A QUANTITATIVE FRAMEWORK FOR SETS OF EXACT APPROXIMATION ORDER BY RATIONAL NUMBERS

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ABSTRACT. In this paper we study a quantitative notion of exactness within Diophantine approximation. Given $\Psi : (0, \infty) \to (0, \infty)$ and $\omega : (0, \infty) \to (0, 1)$ satisfying $\lim_{q\to\infty} \omega(q) = 0$, we study the set of points, which we call $E(\Psi, \omega)$, that are Ψ-well approximable but not $\Psi(1-\omega)$-well approximable. We prove results on the cardinality and dimension of $E(\Psi, \omega)$. In particular we obtain the following general statements:

- For any $\omega : (0, \infty) \to (0, 1)$ and $\tau > 2$ there exists $\Psi : (0, \infty) \to (0, \infty)$ such that $\lim_{q\to\infty} \frac{-\log \Psi(q)}{\log q} = \tau$ and $E(\Psi, \omega) \neq \emptyset$.
- Under natural monotonicity assumptions on Ψ and ω, we prove that if ω decays to zero sufficiently slowly (in a way that depends upon Ψ) then $E(\Psi, \omega)$ is uncountable. Moreover, under further natural assumptions on Ψ we can calculate the Hausdorff dimension of $E(\Psi, \omega)$.

Our main result demonstrates a new threshold for the behaviour of $E(\Psi, \omega)$. A particular instance of this threshold is illustrated by considering functions of the form $\Psi_\tau(q) = q^{-\tau}$ when $\tau \in \mathbb{N}_{\geq 3}$. For these functions we prove the following:

- If $\omega(q) = Cq^{-\tau(\tau-1)}$ for some sufficiently large C or $\omega(q) = q^{-\tau'}$ for some $\tau' < \tau(\tau - 1)$, then $E(\Psi_\tau, \omega)$ is uncountable and we calculate its Hausdorff dimension.
- If $\omega(q) < cq^{-\tau(\tau-1)}$ for some $c \in (0, 1)$ for all q sufficiently large then $E(\Psi_\tau, \omega) = \emptyset$.

1. INTRODUCTION

Diophantine approximation in $\mathbb{R}$ is concerned primarily with how well real numbers can be approximated by rational numbers. This can be made precise within the following general framework: Given a non-increasing function $\Psi : (0, \infty) \to (0, \infty)$ we say that $x \in \mathbb{R}$ is Ψ-approximable if

$$\left| x - \frac{p}{q} \right| \leq \Psi(q)$$

for i.m. $(p, q) \in \mathbb{Z} \times \mathbb{N}$. Here and throughout we use i.m. as a shorthand for infinitely many. We denote the set of Ψ-approximable numbers by $W(\Psi)$. The sets $W(\Psi)$ are of central interest in Diophantine approximation and their properties are well understood.

Suppose $x \in W(\Psi)$ for some Ψ, it is natural to ask whether Ψ is in a sense the optimal approximation function for x. This problem was first considered from a

2020 *Mathematics Subject Classification.* 11J04, 11J70, 11J83.

Key words and phrases. Diophantine approximation, Exact approximation order, Fractal geometry.

metric perspective by Güting in [19]. Given $\tau \geq 2$ let

$$W(\tau) = \left\{ x \in \mathbb{R} : \left| x - \frac{p}{q} \right| \leq q^{-\tau} \text{ for i.m. } (p,q) \in \mathbb{Z} \times \mathbb{N} \right\},$$

and for $x \in \mathbb{R}$ let $V(x) = \sup\{\tau : x \in W(\tau)\}$. Güting proved in [19] that for $\tau \geq 2$ we have

$$\dim_H(\{x : V(x) = \tau\}) = \dim_H(W(\tau)) = \frac{2}{\tau},$$

where $\dim_H$ denotes the Hausdorff dimension. That is, the set of points for which τ is the optimal exponent of approximation has the same Hausdorff dimension as the set of points that are τ-approximable. Note that the formula $\dim_H(W(\tau)) = \frac{2}{\tau}$ was established prior to this independently by Jarnik [23] and Besicovitch [7]. In [5] Beresnevich, Dickinson and Velani proved an analogue of Güting's result where the quality of approximation is viewed at a finer logarithmic scale. The authors of [5] also introduced the following analogue of the set of badly approximable numbers: Given a non-increasing $\Psi : (0, \infty) \to (0, \infty)$ we say that $x \in \mathbb{R}$ is Ψ-badly approximable if $x \in W(\Psi)$ and there exists $c > 0$ such that for all sufficiently large $q \in \mathbb{N}$ we have

$$\left| x - \frac{p}{q} \right| \geq c\Psi(q)$$

for all $p \in \mathbb{Z}$. We let $\mathrm{Bad}(\Psi)$ denote the set of Ψ-badly approximable points. If $x \in \mathrm{Bad}(\Psi)$ then Ψ can be viewed as an optimal approximation function for x because $x \notin W(\Psi')$ for any Ψ' satisfying $\Psi' = o(\Psi)$. In [5] the authors posed the problem of determining the Hausdorff dimension of $\mathrm{Bad}(\Psi)$ for a general Ψ. Note that for any function Ψ tending to zero, by Cassels' Scaling Lemma [11], we have that $\mathrm{Bad}(\Psi)$ is a Lebesgue nullset. In this paper, the sets we will be interested in will be a subset of $\mathrm{Bad}(\Psi)$ for some Ψ. Hence they are nullsets and so when considering the size of these sets we will use Hausdorff dimension.

A more precise framework for understanding those points for which Ψ is the optimal approximation function is provided by the notion of Ψ-exact approximation order: We say that x has Ψ-exact approximation order if $x \in W(\Psi)$ and there exists a function $c_x : \mathbb{N} \to (0,1)$ depending on x which satisfies $\lim_{q\to\infty} c_x(q) = 1$ and

$$\left| x - \frac{p}{q} \right| \geq c_x(q)\Psi(q)$$

for all $(p,q) \in \mathbb{Z} \times \mathbb{N}$. We let $E(\Psi)$ denote the set of points with Ψ-exact approximation order. If $x \in E(\Psi)$ then Ψ can be viewed as the optimal approximation function for x because $x \in W(\Psi)$ but $x \notin W(c\Psi)$ for any $c \in (0,1)$. By definition, for any Ψ we have the inclusions

$$E(\Psi) \subset \mathrm{Bad}(\Psi) \subset W(\Psi).$$

In the special case where $\Psi(q) = q^{-2}$ then $W(\Psi)$ has full Lebesgue measure and $\mathrm{Bad}(\Psi)$, which is now the classical set of badly approximable points, has zero Lebesgue measure and full Hausdorff dimension. The set $E(\Psi)$ is empty. It was shown by

Moreria that if one considers $\Psi_c(q) = cq^{-2}$, then $E(\Psi_c)$ has positive Hausdorff dimension for c sufficiently small [30, Theorem 2]. In the case where $\Psi(q) = o(q^{-2})$ these sets exhibit different behaviour. Jarnik showed in [23, Satz. 6 (p.539)] that when $\Psi(q) = o(q^{-2})$ and non-increasing then $E(\Psi)$ is non-empty. In [8] Bugeaud answered the problem posed by Beresnevich, Dickinson and Velani in [5] in a strong sense and proved that when Ψ is non-increasing and satisfies some additional decay assumptions then $\dim_H(E(\Psi)) = \dim_H(\mathrm{Bad}(\Psi)) = \dim_H(W(\Psi))$. These additional assumptions were subsequently removed in a follow up paper by Bugeaud and Moreira [10, Theorem 4] who showed that when Ψ is non-increasing and $\Psi(q) = o(q^{-2})$, then $\dim_H(E(\Psi)) = \dim_H(\mathrm{Bad}(\Psi)) = \dim_H(W(\Psi))$. For more on this topic we refer the reader to [9, 10, 30] and the references therein. For more on the higher dimensional version of this problem we refer the reader to [3, 17, 28, 34], and for this problem in other settings see [4, 6, 22, 32, 37, 38]. One can also consider other notions of the size of $E(\Psi)$ such as the packing dimension [33], or Fourier dimension [15, 16].

1.1. **A quantitative framework for exactness.** We are concerned with providing a more quantitative framework for understanding those points for which Ψ is the optimal approximation function. Given a non-increasing function $\Psi : (0, \infty) \to (0, \infty)$ and a function $\omega : (0, \infty) \to (0, 1)$ satisfying $\lim_{q\to\infty} \omega(q) = 0$, we associate the set of $x \in \mathbb{R}$ satisfying $x \in W(\Psi)$ and for all $q \in \mathbb{N}$ sufficiently large we have

$$\left| x - \frac{p}{q} \right| \geq (1 - \omega(q))\, \Psi(q)$$

for all $p \in \mathbb{Z}$. We say that a point is of exact (Ψ, ω)-order of approximation if it satisfies these two properties, and denote the set of points by $E(\Psi, \omega)$. If $x \in E(\Psi, \omega)$ then Ψ is the optimal approximation function for Ψ in the sense that $x \in W(\Psi)$ but $x \notin W((1 - \omega)\Psi)$. Note that unlike in the definition of $E(\Psi)$ the function ω is uniform over $x \in \mathbb{R}$ and so provides a more quantitative framework by which we can assert that Ψ is the optimal approximation function. We note that it is a consequence of our assumption $\lim_{q\to\infty} \omega(q) = 0$ that for any Ψ and ω we have the inclusions

$$E(\Psi, \omega) \subset E(\Psi) \subset \mathrm{Bad}(\Psi) \subset W(\Psi).$$

In [1] Akhunzhanov and Moschevitin introduced the related set of points $x \in \mathbb{R}$ satisfying $x \in W(\Psi(1 + \omega))$ and for all $q \in \mathbb{N}$ sufficiently large we have

$$\left| x - \frac{p}{q} \right| \geq \Psi(q)(1 - \omega(q))$$

for all $p \in \mathbb{Z}$. This can be considered as a symmetric version of our set $E(\Psi, \omega)$. They proved that if $q^2\Psi(q)$ is non-increasing and $\omega(q) \geq Cq^2\Psi(q)$ for all q sufficiently large, for some determined fixed constant $C > 0$, then the corresponding set is non-empty. A later paper by Akhunzhanov [2] improved this further by showing that the same conclusion holds under the weaker assumption that $\omega(q) \geq C\left(q^2\Psi(q)\right)^2$ for all q sufficiently large. Note that this work also studied a higher dimensional analogue of this problem, and in particular implies [3, Theorem 2] in the symmetric setting.

The aforementioned results motivate the following question:

Q1. For which Ψ and ω is $E(\Psi,\omega)$ non-empty?

At first glance it might seem reasonable to expect that $E(\Psi,\omega)$ is non-empty for every pair (Ψ,ω). For each Ψ the set $W(\Psi)$ is reasonably well distributed within the unit interval (for example, each $W(\Psi)$ has full packing dimension and satisfies the large intersection property [14]), so one could expect that changing the order of Ψ, by however small an amount, would lead to an increase in points in the set. Indeed, the non-emptiness of the sets $E(\Psi)$ supports this idea. However, this intuitive guess turns out to be incorrect. The obstruction to $E(\Psi,\omega)$ being non-empty comes from the fact $E(\Psi,\omega)$ is not just a lim sup set, it is the intersection of a lim sup set with a lim inf set. For all q sufficiently large, we have to be careful to rule out rational approximations that satisfy $|x-\frac{p}{q}|<\Psi(q)\,(1-\omega(q))$. In Theorem 1.6 below we show that there are large families of pairs of functions (Ψ,ω) such that $E(\Psi,\omega)=\emptyset$. In other words, we show that there exists functions $\Psi,\Phi:(0,\infty)\to(0,\infty)$ such that

$$\Psi(q)<\Phi(q) \quad \text{for all} \quad q\in(0,\infty) \quad \text{and} \quad W(\Psi)=W(\Phi).$$

See also Theorem 1.5 below. During the preparation of this paper a preprint of Moshchevitin and Pitcyn [31] was released which shows a similar result. We reserve further comments to Remark 1.8 below. As highlighted in [31] the only previously known approximation functions where this phenomenon occurs is when one considers approximation functions related to the gaps in the Markoff-Lagrange spectrum [12, 29], for example $\Psi(q)=\frac{1}{\sqrt{5}q^2}$ and $\Phi(q)=\frac{1}{(\sqrt{5}+\varepsilon)q^2}$ for $\varepsilon>0$. See also [20].

If the set $E(\Psi,\omega)$ is non-empty, one can ask about the finer metric properties of $E(\Psi,\omega)$. We can easily see via Cassels' Scaling Lemma that $E(\Psi,\omega)$ is always a Lebesgue nullset, so we consider its Hausdorff dimension. It is an immediate consequence of the definition of $E(\Psi,\omega)$ that if $x\in E(\Psi,\omega)\cap[0,1]$ then x will belong to one of $2(q+1)$ intervals of size $\Psi(q)\omega(q)$ for infinitely many $q\in\mathbb{N}$. The following upper bound for the Hausdorff dimension of $E(\Psi,\omega)$ follows from this observation:

$$\dim_H(E(\Psi,\omega))\leq\inf\left\{s\geq 0:\sum_{q=1}^{\infty}q(\Psi(q)\omega(q))^s<\infty\right\}. \tag{1.1}$$

This leads us to our next question:

Q2. For which Ψ and ω do we have equality in (1.1)?

In Theorems 1.10-1.11 below we give sufficient conditions on ω (depending on Ψ) ensuring that we have equality in (1.1). Jarnik's theorem [23], together with [10], imply that for a general family of Ψ, the Hausdorff dimension of $E(\Psi),\mathrm{Bad}(\Psi)$ and $W(\Psi)$ coincide with the right hand side of (1.1) when ω is taken to be a strictly positive constant function. Thus, if ω decays to zero sufficiently quickly, we should not expect the Hausdorff dimension of $E(\Psi,\omega)$ to coincide with that of $E(\Psi),\mathrm{Bad}(\Psi)$ and $W(\Psi)$. Conversely, if ω decays to zero sufficiently slowly then the dimensions should coincide. This is shown to be the case in Corollary 1.13 below.

In this paper we address both Q1 and Q2. In the special case when Ψ satisfies an additional divisibility criteria, then we obtain complete answers to both of these questions (see Theorems 1.5 and 1.11).

1.2. **Main results: Q1.** We prove the following results on $E(\Psi,\omega)$.

