examples of dangerous and non-dangerous pairs in Figure 14. Dangerous pairs are the only "interesting" pairs in the following sense. Later on, we will apply Lemma 40, to reduce the problem only to pairs so that in every strict alternating cycle, the elements b are ordered left-to-right in the sense introduced in Subsection 4.4. We show that, assuming this property, the non-dangerous pairs have low dimension.

Proposition 44. *Let $I' \subseteq I$ be a set of pairs that are not dangerous such that for every strict alternating cycle $((a_1, b_1), \ldots, (a_k, b_k))$ in P contained in I', for all $\alpha, \beta \in [k]$ with $\alpha \neq \beta$, either b_α left of b_β or b_α right of b_β. Then,* $\dim_P(I') \leqslant 2$.

Proof. Let I'_1 be all the pairs in I' that do not satisfy (d1) and let I'_2 be all the pairs in I' that do not satisfy (d2). Note that $I' = I'_1 \cup I'_2$. We claim that I'_1 and I'_2 are reversible. We will prove that I'_1 is reversible, the proof for I'_2 is symmetric. Suppose to the contrary that I'_1 is not reversible, and so by Proposition 3, I'_1 contains a strict alternating cycle $((a_1, b_1), \ldots, (a_k, b_k))$. Let $\alpha \in [k]$. By the assumption, either b_α is left of $b_{\alpha+1}$ or $b_{\alpha+1}$ is left of b_α (we consider the indices cyclically, e.g. $b_{k+1} = b_1$). Since (a_α, b_α) does not satisfy (d1), $a_\alpha \leqslant b_{\alpha+1}$ in P we have that $b_{\alpha+1}$ is right of b_α. However, this cannot hold cyclically for all $\alpha \in [k]$. The contradiction completes the proof. □

Proposition 45. *Let $(a, b) \in I$ be a dangerous pair and let $d \in B$ such that either d is left of b or b is left of d. Then $a \notin \operatorname{shad}(d)$.*

Proof. Assume that d is left of b. When b is left of d, the proof is symmetric. Let $b'' \in B$ witness (d2) for (a, b). Since $b'' \notin \operatorname{shad}(b)$ and $a \leqslant b''$ in P, by Observation 22, we obtain $a \notin \operatorname{shad}(b)$.

Suppose to the contrary that $a \in \operatorname{shad}(d)$. Since $a \leqslant b''$ in P, there is a witnessing path W from a to b'' in P. Let u be the first element in B on W. It follows that $u \in \operatorname{shad}(d)$. Moreover, since $a \notin \operatorname{shad}(b)$ and $a \leqslant u$ in P, by Observation 22, we obtain $u \notin \operatorname{shad}(b)$. Since d left of b, by Proposition 36.(*L*), u is left of b. Since u is left of b and b is left of b'', u is left of b'' (by Proposition 37). In particular, $u \notin \operatorname{shad}(b'')$, however, $u \leqslant b''$ in P, which is a contradiction with Proposition 33, hence, indeed, $a \notin \operatorname{shad}(d)$. □

Corollary 46. *Let $(a, b) \in I$ be a dangerous pair, let (d1) be witnessed by $b' \in B$, and let (d2) be witnessed by $b'' \in B$. Then*

$$a \notin \operatorname{shad}(b'), \quad a \notin \operatorname{shad}(b''), \quad \textit{and} \quad a \notin B.$$

Proof. The first two items are direct corollaries of Proposition 45. For the last item, if $a \in B$, then since $a \leqslant b'$ in P, by Proposition 33, $\operatorname{shad}(a) \subseteq \operatorname{shad}(b')$, which contradicts $a \notin \operatorname{shad}(b')$, and proves that $a \notin B$. □

5.4. **Interface.** In order to capture the consequences of the theory developed in this section, we introduce the following notion. We say that an instance $(P, x_0, G, e_{-\infty}, I)$ is a *good instance* if

(I6) $a \notin \operatorname{shad}(b)$ for every $(a, b) \in I$;
(I7) for every strict alternating cycle $((a_1, b_1), \ldots, (a_k, b_k))$ in P contained in I, for all distinct $\alpha, \beta \in [k]$, either b_α is left of b_β or b_α is right of b_β;

(I8) every pair in I is dangerous.

Next, we show that for every instance, there is a good instance such that the dimension does not vary much.

Corollary 47. *For every instance $(P, x_0, G, e_{-\infty}, I)$, there exists a good instance $(P', x_0', G', e_{-\infty}', I')$ such that P' is a convex subposet of P, $I' \subseteq I$, and*

$$\dim_P(I) \leqslant 2\dim_{P'}(I') + 6.$$

Proof. Let $\mathcal{I} = (P, x_0, G, e_{-\infty}, I)$ be an instance. For every pair (j, x) where j is a nonnegative integer and x is an element of P, let

$$I(j, x) = \{(a, b) \in I : (a, b) \text{ is risky and the address of } (a, b) \text{ is } (j, x)\}.$$

We define

$$I_\theta = \bigcup\{I(j, x) : \ j \text{ with } j \equiv \theta \bmod 2 \text{ and } x \text{ in } P\}, \text{ for each } \theta \in \{0, 1\} \text{ and}$$
$$I_2 = \{(a, b) \in I : \ (a, b) \text{ is not risky}\}.$$

Note that I_0, I_1, I_2 are pairwise disjoint and their union is I. Hence, applying Proposition 38, we fix $\theta \in \{0, 1\}$ such that

$$\dim_P(I) \leqslant \dim_P(I_0) + \dim_P(I_1) + \dim_P(I_2) \leqslant 2\dim_P(I_\theta) + 2.$$

By Lemma 40, for every strict alternating cycle in P contained in I_θ, there exists (j, x) such that all the pairs in the alternating cycle are in $I(j, x)$. Therefore, applying Proposition 5, we fix a pair (j, x) such that $\dim_P(I_\theta) = \dim_P(I(j, x))$. Let $J = I(j, x)$.

Let P' be the subposet of P induced by all elements p in P with $p \not< x$ in P. Note that P' is a convex subposet of P. Indeed, if p, q, r in P with $p \leqslant q \leqslant r$ in P and p, r in P', then $r \not< x$ in P implies $q \not< x$ in P, and so, q is in P'. Since P' is a convex subposet of P, we can set G' to be the plane graph isomorphic to the cover graph of P' inherited from G. Since x_0 lies in the exterior face of G and the only element of $W_L(x)$ in P that is in P' is x, the element x lies in the exterior face of G'. We set $x_0' = x$. Finally, let $e_{-\infty}' = e_{-\infty}$ when $x_0 = x$ and let $e_{-\infty}$ be the last edge of $W_L(x)$ otherwise.

We claim that $\mathcal{I}' = (P', x_0', G', e_{-\infty}', J)$ satisfies (I1)–(I7). Items (I1), (I2), (I3), and (I4) are clear from the definitions. Note that $J \subseteq \text{Inc}(P')$. Indeed, for every $(a, b) \in J$, the address of (a, b) is (j, x), and so, $x < b$ in P; it follows that $a \not< x$ in P, and finally both a and b are in P'. Since $x < b$ in P for every $(a, b) \in J$, we find that J is singly constrained by x in P. This implies (I5). Let b be an element of P' with $x < b$ in P. Note that $\text{shad}_j(b)$ with respect to $\mathcal{I}$ is equal to $\text{shad}_0(b)$ with respect to $\mathcal{I}'$. Since for every $(a, b) \in J$, the escape number with respect to $\mathcal{I}$ is j, we have $a \notin \text{shad}_j(b)$ with respect to $\mathcal{I}$, and so, $a \notin \text{shad}_0(b)$ with respect to $\mathcal{I}'$. This proves (I6). Finally, (I7) follows from Lemma 40 and Corollary 34.

In the last step, we enforce the condition (I8) on a slightly smaller instance. Let I' be the set of all the pairs in J that are dangerous with respect to $\mathcal{I}'$. By (I7) and Proposition 44, we have $\dim_{P'}(J \setminus I') \leqslant 2$. Note that replacing the set of pairs in an instance by its subset preserves conditions (I1)–(I7). It follows that $(P', x_0', G', e_{-\infty}', I')$ is a good instance. Moreover, P' is a convex subposet of P, $I' \subseteq I$, and

$$\begin{aligned}\dim_P(I) \leqslant 2\dim_P(I_\theta) + 2 = 2\dim_{P'}(J) + 2 &\leqslant 2(\dim_{P'}(I') + \dim_{P'}(J \setminus I')) + 2\\ &\leqslant 2\dim_{P'}(I') + 6.\end{aligned}$$

□

For a given set X and $I \subseteq X \times X$, we define

$$\pi_1(I) = \{a \in X : \text{ there exists } b' \in X \text{ with } (a, b') \in I\} \text{ and}$$
$$\pi_2(I) = \{b \in X : \text{ there exists } a' \in X \text{ with } (a', b) \in I\}.$$

We say that a good instance $(P, x_0, G, e_{-\infty}, I)$ is *maximal* if

(I9) I is the set of all pairs $(a, b) \in \mathrm{Inc}(P)$ such that $a \in \pi_1(I)$, $b \in \pi_2(I)$, $a \notin \mathrm{shad}(b)$, and (a, b) is dangerous.

Note that it is not immediate if a good instance is "contained" in some maximal good instance. This issue is resolved in the next proposition.

Proposition 48. *For every good instance $(P, x_0, G, e_{-\infty}, I)$, there exists $I^+ \subseteq \mathrm{Inc}(P)$ such that $I \subseteq I^+ \subseteq \mathrm{Inc}(P) \cap (\pi_1(I) \times \pi_2(I))$ and $(P, x_0, G, e_{-\infty}, I^+)$ is a maximal good instance.*

Proof. Let $\mathcal{I} = (P, x_0, G, e_{-\infty}, I)$ be a good instance and let I^+ be the set of all pairs $(a, b) \in \mathrm{Inc}(P)$ such that there exist elements a', b' in P with $(a', b), (a, b') \in I$, $a \notin \mathrm{shad}(b)$, and (a, b) is dangerous. Since $\mathcal{I}$ satisfies (I6) and (I8), we have $I \subseteq I^+$. Let $\mathcal{I}^+ = (P, x_0, G, e_{-\infty}, I^+)$. It is clear that $\mathcal{I}^+$ satisfies (I1)–(I6) and (I8). By definition, $\mathcal{I}^+$ also satisfies (I9). Thus, it suffices to argue that $\mathcal{I}^+$ satisfies (I7). Let $((a_1, b_1), \ldots, (a_k, b_k))$ be a strict alternating cycle in P contained in I^+. We plan to apply Lemma 40. To this end, note that since $a \notin \mathrm{shad}(b)$ for every $(a, b) \in I^+$, the escape number of each pair $(a, b) \in I^+$ is 0. Therefore, by Lemma 40 applied in the context of $\mathcal{I}^+$, for all distinct $\alpha, \beta \in [k]$, either b_α is left of b_β or b_α is right of b_β. This shows that $\mathcal{I}^+$ satisfies (I7), and thus, ends the proof. □

6. Topology of a maximal good instance

The tools built so far give ground for the first two steps of the proof of Theorem 1. The unfolding technique presented in the preliminaries (see Lemma 12) allows to reduce the statement of Theorem 1 to a statement on dimension of an instance (see Corollary 13). In Section 5, we saw how to further reduce the problem to a statement on a good instance (see Corollary 47) and a maximal good instance (see Proposition 48). It remains to prove the following bound on dimension in maximal good instances.

Theorem 49. *For every maximal good instance $(P, x_0, G, e_{-\infty}, I)$, we have*

$$\dim_P(I) \leqslant 16\,\mathrm{se}_P(I)^6 \cdot (\mathrm{se}_P(I) + 3)^2.$$

Before we prove Theorem 49, we need some more theory. In this section, we work with a fixed maximal good instance $(P, x_0, G, e_{-\infty}, I)$.

6.1. Exposed paths.

Let

$$B = \{b \text{ in } P : x_0 \leqslant b \text{ in } P\} \text{ and}$$
$$A = \{a \text{ in } P : \text{there exists } (a, b) \in I\}.$$

Since every $(a, b) \in I$ is dangerous (see (I8)), by Corollary 46, $A \cap B = \emptyset$. Let $a \in A$ and $b \in B$ with $a < b$ in P. We say that a witnessing path W from a to b in P is *exposed* if b is the only element of W in B. For every $a \in A$, we define

$$Y(a) = \{y \in B : a < y \text{ in } P \text{ and } a \notin \mathrm{shad}(y)\} \text{ and}$$
$$Z(a) = \{z \in Y(a) : \text{there exists an exposed witnessing path from } a \text{ to } z \text{ in } P\}.$$

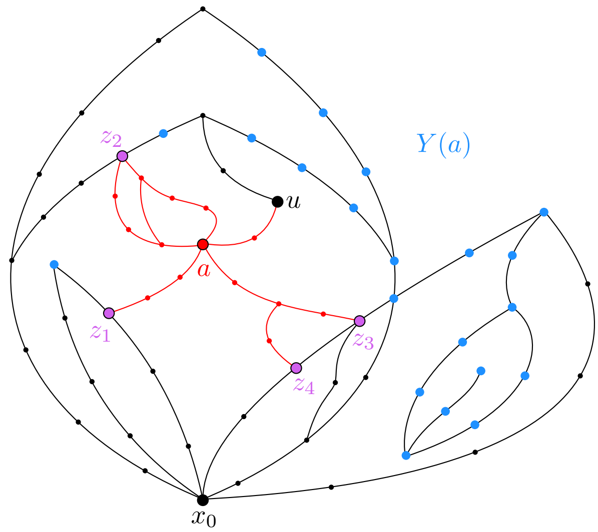


FIGURE 19. $Z(a) = \{z_1, z_2, z_3, z_4\}$. Note that $u \notin Y(a)$ as $a \in \text{shad}(u)$. Note also that $Z(a)$ is not an antichain in P. We mark the elements in $Y(a)$ with blue.

Observe that every element in $Y(a)$ generates at least one element in $Z(a)$. Indeed, let $y \in Y(a)$ and let W be a witnessing path from a to y in P. Let z be the first element in B in W. Since $z \leqslant y$ in P, we have $\text{shad}(z) \subseteq \text{shad}(y)$ (by Proposition 33), and so, $z \in Y(a)$. Moreover, $a[W]z$ is an exposed witnessing path from a to z in P, and so, $z \in Z(a)$. Note that when $z \in Z(a)$, there can be a witnessing path from a to z that is not exposed. In particular, the set $Z(a)$ is not necessarily an antichain. See Figure 19.

Before, we proceed with the material on exposed paths and elements in $Z(a)$, we state a technical corollary of Corollary 46. It follows since (by (I8)) all the pairs in I are dangerous. We will use it extensively.

Corollary 50. *Let $(a, b) \in I$ and let $d \in B$ such that either b is left of d or d is left of b. Then, $a \notin \text{shad}(d)$. Moreover, if $a < d$ in P, then $d \in Y(a)$.*

Proposition 51. *Let $a \in A$, $b \in B$, $y \in Y(a)$, and $z, z' \in Z(a)$. Then,*

(i) *if $a \notin \text{shad}(b)$, then $z \notin \text{int}\,\text{shad}(b)$;*
(ii) *if $z \leqslant y$ in P, then $z \in \partial\,\text{shad}(y)$;*
(iii) $\text{sd}(z) = 0$;
(iv) *if $z \parallel z'$ in P, then either z is left of z' or z is right of z'.*

Proof. In order to prove (i), suppose that $a \notin \text{shad}(b)$ and $z \in \text{shad}(b)$. We argue that z must be in the boundary of $\text{shad}(b)$. Since $z \in Z(a)$, there is an exposed witnessing path W from a to z in P. Note that W intersects $\partial\,\text{shad}(b)$. The only element of W in B is z, and all the elements of $\partial\,\text{shad}(b)$ are in B, hence, $z \in \partial\,\text{shad}(b)$. This completes the proof of (i).

In order to prove (ii) suppose that $z \leqslant y$ in P. By Proposition 33, we have $\text{shad}(z) \subseteq \text{shad}(y)$. In particular, $z \in \text{shad}(y)$. By (i), z lies on the boundary of $\text{shad}(y)$, which proves (ii). Item (iii) follows from (ii) and Proposition 29.(iii). Finally, suppose that $z \parallel z'$ in P. Since $z' \in Z(a) \subseteq Y(a)$, $a \notin \text{shad}(z)$. If $z' \in \text{shad}(z)$, then $z' \in \text{int}\,\text{shad}(z)$, which is a contradiction with (i). It follows that $z' \notin \text{shad}(z)$, and symmetrically, we obtain $z \notin \text{shad}(z')$. Therefore, by Corollary 34, either z is left of z' or z is right of z', which proves (iv). □

Proposition 52. *Let $a \in A$, $b \in B$ with $a \parallel b$ in P and $a \notin \text{shad}(b)$. Let $y, y' \in Y(a)$ with $y' \leqslant y$ in P.*

(L) If y is left of b, then y' is left of b.
(R) If y is right of b, then y' is right of b.

Proof. We prove only (L) as the proof of (R) is symmetric. Assume that y is left of b. Since $a \notin \text{shad}(b)$, by Observation 22, we have $y' \notin \text{shad}(b)$. Furthermore, since $y' \leqslant y$ in P, we have, $y' \in \text{shad}(y)$ (by Proposition 33). Finally, by Proposition 36.(L), since y is left of b, we obtain that y' is left of b. □

Corollary 53. *Let $(a, b) \in I$ and let $y, y' \in B$ with $a < y' \leqslant y$ in P.*

(L) If y is left of b, then y' is left of b.
(R) If y is right of b, then y' is right of b.

Proof. We prove only (L) as the proof of (R) is symmetric. Assume that y is left of b. By (I8), (a, b) is dangerous, and so by Proposition 45, $a \notin \text{shad}(y)$. Since $y' \leqslant y$ in P, we have $\text{shad}(y') \subseteq \text{shad}(y)$ (by Proposition 33). It follows that $a \notin \text{shad}(y')$. In particular, $y, y' \in Y(a)$. By (I6), $a \notin \text{shad}(b)$. Finally, we apply Proposition 52.(L) to obtain that y' is left of b. □

For every $a \in A$, let $\mathcal{M}(a)$ be the family of all the paths of the form $M = x_0[U]z[V]a$, where

- $z \in Z(a)$ (we call the element z the *peak* of M),
- U is a witnessing path from x_0 to z in P,
- V is an exposed witnessing path from a to z in P.

Since every path in $\mathcal{M}(a)$ starts in x_0 and ends in a, no path in $\mathcal{M}(a)$ contains another path from $\mathcal{M}(a)$ as a subpath. Therefore, by Observation 10, the relation of being left linearly orders $\mathcal{M}(a)$. Note also that $\mathcal{M}(a)$ is nonempty as there exists $b \in B$ such that $(a, b) \in I$, so by (I8) and (d1), there exists $b' \in B$ with $a < b'$ in P, thus, $b' \in Y(a)$, and therefore, $Z(a)$ is nonempty. In particular, $\mathcal{M}(a)$ has a minimum and a maximum element. Let $M_L(a) \in \mathcal{M}(a)$ be such that $M_L(a)$ is left of M for every $M \in \mathcal{M}(a)$ with $M \neq M_L(a)$, and let $M_R(a) \in \mathcal{M}(a)$ be such that $M_R(a)$ is right of M for every $M \in \mathcal{M}(a)$ with $M \neq M_R(a)$. Moreover, denote by $z_L(a)$ the peak of $M_L(a)$ and by $z_R(a)$ the peak of $M_R(a)$. See Figure 20.

We finish this subsection with a series of simple and intuitive propositions on properties of $M_L(a)$ and $z_L(a)$ (resp. $M_R(a)$ and $z_R(a)$) for $a \in A$. Proposition 54 says that $M_L(a)$ contains $W_L(z_L(a))$. Proposition 55 gives x_0-consistency of paths in $\{M_L(a) : a \in A\}$. Propositions 56 and 57 describe the position of $z_L(a)$ within $Z(a)$ and with respect to b when $(a, b) \in I$.

Proposition 54. *Let $a \in A$. Then*

$$x_0[M_L(a)]z_L(a) = W_L(z_L(a)) \quad \text{and} \quad x_0[M_R(a)]z_R(a) = W_R(z_R(a)).$$

Proof. We prove only the first identity as the proof of the second one is symmetric. Suppose to the contrary that $x_0[M_L(a)]z_L(a) \neq W_L(z_L(a))$. Note that both paths are witnessing paths from x_0 to $z_L(a)$. Thus, $W_L(z_L(a))$ is left of $x_0[M_L(a)]z_L(a)$. Consider the path $M =$

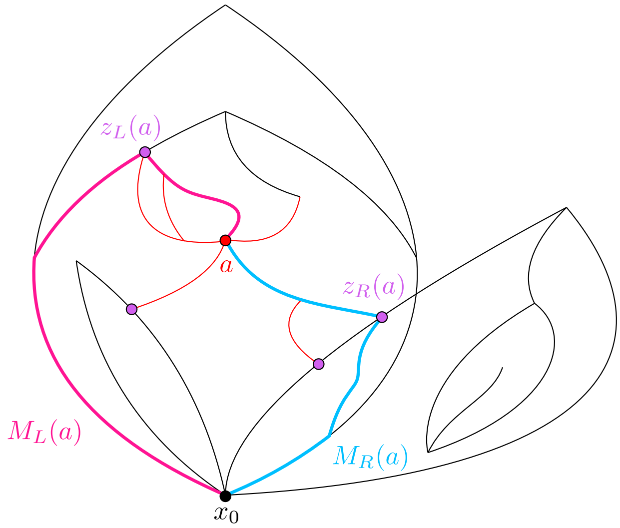


FIGURE 20. The paths $M_L(a)$ and $M_R(a)$ drawn in the same poset as in Figure 19.

$x_0[W_L(z_L(a))]z_L(a)[M_L(a)]a$. It is immediate to see that $M \in \mathcal{M}(a)$ and M is left of $M_L(a)$, which is a contradiction. □

Proposition 55. *Let $a_1, a_2 \in A$.*

(L) *The paths $M_L(a_1)$ and $M_L(a_2)$ are x_0-consistent.*
(R) *The paths $M_R(a_1)$ and $M_R(a_2)$ are x_0-consistent.*

Proof. We prove only (L) as the proof of (R) is symmetric. Let $z_i = z_L(a_i)$ and $M_i = M_L(a_i)$ for each $i \in [2]$. If one of the paths M_1, M_2 is a subpath of the other, then the assertion follows. Hence, assume without loss of generality that M_1 is left of M_2. Let $u = \text{gcpe}(M_1, M_2)$. Suppose to the contrary that the paths are not x_0-consistent. Then, there exists an element v common to both paths with v not in $x_0[M_1]u = x_0[M_2]u$. Note that by Proposition 54 and Proposition 15.(L), $v \notin B$. The path $M = x_0[M_1]v[M_2]a_2$ is left of M_2. We claim that $M \in \mathcal{M}(a_2)$, which contradicts the definition of M_2. The path $a_1[M_L(a_1)]v[M_L(a_2)]a_2$ does not have elements in B, hence, $a_1 \notin \text{shad}(z_1)$ implies $a_2 \notin \text{shad}(z_1)$. Since additionally the path $z_1[M_1]v[M_2]a_2$ is an exposed witnessing path from a_2 to z_1 in P, we have $z_1 \in Z(a_2)$. Moreover, $M = x_0[M_1]z_1[M_1]v[M_2]a_2$, and so, $M \in \mathcal{M}(a_2)$. □

Proposition 56. *Let $a \in A$ and let $z \in Z(a)$.*

(L) *Either $z_L(a)$ is left of z or $z_L(a)$ and z are comparable in P.*
(R) *Either $z_R(a)$ is right of z or $z_R(a)$ and z are comparable in P.*

Proof. We prove only (L) as the proof of (R) is symmetric. Assuming that $z \parallel z_L(a)$ in P, it suffices to prove that $z_L(a)$ is left of z. By Proposition 51.(iv), either $z_L(a)$ is left of z or z is left of $z_L(a)$, hence, it is enough to prove that $W_L(z_L(a))$ is left of $W_L(z)$. Let $M = x_0[W_L(z)]z[V]a$, where V is an exposed witnessing path from a to z in P. Note that $M \in \mathcal{M}(a)$ and z is the peak of M. If $M_L(a) = M$, then $z_L(a) = z$, which contradicts the assumption that $z \parallel z_L(a)$ in P. Thus, $M_L(a)$ is left of M. Let $w = \text{gcpe}(M_L(a), M)$. If

$w \notin B$, then the peaks of $M_L(a)$ and M coincide, that is, $z_L(a) = z$, which is a contradiction again, hence, $w \in B$. Therefore, $W_L(z_L(a))$ is left of $W_L(z)$, which ends the proof. □

Proposition 57. *Let $(a, b) \in I$. Then $z_L(a)$ is left of b, and $z_R(a)$ is right of b.*

Proof. We will show that $z_L(a)$ is left of b. The proof that $z_R(a)$ is right of b is symmetric. The pair (a, b) is a dangerous pair by (I8). By (d1), there exists $b' \in B$ with $a < b'$ in P such that b' is left of b. By Corollary 46, $a \notin \mathrm{shad}(b')$. It follows that $b' \in Y(a)$. Let $z' \in Z(a)$ with $z' \leqslant b'$ in P. By Corollary 53, z' is left of b. By (I6), $a \notin \mathrm{shad}(b)$. Since $a < z_L(a)$ in P, by Observation 22, we have

$$z_L(a) \notin \mathrm{shad}(b). \tag{5}$$

By Proposition 56.(*L*), either $z_L(a)$ is left of z' or $z_L(a)$ and z' are comparable in P. If $z_L(a)$ is left of z', then by transitivity (Proposition 37), $z_L(a)$ is left of b. Thus, we assume that $z_L(a)$ and z' are comparable in P. First, assume that $z_L(a) \leqslant z'$ in P. Since $(a, b) \in I$, $a \leqslant z_L(a) \leqslant z' \leqslant b'$ in P, and b' is left of b, by Corollary 53.(*L*), we obtain that $z_L(a)$ is left of b.

Finally, assume that $z' < z_L(a)$ in P. In particular, $z' \in \mathrm{shad}(z_L(a))$ (by Proposition 33), however, by Proposition 51.(i), z' does not lie in the interior of $\mathrm{shad}(z_L(a))$, thus, z' is an element of $W_L(z_L(a)) \cup W_R(z_L(a))$. Next, we claim that

$$b \notin \mathrm{shad}(z_L(a)). \tag{6}$$

Suppose to the contrary that $b \in \mathrm{shad}(z_L(a))$. By Proposition 30.(*R*), $W_R(b)$ is not right of $W_R(z_L(a))$. If z' is an element of $W_R(z_L(a))$, then $W_R(z')$ is a subpath of $W_R(z_L(a))$, although, since z' is left of b, this implies that $W_R(b)$ is right of $W_R(z_L(a))$, which is a contradiction. Therefore, z' is an element of $W_L(z_L(a))$ and it is not an element of $W_R(z_L(a))$. In other words, z' is strictly on the left side of some shadow block $\mathcal{B}$ of $\mathrm{shad}(z_L(a))$. The path $M_L(a)$ is left of $M = x_0[W_L(z')]z'[W]a$, where W is an exposed witnessing path from a to z' in P. By Proposition 54, $x_0[M_L(a)]z_L(a) = W_L(z_L(a))$. It follows that $x_0[M_L(a)]z' = x_0[M]z'$ and the edges following z' in $M_L(a)$ and M are distinct. Moreover, by (sb3), the first edge of $z'[M]a = W$ is in the interior of $\mathcal{B}$. Since the path W is exposed, all the vertices of W except z' are in the interior of $\mathcal{B}$, and so, in $\mathrm{shad}(z_L(a))$. In particular, $a \in \mathrm{shad}(z_L(a))$, which is a contradiction. This contradiction completes the proof of (6), that is, $b \notin \mathrm{shad}(z_L(a))$. Recall that also $z_L(a) \notin \mathrm{shad}(b)$ (by (5)). Therefore, by Corollary 34, either $z_L(a)$ is left of b or b is left of $z_L(a)$. Recall that z' is an element of $W_L(z_L(a)) \cup W_R(z_L(a))$. If z' lies on $W_L(z_L(a))$, then $W_L(z')$ is a subpath of $W_L(z_L(a))$, and since z' is left of b, $W_L(z_L(a)))$ is left of $W_L(b)$, which implies that $z_L(a)$ is left of b. Similarly, if z' lies on $W_R(z_L(a))$, then $W_R(z')$ is a subpath of $W_R(z_L(a))$, and since z' is left of b, $W_R(z_L(a)))$ is left of $W_R(b)$, which implies that $z_L(a)$ is left of b. □

6.2. **Regions.** Let $a \in A$, $u, v \in Z(a)$ with u left of v, let U be an exposed witnessing path from a to u, let V be an exposed witnessing path from a to v, and let

$$\begin{aligned} q &= \text{the maximal common element of } W_L(u) \text{ and } W_R(v) \text{ in } P,\\ m &= \text{the maximal common element of } U \text{ and } V \text{ in } P,\\ \gamma &= q[W_R(v)]v[V]m[U]u[W_L(u)]q. \end{aligned}$$

Note that γ is a simple closed curve. See Figure 21.

Since u is left of v, we have $q \notin \{u, v\}$, and in particular, both $q[W_L(u)]u$ and $q[W_R(v)]v$ contain at least one edge. Let e be the first edge of $q[W_L(u)]u$ and f be the first edge of $q[W_R(v)]v$. Note that the only common element of γ and $x_0[W_L(u)]q$ is q. By Proposition 18,

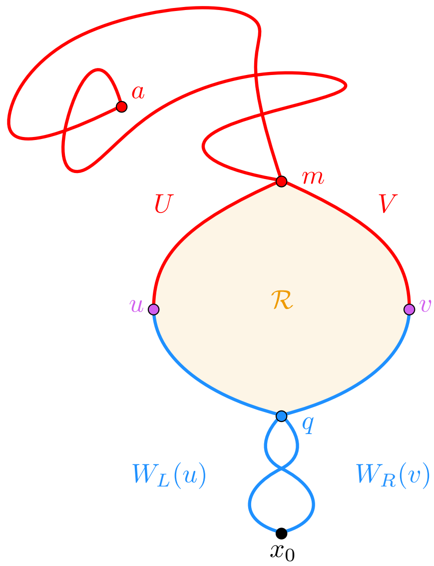


FIGURE 21. The region $\mathcal{R} = \mathcal{R}(a, z_1, z_2, W_1, W_2)$. Note that a can be in the interior of $\mathcal{R}$, in the exterior of $\mathcal{R}$, or equal to m.

f follows e in the counterclockwise orientation of γ. In other words, the cyclic ordering of the edges in $q[W_R(v)]v[V]m[U]u[W_L(u)]q$ is the counterclockwise orientation of γ.

We define $\mathcal{R}(a, u, v, U, V)$ to be the region of γ. Let $\mathcal{R} = \mathcal{R}(a, u, v, U, V)$. We say that q is the *lower-min* of $\mathcal{R}$ and m is the *upper-min* of $\mathcal{R}$. We associate with $\mathcal{R}$ two paths $\gamma_L = x_0[W_L(u)]u[U]m$ and $\gamma_R = x_0[W_R(v)]v[V]m$. We say that the elements of $q[\gamma_L]m$ and $q[\gamma_R]m$ are on the *left side* of $\mathcal{R}$ and on the *right side* of $\mathcal{R}$ respectively. All elements on the left (right) side of $\mathcal{R}$ except q and m are said to be *strictly* on the left (right) side of $\mathcal{R}$.

For convenience, we use the following notation. A tuple $(a, u, v, U, V, \mathcal{R}, q, m, \gamma_L, \gamma_R)$ is a *region tuple* if $a \in A$, $u, v \in Z(a)$ with u left of v, U and V are exposed witnessing paths in P from a to u and a to v respectively, $\mathcal{R} = \mathcal{R}(a, u, v, U, V)$, q is the lower-min of $\mathcal{R}$, m is the upper-min of $\mathcal{R}$, $\gamma_L = x_0[W_L(u)]u[U]m$, and $\gamma_R = x_0[W_R(v)]v[V]m$.

Let $(a, u, v, U, V, \mathcal{R}, q, m, \gamma_L, \gamma_R)$ be a region tuple. Given an edge incident to the boundary of a region, we characterize when it lies in the region. Note similarities to (sb1)–(sb4) and recall Figure 9.

Let w be an element in $\partial\mathcal{R}$. If w is on the left side of $\mathcal{R}$, let e_L^+ and e_L^- be, respectively, the edges (provided they exist) immediately after and immediately before w on the path γ_L. Also, if w is on the right side of $\mathcal{R}$, let e_R^+ and e_R^- be, respectively, the edges (provided they exist) immediately after and immediately before w on γ_R. By Observation 14, an edge e incident to w lies in $\mathcal{R}$ if and only if one of the following holds

(rg1) $w = q$ and $e_L^+ \preccurlyeq e \preccurlyeq e_R^+$ in the q-ordering;
(rg2) $w = m$ and $e_R^- \preccurlyeq e \preccurlyeq e_L^-$ in the m-ordering;
(rg3) w is strictly on the left side and $e_L^+ \preccurlyeq e \preccurlyeq e_L^-$ in the w-ordering;
(rg4) w is strictly on the right side and $e_R^- \preccurlyeq e \preccurlyeq e_R^+$ in the w-ordering.

Assume that $w = q$. If $u = x_0$, we set $e_L^- = e_R^- = e_{-\infty}$. Otherwise, we set e_L^- and e_R^- to be the edges preceding w in $W_L(u)$ and $W_R(v)$ respectively. We claim that

(rg5) $e_R^+ \prec e_R^- \preccurlyeq e_L^- \prec e_L^+$ in the q-ordering.

When $q = x_0$, by (I4), we know that $e_{-\infty}$ does not lie in $\mathcal{R}$, and so, (rg5) holds. Next, assume that $q \neq x_0$. Let W be a witnessing path from x_0 to q in P. This path can intersect $\partial\mathcal{R}$ only in q. Recall that $x_0 \notin \operatorname{int}\mathcal{R}$. Therefore, the last edge e of W does not lie in $\mathcal{R}$. Applying the above to $W = x_0[W_L(u)]q = W_L(q)$ and $W = x_0[W_R(v)]q = W_R(q)$, we obtain that e_L^- and e_R^- do not lie in $\mathcal{R}$. In particular, by (rg1), $e_R^+ \prec e_L^- \prec e_L^+$ and $e_R^+ \prec e_R^- \prec e_L^+$ in the q-ordering. These two combined give two possibilities: $e_R^+ \prec e_R^- \preccurlyeq e_L^- \prec e_L^+$ or $e_R^+ \prec e_L^- \prec e_R^- \prec e_L^+$ in the q-ordering. The former coincides with (rg5), thus, suppose to the contrary that the latter holds. Reordering the edges, we obtain $e_R^- \prec e_L^+ \prec e_L^-$ in the q-ordering. This, by (sb2), yields that e_L^+ lies in the terminal block $\mathcal{B}$ of $\operatorname{shad}(q)$. Since $q[W_L(u)]u$ and $\partial\mathcal{B}$ intersect only in q, and since the first edge of $q[W_L(u)]u$ lies in $\mathcal{B}$, we conclude that $q[W_L(u)]u$ lies in $\mathcal{B}$. In particular, $u \in \operatorname{int}\mathcal{B}$, Thus, by Proposition 29.(vi), we have $\operatorname{shad}(q) = \operatorname{shad}(u)$. Since $\operatorname{shad}(q) = \operatorname{shad}(u)$ and $u \in \operatorname{int}\operatorname{shad}(q)$, we obtain that $\operatorname{sd}(u) > 0$. However, $u \in Z(a)$, so $\operatorname{sd}(u) > 0$ contradicts Proposition 51.(iii). Thus, (rg5) follows.

Within the remaining statements of this subsection, we characterize where an element $b \in B$ lies relative to $\mathcal{R}$. In Proposition 58, we show that if $b \in \operatorname{shad}(q) \setminus \{q\}$, then $b \notin \mathcal{R}$. Proposition 59 is an analog of Corollary 24 for regions defined as above: we show that a witnessing path with both endpoints in $\mathcal{R}$ lies entirely in $\mathcal{R}$. In Proposition 60, we give equivalent conditions for being in $\mathcal{R}$. Corollary 61 is a simpler although less precise statement inferred from Proposition 60. See Figure 22.

Proposition 58. *Let $(a, u, v, U, V, \mathcal{R}, q, m, \gamma_L, \gamma_R)$ be a region tuple and let $b \in B$. If $b \in \mathcal{R} \cap \operatorname{shad}(q)$, then $b = q$.*

Proof. Suppose that $b \in \mathcal{R}$ and $b \neq q$. We will prove that $b \notin \operatorname{shad}(q)$. Let W be a witnessing path from x_0 to b in P. We have $b \in \mathcal{R}$ and $x_0 \notin \operatorname{int}\mathcal{R}$. Thus, W intersects $\partial\mathcal{R}$. Let w be the last element of W in $\partial\mathcal{R}$. That is, $w[W]b \subseteq \mathcal{R}$. Since $w \in B$, we get that w lies in $q[W_L(u)]u \cup q[W_R(v)]v$. If w lies in $q[W_L(u)]u$, let $W' = q[W_L(u)]w[W]b$ and if w lies in $w[W_R(v)]v$, let $W' = q[W_R(v)]w[W]b$. Since $b \neq q$, the path W' contains at least one edge. Let e be the first edge of W'. Recall that e lies in $\mathcal{R}$.