Theorem 1.1. *Suppose that* $\lim_{q\to\infty} q^2\Psi(q) = \lim_{q\to\infty}\omega(q) = 0$ *and* $q \to q^2\Psi(q)$ *is non-increasing. Furthermore, suppose that*

$$\omega(q) \geq \frac{45}{\Psi(q)}\Psi\left(\frac{1}{2q\Psi(q)}\right) \tag{1.2}$$

for all q *sufficiently large. Then* $E(\Psi,\omega)$ *is uncountable.*

Theorem 1.1 allows us to conclude that $E(\Psi,\omega) \neq \emptyset$ and uncountable under a weaker assumption on ω then that imposed by Akhunzhanov in [2]. The following corollary makes our improvement on their result clear in the case of power functions.

Corollary 1.2. *Suppose that* $\Psi(q) = q^{-\tau}$ *for some* $\tau > 2$ *and that* $\omega(q) = Cq^{-\tau(\tau-2)}$ *for some* $C > 0$. *Then* $E(\Psi,\omega)$ *is uncountable if* C *is sufficiently large.*

Remark 1.3. If ω,ω' satisfy $\omega(q) \leq \omega'(q)$ for all q sufficiently large then $E(\Psi,\omega) \subseteq E(\Psi,\omega')$ for any Ψ. Thus Theorem 1.1 is equivalent to the following statement: If Ψ satisfies the conditions of Theorem 1.1 and $\omega : (0,\infty) \to (0,\infty)$ is given by

$$\omega(q) = \frac{45}{\Psi(q)}\Psi\left(\frac{1}{2q\Psi(q)}\right),$$

then $E(\Psi,\omega)$ is uncountable. This may be viewed as the correct way to present Theorem 1.1 However, to avoid this theorem being misinterpreted as applying to only one function ω, we chose to formulate it with condition (1.2). We will adopt this convention throughout.

Remark 1.4. Equation (1.2) may not appear natural. To explain where this bound comes from and to aid the reader with our later proofs, we provide a brief heuristic idea for where this condition appears. It is good to keep in mind the ideas discussed in the following when reading the technical proofs that appear in Section 4. Let Ψ satisfy the assumptions of Theorem 1.1 and let us try to see what sort of bound we should expect on ω. Suppose $E(\Psi,\omega) \neq \emptyset$ and let $x \in E(\Psi,\omega)$. Then there exists $(p,q) \in \mathbb{Z}\times\mathbb{N}$, with q sufficiently large so that $\Psi(q) < \frac{1}{2q^2}$, for which

$$(1-\omega(q))\Psi(q) \leq \left|x - \frac{p}{q}\right| \leq \Psi(q)\,.$$

Since $\Psi(q) < \frac{1}{2q^2}$ we must have that $\frac{p}{q} = \frac{p_n}{q_n}$ is a convergent of x by Legendre's theorem (see Lemma 2.3) and the next partial quotient a_{n+1} must be of the order $\frac{1}{q^2\Psi(q)}$ (see Section 2 for definitions and relevant background on continued fractions). Note that a_{n+1} can not be too much bigger than this, otherwise we would have $\left|x - \frac{p}{q}\right| < (1-\omega(q))\Psi(q)$, which is not allowed. Using properties of continued fractions and our bound on a_{n+1}, we have that the denominator q_{n+1} of the next convergent of x must be of the order $\frac{1}{q\Psi(q)}$. Now let us suppose we are in the worst case scenario where $\frac{p_{n+1}}{q_{n+1}}$ lies in the interval $[\frac{p}{q}+(1-\omega(q))\Psi(q), \frac{p}{q}+\Psi(q)]$. Hence, since x also belongs to this interval, we have that

$$\left| x - \frac{p_{n+1}}{q_{n+1}} \right| < \omega(q)\Psi(q)\,. \tag{1.3}$$

Now, considering the next partial quotient a_{n+2}, we must suppose again that a_{n+2} is bounded from above by some constant multiple of $\frac{1}{q_{n+1}^2\Psi(q_{n+1})}$, otherwise $\frac{p_{n+1}}{q_{n+1}}$ will approximate x too well. Using properties of convergents of continued fractions and our bound on a_{n+2}, we have that

$$\left| x - \frac{p_{n+1}}{q_{n+1}} \right| \geq \frac{1}{3a_{n+2}q_{n+1}^2} \gg \Psi(q_{n+1}) \gg \Psi\left(\frac{1}{q\Psi(q)}\right)\,.$$

Combining this with (1.3) we must have that

$$\omega(q)\Psi(q) \gg \Psi\left(\frac{1}{q\Psi(q)}\right)$$

which is essentially our condition (1.2).

Note that a crucial part in the above heuristics was that the next convergent of our point of interest lies inside the interval to be avoided. If we impose suitable divisibility conditions on our approximation function Ψ we can do much better than assuming this worst case scenario. To that end, we say a function Ψ satisfies the *square divisibility condition* if for every $q\in\mathbb{N}$ we have $\Psi(q)^{-1}=0 \mod q^2$. Examples of functions satisfying the square divisibility condition include $\Psi(q)=q^{-\tau}$ for some $\tau\in\mathbb{N}_{\geq 2}$. Moreover, for any $\tau\geq 2$ there exists a function Ψ satisfying the square divisibility condition and $\lim_{q\to\infty}\frac{-\log\Psi(q)}{\log q}=\tau$. Restricting to approximation functions satisfying the square divisibility condition we are able to prove the following result.

Theorem 1.5. *Suppose that $\lim\limits_{q\to\infty} q^2\Psi(q) = \lim\limits_{q\to\infty}\omega(q)=0$ and $q\to q^2\Psi(q)$ is non-increasing. Furthermore, suppose that Ψ satisfies the square divisibility condition. Then the following statements hold:*

(1) $E(\Psi,\omega)=\emptyset$ if

$$\omega(q) < \frac{1-\omega\left(\frac{1}{\Psi(q)}\right)}{\Psi(q)}\Psi\left(\frac{1}{\Psi(q)}\right)$$

for all q sufficiently large. In particular, if

$$\omega(q) < \frac{c}{\Psi(q)}\Psi\left(\frac{1}{\Psi(q)}\right)$$

for all q sufficiently large for some $c\in(0,1)$, then $E(\Psi,\omega)=\emptyset$.

(2) $E(\Psi,\omega)$ is non-empty and uncountable if

$$\omega(q) \geq \frac{45}{16\Psi(q)}\Psi\left(\frac{1}{8\Psi(q)}\right)$$

for all q sufficiently large.

The second statement in Theorem 1.5 allows us to conclude that $E(\Psi,\omega)$ is uncountable under a weaker assumption on ω than that appearing in Theorem 1.1. The first statement of this theorem shows that for certain approximation functions Ψ the set $E(\Psi,\omega)$ is empty if ω decays to zero sufficiently quickly. Note that for many natural choices of Ψ the thresholds for non-empty and empty appearing in Theorem 1.5 differ by a multiplicative constant. We refer the reader to Corollary 1.12 where we specialise to functions of the form $q \to q^{-\tau}$. In this special case, the critical threshold identified by Theorem 1.5 is particularly clear.

Statement 1 from Theorem 1.5 can be deduced from the following more general result. This result shows that the family of approximation functions satisfying the square divisibility condition are not the only ones that require a bound on the decay rate of ω to ensure that $E(\Psi,\omega)$ is non-empty.

Theorem 1.6. *Assume that Ψ is such that there exists functions $d:(0,\infty)\to(0,\infty)$ and $e:(0,\infty)\to(0,\infty)$ such that $\Psi=\frac{d}{e}$ and $d(q),e(q)\in\mathbb{N}$ for all $q\in\mathbb{N}$. Then the following statements are true:*

(1) If $\omega:(0,\infty)\to(0,\infty)$ is such that

$$\omega(q) < \frac{1-\omega(qe(q))}{\Psi(q)}\Psi(qe(q))$$

for all q sufficiently large, then $E(\Psi,\omega)=\emptyset$.

(2) Assume $e(q)=0 \mod q$ for all $q\in\mathbb{N}$. If $\omega:(0,\infty)\to(0,\infty)$ is such that

$$\omega(q) < \frac{1-\omega(e(q))}{\Psi(q)}\Psi(e(q))$$

for all q sufficiently large, then $E(\Psi,\omega)=\emptyset$.

Remark 1.7. If Ψ satisfies the square divisibility condition then we define $d:(0,\infty)\to(0,\infty)$ and $e:(0,\infty)\to(0,\infty)$ according to the rules $d(q)=1$ and $e(q)=\frac{1}{\Psi(q)}$ for all q. With this formulation it is now clear that the second statement of Theorem 1.6 implies the first statement of Theorem 1.5.

Remark 1.8. In the preprint of Moshchevitin and Pitcyn [31] it is shown that $E(\Psi,\omega)=\emptyset$ for a range of approximation functions Ψ and ω. In particular [31, Theorem 2] considers approximation functions of the same form as those appearing in Theorem 1.6. Indeed, there is a significant overlap between these two statements. For example, when $\Psi_\tau(q)=q^{-\tau}$ for some $\tau\in\mathbb{N}_{\geq 3}$, both of these statements imply that for any $\varepsilon>0$ we have $E(\Psi_\tau,\omega)=\emptyset$ if $\omega(q)\leq(1-\varepsilon)q^{-\tau(\tau-1)}$ for all q sufficiently large. That being said, our proofs are different.

The proof of Theorem 1.6 is straightforward and is given in Section 3. We will briefly highlight here the mechanism that leads to $E(\Psi,\omega)$ being empty. If $x \in E(\Psi,\omega)$ then x is contained in an interval of length $\omega(q)\Psi(q)$ and this interval has either $\frac{p}{q}+\Psi(q)$ or $\frac{p}{q}-\Psi(q)$ as an endpoint for infinitely many $(p,q)\in\mathbb{Z}\times\mathbb{N}$. If $\frac{p}{q}+\Psi(q)$ or $\frac{p}{q}-\Psi(q)$ is a rational number then this observation provides a rational approximation to x with error $\omega(q)\Psi(q)$. If $\omega\cdot\Psi$ is too small relative to the denominators of these newly constructed rationals then x cannot be in $E(\Psi,\omega)$. It is this observation that we exploit in our proof of Theorem 1.6.

Theorem 1.1 provides a lower bound for ω (depending upon Ψ) which guarantees that $E(\Psi,\omega)$ is uncountable. Contrasting this, Theorem 1.6 shows that for Ψ satisfying $\Psi(q)\in\mathbb{Q}$ for all $q\in\mathbb{N}$ there exists an upper bound for ω (again, depending upon Ψ) that guarantees $E(\Psi,\omega)=\emptyset$. In addition, the preprint [31] provides other families of functions Ψ for which there exists an ω such that $E(\Psi,\omega)=\emptyset$. These observations may lead one to ask whether there exists some upper bound for ω (depending only on the decay rate of Ψ) that guarantees $E(\Psi,\omega)=\emptyset$ without any additional assumptions on Ψ. The following theorem demonstrates that this is not the case.

Theorem 1.9. *For any $\omega:(0,\infty)\to(0,\infty)$ and $\tau>2$ there exists a non-increasing function $\Psi:(0,\infty)\to(0,\infty)$ such that $\lim_{q\to\infty}\frac{-\log\Psi(q)}{\log q}=\tau$ and $E(\Psi,\omega)\neq\emptyset$.*

The following question that naturally arises from Theorems 1.5 and 1.6.

Question. *Does there exist a non-increasing $\Psi:(0,\infty)\to(0,\infty)$ and $\omega:(0,\infty)\to(0,\infty)$ such that*

$$\omega(q)\geq\frac{1}{\Psi(q)}\Psi\left(\frac{1}{\Psi(q)}\right)$$

for all sufficiently large q, yet $E(\Psi,\omega)=\emptyset$?

1.3. **Main results: Q2.** In response to Q2 we are able to prove the following statements. Essentially, they show that if we impose the additional assumption $\sum_{q=1}^{\infty}q\Psi(q)<\infty$, then we can replace the uncountability conclusion in Theorems 1.1 and 1.5 with a stronger dimension conclusion.

Theorem 1.10. *Assume that Ψ and ω are such that $q\to q^2\Psi(q)$ and $q\to q^2\Psi(q)\omega(q)$ are non-increasing, $\sum_{q=1}^{\infty}q\Psi(q)<\infty$ and $\lim_{q\to\infty}\omega(q)=0$. Let us also assume that*

$$\omega(q)\geq\frac{45}{\Psi(q)}\Psi\left(\frac{1}{2q\Psi(q)}\right)$$

for all q sufficiently large. Then

$$\dim_H(E(\Psi,\omega))=\inf\left\{s\geq 0:\sum_{q=1}^{\infty}q(\Psi(q)\omega(q))^s<\infty\right\}.$$

Note that the conclusion of Theorem 1.10 can also be written in terms of the lower order at infinity of the function $\Psi\cdot\omega$. That is, if Ψ and ω satisfy the assumptions of Theorem 1.10 and we let $\lambda(\Psi,\omega)$ denote the lower order at infinity of $\Psi\cdot\omega$, then $\dim_H(E(\Psi,\omega)) = \frac{2}{\lambda(\Psi,\omega)}$. See Proposition 6.1 and § 6 for the definitions and details.

In the case of approximation functions satisfying the square divisibility condition we can again allow ω to have a faster rate of decay.

Theorem 1.11. *Assume that Ψ and ω are such that $q \to q^2\Psi(q)$ and $q \to q^2\Psi(q)\omega(q)$ are non-increasing, Ψ satisfies the square divisibility condition, $\sum_{q=1}^{\infty} q\Psi(q) < \infty$, and $\lim_{q\to\infty}\omega(q) = 0$. Let us also assume that*

$$\omega(q) \geq \frac{45}{16\Psi(q)}\Psi\left(\frac{1}{8\Psi(q)}\right)$$

for all q sufficiently large. Then

$$\dim_H(E(\Psi,\omega)) = \inf\left\{s \geq 0 : \sum_{q=1}^{\infty} q(\Psi(q)\omega(q))^s < \infty\right\}.$$

Focusing on the special case of functions of the form $q \to q^{-\tau}$ we see that Theorems 1.5, 1.10 and 1.11 imply the following corollary. The second part of this statement simplifies the critical threshold identified in Theorem 1.5.

Corollary 1.12. *Let $\Psi : (0,\infty) \to (0,\infty)$ be given by $\Psi(q) = q^{-\tau_1}$ for some $\tau_1 > 2$. Then the following statements are true:*

1. *If $\omega : (0,\infty) \to (0,\infty)$ is given by $\omega(q) = q^{-\tau_2}$ for some $\tau_2 < \tau_1(\tau_1 - 2)$, then $\dim_H(E(\Psi,\omega)) = \frac{2}{\tau_1+\tau_2}$.*
2. *If $\tau_1 \in \mathbb{N}_{\geq 3}$ and $\omega : (0,\infty) \to (0,\infty)$ is given by $\omega(q) = q^{-\tau_2}$ for some $\tau_2 < \tau_1(\tau_1 - 1)$, then $\dim_H(E(\Psi,\omega)) = \frac{2}{\tau_1+\tau_2}$.*
3. *If $\tau_1 \in \mathbb{N}_{\geq 3}$ and $\omega : (0,\infty) \to (0,\infty)$ is given by $\omega(q) = Cq^{-\tau_1(\tau_1-1)}$ for some $C > \frac{45}{16}\cdot 8^{\tau_1}$, then $\dim_H(E(\Psi,\omega)) = \frac{2}{\tau_1^2}$.*
4. *If $\tau_1 \in \mathbb{N}_{\geq 3}$ and $\omega : (0,\infty) \to (0,\infty)$ is given by $\omega(q) = cq^{-\tau_1(\tau_1-1)}$ for some $c \in (0,1)$, then $E(\Psi,\omega) = \emptyset$.*

Taking ω to be a function that decays to zero sufficiently slowly, e.g. logarithmically, it is possible to obtain the following quantitative improvement on [8, Theorem 1]. This is an immediate consequence of Theorem 1.10 and Proposition 6.1 proved below.

Corollary 1.13. *Assume that Ψ and ω are such that $q \to q^2\Psi(q)$ and $q \to q^2\Psi(q)\omega(q)$ are non-increasing, $\sum_{q=1}^{\infty} q\Psi(q) < \infty$ and $\lim_{q\to\infty}\omega(q) = 0$. Suppose in addition that $\lim_{q\to\infty}\frac{-\log\omega(q)}{\log q} = 0$. Then*

$$\dim_H(E(\Psi,\omega)) = \dim_H(E(\Psi)) = \dim_H(\mathrm{Bad}(\Psi)) = \dim_H(W(\Psi)).$$

From a technical perspective the main innovation of this paper is a detailed analysis of the set of x satisfying

$$\Psi(q)(1-\omega(q)) \leq \left|x - \frac{p}{q}\right| \leq \Psi(q) \tag{1.4}$$

for some Ψ, ω and $p/q \in \mathbb{Q}$. This analysis is given in Section 4. We show that under suitable assumptions the set of x satisfying (1.4) can be approximated from within by a set that has a simple symbolic description in terms of the continued fraction expansion. Crucially, this symbolic description can be used to rule out unwanted rational solutions to $|x - p'/q'| < \Psi(q')(1 - \omega(q'))$. Under suitable assumptions we can guarantee that this approximating set has comparable Lebesgue measure to the set of x satisfying (1.4). This property will be important when it comes to proving Hausdorff dimension statements.

The rest of the paper is structured as follows. In Section 2 we recall some standard results from continued fraction theory that we will use in our proofs. We also formulate a result that gives an equivalent definition of the set $E(\Psi, \omega)$ in terms of continued fractions, see Proposition 2.5. In Section 4 we will prove several key technical results, Lemmas 4.1, 4.6, and 4.7, showing that under suitable assumptions on Ψ and ω, we can build suitable subsets of $E(\Psi, \omega)$ that have a simple description in terms of the partial quotients of the continued fraction expansion. In Section 5 we prove Theorems 1.1, 1.5 and 1.9, and in Section 6 we prove Theorems 1.10 and 1.11. Theorem 1.6 is proven in a different manner to our other results. Its proof can be found in Section 3.

Notation. Throughout this paper we will adopt the following notation conventions. Given two positive real-valued functions $f, g : X \to \mathbb{R}$ defined on some set X, we write $f \ll g$ if there exists $C > 0$ such that $f(x) \le Cg(x)$ for all $x \in X$. Moreover, given a digit a belonging to some alphabet set $\mathcal{A}$ and $n \in \mathbb{N}$, we denote by $a^n = (\overbrace{a, \ldots, a}^{\times n})$ the word given by the n-fold concatenation of a with itself. We similarly define a^∞ to be the infinite word whose digits consist only of the digit a.

2. Preliminaries

2.1. **Continued fractions and fundamental intervals.** In this section, we recall some classical results on continued fractions, many of which can be found in Khintchine's book [25]. The theory of continued fractions is deeply related to many sets appearing in Diophantine approximation. For instance, one can relate the set of badly approximable points, Ψ-well approximable points, and Ψ-Dirichlet improvable points to properties of the continued fraction expansion of a point, see for example [25, 27]. We prove a key proposition below that gives an equivalent condition for belonging to $E(\Psi, \omega)$ in terms of the continued fraction expansion.

We can write any $x \in \mathbb{R} \setminus \mathbb{Q}$ in continued fraction expansion form:

$$x = a_0(x) + \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cfrac{1}{\ddots}}} = [a_0(x); a_1(x), a_2(x), \ldots],$$

for unique $a_0(x) \in \mathbb{Z}$ and $a_i(x) \in \mathbb{N}$ for all $i \ge 1$. We call the entries in (a_i) the partial quotients of x. When we want to emphasise that some partial quotients are the partial quotients of a real number x, we may write $a_i(x) = a_i$. For $x \in \mathbb{Q}$ a similar continued fraction can be defined, however it is not necessarily unique. We will make use of the following lemma. For a proof see Chapter 1 of Schmidt's book [35].

Lemma 2.1. *Every rational number $\frac{p}{q} \in \mathbb{Q}$ has a unique continued fraction expansion $\frac{p}{q} = [a_0; a_1, \ldots, a_n]$ where $n \in \mathbb{N}$ is even. Furthermore, if $c > b > 0$ we have*

$$[a_0; a_1, \ldots, a_n] < [a_0; a_1, \ldots, a_n, c] < [a_0; a_1, \ldots, a_n, b]\,.$$

Similarly, every $\frac{p}{q} \in \mathbb{Q}$ also has a unique continued fraction expansion $\frac{p}{q} = [a_0; a_1, \ldots, a_m]$ where $m \in \mathbb{N}$ is odd, and for $c > b > 0$ we have

$$[a_0; a_1, \ldots, a_m] > [a_0; a_1, \ldots, a_m, b] > [a_0; a_1, \ldots, a_m, c]\,.$$

Unless the expansion is explicitly given, when we speak of the continued fraction expansion of a rational number we will often mean the even continued fraction expansion given by the first part of Lemma 2.1. Given $x \in \mathbb{R}$ and $n \in \mathbb{N}$ we call

$$\frac{p_n}{q_n} := [a_0(x); a_1(x), \ldots, a_n(x)]$$

the n-th convergent of x. Given the partial quotients of a real number $x \in (0, 1)$ we can inductively define the numerator and denominator of the convergents by setting $p_0 = q_{-1} = 0$, $q_0 = p_{-1} = 1$ and then

$$q_{n+1} = a_{n+1} q_n + q_{n-1}\,, \quad p_{n+1} = a_{n+1} p_n + p_{n-1}\,. \tag{2.1}$$

When manipulating a continued fraction expansion it is often useful to write $q_n(\mathbf{a})$, for an n-tuple of positive integers $\mathbf{a} \in \mathbb{N}^n$, to make explicit what sequence of partial quotients we are using to define q_n. In addition to this, if q_n is the denominator of the nth convergent of x we may write $q_n(x)$ to emphasise the dependence on x. While we assume all partial quotients are strictly positive, we will on occasion undertake some arithmetic operations on the partial quotients with the result being that some terms may equal zero. In these instances, we use the formula $[\ldots, b, 0, c, \ldots] = [\ldots, b+c, \ldots]$, which can easily be verified, to revert back to the case of strictly positive entries.

The following lemma shows how the quality of approximation provided by a convergent is determined by the partial quotients. For a proof of this lemma see [26].

Lemma 2.2. *Let $x \in \mathbb{R}$. Then for any $n \geq 1$, if the nth convergent $\frac{p_n}{q_n}$ is distinct from x, then*

$$\frac{1}{3a_{n+1}q_n^2} < \frac{1}{(a_{n+1}+2)q_n^2} < \left| x - \frac{p_n}{q_n} \right| < \frac{1}{a_{n+1}q_n^2}.$$

For the Ψ we will consider, the following result due to Legendre implies that all but finitely many of the rational Ψ-approximations of a point will be given by its convergents. For a proof see [21].

Lemma 2.3. *Let $x \in \mathbb{R}$ and suppose $(p, q) \in \mathbb{Z} \times \mathbb{N}$ is such that*

$$\left| x - \frac{p}{q} \right| < \frac{1}{2q^2}.$$

Then $\frac{p}{q}$ is a convergent of x.

When the constant $1/2$ appearing on the right hand side of the inequality in Lemma 2.3 is replaced with 1 we have the following result due to Grace [18]. This result tells us that if a rational approximation is sufficiently good, then it is either a convergent, or its continued fraction expansion closely resembles that of the point it is approximating.

Lemma 2.4 ([18])**.** *Let $x = [0; a_1, a_2, \ldots]$ be a real number and $(p, q) \in \mathbb{Z} \times \mathbb{N}$ be such that*

$$\left| x - \frac{p}{q} \right| < \frac{1}{q^2} .$$

Then, there exists an integer n such that

$$\frac{p}{q} \in \{[0; a_1, \ldots, a_n], [0; a_1, \ldots, a_n, 1], [0; a_1, \ldots, a_n, a_{n+1} - 1]\} .$$

Armed with the result of Legendre and several key properties of continued fractions, we are able to prove the following key proposition. This result provides a useful equivalent definition for a set $E(\Psi, \omega)$.

Proposition 2.5. *Let $\Psi : (0, \infty) \to (0, \infty)$ be such that $\Psi(q) < \frac{1}{2q^2}$ for all sufficiently large q. Let $x = [a_0; a_1, a_2, \ldots]$ and $\frac{p_n}{q_n}$ denote the n-th convergent of x for each $n \in \mathbb{N}$. Then $x \in E(\Psi, \omega)$ if and only if*

(i) *for all $n \in \mathbb{N}$ sufficiently large*

$$[a_{n+1}; a_{n+2}, \ldots] \leq \frac{1}{q_n^2 \Psi(q_n)(1 - \omega(q_n))} - \frac{q_{n-1}}{q_n} ,$$

and

(ii) *for infinitely many $n \in \mathbb{N}$*

$$[a_{n+1}; a_{n+2}, \ldots] \in \left[\frac{1}{q_n^2 \Psi(q_n)} - \frac{q_{n-1}}{q_n}, \frac{1}{q_n^2 \Psi(q_n)(1 - \omega(q_n))} - \frac{q_{n-1}}{q_n} \right] .$$

Remark 2.6. Condition (i) corresponds to the lim inf property in the definition of $E(\Psi, \omega)$, while (ii) corresponds to the lim sup property. Under the assumption $\lim_{q\to\infty} q^2\Psi(q) = \lim_{q\to\infty} \omega(q) = 0$, Lemma 2.2 implies that if x satisfies (ii) and

(i') for every $n \in \mathbb{N}$ sufficiently large either (ii) holds or $a_n \leq \left\lfloor \frac{1}{3q_{n-1}^2 \Psi(q_{n-1})} \right\rfloor$,

then $x \in E(\Psi, \omega)$. We will often find it easier to work with condition (i') instead of (i). Similarly, if x does not satisfy (ii) or

$$a_n(x) > \frac{2}{q_{n-1}^2 \Psi(q_{n-1})} \quad \text{for i.m.} \quad n \in \mathbb{N} ,$$

then $x \notin E(\Psi, \omega)$.

Proof of Proposition 2.5. Clearly our statements are unaffected by a finite number of exceptions. Thus we can suppose $Q \in \mathbb{N}$ is large enough so that $\Psi(q) < \frac{1}{2q^2}$ for all $q > Q$. Furthermore, we can assume n is sufficiently large so that $q_n > Q$. Hence, by

Lemma 2.3, when determining whether x belongs to $E(\Psi,\omega)$ we only need to consider the convergents of x. Write $x=[a_0(x);a_1(x),\ldots]$ and note that, for every $n\in\mathbb{N}$,

$$x=\frac{p_n[a_{n+1}(x);a_{n+2}(x),\ldots]+p_{n-1}}{q_n[a_{n+1}(x);a_{n+2}(x),\ldots]+q_{n-1}},$$

see for example [25, Chapter II. §5]. Using the well known formula $p_{n-1}q_n-q_{n-1}p_n=(-1)^n$ (see for example [25, Theorem 2]), it follows from the above that we have

$$\left|x-\frac{p_n}{q_n}\right|=\frac{|(-1)^n|}{q_n\left(q_n[a_{n+1}(x);a_{n+2}(x),\ldots]+q_{n-1}\right)}=\frac{1}{q_n^2\left([a_{n+1}(x);a_{n+2}(x),\ldots]+\frac{q_{n-1}}{q_n}\right)}.$$

It is then clear that

$$\left|x-\frac{p_n}{q_n}\right|\geq(1-\omega(q_n))\Psi(q_n)\quad\Longleftrightarrow\quad[a_{n+1};a_{n+2},\ldots]\leq\tfrac{1}{q_n^2\Psi(q_n)(1-\omega(q_n))}-\tfrac{q_{n-1}}{q_n}\tag{2.2}$$

and

$$(1-\omega(q_n))\Psi(q_n)\leq\left|x-\frac{p_n}{q_n}\right|\leq\Psi(q_n)$$
$$\Longleftrightarrow\quad[a_{n+1};a_{n+2},\ldots]\in\left[\tfrac{1}{q_n^2\Psi(q_n)}-\tfrac{q_{n-1}}{q_n},\,\tfrac{1}{q_n^2\Psi(q_n)(1-\omega(q_n))}-\tfrac{q_{n-1}}{q_n}\right].\tag{2.3}$$

Our result now follows from (2.2), (2.3) and our earlier observation that belonging to $E(\Psi,\omega)$ is determined by the convergents of x. $\square$

2.1.1. *Fundamental intervals.* The natural next step is to partition the unit interval into suitable subsets that satisfy the conditions appearing in Proposition 2.5. To do this we use the notion of fundamental intervals. Given $\mathbf{b}=(b_0,b_1,\ldots,b_n)\in\mathbb{N}^{n+1}$ we associate the fundamental interval

$$I_n(b_0;b_1,\ldots,b_n):=\{x\in\mathbb{R}:a_0(x)=b_0,\ a_1(x)=b_1,\ \ldots,\ a_n(x)=b_n\}.$$

We emphasise that unlike many articles on Diophantine approximation our fundamental intervals are not confined to $[0,1]$ because we allow for a choice of a_0. This added flexbility will simplify our proofs in Section 4. When it is notationally more convenient, we may denote a fundamental interval in several different ways; if $\mathbf{b}\in\mathbb{N}^{n+1}$ let $I(\mathbf{b})=I_n(b_0;b_1,\ldots,b_n)$ and, for $j\leq n$, let $I_j(\mathbf{b})=I_j(b_0;b_1,\ldots,b_j)$.