Let e_L^+ and e_R^+ be, respectively, the edges following q in $W_L(u)$ and $W_R(v)$. If $q = x_0$, we set $e_L^- = e_R^- = e_{-\infty}$. Otherwise, we set e_L^- and e_R^- to be, respectively, the edges preceding q in $W_L(u)$ and $W_R(v)$. By (rg5) and (rg1),

$$e_L^- \prec e_L^+ \preccurlyeq e \preccurlyeq e_R^+ \prec e_R^- \text{ in the } q\text{-ordering.}$$

By (sb2), e does not lie in $\operatorname{shad}(q)$. Since all elements of W' are greater than q in p, the element q is the only common element of W' and $\operatorname{shad}(q)$. In particular, $b \notin \operatorname{shad}(q)$, which ends the proof. □

Proposition 59. *Let $(a, u, v, U, V, \mathcal{R}, q, m, \gamma_L, \gamma_R)$ be a region tuple. Let W be a witnessing path in P with all elements in B. If both endpoints of W are in $\mathcal{R}$, then $W \subseteq \mathcal{R}$.*

Proof. Suppose to the contrary that the endpoints of W are in $\mathcal{R}$ but W is not contained in $\mathcal{R}$. Let r and s be elements in the intersection of W and $\partial\mathcal{R}$ with $r \leqslant s$ in P such that $r[W]s$ is in the exterior of $\mathcal{R}$ except for the elements r and s. Let e be the first edge of $r[W]s$. In particular, e does not lie in $\mathcal{R}$. Since all elements of W are in B, we have that r and s lie in $u[W_L(u)]q[W_R(v)]v$.

First, assume that r and s lie strictly on the left side of $\mathcal{R}$. By (rg3), $x_0[W_L(u)]r[W]s[W_L(u)]u$ is left of $W_L(u)$, which is a contradiction. The argument when r and s lie strictly on the right side of $\mathcal{R}$ is symmetric. Now, assume that r lies strictly on the left side of $\mathcal{R}$ and s lies

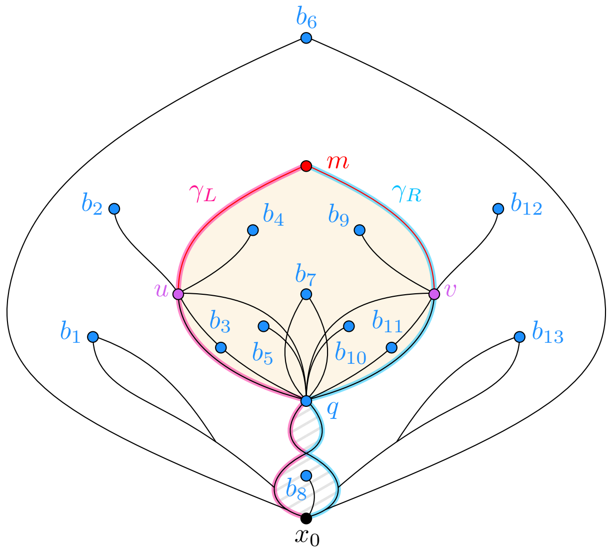


FIGURE 22. Examples of elements in B in different locations with respect to the region $\mathcal{R}(a, u, v, U, V)$.

strictly on the right side of $\mathcal{R}$. By (rg3), $x_0[W_L(u)]r[W]s[W_R(v)]v$ is left of $W_L(u)$, which is a contradiction with the fact that u is left of v. The argument when r lies strictly on the left side of $\mathcal{R}$ and s lies strictly on the right side of $\mathcal{R}$ is symmetric.

Finally, assume that $r = q$ and s lies strictly on the left side of $\mathcal{R}$. (Note that when $r = q$ and s lies strictly on the right side of $\mathcal{R}$, the proof is symmetric.) Let e_L^+ and e_R^+ be the edges following q in $W_L(u)$ and $W_R(v)$ respectively. If $q = x_0$, let $e_L^- = e_R^- = e_{-\infty}$, otherwise, let e_L^- and e_R^- be the edges preceding q in $W_L(u)$ and $W_R(v)$ respectively. We split the reasoning into cases depending on the position of e in the q-ordering.

Since e is outside $\mathcal{R}$, by (rg1), $e_R^+ \prec e \prec e_L^+$ in the q-ordering. If $e_L^- \prec e \prec e_L^+$ in the q-ordering, then $x_0[W_L(q)]q[W]s[W_L(u)]u$ is left of $W_L(u)$, which is a contradiction. If $e_R^+ \prec e \prec e_R^-$ in the q-ordering, then $x_0[W_R(q)]q[W]s[W_L(u)]u$ is right of $W_R(v)$, which is a contradiction with the fact that u is left of v. Finally, assume that $e_R^- \prec e \prec e_L^-$ in the q-ordering. Then by (sb2), e lies in the terminal block $\mathcal{B}$ of $\mathrm{shad}(q)$. In particular, $q[W]s[W_L(u)]u$ lies in $\mathcal{B}$, and so, $u \in \mathcal{B} \subseteq \mathrm{shad}(q)$, which contradicts Proposition 58. □

Proposition 60. *Let $(a, u, v, U, V, \mathcal{R}, q, m, \gamma_L, \gamma_R)$ be a region tuple and let $b \in B$ with $b \notin \mathrm{shad}(q) \setminus \{q\}$. Then*

(i) *$b \in \partial\mathcal{R}$ if and only if b lies in $q[W_L(u)]u \cup q[W_R(v)]v$.*
(ii) *$b \in \mathrm{int}\,\mathcal{R}$ if and only if $W_L(b)$ is right of γ_L and $W_R(b)$ is left of γ_R.*
(iii) *$b \notin \mathcal{R}$ if and only if $W_L(b)$ is left of γ_L or $W_R(b)$ is right of γ_R.*

Proof. Recall that the boundary of $\mathcal{R}$ is the union of four paths: $q[W_L(u)]u$, $q[W_R(v)]v$, $u[U]m$, and $v[V]m$. Since $b \in B$ and the two paths U, V are exposed, item (i) follows.

Note that exactly one of the following three conditions holds: the right side of (ii), the right side of (iii), $W_L(b)$ is contained in γ_L or $W_R(b)$ is contained in γ_R. The last option is possible only if $b \in \partial\mathcal{B}$ holds or $b \in \mathrm{shad}(q) \setminus \{q\}$. However, we assumed $b \notin \mathrm{shad}(q) \setminus \{q\}$. Therefore, in order to conclude the proof of the proposition, we will show the right-to-left direction of each of the equivalences: (ii) and (iii).

We proceed with the right-to-left implication in (ii). Assume that $W_L(b)$ is right of γ_L and $W_R(b)$ is left of γ_R. Note that this does not hold for $b = q$, so $b \neq q$. Since we have already proved (i), we may additionally assume $b \notin \partial\mathcal{R}$. Our first claim is that $W_L(q)$ is a subpath of $W_L(b)$ and $W_R(q)$ is a subpath of $W_R(b)$. Recall that $W_L(q)$ is a subpath of γ_L and $W_R(q)$ is a subpath of γ_R. Therefore, we have

- either $W_L(q)$ is a subpath of $W_L(b)$ or $W_L(q)$ is left of $W_L(b)$, and
- either $W_R(q)$ is a subpath of $W_R(b)$ or $W_R(q)$ is right of $W_R(b)$.

The situation, where in both bullets the first option occurs, is exactly the statement of our claim. We will prove that this is the only possible situation. If $W_L(q)$ is a subpath of $W_L(b)$, then in particular, $q \in \operatorname{shad}(b)$, which by Proposition 30.(*R*) yields that $W_R(q)$ is not right of $W_R(b)$, and so, $W_R(q)$ is a subpath of $W_R(b)$. Symmetrically, if $W_R(q)$ is a subpath of $W_R(b)$, then $W_L(q)$ is a subpath of $W_L(b)$. Finally, assume that in both bullets the second option occurs, that is, $W_L(q)$ is left of $W_L(b)$ and $W_R(q)$ is right of $W_R(b)$. It follows that (b, q) is an inside pair. Therefore, by Proposition 31, $b \in \operatorname{shad}(q)$, which is a contradiction. In summary, we proved that in both bullets the first option occurs.

Let W be a witnessing path from q to b in P. If all elements of W are in $\mathcal{R}$, then $b \in \mathcal{R}$, and so, $b \in \operatorname{int}\mathcal{R}$ as desired. Now consider an element w in W in the boundary of $\mathcal{R}$. Since $b \notin \partial\mathcal{R}$, we have $w \neq b$. Let e be the edge of W following w. We claim that e lies in $\mathcal{R}$, which will conclude the proof. Since $w \in B$, w lies in $W_L(u) \subseteq \gamma_L$ or in $W_R(v) \subseteq \gamma_R$. If w is an element of γ_L, then let e_L^-, e_L^+ be the edges immediately preceding and following w in γ_L (when $w = x_0$, we set $e_L^- = e_{-\infty}$). If w is an element of γ_R, then let e_R^-, e_R^+ be the edges immediately preceding and following w in γ_R (when $w = x_0$, we set $e_R^- = e_{-\infty}$). First, suppose that $w = q$. Recall that $e_R^+ \prec e_R^- \preccurlyeq e_L^- \prec e_L^+$ in the q-ordering (by (rg5)). The path $W_L(u)$ is either left of $W_L(b)$ or is a subpath of $W_L(b)$. The path $W_L(b)$ is either left of $W' = x_0[W_L(b)]q[W]b$ or is equal to W'. It follows that $W_L(u)$ is either left of W' or is a subpath of W'. Symmetrically $W_R(v)$ is either right of $W'' = x_0[W_R(b)]q[W]b$ or is a subpath of W''. In particular, $e_L^- \prec e_L^+ \preccurlyeq e$ and $e \preccurlyeq e_R^+ \prec e_R^-$ in the q-ordering. Therefore, either $e_L^+ \preccurlyeq e \preccurlyeq e_R^+$ or $e_R^- \preccurlyeq e \preccurlyeq e_L^-$ in the q-ordering. In the latter case, by (sb2), we obtain that the edge e lies in $\operatorname{shad}(q)$, and therefore, the whole path $q[W]b$ lies in $\operatorname{shad}(q)$, which forces $b \in \operatorname{shad}(q)$ and contradicts the assumption. It follows that $e_L^+ \preccurlyeq e \preccurlyeq e_R^+$ in the q-ordering, and so, by (rg1), e lies in $\mathcal{R}$, as desired. Next, assume that w is strictly on the left side (the case, where w is strictly on the right side is symmetric). The path γ_L is left of $W_L(b)$, which is equal to or left of the path $x_0[W_L(u)]w[W]b$. Since in all three paths the same edge, namely e_L^-, precedes w, we have $e_L^+ \preccurlyeq e \prec e_L^-$ in the w-ordering, and so, by (rg3), e lies in $\mathcal{R}$ as desired. This ends the proof of the right-to-left implication in (ii).

Finally, we prove the right-to-left implication in (iii). That is, if $W_L(b)$ is left of γ_L or $W_R(b)$ is right of γ_R, then $b \notin \mathcal{R}$. We will prove that $W_L(b)$ left of γ_L implies $b \notin \mathcal{R}$. The proof that $W_R(b)$ right of γ_R implies $b \notin \mathcal{R}$ is symmetric.

Assume that $W_L(b)$ is left of γ_L. Let $w = \operatorname{gcpe}(W_L(b), \gamma_L)$ and let $e = ww'$ be the edge following w in $W_L(b)$. We claim that the interior of e is disjoint from $\mathcal{R}$. Let e_L^-, e_L^+ be the edges immediately preceding and following w in γ_L (when $w = x_0$, we set $e_L^- = e_{-\infty}$). In the case where $w < q$ in P, we have $w \in \operatorname{shad}(q)$ and $w \neq q$. Thus, by Proposition 58, $w \notin \mathcal{R}$, hence, the interior of e is disjoint from $\mathcal{R}$ as desired. Next, assume that $q \leqslant w$ in P. Since $W_L(b)$ is left of γ_L, we have $e_L^- \prec e \prec e_L^+$ in the q-ordering. Therefore, by (rg1) or (rg3), the interior of e is disjoint from $\mathcal{R}$. However, if $b \in \mathcal{R}$, then by Proposition 59, $w[W_L(b)]b$ lies in $\mathcal{R}$, and so, e lies in $\mathcal{R}$. This shows that $b \notin \mathcal{R}$ as desired. □

For future reference, we state the following useful corollary.

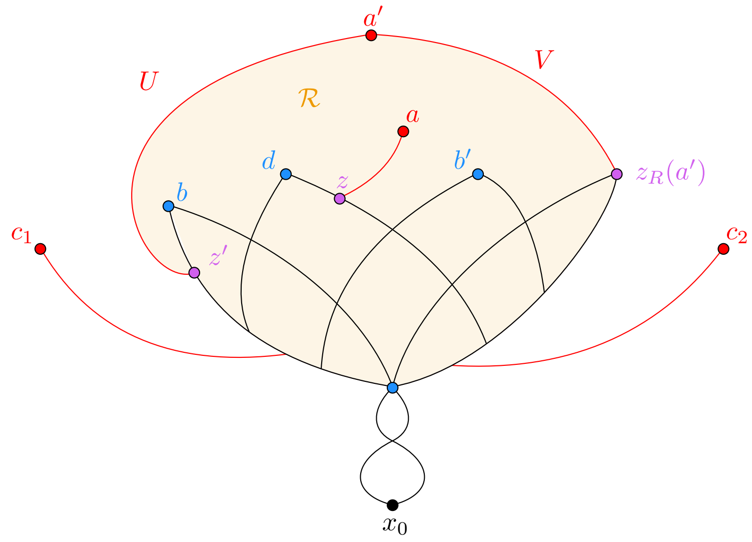


FIGURE 23. An illustration of the statement of Proposition 62.(L). For simplicity, we do not always mark such elements. Note that the assumptions $a \parallel b$, $a \parallel b'$, and $a' \parallel d$ in P are necessary. The necessity of the first two is depicted with elements c_1 and c_2 respectively. For the last, note that if $a' \leqslant d$ in P, and element from outside the region could send a comparability to d through V.

Corollary 61. *Let $(a, u, v, U, V, \mathcal{R}, q, m, \gamma_L, \gamma_R)$ be a region tuple and let $b \in B$.*

(i) *If u is left of b and b is left of v, then, $b \in \operatorname{int} \mathcal{R}$.*
(ii) *If b is left of u or v is left of b, then $b \notin \mathcal{R}$.*

Proof. Since $q < u$ in P, we have $\operatorname{shad}(q) \subseteq \operatorname{shad}(u)$ (by Proposition 33). Assume that u is left of b. In particular, $b \notin \operatorname{shad}(u)$, and so, $b \notin \operatorname{shad}(q)$. Since $W_L(u)$ is left of $W_L(b)$, we have that γ_L is left of $W_L(b)$. Since b is left of v, we have that $W_R(b)$ is left of γ_R. Therefore, by Proposition 60.(ii), $b \in \operatorname{int} \mathcal{R}$, which completes the proof of (i).

Similarly as above, if b is left of u, $b \notin \operatorname{shad}(u)$, and $b \notin \operatorname{shad}(q)$; and if v is left of b, then $b \notin \operatorname{shad}(v)$, and $b \notin \operatorname{shad}(q)$. If b is left of u, then $W_L(b)$ is left of γ_L, and $b \notin \mathcal{R}$ by Proposition 60.(iii). Symmetrically, if b is right of v, then $W_R(b)$ is right of γ_R, and $b \notin \mathcal{R}$ by Proposition 60.(iii). This gives (ii). ☐

We conclude this section with a technical statement that will be used repetitively in the proof of Lemma 76 (Coloring Lemma).

Proposition 62. *Let $(a, b), (a', b') \in I$ and $d \in B$ with b left of d and d left of b'.*

(L) Assume that $a < d$, $a \parallel b'$, and $a' \parallel d$ in P. Let $z' \in Z(a')$ with $z' \leqslant b$ in P. Let U and V be exposed witnessing paths from a' to z' and from a' to $z_R(a')$ in P, respectively. Then, z' is left of $z_R(a')$ and

$$a \in \operatorname{int} \mathcal{R}(a', z', z_R(a'), U, V).$$

(R) Assume that $a' < d$, $a' \parallel b$, and $a \parallel d$ in P. Let $z \in Z(a)$ with $z \leqslant b'$ in P. Let U and V be exposed witnessing paths from a to $z_L(a)$ and from a to z in P, respectively. Then, $z_L(a)$ is left of z and

$$a' \in \operatorname{int} \mathcal{R}(a, z_L(a), z, U, V).$$

Proof. We prove only (L) as the proof of (R) is symmetric. See Figure 23. Recall that b is left of d, d is left of b', and the pairs (a,b) and (a',b') are dangerous (by (I8)). Therefore, by Proposition 45 (and transitivity – Proposition 37),

$$a \notin \text{shad}(b'),\ a \notin \text{shad}(d),\ a' \notin \text{shad}(b), \text{ and } a' \notin \text{shad}(d).$$

In particular, $d \in Y(a)$, and so, we can fix $z \in Z(a)$ such that $z \leqslant d$ in P. Altogether, we have

$$z', b \in Y(a') \text{ and } z, d \in Y(a).$$

Since $\text{shad}(z') \subseteq \text{shad}(b)$ (by Proposition 33), and $a \notin \text{shad}(b)$ (by (I6)), we also obtain

$$a \notin \text{shad}(z').$$

Claim 63. z' is left of d, z' is left of z, and z is left of b'.

Proof. First, we prove that z' is left of d. By assumption, $a' \parallel d$ and $a' < z' \leqslant b$ in P. We have already proved that $a' \notin \text{shad}(d)$ and $z', b \in Y(a')$. Finally, b is left of d, hence, by Proposition 52.(*L*), z' is left of d.

Next, we prove that z' is left of z. Since $a \parallel b$ in P, we have $a \parallel z'$ in P. Moreover, $a < z \leqslant d$ in P. We have already proved that $a \notin \text{shad}(z')$, $z, d \in Y(a)$, and that z' is left of d. Therefore, by Proposition 52.(*R*), z' is left of z.

Finally, we prove that z is left of b'. By assumption, $a \parallel b'$ and $a < z \leqslant d$ in P. We have already proved that $a \notin \text{shad}(b')$ and $z, d \in Y(a)$. Finally, d is left of b', hence, by Proposition 52.(*L*), z is left of b'. This completes the proof of the claim. $\triangleleft$

Since b' is left of $z_R(a')$ (by Proposition 57), Claim 63 yields that z' is left of z and z is left of $z_R(a')$. In particular, z' is left of $z_R(a')$, and so, we can define $\mathcal{R} = \mathcal{R}(a', z', z_R(a'), U, V)$. Moreover, by Corollary 61.(i), $z \in \text{int}\,\mathcal{R}$. Let W be an exposed witnessing path from a to z. Suppose to the contrary that $a \notin \text{int}\,\mathcal{R}$. Then, W intersects $\partial\mathcal{R}$ in an element not in B, say u. It follows that $a' \leqslant u < z \leqslant d$ in P, which is a contradiction. This ends the proof. $\square$

6.3. **Regular sequences.** The goal of the next two subsections is to classify all alternating cycles of size 2 with pairs in I. Eventually, we will split them into four types, see Corollary 70.

We say that a sequence $((a_1,b_1),\ldots,(a_k,b_k))$ is *regular* if $(a_i,b_i) \in I$ for every $i \in [k]$ and b_i is left of b_{i+1} for every $i \in [k-1]$. Since the "left of" relation is transitive, see Proposition 37, we note that in a regular sequence $((a_1,b_1),\ldots,(a_k,b_k))$, the elements $b_1,\ldots,b_k$ are linearly ordered by the "left of" relation. Observe also that a subsequence of a regular sequence is regular.

We list four key properties that a regular sequence $((a_1,b_1),(a_2,b_2))$ may or may not satisfy. The first letter of the name of a property indicates if it concerns leftmost paths (L) or rightmost paths (R). The ordering of the numbers 1 and 2 indicates the relative positions of a_1 and a_2 in the drawing, reading from left to right. See Figure 24. Properties labeled without a star are simple, while those with a star are more involved (and in fact imply respective simpler statements, see Observation 65).

($L12$) $M_L(a_1)$ is left of $M_L(a_2)$.

($L21^\star$) There exist $u \in Z(a_2)$ and an exposed witnessing path U from a_2 to u in P such that
- u is an element of both $W_L(z_L(a_1))$ and $W_L(b_1)$,
- $x_0[W_L(z_L(a_1))]u[U]a_2$ is left of $W_L(z_L(a_1))$.

($R12$) $M_R(a_2)$ is right of $M_R(a_1)$.

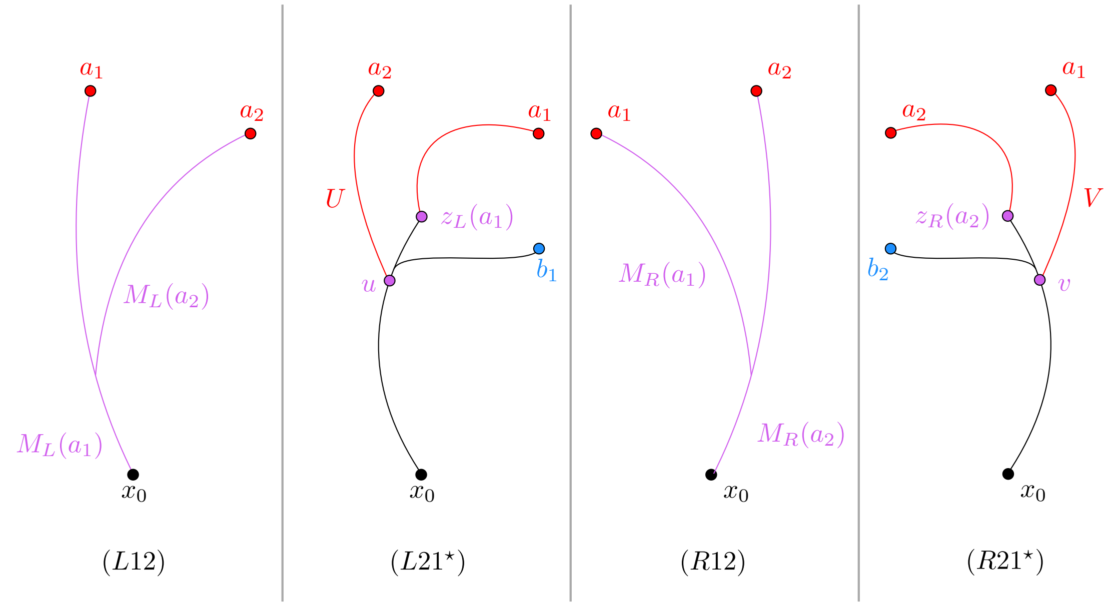


FIGURE 24. In each part of the figure, a regular sequence $((a_1, b_1), (a_2, b_2))$ satisfies one of the four properties.

$(R21^\star)$ There exist $v \in Z(a_1)$ and an exposed witnessing path V from a_1 to v in P such that
- v is an element of both $W_R(z_R(a_2))$ and $W_R(b_2)$,
- $x_0[W_R(z_R(a_2))]v[V]a_1$ is right of $W_R(z_R(a_2))$.

If a regular sequence $\sigma = ((a_1, b_1), (a_2, b_2))$ satisfies $(L21^\star)$, then for all $u \in Z(a_2)$ and U as in $(L21^\star)$, we say that (u, U) *witnesses* $(L21^\star)$ for σ. Sometimes, we omit U and just say that u witnesses $(L21^\star)$ for σ. Symmetrically, if σ satisfies $(R21^\star)$, then for all $v \in Z(a_1)$ and V as in $(R21^\star)$, we say that (v, V) *witnesses* $(R21^\star)$ for σ.

Note that for a regular sequence $((a_1, b_1), (a_2, b_2))$, the satisfaction of each of the properties does not depend on both b_1 and b_2, i.e. for each of the properties, one or both of these elements can be replaced by other elements in B so that the sequence is still regular, and then the property is still satisfied. More precisely, we have the following.

Observation 64. *Let $\sigma = ((a_1, b_1), (a_2, b_2))$ be regular, and let $b_1', b_2' \in B$.*

(i) *If σ satisfies $(L12)$ and $\sigma' = ((a_1, b_1'), (a_2, b_2'))$ is regular, then σ' satisfies $(L12)$.*
(ii) *If σ satisfies $(L21^\star)$ and $\sigma' = ((a_1, b_1), (a_2, b_2'))$ is regular, then σ' satisfies $(L21^\star)$.*
(iii) *If σ satisfies $(R12)$ and $\sigma' = ((a_1, b_1'), (a_2, b_2'))$ is regular, then σ' satisfies $(R12)$.*
(iv) *If σ satisfies $(R21^\star)$ and $\sigma' = ((a_1, b_1'), (a_2, b_2))$ is regular, then σ' satisfies $(L21^\star)$.*

In some sense, $(L21^\star)$ complements $(L12)$ as if $(L21^\star)$ holds then $(L12)$ cannot hold. The same applies to $(R21^\star)$ and $(R12)$. We highlight this in the next observation.

Observation 65. *Let $\sigma = ((a_1, b_1), (a_2, b_2))$ be regular.*

(L) *If σ satisfies $(L21^\star)$, then $a_2 < b_1$ in P and $M_L(a_2)$ is left of $M_L(a_1)$.*
(R) *If σ satisfies $(R21^\star)$, then $a_1 < b_2$ in P and $M_R(a_1)$ is right of $M_R(a_2)$.*

The following proposition shows that $(L21^\star)$ (and $(R21^\star)$) implies a stronger property.

Proposition 66. *Let $\sigma = ((a_1, b_1), (a_2, b_2))$ be regular.*

(*L*) *If $u \in Z(a_2)$ witnesses (L21⋆) for σ, then for every $z \in Z(a_1)$ such that z is left of b_1, the element u lies in $W_L(z)$.*
(*R*) *If $v \in Z(a_1)$ witnesses (R21⋆) for σ, then for every $z \in Z(a_2)$ such that z is right of b_2, the element v lies in $W_R(z)$.*

Proof. We prove only (*L*). The proof of (*R*) is symmetric. Assume that σ satisfies (L21⋆) which is witnessed by $u \in Z(a_2)$, and let $z \in Z(a_1)$ such that z is left of b_1. Recall that u lies in both $W_L(z_L(a_1))$ and $W_L(b_1)$. We consider cases of how the paths $W_L(z_L(a_1))$ and $W_L(z)$ relate to each other.

If $W_L(z_L(a_1))$ is a subpath of $W_L(z)$, then the assertion holds trivially as u lies in $W_L(z_L(a_1))$. If $W_L(z)$ is a subpath of $W_L(z_L(a_1))$, then it suffices to argue that $u < z$ in P. Indeed, otherwise $a_1 < z \leqslant u \leqslant b_1$ in P, which is a contradiction. It follows that u lies in $W_L(z)$.

Next assume that $W_L(z_L(a_1))$ is left of $W_L(z)$. Since z is left of b_1, in particular, $W_L(z)$ is left of $W_L(b_1)$. Again, recall that u lies in both $W_L(z_L(a_1))$ and $W_L(b_1)$. All this allows us to apply Corollary 16 (the considered paths are pairwise x_0-consistent by Proposition 15.(*L*)), to obtain that u indeed lies in $W_L(z)$.

Finally, assume that $W_L(z)$ is left of $W_L(z_L(a_1))$. Consider the path $M = x_0[W_L(z)]z[W]a_1$, where W is an exposed witnessing path from a_1 to z in P. Clearly, $M \in \mathcal{M}(a_1)$. Since $x_0[M_L(a_1)]z_L(a_1) = W_L(z_L(a_1))$ by Proposition 54, the path M is left of $M_L(a_1)$, which contradicts the definition of $M_L(a_1)$, and shows that this case does not occur. □

Next, we show that each of the four properties is transitive.

Proposition 67. *Let $((a_1, b_1), (a_2, b_2), (a_3, b_3))$ be regular. Let $\sigma_{12} = ((a_1, b_1), (a_2, b_2))$, $\sigma_{23} = ((a_2, b_2), (a_3, b_3))$, and $\sigma_{13} = ((a_1, b_1), (a_3, b_3))$.*

(i) *If σ_{12} and σ_{23} satisfy (L12), then σ_{13} satisfies (L12).*
(ii) *If σ_{12} and σ_{23} satisfy (L21⋆), then σ_{13} satisfies (L21⋆).*
(iii) *If σ_{12} and σ_{23} satisfy (R12), then σ_{13} satisfies (R12).*
(iv) *If σ_{12} and σ_{23} satisfy (R21⋆), then σ_{13} satisfies (R21⋆).*

Proof. The arguments for items (i) and (iii) are symmetric so we only include a proof of the former. The same applies to items (ii) and (iv).

We start with an argument for (i). Suppose that σ_{12} and σ_{23} satisfy (L12). Therefore, $M_L(a_1)$ is left of $M_L(a_2)$ and $M_L(a_2)$ is left of $M_L(a_3)$. By Proposition 9, this relation is transitive so $M_L(a_1)$ is left of $M_L(a_3)$, as desired.

Now we proceed with the proof (ii). Suppose that σ_{12} and σ_{23} satisfy (L21⋆). Let $u \in Z(a_2)$ and U be an exposed witnessing path from a_2 to u in P such that (u, U) witnesses (L21⋆) for σ_{12} and let $u' \in Z(a_3)$ and U' be an exposed witnessing path from a_3 to u' in P such that (u', U') witnesses (L21⋆) for σ_{23}. We claim that (u', U') witnesses (L21⋆) for σ_{13}.

Since $(a_2, b_2) \in I$, $a_2 < u \leqslant b_1$ in P, and b_1 left of b_2, we can apply Corollary 53.(*L*) and conclude that u is left of b_2. Now, since σ_{23} is regular, u' witnesses (L21⋆) for σ_{23}, $u \in Z(a_2)$, and u left of b_2, we can apply Proposition 66.(*L*) to σ_{23} and conclude that u' lies in $W_L(u)$. Since u witnesses (L21⋆) for σ_{12}, u lies in both $W_L(z_L(a_1))$ and $W_L(b_1)$, and so, u' lies in both $W_L(z_L(a_1))$ and $W_L(b_1)$.

Finally, to conclude that (u', U') witnesses $(L21^\star)$ for σ_{13} and end the proof, it suffices to show that $M' = x_0[W_L(z_L(a_1))]u'[U']a_3$ is left of $W_L(z_L(a_1))$. First, note that $M' = x_0[W_L(z_L(a_2))]u'[U']a_3$, thus, by $(L21^\star)$ for σ_{23}, M' is left of $W_L(z_L(a_2))$. By Proposition 54, $W_L(z_L(a_2))$ is a subpath of $M_L(a_2)$, hence, $W_L(z_L(a_2))$ is either a subpath of $M = x_0[W_L(z_L(a_1))]u[U]a_2$ or is left of M. By $(L21^\star)$ for σ_{12}, M is left of $W_L(z_L(a_1))$. It follows that M' is left of $W_L(z_L(a_1))$, which completes the proof of (ii). □

6.4. **Classifying alternating cycles of size 2.** Let $((a_1, b_1), (a_2, b_2))$ be an alternating cycle in P with both pairs in I. Note that by (I7) either $((a_1, b_1), (a_2, b_2))$ or $((a_2, b_2), (a_1, b_1))$ is regular. The goal of this subsection is to classify regular alternating cycles of size 2 in P into one of the four types. To this end, we associate some regions to each regular alternating cycle of size 2. First, we need a simple property that is a corollary of Corollary 50.

Corollary 68. *Let $\sigma = ((a_1, b_1), (a_2, b_2))$ be a regular alternating cycle in P. Then, $b_2 \in Y(a_1)$ and $b_1 \in Y(a_2)$.*

Let $\sigma = ((a_1, b_1), (a_2, b_2))$ be a regular alternating cycle in P. By Corollary 68, $b_2 \in Y(a_1)$ and $b_1 \in Y(a_2)$. Thus, we can fix $z_1 \in Z(a_1)$ and $z_2 \in Z(a_2)$ such that $z_1 \leqslant b_2$ and $z_2 \leqslant b_1$ in P. By Proposition 57, $z_L(a_1)$ is left of b_1 and by Corollary 53.(R), z_1 is right of b_1. Therefore, by transitivity, $z_L(a_1)$ is left of z_1. Symmetrically, we obtain that z_2 is left of $z_R(a_2)$. Let W_1 be an exposed witnessing path from a_1 to z_1 in P, and let W_2 be an exposed witnessing path from a_2 to z_2 in P. Let

$$\mathcal{R}_1 = \mathcal{R}(a_1, z_L(a_1), z_1, a_1[M_L(a_1)]z_L(a_1), W_1),$$
$$\mathcal{R}_2 = \mathcal{R}(a_2, z_2, z_R(a_2), W_2, a_2[M_R(a_2)]z_R(a_2)).$$

We say that $\mathcal{R}_1$ is a *left region* of σ and $\mathcal{R}_2$ is a *right region* of σ. Observe that by Corollary 61.(i),

$$b_1 \in \operatorname{int} \mathcal{R}_1 \text{ and } b_2 \in \operatorname{int} \mathcal{R}_2. \tag{7}$$

In the next proposition, we connect relative positions of a_1, a_2 and $\mathcal{R}_1, \mathcal{R}_2$ with the properties $(L12)$, $(L21^\star)$, $(R12)$, and $(R21^\star)$. See Figure 25.

Proposition 69. *Let $\sigma = ((a_1, b_1), (a_2, b_2))$ be a regular alternating cycle. Let $\mathcal{R}_1$ be a left region of σ and let $\mathcal{R}_2$ be a right region of σ.*

(i) *If $a_2 \in \operatorname{int} \mathcal{R}_1$, then σ satisfies $(L12)$.*
(ii) *If $a_2 \notin \mathcal{R}_1$, then σ satisfies $(L21^\star)$.*
(iii) *If $a_1 \in \operatorname{int} \mathcal{R}_2$, then σ satisfies $(R12)$.*
(iv) *If $a_1 \notin \mathcal{R}_2$, then σ satisfies $(R21^\star)$.*

Proof. We only prove statements (i) and (ii). The proofs of the other statements are symmetric. Let $z_1 \in Z(a_1)$ such that $z_1 \leqslant b_2$ in P and let W_1 be an exposed witnessing path from a_1 to z_1 in P such that $\mathcal{R}_1 = \mathcal{R}(a_1, z_L(a_1), z_1, a_1[M_L(a_1)]z_L(a_1), W_1)$. Let q be the lower-min of $\mathcal{R}_1$, and let m be the upper-min of $\mathcal{R}_1$. Let $\gamma_L = x_0[W_L(z_L(a_1))]z_L(a_1)[M_L(a_1)]m$ and $\gamma_R = x_0[W_R(z_1)]z_1[W_1]m$.