Note that fundamental intervals are indeed intervals. They will act as building blocks in the proofs of each of our theorems.

A useful fact when working with fundamental intervals follows from Lemma 2.1: A fundamental interval $I_n(\mathbf{b})$ is partitioned into countably many $(n+1)$-th level fundamental intervals $I_{n+1}(\mathbf{b}k)$ where $k\in\mathbb{N}$. If n is even then the k appearing in $I_{n+1}(\mathbf{b}k)$ is increasing as we move from right to left within $I_n(\mathbf{b})$. Similarly, if n is odd then k increases as we move from left to right within $I_n(\mathbf{b})$.

The following lemma provides a useful description of fundamental intervals. Here and throughout this paper, we let $\mathcal{L}$ denote the Lebesgue measure on $\mathbb{R}$.

Lemma 2.7. *Given* $\mathbf{b} = (b_0, b_1, \ldots, b_n) \in \mathbb{N}^n$ *let* $\frac{p_n}{q_n} = [0; b_1, \ldots, b_n]$ *and* $\frac{p_{n-1}}{q_{n-1}} = [0; b_1, \ldots, b_{n-1}]$. *Then*

$$I_n(b_0; b_1, \ldots, b_n) = \left[b_0 + \frac{p_n}{q_n}, b_0 + \frac{p_n + p_{n-1}}{q_n + q_{n-1}}\right]$$

if n is even and

$$I_n(b_0; b_1, \ldots, b_n) = \left[b_0 + \frac{p_n + p_{n-1}}{q_n + q_{n-1}}, b_0 + \frac{p_n}{q_n}\right]$$

if n is odd. Moreover, we have

$$\frac{1}{2q_n^2} \leq \mathcal{L}(I_n(b_0; b_1, \ldots, b_n)) \leq \frac{1}{q_n^2}$$

Proof. Proofs of the $I_n(b_0; b_1, \ldots, b_n)$ identities can be found in [26, Chapter III, § 12.]. Our bounds for $\mathcal{L}(I_n(b_0; b_1, \ldots, b_n))$ follows from these identities and the well known fact that $|p_{n-1}q_n - q_{n-1}p_n| = 1$ for all $n \in \mathbb{N}$. □

The following lemma gives bounds on the Lebesgue measure of naturally occurring subsets of fundamental intervals. For a proof see [8, Lemma 3].

Lemma 2.8. *Let* $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{N}^n$ *and* $A \geq 3$. *Then for any* $a \in \mathbb{Z}$

$$\mathcal{L}\left(\{x \in I_n(a; \mathbf{b}) : a_{n+1}(x) \leq A\}\right) \geq \left(1 - \frac{3}{A}\right) \mathcal{L}(I_n(a; \mathbf{b}))$$

Lastly, we have the following lemma which shows that the length of a fundamental interval and the denominator of a convergent are almost multiplicative functions.

Lemma 2.9. *Let* $\mathbf{a} = (a_1, \ldots, a_m), \mathbf{b} = (b_1, \ldots, b_n)$ *be finite words. Then*

$$\frac{q_m(\mathbf{a})q_n(\mathbf{b})}{2} \leq q_{m+n}(\mathbf{ab}) \leq 4q_m(\mathbf{a})q_n(\mathbf{b}).$$

Moreover, for any $a \in \mathbb{Z}$ we have

$$\frac{\mathcal{L}(I_m(0, \mathbf{a}))\mathcal{L}(I_n(0, \mathbf{b}))}{16} \leq \mathcal{L}(I_{m+n}(a, \mathbf{ab})) \leq 64\mathcal{L}(I_m(0, \mathbf{a}))\mathcal{L}(I_n(0, \mathbf{b})).$$

In the special case when $\mathbf{b} = (b)$ is a word of length 1 we have the following bound for any $a \in \mathbb{Z}$

$$\mathcal{L}(I_{m+1}(a; \mathbf{a}b)) \leq 4\mathcal{L}\left(I_{m+1}(a; \mathbf{a}(b+1))\right).$$

Proof. The first two statements follow from [24, Lemmas 2.5 and 2.6]. For the final statement when $\mathbf{b} = (b)$ is a word of length 1, we observe that for any $a \in \mathbb{Z}$ we have

$$\mathcal{L}(I_{m+1}(a; \mathbf{a}b)) = \frac{1}{(q_m b + q_{m-1})(q_m(b+1) + q_{m-1})}$$

where $\frac{p_m}{q_m} = [0; a_1, \ldots, a_m]$. This follows from Lemma 2.7 and (2.1). Thus

$$\mathcal{L}(I_{m+1}(a; \mathbf{a}(b+1))) = \frac{(q_m b + q_{m-1})}{(q_m(b+2) + q_{m-1})} \mathcal{L}(I_{m+1}(a; \mathbf{a}b)).$$

Now using the fact $0 \leq \frac{q_{m-1}}{q_m} \leq 1$ for all m and $b \geq 1$, we have

$$\mathcal{L}(I_{m+1}(a;\mathbf{a}(b+1))) \geq \frac{b}{b+3}\mathcal{L}(I_{m+1}(a;\mathbf{a}b)) \geq \frac{\mathcal{L}(I_{m+1}(a;\mathbf{a}b))}{4}.$$

This completes our proof.

□

There are some instances when we will need to know about the relative size of fundamental intervals that are close in a geometric sense, but are not nested within the same previous level fundamental interval.

Lemma 2.10. *Let $\mathbf{a} = (a_1, \dots, a_n)$ be a finite word and let $c \in \mathbb{N}_{\geq 2}$. Then for any $b \in \mathbb{N}$ we have that*

$$\mathcal{L}\left(I_{n+2}(0;\mathbf{a},c,b)\right) \geq \mathcal{L}\left(I_{n+3}(0;\mathbf{a},c-1,1,b)\right).$$

Proof. By Lemma 2.7, (2.1), and the well-known property that $|p_n q_{n+1} - p_{n+1}q_n| = 1$ for every $n \in \mathbb{N}$, we have that

$$\begin{aligned}\mathcal{L}\left(I_{n+2}(0;\mathbf{a},c,b)\right) =& \frac{1}{q_{n+2}(\mathbf{a}cb)\left(q_{n+2}(\mathbf{a}cb)+q_{n+1}(\mathbf{a}c)\right)}\\ =& \frac{1}{\left((bc+1)q_n(\mathbf{a})+bq_{n-1}(\mathbf{a})\right)\left(((b+1)c+1)q_n(\mathbf{a})+(b+1)q_{n-1}(\mathbf{a})\right)},\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}&\mathcal{L}(I_{n+3}(0;\mathbf{a},c-1,1,b))\\ =& \frac{1}{q_{n+3}(\mathbf{a}(c-1)1b)\left(q_{n+3}(\mathbf{a}(c-1)1b)+q_{n+2}(\mathbf{a}(c-1)1)\right)}\\ =& \frac{1}{\left(((b+1)c-1)q_n(\mathbf{a})+(b+1)q_{n-1}(\mathbf{a})\right)\left(((b+2)c-1)q_n(\mathbf{a})+(b+2)q_{n-1}(\mathbf{a})\right)}.\end{aligned} \tag{2.5}$$

Considering the first bracket in the denominator of (2.5) we see that

$$\begin{aligned}((b+1)c-1)q_n(\mathbf{a})+(b+1)q_{n-1}(\mathbf{a}) &= (bc+1)q_n(\mathbf{a})+bq_{n-1}(\mathbf{a})+(c-2)q_n(\mathbf{a})+q_{n-1}(\mathbf{a})\\ &\geq (bc+1)q_n(\mathbf{a})+bq_{n-1}(\mathbf{a})\end{aligned}$$

since $c \geq 2$ and $q_{n-1}(\mathbf{a}) \geq 0$ for any $n \in \mathbb{N}$. This is the first bracket in the denominator of (2.4). The same inequality holds when comparing the second brackets appearing in each of the denominators. Thus (2.5) is less than or equal to (2.4) as claimed. □

2.2. **Properties of convergent series and Cantor sets.** The following lemma is well known. For a proof see [8, Lemma 5].

Lemma 2.11. *Let $\Psi : (0,\infty) \to (0,\infty)$ be non-increasing. If $\sum_{q=1}^{\infty} \Psi(q)$ converges then for any $c > 0$ and $M > 1$ the series $\sum_{x=1}^{\infty} M^x \Psi(cM^x)$ converges.*

We now recall a standard lower bound for the Hausdorff dimension of Cantor sets. Let $[0,1]=E_0\subset E_1\subset E_2\subset\cdots$ be a nested sequence of sets such that each E_k consists of finitely many disjoint intervals. Moreover, assume that each interval in E_k contains at least two intervals in E_{k+1}, and that the maximal length of an interval in E_k converges to zero as k tends to infinity. Then the set

$$E=\bigcap_{k=0}^{\infty}E_k$$

is a Cantor set. The following lower bound for the Hausdorff dimension of E follows from Falconer [13, Example 4.6] and the remarks following [8, Proposition 1].

Proposition 2.12. *Let $\{E_k\}$ be as above. Suppose that each interval in E_{k-1} contains at least m_k intervals in E_k which are separated by gaps of size at least ε_k, where $0<\varepsilon_{k+1}<\varepsilon_k$. Moreover, suppose that the intervals of E_k are of length at least δ_k and that $m_{k+1}\varepsilon_{k+1}>c\delta_k$ for some $c>0$. Then*

$$\dim_H(E)\geq\liminf_{k\to\infty}\frac{\log m_k}{-\log\delta_k}.$$

3. Proof of empty statements

Proof of Theorem 1.6. We will prove statement (1) in detail and then explain the minor changes needed to prove statement (2).

Let Ψ and ω satisfy the assumptions of statement (1). For the purpose of obtaining a contradiction, assume $E(\Psi,\omega)$ is non-empty. Let $x\in E(\Psi,\omega)$, then for infinitely many $(p,q)\in\mathbb{Z}\times\mathbb{N}$ we have

$$\Psi(q)\left(1-\omega(q)\right)\leq\left|x-\frac{p}{q}\right|\leq\Psi(q). \tag{3.1}$$

Let us temporarily assume that $(p,q)\in\mathbb{Z}\times\mathbb{N}$ satisfies (3.1) and we have $x-\frac{p}{q}>0$. For such a (p,q) we observe the following equivalences

$$\begin{aligned}\Psi(q)\left(1-\omega(q)\right)\leq\left|x-\frac{p}{q}\right|\leq\Psi(q)&\iff-\Psi(q)\omega(q)\leq x-\frac{p}{q}-\Psi(q)\leq 0\\ &\iff-\Psi(q)\omega(q)\leq x-\frac{p}{q}-\frac{d(q)}{e(q)}\leq 0\\ &\iff-\Psi(q)\omega(q)\leq x-\frac{pe(q)+d(q)q}{qe(q)}\leq 0.\end{aligned}$$

Letting $p'=pe(q)+d(q)q$, the above implies that

$$\left|x-\frac{p'}{qe(q)}\right|\leq\Psi(q)\omega(q). \tag{3.2}$$

It can similarly be shown that (3.2) holds for some $p'\in\mathbb{Z}$ when (3.1) is satisfied by some $(p,q)\in\mathbb{Z}\times\mathbb{N}$ satisfying $x-\frac{p}{q}<0$. Thus if $x\in E(\Psi,\omega)$ then (3.2) is satisfied by i.m. $(p', q) \in \mathbb{Z} \times \mathbb{N}$. The key observation is that (3.2) is a rational approximation to x. Therefore, since $x \in E(\Psi, \omega)$, it follows from (3.2) that we must have

$$\Psi(qe(q))(1-\omega(qe(q)) \leq \left| x - \frac{p'}{qe(q)} \right| \leq \Psi(q)\omega(q) \tag{3.3}$$

for infinitely many values of q. Comparing the left hand term and right hand term of (3.3), we see that this contradicts our assumption.

To prove statement (2) we will use the fact that $\frac{p}{q} + \frac{d(q)}{e(q)}$ can be written as a fraction whose denominator is $e(q)$ for any $(p, q) \in \mathbb{Z} \times \mathbb{N}$. This is a consequence of our assumption that $e(q) = 0 \mod q$ for all $q \in \mathbb{N}$. As a result of this, if we proceed as in the the proof of statement (1) and let $x \in E(\Psi, \omega)$, then for infinitely many $(p', q) \in \mathbb{Z} \times \mathbb{N}$ we would have

$$\left| x - \frac{p'}{e(q)} \right| \leq \Psi(q)\omega(q).$$

This is our analogue of (3.2). The rest of the argument is now identical. □

4. Technical results

In this section we establish the main technical results that will allow us to prove our theorems. Roughly speaking, we will show that for a given $(p, q) \in \mathbb{Z} \times \mathbb{N}$, under the assumptions of our theorems, it is possible to construct a well-behaved subset of

$$\left\{ x : (1-\omega(q))\Psi(q) \leq x - \frac{p}{q} < \Psi(q) \right\}. \tag{4.1}$$

That is, using fundamental intervals, we build subsets whose partial quotients up to some n satisfy the conditions appearing in Proposition 2.5. The main results of this section are Lemmas 4.1, 4.6 and 4.7.

Throughout this section we let $\Psi : (0, \infty) \to (0, \infty)$ and $\omega : (0, \infty) \to (0, 1)$ be such that

$$q \to q^2\Psi(q) \text{ is non-increasing and } \lim_{q\to\infty} q^2\Psi(q) = \lim_{q\to\infty} \omega(q) = 0. \tag{4.2}$$

4.1. **Technical results for Theorems 1.1 and 1.10.**

Before we state the main result of this subsection we introduce some terminology. We say that a finite set of words $\{\mathbf{b}_l\}_{l=1}^{L}$ is consecutive if they are all of the same length and either $\{\mathbf{b}_l\}_{l=1}^{L} = \{b\}_{b=M_1}^{M_2}$ for some $M_1, M_2 \in \mathbb{N}$ or if there exists a word $\mathbf{a}$ such that $\{\mathbf{b}_l\}_{l=1}^{L} = \{\mathbf{a}b\}_{b=M_1}^{M_2}$ for some $M_1, M_2 \in \mathbb{N}$. The main result of this subsection is the following lemma.

Lemma 4.1. *Let Ψ and ω satisfy (4.2). Assume that*

$$\omega(q) \geq \frac{45}{\Psi(q)} \Psi\left(\frac{1}{2q\Psi(q)}\right)$$

for all q sufficiently large. Let $(p,q)\in\mathbb{Z}\times\mathbb{N}$ and $\frac{p}{q}=[0;a_1,\ldots,a_n]$ for n even. Then if q is sufficiently large, there exists a finite set of consecutive words $\{\mathbf{b}_l\}_{l=1}^{L}$ such that

$$\bigcup_{l=1}^{L} I(0;a_1,\ldots,a_n,\mathbf{b}_l)\subset\left\{x:\Psi(q)(1-\omega(q))\leq x-\frac{p}{q}<\Psi(q)\right\} \tag{4.3}$$

and

$$\mathcal{L}\left(\bigcup_{l=1}^{L} I(0;a_1,\ldots,a_n,\mathbf{b}_l))\right)\geq\frac{\omega(q)\Psi(q)}{6400}\,. \tag{4.4}$$

Moreover, if each $\mathbf{b}_l$ has length $m\geq 2$, then

$$b_i\in\left\{1,2,\ldots,\left\lfloor\frac{1}{3q_{n+i}^2\Psi(q_{n+i})}\right\rfloor\right\}\qquad 1\leq i\leq m-1\,, \tag{4.5}$$

where $\frac{p_{n+i}}{q_{n+i}}=[0;a_1,\ldots,a_n,b_0,b_1,\ldots,b_{i-1}]$ for $1\leq i\leq m-1$.

Instead of working directly with the set in (4.1), we will work with the set appearing in Proposition 2.5 (ii). The following lemma can be viewed as a finite version of Proposition 2.5 with a fixed prefix word.

Lemma 4.2. *Let Ψ and ω satisfy (4.2). Let $(p,q)\in\mathbb{Z}\times\mathbb{N}$ be such that $\frac{p}{q}\in(0,1)$ and $\frac{p}{q}=[0;a_1,\ldots,a_n]$ for n even. Then if q is sufficiently large, we have*

$$(1-\omega(q))\Psi(q)\leq x-\frac{p}{q}\leq\Psi(q) \tag{4.6}$$

if and only if

$$x=[0;a_1,\ldots,a_n,a_{n+1}(x),a_{n+2}(x),\ldots] \tag{4.7}$$

with

$$[a_{n+1}(x);a_{n+2}(x),\ldots]\in\left[\frac{1}{q^2\Psi(q)}-\frac{q_{n-1}}{q},\frac{1}{q^2\Psi(q)(1-\omega(q))}-\frac{q_{n-1}}{q}\right]. \tag{4.8}$$

Proof. The proof is similar to the proof of Proposition 2.5. Note that since $\lim_{q\to\infty}q^2\Psi(q)=0$ we have $\Psi(q)\leq\frac{1}{2q^2}$ for all q sufficiently large, and so by Lemma 2.3 if $0<x-p/q\leq\Psi(q)$ then p/q is a convergent of x. Furthermore, since n is even and we are looking at approximations of x exclusively from rationals to the left of x, it is a consequence of Lemma 2.1 and Lemma 2.3 that the first n partial quotients of x must agree with the partial quotients of $\frac{p}{q}$, thus giving us (4.7). The equivalence between (4.6) and (4.8) now follows from the same argument as given in the proof of Proposition 2.5. □

Instead of working with the set in (4.1), we will work with the set provided by Lemma 4.2. For this reason, we introduce some additional notation. Given Ψ, ω and $p/q=[0;a_1,\ldots,a_n]$ for n even, let

$$S(p/q)=S(\Psi,\omega,p/q):=\left[\frac{1}{q^2\Psi(q)}-\frac{q_{n-1}}{q},\frac{1}{q^2\Psi(q)(1-\omega(q))}-\frac{q_{n-1}}{q}\right].$$

Note that $S(p/q)$ is the set appearing in Proposition 2.5 (ii). For convenience we will often suppress the dependence on Ψ and ω from our notation and just write $S(p/q)$. We construct a well-behaved subset of $S(p/q)$ in Lemma 4.5. Lemma 4.1 will essentially follow from Lemma 4.2 and Lemma 4.5. Before we prove Lemma 4.5 we establish some basic properties of $S(p/q)$. This is the content of the next two lemmas.

Lemma 4.3. *Let Ψ and ω satisfy (4.2). Let $(p,q)\in\mathbb{Z}\times\mathbb{N}$ and $\frac{p}{q}=[0;a_1,\ldots,a_n]$ for n even. Then if q is sufficiently large, for any $b\in\mathbb{N}$ such that $[b,b+1]\cap S(p/q)\neq\emptyset$ we have*

$$\frac{1}{2q^2\Psi(q)}\leq b\leq\frac{2}{q^2\Psi(q)}.$$

Proof. These inequalities immediately follow our assumptions and $0\leq\frac{q_{n-1}}{q}\leq 1$. $\square$

Note that belonging to $S(p/q)$ infinitely often is not enough to guarantee an element belongs to $E(\Psi,\omega)$, we also need to ensure Proposition 2.5 (i) is satisfied. As stated in Remark 2.6 it is easier to work with condition (i'). We impose upper bounds on the partial quotients via the following sets: Given $(p,q)\in\mathbb{Z}\times\mathbb{N}$ where $\frac{p}{q}=[0;a_1,\ldots,a_n]$ for n even, an integer $b_0\in\mathbb{N}$, and a possibly empty word $\mathbf{b}=(b_1,\ldots,b_m)$, we define

$$B(p/q,b_0,\mathbf{b}):=\left\{I_{m+1}(b_0;\mathbf{b}c_1):c_1\in\left\{\left\lceil\frac{1}{3q_{n+m+1}^2\Psi(q_{n+m+1})}\right\rceil,\ldots\right\}\right\},$$

where q_{n+m+1} is given by $\frac{p_{n+m+1}}{q_{n+m+1}}=[0;a_1,\ldots,a_n,b_0,b_1,\ldots,b_m]$. We also let

$$\mathcal{B}(p/q,b_0,\mathbf{b}):=\bigcup_{I\in B(p/q,b_0,\mathbf{b})}I\,. \tag{4.9}$$

We emphasise that $\mathcal{B}(p/q,b_0,\mathbf{b})$ is an interval. The significance of this set is that if

$$x\in I_{n+m+2}(0;a_1,\ldots,a_n,b_0,\mathbf{b}c_1)\text{ for some }c_1<\left\lceil\frac{1}{3q_{n+m+1}^2\Psi(q_{n+m+1})}\right\rceil,$$

then

$$\left|x-\frac{p_{n+m+1}}{q_{n+m+1}}\right|>\Psi(q_{n+m+1}).$$

This follows from Lemma 2.2. We will on occasion informally refer to the sets of the form $\mathcal{B}(p/q,b_0,\mathbf{b})$ as bad sets. This is because in our proofs they will consist of those fundamental intervals that we are trying to avoid. Given a word $\mathbf{b}=(b_0,b_1,\cdots,b_m)$ we will also adopt the notational convention that $B(p/q,\mathbf{b})=B(p/q,b_0,b_1\cdots b_m)$ and $\mathcal{B}(p/q,\mathbf{b})=\mathcal{B}(p/q,b_0,b_1\cdots b_m)$, i.e. the first entry in $\mathbf{b}$ determines the integer and the remainder determines the finite word appearing in the definition of these sets.

The following lemma shows that under the assumptions of Theorems 1.1 and 1.10, the set $S(p/q)$ will always be much larger than $\mathcal{B}(p/q,b_0,\mathbf{b})$ for any $\mathbf{b}$ and b_0 such that $[b_0,b_0+1]\cap S(p/q)\neq\emptyset$.

Lemma 4.4. *Let Ψ and ω satisfy (4.2). Assume that*

$$\omega(q)\geq\frac{45}{\Psi(q)}\Psi\left(\frac{1}{2q\Psi(q)}\right)$$

for all q sufficiently large. Let $(p,q)\in\mathbb{Z}\times\mathbb{N}$ and $p/q=[0;a_1,\ldots,a_n]$ for n even. Then if q is sufficiently large, for $b_0\in\mathbb{N}$ satisfying $[b_0,b_0+1]\cap S(p/q)\neq\emptyset$ and a word $\mathbf{b}$ we have

$$\mathcal{L}(S(p/q))\geq 20\mathcal{L}(\mathcal{B}(p/q,b_0,\mathbf{b})).$$

Proof. Let $(p,q)\in\mathbb{Z}\times\mathbb{N}$ be given. Let b_0 satisfy $[b_0,b_0+1]\cap S(p/q)\neq\emptyset$ and $\mathbf{b}=(b_1,\ldots,b_m)$ be a possibly empty word. It is a consequence of Lemma 2.8 that

$$\mathcal{L}(\mathcal{B}(p/q,b_0,\mathbf{b}))\leq 9q_{n+m+1}^2\Psi(q_{n+m+1})\mathcal{L}(I_m(b_0;\mathbf{b})). \tag{4.10}$$

Using the fact $q\to q^2\Psi(q)$ is non-increasing we see that the right hand side of the inequality above is maximised when $\mathbf{b}$ is the empty word. Thus

$$\mathcal{L}(\mathcal{B}(p/q,b_0,\mathbf{b}))\leq 9q_{n+1}^2\Psi(q_{n+1}) \tag{4.11}$$

for any word $\mathbf{b}$. By Lemma 4.3 we know that $b_0\geq\frac{1}{2q^2\Psi(q)}$. Using this inequality together with the inequality $q_{n+1}\geq b_0q$, which follows from (2.1), we have $q_{n+1}\geq\frac{1}{2q\Psi(q)}$. Using this bound together with the fact $q\to q^2\Psi(q)$ is non-increasing, the following measure bound follows from (4.11)

$$\mathcal{L}(\mathcal{B}(p/q,b_0,\mathbf{b}))\leq\frac{9}{4q^2\Psi(q)^2}\Psi\left(\frac{1}{2q\Psi(q)}\right). \tag{4.12}$$

The following bound for the measure of $S(p/q)$ follows immediately from the definition:

$$\mathcal{L}(S(p/q))=\frac{\omega(q)}{q^2\Psi(q)(1-\omega(q))}\geq\frac{\omega(q)}{q^2\Psi(q)}. \tag{4.13}$$

Thus our result would follow from (4.12) and (4.13) if the following inequality held

$$\frac{\omega(q)}{q^2\Psi(q)}\geq\frac{180}{4q^2\Psi(q)^2}\Psi\left(\frac{1}{2q\Psi(q)}\right).$$

After rearranging this inequality and cancelling some terms, we see that it is satisfied for all q sufficiently large by our assumption. Thus our result follows. □

The following lemma builds a well-behaved subset of $S(p/q)$.

Lemma 4.5. *Let Ψ and ω satisfy (4.2). Assume that*

$$\omega(q)\geq\frac{45}{\Psi(q)}\Psi\left(\frac{1}{2q\Psi(q)}\right)$$

for all q sufficiently large. Let $(p,q)\in\mathbb{Z}\times\mathbb{N}$ and $\frac{p}{q}=[0;a_1,\ldots,a_n]$ for n even. Then if q is sufficiently large, there exists a finite set of consecutive words $\{\mathbf{b}_l\}_{l=1}^L$ such that

$$\bigcup_{l=1}^{L}I(\mathbf{b}_l)\subset S(p/q)\quad\text{and}\quad\mathcal{L}\left(\bigcup_{l=1}^{L}I(\mathbf{b}_l)\right)\geq\frac{9\mathcal{L}(S(p/q))}{100}. \tag{4.14}$$

Moreover, if each consecutive word has length $m\geq 2$, then

$$b_i\in\left\{1,2,\ldots,\left\lfloor\frac{1}{3q_{n+i}^2\Psi(q_{n+i})}\right\rfloor\right\}\quad 1\leq i\leq m-1, \tag{4.15}$$

where $\frac{p_{n+i}}{q_{n+i}} = [0; a_1, \ldots, a_n, b_0, b_1, \ldots, b_{i-1}]$ *for* $1 \leq i \leq m-1$.

Proof. Our proof is split into a number of cases that are listed below. Many of these cases follow by similar arguments, and so for the purpose of brevity, we will on occasion only give an outline of the argument.

Case 1. - There exists $b_0 \in \mathbb{N}$ such that $I_0(b_0) \subseteq S(p/q)$. Let M_1 be the smallest natural number such that $I_0(M_1) \subset S(p/q)$ and let M_2 be the largest natural number satisfying $I_0(M_2) \subseteq S(p/q)$. M_1 and M_2 are well defined because of our assumption. Then we have

$$\bigcup_{b=M_1}^{M_2} I_0(b) \subseteq S(p/q).$$

It follows from a simple argument using the definitions of M_1 and M_2 that we have the following measure bound

$$\mathcal{L}\left(\bigcup_{b=M_1}^{M_2} I_0(b)\right) \geq \frac{\mathcal{L}(S(p/q))}{3}.$$

Taking our set of consecutive words to be $\{b\}_{b=M_1}^{M_2}$ completes our proof in this case.

Case 2. - There exists no $b_0 \in \mathbb{N}$ such that $I_0(b_0) \subseteq S(p/q)$ and there exists $b_0 \in \mathbb{N}$ such that $I_0(b_0) \cap S(p/q) \neq \emptyset$ and $I_0(b_0+1) \cap S(p/q) \neq \emptyset$. It follows from our assumptions that $S(p/q) \subset I_0(b_0) \cup I_0(b_0+1)$, otherwise we would be in Case 1. Thus either

$$\mathcal{L}(S(p/q) \cap I_0(b_0)) \geq \frac{\mathcal{L}(S(p/q))}{2} \tag{4.16}$$

or

$$\mathcal{L}(S(p/q) \cap I_0(b_0+1)) \geq \frac{\mathcal{L}(S(p/q))}{2}. \tag{4.17}$$

We treat each of these subcases individually.