We start with an argument for (i). Assume that $a_2 \in \operatorname{int} \mathcal{R}_1$. We shall prove that σ satisfies $(L12)$, that is, $M_L(a_1)$ is left of $M_L(a_2)$. Since $a_2 \in \operatorname{int} \mathcal{R}_1$ and $x_0 \notin \operatorname{int} \mathcal{R}_1$, $M_L(a_2)$ contains an element in $\partial \mathcal{R}_1$. Let u be the first such element in $a_2[M_L(a_2)]x_0$. First, suppose that u lies in $z_L(a_2)[M_L(a_2)]x_0$ and $u \neq z_L(a_2)$. In this case, $z_L(a_2) \in \operatorname{int} \mathcal{R}_1$, thus, by Proposition 60.(ii), γ_L is left of $W_L(z_L(a_2))$, which implies $M_L(a_1)$ left of $M_L(a_2)$ as desired. Next, suppose that u lies in $a_2[M_L(a_2)]z_L(a_2)$. Note that each element v of γ_R satisfies $v \leqslant z_1 \leqslant b_2$ in P. Therefore, u does not lie in γ_R as this would imply $a_2 \leqslant u \leqslant b_2$ in P, which is a contradiction. It follows

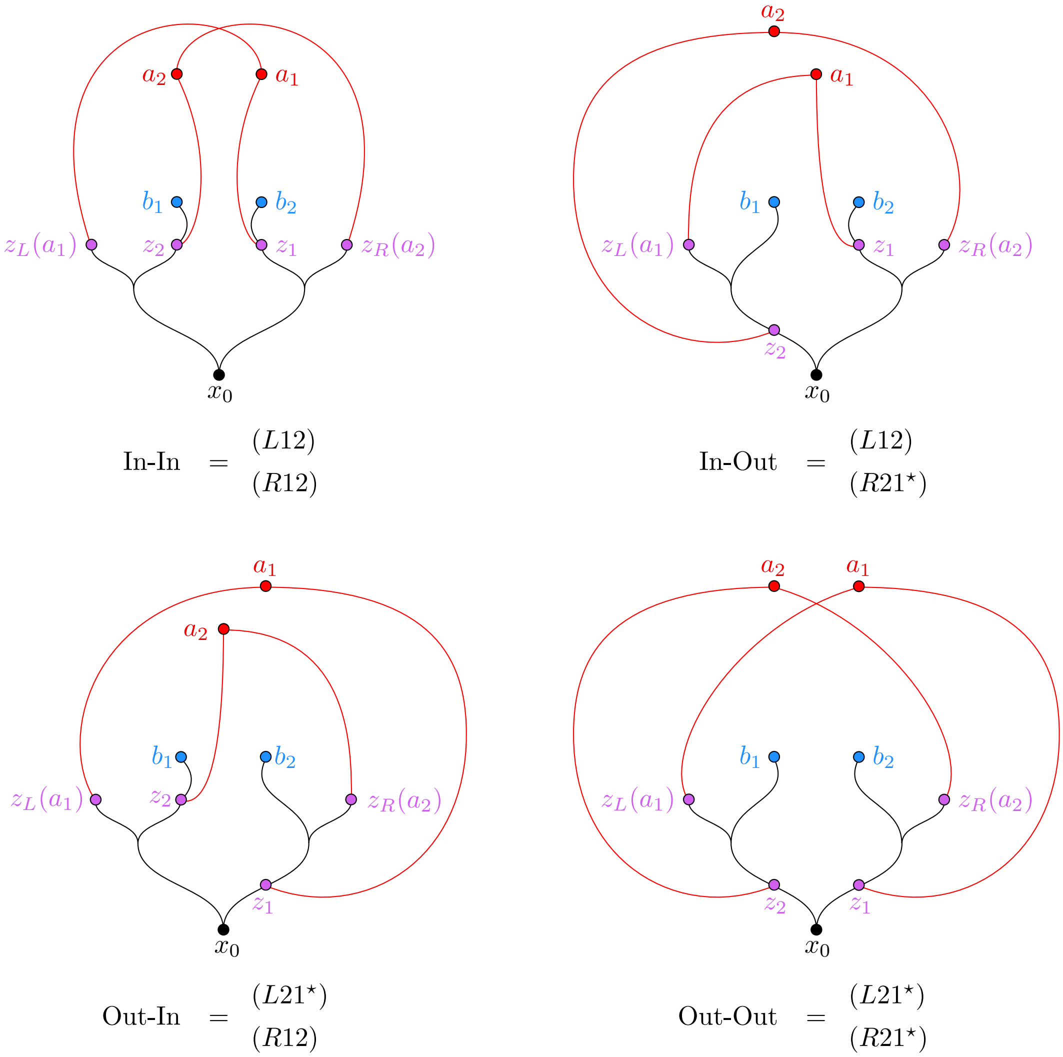


FIGURE 25. In each part of the figure, we depict a left and a right region for a regular alternating cycle $((a_1, b_1), (a_2, b_2))$. We exhaust all possible combinations of the satisfaction of the statements $a_2 \in \mathcal{R}_1$ and $a_1 \in \mathcal{R}_2$. In this way, we obtain a classification of all alternating cycles of size 2 in P with both pairs in I, see the definitions after Corollary 70.

that u lies strictly on the left side of $\mathcal{R}_1$. Since the last edge of $a_2[M_L(a_2)]u$ has its interior contained in $\operatorname{int}\mathcal{R}_1$, by (rg3) and Proposition 55.(L), $M_L(a_1)$ is left of $M_L(a_2)$. This completes the proof of (i).

Now, we prove (ii). Assume that $a_2 \notin \mathcal{R}_1$. We shall prove that σ satisfies (L21*). Since $a_2 < b_1$ in P, there is a witnessing path W from a_2 to b_1 in P. Note that $a_2 \notin \mathcal{R}_1$ and $b_1 \in \operatorname{int}\mathcal{R}_1$ (by (7)), so W must intersect $\partial\mathcal{R}_1$. Note that W is disjoint from γ_R as otherwise a common element z of the two paths would certify $a_2 \leqslant z \leqslant z_1 \leqslant b_2$ in P, which is a contradiction. Also, W is disjoint from $m[M_L(a_1)]z_L(a_1)$ as otherwise a common element z of the two paths would certify $a_1 \leqslant m \leqslant z \leqslant b_1$ in P, which is a contradiction. Therefore, W intersects $\partial\mathcal{R}_1$ only in W', where W' is $q[W_L(z_L(a_1))]z_L(a_1)$ with the elements q and $z_L(a_1)$ removed.

Let u be the first element of $a_2[W]b_1$ in B, and let v be the first element of $a_2[W]b_1$ in $\partial\mathcal{R}_1$. Thus, as we argued v lies in W'. Clearly, $u \leqslant v$ in P. We claim that $u = v$. Suppose to the contrary that $u \neq v$. In particular, since $a_2 \notin \mathcal{R}_1$, we have $u \notin \mathcal{R}_1$. Then, by Proposition 60.(iii), $W_L(u)$ is left γ_L or $W_R(u)$ is right of γ_R. First, assume that $W_L(u)$ is left of γ_L. Since v lies in W', we have $v < z_L(a_1)$ in P, hence, $z_L(a_1)$ does not lie in $W_L(u)$. It follows that $W_L(u)$ is left of $W_L(z_L(a_1))$. However, the path $x_0[W_L(u)]u[W]v[W_L(z_L(a_1))]z_L(a_1)$ is now left of $W_L(z_L(a_1))$, which is a contradiction. Next, assume that $W_R(u)$ is right of γ_R. If z_1 lies in $W_R(u)$, then $z_1 \leqslant u < v < z_L(a_1)$ in P, which contradicts that $z_L(a_1)$ is left of z_1, thus, z_1 does not lie in $W_R(u)$. It follows that $W_R(u)$ is right of $W_R(z_1)$. Moreover, $x_0[W_R(u)]u[W]v[W_L(z_L(a_1))]z_L(a_1)$ is right of $W_R(z_1)$, which is right of $W_R(z_L(a_1))$. This yields existence of a witnessing path from x_0 to $z_L(a_1)$ right of $W_R(z_L(a_1))$ in P, which is a contradiction. We conclude that indeed $u = v$.

Let $V = a_2[W]v$. We claim that (v, V) witnesses ($L21^\star$) for σ. As proved, V is an exposed witnessing path from a_2 to v in P, and so, $v \in Z(a_2)$. Since v is an element of W', v lies in $W_L(z_L(a_1))$. Let $w = \mathrm{gcpe}(W_L(z_L(a_1)), W_L(b_1))$. If $w < v$ in P, then since $W_L(z_L(a_1))$ is left of $W_L(b_1)$, we have $x_0[W_L(z_L(a_1))]v[W]b_1$ left of $W_L(b_1)$, which is false. Thus, $v \leqslant w$ in P, which implies that v lies in $W_L(b_1)$. Finally, since v is strictly on the left side of $\mathcal{R}_1$, by (rg3), $x_0[W_L(z_L(a_1))]v[V]a_2$ is left of $W_L(z_L(a_1))$. This completes the proof that σ satisfies ($L21^\star$). □

Let $\sigma = ((a_1, b_1), (a_2, b_2))$ be a regular alternating cycle in P. Fix a left region $\mathcal{R}_1$ of σ and a right region $\mathcal{R}_2$ of σ. Since $a_1 \parallel a_2$ in P, we have $a_1 \notin \partial\mathcal{R}_2$ and $a_2 \notin \partial\mathcal{R}_1$. Therefore, Proposition 69 implies that σ must satisfy either ($L12$) or ($L21^\star$) and either ($R12$) or ($R21^\star$). By Observation 65, σ cannot satisfy both ($L12$) and ($L21^\star$) and symmetrically σ cannot satisfy both ($R12$) and ($R21^\star$). All this leads to the following conclusion.

Corollary 70. *Let $\sigma = ((a_1, b_1), (a_2, b_2))$ be a regular alternating cycle in P with both pairs in I.*

(i)
$$\sigma \text{ satisfies } (L12) \iff a_2 \in \operatorname{int}\mathcal{R}_1 \text{ for every left region } \mathcal{R}_1 \text{ of } \sigma \iff a_2 \in \operatorname{int}\mathcal{R}_1 \text{ for some left region } \mathcal{R}_1 \text{ of } \sigma.$$

(ii)
$$\sigma \text{ satisfies } (L21^\star) \iff a_2 \notin \mathcal{R}_1 \text{ for every left region } \mathcal{R}_1 \text{ of } \sigma \iff a_2 \notin \mathcal{R}_1 \text{ for some left region } \mathcal{R}_1 \text{ of } \sigma.$$

(iii)
$$\sigma \text{ satisfies } (R12) \iff a_1 \in \operatorname{int}\mathcal{R}_2 \text{ for every right region } \mathcal{R}_2 \text{ of } \sigma \iff a_1 \in \operatorname{int}\mathcal{R}_2 \text{ for some right region } \mathcal{R}_2 \text{ of } \sigma.$$

(iv)
$$\sigma \text{ satisfies } (R21^\star) \iff a_1 \notin \mathcal{R}_2 \text{ for every right region } \mathcal{R}_2 \text{ of } \sigma \iff a_1 \notin \mathcal{R}_2 \text{ for some right region } \mathcal{R}_2 \text{ of } \sigma.$$

Therefore, all regular alternating cycles of size 2 can be classified into four types. We say that a regular alternating cycle σ of size 2 in P with both pairs in I is:

In-In if σ satisfies ($R12$) and ($L12$);
In-Out if σ satisfies ($R12$) and ($L21^\star$);
Out-In if σ satisfies ($R21^\star$) and ($L12$);
Out-Out if σ satisfies ($R21^\star$) and ($L21^\star$).

See again Figure 25.

7. Auxiliary oriented graphs: definitions and the coloring lemma

An *orientation* of a graph G is a function assigning to each edge $\{u, v\}$ of G one of the pairs: (u, v) or (v, u). An *oriented graph* is a graph with a fixed orientation. The *vertex set* of an oriented graph H is the vertex set of the underlying graph, while the *edge set* of H is the set of all (u, v) such that $\{u, v\}$ is an edge in the underlying graph mapped to (u, v) by the orientation. We say that an edge (u, v) of an oriented graph H is an edge *from* u *to* v in H.

A *directed path* is an oriented graph such that $\{v_0, \ldots, v_k\}$ is its vertex set and $\{(v_{i-1}, v_i) : i \in [k]\}$ is its edge set, where $v_0, \ldots, v_k$ are pairwise distinct. This directed path *starts* in v_0 and *ends* in v_k. The number of vertices of a directed path is its *order*. A *directed cycle* is an oriented graph with at least three vertices such that removing each of its edges gives a directed path. An oriented graph is *acyclic* if it contains no directed cycle.

When H is an acyclic oriented graph, we define max-path(H) as the maximum order of a directed path in H. Moreover, for a vertex v of H, let max-start-path(H, v) be the maximum order of a directed path in H starting in v.

In this section, we fix a maximal good instance $(P, x_0, G, e_{-\infty}, I)$. We study six auxiliary oriented graphs – H_{OO}, H_{IIL}, H_{IIR}, H_{IILR}, H_{IO}, and H_{OI} – each with vertex set I and edges given by carefully chosen regular alternating cycles of size 2. All six oriented graphs are acyclic. Four of them – H_{OO}, H_{IIL}, H_{IIR}, and H_{IILR} – have the maximum order of a directed path at most $\text{se}_P(I)$. The graphs H_{IO} and H_{OI} do not admit such a strong bound in general, and instead, we present a much more subtle statement. We state each of these results (Propositions 71 to 74 and Lemma 75) in this section, however, we postpone the proofs to Section 8. These results give ground for the coloring κ of I in which $\kappa((a, b))$ for $(a, b) \in I$ is defined according to the order of the longest directed path starting or ending in (a, b) in all six oriented graphs. The number of colors used by κ on I will be in $\mathcal{O}(\text{se}_P(I)^8)$. Finally, the key statement of this section, the Coloring Lemma, see Lemma 76, shows that pairs in I with the same color under κ form a reversible set in P. This completes the proof of Theorem 49 up to the postponed proofs.

As mentioned above, I is the vertex set of each of the six auxiliary oriented graphs defined in this section. Moreover, from the definitions, it follows that each edge $((a_1, b_1), (a_2, b_2))$ in these graphs is a regular sequence. In particular, b_1 is left of b_2. Therefore, by the transitivity of the "left of" relation (see Proposition 37), each of the six oriented graphs is acyclic.

Let H_{OO} be the oriented graph with the vertex set I and $\sigma = ((a_1, b_1), (a_2, b_2))$ is an edge in H_{OO} if σ is a regular Out-Out alternating cycle in P.

Proposition 71. max-path$(H_{\text{OO}}) \leqslant \text{se}_P(I)$.

Let H_{II} be the oriented graph with the vertex set I and $\sigma = ((a_1, b_1), (a_2, b_2))$ is an edge in H_{II} if σ is a regular In-In alternating cycle in P. The argument in the Coloring Lemma requires us to deal with certain supergraphs of H_{II}, namely H_{IIL} and H_{IIR}.

Let H_{IIL} be the oriented graph with the vertex set I and $\sigma = ((a_1, b_1), (a_2, b_2))$ is an edge in H_{IIL} if σ is an edge in H_{II} or there exists a *witness* $t \in B$ for σ such that

(iil1) $(a_2, t) \in I$ and $((a_1, b_1), (a_2, t))$ is an edge in H_{II},
(iil2) t is left of b_2, and
(iil3) $a_1 \parallel b_2$ in P.

Proposition 72. max-path$(H_{\text{IIL}}) \leqslant \text{se}_P(I)$.

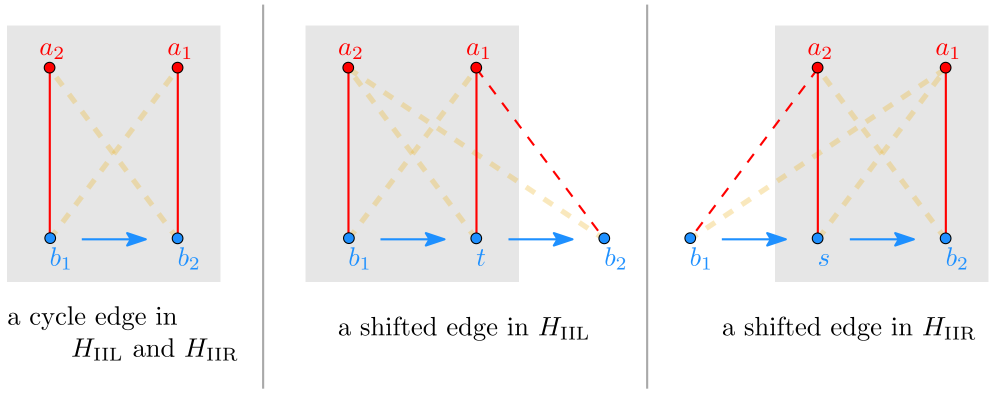


FIGURE 26. Schematic drawings of edges in H_{IIL} and H_{IIR}. Dashed lines indicate incomparabilities in P. With yellow lines, we denote incomparable pairs that are surely in I. Gray regions indicate In-In alternating cycles in P. Blue arrows indicate the "left of" relation. We use the same drawing conventions in several forthcoming figures.

Let H_{IIR} be the oriented graph with the vertex set I and $\sigma = ((a_1, b_1), (a_2, b_2))$ is an edge in H_{IIR} if σ is an edge in H_{II} or there exists *witness* $s \in B$ for σ such that

(iir1) $(a_1, s) \in I$ and $((a_1, s), (a_2, b_2))$ is an edge in H_{II},
(iir2) b_1 is left of s, and
(iir3) $a_2 \parallel b_1$ in P.

Proposition 73. max-path$(H_{\text{IIR}}) \leqslant \text{se}_P(I)$.

Let σ be an an edge in H_{IIL} (in H_{IIR}, respectively). If σ is an edge in H_{II}, then we say that σ is a *cycle* edge in H_{IIL} (in H_{IIR}); otherwise, we say that σ is a *shifted* edge in H_{IIL} (in H_{IIR}). See Figure 26 for a schematic drawing of edges in H_{IIL} and H_{IIR}. See also Figure 39 later in the paper.

Let H_{IILR} be the oriented graph with the vertex set I and $\sigma = ((a_1, b_1), (a_2, b_2))$ is an edge in H_{IILR} if there exists a *witness* (s, t) with $s, t \in B$ for σ such that

(iilr1) $(a_1, s), (a_2, t) \in I$ and $((a_1, s), (a_2, t))$ is an edge in H_{II},
(iilr2) b_1 is left of s and t is left of b_2, and
(iilr3) $a_1 \parallel b_2$ and $a_2 \parallel b_1$ in P.

See Figure 27 for a schematic drawing of edges in H_{IILR}.

Proposition 74. max-path$(H_{\text{IILR}}) \leqslant \text{se}_P(I)$.

Let H_{IO} be the oriented graph with the vertex set I and $\sigma = ((a_1, b_1), (a_2, b_2))$ is an edge in H_{IO} if σ is regular and satisfies $(R12)$ and $(L21^\star)$.

Let $\sigma = ((a_1, b_1), (a_2, b_2))$ be an edge in H_{IO}. Note that by Observation 65.(L), $a_2 < b_1$ in P. We say that σ is a *cycle* edge in H_{IO} if $a_1 < b_2$ in P, in other words, if σ is a regular In-Out alternating cycle in P. We say that σ is a *shifted* edge in H_{IO} if there exists a *witness* $t \in B$ for σ such that

(io1) $(a_2, t) \in I$ and $((a_1, b_1), (a_2, t))$ is a regular In-Out alternating cycle in P,
(io2) t is left of b_2, and
(io3) $a_1 \parallel b_2$ in P.

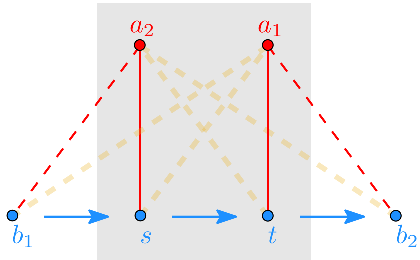


FIGURE 27. Schematic drawing of an edge in H_{IILR}.

See Figure 28 for a schematic drawing of edges in H_{IO}. See also Figure 44 later in the paper.

Let H_{OI} be the directed graph with the vertex set I and $\sigma = ((a_1, b_1), (a_2, b_2))$ is an edge in H_{OI} if σ satisfies $(L12)$ and $(R21^\star)$.

Let $\sigma = ((a_1, b_1), (a_2, b_2))$ be an edge in H_{OI}. Note that by Observation 65.(R), $a_1 < b_2$ in P. We say that σ is a *cycle* edge in H_{OI} if $a_2 < b_1$ in P, in other words, if σ is a regular Out-In alternating cycle in P. We say that σ is a *shifted* edge in H_{OI} if there exists a *witness* $t \in B$ for σ such that

(oi1) $(a_1, t) \in I$ and $((a_1, t), (a_2, b_2))$ is a regular Out-In alternating cycle in P,
(oi2) b_1 is left of t, and
(oi3) $a_2 \parallel b_1$ in P.

Let $H \in \{H_{\mathrm{IO}}, H_{\mathrm{OI}}\}$. We assign a weight of 0 or 1 to each edge of H. All cycle edges and shifted edges in H are of weight 1 and all the remaining edges in H are of weight 0. The *weight* of a path in H is the sum of the weights of its edges. Furthermore, for a vertex v of H, max-start-weight(H, v) is the maximum weight of a directed path in H starting in v and max-end-weight(H, v) is the maximum weight of a directed path in H ending in v.

Contrary to the four previously defined oriented graphs, the maximum order of a directed path in H_{IO} and H_{OI} cannot be bounded in terms of $\mathrm{se}_P(I)$. Also, the maximum weight of a directed path cannot be bounded in terms of $\mathrm{se}_P(I)$ (see Figure 45 later in the paper). Within Lemma 75, we prove a weaker property that is good enough for our purposes in Lemma 76.

Lemma 75. *Let* $m = 2\,\mathrm{se}_P(I) \cdot (2\,\mathrm{se}_P(I) + 6)$.

(i) *If* $((a, b), (a', b'))$ *is an edge of weight* 1 *in* H_{IO}*, then*
$$\text{max-start-weight}(H_{\mathrm{IO}}, (a, b)) \not\equiv \text{max-start-weight}(H_{\mathrm{IO}}, (a', b')) \bmod m.$$
(ii) *If* $((a, b), (a', b'))$ *is an edge of weight* 1 *in* H_{OI}*, then*
$$\text{max-end-weight}(H_{\mathrm{OI}}, (a, b)) \not\equiv \text{max-end-weight}(H_{\mathrm{OI}}, (a', b')) \bmod m.$$

We are ready to define the final coloring of I. Let
$$s = \mathrm{se}_P(I) \quad \text{and} \quad m = 2s \cdot (2s + 6).$$

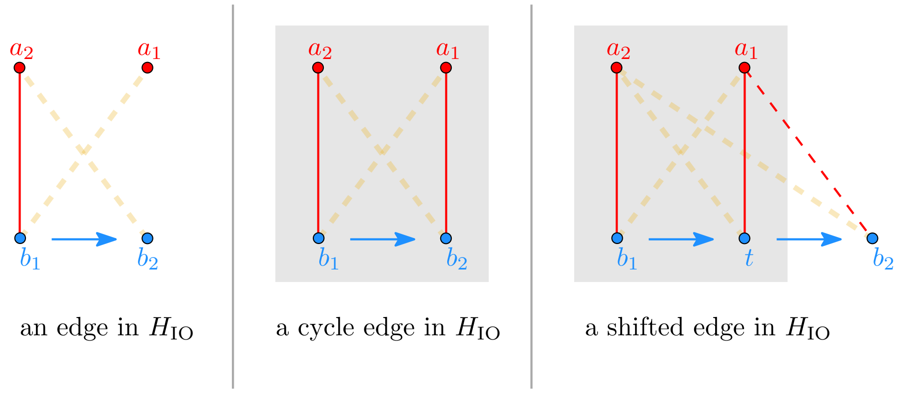


FIGURE 28. Schematic drawings of edges in H_{IO}. Gray regions indicate In-Out alternating cycles in P.

We define κ to be a function such that for every $(a, b) \in I$,

$$\begin{aligned}
\kappa((a,b))_1 &= \text{max-start-path}(H_{\mathrm{OO}}, (a,b)),\\
\kappa((a,b))_2 &= \text{max-start-path}(H_{\mathrm{IIL}}, (a,b)),\\
\kappa((a,b))_3 &= \text{max-start-path}(H_{\mathrm{IIR}}, (a,b)),\\
\kappa((a,b))_4 &= \text{max-start-path}(H_{\mathrm{IILR}}, (a,b)),\\
\kappa((a,b))_5 &= \text{max-start-weight}(H_{\mathrm{IO}}, (a,b)) \bmod m,\\
\kappa((a,b))_6 &= \text{max-end-weight}(H_{\mathrm{OI}}, (a,b)) \bmod m,
\end{aligned}$$

and

$$\kappa((a,b)) = (\kappa((a,b))_i)_{i \in [6]}.$$

Let $\mathrm{Im}(\kappa)$ be the image of κ. Altogether we have

$$\begin{aligned}
|\operatorname{Im}(\kappa)| \leqslant\ & \text{max-path}(H_{\mathrm{OO}}) \cdot \text{max-path}(H_{\mathrm{IIL}})\\
&\cdot \text{max-path}(H_{\mathrm{IIR}}) \cdot \text{max-path}(H_{\mathrm{IILR}}) \cdot m \cdot m\\
\leqslant\ & s^4 \cdot (2s \cdot (2s+6))^2\\
\leqslant\ & 16 s^6 \cdot (s+3)^2 = \mathcal{O}\left(s^8\right),
\end{aligned}$$

where the second inequality follows from Propositions 71 to 74. The Coloring Lemma, proved below assuming Lemma 75, states that incomparable pairs from I with the same color under κ form a reversible set in P.

Altogether, we obtain

$$\dim_P(I) \leqslant |\operatorname{Im}(\kappa)| \leqslant 16\, \mathrm{se}_P(I)^6 \cdot (\mathrm{se}_P(I) + 3)^2,$$

which will complete the proof of Theorem 49.

Lemma 76 (Coloring lemma). *For every $\xi \in \operatorname{Im}(\kappa)$, the set $\{(a,b) \in I \mid \kappa((a,b)) = \xi\}$ is reversible.*

Proof assuming Lemma 75. Let $\xi \in \operatorname{Im}(\kappa)$ and $\xi = (\xi_i)_{i \in [6]}$. Let $I_\xi = \{(a,b) \in I \mid \kappa((a,b)) = \xi\}$. We show that I_ξ is reversible. Suppose otherwise and let $((a_1, b_1), \ldots, (a_k, b_k))$ be a strict alternating cycle in P with all the pairs in I_ξ. The indices in $[k]$ are considered cyclically, e.g. $a_{k+1} = a_1$.

Claim 77. For all $\alpha, \beta \in [k]$ and $H \in \{H_{\mathrm{IO}}, H_{\mathrm{OI}}\}$, there is no edge of weight 1 from (a_α, b_α) to (a_β, b_β) in H.

Proof. If there is an edge edge of weight 1 from (a_α, b_α) to (a_β, b_β) in some $H \in \{H_{\mathrm{IO}}, H_{\mathrm{OI}}\}$, then by Lemma 75, either $\kappa((a_\alpha, b_\alpha))_5 \neq \kappa(a_\beta, b_\beta))_5$ or $\kappa((a_\alpha, b_\alpha))_6 \neq \kappa(a_\beta, b_\beta))_6$, in particular, $\kappa((a_\alpha, b_\alpha)) \neq \kappa((a_\beta, b_\beta))$, which is a contradiction. □

Claim 78. For all $\alpha, \beta \in [k]$ and $H \in \{H_{\mathrm{OO}}, H_{\mathrm{IIL}}, H_{\mathrm{IIR}}, H_{\mathrm{IILR}}\}$, there is no edge from (a_α, b_α) to (a_β, b_β) in H.

Proof. Suppose otherwise, and let $\alpha, \beta \in [k]$ and $H \in \{H_{\mathrm{OO}}, H_{\mathrm{IIL}}, H_{\mathrm{IIR}}, H_{\mathrm{IILR}}\}$ be such that $((a_\alpha, b_\alpha), (a_\beta, b_\beta))$ is an edge in H. Clearly,

$$\text{max-start-path}(H, (a_\alpha, b_\alpha)) > \text{max-start-path}(H, (a_\beta, b_\beta)).$$

Let $i \in [4]$ be such that $\kappa((a, b))_i = \text{max-start-path}(H, (a, b))$ for every $(a, b) \in I$. We obtain,

$$\xi_i = \kappa((a_\alpha, b_\alpha))_i > \kappa((a_\beta, b_\beta))_i = \xi_i.$$

This contradiction completes the proof of the claim. ◁

By (I7), the set $\{b_1, \ldots, b_k\}$ is linearly ordered by the "left of" relation.

Claim 79. $k \neq 2$.

Proof. Suppose that $k = 2$. Without loss of generality assume that b_1 is left of b_2. Then, $((a_1, b_1), (a_2, b_2))$ is a regular alternating cycle in P. Moreover, by Corollary 70, $((a_1, b_1), (a_2, b_2))$ is an alternating cycle of one of four types: In-In, In-Out, Out-In, or Out-Out. It follows that $((a_1, b_1), (a_2, b_2))$ is an edge in H_{IIL} (and H_{IIR}) or is an edge in H_{OO} or is an edge of weight 1 in H_{IO} or is an edge of weight 1 in H_{OI}. This contradicts Claim 78 or Claim 77. ◁

Recall that pairs (a_α, b_α) are in I, for all $\alpha \in [k]$. We prove that some pairs (a_α, b_β) for $\alpha, \beta \in [k]$ with $\alpha \neq \beta$ are also in I. The next claim follows directly from Corollary 50.

Claim 80. For all distinct $\alpha, \beta \in [k]$, we have $a_\alpha \notin \text{shad}(b_\beta)$ and $b_{\alpha+1} \in Y(a_\alpha)$.

Claim 81. Let $\alpha, \beta \in [k]$. If $b_{\alpha+1}$ is left of b_β and b_β is left of b_α, then $(a_\alpha, b_\beta) \in I$. Symmetrically, if b_α is left of b_β and b_β is left of $b_{\alpha+1}$, then $(a_\alpha, b_\beta) \in I$.

Proof. Assume that $b_{\alpha+1}$ is left of b_β and b_β is left of b_α. The proof in the case where b_α is left of b_β and b_β is left of $b_{\alpha+1}$ is symmetric. Note that $a_\alpha \in \pi_1(I)$ and $b_\beta \in \pi_2(I)$. Thus, by (I9), it suffices to show that $a_\alpha \notin \text{shad}(b_\beta)$, and that (a_α, b_β) is dangerous. The former follows from Claim 80. For the latter, note that $b_{\alpha+1}$ and $z_R(a_\alpha)$ witness that (a_α, b_β) is a dangerous pair. Indeed, $a_\alpha < b_{\alpha+1}$ and $a_\alpha < z_R(a_\alpha)$ in P; $b_{\alpha+1}$ is left of b_β (by assumption); and b_β is left of $z_R(a_\alpha)$ (by Proposition 57 and transitivity). Altogether, we obtain that $(a_\alpha, b_\beta) \in I$. ◁

Claim 82. There is no $\alpha \in [k]$ such that $b_{\alpha+1}$ is left of $b_{\alpha+2}$ and $b_{\alpha+2}$ is left of b_α.

Proof. Suppose to the contrary that there exists $\alpha \in [k]$ witnessing that the claim is false. Without loss of generality, assume that $\alpha = k$. Namely, b_1 is left of b_2 and b_2 is left of b_k. See Figure 29. We aim to show that $\sigma = ((a_1, b_1), (a_k, b_k))$ is a shifted edge in H_{IIL} witnessed by b_2 or a shifted edge in H_{IO} witnessed by b_2.

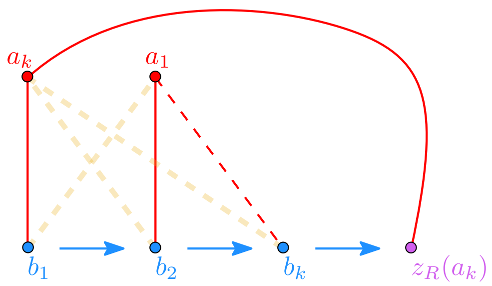


FIGURE 29. Schematic drawing of the situation in Claim 82. In the proof, we show that $(a_k, b_2) \in I$. Next, we show that $((a_1, b_1), (a_k, b_k))$ is a shifted edge in H_{IIL} or a shifted edge in H_{IO}, which is witnessed by b_2.

Since b_1 is left of b_2 and b_2 is left of b_k, by Claim 81, we obtain $(a_k, b_2) \in I$. Note that $\sigma' = ((a_1, b_1), (a_k, b_2))$ is a regular alternating cycle in P. Since $((a_1, b_1), \ldots, (a_k, b_k))$ is strict, we have $a_1 \parallel b_k$ in P (as $k \neq 2$ by Claim 79). Next, since b_2 is left of b_k and $a_1 \parallel b_k$ in P, items (iil2), (iil3), (io2), and (io3) are satisfied for σ witnessed by b_2. We show that either (iil1) or (io1) is also satisfied for σ witnessed by b_2. Namely, we show that σ' is either an In-In alternating cycle or an In-Out alternating cycle.

We prove that σ' satisfies $(R12)$. By Claim 80, $b_1 \in Y(a_k)$, and so, we can fix $z_k \in Z(a_k)$ such that $z_k \leqslant b_1$ in P. We have $(a_1, b_1), (a_k, b_k) \in I$, $b_2 \in B$, b_1 left of b_2, b_2 left of b_k, and $a_1 < b_2$, $a_1 \parallel b_k$, $a_k \parallel b_2$ in P. Altogether, we can apply Proposition 62.(L) to obtain that z_k is left of $z_R(a_k)$ and $a_1 \in \operatorname{int} \mathcal{R}$ where $\mathcal{R} = \mathcal{R}(a_k, z_k, z_R(a_k), U, a_k[M_R(a_k)]z_R(a_k))$ and U is an exposed witnessing path from a_k to z_k in P. Note that $\mathcal{R}$ is a right region of σ'. Therefore, by Proposition 69.(iii), σ' satisfies $(R12)$.

Finally, by Corollary 70, σ' satisfies either $(L12)$ or $(L21^\star)$. In the former case, σ' is an In-In alternating cycle and in the latter case, σ' is an In-Out alternating cycle. As discussed before, this shows that $\sigma = ((a_1, b_1), (a_k, b_k))$ is a shifted edge in H_{IIL} or a shifted edge in H_{IO}. The former contradicts Claim 78 and the latter contradicts Claim 77. $\lhd$

The next claim is a symmetric statement to Claim 82.

Claim 83. There is no $\alpha \in [k]$ such that b_α is left of $b_{\alpha+2}$ and $b_{\alpha+2}$ is left of $b_{\alpha+1}$.

Proof. Suppose to the contrary that there exists $\alpha \in [k]$ witnessing that the claim is false. Without loss of generality, assume that $\alpha = 1$. Namely, b_1 is left of b_3 and b_3 is left of b_2. See Figure 30. We aim to show that $\sigma = ((a_1, b_1), (a_2, b_2))$ is a shifted edge in H_{IIR} witnessed by b_3 or a shifted edge in H_{OI} witnessed by b_3.

Since b_1 is left of b_3 and b_3 is left of b_2, by Claim 81, we obtain $(a_1, b_3) \in I$. Note that $\sigma' = ((a_1, b_3), (a_2, b_2))$ is a regular alternating cycle in P. Since $((a_1, b_1), \ldots, (a_k, b_k))$ is strict, we have $a_2 \parallel b_1$ in P (as $k \neq 2$ by Claim 79). Next, since b_1 is left of b_3 and $a_2 \parallel b_1$ in P, items (iir2), (iir3), (oi2), and (oi3) are satisfied for σ witnessed by b_3. We show that either (iir1) or (oi1) is also satisfied for σ witnessed by b_3. Namely, we show that σ' is either an In-In alternating cycle or an Out-In alternating cycle.

We prove that σ' satisfies $(L12)$. By Claim 80, $b_2 \in Y(a_1)$, and so, we can fix $z_1 \in Z(a_1)$ such that $z_1 \leqslant b_2$ in P. We have $(a_1, b_1), (a_2, b_2) \in I$, $b_3 \in B$, b_1 left of b_3, b_3 left of b_2, and $a_2 < b_3$, $a_2 \parallel b_1$, $a_1 \parallel b_3$ in P. Altogether, we can apply Proposition 62.(R) to obtain that $z_L(a_1)$ is left of z_1) and $a_2 \in \operatorname{int} \mathcal{R}$ where $\mathcal{R} = \mathcal{R}(a_1, z_L(a_1), z_1, a_1[M_L(a_1)]z_L(a_1), V)$ and V is

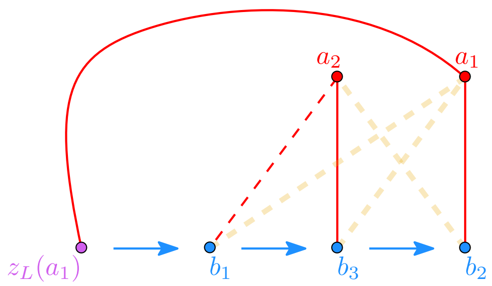


FIGURE 30. Schematic drawings of the situation in Claim 83. In the proof, we show that $(a_1, b_3) \in I$. Next, we show that $((a_1, b_1), (a_2, b_2))$ is a shifted edge in H_{IIR} or a shifted edge in H_{OI}, which is witnessed by b_3.

an exposed witnessing path from a_1 to z_1 in P. Note that $\mathcal{R}$ is a left region of σ'. Therefore, by Proposition 69.(i), σ' satisfies $(L12)$.

Finally, by Corollary 70, σ' satisfies either $(R12)$ or $(R21^\star)$. In the former case, σ' is an In-In alternating cycle and in the latter case, σ' is an Out-In alternating cycle. As discussed before, this shows that $\sigma = ((a_1, b_1), (a_2, b_2))$ is a shifted edge in H_{IIR} or a shifted edge in H_{OI}. The former contradicts Claim 78 and the latter contradicts Claim 77. ◁

We continue the proof, assuming without loss of generality that b_1 is left of b_i for every $i \in [k]$ with $i \neq 1$.

Claim 84. $k \neq 3$.

Proof. Suppose to the contrary that $k = 3$. Note that either b_2 is left of b_3 or b_3 is left of b_2. The former contradicts Claim 82 and the latter contradicts Claim 83. ◁

Therefore, from now on, we assume $k \geqslant 4$. Note that by Claim 83, b_3 is not left of b_2 and by Claim 82, b_2 is not left of b_k. Hence, b_1 is left of b_k and b_k is left of b_2 and b_2 is left of b_3. It follows that there exists $\alpha \in \{3, \ldots, k-1\}$ such that

$$b_1 \text{ is left of } b_{\alpha+1},\ b_{\alpha+1} \text{ left of } b_2, \text{ and } b_2 \text{ left of } b_\alpha. \tag{8}$$

Claim 85. $((a_1, b_1), (a_\alpha, b_\alpha))$ is an edge in H_{IILR}.

Proof. See Figure 31. We show that $(b_{\alpha+1}, b_2)$ is a witness that $((a_1, b_1), (a_\alpha, b_\alpha))$ is an edge in H_{IILR}. Since b_1 is left of $b_{\alpha+1}$ and b_2 is left of b_α, (iilr2) is satisfied. Since $\alpha \notin \{2, k\}$, we have $a_\alpha \parallel b_1$ and $a_1 \parallel b_\alpha$ in P, and thus, (iilr3) is satisfied. It remains to show that (iilr1) holds, that is, $(a_1, b_{\alpha+1}), (a_\alpha, b_2) \in I$ and $\sigma' = ((a_1, b_{\alpha+1}), (a_\alpha, b_2))$ is a regular In-In alternating cycle. The first part follows from (8) and Claim 81. In particular, σ' is a regular alternating cycle in P. We continue arguing that σ' satisfies $(L12)$ and $(R12)$.

By Claim 80, $b_{\alpha+1} \in Y(a_\alpha)$, and so, we can fix $z_\alpha \in Z(a_\alpha)$ with $z_\alpha \leqslant b_{\alpha+1}$ in P. We have $(a_1, b_{\alpha+1}), (a_\alpha, b_\alpha) \in I$, $b_2 \in B$ with $b_{\alpha+1}$ left of b_2 and b_2 left of b_α, and $a_1 \leqslant b_2$ in P. Since $\alpha \notin \{1, 2\}$, we have $a_1 \parallel b_\alpha$ and $a_\alpha \parallel b_2$ in P. Altogether, we can apply Proposition 62.(L) to obtain that z_α is left of $z_R(a_\alpha)$ and $a_1 \in \operatorname{int} \mathcal{R}$ where $\mathcal{R} = \mathcal{R}(a_\alpha, z_\alpha, z_R(a_\alpha), U, a_\alpha[M_R(a_\alpha)]z_R(a_\alpha))$ and U is an exposed witnessing path from a_α to z_α in P. Note that $\mathcal{R}$ is a right region of σ'. Therefore, by Proposition 69.(iii), σ' satisfies $(R12)$.

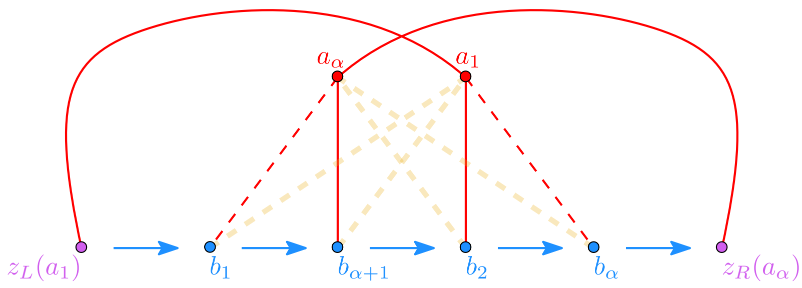


FIGURE 31. Schematic drawing of the situation in Claim 85. In the proof, we show that $(a_1, b_{\alpha+1}), (a_\alpha, b_2) \in I$. Next, we show that $((a_1, b_1), (a_\alpha, b_\alpha))$ is an edge in H_{IILR} which is witnessed by $(b_{\alpha+1}, b_2)$.

By Claim 80, $b_2 \in Y(a_1)$, and so, we can fix $z_1 \in Z(a_1)$ with $z_1 \leqslant b_2$ in P. We have $(a_1, b_1), (a_\alpha, b_2) \in I$, $b_{\alpha+1} \in B$ with b_1 left of $b_{\alpha+1}$ and $b_{\alpha+1}$ left of b_2, and $a_\alpha \leqslant b_{\alpha+1}$ in P. Since $\alpha \notin \{1, k\}$, we have $a_\alpha \parallel b_1$ and $a_1 \parallel b_{\alpha+1}$ in P. Altogether, we can apply Proposition 62.(R) to obtain that $z_L(a_1)$ is left of z_1 and $a_\alpha \in \operatorname{int} \mathcal{R}$ where $\mathcal{R} = \mathcal{R}(a_1, z_L(a_1), z_1, a_1[M_L(a_1)]z_L(a_1), V)$ and V is an exposed witnessing path from a_1 to z_1 in P. Note that $\mathcal{R}$ is a left region of σ'. Therefore, by Proposition 69.(i), σ' satisfies (L12).

This completes the proof that $\sigma' = ((a_1, b_{\alpha+1}), (a_\alpha, b_2))$ is a regular In-In alternating cycle. Thus, $(b_{\alpha+1}, b_2)$ is a witness that $((a_1, b_1), (a_\alpha, b_\alpha))$ is an edge in H_{IILR}. $\lhd$

Claims 85 and 78 contradict each other. This completes the proof of the lemma. $\square$

8. AUXILIARY ORIENTED GRAPHS: THE PROOFS

The goal of this section is to prove all the statements given without proofs in Section 7 (Propositions 71 to 74 and Lemma 75). To this end, we again fix a maximal good instance $(P, x_0, G, e_{-\infty}, I)$ throughout the section.

8.1. **More on regions.** In this subsection, we collect technical statements on regions that are used later.

Proposition 86. *Let $(a, u, v, U, V, \mathcal{R}, q, m, \gamma_L, \gamma_R)$ be a region tuple.*

(L) $z_L(a) \notin \operatorname{int} \mathcal{R}$, *and moreover, if* $z_L(a) \not\leqslant u$ *in* P, *then* $z_L(a) \notin \mathcal{R}$.
(R) $z_R(a) \notin \operatorname{int} \mathcal{R}$, *and moreover, if* $z_R(a) \not\leqslant v$ *in* P, *then* $z_R(a) \notin \mathcal{R}$.

Proof. We prove only (L) as the proof of (R) is symmetric. Suppose to the contrary that $z_L(a) \in \operatorname{int} \mathcal{R}$. Then, by Proposition 60.(ii), $W_L(z_L(a))$ is right of γ_L. It follows that $W_L(z_L(a))$ is right of $M = x_0[W_L(u)]u[U]a$. Hence, by Proposition 54, $M_L(a)$ is right of M. Since $M \in \mathcal{M}(a)$, this is a contradiction, and thus, we obtain $z_L(a) \notin \operatorname{int} \mathcal{R}$.

Now, assume that $z_L(a) \not\leqslant u$ in P, and suppose to the contrary that $z_L(a) \in \mathcal{R}$. We already proved that $z_L(a) \notin \operatorname{int} \mathcal{R}$, hence, $z_L(a) \in \partial\mathcal{R}$. Moreover, since $z_L(a) \not\leqslant u$ in P, the element $z_L(a)$ does not lie in $W_L(u)$, which implies that $z_L(a)$ lies in $W_R(v)$. In particular, $z_L(a) \in \operatorname{shad}(v)$. On the other hand, we claim that $z_L(a) \notin \operatorname{shad}(u)$. Indeed, since $u \in Z(a)$, $a \notin \operatorname{shad}(u)$, and so, by Proposition 51.(i), $z_L(a)$ does not lie in the interior of $\operatorname{shad}(u)$ (since $z_L(a) \not\leqslant u$ in P, $z_L(a)$ is not on the boundary of $\operatorname{shad}(u)$ either). Therefore, since u is left

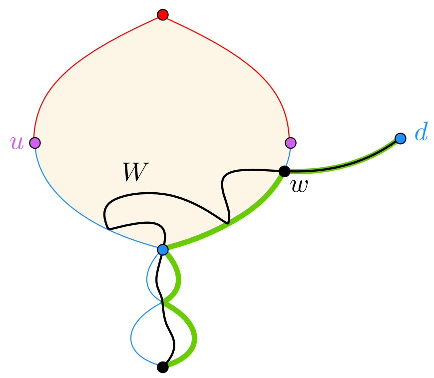


FIGURE 32. An illustration of the statement of Proposition 89.(L). We show that w lies in γ_R and γ_R is left of the path marked in green in the figure.

of v, by Proposition 36.(R), $z_L(a)$ is right of u. In particular, $M = x_0[W_L(u)]u[U]a$ is left of $M_L(a)$, which along with $M \in \mathcal{M}(a)$ is a contradiction, and ends the proof. □

Proposition 87. *Let $(a, u, v, U, V, \mathcal{R}, q, m, \gamma_L, \gamma_R)$ be a region tuple. Let $b \in B$ with $b \in \mathcal{R}$. Then, q lies in both $W_L(b)$ and $W_R(b)$.*

Proof. We only prove that q lies in $W_L(b)$ as the proof that q lies in $W_R(b)$ is symmetric. Since $b \in \mathcal{R}$ and $x_0 \notin \operatorname{int} \mathcal{R}$, the path $W_L(b)$ intersects $\partial\mathcal{R}$, and so, it intersects $u[W_L(u)]q[W_R(v)]v$. Let w be an element of $W_L(b)$ in $\partial\mathcal{R}$. If w lies in $u[W_L(u)]q$, then by Proposition 15.(L), $x_0[W_L(u)]w = x_0[W_L(b)]w$, and so, q lies in $W_L(b)$ as desired. Thus, assume that w lies in $q[W_R(v)]v$ and $w \neq q$. Consider the path $W = x_0[W_L(q)]q[W_R(v)]w[W_L(b)]b$. If $W_L(b) = W$, then the assertion holds, thus, assume that $W_L(b) \neq W$. It follows that $W_L(b)$ is left of W.

Next, we study the relation between $W_L(q)$ and $W_L(b)$. Since $q \leqslant b$ in P, it is not possible that $W_L(b)$ is a strict subpath of $W_L(q)$. If $W_L(q)$ is a subpath of $W_L(b)$, then q lies in $W_L(b)$, as desired. If $W_L(q)$ is left of $W_L(b)$, then W is left of $W_L(b)$, which is false. Finally, if $W_L(b)$ is left of $W_L(q)$, the path $x_0[W_L(b)]w[W_R(v)]v$ is left of $W_L(u) = x_0[W_L(q)]q[W_L(u)]u$, which contradicts the fact that u is left of v. This ends the proof. □

Corollary 88. *Let $(a, u, v, U, V, \mathcal{R}, q, m, \gamma_L, \gamma_R)$ be a region tuple. Let $u', v' \in B$ and let q' be the maximal common element of $W_L(u')$ and $W_R(v')$ in P. If $u', v' \in \mathcal{R}$, then, $u'[W_L(u')]q'[W_R(v')]v' \subseteq \mathcal{R}$.*

Proof. Assume that $u', v' \in \mathcal{R}$. By Proposition 87, q lies in both $W_L(u')$ and $W_R(v')$. In particular, $q \leqslant q'$ in P and by Proposition 59, both paths $q[W_L(u')]u'$ and $q[W_R(v')]v'$ are contained in $\mathcal{R}$. This completes the proof. □

Proposition 89. *Let $(a, u, v, U, V, \mathcal{R}, q, m, \gamma_L, \gamma_R)$ be a region tuple. Let $d \in B$ with $d \notin \mathcal{R}$ and let W be a witnessing path from x_0 to d in P that intersects $\mathcal{R}$. Let w be the last element of W in $\mathcal{R}$.*

(L) If u is left of d, then w lies in γ_R and γ_R is left of $x_0[W_R(v)]w[W]d$.
(R) If v is right of d, then w lies in γ_L and γ_L is right of $x_0[W_L(u)]w[W]d$.

Proof. We prove only (L) as the proof of (R) is symmetric. See Figure 32 for an illustration of the statement. Note that by definition $w \in B$ and $w \in \partial\mathcal{R}$. Let e be the edge following w in W. Note that e does not lie in $\mathcal{R}$. If w is an element of γ_L, then let e_L^-, e_L^+ be the edges immediately preceding and following w in γ_L (when $w = x_0$, we set $e_L^- = e_{-\infty}$). If w is an element of γ_R, then let e_R^-, e_R^+ be the edges immediately preceding and following w in γ_R (when $w = x_0$, we set $e_R^- = e_{-\infty}$).

First, suppose that w lies in $W_L(u)$ and $e_L^- \prec e \prec e_L^+$ in the w-ordering. This implies that $W' = x_0[W_L(u)]w[W]d$ is left of γ_L. In particular, either $W_L(u)$ is a subpath of W' or W' is left of $W_L(u)$. In both cases, there is a contradiction with the fact that u is left of d.

Second, note that if w lies in $W_R(v)$ and $e_R^+ \prec e \prec e_R^-$ in the w-ordering, then, γ_R is left of $x_0[W_R(v)]w[W]d$, so the statement of the proposition holds.

Since $w \in B$ and $w \in \partial\mathcal{R}$ we conclude that w lies in $u[W_L(u)]q[W_R(v)]v$. If w lies in $u[W_L(u)]q$ and $w \neq q$, then by (rg3), $e_L^- \prec e \prec e_L^+$ in the w-ordering, which leads to a contradiction as discussed above. If w lies in $q[W_R(v)]v$ and $w \neq q$, then by (rg4), $e_R^+ \prec e \prec e_R^-$ in the w-ordering, which yields the statement as discussed above. Therefore, the only remaining case to consider is when $w = q$. In this case, all four edges e_L^-, e_L^+, e_R^-, e_R^+ are well-defined and $e \notin \{e_L^-, e_L^+, e_R^-, e_R^+\}$. We have four possibilities: $e_L^- \prec e \prec e_L^+$, $e_L^+ \prec e \prec e_R^+$, $e_R^+ \prec e \prec e_R^-$, $e_R^- \prec e \prec e_L^-$ in the w-ordering. The first case leads to a contradiction, while the third case yields the statement as discussed in a paragraph above. The second case contradicts (rg1). Finally, in the fourth case, by (sb2), we obtain that the edge e lies in $\mathrm{shad}(q)$, and in particular, the whole path $q[W]d$ lies in $\mathrm{shad}(q)$, which forces $d \in \mathrm{shad}(q) \subseteq \mathrm{shad}(u)$ and contradicts the assumption that u is left of d. □

Proposition 90. *Let $(a, u, v, U, V, \mathcal{R}, q, m, \gamma_L, \gamma_R)$ be a region tuple. Let $c \in A$ and $b \in B$ with $b, c \in \mathcal{R}$. Let $z \in Z(c)$ and W be an exposed witnessing path from c to z in P such that $W \subseteq \mathcal{R}$.*

(L) *Let w be an element of $W_R(v)$ with $q \leqslant w$ in P.*
   (i) *If $b \notin \mathrm{shad}(w)$, then either $W_L(w)$ is a subpath of $W_L(b)$ or $W_L(b)$ is left of $W_L(w)$.*
   (ii) *If $c \notin \mathrm{shad}(w)$, then $W_L(w)$ is not left of $W_L(z)$.*
   (iii) *If $c \notin \mathrm{shad}(w)$, then $\mathrm{gcpe}(W_L(u), W_L(w))$ lies in $W_L(z)$.*

(R) *Let w be an element of $W_L(u)$ with $q \leqslant w$ in P.*
   (i) *If $b \notin \mathrm{shad}(w)$, then either $W_R(w)$ is a subpath of $W_R(b)$ or $W_R(b)$ is right of $W_R(w)$.*
   (ii) *If $c \notin \mathrm{shad}(w)$, then $W_R(w)$ is not right of $W_R(z)$.*
   (iii) *If $c \notin \mathrm{shad}(w)$, then $\mathrm{gcpe}(W_R(v), W_R(w))$ lies in $W_R(z)$.*

Proof. We prove only (L) as the proof of (R) is symmetric. See Figure 33 for an illustration of the statement.

Let $u' = \mathrm{gcpe}(W_L(u), W_L(w))$. Note that by Proposition 87, q lies in $W_L(w)$, and so, $q \leqslant u'$ in P. Let

$$\gamma_1 = u'[W_L(w)]w[\gamma_R]m[\gamma_L]u' \text{ and}$$
$$\gamma_2 = u'[W_L(u)]q[W_R(v)]w[W_L(w)]u'.$$

Moreover, let Γ_1 be the region of γ_1 and Γ_2 be the region of γ_2. See Figure 34. Note that all segments of γ_1 and γ_2 in the definition above, except $u'[W_L(w)]w$ are contained in $\partial\mathcal{R}$. Since $u'[W_L(w)]w \subseteq \mathcal{R}$ (by Proposition 59), we have $\gamma_1, \gamma_2 \subseteq \mathcal{R}$, and so, by Proposition 7, $\Gamma_1, \Gamma_2 \subseteq \mathcal{R}$. Observe also that $\Gamma_1 \cup \Gamma_2 = \mathcal{R}$ and $\Gamma_1 \cap \Gamma_2 = u'[W_L(w)]w$. Note that $\gamma_2 =$

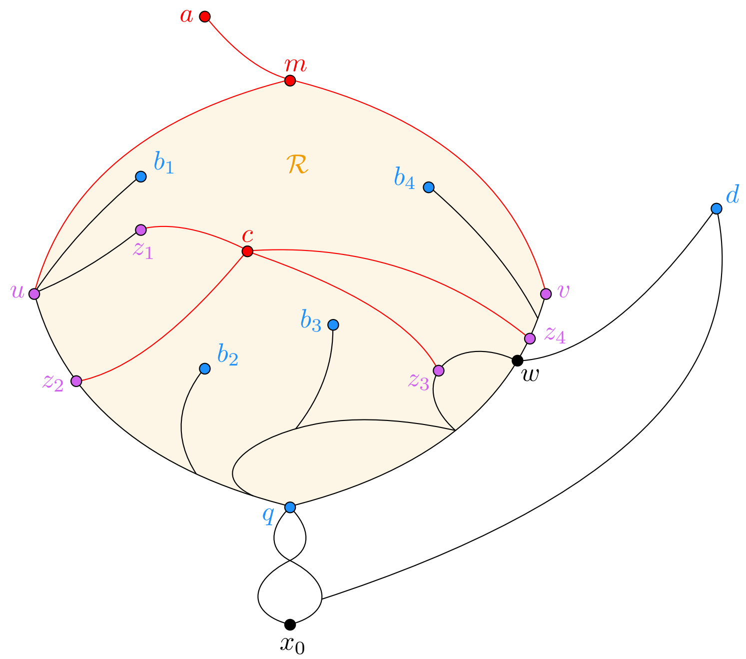


FIGURE 33. An illustration of the statement of Proposition 90.(*L*). We depict several possibilities of how z (z_1, z_2, z_3, and z_4) and b (b_1, b_2, b_3, and b_4) can be positioned in the region. Later, in Proposition 91, we show that every element of B common to $\mathcal{R}$ and $\text{shad}(d)$ lies in $\text{shad}(w)$.

$q[W_R(w)]w[W_L(w)]q$, and so, $\Gamma_2 \subseteq \text{shad}(w)$. Moreover, since q is a common element of w we have that $\text{shad}(w) = \text{shad}(q) \cup \Gamma_2$.

First, we prove (i). Assume that $b \notin \text{shad}(w)$. Since $b \in \mathcal{R}$ and $b \notin \text{shad}(w)$, we have $b \in \Gamma_1$. We will prove that either $W_L(w)$ is a subpath of $W_L(b)$ or $W_L(b)$ is left of $W_L(w)$.

We have $b \in \Gamma_1$ and $q \in \Gamma_2$. By Proposition 87, q lies in $W_L(b)$ and by Proposition 59, $q[W_L(b)]b \subseteq \mathcal{R}$. Therefore, $q[W_L(b)]b$ intersects $\Gamma_1 \cap \Gamma_2 = u'[W_L(w)]w$ – let w' be the maximal element of P in this intersection. The paths $W_L(b)$ and $W_L(w)$ are x_0-consistent (by Proposition 15.(*L*)), and so, $w' = \text{gcpe}(W_L(b), W_L(w))$.

If $w' = w$, then $W_L(w)$ is a subpath of $W_L(b)$ as desired. Hence, assume $w' \neq w$. Note that $w' \in \text{shad}(w)$, thus, $w' \neq b$. Let e be the edge following w' in $W_L(b)$. Observe that e lies in Γ_1. If $w' = q = x_0$, then let $e^- = e_{-\infty}$. Otherwise, let e^- be the edge immediately preceding w' in $W_L(w)$. Since $w' \neq w$, we can define e^+ to be the edge immediately following w' in $W_L(w)$. We claim that

$$e^- \prec e \prec e^+ \text{ in the } w'\text{-ordering.} \tag{9}$$

Note that (9) and $w' = \text{gcpe}(W_L(b), W_L(w))$ implies that $W_L(b)$ is left of $W_L(w)$ which will complete the proof of (i).

Note that $u' < u$ in P as u and v are incomparable in P. We define f^+ to be the edge immediately following u' in $W_L(u)$. Finally, let g^+ be the first edge of $q[W_R(v)]v$. When $w' \neq u'$, e^+ follows e^- in γ_1, and so, we obtain (9) by Observation 14. Now, suppose that $w' = u'$. In this case, e^+ follows f^+ in γ_1, and so, again by Observation 14, we have

$$f^+ \prec e \prec e^+ \text{ in the } w'\text{-ordering.} \tag{10}$$

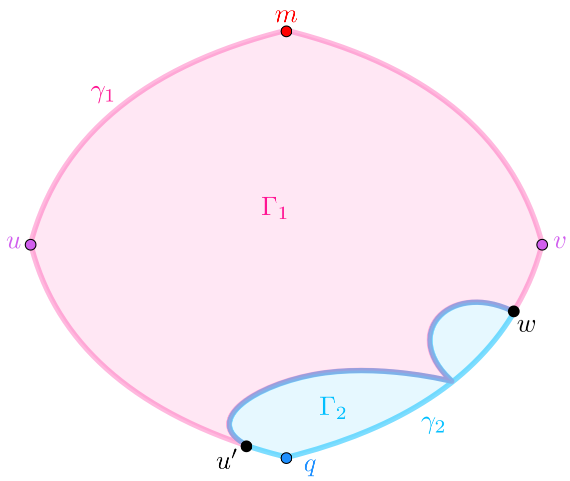


FIGURE 34. The curves γ_1 and γ_2 and the regions Γ_1 and Γ_2.

Since $w'[W_L(w)]w \subseteq \mathcal{R}$, the edge e^+ lies in $\mathcal{R}$. If $w' \neq q$, then by (rg3), we have $f^+ \prec e^+ \prec e^-$ in the w'-ordering, and combining with (10), we obtain (9). Finally, assume that $w' = u' = q$. Then, e^- does not lie in $\mathcal{R}$, and so, by (rg1), we have $g^+ \prec e^- \prec f^+$ in the w'-ordering and $f^+ \prec e^+ \prec g^+$ in the w'-ordering. It follows that $e^+ \prec e^- \prec f^+$ in the w'-ordering. Again combining with (10), we obtain (9). This completes the proof of (9) and as discussed of (i).

For the proof of (ii) and (iii) assume that $c \notin \text{shad}(w)$. First, we claim that

$$W \subseteq \Gamma_1. \tag{11}$$

Indeed, since $c \in \Gamma_1$, W is an exposed path contained in $\mathcal{R}$, $\Gamma_1 \subseteq \mathcal{R}$, and the parts of the boundaries of Γ_1 and $\mathcal{R}$ that are not in B are the same, we have $W \subseteq \Gamma_1$, as desired.

Now, we prove (ii). Since $c \in \mathcal{R}$ and $c \notin \text{shad}(w)$, we have $c \in \Gamma_1$. Suppose to the contrary that $W_L(w)$ is left of $W_L(z)$. We claim that $z \notin \text{shad}(w)$. Since $W \subseteq \Gamma_1$ (by (11)), either $z \in \text{int}\,\Gamma_1$ or $z \in \partial\Gamma_1$. First, suppose that $z \in \text{int}\,\Gamma_1$. It follows that $z \notin \Gamma_2$. Additionally, by Proposition 58, $z \notin \text{shad}(q)$. Recall that $\text{shad}(w) = \text{shad}(q) \cup \Gamma_2$. This implies $z \notin \text{shad}(w)$, so the claim holds in this case. Next, assume that $z \in \partial\Gamma_1$. If $z \in \text{shad}(w)$, then $z \in \partial\Gamma_1 \cap \text{shad}(w) = \partial\Gamma_1 \cap \Gamma_2 = u'[W_L(w)]w$, which yields that z lies in $W_L(w)$, which contradicts $W_L(w)$ left of $W_L(z)$. This shows that indeed $z \notin \text{shad}(w)$. Now, we can apply (i) for $b = z$. It follows that either $W_L(w)$ is a subpath of $W_L(z)$ or $W_L(z)$ is left of $W_L(w)$. In both cases, we obtain a contradiction, which completes the proof of (ii).

Finally, we prove (iii), namely, we prove that u' lies in $W_L(z)$. Since $W \subseteq \Gamma_1$ (by (11)), we have $z \in \Gamma_1$. On the other hand, $q \in \Gamma_2$. By Proposition 87, q lies in $W_L(z)$ and by Proposition 59, $q[W_L(z)]z \subseteq \mathcal{R}$. Therefore, $q[W_L(z)]z$ intersects $\Gamma_1 \cap \Gamma_2 = u'[W_L(w)]w$. In particular, by x_0-consistency of $W_L(w)$ and $W_L(z)$ (Proposition 15.(L)), u' lies in $W_L(z)$, which completes the proof. $\square$

Proposition 91. *Let* $(a, u, v, U, V, \mathcal{R}, q, m, \gamma_L, \gamma_R)$ *be a region tuple. Let* $d \in B$ *with* $d \notin \mathcal{R}$*.*

(L) *Let* u *be left of* d *and let* w *be the maximal common element of* $W_L(d)$ *and* $W_R(v)$ *in* P*. For every* $b \in B$ *with* $b \in \mathcal{R}$ *and* $b \in \text{shad}(d)$*, we have* $b \in \text{shad}(w)$*. Moreover, if such* $b \in B$ *exists, then* $q \leqslant w$ *in* P*.*

(R) *Let* v *be right of* d *and* w *be the maximal common element of* $W_R(d)$ *and* $W_L(u)$ *in* P*. For every* $b \in B$ *with* $b \in \mathcal{R}$ *and* $b \in \text{shad}(d)$*, we have* $b \in \text{shad}(w)$*. Moreover, if such* $b \in B$ *exists, then* $q \leqslant w$ *in* P*.*

Proof. We prove only (L) as the proof of (R) is symmetric. See again Figure 33. Let $b \in B$ with $b \in \mathcal{R}$ and $b \in \text{shad}(d)$.

First, we show the "moreover" part, that is, $q \leqslant w$ in P. Since u is left of d, we have $W_L(u)$ left of $W_L(d)$. Moreover, $W_L(q)$ is a subpath of $W_L(u)$, hence, either $W_L(q)$ is left of $W_L(d)$ or $W_L(q)$ is a subpath of $W_L(d)$. In the latter case, q lies in both $W_L(d)$ and $W_L(w)$, hence, $q \leqslant w$ in P, as desired. Thus, we may assume that $W_L(q)$ is left of $W_L(d)$. Since $b \in \text{shad}(d)$, by Proposition 30.(L), $W_L(b)$ is not left of $W_L(d)$. This translates into one of the three options: $W_L(d)$ is a subpath of $W_L(b)$ or $W_L(b)$ is a subpath of $W_L(d)$ or $W_L(d)$ is left of $W_L(b)$. Since $b \in \mathcal{R}$, by Proposition 87, q lies in $W_L(b)$, and so, $W_L(q)$ is a subpath of $W_L(b)$. Each of the options "$W_L(d)$ is a subpath of $W_L(b)$" and "$W_L(d)$ is left of $W_L(b)$" implies that $W_L(q)$ is left of $W_L(b)$, which is false. Thus, the third option holds: $W_L(b)$ is a subpath of $W_L(d)$. It follows that $W_L(q)$ is a subpath of $W_L(d)$, and so, $q \leqslant w$ in P, as desired.

Now, we prove the main statement. To this end, suppose to the contrary that $b \notin \text{shad}(w)$. By Proposition 90.(L).(i), either $w < b$ in P or $W_L(b)$ is left of $W_L(w)$. In the latter case, we obtain $W_L(b)$ left of $W_L(d)$, which contradicts $b \in \text{shad}(d)$ by Proposition 30.(L). Thus, assume that $w < b$ in P and let W be a witnessing path from w to b in P.

Note that $w \neq d$ as $w \in \mathcal{R}$ and $d \notin \mathcal{R}$. Let e^- and e^+ be the edges immediately preceding and following w in $W_L(d)$ (when $w = x_0$, we set $e^- = e_{-\infty}$). Let f^- and f^+ be the edges immediately preceding and following w in γ_R (when $w = x_0$, we set $f^- = e_{-\infty}$). Let e be the edge of W incident to w. See Figure 35.

Note that

$$f^- \prec f^+ \prec e^+ \text{ in the } w\text{-ordering} \qquad \text{by Proposition 89.}(L), \tag{12}$$

$$f^- \prec e \preccurlyeq f^+ \text{ in the } w\text{-ordering} \qquad \text{by (rg3) or (rg1).} \tag{13}$$

Next, we claim that

$$f^- \preccurlyeq e^- \prec f^+ \text{ in the } w\text{-ordering.} \tag{14}$$

First, assume that $q < w$ in P. Then, since $q, w \in \mathcal{R}$, by Proposition 59, $q[W_L(w)]w \subseteq \mathcal{R}$, and so, $e^- \subseteq \mathcal{R}$. Thus, by (rg4), (14) follows. Finally, assume that $q = w$. Then, f^+ does not lie in $\text{shad}(q)$, and so, by (sb2), (14) follows.

By (13) and (14), either $f^- \prec e \prec e^-$ in the w-ordering or $e^- \prec e \preccurlyeq f^+$ in the w-ordering. The former implies (by (sb2)) that e lies in $\text{shad}(w)$, which yields $b \in \text{shad}(w)$, which is a contradiction. Therefore, the latter holds. Combining it with (12), (13), and (14), we obtain $e^- \prec e \preccurlyeq f^+ \prec e^+$ in the w-ordering. This implies that $x_0[W_L(w)]w[W]b$ is left of $W_L(d)$, and thus, $W_L(b)$ is left of $W_L(d)$. However, this is a contradiction with $b \in \text{shad}(d)$ by Proposition 30.(L). This completes the proof. □

8.2. **Out-Out oriented graphs.** In this subsection, we prove Proposition 71, which states that $\text{max-path}(H_{OO}) \leqslant \text{se}_P(I)$. See an example of a path in H_{OO} in Figure 36.

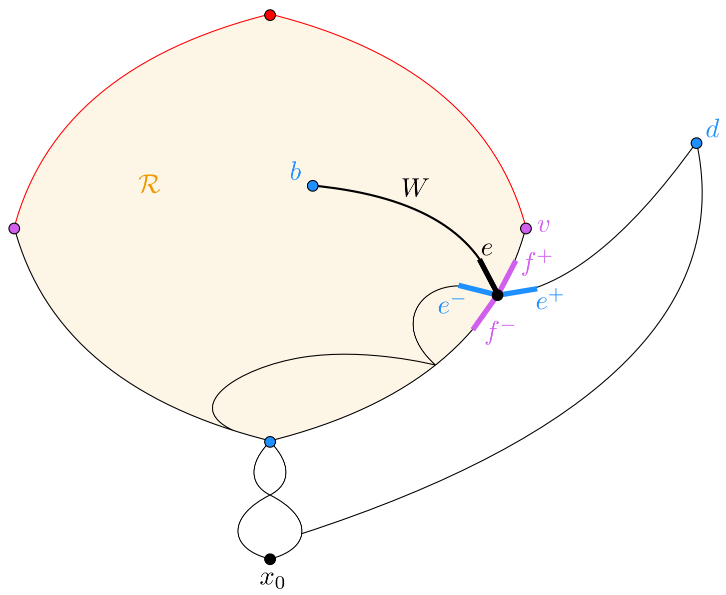


FIGURE 35. We prove that the edges e, e^-, e^+, f^-, and f^+ are in the order depicted in the figure around w.

Proof of Proposition 71. If max-path$(H_{\mathrm{OO}}) \leqslant 1$, then there is nothing to prove as $\mathrm{se}_P(I) \geqslant 1$. Thus, we can assume max-path$(H_{\mathrm{OO}}) \geqslant 2$. Let $((a_1, b_1), \ldots, (a_n, b_n))$ be a path in H_{OO} where n is an integer with $n \geqslant 2$. By Proposition 67, $((a_i, b_i), (a_j, b_j))$ satisfies $(L21^\star)$ and $(R21^\star)$ for every $i, j \in [n]$ with $i < j$. Therefore, by Observation 65, $a_j < b_i$ and $a_i < b_j$ in P. It follows that $J = \{(a_i, b_i) : i \in [n]\}$ induces a standard example in P. Since $J \subseteq I$ and $|J| = n$, we obtain $n \leqslant \mathrm{se}_P(I)$, as desired. □

8.3. **In-In oriented graphs.** In this subsection, we prove Propositions 72 to 74. Namely, we prove that max-path$(H) \leqslant \mathrm{se}_P(I)$ for each $H \in \{H_{\mathrm{IIL}}, H_{\mathrm{IIR}}, H_{\mathrm{IILR}}\}$. In fact, we will prove that max-path$(H) \leqslant$ max-path(H_{II}), and then that max-path$(H_{\mathrm{II}}) \leqslant \mathrm{se}_P(I)$. We begin with the latter, see Proposition 94.

For every $a \in A$, by Proposition 57, $z_L(a)$ is left of $z_R(a)$, and hence, we can define

$$\mathcal{R}(a) = \mathcal{R}(a, z_L(a), z_R(a), z_L(a)[M_L(a)]a, z_R(a)[M_R(a)]a).$$

Proposition 92. *Let $((a_1, b_1), (a_2, b_2))$ be a regular In-In alternating cycle. Then, $\mathcal{R}(a_1) = \mathcal{R}(a_2)$.*

Proof. Let $\sigma = ((a_1, b_1), (a_2, b_2))$ and let m_1 and m_2 be the upper-mins of $\mathcal{R}(a_1)$ and $\mathcal{R}(a_2)$, respectively. First, we argue that

$$m_2 \text{ lies in } M_L(a_1) \quad \text{and} \quad m_1 \text{ lies in } M_R(a_2). \tag{15}$$

We will prove only the first part of the statement: m_2 lies in $M_L(a_1)$, as the proof of the second part is symmetric. Let $u = \mathrm{gcpe}(M_L(a_1), M_L(a_2))$. Note that both m_2 and u lie in $M_L(a_2)$. There are two exclusive possibilities: $x_0[M_L(a_2)]m_2$ is a subpath of $x_0[M_L(a_2)]u$ or $x_0[M_L(a_2)]u$ is a proper subpath of $x_0[M_L(a_2)]m_2$. We argue that the former always holds. Assume to the contrary that $x_0[M_L(a_2)]u$ is a proper subpath of $x_0[M_L(a_2)]m_2$. Since σ

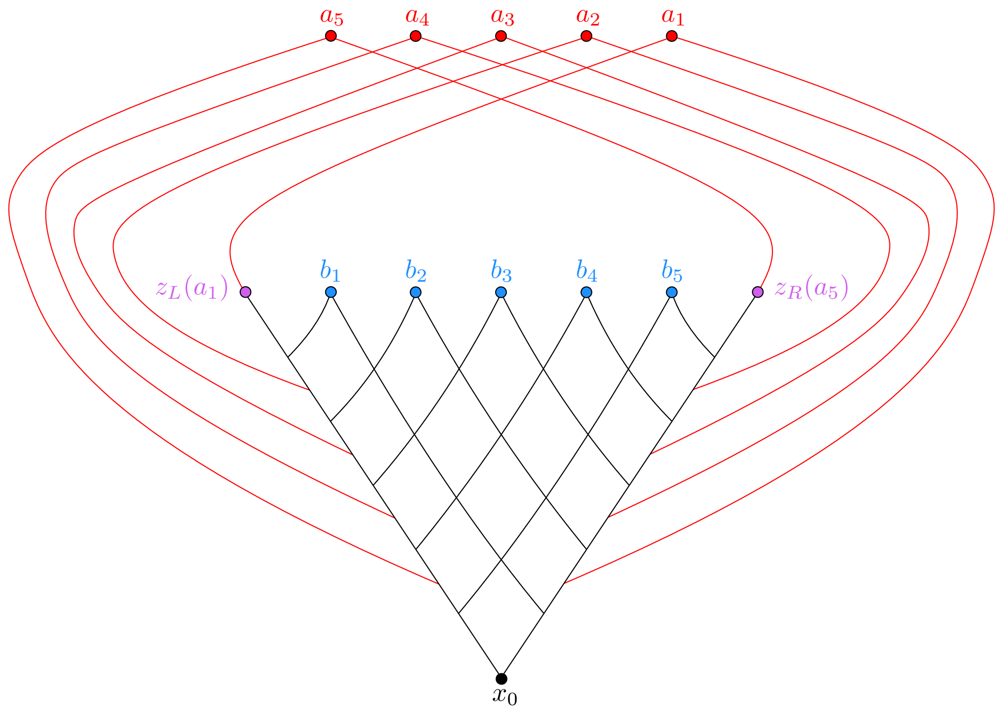


FIGURE 36. $((a_1, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_4), (a_5, b_5))$ is a path in H_{OO}.

satisfies $(L12)$, $M_L(a_1)$ is left of $M_L(a_2)$. Let e be the edge following u in $M_L(a_1)$ – it exists because $a_1 \parallel a_2$ in P.

We claim that

$$e \text{ does not lie in } \mathcal{R}(a_2). \tag{16}$$

Let q_2 be the lower-min of $\mathcal{R}(a_2)$. Note that both u and q_2 lie in $M_L(a_2)$. If $u < q_2$ in P, then by Proposition 58, $u \notin \mathcal{R}(a_2)$, and so, e does not lie in $\mathcal{R}(a_2)$, as claimed. If $u = q_2$, then since $M_L(a_1)$ is left of $M_L(a_2)$, by (rg1), e does not lie in $\mathcal{R}(a_2)$, as claimed. Finally, otherwise, u lies strictly on the left side of $\mathcal{R}(a_2)$, and so, by (rg3), e does not lie in $\mathcal{R}(a_2)$, as claimed. This way, we obtain (16).