Case 2a. (4.16) *holds.* It follows from our assumptions that $S(p/q)$ contains the right end point of $I_0(b_0)$. Thus $S(p/q) \cap I_1(b_0; 1) \neq \emptyset$. If $I_1(b_0; 1) \subset S(p/q) \cap I_0(b_0)$ then we can take $\{b_0 1\}$ to be our set of words, as in this case

$$\frac{\mathcal{L}(S(p/q))}{2} \leq \mathcal{L}(S(p/q) \cap I_0(b_0)) \leq \mathcal{L}(I_0(b_0)) = 2\mathcal{L}(I_1(b_0; 1)),$$

where in the final equality we used the fact that $\mathcal{L}(I_1(b_0; 1)) = 1/2$ for all $b_0 \in \mathbb{Z}$. Thus (4.14) and (4.15) are satisfied. Now suppose $S(p/q) \cap I_0(b_0) \subset I_1(b_0; 1)$. The fundamental intervals $\{I_2(b_0; 1, n)\}_{n=1}^{\infty}$ are all contained in $I_1(b_0; 1)$ and are increasing in the sense that the right end point of $I_2(b_0; 1, n)$ is the left end point of $I_2(b_0; 1, n+1)$ for all $n \in \mathbb{N}$. Thus, since $S(p/q)$ contains the right end point of $I_1(b_0; 1)$ it intersects

$I_2(b_0;1,n)$ for all n sufficiently large. Therefore, $S(p/q)\cap\mathcal{B}(p/q,b_0,1)\neq\emptyset$. It follows now from our assumption $S(p/q)\cap I_0(b_0)\subset I_1(b_0;1)$, (4.16) and Lemma 4.4 that

$$\mathcal{B}(p/q,b_0,1)\subseteq S(p/q)\cap I_1(b_0;1) \tag{4.18}$$

and

$$10\mathcal{L}(\mathcal{B}(p/q,b_0,1))\leq \mathcal{L}(S(p/q)\cap I_1(b_0;1))\,. \tag{4.19}$$

Thus

$$10\mathcal{L}\left(I_2\left(b_0;1,\left\lceil\frac{1}{3q_{n+2}^2\Psi(q_{n+2})}\right\rceil\right)\right)\leq \mathcal{L}(S(p/q)\cap I_1(b_0;1))\,. \tag{4.20}$$

It follows now from (4.18), (4.20) and Lemma 2.9 that

$$I_2\left(b_0;1,\left\lfloor\frac{1}{3q_{n+2}^2\Psi(q_{n+2})}\right\rfloor\right)\subseteq S(p/q)\,. \tag{4.21}$$

Let $M\geq 1$ be as small as possible that

$$\bigcup_{b_2=M}^{\left\lfloor\frac{1}{3q_{n+2}^2\Psi(q_{n+2})}\right\rfloor} I_2\left(b_0;1,b_2\right)\subset S(p/q).$$

Note that M is well defined by (4.21). If $M=1$, then we would have $I_1(b_0;1)\subset S(p/q)\cap I_0(b_0)$, and so we can apply the argument given at the beginning of this case. If $M\geq 2$ then we can take our set of words to be $\{b_01b_2\}_{b_2=M}^{\left\lfloor\frac{1}{3q_{n+2}^2\Psi(q_{n+2})}\right\rfloor}$ since by Lemma 2.9, (4.16), (4.19), and the minimality of M, we have

$$\begin{aligned}
5\mathcal{L}\left(\bigcup_{b_2=M}^{\left\lfloor\frac{1}{3q_{n+2}^2\Psi(q_{n+2})}\right\rfloor} I_2\left(b_0;1,b_2\right)\right) &\geq \mathcal{L}\left(\bigcup_{b_2=M-1}^{\left\lfloor\frac{1}{3q_{n+2}^2\Psi(q_{n+2})}\right\rfloor} I_2\left(b_0;1,b_2\right)\right)\\
&\geq \mathcal{L}\big((S(p/q)\cap I_1(b_0;1))\setminus\mathcal{B}(p/q,b_0,1)\big)\\
&\geq \mathcal{L}(S(p/q)\cap I_1(b_0;1))-\frac{\mathcal{L}(S(p/q)\cap I_1(b_0;1))}{10}\\
&=\frac{9\mathcal{L}(S(p/q)\cap I_1(b_0;1))}{10}\\
&\geq\frac{9\mathcal{L}(S(p/q))}{20}\,. \qquad (4.22)
\end{aligned}$$

Thus

$$\mathcal{L}\left(\bigcup_{b_2=M}^{\left\lfloor\frac{1}{3q_{n+2}^2\Psi(q_{n+2})}\right\rfloor} I_2\left(b_0;1,b_2\right)\right)\geq\frac{9\mathcal{L}((S(p/q))}{100}\,.$$

This completes our proof in this case.

Case 2b. (4.17) *holds.* The fundamental intervals $\{I_1(b_0+1;n)\}$ are all contained in $I_0(b_0+1)$ and are decreasing in the sense that the left end point of $I_1(b_0+1;n)$ is the right endpoint of $I_1(b_0+1;n+1)$ for all $n \in \mathbb{N}$. It follows from this observation and (4.17) that

$$I_1(b_0+1;n) \subset S(p/q)$$

for all n sufficiently large. Using this observation together with similar arguments used in Case 2a, it can be shown that there exists $M \in \mathbb{N}$ such that

$$\bigcup_{b_1=M}^{\left\lfloor \frac{1}{3q_{n+1}^2\Psi(q_{n+1})} \right\rfloor} I_1\left(b_0+1;b_1\right) \subset S(p/q).$$

and

$$\mathcal{L}\left(\bigcup_{b_1=M}^{\left\lfloor \frac{1}{3q_{n+1}^2\Psi(q_{n+1})} \right\rfloor} I_1\left(b_0+1;b_1\right)\right) \geq \frac{9\mathcal{L}(S(p/q))}{100}.$$

Taking our set of words to be $\{(b_0+1)b_1\}_{b_1=M}^{\left\lfloor \frac{1}{3q_{n+1}^2\Psi(q_{n+1})} \right\rfloor}$ completes our proof.

Case 3. - There exists $b_0 \in \mathbb{N}$ **such that** $S(p/q) \subset I_0(b_0)$. We split this case into further subcases.

Case 3a - $S(p/q) \cap \mathcal{B}(p/q,b_0) \neq \emptyset$. Replicating the arguments given in Case 2a we can show that $S(p/q) \cap \mathcal{B}(p/q,b_0) \neq \emptyset$ implies

$$I_1\left(b_0, \left\lfloor \frac{1}{3q_{n+1}^2\Psi(q_{n+1})} \right\rfloor\right) \subset S(p/q).$$

We then choose the smallest M such that

$$\bigcup_{b_1=M}^{\left\lfloor \frac{1}{3q_{n+1}^2\Psi(q_{n+1})} \right\rfloor} I_1(b_0,b_1) \subset S(p/q).$$

Taking our set of words to be $\{b_0b_1\}_{b_1=M}^{\left\lfloor \frac{1}{3q_{n+1}^2\Psi(q_{n+1})} \right\rfloor}$, we can replicate the argument used in Case 2a to see that our conclusion is satisfied.

Case 3b - $S(p/q) \cap \mathcal{B}(p/q,b_0) = \emptyset$ *and there exists* b_1 *such that* $I_1(b_0;b_1) \subset S(p/q)$. If we let M_1 and M_2 be respectively the minimum and maximum values of m for which $I_1(b_0;m) \subset S(p/q)$, then by the same reasoning as in Case 1, with a slight modification taking into account the fact that consecutive 1st level fundamental intervals are of different sizes (Lemma 2.9 bounds the change in length), it can be shown that

$\{b_0b_1\}_{b_1=M_1}^{M_2}$ satisfies our desired conclusion.

Case 3c - $S(p/q)\cap\mathcal{B}(p/q,b_0)=\emptyset$, there exists no b_1 such that $I_1(b_0;b_1)\subset S(p/q)$ and there exists b_1 such that $I_1(b_0;b_1)\cap S(p/q)\neq\emptyset$ and $I_1(b_0;b_1+1)\cap S(p/q)\neq\emptyset$. In this case, it follows from our assumptions that $S(p/q)\subset I_1(b_0;b_1)\cup I_1(b_0;b_1+1)$. Thus we must have either

$$\frac{\mathcal{L}(S(p/q))}{2}\leq\mathcal{L}(S(p/q)\cap I_1(b_0;b_1))\quad\text{or}\quad\frac{\mathcal{L}(S(p/q))}{2}\leq\mathcal{L}(S(p/q)\cap I_1(b_0;b_1+1)).$$

We now construct our set of words using the same arguments as used in Case 2a.

Case 3d - $S(p/q)\cap\mathcal{B}(p/q,b_0)=\emptyset$ and there exists b_1 such that $S(p/q)\subset I_1(b_0;b_1)$. Since $S(p/q)\cap\mathcal{B}(p/q,b_0)=\emptyset$ we know that b_1 satisfies (4.15). Now at this stage four possible events can occur:

(1) $S(p/q)\cap\mathcal{B}(p/q,b_0,b_1)\neq\emptyset$.
(2) $S(p/q)\cap\mathcal{B}(p/q,b_0,b_1)=\emptyset$ and there exists b_2 such that $I_2(b_0;b_1,b_2)\subset S(p/q)$.
(3) $S(p/q)\cap\mathcal{B}(p/q,b_0,b_1)=\emptyset$, there exists no b_2 such that $I_2(b_0;b_1,b_2)\subset S(p/q)$ and there exists b_2 such that $I_2(b_0;b_1,b_2)\cap S(p/q)\neq\emptyset$ and $I_2(b_0;b_1,b_2+1)\cap S(p/q)\neq\emptyset$.
(4) $S(p/q)\cap\mathcal{B}(p/q,b_0,b_1)=\emptyset$ and there exists b_2 such that $S(p/q)\subset I_2(b_0;b_1,b_2)$.

If (1) holds then the arguments outlined in Case 3a can be applied. Similarly, if (2) or (3) hold then the arguments in Case 3b and Case 3c can be applied respectively. If (4) holds then since $S(p/q)\cap\mathcal{B}(p/q,b_0,b_1)=\emptyset$ we know that b_2 satisfies (4.15) and we are essentially back at the beginning of Case 3d. We can then define an analogous list of cases (1)-(4) that describe the behaviour of $S(p/q)$ within $I_2(b_0;b_1,b_2)$. We can then proceed by an analogous analysis. Since the maximum length of an m-th level fundamental interval tends to zero as m increases, it is clear that $S(p/q)$ cannot be contained in a fundamental interval $I_m(b_0;b_1,\dots b_m)$ for arbitrarily large values of m. Thus this process must eventually yield a word $\mathbf{b}=(b_1,\dots,b_m)$ such that (4.15) holds for $1\leq i\leq m$ and one of the following three events occur:

(1') $S(p/q)\cap\mathcal{B}(p/q,b_0,\mathbf{b})\neq\emptyset$.
(2') $S(p/q)\cap\mathcal{B}(p/q,b_0,\mathbf{b})=\emptyset$ and there exists b such that $I_{m+1}(b_0;\mathbf{b}b)\subset S(p/q)$.
(3') $S(p/q)\cap\mathcal{B}(p/q,b_0,\mathbf{b})=\emptyset$, there exists no b such that $I_2(b_0;\mathbf{b}b)\subset S(p/q)$ and there exists b such that $I_2(b_0;\mathbf{b}b)\cap S(p/q)\neq\emptyset$ and $I_2(b_0;\mathbf{b}(b+1))\cap S(p/q)\neq\emptyset$.

Applying the same arguments as those outlined in Cases 3a, 3b and 3c we can then construct our desired set of words and complete our proof. □

Equipped with Lemma 4.2 we can now prove Lemma 4.1.

Proof of Lemma 4.1. Let $(p,q)\in\mathbb{Z}\times\mathbb{N}$ and q be large. Let $\{\mathbf{b}_l\}_{l=1}^{L}$ be as in Lemma 4.5. Then (4.3) follows from Lemma 4.2. Equation (4.5) follows from (4.15). It remains to prove the measure bound (4.4). It follows from an application of the mean

value theorem applied to the map $x \to \frac{1}{x}$ and Lemma 4.3 that when considering the left shift of the words $\{0\mathbf{b}_l\}_{l=1}^{L}$ we have that
(4.23)

$$\mathcal{L}\left(\bigcup_{l=1}^{L} I(0;\mathbf{b}_l)\right) \geq \frac{q^4\Psi(q)^2}{9}\mathcal{L}\left(\bigcup_{l=1}^{L} I(\mathbf{b}_l)\right) \geq \frac{q^4\Psi(q)^2}{9}\frac{9\mathcal{L}(S(p/q))}{100} \geq \frac{q^2\omega(q)\Psi(q)}{200}.$$

Thus applying Lemma 2.7, Lemma 2.9 and (4.23) we have

$$\begin{aligned}\mathcal{L}\left(\bigcup_{l=1}^{L} I(0;a_1,\ldots,a_n,\mathbf{b}_l)\right) &\geq \frac{1}{16}\mathcal{L}(I(0;a_1,\ldots,a_n))\mathcal{L}\left(\bigcup_{l=1}^{L} I(0;\mathbf{b}_l)\right)\\ &\geq \frac{1}{32q^2}\cdot\frac{q^2\omega(q)\Psi(q)}{200}\\ &= \frac{\omega(q)\Psi(q)}{6400}.\end{aligned}$$

This completes our proof. □

4.2. **Technical results for Theorems 1.5 and 1.11.** The purpose of this subsection is to prove the following two lemmas. Both of these lemmas assume that a function Ψ satisfies the square divisibility condition. We recall that Ψ satisfies the square divisibility condition if for every $q \in \mathbb{N}$ we have $\Psi(q)^{-1} = 0 \mod q^2$.

Lemma 4.6. *Let Ψ and ω satisfy (4.2). Furthermore suppose that Ψ satisfies the square divisibility condition and*

$$\omega(q) \geq \frac{45}{16\Psi(q)}\Psi\left(\frac{1}{8\Psi(q)}\right)$$

for all q sufficiently large. Let $(p,q) \in \mathbb{Z}\times\mathbb{N}$ and $\frac{p}{q} = [0;a_1,\ldots,a_n]$ for n even where $a_1, a_n \geq 2$. Then if q is sufficiently large, we have that

$$I\left(\frac{1}{q^2\Psi(q)}-1;1,a_n-1,a_{n-1},\ldots,a_2,a_1-1,1,\left\lfloor\frac{1}{3q_*^2\Psi(q_*)}\right\rfloor\right) \subseteq S(p/q)\,,$$

where $\frac{p_*}{q_*} = [0;a_1,\ldots,a_n,\frac{1}{q^2\Psi(q)}-1,1,a_n-1,a_{n-1},\ldots,a_2,a_1-1,1]$. Equivalently,

$$\begin{aligned}&I\left(0;a_1,\ldots,a_n,\tfrac{1}{q^2\Psi(q)}-1,1,a_n-1,a_{n-1},\ldots,a_2,a_1-1,1,\left\lfloor\frac{1}{3q_*^2\Psi(q_*)}\right\rfloor\right)\\ &\qquad\qquad\subset \left\{x : \Psi(q)(1-\omega(q)) \leq x-\frac{p}{q} \leq \Psi(q)\right\}.\end{aligned}$$

Lemma 4.6 is the main technical tool that allows us to prove the non-empty part of Theorem 1.5. The result still holds without the assumption $a_1, a_n \geq 2$, however, we include this assumption as it simplifies our proofs and is sufficient for our needs.