Since σ is In-In, it satisfies $(R12)$ and since $\mathcal{R}(a_2)$ is a right region of σ, by Corollary 70.(iii), we have $a_1 \in \operatorname{int} \mathcal{R}(a_2)$. Since e does not lie in $\mathcal{R}(a_2)$ (by (16)) and $a_1 \in \operatorname{int} \mathcal{R}(a_2)$, the path $u[M_L(a_1)]a_1$ intersects $\partial\mathcal{R}(a_2)$ in an element distinct from u, let v be such an element. Since the paths $M_L(a_1)$ and $M_L(a_2)$ are x_0-consistent (by Proposition 55.(L)), v does not lie in $M_L(a_2)$, and hence, it lies in $M_R(a_2)$. Let $M = x_0[M_L(a_2)]u[M_L(a_1)]v[M_R(a_2)]a_2$ and note that M is left of $M_L(a_2)$. To conclude (15), it suffices to show that

$$M \in \mathcal{M}(a_2). \tag{17}$$

Indeed, if $M \in \mathcal{M}(a_2)$, then M being left of $M_L(a_2)$ contradicts the definition of $M_L(a_2)$ and completes the proof of (15).

We split the proof of (17) into two cases: $v \in B$ and $v \notin B$. First, suppose that $v \in B$. Then, $M = x_0[M_L(a_1)]v[M_R(a_2)]z_R(a_2)[M_R(a_2)]a_2$. Since $z_R(a_2) \in Z(a_2)$, $x_0[M_L(a_1)]v[M_R(a_2)]z_R(a_2)$ is a witnessing path in P, and $z_R(a_2)[M_R(a_2)]a_2$ is an exposed witnessing path in P, $M \in \mathcal{M}(a_2)$. Next, suppose that $v \notin B$. We have $M = x_0[M_L(a_1)]z_L(a_1)[M_L(a_1)]v[M_R(a_2)]a_2$. Note that $x_0[M_L(a_1)]z_L(a_1)$ is a witnessing path in P and $z_L(a_1)[M_L(a_1)]v[M_R(a_2)]a_2$ is an exposed witnessing path in P. To obtain that

$M \in \mathcal{M}(a_2)$, it suffices to show that $z_L(a_1) \in Z(a_2)$. It is clear that there is an exposed witnessing path from a_2 to $z_L(a_1)$ in P (the path $a_2[M_R(a_2)]v[M_L(a_1)]z_L(a_1)$). Thus, it remains to prove that $a_2 \notin \text{shad}(z_L(a_1))$. Observe that none of the elements of the paths $a_1[M_L(a_1)]v$ and $a_2[M_R(a_2)]v$ are in B. Therefore, these paths do not intersect $\partial\,\text{shad}(z_L(a_1))$. In particular, either all a_1, v, and a_2 are in $\text{shad}(z_L(a_1))$ or none of them. Since $z_L(a_1) \in Z(a_1)$ by definition, the latter holds. We conclude that $a_2 \notin \text{shad}(z_L(a_1))$, and so, $z_L(a_1) \in Z(a_2)$, and so, $M \in \mathcal{M}(a_2)$, and finally, (17) holds. This completes the proof of (15).

Since $M_L(a_1), M_L(a_2)$ and $M_R(a_1), M_R(a_2)$ are x_0-consistent (by Proposition 55), we obtain

$$x_0[M_L(a_2)]m_2 = x_0[M_L(a_1)]m_2 \quad \text{and} \quad x_0[M_R(a_1)]m_1 = x_0[M_R(a_2)]m_1. \tag{18}$$

Since m_1 and m_2 lie in $M_L(a_1)$, m_1 and m_2 are comparable in P.

Suppose first that $m_1 < m_2$ in P. Recall that m_2 and m_1 lie in $M_R(a_2)$. Since $m_1 < m_2$ in P we conclude that m_2 lies in $x_0[M_R(a_2)]m_1$, which by (18) implies that m_2 lies in $M_R(a_1)$. All this together means that m_2 is a common element of $M_L(a_1)$ and $M_R(a_1)$. However, this and $m_1 < m_2$ in P contradicts the choice of m_1. Similarly, the assumption of $m_2 < m_1$ in P also leads to a contradiction. We conclude that $m_1 = m_2$, and so, $x_0[M_L(a_1)]m_1 = x_0[M_L(a_2)]m_2$ and $x_0[M_R(a_1)]m_1 = x_0[M_R(a_2)]m_2$. Thus, $\mathcal{R}(a_1) = \mathcal{R}(a_2)$. □

Now, we prove the transitivity of regular In-In alternating cycles, which will give Proposition 94, that is, $\text{max-path}(H_{\text{II}}) \leqslant \text{se}_P(I)$. See examples of paths in H_{II} in Figure 37.

Proposition 93. *Let $((a_1, b_1), (a_2, b_2))$ and $((a_2, b_2), (a_3, b_3))$ be regular In-In alternating cycles. Then, $((a_1, b_1), (a_3, b_3))$ is a regular In-In alternating cycle.*

Proof. Let $\sigma = ((a_1, b_1), (a_3, b_3))$. Since b_1 is left of b_2 and b_2 is left of b_3, b_1 is left of b_3 (by Proposition 37), hence, σ is regular. By Proposition 67.(i) and (iii), σ satisfies $(L12)$ and $(R12)$. Therefore, it suffices to prove that σ is an alternating cycle, that is, $a_1 < b_3$ and $a_3 < b_1$ in P. We only prove that $a_1 < b_3$ in P, the proof of the other inequality is symmetric.

By Corollary 68, $b_1, b_3 \in Y(a_2)$, thus, we can fix $z_1, z_3 \in Z(a_2)$ such that $z_1 \leqslant b_1$ and $z_3 \leqslant b_3$ in P. Let W_1 and W_3 be exposed witnessing paths in P from a_2 to z_1 and z_3 respectively. Since b_1 is left of b_2 and b_2 is left of b_3, by Corollary 53, z_1 is left of b_2 and b_2 is left of z_3. In particular, z_1 is left of z_3, and so, we can define the region $\mathcal{S} = \mathcal{R}(a_2, z_1, z_3, W_1, W_3)$. We consider two cases: either $a_1 \in \mathcal{S}$ or $a_1 \notin \mathcal{S}$. Note that neither of the cases leads to a contradiction, see Figure 38.

First, assume that $a_1 \in \mathcal{S}$. By Proposition 92, $\mathcal{R}(a_1) = \mathcal{R}(a_2) = \mathcal{R}(a_3)$, and so, $z_L(a_1) = z_L(a_2) = z_L(a_3) =: z$. Note that $a_1, a_2, a_3 \leqslant z$ in P, hence, $z \nleqslant b_1, b_3$ in P, and thus, $z \nleqslant z_1, z_3$ in P. Therefore, by Proposition 86, $z \notin \mathcal{S}$. Let W be an exposed witnessing path from a_1 to z in P. Since $a_1 \in \mathcal{S}$ and $z \notin \mathcal{S}$, there is an element w of W in $\partial\mathcal{S}$. Since W is an exposed path and $a_1 \parallel b_1$ in P, w lies in W_3. It follows that $a_1 < w < z_3 \leqslant b_3$ in P, as desired.

Finally, assume that $a_1 \notin \mathcal{S}$. Since z_1 is left of b_2 and b_2 is left of z_3, by Corollary 61.(i), we have $b_2 \in \text{int}\,\mathcal{S}$. Let W be a witnessing path from a_1 to b_2 in P. Since $a_1 \notin \mathcal{S}$ and $b_2 \in \mathcal{S}$, there is an element w of W in $\partial\mathcal{S}$. Since $a_1 \parallel b_1$ and $a_2 \parallel b_2$ in P, w lies in $W_R(z_3)$. It follows that $a_1 < w \leqslant z_3 \leqslant b_3$ in P. □

Proposition 94. $\text{max-path}(H_{\text{II}}) \leqslant \text{se}_P(I)$.

Proof. If $\text{max-path}(H_{\text{II}}) \leqslant 1$, then there is nothing to prove as $\text{se}_P(I) \geqslant 1$. Thus, we can assume $\text{max-path}(H_{\text{II}}) \geqslant 2$. Let $((a_1, b_1), \dots, (a_n, b_n))$ be a path in H_{II} where n is an integer with $n \geqslant 2$. By Proposition 93, $((a_i, b_i), (a_j, b_j))$ is an alternating cycle for every $i, j \in [n]$

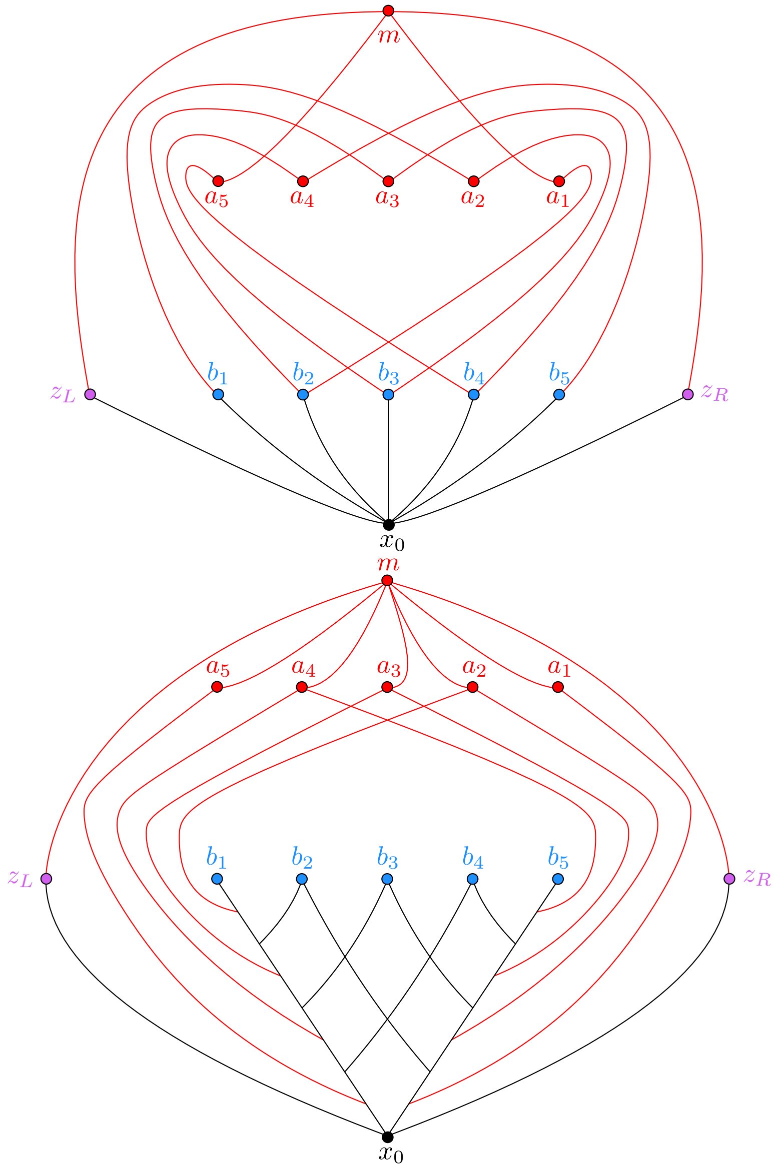


FIGURE 37. In both figures $((a_1, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_4), (a_5, b_5))$ is a path in H_{II}. We also have $\mathcal{R}(a_1) = \mathcal{R}(a_2) = \mathcal{R}(a_3) = \mathcal{R}(a_4) = \mathcal{R}(a_5)$ in both figures – recall Proposition 92. In the proof of Proposition 93, there are two cases: $a_1 \in \mathcal{S}$ and $a_1 \notin \mathcal{S}$. Both are possible, we depict the former in the top part of the figure and the latter in the bottom part.

with $i < j$. It follows that $J = \{(a_i, b_i) : i \in [n]\}$ induces a standard example in P. Since $J \subseteq I$ and $|J| = n$, we obtain $n \leqslant \mathrm{se}_P(I)$, as desired. □

Next, we show that every directed path in H_{IIL} and H_{IIR} can be "transformed" into a directed path in H_{II}. We gave a schematic illustration of edges of these directed graphs in Figure 26. See Figure 39 for a more precise illustration. See also Figure 40 for an example of a path in H_{IIL} consisting of only shifted edges.

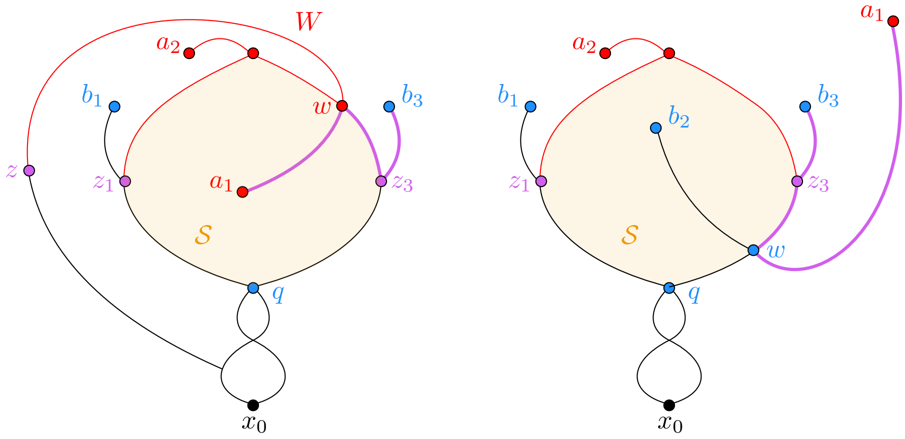


FIGURE 38. An illustration of the proof of Proposition 93. On the left, we depict the case where $a_1 \in \mathcal{S}$ and on the right, we depict the case where $a_1 \notin \mathcal{S}$. We mark with a purple curve the final comparability $a_1 < b_3$ in P.

Proposition 95.

(*L*) *Let* $((a_1, b_1), (a_2, b_2))$ *be a shifted edge in* H_{IIL} *with a witness* $t \in B$ *and let* $((a_2, b_2), (a_3, b_3))$ *be a cycle edge in* H_{IIL}*. Then,* $((a_2, t), (a_3, b_3))$ *is a cycle edge in* H_{IIL}*.*

(*R*) *Let* $((a_1, b_1), (a_2, b_2))$ *be a cycle edge in* H_{IIR} *and let* $((a_2, b_2), (a_3, b_3))$ *be a shifted edge in* H_{IIR} *with a witness* $s \in B$*. Then,* $((a_1, b_1), (a_2, s))$ *is a cycle edge in* H_{IIR}*.*

Proof. We only prove (*L*) as the proof of (*R*) is symmetric. See Figure 41 for an illustration of the proof. Let $\sigma = ((a_2, t), (a_3, b_3))$. By (iil1), $(a_2, t) \in I$. Since $((a_1, b_1), (a_2, t))$ and $((a_2, b_2), (a_3, b_3))$ are edges in H_{II}, in particular b_1 is left of t and b_2 is left of b_3. By (iil2), t is left of b_2. Altogether,

$$b_1 \text{ is left of } t,\ t \text{ is left of } b_2, \text{ and } b_2 \text{ is left of } b_3.$$

In particular, σ is regular. Since $((a_2, b_2), (a_3, b_3))$ is an In-In alternating cycle in P, by Observation 64.(i) and (iii), σ satisfies (*L*12) and (*R*12). It remains to argue that σ is an alternating cycle in P, that is, $a_2 < b_3$ and $a_3 < t$ in P. The former follows from the fact that $((a_2, b_2), (a_3, b_3))$ is an alternating cycle in P. Therefore, to conclude the proof it suffices to show that $a_3 < t$ in P. Suppose to the contrary that $a_3 \parallel t$ in P.

By Corollary 68, $b_1, b_3 \in Y(a_2)$ and $t \in Y(a_1)$. Thus, we can fix $z_1, z_3 \in Z(a_2)$ and $z_t \in Z(a_1)$ such that $a_2 < z_1 \leqslant b_1$, $a_2 < z_3 \leqslant b_3$, and $a_1 < z_t \leqslant t$ in P.

Claim 96. b_1 is left of z_t, z_1 is left of z_t, z_t is left of b_2, and b_2 is left of z_3.

Proof. Since $(a_1, b_1) \in I$ and $a_1 < z_t \leqslant t$ in P, by Corollary 53.(*R*), b_1 is left of z_t. Since $(a_2, b_2) \in I$ and $a_2 < z_3 \leqslant b_3$ in P, by Corollary 53.(*R*), b_2 is left of z_3.

We have $(a_1, b_1) \in I$ and b_1 left of b_2, thus, by Corollary 50, $a_1 \notin \mathrm{shad}(b_2)$. Additionally, $a_1 \parallel b_2$ in P (by (iil3)) and t is left of b_2. Therefore, by Proposition 52.(*R*), z_t is left of b_2.

It remains to show that z_1 is left of z_t. Since $a_2 \parallel t$ and $z_t \leqslant t$ in P, we have $a_2 \parallel z_t$ in P. Since $(a_2, t) \in I$, we have $a_2 \notin \mathrm{shad}(t)$ by (I6). By Proposition 33, $\mathrm{shad}(z_t) \subseteq \mathrm{shad}(t)$, and

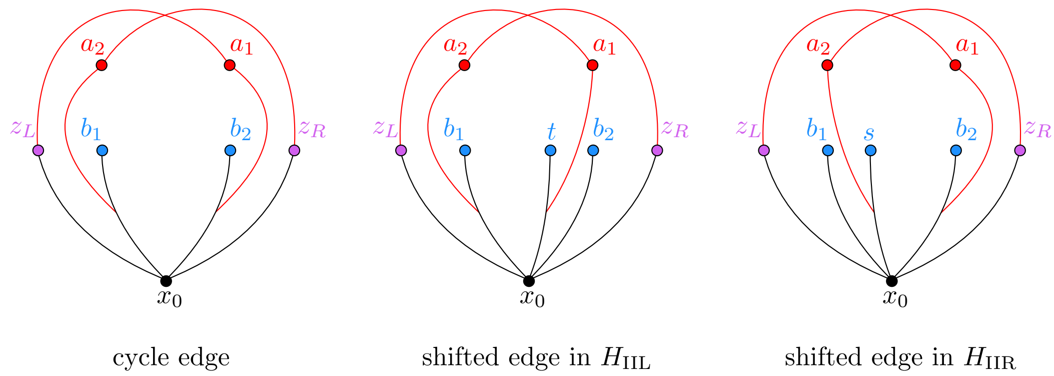


FIGURE 39. Edges in H_{IIL} and H_{IIR}. Note that in all the three cases $z_L = z_L(a_1) = z_L(a_2)$ and $z_R = z_R(a_1) = z_R(a_2)$ as shown in Proposition 92.

so, $a_2 \notin \operatorname{shad}(z_t)$. Moreover, $z_1, b_1 \in Y(a_2)$. Thus, Proposition 52.(*L*) gives that z_1 is left of z_t. ◁

Let W_1 be an exposed witnessing path from a_2 to z_1 in P and let W_3 be an exposed witnessing path from a_2 to z_3 in P. By Claim 96 we have z_1 is left of z_3, thus the region $\mathcal{R} = \mathcal{R}(a_2, z_1, z_3, W_1, W_3)$ is well defined.

Claim 97. $a_1 \in \operatorname{int} \mathcal{R}$.

Proof. Since z_1 is left of z_t and z_t is left of z_3 (by Claim 96) Corollary 61.(i) implies $z_t \in \operatorname{int} \mathcal{R}$. Suppose to the contrary that $a_1 \notin \operatorname{int} \mathcal{R}$ and let U be an exposed witnessing path from a_1 to z_t in P. Since $z_t \in \operatorname{int} \mathcal{R}$ and $a_1 \notin \operatorname{int} \mathcal{R}$, there exists an element u of U with $u \in \partial \mathcal{R}$. Since U is an exposed path, $u \notin B$. If u lies on the left side of $\mathcal{R}$, then $a_1 \leqslant u < z_1 \leqslant b_1$ in P, which is false. If u lies on the right side of $\mathcal{R}$, then $a_2 \leqslant u < z_t \leqslant t$ in P, which is also false. ◁

By (iil1), $((a_1, b_1), (a_2, t))$ is an edge in H_{II} and by assumption $((a_2, b_2), (a_3, b_3))$ is an edge in H_{II}. Hence, by Proposition 92, $\mathcal{R}(a_1) = \mathcal{R}(a_2) = \mathcal{R}(a_3)$. Let $z_0 = z_L(a_1) = z_L(a_2) = z_L(a_3)$.

Claim 98. $z_0 \notin \mathcal{R}$.

Proof. Note that $z_0 \not\leqslant z_1$ in P. Indeed, otherwise, $a_1 < z_L(a_1) = z_0 \leqslant z_1 \leqslant b_1$ in P. Since $z_0 = z_L(a_2)$, by Proposition 86.(*L*), $z_0 \notin \mathcal{R}$, as desired. ◁

Let U and V be exposed witnessing paths in P from a_1 to z_t and from a_1 to z_0, respectively. By Claims 97 and 98, $a_1 \in \operatorname{int} \mathcal{R}$ $z_0 \notin \mathcal{R}$, hence, V intersects $\partial \mathcal{R}$. Since V is exposed and $a_1 \parallel z_1$ in P, V intersects W_3. Let v be an element of this intersection. Note that $v \notin B$.

Claim 99. $z_3 \in Z(a_1)$.

Proof. The path $a_1[V]v[W_3]z_3$ is an exposed witnessing path in P, thus, it suffices to show that $a_1 \notin \operatorname{shad}(z_3)$. Observe that none of the elements of the paths $a_2[W_3]v$ and $a_1[V]v$ are in B. Therefore, these paths do not intersect $\partial \operatorname{shad}(z_3)$. In particular, either all a_1, v, and a_2 are in $\operatorname{shad}(z_3)$ or none of them. Since $z_3 \in Z(a_2)$, the latter holds. We conclude that $a_1 \notin \operatorname{shad}(z_3)$, and so, $z_3 \in Z(a_1)$. ◁

Since z_t is left of z_3 (by Claim 96), $z_t \in Z(a_1)$, and $z_3 \in Z(a_1)$ (by Claim 99), we can define the region
$$\mathcal{S} = \mathcal{R}(a_1, z_t, z_3, U, a_1[V]v[W_3]z_3).$$

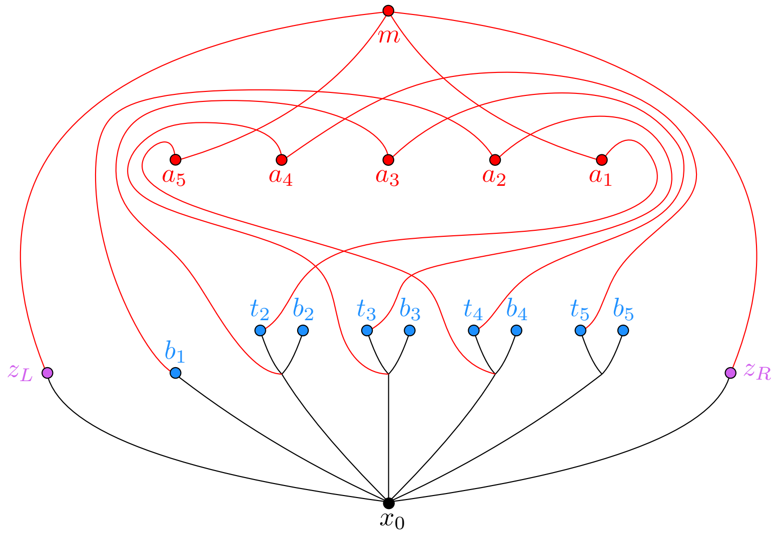


FIGURE 40. $((a_1, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_4), (a_5, b_5))$ is a path in H_{IIL}. All the edges are shifted edges. For every $i \in \{2, 3, 4, 5\}$, t_i is a witness for $((a_{i-1}, b_{i-1}), (a_i, b_i))$. Note that $\{(a_1, b_1), (a_2, t_2), (a_3, t_3), (a_4, t_4), (a_5, t_5)\}$ induces a standard example of order 5 in P.

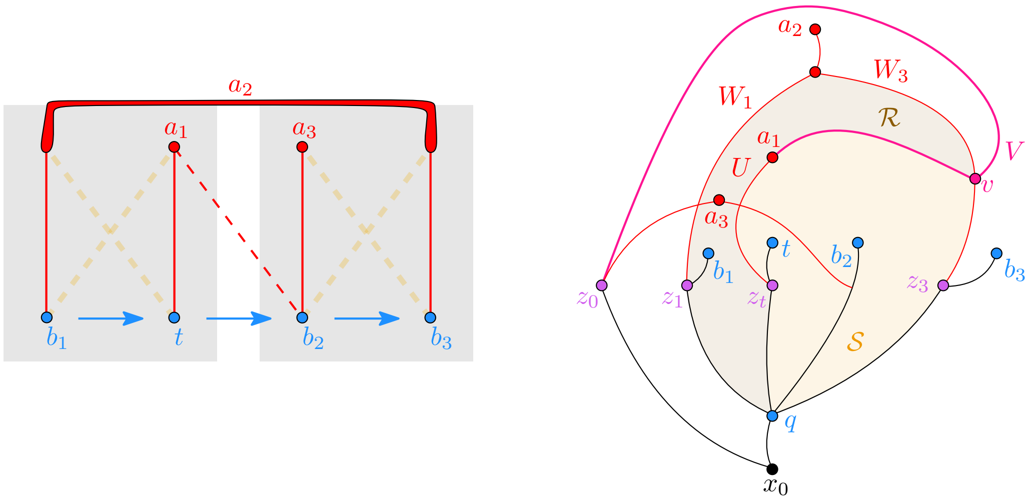


FIGURE 41. An illustration of the proof of Proposition 95. On the left-hand side, we show a schematic summary of the assumptions (drawing conventions are the same as in Figure 26). On the right-hand side, we illustrate the objects occurring in the proof of the proposition.

Since z_t is left of b_2 and b_2 is left of z_3 (by Claim 96), by Corollary 61.(i), $b_2 \in \operatorname{int} \mathcal{S}$. Since z_0 is left of z_t (by Claim 96 and Proposition 37), by Corollary 61.(ii), $z_0 \notin \mathcal{S}$. We consider two cases based on the location of a_3.

First, assume that $a_3 \in \operatorname{int} \mathcal{S}$.[7] Consider a witnessing path W from a_3 to z_0 in P. Since $a_3 \in \operatorname{int} \mathcal{S}$ and $z_0 \notin \mathcal{S}$, the path W has an element w in $\partial\mathcal{S}$. If w lies on the right side of $\mathcal{S}$, then $a_3 \leqslant w \leqslant z_3 \leqslant b_3$ in P, which is false. Therefore, w lies on the left side of $\mathcal{S}$, which implies $a_3 \leqslant w \leqslant z_t \leqslant t$ in P, as desired.

Finally, assume that $a_3 \notin \operatorname{int} \mathcal{S}$. Consider a witnessing path W from a_3 to b_2 in P. Since $a_3 \notin \operatorname{int} \mathcal{S}$ and $b_2 \in \operatorname{int} \mathcal{S}$, the path W has an element w in $\partial\mathcal{S}$. If w lies on the right side of $\mathcal{S}$, then $a_3 \leqslant w \leqslant z_3 \leqslant b_3$ in P, which is false. Therefore, w lies on the left side of $\mathcal{S}$, which implies $a_3 \leqslant w \leqslant z_t \leqslant t$ in P. This completes the proof. □

Corollary 100.

(*L*) *For all positive integers n and for each path $((a_1, b_1), \ldots, (a_n, b_n))$ in H_{IIL}, there exist $t_2, \ldots, t_n \in B$ such that $((a_1, b_1), (a_2, t_2), \ldots, (a_n, t_n))$ is a path in H_{II}.*

(*R*) *For all positive integers n and for each path $((a_1, b_1), \ldots, (a_n, b_n))$ in H_{IIR}, there exist $s_1, \ldots, s_{n-1} \in B$ such that $((a_1, s_1), \ldots, (a_{n-1}, s_{n-1}), (a_n, b_n))$ is a path in H_{II}.*

Proof. We only prove (*L*) as the proof of (*R*) is symmetric. We proceed by induction on n. When $n = 1$, the statement is vacuously satisfied. Suppose that $n \geqslant 2$ and let $((a_1, b_1), \ldots, (a_n, b_n))$ be a path in H_{IIL}. By induction, there exist $t_3, \ldots t_n$ such that $((a_2, b_2), (a_3, t_3), \ldots, (a_n, t_n))$ is a path in H_{II}. If $((a_1, b_1), (a_2, b_2))$ is a cycle edge in H_{IIL}, then it suffices to set $t_2 = b_2$. Thus, assume that $((a_1, b_1), (a_2, b_2))$ is a shifted edge in H_{IIL}, say witnessed by $t \in B$. We claim that the assertion is satisfied for $t_2 = t$. By (iil1), $((a_1, b_1), (a_2, t))$ is an edge in H_{II}. Hence, when $n = 2$, the statement holds. When $n \geqslant 3$, by Proposition 95.(*L*), $((a_2, t), (a_3, t_3))$ is an edge in H_{II}. This ends the proof. □

Proof of Propositions 72 and 73. Corollary 100 implies that

$$\text{max-path}(H_{\mathrm{IIL}}) \leqslant \text{max-path}(H_{\mathrm{II}}) \quad \text{and} \quad \text{max-path}(H_{\mathrm{IIR}}) \leqslant \text{max-path}(H_{\mathrm{II}}).$$

Proposition 94 states that $\text{max-path}(H_{\mathrm{II}}) \leqslant \text{se}_P(I)$, and so, we obtain

$$\text{max-path}(H_{\mathrm{IIL}}) \leqslant \text{se}_P(I) \quad \text{and} \quad \text{max-path}(H_{\mathrm{IIR}}) \leqslant \text{se}_P(I).$$ □

In the last part of this subsection, we study H_{IILR}. In Figure 42, we show an example of a path in H_{IILR}. Also, recall a schematic drawing of an edge in H_{IILR} in Figure 27. We aim to prove Proposition 74, namely, that $\text{max-path}(H_{\mathrm{IILR}}) \leqslant \text{se}_P(I)$.

Proposition 101. *Let $\sigma_{12} = ((a_1, b_1), (a_2, b_2))$ and $\sigma_{23} = ((a_2, b_2), (a_3, b_3))$ be two edges in H_{IILR} witnessed by (s_1, t_2) and (s_2, t_3) respectively. Then, $\sigma = ((a_1, s_1), (a_2, s_2))$ is an edge in H_{II}.*

Proof. See Figure 43 for an illustration of the proof. By (iilr1), $(a_1, s_1) \in I$ and $(a_2, s_2) \in I$. By Observation 64.(i) and (iii), $((a_1, s_1), (a_2, s_2))$ satisfies (*L*12) and (*R*12). By (iilr1) and (iilr2), for each $i \in [2]$,

$$b_i \text{ is left of } s_i,\ s_i \text{ is left of } t_{i+1}, \text{ and } t_{i+1} \text{ is left of } b_{i+1}.$$

In particular, s_1 is left of s_2. By (iilr1), $a_2 < s_1$ in P. Therefore, it suffices to show that $a_1 < s_2$ in P.

[7] In fact, with more careful analysis, one can show that $a_3 \in \operatorname{int} \mathcal{S}$ leads to a contradiction.

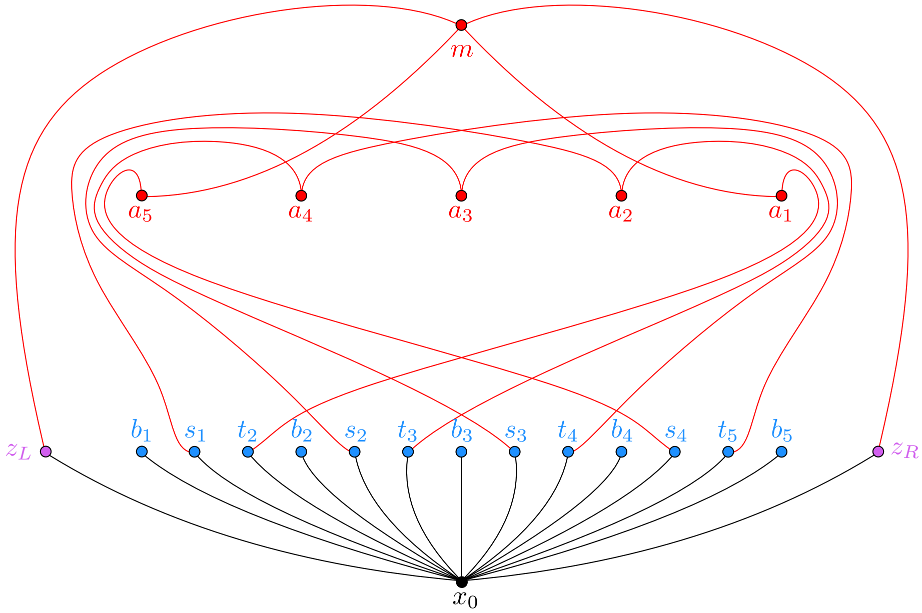


FIGURE 42. $((a_1, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_4), (a_5, b_5))$ is a path in H_{IILR}. For every $i \in [4]$, (s_i, t_{i+1}) is a witness for $((a_i, b_i), (a_{i+1}, b_{i+1}))$. Note that $\{(a_1, s_1), (a_2, s_2), (a_3, s_3), (a_4, s_4)\}$ induces a standard example of order 4 in P.

Let $i \in [2]$. By Corollary 68, $t_{i+1} \in Y(a_i)$ and $s_i \in Y(a_{i+1})$. Thus, we can fix $u_{i+1} \in Z(a_{i+1})$ and $v_i \in Z(a_i)$ such that $u_{i+1} \leqslant s_i$ and $v_i \leqslant t_{i+1}$ in P. Additionally, let $u_1 = z_L(a_1)$ and $v_3 = z_R(a_3)$. For every $i \in [3]$, let U_i and V_i be exposed witnessing paths in P from a_i to u_i and from a_i to v_i respectively. By Proposition 57,

$$u_1 \text{ is left of } b_1 \text{ and } b_3 \text{ is left of } v_3. \tag{19}$$

Claim 102. b_i is left of u_{i+1}, u_{i+1} is left of v_i, and v_i is left of b_{i+1}, for every $i \in [2]$.

Proof. Let $i \in [2]$. Since $(a_i, b_i), (a_{i+1}, b_{i+1}) \in I$ and b_i is left b_{i+1}, by Corollary 50,

$$a_i \notin \operatorname{shad}(b_{i+1}) \text{ and } a_{i+1} \notin \operatorname{shad}(b_i).$$

Since $a_{i+1} \parallel b_i$ in P (by (iilr3)), $u_{i+1}, s_i \in Y(a_{i+1})$, $u_{i+1} \leqslant s_i$ in P, and b_i is left of s_i, by Proposition 52.(*R*), b_i is left of u_{i+1}. Since $a_i \parallel b_{i+1}$ in P (by (iilr3)), $v_i, t_{i+1} \in Y(a_i)$, $v_i \leqslant t_{i+1}$ in P, and t_{i+1} is left of b_{i+1}, by Proposition 52.(*L*), v_i is left of b_{i+1}.

It remains to show that u_{i+1} is left of v_i. Since $(a_{i+1}, t_{i+1}) \in I$, $a_{i+1} < u_{i+1} \leqslant s_i$ in P, and s_i is left of t_{i+1} by Corollary 53.(*L*), u_{i+1} is left of t_{i+1}. Since $a_i \parallel s_i$ and $u_{i+1} \leqslant s_i$ in P, we have $a_i \parallel u_{i+1}$ in P. Since $(a_i, s_i) \in I$, we have $a_i \notin \operatorname{shad}(s_i)$. By Proposition 33, $\operatorname{shad}(u_{i+1}) \subseteq \operatorname{shad}(s_i)$, and so, $a_i \notin \operatorname{shad}(u_{i+1})$. Moreover, $v_i, t_{i+1} \in Y(a_i)$. It follows that we can apply Proposition 52.(*R*) to obtain that u_{i+1} is left of v_i. $\lhd$

Since u_1 is left of b_1 (by (19)) and b_1 is left of v_1 (by Claim 102), we obtain that u_1 is left of v_1. Moreover, by Claim 102, u_2 is left of v_2. Thus, for each $i \in [2]$ we can define regions

$$\mathcal{S}_i = \mathcal{R}(a_i, u_i, v_i, U_i, V_i).$$

Claim 103. $a_{i+1} \in \operatorname{int} \mathcal{S}_i$ and $U_{i+1} \subseteq \operatorname{int} \mathcal{S}_i$ for every $i \in [2]$.

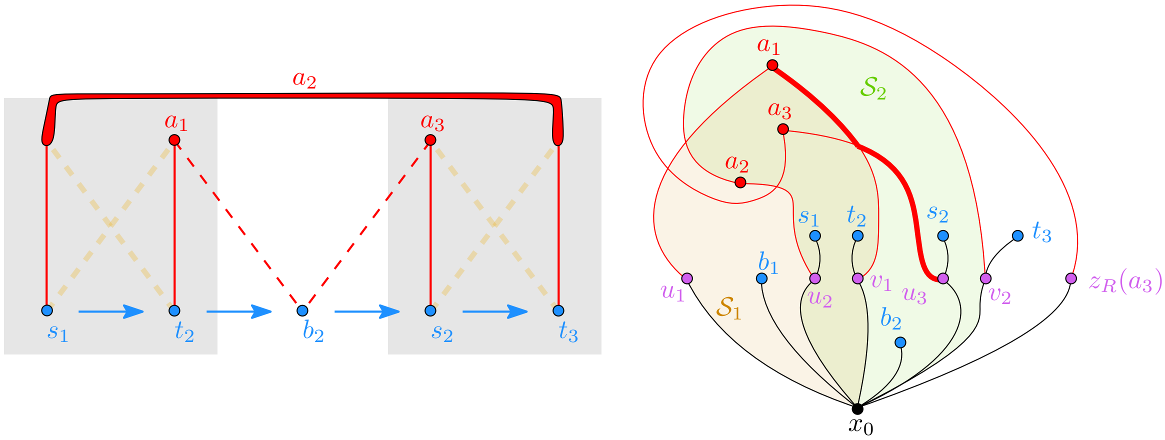


FIGURE 43. An illustration of the proof of Proposition 101. On the left-hand side, we show a schematic summary of the assumptions (drawing conventions are the same as in Figure 26). On the right-hand side, we illustrate the objects occurring in the proof of the proposition.

Proof. Let $i \in [2]$. By (19) and Claim 102, u_i is left of u_{i+1} and u_{i+1} is left of v_i. Thus, by Corollary 61.(i), $u_{i+1} \in \operatorname{int} \mathcal{S}_i$. Note that to prove the claim it suffices to show that $U_{i+1} \subseteq \operatorname{int} \mathcal{S}_i$. Since $u_{i+1} \in \operatorname{int} \mathcal{S}_i$, it is enough to show that U_{i+1} is disjoint from $\partial \mathcal{S}_i$. Suppose to the contrary that w is an element in the intersection of U_{i+1} and $\partial \mathcal{S}_i$. Since U_{i+1} is an exposed path, we have $w \notin B$, which implies that w lies in $U_i \cup V_i$. This yields $a_i \leqslant w < u_{i+1} \leqslant s_i$ in P, which is false. $\lhd$

Claim 104. $a_3 \in \operatorname{int} \mathcal{S}_1$.

Proof. Note that v_2 is left of b_3 (by Claim 102), and b_3 is left of v_3 (by (19)). Hence, v_2 is left of v_3 (by Proposition 37), and so by Corollary 61.(ii), $v_3 \notin \mathcal{S}_2$. We have $a_3 \in \operatorname{int} \mathcal{S}_2$ (by Claim 103) and $v_3 \notin \mathcal{S}_2$, hence, V_3 must intersect $\partial \mathcal{S}_2$, say in w. Since V_3 is exposed, w lies either in U_2 or in V_2. The latter is not possible as otherwise, $a_3 \leqslant w < v_2 \leqslant t_3$ in P. Thus, w lies in U_2. By Claim 103, $U_2 \subseteq \operatorname{int} \mathcal{S}_1$, hence, $w \in \operatorname{int} \mathcal{S}_1$. We argue that $a_3[V_3]w \subseteq \operatorname{int} \mathcal{S}_1$, which suffices to conclude the proof of the claim. Otherwise, there exists w' in $a_3[V_3]w$ such that $w' \in \partial \mathcal{S}_1$. Moreover, $w' \notin B$ as V_3 is exposed. It follows that $a_1 \leqslant w' \leqslant w < u_2 \leqslant s_1$ in P, which is a contradiction. $\lhd$

Recall that we needed to prove that $a_1 < s_2$ in P. Since v_1 is left of u_3 (by Claim 102), by Corollary 61.(ii), $u_3 \notin \mathcal{S}_1$. However, by Claim 104, $a_3 \in \operatorname{int} \mathcal{S}_1$, hence, U_3 intersects $\partial \mathcal{S}_1$, say in w. Since U_3 is exposed, w lies in $U_1 \cup V_1$. In particular, $a_1 \leqslant w < u_3 \leqslant s_2$ in P. $\square$

Proof of Proposition 74. If $\text{max-path}(H_{\text{IILR}}) \leqslant 1$, then the assertion is clear. Thus, we can assume that $\text{max-path}(H_{\text{IILR}}) \geqslant 2$. Let $((a_1, b_1), \ldots, (a_n, b_n))$ be a path in H_{IILR} where n is an integer with $n \geqslant 2$. For each $i \in [n-1]$, let (s_i, t_{i+1}) with $s_i, t_{i+1} \in B$ be a witness for the edge $((a_i, b_i), (a_{i+1}, b_{i+1}))$ in H_{IILR}. By Proposition 101, for each $i \in [n-2]$, $((a_i, s_i), (a_{i+1}, s_{i+1}))$ is an edge in H_{II}. Moreover, $((a_{n-1}, s_{n-1}), (a_n, t_n))$ is an edge in H_{II} by (iilr1). Therefore, $((a_1, s_1), \ldots, (a_{n-1}, s_{n-1}), (a_n, t_n))$ is a path in H_{II}, and so, by Proposition 94, $\text{max-path}(H_{\text{IILR}}) \leqslant \text{max-path}(H_{\text{II}}) \leqslant \text{se}_P(I)$, which completes the proof. $\square$

8.4. **In-Out and Out-In oriented graphs.** In this subsection, we prove Lemma 75, which concerns H_{IO} and H_{OI}. We restate the lemma below. Note that we give a proof only in the

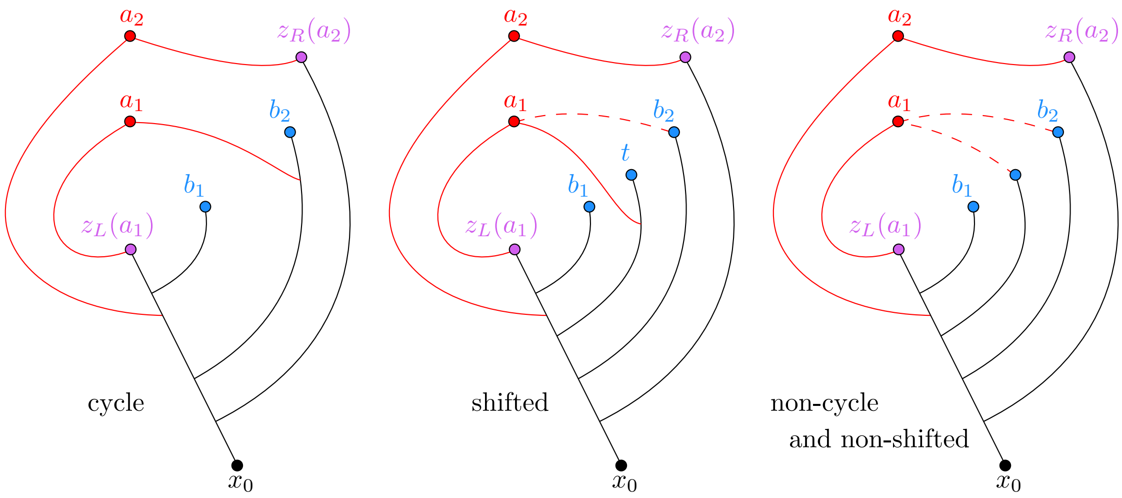


FIGURE 44. $((a_1, b_1), (a_2, b_2))$ are edges in H_{IO}.

case of H_{IO}, and the proof for H_{OI} is symmetric. In Figure 28, we gave a schematic drawing of edges in H_{IO} See Figure 44 for a more precise illustration. In Figure 45, we give an example of a path in H_{IO}.

Lemma 75. *Let $m = 2\,\mathrm{se}_P(I) \cdot (2\,\mathrm{se}_P(I) + 6)$.*

(i) *If $((a, b), (a', b'))$ is an edge of weight 1 in H_{IO}, then*
$$\text{max-start-weight}(H_{\mathrm{IO}}, (a, b)) \not\equiv \text{max-start-weight}(H_{\mathrm{IO}}, (a', b')) \bmod m.$$
(ii) *If $((a, b), (a', b'))$ is an edge of weight 1 in H_{OI}, then*
$$\text{max-end-weight}(H_{\mathrm{OI}}, (a, b)) \not\equiv \text{max-end-weight}(H_{\mathrm{OI}}, (a', b')) \bmod m.$$

Proof. We only prove the statement for H_{IO} (i.e. (i)), the proof for H_{OI} (i.e. (ii)) is symmetric. Let $((a, b), (a', b'))$ be an edge of weight 1 in H_{IO}. Clearly,
$$\text{max-start-weight}(H_{\mathrm{IO}}, (a, b)) > \text{max-start-weight}(H_{\mathrm{IO}}, (a, b)).$$
It suffices to show that
$$\text{max-start-weight}(H_{\mathrm{IO}}, (a', b')) > \text{max-start-weight}(H_{\mathrm{IO}}, (a, b)) - m. \tag{20}$$
See Figure 46 for a high-level idea of the proof.

If max-start-weight$(H_{\mathrm{IO}}, (a, b)) \leqslant m - 1$, then the assertion is clear. Thus, assume that max-start-weight$(H_{\mathrm{IO}}, (a, b)) \geqslant m$. Let
$$((a_1, b_1), \ldots, (a_n, b_n)) \text{ be a path in } H_{\mathrm{IO}}$$
witnessing max-start-weight$(H_{\mathrm{IO}}, (a, b))$, that is, $(a_1, b_1) = (a, b)$ and the number of edges of weight one in this path is max-start-weight$(H_{\mathrm{IO}}, (a, b))$. Note that n can be arbitrarily large. For each $i \in [n - 1]$, let $\sigma_i = ((a_i, b_i), (a_{i+1}, b_{i+1}))$ and if σ_i is a shifted edge in H_{IO}, then let $t_{i+1} \in B$ be a witness for σ_i.

For convenience, we define
$$W^\star = W_L(z_L(a_1)).$$
For every $d \in B$, define
$$\pi(d) = \mathrm{gcpe}(W^\star, W_L(d)).$$

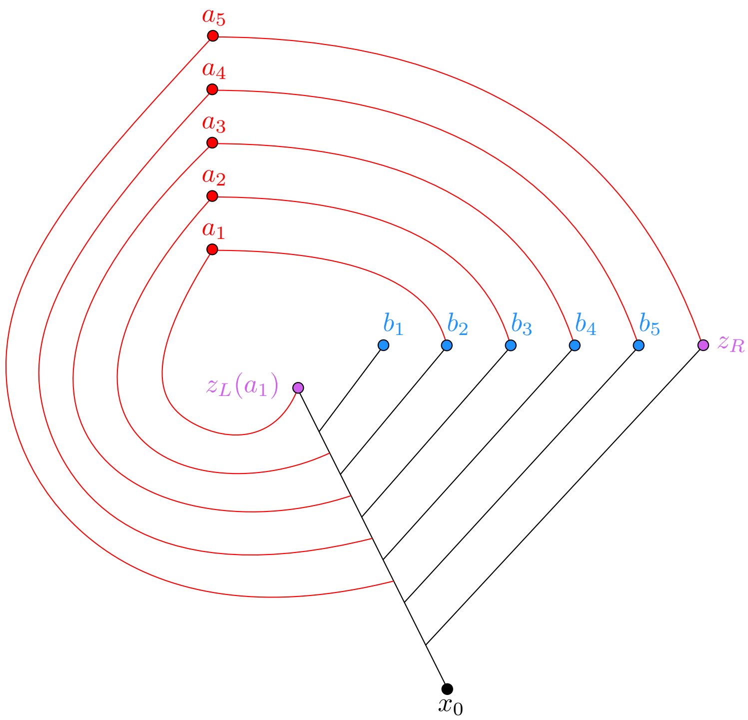


FIGURE 45. $((a_1, b_1), (a_2, b_2), (a_3, b_3), (a_4, b_4), a_5, b_5))$ is a path in H_{IO}. Note that in this poset there is no S_3.

Note that the element $\pi(d)$ has an equivalent description when d is right of $z_L(a_1)$, see Claim 117.

We inductively construct elements $u_1, \ldots, u_{n-1}$ and witnessing paths $U_1, \ldots, U_{n-1}$ in P such that for every $i \in [n-1]$,

(u1) $u_1 = z_L(a_1)$,
(u2) $u_i \in Z(a_i)$,
(u3) u_i lies in $W^\star$,
(u4) u_i is left of b_i,
(u5) $u_i \leqslant \pi(b_{i-1})$ in P if $i > 1$,
(u6) U_i is an exposed witnessing path from a_i to u_i in P,
(u7) $x_0[W^\star]u_i[U_i]a_i$ is left of $W^\star$ if $i > 1$.

See Figure 47. Let $i \in [n-1]$. Since u_i and $\pi(b_i)$ both lie in $W^\star$ (by (u1) and (u3)), the elements u_i and $\pi(b_i)$ are comparable in P. However, $u_i \leqslant \pi(b_i)$ in P leads to a contradiction: $a_i < u_i \leqslant \pi(b_i) \leqslant b_i$ in P (the first comparability follows from (u2)). Thus, (u1)–(u3) imply that for every $i \in [n-1]$,

(u8) $\pi(b_i) < u_i$ in P.

We proceed with the construction of $u_1, \ldots, u_{n-1}$ and $U_1, \ldots, U_{n-1}$. First, let $u_1 = z_L(a_1)$ and let U_1 be an exposed witnessing path from a_1 to u_1 in P. Items (u1), (u2), (u3), and (u6) are clearly satisfied for $i = 1$ and items (u5) and (u7) are vacuously true for $i = 1$. Since $(a_1, b_1) \in I$, by Proposition 57, $u_1 = z_L(a_1)$ is left of b_1, so (u4) also holds.

Let $i \in [n-1]$ with $i > 1$ and assume that $u_1, \ldots, u_{i-1}, U_1, \ldots, U_{i-1}$ are already defined and satisfy all the items. Let $u_i \in Z(a_i)$ and let U_i be an exposed witnessing path from a_i to u_i

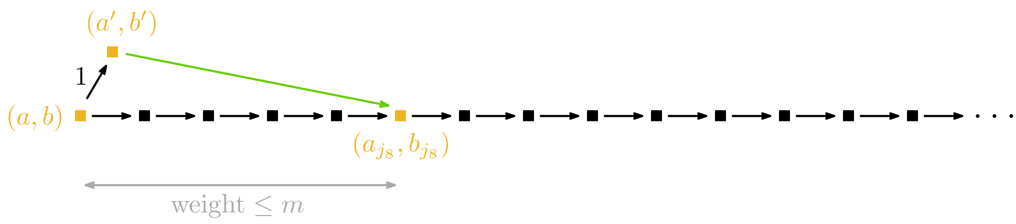


FIGURE 46. We depict a high-level idea of the proof of Lemma 75. The arrows indicate edges in H_{IO}. We fix a heavy path starting in (a,b). The main goal is to find a pair on this path that is "quite close in the path" to (a,b), and there is an edge (in green) from (a',b') to this pair. In the figure, this pair is called (a_{j_8},b_{j_8}), which is the notation following the proof.

in P such that (u_i,U_i) witnesses $(L21^\star)$ for σ_{i-1}. Immediately from this definition items (u2) and (u6) hold.

Since $u_{i-1} \in Z(a_{i-1})$ (by (u2)), u_{i-1} is left of b_{i-1} (by (u4)), and σ_{i-1} satisfies $(L21^\star)$, Proposition 66.(L) implies that u_i lies in $W_L(u_{i-1})$. Since u_{i-1} lies in $W^\star$ (by (u3)), $W_L(u_{i-1})$ is a subpath of $W^\star$. In particular, u_i lies in $W^\star$ so (u3) holds for i. Since $(a_i,b_i) \in I$, $a_i \leqslant u_i \leqslant b_{i-1}$ in P, and b_{i-1} is left of b_i, by Corollary 53.(L), we obtain that u_i is left of b_i, thus, (u4) holds for i.

Recall that (u_i,U_i) witnesses $(L21^\star)$ for σ_{i-1}. Since u_i lies in $W^\star$ (by (u3)) and u_i lies in $W_L(b_{i-1})$, we obtain that $u_i \leqslant \pi(b_{i-1})$ in P, and so, (u5) holds for i. The element u_i also lies in $W_L(z_L(a_{i-1}))$. Thus, $x_0[W_L(z_L(a_{i-1}))]u_i = x_0[W^\star]u_i$. Finally, $M = x_0[W_L(z_L(a_{i-1}))]u_i[U_i]a_i = x_0[W^\star]u_i[U_i]a_i$ is left of $W_L(z_L(a_{i-1}))$ (again by $(L21^\star)$). Item (u7) is equivalent to saying that M is left of $W^\star$. Thus, we obtained (u7) for i when $i=2$. Next, we assume that $i>2$. Since $u_{i-1} \in Z(a_{i-1})$ (by (u2)), $W_L(z_L(a_{i-1}))$ is either left of $M' = x_0[W_L(u_{i-1})]u_{i-1}[U_{i-1}]a_{i-1} = x_0[W^\star]u_{i-1}[U_{i-1}]a_{i-1}$ or is a subpath of M'. If $W_L(z_L(a_{i-1}))$ is left of M', then by transitivity, M is left of M'. Similarly, if $W_L(z_L(a_{i-1}))$ is a subpath of M', then M is left of M' as well. Item (u7) for i follows now from the fact that by (u7) for $i-1$ (recall $i>2$), M' is left of $W^\star$. This completes the construction.

Claim 105. For all $i,j \in [n-1]$ with $i<j$,

(i) $u_j < u_i$ in P,
(ii) $W_L(u_j)$ is a proper subpath of $W_L(u_i)$,
(iii) $a_i \parallel u_j$ in P.

Proof. Let $i,j \in [n-1]$ with $i<j$. By (u5) and (u8), $u_j \leqslant \pi(b_{j-1}) < u_{j-1}$ in P, thus, (i) follows from a simple induction. Item (ii) follows from the fact that both u_i and u_j lie in $W^\star$ (by (u3)) and (i). For the proof of (iii) suppose to the contrary that $a_i < u_j$ or $u_j \leqslant a_i$ in P. Note that the latter can not hold as $a_i \notin B$ and $u_j \in B$. Thus, we assume $a_i < u_j$ in P. Then, $a_i < u_j \leqslant u_{i+1} \leqslant \pi(b_i) \leqslant b_i$ in P, where the second inequality follows from (i) and the third inequality follows from (u8). Clearly, $a_i < b_i$ in P is a contradiction, which completes the proof. $\triangleleft$

The next claim helps us to define a handful of useful regions.

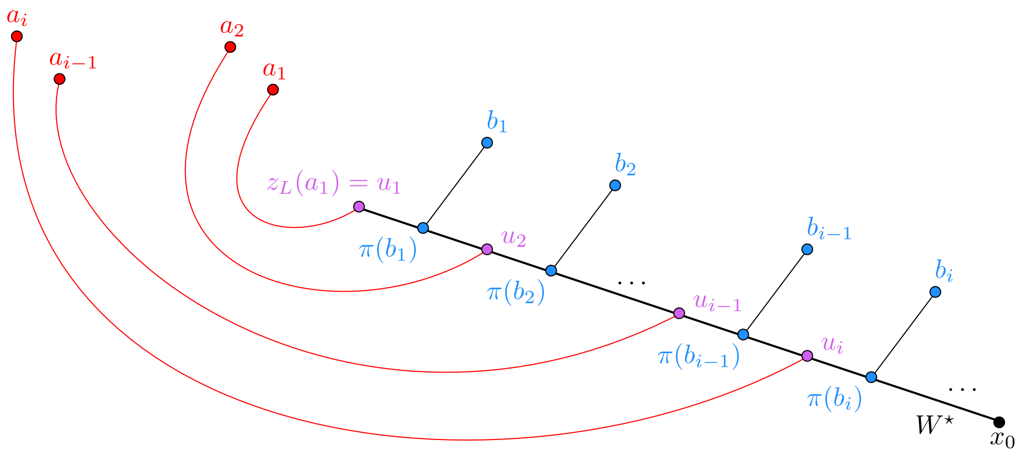


FIGURE 47. An illustration of items (u1)–(u8).

Claim 106. For all $i \in [n-1]$,

(i) $a_i \notin \mathrm{shad}(b_{i+1})$,
(ii) $b_{i+1} \in Y(a_i)$ if σ_i is a cycle edge,
(iii) $t_{i+1} \in Y(a_i)$ if σ_i is a shifted edge.

Proof. Let $i \in [n-1]$. Recall that $(a_i, b_i) \in I$. The element b_i is left of b_{i+1} and when σ_i is a shifted edge, b_i is left of t_{i+1} (by (io1)). Therefore, by Corollary 50, $a_i \notin \mathrm{shad}(b_{i+1})$ and when σ_i is a shifted edge, $a_i \notin \mathrm{shad}(t_{i+1})$. Items (ii) and (iii) follow immediately since $a_i < b_{i+1}$ in P if σ_i is a cycle edge and $a_i < t_{i+1}$ in P if σ_i is a shifted edge. $\triangleleft$

Let
$$E = \{i \in [n-1] : \ \sigma_i \text{ is of weight } 1\}.$$
Claim 106 allows for the following definition. For each $i \in E$, let v_i be an element of P such that
$$v_i \in Z(a_i) \text{ and } \begin{cases} v_i \leqslant b_{i+1} \text{ in } P & \text{if } \sigma_i \text{ is a cycle edge,} \\ v_i \leqslant t_{i+1} \text{ in } P & \text{if } \sigma_i \text{ is a shifted edge.} \end{cases}$$
Additionally, for every $i \in E$, let V_i be an exposed witnessing path from a_i to v_i in P. See an illustration in Figure 48.

Claim 107. For all $i \in E$,

(i) b_i is left of v_i,
(ii) u_i is left of v_i,
(iii) v_i is left of b_{i+1} if σ_i is a shifted edge,
(iv) $a_{i+1} \parallel v_i$ in P and $a_{i+1} \notin \mathrm{shad}(v_i)$.

Proof. Let $i \in E$. If σ_i is a cycle edge, then since $(a_i, b_i) \in I$, $a_i < v_i \leqslant b_{i+1}$ in P, and b_i is left of b_{i+1}, by Corollary 53.(R), b_i is left of v_i. Assume that σ_i is a shifted edge. Since $(a_i, b_i) \in I$, $a_i < v_i \leqslant t_{i+1}$ in P, and b_i is left of t_{i+1} (by (io1)), by Corollary 53.(R), b_i is left of v_i. This completes the proof of (i), and also (ii) since u_i is left of b_i by (u4).

For the proof of (iii), assume that σ_i is a shifted edge. By (io3), $a_i \parallel b_{i+1}$ in P and by Claim 106.(i), $a_i \notin \mathrm{shad}(b_{i+1})$. By Claim 106.(iii), $v_i, t_{i+1} \in Y(a_i)$. Altogether, by Proposition 52.(L), v_i is left of b_{i+1}, which implies (iii).

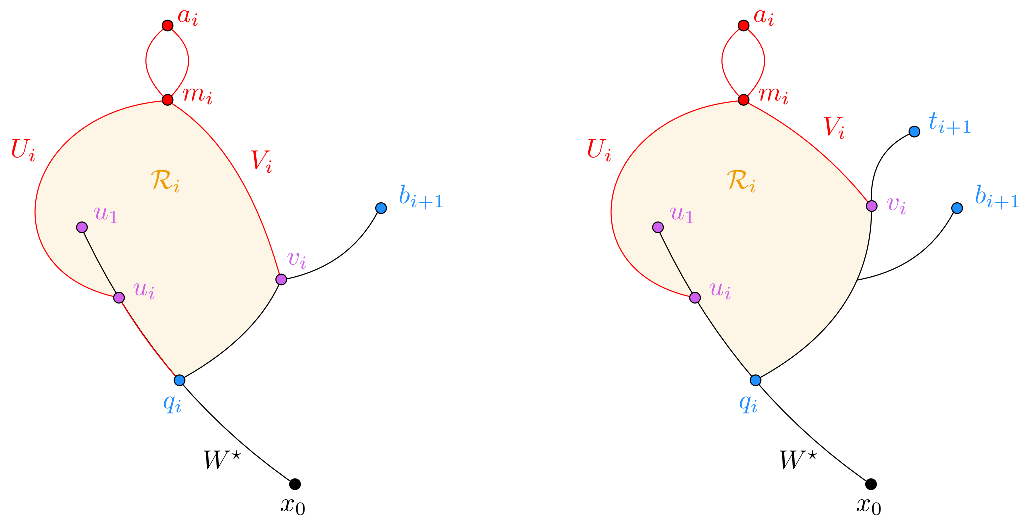


FIGURE 48. The region $\mathcal{R}_i$: on the left when σ_i is a cycle edge, and on the right when σ_i is a shifted edge. We prove in Claim 109 that $u_i[W^\star]u_1$ indeed lies in $\mathcal{R}_i$ as depicted.

Let $d = b_{i+1}$ when σ_i is a cycle edge and $d = t_{i+1}$ when σ_i is a shifted edge. Note that $(a_{i+1}, d) \in I$ (by (io1)) and $v_i \leqslant d$ in P. In particular, $a_{i+1} \parallel d$ in P and $a_{i+1} \notin \text{shad}(d)$ (by (I6)). Since $a_{i+1} \notin B$ and $v_i \in B$, we have $v_i \nleqslant a_{i+1}$ in P. Since $v_i \leqslant d$ in P, we also have $a_{i+1} \nleqslant v_i$ in P. Therefore, $a_{i+1} \parallel v_i$ in P. Since $\text{shad}(v_i) \subseteq \text{shad}(d)$ (by Proposition 33), we also obtain $a_{i+1} \notin \text{shad}(v_i)$. This completes the proof of (iv). ◁

The following is a simple corollary of Claim 107.

Claim 108. For all $i \in [n-1]$ and $j \in E$ with $i \leqslant j$,

(i) u_i is left of b_j,
(ii) b_i is left of v_j,
(iii) u_i is left of v_j.

Proof. Let $i \in [n-1]$ and $j \in E$ with $i \leqslant j$. By (u4), u_i is left of b_i and by Claim 107.(i), b_j is left of v_j. Additionally, b_i is left of or is equal to b_j. Now, the transitivity of the "left of" relation (Proposition 37) gives all the items. ◁

By Claim 107.(ii), u_i is left of v_i for every $i \in E$. Moreover, by (u2) and the definition of v_i, we have $u_i, v_i \in Z(a_i)$. Therefore, for every $i \in E$, we can define the region

$$\mathcal{R}_i = \mathcal{R}(a_i, u_i, v_i, U_i, V_i).$$

Next, for each $i \in E$, let q_i and m_i be the lower-min and the upper-min of $\mathcal{R}_i$ respectively, and let $\gamma_{L,i} = x_0[W_L(u_i)]u_i[U_i]m_i$ and $\gamma_{R,i} = x_0[W_R(v_i)]v_i[V_i]m_i$. See Figure 48 again.

Claim 109. For all $j \in E$, we have $u_j[W^\star]u_1 \subseteq \text{int}\,\mathcal{R}_j \cup \{u_j\}$. In particular, for all $i \in [n-1]$ and $j \in E$ with $i < j$, we have $u_i \in \text{int}\,\mathcal{R}_j$.

Proof. Let $j \in E$. Note that $q_j < u_j < u_1$ in P (by Claim 105.(i)). Since $u_j \in \mathcal{R}_j$, by Proposition 58, $u_j \notin \text{shad}(q_j)$. In particular, $u_j < u_1$ and $u_j \nleqslant q_j$ in P, thus, by Observation 22, we have $u_1 \notin \text{shad}(q_j)$. We argue that $\gamma_{L,j}$ is left of $W^\star = W_L(u_1)$ and $W_R(u_1)$ is left of $\gamma_{R,j}$. This will show that $u_1 \in \text{int}\,\mathcal{R}_j$ by Proposition 60.(ii). The path $\gamma_{L,j}$ is left of $W^\star$ by (u7).

By Claim 108.(iii), u_1 is left of v_j, thus, $W_R(u_1)$ is left of $W_R(v_j)$, and so, $W_R(u_1)$ is left of $\gamma_{R,j}$. As noted before, we obtain that $u_1 \in \operatorname{int} \mathcal{R}_j$. Consider the path $u_j[W^\star]u_1$. This path is disjoint from $q_j[\gamma_{R,j}]v_j$ as $u_j \parallel v_j$ in P (by Claim 108.(iii)). Moreover, this path does not intersect $q_j[\gamma_{L,j}]u_j$ in any element other than u_j as this would yield a directed cycle in P. Since all elements of $u_j[W^\star]u_1$ are in B, it follows that $W \subseteq \operatorname{int} \mathcal{R}_j \cup \{u_j\}$, as desired. The "in particular" statement follows since for all $i \in [n-1]$ with $i < j$, the element u_i lies in $u_j[W^\star]u_1$ and $u_i \neq u_j$ since u_i lies in $W^\star$ (by (u3)) and $u_j < u_i$ in P (by Claim 105.(i)). $\lhd$

In the next claim, we show that for all $i, j \in [n-1]$ with $i < j$, if σ_i is of weight 1, then either $v_j \in \operatorname{int} \mathcal{R}_i$ or $v_j \notin \mathcal{R}_i$, in other words, we show that $v_j \notin \partial \mathcal{R}_i$. Later, we use this to define a 2-coloring of such pairs of indices. This coloring will enable us to break down the problem into two subproblems with stronger assumptions.

Claim 110. For all $i, j \in E$ with $i < j$,

(i) $v_j \not\leqslant v_i$ in P;
(ii) if σ_i is a shifted edge, then $v_j \notin \mathcal{R}_i$;
(iii) if σ_i is a cycle edge, then $v_j \notin \partial \mathcal{R}_i$;
(iv) if σ_i is a cycle edge and $v_j \in \operatorname{int} \mathcal{R}_i$, then v_i lies in $W_R(v_j)$;
(v) if σ_i is a cycle edge and $v_j \in \operatorname{int} \mathcal{R}_i$, then v_i lies in $W_R(b_{i+1})$.

Proof. See Figure 49 for an illustration. Let $i, j \in E$ with $i < j$. Suppose to the contrary that $v_j \leqslant v_i$ in P. Let $d_{i+1} = b_{i+1}$ when σ_i is a cycle edge and $d_{i+1} = t_{i+1}$ when σ_i is a shifted edge. If $i + 1 = j$ then $a_{i+1} < v_{i+1} \leqslant v_i \leqslant d_{i+1}$ in P, which is a clear contradiction. Thus, we assume $i + 1 < j$. Now we have that d_{i+1} is left of b_j (by (io2) when $d_{i+1} = t_{i+1}$). Since $v_j \leqslant v_i$ in P, we have, $v_j \in \operatorname{shad}(v_i) \subseteq \operatorname{shad}(d_{i+1})$ (by Proposition 33). By Claim 107.(i), b_j is left of v_j, thus, $v_j \notin \operatorname{shad}(b_j)$. Therefore, we can apply Proposition 36.(*L*), to obtain that v_j is left of b_j, which is again a contradiction that completes the proof of (i).

For the proof of (ii) observe that if σ_i is a shifted edge, then v_i is left of b_{i+1} (by Claim 107.(iii)), b_{i+1} is either left of b_j or equal to b_j, and b_j is left of v_j (by Claim 107.(i)), hence, v_i is left of v_j. Therefore, by Corollary 61.(ii), $v_j \notin \mathcal{R}_i$.

From now on, assume that σ_i is a cycle edge. All elements of $\partial \mathcal{R}_i$ that are in B are in $W_L(u_i) \cup W_R(v_i)$. Since $u_i \parallel v_j$ in P (by Claim 108.(iii)), v_j does not lie in $W_L(u_i)$. By (i), v_j does not lie in $W_R(v_i)$. This completes the proof of (iii).

From now on, assume also that $v_j \in \operatorname{int} \mathcal{R}_i$. Let $w = \operatorname{gcpe}(W_R(v_j), \gamma_{R,i})$. In particular, w lies in $W_R(v_i)$. If $w = v_i$, then the assertion holds. Thus, suppose to the contrary that $w < v_i$ in P. Since $v_j \in \operatorname{int} \mathcal{R}_i$, by Proposition 60.(ii), $W_R(v_j)$ is left of $\gamma_{R,i}$, and so, of $W_R(v_i)$. Let W be a witnessing path from v_i to b_{i+1} in P. Note that $x_0[W_R(v_i)]v_i[W]b_{i+1}$ is either left of or equal to $W_R(b_{i+1})$. It follows that $W_R(v_j)$ is left of $W_R(b_{i+1})$. However, b_{i+1} is left of v_j by Claim 108.(ii), thus, we obtained a contradiction. This proves (iv).

The path $x_0[W_R(v_i)]v_i[W]b_{i+1}$ is either left of or equal to $W_R(b_{i+1})$. Observe that either v_i lies in $W_R(b_{i+1})$ or $W_R(v_i)$ is left of $W_R(b_{i+1})$. Thus, suppose the latter. Recall that b_{i+1} is left of v_j, and in particular, $W_R(b_{i+1})$ is left of $W_R(v_j)$. We obtain that $W_R(v_i)$ is left of $W_R(v_j)$, which is false by (iv). This completes the proof of (v). $\lhd$

Recall that $|E| \geqslant m$ and let E_0 be the m least integers of E. Let $i, j \in E_0$ with $i < j$. By Claim 110, either $v_j \in \operatorname{int} \mathcal{R}_i$ or $v_j \notin \mathcal{R}_i$. See Figure 49 again. Let

$$\phi(i,j) = \begin{cases} \text{in} & \text{if } v_j \in \operatorname{int} \mathcal{R}_i, \\ \text{out} & \text{if } v_j \notin \mathcal{R}_i. \end{cases}$$

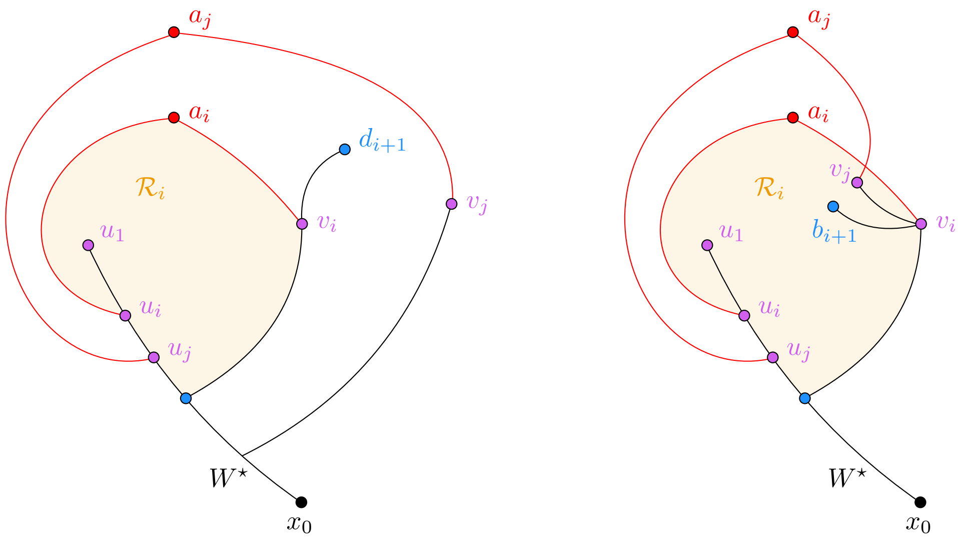


FIGURE 49. An illustration of the statement of Claim 110. Note that both cases $v_j \notin \mathcal{R}_i$ and $v_j \in \mathcal{R}_i$ are possible. However, the latter is possible only in the case of a cycle edge.

The mapping ϕ can be seen as a 2-coloring of $\binom{E_0}{2}$. Consider two auxiliary digraphs H_{in} and H_{out} with vertex sets E_0. For all $i, j \in E_0$, (i, j) is an edge in H_{in} if $i < j$ and $\phi(i, j) = \text{in}$, and (i, j) is an edge in H_{out} if $i < j$ and $\phi(i, j) = \text{out}$. Note that H_{in} and H_{out} are acyclic. Next, for every $i \in E_0$, let

$$\phi'(i) = (\text{max-start-path}(H_{\text{in}}, i), \text{max-start-path}(H_{\text{out}}, i)).$$

We claim that ϕ' is injective. Indeed, let $i, j \in E_0$ with $i < j$. If $\phi(i, j) = \text{in}$, then for every path $j_1 \cdots j_m$ in H_{in} with $j_1 = j$, $ij_1 \cdots j_m$ is also a path in H_{in}. It follows that $\text{max-start-path}(H_{\text{in}}, i) > \text{max-start-path}(H_{\text{in}}, j)$. Analogously, if $\phi(i, j) = \text{out}$, we obtain $\text{max-start-path}(H_{\text{out}}, i) > \text{max-start-path}(H_{\text{out}}, j)$. This shows that ϕ' is injective.

Since ϕ' is an injective function from E_0 to $[\text{max-path}(H_{\text{in}})] \times [\text{max-path}(H_{\text{out}})]$ we have

$$|E_0| \leqslant \text{max-path}(H_{\text{in}}) \cdot \text{max-path}(H_{\text{out}}).$$

In the next claim, we show that $\text{max-path}(H_{\text{in}}) \leqslant \text{se}_P(I)$. This together with $m = |E_0|$ and the inequality above imply

$$2(2\,\text{se}_P(I) + 6) = m/\,\text{se}_P(I) \leqslant \text{max-path}(H_{\text{out}}). \tag{21}$$

Claim 111. $\text{max-path}(H_{\text{in}}) \leqslant \text{se}_P(I)$.