Lemma 4.7. *Let Ψ and ω satisfy (4.2). Furthermore suppose that Ψ satisfies the square divisibility condition and*

$$\omega(q) \geq \frac{45}{16\Psi(q)}\Psi\left(\frac{1}{8\Psi(q)}\right)$$

for all q sufficiently large. Let $(p,q)\in\mathbb{Z}\times\mathbb{N}$ and $\frac{p}{q}=[0;a_1,\dots,a_n]$ for n even, where the partial quotients satisfy

$$a_i\le\frac{1}{6q^2\Psi(q)}\,,\quad 1\le i\le n\,.$$

Then if q is sufficiently large, there exists a finite set of consecutive words $\{\mathbf{b}_l\}_{l=1}^{L}$ such that

$$\bigcup_{l=1}^{L} I(0;a_1,\dots,a_n,\mathbf{b}_l)\subset\left\{x:\Psi(q)(1-\omega(q))\le x-\frac{p}{q}<\Psi(q)\right\}\tag{4.24}$$

and

$$\mathcal{L}\left(\bigcup_{l=1}^{L} I(0;a_1,\dots,a_n,\mathbf{b}_l)\right)\ge\frac{\omega(q)\Psi(q)}{10^8}.\tag{4.25}$$

Moreover, if each $\mathbf{b}_l$ has length $m\ge 2$, then

$$b_i\in\left\{1,2,\dots,\left\lfloor\frac{1}{3q_{n+i}^2\Psi(q_{n+i})}\right\rfloor\right\}\quad 1\le i\le m-1\,,\tag{4.26}$$

where $\frac{p_{n+i}}{q_{n+i}}=[0;a_1,\dots,a_n,b_0,b_1,\dots,b_{i-1}]$ for $1\le i\le m-1$.

Lemma 4.7 will allow us to prove Theorem 1.11. It is an analogue of Lemma 4.1 under the square divisibility assumption. We highlight the additional assumption appearing in Lemma 4.7 that $a_i\le\frac{1}{6q^2\Psi(q)}$ for $1\le i\le n$. This condition imposes additional technicalities in our proof of Theorem 1.11 in Section 6 that are not present in the proof of Theorem 1.10.

When Ψ satisfies the square divisibility condition we have a much better understanding of where $S(p/q)$ lies in relation to the partition of the unit interval given by the fundamental intervals. In this subsection we utilise this property. We begin with the following lemma which gives a useful expression for the left endpoint of $S(p/q)$.

Lemma 4.8. *Let Ψ and ω satisfy (4.2). Furthermore suppose that Ψ satisfies the square divisibility condition. Let $(p,q)\in\mathbb{Z}\times\mathbb{N}$ and $\frac{p}{q}=[0;a_1,\dots,a_n]$ with n even. Then the left endpoint of $S(p/q)$ is equal to the rational number*

$$\left[\frac{1}{q^2\Psi(q)}-1;1,a_n-1,a_{n-1},\dots,a_2,a_1-1,1\right].$$

Furthermore, there exists $\mathbf{c}=(c_0,c_1,\dots,c_m)$ with m even that satisfies the following properties

- *(i)* $[c_0;c_1,\dots,c_m]=\left[\frac{1}{q^2\Psi(q)}-1;1,a_n-1,a_{n-1},\dots,a_2,a_1-1,1\right]$,
- *(ii)* $c_0=\frac{1}{q^2\Psi(q)}-1$,
- *(iii)* $0<c_i\le\max_{1\le i\le n}a_i+1$ *for every $1\le i\le m$.*

Remark 4.9. We remark that in the first expression for the left endpoint of $S(p/q)$ there are an even number of digits after the integer part $\frac{1}{q^2\Psi(q)}-1$. The problem with this expression is that it may produce digits that take the value zero, i.e. $a_n - 1$ and $a_1 - 1$ may both equal zero. This is why we have to introduce the word $\mathbf{c}$.

Proof of Lemma 4.8. Recall the formulas

$$\frac{q_{n-1}}{q_n} = [0; a_n, \dots, a_1] \quad \text{and} \quad -[0; b_1, \dots, b_n] = [-1; 1, b_1 - 1, b_2, \dots, b_n] \,.$$

Both are standard results in continued fraction theory, see for example [36]. It is then clear to see that since $\frac{1}{q^2\Psi(q)} \in \mathbb{N}$, the left endpoint of $S(p/q)$ is

$$\begin{aligned} \frac{1}{q^2\Psi(q)} - \frac{q_{n-1}}{q} &= \left[\frac{1}{q^2\Psi(q)} - 1; 1, a_n - 1, a_{n-1}, \dots, a_2, a_1\right] \\ &= \left[\frac{1}{q^2\Psi(q)} - 1; 1, a_n - 1, a_{n-1}, \dots, a_2, a_1 - 1, 1\right] . \end{aligned}$$

Define the word $\mathbf{c}$ according to the rule

$$\mathbf{c} = \begin{cases} \left(\frac{1}{q^2\Psi(q)} - 1, 1, a_n - 1, a_{n-1}, \dots, a_2, a_1 - 1, 1\right) & \text{if } a_1, a_n \geq 2\,, \\ \left(\frac{1}{q^2\Psi(q)} - 1, 1 + a_{n-1}, a_{n-2} \dots, a_2, a_1 - 1, 1\right) & \text{if } a_1 \geq 2 \text{ and } a_n = 1\,, \\ \left(\frac{1}{q^2\Psi(q)} - 1, 1, a_n - 1, a_{n-1}, \dots, a_3, a_2 + 1\right) & \text{if } a_1 = 1 \text{ and } a_n \geq 2\,, \\ \left(\frac{1}{q^2\Psi(q)} - 1, 1 + a_{n-1}, a_{n-2}, \dots, a_3, a_2 + 1\right) & \text{if } a_1 = 1 \text{ and } a_n = 1\,. \end{cases}$$

Note that in all of the above cases $c_0 = \frac{1}{q^2\Psi(q)} - 1$ so (ii) holds. It can easily be verified that each of the digits c_i satisfy (iii). By using the formula $[\dots, b, 0, c, \dots] = [\dots, b + c, \dots]$ it is easy to see that

$$[\mathbf{c}] = \left[\frac{1}{q^2\Psi(q)} - 1; 1, a_n - 1, a_{n-1}, \dots, a_2, a_1 - 1, 1\right] .$$

So (iii) holds. Finally, it follows our formula for $\mathbf{c}$ that the number of digits appearing after $\frac{1}{q^2\Psi(q)} - 1$ is $n + 2$, n, or $n - 2$. Since n is even, this implies m is even. □

We need the following modified version of Lemma 4.4.

Lemma 4.10. *Let Ψ and ω satisfy (4.2). Furthermore suppose that Ψ satisfies the square divisibility condition and*

$$\omega(q) \geq \frac{45}{16\Psi(q)} \Psi\left(\frac{1}{8\Psi(q)}\right)$$

for all q sufficiently large. Let $(p, q) \in \mathbb{Z} \times \mathbb{N}$ and $\frac{p}{q} = [0; a_1, \dots, a_n]$ for n even. Then if q is sufficiently large, we have

$$\mathcal{L}(S(p/q)) \geq 20\mathcal{L}\left(\mathcal{B}\left(p/q, \mathbf{c}\right)\right)$$

where $\mathbf{c}$ is as in Lemma 4.8.

Proof. By Lemma 2.8 we have that

$$\mathcal{L}\left(\mathcal{B}\left(p/q,\mathbf{c}\right)\right)\leq 9q_*^2\Psi(q_*)\mathcal{L}\left(I\left(\mathbf{c}\right)\right) \tag{4.27}$$

where $\frac{p_*}{q_*}=[0,a_1\ldots,a_n,\mathbf{c}]$. Using Lemma 2.9, the fact that $1-\frac{q_{n-1}}{q_n}=\frac{q_n-q_{n-1}}{q_n}=[0;c_1,\ldots,c_m]$ is in reduced form, and Lemma 4.8 (ii), we have that

$$q_*=q_{n+1+m}\left((a_1,\ldots,a_n)\left(\frac{1}{q^2\Psi(q)}-1\right)(c_1,\ldots,c_m)\right)\geq\frac{1}{8\Psi(q)}. \tag{4.28}$$

Combining (4.27), (4.28), the non-increasing property of $q\mapsto q^2\Psi(q)$, and Lemma 2.7, we have

$$\mathcal{L}\left(\mathcal{B}\left(p/q,\mathbf{c}\right)\right)\leq\frac{9}{64q^2\Psi(q)^2}\Psi\left(\frac{1}{8\Psi(q)}\right). \tag{4.29}$$

The rest of this proof is identical to the latter part of the proof of Lemma 4.4. We use the simple lower bound $\mathcal{L}(S(p/q))\geq\frac{\omega(q)}{q^2\Psi(q)}$ and then compare it with the upper bound for $\mathcal{L}\left(\mathcal{B}\left(p/q,\mathbf{c}\right)\right)$ obtained above to see, via our lower bound on the decay rate of ω, that $\mathcal{L}\left(S(p/q)\right)\geq 20\mathcal{L}\left(\mathcal{B}\left(p/q,\mathbf{c}\right)\right)$ as required.

□

Equipped with Lemma 4.10 we can now prove Lemma 4.6.

Proof of Lemma 4.6. For ease of notation write

$$\mathbf{b}=\left(\mathbf{c},\left\lfloor\frac{1}{3q_*^2\Psi(q_*)}\right\rfloor\right)$$

where $\mathbf{c}$ is given by

$$\mathbf{c}=\left(\frac{1}{q^2\Psi(q)}-1,1,a_n-1,a_{n-1},\ldots,a_2,a_1-1,1\right).$$

By our assumption $a_1,a_n\geq 2$, all of the digits in $\mathbf{c}$ are strictly positive integers. Note that the left endpoint of $I(\mathbf{b})$ is the right endpoint of $\mathcal{B}\left(p/q,\mathbf{c}\right)$, and $\mathcal{B}(p/q,\mathbf{c})$ has the same left endpoint as $S(p/q)$ by Lemma 4.8. By Lemma 2.9 we have the bound

$$\mathcal{L}\left(I(\mathbf{b})\cup\mathcal{B}\left(p/q,\mathbf{c}\right)\right)\leq 5\mathcal{L}\left(\mathcal{B}\left(p/q,\mathbf{c}\right)\right),$$

and so by Lemma 4.10, $S(p/q)$ is sufficiently large, relative to $\mathcal{B}\left(p/q,\mathbf{c}\right)$, to guarantee that

$$S(p/q)\supseteq I(\mathbf{b})\cup\mathcal{B}\left(p/q,\mathbf{c}\right).$$

Hence $I(\mathbf{b})\subset S(p/q)$ as claimed.

□

The following lemma is important because it tells us in a quantifiable way that $S(p/q)$ avoids fundamental intervals with big partial quotients. In this lemma, given $x\in\mathbb{R}$ and $A\subset\mathbb{R}$, we let $d(x,A)$ denote the Euclidean distance between x and A.

Lemma 4.11. *Let Ψ and ω satisfy (4.2). Furthermore suppose that Ψ satisfies the square divisibility condition and*

$$\omega(q) \geq \frac{45}{16\Psi(q)}\Psi\left(\frac{1}{8\Psi(q)}\right)$$

for all q sufficiently large. Let $(p,q) \in \mathbb{Z}\times\mathbb{N}$ and $\frac{p}{q} = [0; a_1, \dots, a_n]$ for n even, where the partial quotients satisfy

$$a_i \leq \frac{1}{6q^2\Psi(q)}, \quad 1 \leq i \leq n\,. \tag{4.30}$$

Then for $\mathbf{c} = (c_0, \dots, c_m)$ as in Lemma 4.8, if q is sufficiently large and $0 \leq j \leq m-1$, then the following statements hold:

(i) If j is odd, then

$$d\left([c_0; c_1, \dots, c_j, c_{j+1}+1], \mathcal{B}(p/q, (c_0, \dots, c_j))\right) > \frac{1}{640}\mathcal{L}(\mathcal{B}(p/q, (c_0, \dots, c_j)))\,,$$

(ii) If j is even and $c_{j+1} = 1$, then

$$d\left([c_0; c_1, \dots, c_{j+1}, c_{j+2}+1], \mathcal{B}(p/q, (c_0, \dots, c_j, c_{j+1}))\right) > \frac{1}{40960}\mathcal{L}(\mathcal{B}(p/q, (c_0, \dots, c_j)))\,.$$

Proof. Assume for now that j is odd, we will deal with case (ii) later. To prove this statement we will focus on a sub-collection of the fundamental intervals lying between $[c_0; c_1, \dots, c_j, c_{j+1}+1]$ and $\mathcal{B}(p/q, (c_0, \dots, c_j)$. The proof of our result will then rely upon two calculations. The first is a lower bound for the number of fundamental intervals (at level $j+1$) in our subcollection. The second calculation yields a lower bound for the size of such a fundamental interval.