Proof. Let $i_1 \cdots i_p$ be a path in H_{in}. An example of such a path is shown in Figure 50. We argue that $\{(a_{i_j}, b_{i_j}) : j \in [p]\}$ induces a standard example in P, i.e., $a_{i_k} < b_{i_\ell}$ in P for all distinct $k, \ell \in [p]$. This will show that $p \leqslant \text{se}_P(I)$ By Claim 105.(i) and (u5), for all $k, \ell \in [p]$ with $k < \ell$, we have $a_{i_\ell} < u_{i_\ell} \leqslant u_{i_k+1} \leqslant \pi(b_{i_k}) \leqslant b_{i_k}$ in P. It remains to argue that $a_{i_k} < b_{i_\ell}$ in P for all $k, \ell \in [p]$ with $k < \ell$.

Let $j \in [p-1]$. Since (i_j, i_{j+1}) is an edge in H_{in}, we have $v_{i_{j+1}} \in \text{int}\,\mathcal{R}_{i_j}$. By Claim 110.(ii) and since $i_j \in E$ (that is, σ_{i_j} is of weight 1), σ_{i_j} is a cycle edge. Therefore, by Claim 110.(iv), v_{i_j} lies in $W_R(v_{i_{j+1}})$. In other words, $W_R(v_{i_j})$ is a subpath of $W_R(v_{i_{j+1}})$. Now, let $k, \ell \in [p]$ with $k < \ell$. By transitivity of "being a subpath" relation, $W_R(v_{i_k})$ is a subpath of $W_R(v_{i_\ell})$, and in particular, v_{i_k} lies in $W_R(v_{i_\ell})$. By Claim 110.(v), v_{i_k} lies in $W_R(b_{i_k+1})$ and v_{i_ℓ} lies in

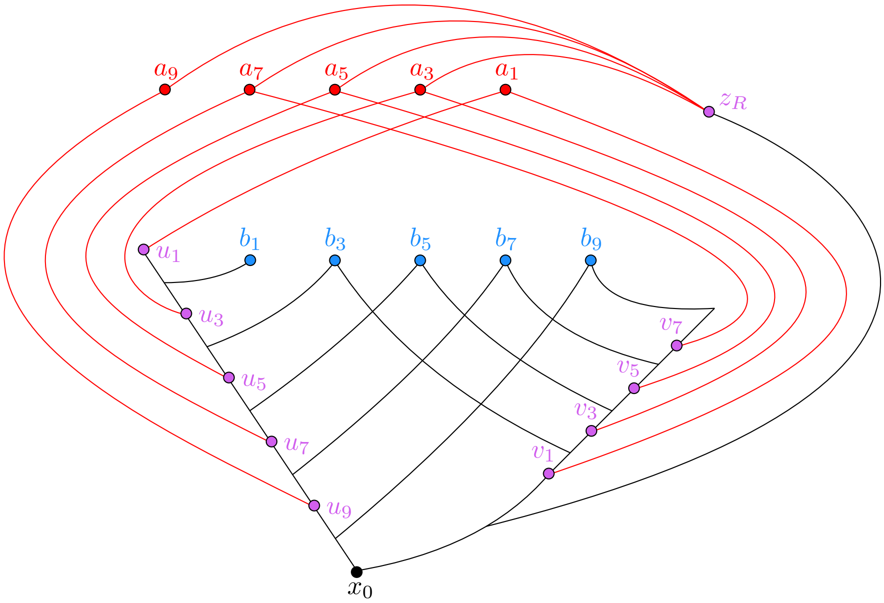


FIGURE 50. 13579 is a path in H_{in}. We do not draw other elements of the poset to keep the figure clearer. Note that $\{(a_i, b_i) : i \in \{1, 3, 5, 7, 9\}\}$ induces a standard example in P.

$W_R(b_{i_\ell+1})$. If $i_k + 1 = i_\ell$, we obtain $a_{i_k} < v_{i_k} \leqslant b_{i_k+1} = b_{i_\ell}$ in P, which yields the assertion, thus, suppose that $i_k + 1 < i_\ell$. Since v_{i_k} lies in $W_R(v_{i_\ell})$ and v_{i_ℓ} lies in $W_R(b_{i_\ell+1})$, v_{i_k} lies in $W_R(b_{i_\ell+1})$. Consider the paths $W_R(b_{i_k+1})$, $W_R(b_{i_\ell})$, and $W_R(b_{i_\ell+1})$. They are pairwise x_0-consistent (by Proposition 15.(*R*)). Moreover, since b_{i_k+1} is left of b_{i_ℓ} and b_{i_ℓ} is left of $b_{i_\ell+1}$, we have that $W_R(b_{i_k+1})$ is left of $W_R(b_{i_\ell})$ and $W_R(b_{i_\ell})$ is left of $W_R(b_{i_\ell+1})$. We argued that v_{i_k} lies in $W_R(b_{i_k+1})$ and $W_R(b_{i_\ell+1})$. Therefore, by Corollary 16, v_{i_k} lies in $W_R(b_{i_\ell})$. This implies that $a_{i_k} < v_{i_k} \leqslant b_{i_\ell}$ in P, which ends the proof. $\lhd$

In the next part of the proof, we assign a region to each edge in H_{out}. These regions are helpful in studying the structure of paths in H_{out}. Let (i, j) be an edge in H_{out}.[8] In the next paragraph, we define $v_{i,j}$ and a witnessing path $V_{i,j}$ from a_{i+1} to $v_{i,j}$ in P. We split the definition into cases, where the case with a higher number excludes cases with lower numbers. See Figure 51.

Case 1: $j = i + 1$.

We set $v_{i,j} = v_{i+1} = v_j$ and $V_{i,j} = V_{i+1} = V_j$.

Case 2: $U_{i+1} \cap V_j \neq \emptyset$.

Let v be the first element of $u_{i+1}[U_{i+1}]a_{i+1}$ in V_j. We set $v_{i,j} = v_j$ and $V_{i,j} = a_{i+1}[U_{i+1}]v[V_j]v_j$.

Case 3: $a_{i+1}[M_R(a_{i+1})]z_R(a_{i+1}) \cap V_j \neq \emptyset$.

Let v be the first element of $a_{i+1}[M_R(a_{i+1})]z_R(a_{i+1})$ in V_j. We set $v_{i,j} = v_j$ and $V_{i,j} = a_{i+1}[M_R(a_{i+1})]v[V_j]v_j$.

Case 4: otherwise.

We set $v_{i,j} = z_R(a_{i+1})$ and $V_{i,j} = a_{i+1}[M_R(a_{i+1})]z_R(a_{i+1})$.

[8]In fact, for many statements and definitions we only need $i, j \in E$ with $i < j$. However, there is no point in such generality. We mark the moment when we use the fact that (i, j) is an edge in H_{out} with another footnote.

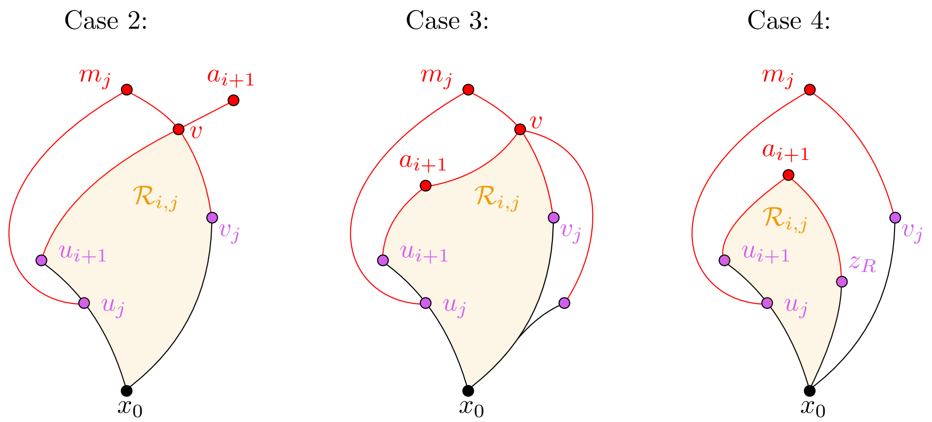


FIGURE 51. Illustration of some cases of the definition of $\mathcal{R}_{i,j}$.

This completes the definition of $v_{i,j}$ and $V_{i,j}$. Note that $v_{i,j} \in \{v_j, z_R(a_{i+1})\}$ and $V_{i,j}$ is an exposed witnessing path from a_{i+1} to $v_{i,j}$ in P. Moreover, we have the following.

Claim 112. Let (i,j) be an edge in H_{out}.

(i) u_i is left of $v_{i,j}$,
(ii) u_{i+1} is left of $v_{i,j}$,
(iii) $v_{i,j} \in Z(a_{i+1})$.

Proof. For the proofs of (i) and (ii), let $k \in \{i, i+1\}$. We have u_k left of v_j by Claim 108.(iii). Furthermore, b_k is left of or equal to b_{i+1} and b_{i+1} is left of $z_R(a_{i+1})$ (by Proposition 57). Since u_k is left of b_k (by (u4)), u_k is left of $z_R(a_{i+1})$, as desired. For the proof of (iii), note that $z_R(a_{i+1}) \in Z(a_{i+1})$ by definition, and so, we assume that $v_{i,j} = v_j$. Since $(a_{i+1}, b_{i+1}) \in I$ and b_{i+1} is left of v_j (by Claim 108.(ii)), by Corollary 50, $a_{i+1} \notin \text{shad}(v_j)$. Finally, recall that $V_{i,j}$ is an exposed witnessing path from a_{i+1} to v_j in P, and so, $v_j \in Z(a_{i+1})$, as desired. $\lhd$

Let (i,j) be an edge in H_{out}. Recall that u_{i+1} is left of $v_{i,j}$ (by Claim 112.(ii)), $u_{i+1} \in Z(a_{i+1})$ (by (u2)), and $v_{i,j} \in Z(a_{i+1})$ (by Claim 112.(iii)). Therefore, we can define

$$\mathcal{R}_{i,j} = \mathcal{R}(a_{i+1}, u_{i+1}, v_{i,j}, U_{i+1}, V_{i,j}).$$

See Figure 51 again. Let $q_{i,j}$ and $m_{i,j}$ be the lower-min and the upper-min of $\mathcal{R}_{i,j}$. The goal of the next part of this section is to show that if (i,j) is an edge in H_{out}, then $\mathcal{R}_i \subseteq \mathcal{R}_{i,j} \subseteq \mathcal{R}_j$.

Claim 113. Let (i,j) be an edge in H_{out}. Then,

$$\mathcal{R}_{i,j} \subseteq \mathcal{R}_j.$$

Proof. If $j = i+1$, then $\mathcal{R}_{i,j} = \mathcal{R}_j$ and the claim holds. Thus, we assume $j > i+1$. In order to prove the claim, we show that $\partial\mathcal{R}_{i,j} \subseteq \mathcal{R}_j$, which suffices by Proposition 7. Furthermore, if $u_{i+1}[U_{i+1}]m_{i,j}[V_{i,j}]v_{i,j} \subseteq \mathcal{R}_j$, then in particular, $u_{i+1}, v_{i,j} \in \mathcal{R}_j$, so by Corollary 88, $u_{i+1}[W_L(u_{i+1})]q_{i,j}[W_R(v_{i,j})]v_{i,j} \subseteq \mathcal{R}_j$, and therefore, $\partial\mathcal{R}_{i,j} \subseteq \mathcal{R}_j$. Let

$$M = u_{i+1}[U_{i+1}]m_{i,j}[V_{i,j}]v_{i,j}.$$

As discussed, it suffices to argue that

$$M \subseteq \mathcal{R}_j.$$

We split the rest of the proof into cases depending on the applied case in the definition of $\mathcal{R}_{i,j}$. Recall that we assumed $j > i+1$ so Case 1 does not hold.

Case 2. We have

$$M = u_{i+1}[U_{i+1}]v[V_j]v_j,$$

where v is the first element of $u_{i+1}[U_{i+1}]a_{i+1}$ in V_j. Note that $m_j < v$ in P as otherwise, we have $a_{i+1} < v \leqslant m_j < u_j$ in P, which contradicts Claim 105.(iii). Thus, $v[V_j]v_j \subseteq \partial\mathcal{R}_j$, and so, it suffices to show that

$$u_{i+1}[U_{i+1}]v \subseteq \mathcal{R}_j.$$

By Claim 109, $u_{i+1} \in \operatorname{int}\mathcal{R}_j$, thus, all we need to show is that $u_{i+1}[U_{i+1}]v$ intersects with $\partial\mathcal{R}_j$ only in v. By Claim 105.(iii), $a_{i+1} \parallel u_j$ in P, hence, U_{i+1} is disjoint from U_j. Since U_{i+1} is an exposed witnessing path in P and $u_{i+1} \in \operatorname{int}\mathcal{R}_j$, U_{i+1} is disjoint from $u_j[W_L(u_j)]q_j[W_R(v_j)]v_j$. This and the definition of v imply that the only common element of $u_{i+1}[U_{i+1}]v$ and $\partial\mathcal{R}_j$ is v, as desired.

Case 3. We have

$$M = u_{i+1}[U_{i+1}]m[M_R(a_{i+1})]v[V_j]v_j,$$

where v is the first element of $a_{i+1}[M_R(a_{i+1})]z_R(a_{i+1})$ in V_j and m is the last common element of $a_{i+1}[M_R(a_{i+1})]z_R(a_{i+1})$ and $a_{i+1}[U_{i+1}]u_{i+1}$. Note that $m_j < v$ in P as otherwise, we have $a_{i+1} < v \leqslant m_j < u_j$ in P, which contradicts Claim 105.(iii). Thus, $v[V_j]v_j \subseteq \partial\mathcal{R}_j$, and so, it suffices to show that

$$W = u_{i+1}[U_{i+1}]m[M_R(a_{i+1})]v \subseteq \mathcal{R}_j.$$

By Claim 109, $u_{i+1} \in \operatorname{int}\mathcal{R}_j$, thus, all we need to show is that W intersects with $\partial\mathcal{R}_j$ only in v. By Claim 105.(iii), $a_{i+1} \parallel u_j$ in P, hence, W is disjoint from U_j. Since the only element of W in B is u_{i+1} and $u_{i+1} \in \operatorname{int}\mathcal{R}_j$, W is disjoint from $u_j[W_L(u_j)]q_j[W_R(v_j)]v_j$. The path $[U_{i+1}]$ is disjoint from V_j as Case 2 does not hold. All this and the definition of v imply that the only common element of W and $\partial\mathcal{R}_j$ is v, as desired.

Case 4. We have

$$M = u_{i+1}[U_{i+1}]m[M_R(a_{i+1})]z_R(a_{i+1}),$$

where m is the maximal in P common element of $a_{i+1}[M_R(a_{i+1})]z_R(a_{i+1})$ and U_{i+1}. By Claim 109, $u_{i+1} \in \operatorname{int}\mathcal{R}_j$, thus, all we need to show is that M intersects with $\partial\mathcal{R}_j$ only in $z_R(a_{i+1})$. Since Cases 2 and 3 do not hold and by Claim 105.(iii), M is disjoint from $u_j[U_j]m_j[V_j]v_j$. It follows that M can intersect $\partial\mathcal{R}_j$ only in $u_j[W_L(u_j)]q_j[W_R(v_j)]v_j$. Since M is the union of two exposed witnessing paths in P and $u_{i+1} \in \operatorname{int}\mathcal{R}_j$, this intersection may occur only in $z_R(a_{i+1})$, as desired. This completes the proof. $\lhd$

Claim 114. Let (i, j) be an edge in H_{out}. We have

(i) $\gamma_{R,i}$ is left of $W_R(v_{i,j})$,
(ii) $u_i \in \operatorname{int}\mathcal{R}_{i,j}$.

Proof. In order to prove (i), recall that $v_{i,j} \in \{v_j, z_R(a_{i+1})\}$. First, assume that $v_{i,j} = v_j$. By definition of edges in H_{out}, $v_j \notin \mathcal{R}_i$.[9] Thus, by Proposition 60.(iii), either $W_L(v_j)$ is left of $\gamma_{L,i}$ or $\gamma_{R,i}$ is left of $W_R(v_j)$. Since the latter is the desired statement, let us show that the former can not hold. By Claim 105.(ii), $W_L(u_j)$ is a subpath of $W_L(u_i)$. Moreover, recall that $W_L(u_i)$ is a subpath of $\gamma_{L,i}$. It follows that if $W_L(v_j)$ is left of $\gamma_{L,i}$, then either $W_L(v_j)$ is left of $W_L(u_j)$ or $W_L(u_j)$ is a subpath of $W_L(v_j)$. Both are false, since u_j is left of v_j (by Claim 107.(ii)), and so, $W_L(u_j)$ is left of $W_L(v_j)$. This completes the proof of (i) in the case, where $v_{i,j} = v_j$.

Next, assume that $v_{i,j} = z_R(a_{i+1})$. Since $a_{i+1} \parallel v_i$ in P (by Claim 107.(iv)), $z_R(a_{i+1})$ does not lie in $W_R(v_i)$, and in particular, does not lie in $\gamma_{R,i}$. Therefore, either $\gamma_{R,i}$ is left of $W_R(z_R(a_{i+1}))$ or $W_R(z_R(a_{i+1}))$ is left of $\gamma_{R,i}$. Since the former is the desired statement, let us

[9]Note that this is the only place, in this section of the proof, where we call the definition of edges in H_{out}.

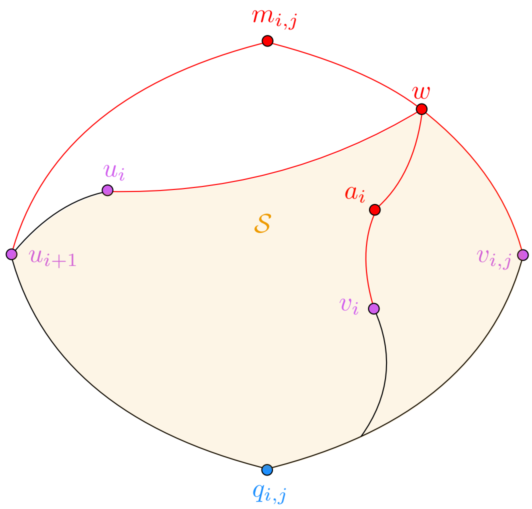


FIGURE 52. Region $\mathcal{R}_{i,j}$ along with objects appearing in the proof of Claim 115.

show that the latter can not hold. Suppose to the contrary that $W_R(z_R(a_{i+1}))$ is left of $\gamma_{R,i}$. The path $\gamma_{R,i}$ is either left of $M_R(a_i)$ or is a subpath of $M_R(a_i)$. It follows that $W_R(z_R(a_{i+1}))$ is left of $M_R(a_i)$. In particular, $M_R(a_{i+1})$ is left of $M_R(a_i)$, however, this is a contradiction with $(R12)$ for σ_i, and so, the proof of (i) is completed.

Let $\gamma_L = x_0[W_L(u_{i+1})]u_{i+1}[U_{i+1}]m_{i,j}$ and $\gamma_R = x_0[W_R(v_{i,j})]v_{i,j}[V_{i,j}]m_{i,j}$. We argue that γ_L is left of $W_L(u_i)$ and $W_R(u_i)$ is left of γ_R. This will show that $u_i \in \operatorname{int} \mathcal{R}_{i,j}$ by Proposition 60.(ii). By Claim 105.(ii), $W_L(u_{i+1})$ is a proper subpath of $W_L(u_i)$ and $W_L(u_i)$ is a subpath of $W^\star$. Hence, by (u7), γ_L is left of $W_L(u_i)$. By Claim 112.(i), u_i is left of $v_{i,j}$, thus, $W_R(u_i)$ is left of $W_R(v_{i,j})$, and so, $W_R(u_i)$ is left of γ_R. As discussed, this ends the proof of (ii). $\lhd$

Claim 115. Let (i, j) be an edge in H_{out}. Then,

$$\mathcal{R}_i \subseteq \mathcal{R}_{i,j}.$$

Proof. In order to prove the claim, we show that $\partial\mathcal{R}_i \subseteq \mathcal{R}_{i,j}$, which suffices by Proposition 7. Moreover, if $u_i[U_i]m_i[V_i]v_i \subseteq \mathcal{R}_{i,j}$, then by Corollary 88, $\partial\mathcal{R}_i \subseteq \mathcal{R}_{i,j}$. Thus, we only argue that

$$u_i[U_i]m_i[V_i]v_i \subseteq \mathcal{R}_{i,j}. \tag{22}$$

By Claim 105.(iii), we have $a_i \parallel u_{i+1}$ in P and by Claim 107.(iv), $a_{i+1} \parallel v_i$ in P. In particular,

$$U_{i+1} \cap (U_i \cup V_i) = \emptyset \text{ and } V_i \cap (U_{i+1} \cup V_{i,j}) = \emptyset. \tag{23}$$

By Claim 114.(ii), $u_i \in \operatorname{int} \mathcal{R}_{i,j}$. Note that we will use this fact extensively.

First, assume that $u_i[U_i]m_i$ and $v_{i,j}[V_{i,j}]m_{i,j}$ are disjoint. Since U_i and V_i are exposed witnessing paths in P, by (23), $u_i[U_i]m_i[V_i]v_i$ can intersect $\partial\mathcal{R}_{i,j}$ only in v_i. This implies (22), which concludes the proof in this case.

For the rest of the proof, we assume that $u_i[U_i]m_i$ and $v_{i,j}[V_{i,j}]m_{i,j}$ intersect. See an illustration in Figure 52. Let w be the first element of $u_i[U_i]m_i$ in $v_{i,j}[V_{i,j}]m_{i,j}$. In order to define a new auxiliary region we argue that

$$u_i \in Z(a_{i+1}). \tag{24}$$

We have u_i left of b_i (by (u4)) and b_i left of b_{i+1}. Now, since $(a_{i+1}, b_{i+1}) \in I$ and u_i is left of b_{i+1}, by Corollary 50, $a_{i+1} \notin \operatorname{shad}(u_i)$. Moreover, $a_{i+1}[V_{i,j}]w[U_i]u_i$ is an exposed witnessing path from a_{i+1} to u_i in P, as desired. This completes the proof of (24).

Since u_i is left of $v_{i,j}$ (by Claim 112.(i)), $u_i \in Z(a_{i+1})$ (by (24)), and $v_{i,j} \in Z(a_{i+1})$ (by Claim 112.(iii)) we can define

$$\mathcal{S} = \mathcal{R}(a_{i+1}, u_i, v_{i,j}, a_{i+1}[V_{i,j}]w[U_i]u_i, a_{i+1}[V_{i,j}]v_{i,j}).$$

Note that by definition, w is the upper-min of $\mathcal{S}$. Since $u_i \in \operatorname{int}\mathcal{R}_{i,j}$, also by the definition of w and (23), we have $u_i[U_i]w \subseteq \mathcal{R}_{i,j}$. On the other hand, $w[V_{i,j}]v_{i,j} \subseteq \partial\mathcal{R}_{i,j} \subseteq \mathcal{R}_{i,j}$. Since $u_i, v_{i,j} \in \mathcal{R}_{i,j}$, Corollary 88 gives $\partial\mathcal{S} \subseteq \mathcal{R}_{i,j}$, and subsequently, Proposition 7 gives

$$\mathcal{S} \subseteq \mathcal{R}_{i,j}.$$

As noted above, $u_i[U_i]w \subseteq \mathcal{R}_{i,j}$. In particular, if $w[U_i]m_i[V_i]v_i \subseteq \mathcal{S}$, then $w[U_i]m_i[V_i]v_i \subseteq \mathcal{R}_{i,j}$, and so, (22) follows. Hence, all we need to prove that is

$$w[U_i]m_i[V_i]v_i \subseteq \mathcal{S}. \tag{25}$$

First, we show that

$$v_i \in \mathcal{S}. \tag{26}$$

Let $q_{\mathcal{S}}$ be the lower-min of $\mathcal{S}$.[10] If $v_i \in \partial\mathcal{S}$, then (26) clearly holds, thus, assume that $v_i \notin \partial\mathcal{S}$. Since $q_{\mathcal{S}} < u_i$ in P, Proposition 33 implies that $\operatorname{shad}(q_{\mathcal{S}}) \subseteq \operatorname{shad}(u_i)$. Since u_i is left of v_i, $v_i \notin \operatorname{shad}(u_i)$, and so, $v_i \notin \operatorname{shad}(q_{\mathcal{S}})$. We assumed $v_i \notin \partial\mathcal{S}$, thus, Claim 114.(i) implies that $W_R(v_i)$ is left of $W_R(v_{i,j})$. On the other hand, $W_L(u_i)$ is left of $W_L(v_i)$ by Claim 107.(ii). Therefore, by Proposition 60.(ii), $v_i \in \operatorname{int}\mathcal{S}$, which concludes the proof of (26).

Let e be the first edge of $v_i[V_i]m_i$. Next, we show that

$$\text{the interior of } e \text{ lies in } \mathcal{S}. \tag{27}$$

Recall that $v_i \in \mathcal{S}$ (by 26). If $v_i \in \operatorname{int}\mathcal{S}$, then (27) clearly follows, hence, assume that $v_i \in \partial\mathcal{S}$. In other words, v_i is an element of $u_i[W_L(u_i)]q_{\mathcal{S}}[W_R(v_{i,j})]v_{i,j}$. Since u_i is left of v_i, v_i does not lie in $W_L(u_i)$, and so, v_i lies in $W_R(v_{i,j})$ and $v_i \neq q_{\mathcal{S}}$. In other words, v_i lies strictly on the right side of $\mathcal{S}$. Therefore, by Claim 114.(i) and (rg4), the edge e lies in $\mathcal{S}$ and (27) follows.

Finally, we are ready to argue (25). By (26) and (27), it suffices to show that $W = v_i[V_i]m_i[U_i]w$ intersects $\partial\mathcal{S}$ at most in the endpoints. Since U_i and V_i are exposed witnessing paths in P, the only element in B of W is v_i, and so, it suffices to show that W intersects $u_i[U_i]w[V_{i,j}]v_{i,j}$ only in w. The path $m_i[U_i]w$ does not intersect $u_i[U_i]w[V_{i,j}]v_{i,j}$ in an element distinct from w as this yields a directed cycle in P. The path $v_i[V_i]m_i$ is disjoint from $V_{i,j}$ by (23). Finally, $v_i[V_i]m_i$ does not intersect $u_i[U_i]w$ in an element distinct from w as this contradicts the definition of m_i. We showed (25), and so the proof of the claim is complete. $\lhd$

Next, we develop two claims about paths leaving regions: "going in" claim and "going out" claim (see Figure 53).

Claim 116 (Going in). Let (i, j) be an edge of H_{out}. Let $c \in A$ be such that $c \parallel u_i$ in P and let $z \in Z(c)$. If $z \in \operatorname{int}\mathcal{R}_i$, then $c \in \operatorname{int}\mathcal{R}_j$.

Proof. Assume that $z \in \operatorname{int}\mathcal{R}_i$. By Claim 113 and Claim 115, we have

$$\mathcal{R}_i \subseteq \mathcal{R}_{i,j} \subseteq \mathcal{R}_j.$$

[10] In fact, one can prove that $q_{\mathcal{S}} = q_{i,j}$ but we do not need this equality.

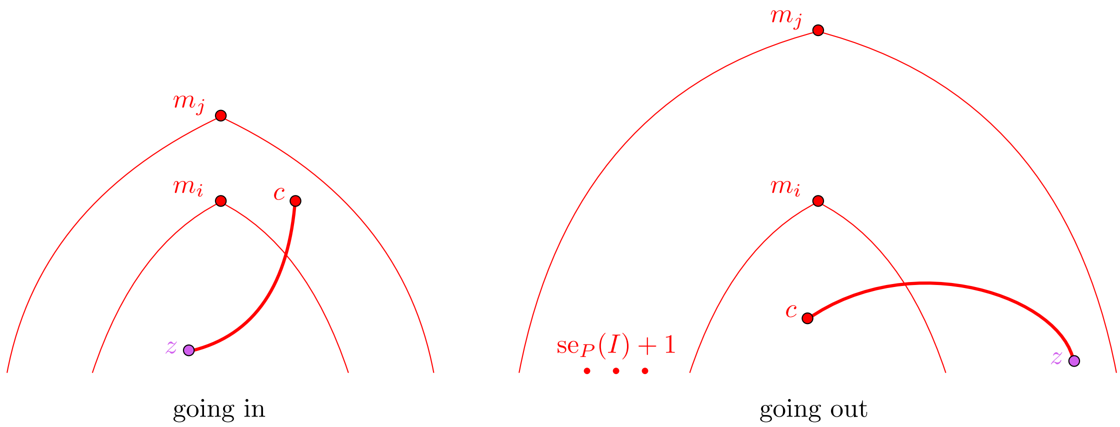


FIGURE 53. "Going in" claim (Claim 116) and "going out" claim (Claim 122) state that exposed witnessing paths can not escape too far while going in or out, respectively. In the figure, we show a schematic drawing of the statements.

In particular, it suffices to show that $c \in \operatorname{int} \mathcal{R}_{i,j}$. Suppose to the contrary that $c \notin \operatorname{int} \mathcal{R}_{i,j}$. Let W be an exposed witnessing path from c to z in P. Since $\mathcal{R}_i \subseteq \mathcal{R}_{i,j}$, $z \in \operatorname{int} \mathcal{R}_i$ and $c \notin \operatorname{int} \mathcal{R}_{i,j}$, there exist elements w_i and $w_{i,j}$ of W such that

$$w_i \in \partial\mathcal{R}_i,\ \ w_{i,j} \in \partial\mathcal{R}_{i,j},\ \text{and}\ c \leqslant w_{i,j} \leqslant w_i < z \text{ in } P.$$

The last inequality is strict as z lies in the interior of $\mathcal{R}_i$. In particular, since W is an exposed witnessing path in P, we have $w_i, w_{i,j} \notin B$. Hence, $w_i \in \partial\mathcal{R}_i$ implies that w_i lies in $u_i[U_i]m_i[V_i]v_i$ and $w_{i,j} \in \partial\mathcal{R}_{i,j}$ implies that $w_{i,j}$ lies in $u_{i+1}[U_{i+1}]m_{i,j}[V_{i,j}]v_{i,j}$. Recall that we assumed $c \parallel u_i$ in P. It follows that w_i does not lie in U_i. Also by Claim 105.(i), $u_{i+1} < u_i$ in P. In particular, $c \not\leqslant u_{i+1}$ in P and since $c \in A$ and $u_{i+1} \in B$ we also have $u_{i+1} \not\leqslant c$ in P. Hence $c \parallel u_{i+1}$ in P. It follows that $w_{i,j}$ does not lie in U_{i+1}. Therefore, w_i lies in V_i and $w_{i,j}$ lies in $V_{i,j}$. We obtain

$$a_{i+1} \leqslant w_{i,j} \leqslant w_i \leqslant v_i \text{ in } P.$$

This contradicts Claim 107.(iv) and completes the proof. $\triangleleft$

The next series of claims leads to the "going out" claim, which is crucial in the final steps of the proof. At the very beginning of the proof of Lemma 75, for each $d \in B$, we defined $\pi(d) = \operatorname{gcpe}(W^\star, W_L(d))$. This element has an equivalent description given that u_1 is left of d.

Claim 117. Let $d \in B$ such that u_1 is left of d. Then,

$$\pi(d) = \text{the maximal element of } W^\star \text{ that is less than } d \text{ in } P.$$

Proof. Let $\pi'(d)$ be the maximal element of $W^\star$ that is less than d in P. The aim is to prove that $\pi(d) = \pi'(d)$. Clearly, both $\pi(d)$ and $\pi'(d)$ lie in $W^\star$ and $\pi(d) \leqslant \pi'(d)$ in P. Suppose to the contrary that $\pi(d) < \pi'(d)$ in P. Let W be a witnessing path from $\pi'(d)$ to d in P. Consider the path $W' = x_0[W^\star]\pi'(d)[W]d$. Since u_1 is left of d and $\pi(d) = \operatorname{gcpe}(W_L(u_1), W_L(d))$, we obtain that W' is left of $W_L(d)$, which is a contradiction. $\triangleleft$

Claim 118. Let $i \in E$ and $d \in B$ such that u_1 is left of d, $u_i \parallel d$ in P, and $d \notin \mathcal{R}_i$. Then, $\pi(d) \leqslant \pi(v_i)$ in P.

Proof. Recall that $W_L(u_i)$ is a subpath of $W^\star$ (by (u3)). Note that since $u_i \parallel d$ in P, $\pi(d)$ does not lie in $u_i[W^\star]u_1$. By Claim 117, $q_i \leqslant \pi(v_i)$ in P. Thus, if $\pi(d) \leqslant q_i$ in P, the assertion holds, and so, we assume that $q_i < \pi(d) < u_i$ in P. In particular, $\pi(d)$ lies strictly on the left side of $\mathcal{R}_i$. Since $W^\star$ is left of $W_L(d)$, the first edge of $\pi(d)[W_L(d)]d$ lies in $\mathcal{R}_i$ (by (rg3)). However, $d \notin \mathcal{R}_i$, and so, $\pi(d)[W_L(d)]d$ intersects $\partial\mathcal{R}_i$ in an element distinct from $\pi(d)$. Since $\pi(d)[W_L(d)]d$ has all elements in B, this path must intersect $u_i[W_L(u_i)]q_i[W_R(v_i)]v_i$. The paths $W_L(u_i)$ and $W_L(d)$ are x_0-consistent (by Proposition 15.(*L*)), thus, $\pi(d)[W_L(d)]d$ does not intersect $W_L(u_i)$ in an element distinct from $\pi(d)$. It follows that $\pi(d)[W_L(d)]d$ intersects $W_R(v_i)$. In particular, $\pi(d) < v_i$ in P. By Claim 117, this implies that $\pi(d) \leqslant \pi(v_i)$ in P, which ends the proof. ◁

Claim 119. Let $i, j, k \in [n-1]$ such that $i < j < k$ and (i, j) is an edge of H_{out}. Then,

(i) $b_k \notin \mathcal{R}_i$,
(ii) $\pi(b_k) \leqslant \pi(v_i)$ in P,
(iii) $a_{k+1} < v_i$ in P.

Proof. We start with the proof of (i). We are going to call Proposition 60 twice for the region $\mathcal{R}_i$ and elements b_k and v_j. To make these calls simpler, we note beforehand that

$$b_k, v_j \notin \text{shad}(q_i). \tag{28}$$

Since $q_i < u_i$ in P, we have $\text{shad}(q_i) \subseteq \text{shad}(u_i)$ (by Proposition 33). Since u_i left of b_k (by Claim 108.(i)) and u_i left of v_j (by Claim 108.(iii)), (28) holds.

We will argue that

$$\gamma_{R,i} \text{ left of } W_R(b_{j+1}). \tag{29}$$

Note that (29) quickly implies (i). Indeed, $W_R(b_{j+1})$ is left of or equal to $W_R(b_k)$ (since $j+1 \leqslant k$), hence, (29) implies that $\gamma_{R,i}$ is left of $W_R(b_k)$. Thus, by Proposition 60.(iii), $b_k \notin \mathcal{R}_i$, as desired.

Now, we prove (29). Since u_i is left of v_j (by Claim 108.(iii)) and $v_j \notin \mathcal{R}_i$ (as (i, j) is an edge in H_{out}), by Proposition 60.(iii),

$$\gamma_{R,i} \text{ is left of } W_R(v_j). \tag{30}$$

Note that $j \in E$ as (i, j) is an edge of H_{out}. In other words, σ_j is an edge of weight 1. If σ_j is a shifted edge, then v_j is left of b_{j+1} (by Claim 107.(iii)), and so, $W_R(v_j)$ is left of $W_R(b_{j+1})$. This and (30) imply (29). If σ_j is a cycle edge, then $v_j \leqslant b_{j+1}$ in P, and so, either $W_R(v_j)$ is a subpath of $W_R(b_{j+1})$ or $W_R(v_j)$ is left of $W_R(b_{j+1})$ (by Proposition 17.(*R*)). This and (30) again imply (29). This completes the proof of (i).

For the proof of (ii), note that $b_k \in B$, u_1 is left of b_k and $u_i \parallel b_k$ in P (by Claim 108.(i)). Therefore, (ii) follows from Claim 118.

Finally, we have

$$a_{k+1} < u_{k+1} \leqslant \pi(b_k) \leqslant \pi(v_i) < v_i \text{ in } P.$$

The inequalities follow respectively from: (u2), (u5), (ii), and the definition of π. This way, we obtain (iii). ◁

In Claim 119, we showed some properties of elements with indices $i, k \in [n-1]$ with $i < k$ such that there exists $j \in [n-1]$ with $i < j < k$ and $(i, j) \in H_{\text{out}}$. We would like to have these properties for every pair of indices. To this end, we fix a long path in H_{out} and focus on every other element of the path.

Fix a path in H_{out} witnessing max-path(H_{out}), and let E_1 be its vertex set. Claims 113 and 115 immediately imply the following.

Claim 120. Let $i, j \in E_1$ with $i < j$. Then, $\mathcal{R}_i \subseteq \mathcal{R}_j$.

Next, let E_2 be every other element of E_1 starting from the first one. In particular, by (21)

$$|E_2| \geqslant |E_1|/2 = \text{max-path}(H_{\text{out}})/2 \geqslant 2\,\text{se}_P(I) + 6. \tag{31}$$

From Claim 119 we obtain the following.

Claim 121. Let $i, j \in E_2$ with $i < j$. Then,

(i) $b_j \notin \mathcal{R}_i$,
(ii) $\pi(b_j) \leqslant \pi(v_i)$ in P,
(iii) $a_{j+1} < v_i$ in P.

Finally, we are ready to state and prove the "going out" claim.

Claim 122 (Going out). Let $i, j \in E_2$ with $i < j$ such that $|\{k \in E_2 : i \leqslant k \leqslant j\}| \geqslant \text{se}_P(I)+1$. Let $c \in A$, $z \in Z(c)$, and let W be an exposed witnessing path from c to z in P. If $c \in \mathcal{R}_i$, then $W \subseteq \mathcal{R}_j$ (in particular, $z \in \mathcal{R}_j$).

Proof. Assume that $c \in \mathcal{R}_i$. Suppose to the contrary that $W \not\subseteq \mathcal{R}_j$. Let $X = \{k \in E_2 : i \leqslant k \leqslant j\}$ and for each $k \in X$, let $d_{k+1} = b_{k+1}$ when σ_k is a cycle edge and $d_{k+1} = t_{k+1}$ when σ_k is a shifted edge. Recall that $(a_{k+1}, d_{k+1}) \in I$ for each $k \in X$ (by (io1) when σ_k is a shifted edge). We show that for all $k, \ell \in X$ with $k < \ell$, we have

$$a_{k+1} < d_{\ell+1} \text{ and } a_{\ell+1} < d_{k+1} \text{ in } P.$$

Therefore, $\{(a_{k+1}, d_{k+1}) : k \in X\}$ induces a standard example of order $|X|$ in P with all the incomparable pairs in I, which is a contradiction as $|X| \geqslant \text{se}_P(I) + 1$.

Fix some $k, \ell \in X$ with $k < \ell$. First, note that by Claim 121.(iii), $a_{\ell+1} < v_k \leqslant d_{k+1}$ in P. The rest of the proof is devoted to arguing that $a_{k+1} < d_{\ell+1}$ in P. Since $k, \ell \in E_2$, we can fix $k' \in E_1$ such that $k < k' < \ell$ and (k, k') is an edge of H_{out}. By Claims 113, 115 and 120,

$$\mathcal{R}_i \subseteq \mathcal{R}_k \subseteq \mathcal{R}_{k,k'} \subseteq \mathcal{R}_{k'} \subseteq \mathcal{R}_\ell \subseteq \mathcal{R}_j.$$

Since $c \in \mathcal{R}_i$, $W \not\subseteq \mathcal{R}_j$, and W is exposed, there exist elements $w_{k,k'}$, w_ℓ, and w_j of W not in B such that

$$w_{k,k'} \in \partial\mathcal{R}_{k,k'},\ w_\ell \in \partial\mathcal{R}_\ell,\ w_j \in \partial\mathcal{R}_j, \text{ and } c \leqslant w_{k,k'} \leqslant w_\ell \leqslant w_j < z \text{ in } P.$$

The element $w_{k,k'}$ lies in $u_{k+1}[U_{k+1}]m_{k,k'}[V_{k,k'}]v_{k,k'}$ and the element w_ℓ lies in $u_\ell[U_\ell]m_\ell[V_\ell]v_\ell$. The former yields $a_{k+1} \leqslant m_{k,k'} \leqslant w_{k,k'}$ in P. Since $k, \ell \in E_2$ and $k < \ell$, we have $k+1 < \ell$. If w_ℓ lies in U_ℓ, then $a_{k+1} \leqslant w_{k,k'} \leqslant w_\ell \leqslant u_\ell \leqslant u_{k+2} \leqslant b_{k+1}$ in P (the last two comparabilities follow from Claim 105.(i) and (u5) respectively), which is a false. Thus, w_ℓ lies in V_ℓ. This yields $w_\ell < v_\ell$ in P. Altogether, we obtain

$$a_{k+1} \leqslant w_{k,k'} \leqslant w_\ell < v_\ell \leqslant d_{\ell+1} \text{ in } P,$$

as desired. ◁

We have $|E_2| \geqslant 2\,\text{se}_P(I) + 6 = 1 + (\text{se}_P(I) + 1) + 1 + (\text{se}_P(I) + 1) + 2$ (by (31)). Hence, we can define the following numbers. Let $j_1, \ldots, j_8 \in E_2$ be such that

$$j_1 < j_2 < j_3 < j_4 < j_5 < j_6 < j_7 < j_8,$$
$$|\{k \in E_2 : j_2 \leqslant k \leqslant j_3\}| \geqslant \text{se}_P(I) + 1,$$
$$|\{k \in E_2 : j_5 \leqslant k \leqslant j_6\}| \geqslant \text{se}_P(I) + 1.$$

The plan is to show that $((a', b'), (a_{j_8}, b_{j_8}))$ is an edge in H_{IO}. This will conclude the proof of Lemma 75, see (39). Recall that $(a, b) = (a_1, b_1)$.

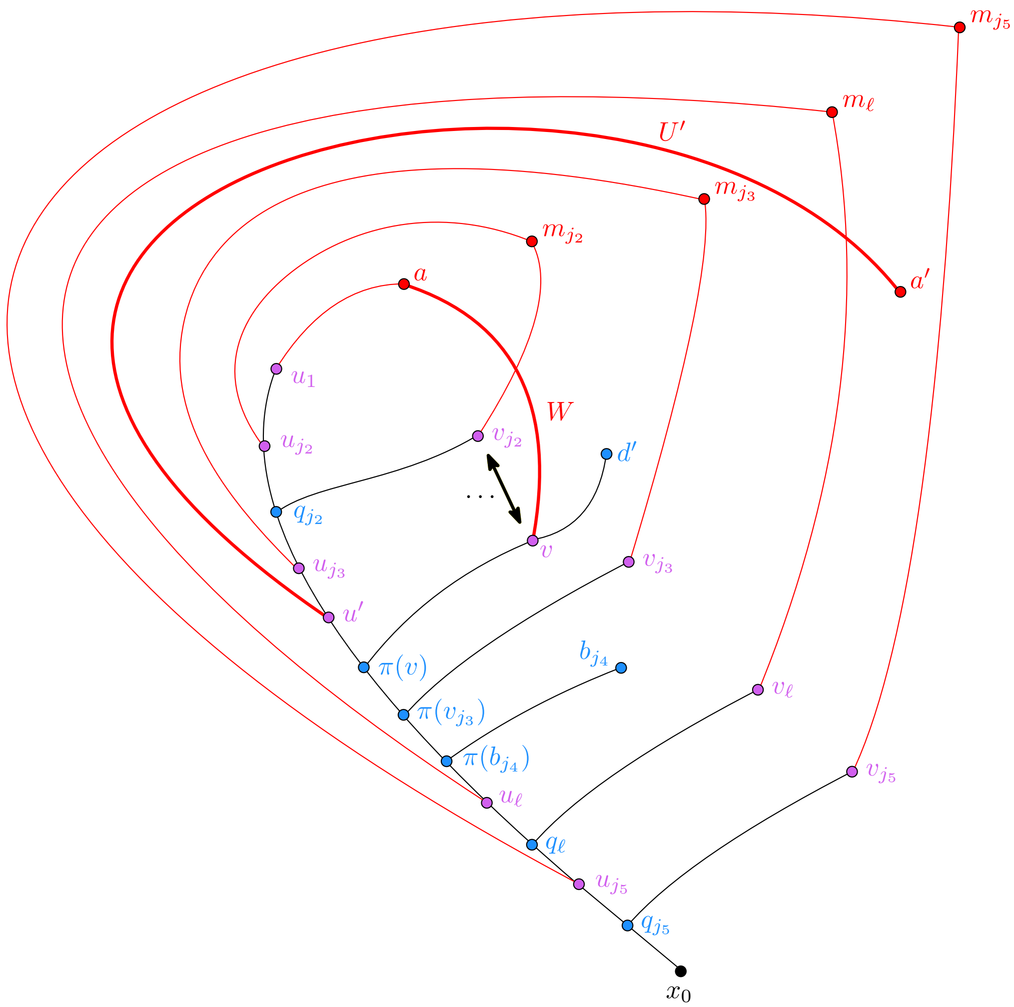


FIGURE 54. An illustration of the proof of Claim 124. We denote by d' either b' or t', depending on the case.

Claim 123. $a \in \mathcal{R}_{j_2}$.

Proof. Let $\ell \in E_1$ be such that $j_1 < \ell < j_2$ and (ℓ, j_2) is an edge in H_{out}. In particular, $1 < \ell$, and so, by Claim 109, $u_1 \in \operatorname{int} \mathcal{R}_\ell$. Additionally, by Claim 105.(iii), $a \parallel u_\ell$ in P. Since $u_1 \in Z(a)$ (by (u2)), we can apply Claim 116 to obtain $a \in \operatorname{int} \mathcal{R}_{j_2}$. $\lhd$

Let $u' \in Z(a')$ and let U' be an exposed witnessing path from a' to u' in P be such that (u', U') witnesses (*L*21*⋆*) for $((a_1, b_1), (a', b'))$. Note that u' lies in $W^\star$.

Claim 124.

(i) $a' \in \mathcal{R}_{j_5}$.
(ii) u_{j_5} lies in $W_L(u')$.

Proof. The proof is illustrated in Figure 54. Recall that $((a,b),(a',b'))$ is an edge of weight 1 in H_{IO}. If it is a shifted edge, let $t' \in B$ be a witness for the edge. Let $v \in B$ be such that

$$v \in Z(a) \text{ and } \begin{cases} v \leqslant b' \text{ in } P & \text{if } ((a,b),(a',b')) \text{ is a cycle edge,} \\ v \leqslant t' \text{ in } P & \text{if } ((a,b),(a',b')) \text{ is a shifted edge.} \end{cases}$$

By Claim 123, we have $a \in \mathcal{R}_{j_2}$. Let W be an exposed witnessing path from a to v in P. Since $|\{k \in E_2 : j_2 \leqslant k \leqslant j_3\}| \geqslant \mathrm{se}_P(I)+1$, by Claim 122, $W \subseteq \mathcal{R}_{j_3}$. Moreover, since $(a,b) \in I$ and b is left of v_{j_3} (by Claim 108.(ii)), by Corollary 50, $a \notin \mathrm{shad}(v_{j_3})$. Since $W \subseteq \mathcal{R}_{j_3}$ and $a \notin \mathrm{shad}(v_{j_3})$, by Proposition 90.(*L*).(iii), we obtain that $\mathrm{gcpe}(W_L(u_{j_3}), W_L(v_{j_3}))$ lies in $W_L(v)$. Since $W_L(u_{j_3})$ is a subpath of $W^\star$ (by Claim 105.(ii)) and $u_{j_3} \parallel v_{j_3}$ in P (by Claim 107.(ii)), we have $\pi(v_{j_3}) = \mathrm{gcpe}(W_L(u_{j_3}), W_L(v_{j_3}))$, and so,

$$\pi(v_{j_3}) \leqslant \pi(v) \text{ in } P. \tag{32}$$

Let $\ell \in E_1$ be such that $j_4 < \ell < j_5$ and (ℓ, j_5) is an edge in H_{out}. We have

$$u_\ell \leqslant u_{j_4+1} \leqslant \pi(b_{j_4}) \leqslant \pi(v_{j_3}) \leqslant \pi(v) < u' \text{ in } P, \tag{33}$$

where the first comparability follows from Claim 105.(i), the second follows from (u5), the third follows from Claim 121.(ii), the fourth follows from (32), and the fifth follows from the facts that u' and $\pi(v)$ lie in $W^\star$ and $a' \parallel v$ in P (by (io1) in the case of a shifted edge). Note that this already implies (ii) since $u_{j_5} \leqslant u_\ell < u'$ in P (by Claim 105.(i) and (33)) and all of these elements lie in $W^\star$ (by (u3)).

By Claim 109, $u_\ell[W^\star]u_1 \subseteq \mathrm{int}\,\mathcal{R}_\ell \cup \{u_\ell\}$. By (33), u' lies in $u_\ell[W^\star]u_1$ and $u' \neq u_\ell$, so we conclude that $u' \in \mathrm{int}\,\mathcal{R}_\ell$. Since $u_\ell \leqslant \pi(v)$ in P (by (33)), we have $a' \parallel u_\ell$ in P. Therefore, by Claim 116, $a' \in \mathrm{int}\,\mathcal{R}_{j_5}$. This completes the proof of (i) and the claim. ◁

Claim 125. Let $z \in Z(a')$ and let W be a witnessing path from a' to z in P. Then, $W \subseteq \mathcal{R}_{j_6}$. In particular, $z \in \mathcal{R}_{j_6}$.

Proof. By Claim 124.(i), $a' \in \mathcal{R}_{j_5}$. Since $j_5, j_6 \in E_2$ and $|\{k \in E_2 : j_5 \leqslant k \leqslant j_6\}| \geqslant \mathrm{se}_P(I)+1$, Claim 122 implies that $a' \in \mathcal{R}_{j_6}$. ◁

Claim 126. $b' \in \mathrm{int}\,\mathcal{R}_{j_6}$.

Proof. We have $q_{j_6} < u_{j_6} < u_2 \leqslant \pi(b) \leqslant b$ in P by Claim 105.(i) and (u5). Thus, $\mathrm{shad}(q_{j_6}) \subseteq \mathrm{shad}(b)$ (by Proposition 33), and so, since b is left of b', we obtain that $b' \notin \mathrm{shad}(q_{j_6})$. Therefore, to show that $b' \in \mathrm{int}\,\mathcal{R}_{j_6}$, it suffices (by Proposition 60.(ii)) to prove that γ_{L,j_6} is left of $W_L(b')$ and $W_R(b')$ is left of γ_{R,j_6}. We only prove the first statement, the proof of the second one is symmetric. By Claim 125, we have $z_L(a') \in \mathcal{R}_{j_6}$. By Proposition 57, $z_L(a')$ is left of b'. Since $z_L(a') \in \mathcal{R}_{j_6}$, by Proposition 60.(iii), either γ_{L,j_6} is left of $W_L(z_L(a'))$ or $W_L(z_L(a'))$ is a subpath of γ_{L,j_6}. In both cases, it follows that γ_{L,j_6} is left of $W_L(b')$ as $W_L(z_L(a'))$ is left of $W_L(b')$. This completes the proof of the claim. ◁

Claim 127. u_{j_6} lies in $W_L(z_L(a'))$.

Proof. Let $U'' = z_L(a')[M_L(a')]a'$ and note that $U'' \subseteq \mathcal{R}_{j_6}$ by Claim 125. In particular, $z_L(a') \in \mathcal{R}_{j_6}$, and so, by Proposition 60, $W_L(z_L(a'))$ is not left of $W_L(u_{j_6})$. If $W_L(u_{j_6})$ is a subpath of $W_L(z_L(a'))$, then the assertion follows. We will show that all the other cases lead to a contradiction. Since $u_{j_6} < u_{j_5}$ in P (by Claim 124.(ii)) and $W_L(u_{j_5})$ is a subpath of $W_L(u')$ (by Claim 124.(ii)), we obtain that

$$W_L(u_{j_6}) \text{ is a subpath of } W_L(u'). \tag{34}$$

If $W_L(u_{j_6})$ is left of $W_L(z_L(a'))$, then by (34), $W_L(u')$ is left of $W_L(z_L(a'))$, which contradicts the definition of $M_L(a')$. Finally, suppose that $W_L(z_L(a'))$ is a proper subpath of $W_L(u_{j_6})$. Note that $q_{j_6} < z_L(a')$ as otherwise $a' < z_L(a') \leqslant q_{j_6} \leqslant b'$ in P (by Claim 126 and Proposition 87), which is false. Altogether, we have $q_{j_6} < z_L(a') < u_{j_6}$ in P. Let e^- and e^+ be the edges of $W_L(u_{j_6})$ immediately preceding and immediately following $z_L(a')$, respectively. Let f be the edge of U'' incident to $z_L(a')$. Since $U'' \subseteq \mathcal{R}_{j_6}$, by (rg3), $e^+ \prec f \prec e^-$ in the $z_L(a')$-ordering. It follows that $W_L(u_{j_6})$ is left of $M_L(a')$. Furthermore, by (34), $W_L(u')$ is left of $M_L(a')$, and so, $x_0[W_L(u')]u'[U']a'$ is left of $M_L(a')$, which is false. This contradiction ends the proof. $\lhd$

Claim 128. b' is left of b_{j_7}.

Proof. The proof is illustrated in Figure 55. First, we prove that

$$b' \notin \text{shad}(b_{j_7}). \tag{35}$$

Let w be the maximal common element of $W_R(v_{j_6})$ and $W_L(b_{j_7})$ in P. Note that $u' \notin \text{shad}(w)$ as otherwise by Claim 124.(ii) and Proposition 33 applied multiple times, $u_{j_6} \in \text{shad}(u_{j_5}) \subseteq \text{shad}(u') \subseteq \text{shad}(w) \subseteq \text{shad}(v_{j_6})$, which contradicts Claim 107.(ii). If $a' \in \text{shad}(w)$, then an exposed witnessing path from a' to u' in P intersects $\partial\,\text{shad}(w)$ in an element of B, and so, exactly in u', which is a contradiction since $u' \notin \text{shad}(w)$. Therefore, $a' \notin \text{shad}(w)$. Let W be an exposed witnessing path from a' to $z_R(a')$ in P. By Claim 125, we have $W \subseteq \mathcal{R}_{j_6}$. Therefore, we can apply Proposition 90.(L).(ii) to obtain that $W_L(w)$ is not left of $W_L(z_R(a'))$. It follows that one of the following options holds: (a) $W_L(z_R(a'))$ is left of $W_L(w)$, (b) $W_L(z_R(a'))$ is a subpath of $W_L(w)$, (c) $W_L(w)$ is a subpath of $W_L(z_R(a'))$.

Recall that the goal is to prove (35). Suppose to the contrary that $b' \in \text{shad}(b_{j_7})$. By Claim 126, $b' \in \mathcal{R}_{j_6}$. By Claim 108.(i), u_{j_6} is left of b_{j_7}. By Claim 121.(i), $b_{j_7} \notin \mathcal{R}_{j_6}$. It follows that we can apply Proposition 91.(L) to obtain that $q_{j_6} \leqslant w$ in P and $b' \in \text{shad}(w)$.

If $W_L(w)$ is a subpath of $W_L(z_R(a'))$ (option (c)), then $b' \in \text{shad}(w) \subseteq \text{shad}(z_R(a'))$ (by Proposition 33), which contradicts b' being left of $z_R(a')$ ($(a', b') \in I$ and Proposition 57). Therefore, we assume that $W_L(z_R(a'))$ is left of $W_L(w)$ (option (a)) or $W_L(z_R(a'))$ is a subpath of $W_L(w)$ (option (b)). Since $W_L(b')$ is left of $W_L(z_R(a'))$ (by Proposition 57), we obtain that $W_L(b')$ is left of $W_L(w)$. Recall that the supposition that we made (i.e. $b' \in \text{shad}(b_{j_7})$) implies $b' \in \text{shad}(w)$. However, Proposition 30.(L) yields that in this case $W_L(b')$ is not left of $W_L(w)$, which is a final contradiction that ends the proof of (35).

Next, we argue that

$$W_R(b') \text{ is left of } W_R(b_{j_7}). \tag{36}$$

Since $b' \in \text{int}\,\mathcal{R}_{j_6}$ (by Claim 126), by Proposition 60.(ii), $W_R(b')$ is left of γ_{R,j_6}. Since $b_{j_7} \notin \mathcal{R}_{j_6}$ (Claim 121.(i)), by Proposition 60.(iii), γ_{R,j_6} is left of $W_R(b_{j_7})$. Altogether, we obtain $W_R(b')$ is left of $W_R(b_{j_7})$, hence, (36) follows. Finally, (36) and Proposition 30.(R) imply that $b_{j_7} \notin \text{shad}(b')$. Furthermore, by Corollary 34 implies that b' is left of b_{j_7}. $\lhd$

Claim 129. $((a', b'), (a_{j_8}, b_{j_8}))$ satisfies $(R12)$.

Proof. We need to prove that $M_R(a')$ is left of $M_R(a_{j_8})$. To this end, we show that

$$M_R(a') \text{ is left of } \gamma_{R,j_6}. \tag{37}$$

Let W' be an exposed witnessing path from a' to $z_R(a')$ in P. By Claim 125, we have $W' \subseteq \mathcal{R}_{j_6}$. In particular, $z_R(a') \in \mathcal{R}_{j_6}$. If $z_R(a') \in \text{int}\,\mathcal{R}_{j_6}$, then by Proposition 60.(ii), $W_R(a')$ is left of γ_{R,j_6}, which implies (37). Thus, we can assume that $z_R(a') \in \partial\mathcal{R}_{j_6}$. If $z_R(a')$

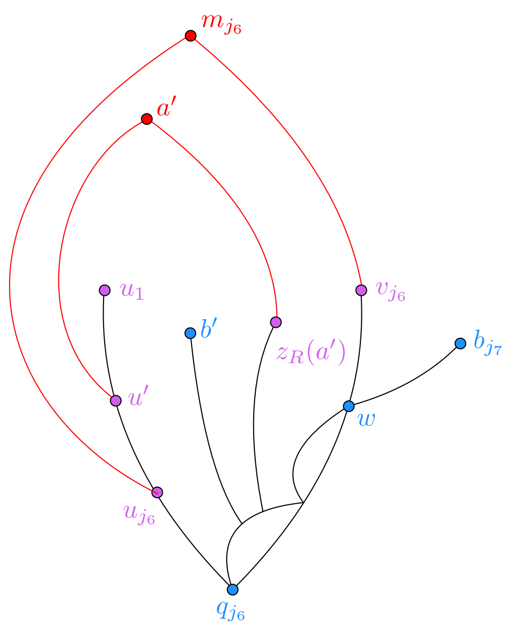


FIGURE 55. An illustration of the proof of Claim 128.

lies in $q_{j_6}[W_L(u_{j_6})]u_{j_6}$, then $z_R(a') \leqslant u_{j_6} \leqslant z_L(a')$ in P (by Claim 127), which contradicts $z_L(a')$ being left of $z_R(a')$ (by Proposition 57). It follows that $z_R(a')$ lies strictly on the right side of $\mathcal{R}_{j_6}$. Recall that $W' \subseteq \mathcal{R}_{j_6}$. Finally, by (rg4), $x_0[W_R(v_{j_6})]z_R(a')[W']a'$ is left of γ_{R,j_6}, which again gives (37).

By Claim 121.(i), $b_{j_8} \notin \mathcal{R}_{j_6}$. It follows that by Proposition 60.(iii), γ_{R,j_6} is left of $W_R(b_{j_8})$. This and (37) gives that $M_R(a')$ is left of $W_R(b_{j_8})$. Finally, $W_R(b_{j_8})$ is left of $W_R(z_R(a_{j_8}))$ by Proposition 57, thus, $M_R(a')$ is left of $M_R(a_{j_8})$. ◁

Claim 130. $((a', b'), (a_{j_8}, b_{j_8}))$ satisfies ($L21^\star$).

Proof. Let $u = u_{j_8}$ and $U = U_{j_8}$. We claim that (u, U) witnesses ($L21^\star$) for $((a', b'), (a_{j_8}, b_{j_8}))$. Note that $u \in Z(a_{j_8})$ (by (u2)) and U is an exposed witnessing path from a_{j_8} to u in P (by (u6)). Recall that by (u3), u and u_{j_6} lie in $W^\star$. First, we show that

$$\pi(b_{j_7}) \leqslant \pi(b') \text{ in } P. \tag{38}$$

By Proposition 57, $W^\star = W_L(z_L(a))$ is left of $W_L(b)$. Thus, since $W_L(b)$ is left of $W_L(b')$, we have $W^\star$ left of $W_L(b')$. By Claim 128, $W_L(b')$ is left of $W_L(b_{j_7})$. Note that by definition, $\pi(b_{j_7})$ lies in both $W^\star$ and $W_L(b_{j_7})$. The paths $W^\star$, $W_L(b')$, and $W_L(b_{j_7})$ are pairwise x_0-consistent (by Proposition 15.(L)). Therefore, by Corollary 16, $\pi(b_{j_7})$ also lies in $W_L(b')$, and so, $\pi(b_{j_7}) \leqslant \pi(b')$ in P, which yields (38).

To get the assertion, we have to prove that u lies in both $W_L(b')$ and $W_L(z_L(a'))$, and that $M = x_0[W_L(u)]u[U]a_{j_8}$ is left of $W_L(z_L(a'))$. Note that by (u5), Claim 105.(i), and (38) we have $u < u_{j_7+1} \leqslant \pi(b_{j_7}) \leqslant \pi(b')$ in P. This implies that u lies in $W_L(b')$, as desired. By Claim 127, u_{j_6} lies in $W_L(z_L(a'))$. Since $u < u_{j_6}$ in P (by Claim 105.(i)), we obtain that u lies in $W_L(z_L(a'))$, as desired. Finally, by (u7), M is left of $W^\star$. Since $W_L(u)$ is a proper subpath of $W_L(u_{j_6})$ (by Claim 105.(ii)), $W_L(u_{j_6})$ is a subpath of $W^\star$, and $W_L(u_{j_6})$ is a subpath of $W_L(z_L(a'))$ (by Claim 127), we obtain that M is left of $W_L(z_L(a'))$, as desired. This ends the proof of the claim. ◁

By definition, $(a', b'), (a_{j_8}, b_{j_8}) \in I$. Moreover, by Claim 128 we have b' is left of b_{j_7} which is left of b_{j_8}, and therefore $((a', b'), (a_{j_8}, b_{j_8}))$ is a regular sequence. Finally, Claims 129 and 130 imply that $((a', b'), (a_{j_8}, b_{j_8}))$ is an edge in H_{IO}. The following concludes the proof (20), and so, of Lemma 75.

$$\begin{aligned} \text{max-start-weight}(H_{\mathrm{IO}}, (a', b')) &\geqslant |\{i : i \in \{j_8, \ldots, n-1\} \text{ and } \sigma_i \text{ is of weight } 1\}| \\ &= |E| - |E \cap [j_8 - 1]| \\ &= |E| - |E \cap [j_8]| + 1 \\ &\geqslant |E| - |E_0| + 1 \\ &> \text{max-start-weight}(H_{\mathrm{IO}}, (a, b)) - m. \end{aligned} \tag{39}$$

The first inequality follows from the fact that $((a', b'), (a_{j_8}, b_{j_8}), (a_{j_8+1}, b_{j_8+1}), \ldots, (a_n, b_n))$ is a path in H_{IO}. The third and fourth lines follow by $j_8 \in E_2 \subseteq E_0$ and therefore $E \cap [j_8] \subseteq E_0$. □

9. Proof of the main result

In this section, we wrap up the proof of the main theorem. In fact, we prove a more general statement (Theorem 131), which immediately implies that the class of posets that are subposets of posets with planar cover graphs is also dim-bounded (Corollary 132). Note that a subposet of a poset with a planar cover graph does not necessarily have a planar cover graph, e.g. standard examples. Finally, note that Theorem 131 implies Theorem 1 by taking $I = \mathrm{Inc}(P)$.

Theorem 131. *For every poset P with a planar cover graph and for every $I \subseteq \mathrm{Inc}(P)$,*

$$\dim_P(I) \leqslant 64s^6 \cdot (s+3)^2 + 12,$$

where $s = \mathrm{se}_P(\mathrm{Inc}(P) \cap (\pi_1(I) \times \pi_2(I)))$.

Proof. Let P be a poset with a planar cover graph and let $I \subseteq \mathrm{Inc}(P)$. By Corollary 13, there exists an instance $(P', x'_0, G', e'_{-\infty}, I')$ such that $P' - x'_0$ is a convex subposet of P and $I' \subseteq I$, or $P' - x'_0$ is a convex subposet of P^{-1} and $I' \subseteq I^{-1}$, and

$$\dim_P(I) \leqslant 2 \dim_{P'}(I').$$

In particular, since $P' - x'_0$ is a subposet of P and $I' \subseteq I$ or $P' - x'_0$ is a subposet of P^{-1} and $I' \subseteq I^{-1}$ we have

$$\mathrm{se}_{P'}(\mathrm{Inc}(P') \cap (\pi_1(I') \times \pi_2(I')) \leqslant \mathrm{se}_P(\mathrm{Inc}(P) \cap (\pi_1(I) \times \pi_2(I)).$$

By Corollary 47, there exists a good instance $(Q, x_0, G, e_{-\infty}, J)$ such that Q is a convex subposet of P', $J \subseteq I'$, and

$$\dim_{P'}(I') \leqslant 2 \dim_Q(J) + 6.$$

In particular, since Q is a subposet of P' and $J \subseteq I'$

$$\mathrm{se}_Q(\mathrm{Inc}(Q) \cap (\pi_1(J) \times \pi_2(J)) \leqslant \mathrm{se}_{P'}(\mathrm{Inc}(P') \cap (\pi_1(I') \times \pi_2(I')).$$

By Proposition 48, there exists $J \subseteq J^+ \subseteq \mathrm{Inc}(Q) \cap (\pi_1(J) \times \pi_2(J))$ such that $(Q, x_0, G, e_{-\infty}, J^+)$ is a maximal good instance. In particular,

$$\begin{aligned} &\dim_Q(J) \leqslant \dim_Q(J^+) \text{ and} \\ &\mathrm{se}_Q(J^+) \leqslant \mathrm{se}_Q(\mathrm{Inc}(Q) \cap (\pi_1(J) \times \pi_2(J)). \end{aligned}$$

By Theorem 49,

$$\dim_Q(J^+) \leqslant 16 \, \mathrm{se}_Q(J^+)^6 \cdot (\mathrm{se}_Q(J^+) + 3)^2.$$

Observe that

$$\begin{aligned}\mathrm{se}_Q(J^+) &\leqslant \mathrm{se}_Q(\mathrm{Inc}(Q)\cap(\pi_1(J)\times\pi_2(J))\\ &\leqslant \mathrm{se}_{P'}(\mathrm{Inc}(P')\cap(\pi_1(I')\times\pi_2(I'))\\ &\leqslant \mathrm{se}_P(\mathrm{Inc}(P)\cap(\pi_1(I)\times\pi_2(I)).\end{aligned}$$

Summarizing, we obtain

$$\begin{aligned}\dim_P(I) &\leqslant 2\dim_{P'}(I')\\ &\leqslant 4\dim_Q(J)+12\\ &\leqslant 4\dim_Q(J^+)+12\\ &\leqslant 4(16\,\mathrm{se}_Q(J^+)^6\cdot(\mathrm{se}_Q(J^+)+3)^2)+12\\ &\leqslant 64s^6\cdot(s+3)^2+12,\end{aligned}$$

where $s=\mathrm{se}(\mathrm{Inc}(P)\cap(\pi_1(I)\times\pi_2(I))$. □

Corollary 132. *The class of posets that are subposets of posets with planar cover graphs is* dim-*bounded.*

Proof. Let $f(s)=64s^6\cdot(s+3)^2+12$ for every positive integer s. Let P be a poset with a planar cover graph and let Q be a subposet of P. Let $I\subseteq\mathrm{Inc}(P)$ be the set of all incomparable pairs (a,b) in P such that both a and b are elements of Q. By definition, $I=\mathrm{Inc}(P)\cap(\pi_1(I)\times\pi_2(I))$. In particular, $I=\mathrm{Inc}(Q)$, $\dim_Q(I)=\dim_P(I)$, and $\mathrm{se}_Q(I)=\mathrm{se}_P(I)$. Therefore, by Theorem 131,

$$\dim(Q)=\dim_Q(I)=\dim_P(I)\leqslant f(\mathrm{se}_P(I))=f(\mathrm{se}_Q(I))=f(\mathrm{se}(Q)).$$ □

References

[1] Heather Smith Blake, Piotr Micek, and William T. Trotter. Boolean dimension and dim-boundedness: Planar cover graph with a zero, 2022. arXiv:2206.06942.

[2] Parinya Chalermsook, Bundit Laekhanukit, and Danupon Nanongkai. Graph products revisited: Tight approximation hardness of induced matching, poset dimension and more. In *Proceedings of the Twenty-Fourth Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, 2013. arXiv:1212.4129.

[3] Reinhard Diestel. *Graph Theory, 6th Edition*, volume 173 of *Graduate texts in mathematics*. Springer, 2024.

[4] Robert P. Dilworth. A decomposition theorem for partially ordered sets. *Annals of Mathematics*, 51(1):161, 1950.

[5] Ben Dushnik and Edwin W. Miller. Partially ordered sets. *American Journal of Mathematics*, 63:600–610, 1941.

[6] Stefan Felsner, Irina Mustaţă, and Martin Pergel. The complexity of the partial order dimension problem – closing the gap. *SIAM Journal on Discrete Mathematics*, 31(1):172–189, 2017. arXiv:1501.01147.

[7] Stefan Felsner, Tamás Mészáros, and Piotr Micek. Boolean dimension and tree-width. *Combinatirica*, 40:655–677, 2020. arXiv:1707.06114.

[8] Stefan Felsner and William T Trotter. Dimension, graph and hypergraph coloring. *Order*, 17:167–177, 2000.

[9] Stefan Felsner, William T. Trotter, and Veit Wiechert. The dimension of posets with planar cover graphs. *Graphs and Combinatorics*, 31(4):927–939, 2014.

[10] Zoltán Füredi, Péter Hajnal, Vojtěch Rödl, and William T. Trotter. Interval orders and shift graphs. *Sets, Graphs and Numbers*, 60:297–313, 1991.

[11] Michael R. Garey and David S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W. H. Freeman, 1979.

[12] Maximilian Gorsky and Michał T. Seweryn. Posets with k-outerplanar cover graphs have bounded dimension, 2021. arXiv:2103.15920.

[13] Toshiro Hiraguchi. On the dimension of orders. *The Scientific Reports of Kanazawa University*, 4:1–20, 1955.

[14] Tony Huynh, Gwenaël Joret, Piotr Micek, Michał T. Seweryn, and Paul Wollan. Excluding a ladder. *Combinatorica*, 42(3):405–432, 2021. arXiv:2002.00496.

[15] Gwenaël Joret, Piotr Micek, Kevin G. Milans, William T. Trotter, Bartosz Walczak, and Ruidong Wang. Tree-width and dimension. *Combinatorica*, 36(4):431–450, 2016. arXiv:1301.5271.

[16] Gwenaël Joret, Piotr Micek, and Veit Wiechert. Sparsity and dimension. *Combinatorica*, 38(5):1129–1148, 2018. Extended abstract in Proceedings of the Twenty-Seventh Annual ACM-SIAM Symposium on Discrete Algorithms (SODA '16). arXiv:1507.01120.

[17] Gwenaël Joret, Piotr Micek, Patrice Ossona de Mendez, and Veit Wiechert. Nowhere dense graph classes and dimension. *Combinatorica*, 39(5):1055–1079, 2019. arXiv:1708.05424.

[18] Gwenaël Joret, Piotr Micek, Michał Pilipczuk, and Bartosz Walczak. Cliquewidth and dimension. In *Proceedings of the 2024 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*, page 1437–1446. Society for Industrial and Applied Mathematics, 2024. arXiv:2308.11950.

[19] Gwenaël Joret, Piotr Micek, and Veit Wiechert. Planar posets have dimension at most linear in their height. *SIAM Journal on Discrete Mathematics*, 31(4):2754–2790, 2017. arXiv:1612.07540.

[20] David Kelly. On the dimension of partially ordered sets. *Discrete Mathematics*, 35(1):135–156, 1981. Special Volume on Ordered Sets, doi.org/10.1016/0012-365X(81)90203-X.

[21] Jakub Kozik, Piotr Micek, and William T. Trotter. Dimension is polynomial in height for posets with planar cover graphs. *Journal of Combinatorial Theory, Series B*, 165:164–196, 2024. arXiv:1907.00380.

[22] Piotr Micek and Veit Wiechert. Topological minors of cover graphs and dimension. *Journal of Graph Theory*, 86(3):295–314, 2017. arXiv:1504.07388.

[23] Walter Schnyder. Planar graphs and poset dimension. *Order*, 5(4):323–343, 1989.

[24] Alex Scott and Paul Seymour. A survey of χ-boundedness. *Journal of Graph Theory*, 95, 2020. arXiv:1812.07500.

[25] Michał T. Seweryn. Improved bound for the dimension of posets of treewidth two. *Discrete Mathematics*, 343(1):111605, 2020. arXiv:1902.01189.

[26] Noah Streib and William T. Trotter. Dimension and height for posets with planar cover graphs. *European Journal of Combinatorics*, 35:474–489, 2014. doi:10.1016/j.ejc.2013.06.017.

[27] William T. Trotter. Order preserving embeddings of aographs. *Congressus Numerantium*, 642:572–579, 1978.

[28] William T. Trotter. *Combinatorics and partially ordered sets: Dimension theory*. Johns Hopkins Series in the Mathematical Sciences. Johns Hopkins University Press, Baltimore, MD, 1992.

[29] William T. Trotter, Jr. and John I. Moore, Jr. The dimension of planar posets. *Journal of Combinatorial Theory, Series B*, 22(1):54–67, 1977. doi.org/10.1016/0095-8956(77)90048-X.

[30] Bartosz Walczak. Minors and dimension. *Journal of Combinatorial Theory, Series B*, 122:668–689, 2017. Extended abstract in Proceedings of the Twenty-Sixth Annual ACM-SIAM Symposium on Discrete Algorithms (SODA '15). arXiv:1407.4066.

[31] Mihalis Yannakakis. The complexity of the partial order dimension problem. *SIAM Journal on Algebraic Discrete Methods*, 3(3):351–358, 1982.