The left endpoint of $\mathcal{B}(p/q, (c_0, \dots, c_j))$ is

$$\left[c_0; c_1, \dots, c_j, \left\lfloor \frac{1}{3q_{n+1+j}^2\Psi(q_{n+1+j})}\right\rfloor\right],$$

where

$$\frac{p_{n+1+j}}{q_{n+1+j}} = [0; a_1, \dots, a_n, c_0, c_1, \dots, c_j]\,.$$

We then have the bound

$$\begin{aligned}
&\#\left\{I(c_0; c_1, \dots, c_j, b) : c_j + 2 \leq b \leq \left\lfloor \frac{1}{3q_{n+1}^2\Psi(q_{n+1})}\right\rfloor\right\}\\
&= \left\lfloor \frac{1}{3q_{n+1}^2\Psi(q_{n+1})}\right\rfloor - c_{j+1} - 1\\
&\geq \frac{1}{4q_{n+1}^2\Psi(q_{n+1})} - \frac{1}{6q^2\Psi(q)} - 2 \qquad \text{(by (4.30) \& Lemma 4.8 } (iii))\\
&\geq \frac{1}{4q_{n+1}^2\Psi(q_{n+1})} - \frac{1}{5q^2\Psi(q)} \qquad \text{(for } q \text{ sufficiently large)}\\
&\geq \frac{1}{20q_{n+1}^2\Psi(q_{n+1})}.
\end{aligned} \tag{4.31}$$

The last inequality is a consequence of $q = q_n$ and the non-increasing property of $q \mapsto q^2\Psi(q)$, so $q^2\Psi(q) \geq q_{n+1}^2\Psi(q_{n+1})$. Note that there may be significantly more fundamental intervals between $[c_0; c_1, \dots, c_j, c_{j+1}+1]$ and $\mathcal{B}(p/q, (c_0, \dots, c_j))$, however, the last partial quotient will be potentially larger, which will affect the next calculation. For every fundamental interval $I\left(c_0; c_1, \dots, c_j, b\right)$ with $b \leq \frac{1}{3q_{n+1}^2\Psi(q_{n+1})}$ we have, by Lemma 2.9, that the length of the fundamental interval is bounded from below by

$$\begin{aligned}\mathcal{L}\left(I\left(c_0; c_1, \dots, c_j, b\right)\right) &\geq \mathcal{L}\left(I_{j+1}\left(c_0; c_1, \dots, c_j, \frac{1}{3q_{n+1}^2\Psi(q_{n+1})}\right)\right) \\ &\geq \frac{1}{32}\frac{1}{\left(\frac{1}{3q_{n+1}^2\Psi(q_{n+1})}\right)^2}\mathcal{L}\left(I_j\left(c_0; c_1, \dots, c_j\right)\right) \end{aligned} \tag{4.32}$$

Multiplying (4.31) by (4.32) we obtain that

$$d\left([c_0; c_1, \dots, c_j, c_{j+1}+1], \mathcal{B}(p/q, (c_0, \dots, c_j))\right) > \tfrac{9}{640}q_{n+1}^2\Psi(q_{n+1})\mathcal{L}\left(I_j\left(c_0; c_1, \dots, c_j\right)\right). \tag{4.33}$$

By (4.10) and the non-increasing property of $q \mapsto q^2\Psi(q)$, we have that

$$\begin{aligned}\mathcal{L}\left(\mathcal{B}\left(p/q, (c_0, c_1, \dots, c_j)\right)\right) &\leq 9q_{n+1+j}^2\Psi(q_{n+1+j})\mathcal{L}\left(I_j\left(c_0; c_1, \dots, c_j\right)\right) \\ &\leq 9q_{n+1}^2\Psi(q_{n+1})\mathcal{L}\left(I_j\left(c_0; c_1, \dots, c_j\right)\right), \end{aligned} \tag{4.34}$$

and so by (4.33) we have

$$d\left([c_0; c_1, \dots, c_j, c_{j+1}+1], \mathcal{B}(p/q, (c_0, \dots, c_j))\right) \geq \frac{1}{640}\mathcal{L}\left(\mathcal{B}\left(p/q, (c_0, c_1, \dots, c_j)\right)\right).$$

This completes the proof in case (i). Now consider case (ii). Since j is even and $j \leq m-2$ the digit c_{j+2} is well-defined. Consider

$$d\left([c_0; c_1, \dots, c_{j+1}, c_{j+2}+1], \mathcal{B}(p/q, (c_0, \dots, c_j, c_{j+1}))\right).$$

Now $j+1$ is odd so we can use (4.31), (4.32), Lemma 2.9 combined with our assumption $c_{j+1} = 1$, and (4.34), to obtain

$$\begin{aligned}& d\left([c_0; c_1, \dots, c_{j+1}, c_{j+2}+1], \mathcal{B}(p/q, (c_0, \dots, c_j, c_{j+1}))\right) \\ >& \frac{9}{640}q_{n+1}^2\Psi(q_{n+1})\mathcal{L}\left(I_{j+1}\left(c_0; c_1, \dots, c_{j+1}\right)\right) \\ >& \frac{9}{40960}q_{n+1}^2\Psi(q_{n+1})\mathcal{L}\left(I_j\left(c_0; c_1, \dots, c_j\right)\right) \\ >& \frac{1}{40960}\mathcal{L}\left(\mathcal{B}\left(p/q, (c_0, c_1, \dots, c_j)\right)\right)\end{aligned}$$

completing the proof in case (ii). □

The following lemma is our analogue of Lemma 4.5 in the context of Theorem 1.11. Its proof is similar, however there is one important distinction. In the proof of Lemma 4.5 the first bad set that $S(p/q)$ can possibly intersect is $\mathcal{B}(p/q, b_0)$. Lemma 4.4 then tells us that $S(p/q)$ is much larger than $\mathcal{B}(p/q, b_0)$, so if the two sets were to intersect only a small amount of $S(p/q)$ would be lost. In the context of Lemma 4.7, it is possible for $\mathcal{B}(p/q, b_0)$ to be significantly larger than $S(p/q)$, and so a large part of $S(p/q)$ can be lost if the two sets intersect. To combat this we use our square divisibility condition which by Lemma 4.8 gives us more detailed information about the location of $S(p/q)$. Moreover, by Lemma 4.11 under mild conditions on the partial quotients of p/q, we can ensure $S(p/q)$ is sufficiently far from a bad set.

Lemma 4.12. *Let Ψ and ω satisfy (4.2). Assume that Ψ satisfies the square divisibility condition and that*

$$\omega(q) \geq \frac{45}{16\Psi(q)}\Psi\left(\frac{1}{8\Psi(q)}\right) \tag{4.35}$$

for all $q \in \mathbb{N}$ sufficiently large. Let $(p,q) \in \mathbb{Z}\times\mathbb{N}$ and $\frac{p}{q} = [0; a_1, \ldots, a_n]$ for n even, where the partial quotients satisfy

$$a_i \leq \frac{1}{6q^2\Psi(q)}\,, \qquad 1 \leq i \leq n\,. \tag{4.36}$$

Then if q is sufficiently large, there exists a finite set of consecutive words $\{\mathbf{b}_l\}_{l=1}^{L}$ such that

$$\bigcup_{l=1}^{L} I(\mathbf{b}_l) \subset S(p/q) \quad \text{and} \quad \mathcal{L}\left(\bigcup_{l=1}^{L} I(\mathbf{b}_l)\right) \geq \frac{\mathcal{L}(S(p/q))}{10^6}. \tag{4.37}$$

Moreover, if each $\mathbf{b}_l$ has length $m \geq 2$, then

$$b_i \in \left\{1, 2, \ldots, \left\lfloor \frac{1}{3q_{n+i}^2\Psi(q_{n+i})}\right\rfloor\right\} \qquad 1 \leq i \leq m-1\,, \tag{4.38}$$

where $\frac{p_{n+i}}{q_{n+i}} = [0; a_1, \ldots, a_n, b_0, b_1, \ldots, b_{i-1}]$ for $1 \leq i \leq m-1$.

Proof. The proof is similar to that of Lemma 4.5 and involves a lengthy case analysis. For the purpose of brevity, when the argument for a case is similar to one that has appeared previously, then we will only provide an outline for why the result follows.

Case 1^*. - There exists $b_0 \in \mathbb{N}$ such that $I_0(b_0) \subseteq S(p/q)$. The proof is the same as that appearing in Lemma 4.5. That is, take the set of consecutive words to be $\{b\}_{b=M_1}^{M_2}$ where $M_1 = \min\{K : I_0(K) \subseteq S(p/q)\}$ and $M_2 = \max\{K : I_0(K) \subseteq S(p/q)\}$. The same calculations as in case 1 of the proof of Lemma 4.5 show this to be a suitable collection of words.

Case 2^*. - There exists no $b_0 \in \mathbb{N}$ such that $I_0(b_0) \subset S(p/q)$ and there exists $b_0 \in \mathbb{N}$ such that $I_0(b_0) \cap S(p/q) \neq \emptyset$ and $I_0(b_0+1) \cap S(p/q) \neq \emptyset$. By Lemma 4.8 (ii) we have that $b_0 = c_0 = \frac{1}{q^2\Psi(q)} - 1$. We split into two subcases.

Case 2^*a. - $\mathcal{L}(S(p/q)) \geq 20\mathcal{L}(\mathcal{B}(p/q, c_0))$. We follow the proof of case 2 appearing in Lemma 4.5. That is, if

(1) $\mathcal{L}(S(p/q) \cap I_0(c_0)) \geq \frac{\mathcal{L}(S(p/q))}{2}$: If $I_1(c_0; 1) \subset S(p/q)$ then we take $\{(c_0, 1)\}$ to be our set of words. If $I_1(c_0; 1)$ is not contained in $S(p/q)$ then we can take our set

of words to be

$$\{(c_0;1,b)\}_{b=M}^{\left\lfloor\frac{1}{3q_{n+2}^2\Psi(q_{n+2})}\right\rfloor} \quad \text{with} \quad M := \min\left\{K : I_2(c_0;1,K) \subset S(p/q)\right\}.$$

(2) $\mathcal{L}(S(p/q) \cap I_0(c_0+1)) \geq \frac{\mathcal{L}(S(p/q))}{2}$: Then we can take our set of words to be

$$\{(c_0+1,b)\}_{b=M}^{\left\lfloor\frac{1}{3q_{n+1}^2\Psi(q_{n+1})}\right\rfloor} \quad \text{with} \quad M := \min\{K : I_1(c_0+1;K) \subset S(p/q)\}.$$

Both sets of consecutive words are shown to satisfy (4.37) and (4.38) in the course of the proof of Lemma 4.5.

Case 2^*b**. -** $\mathcal{L}(S(p/q)) < 20\mathcal{L}(\mathcal{B}(p/q,c_0))$**.** By the assumption of case 2^* we have that $S(p/q)$ contains the right endpoint of $I_0(c_0)$. Consider the following subcases;

Case 2^*bi. - $c_1 > 1$. Since $I_1(c_0;1) \cap S(p/q) \neq \emptyset$ and $S(p/q)$ contains the right endpoint of $I_0(c_0)$, we have

$$\bigcup_{b=1}^{c_1-1} I_1(c_0;b) \subseteq S(p/q).$$

By (4.36), Lemma 4.8 (iii), and the non-increasing property of $q \mapsto q^2\Psi(q)$, the set is disjoint from $\mathcal{B}(p/q,c_0)$. Moreover the collection contains the fundamental interval $I_1(c_0;1)$ and it can be shown, via Lemma 2.9 and our assumption in case 2^*b that

$$\begin{aligned} \mathcal{L}\left(\bigcup_{b=1}^{c_1-1} I_1(c_0;b)\right) \geq \mathcal{L}\left(I_1(c_0;1)\right) &\geq \frac{1}{32}\mathcal{L}\left(I_0(c_0)\right) \\ &\geq \frac{1}{32}\mathcal{L}\left(\mathcal{B}(p/q,c_0)\right) \\ &\geq \frac{1}{640}\mathcal{L}(S(p/q)). \end{aligned} \tag{4.39}$$

Thus (4.37) and (4.38) are satisfied by the collection of words $\{(c_0,b)\}_{b=1}^{c_1-1}$.

Case 2^*bii. - $c_1 = 1$. Since $I_2(c_0;1,c_2) \cap S(p/q) \neq \emptyset$ and $S(p/q)$ contains the right end point of $I_0(c_0)$ we have

$$\bigcup_{b=c_2+1}^{\left\lfloor\frac{1}{3q_{n+2}^2\Psi(q_{n+2})}\right\rfloor} I_2(c_0;1,b) \subseteq S(p/q) \setminus \mathcal{B}(p/q,(c_0,1)).$$

By Lemma 4.11, with $j=0$ and $c_1 = 1$, and our assumption in case 2^*b, we have that

$$\mathcal{L}\left(\bigcup_{b=c_2+1}^{\left\lfloor\frac{1}{3q_{n+2}^2\Psi(q_{n+2})}\right\rfloor} I_2(c_0;1,b)\right) \geq \frac{1}{40960}\mathcal{L}\left(\mathcal{B}(p/q,c_0)\right) \geq \frac{1}{819200}\mathcal{L}\left(S(p/q)\right).$$

Hence (4.37) and (4.38) are satisfied by the collection of words $\{(c_0,1,b)\}_{b=c_2+1}^{\left\lfloor \frac{1}{3q_{n+2}^2\Psi(q_{n+2})}\right\rfloor}$.

Case 3*. - There exists $b_0 \in \mathbb{N}$ such that $I_0(b_0) \supseteq S(p/q)$. Note again, by Lemma 4.8 *(ii)*, that $b_0 = c_0 = \frac{1}{q^2\Psi(q)} - 1$. Similarly to case 3 of Lemma 4.5 we split into further subcases.

*Case 3*a - $S(p/q) \cap \mathcal{B}(p/q, c_0) \neq \emptyset$.* This case is not possible. To see this note that the left endpoint of $S(p/q)$ lies inside $I_1(c_0; c_1)$ by Lemma 4.8. By (4.36) and Lemma 4.8 *(iii)* it is clear that $\mathcal{B}(p/q, c_0)$ is disjoint from $I_1(c_0; c_1)$ and lies to its left. Since the left endpoint of $S(p/q)$ is contained in $I_1(c_0; c_1)$ we must have $S(p/q) \cap \mathcal{B}(p/q, c_0) = \emptyset$.

*Case 3*b - $S(p/q) \cap \mathcal{B}(p/q, c_0) = \emptyset$ and there exists b_1 such that $I_1(c_0; b_1) \subset S(p/q)$.* The argument is the same as in case 3b of Lemma 4.5. That is, the collection of words $\{(c_0, b)\}_{b=M_1}^{M_2}$ with $M_1 = \min\{K : I_1(c_0; K) \subseteq S(p/q)\}$ and $M_2 = \max\{K : I_1(c_0; K) \subseteq S(p/q)\}$, (4.37) and (4.38).

*Case 3*c - $S(p/q) \cap \mathcal{B}(p/q, c_0) = \emptyset$, there exists no b_1 such that $I_1(c_0; b_1) \subset S(p/q)$ and there exists b_1 such that $I_1(c_0; b_1) \cap S(p/q) \neq \emptyset$ and $I_1(c_0; b_1 + 1) \cap S(p/q) \neq \emptyset$.* By Lemma 4.8 we know in this case we must have $b_1 + 1 = c_1$. Moreover, by (4.36), Lemma 4.8(iii), and the non-increasing assumption for $q \to q^2\Psi(q)$ the digit c_1 must satisfy (4.38). We split into further subcases again.

*Case 3*ci - $\mathcal{L}(S(p/q)) \geq 20\mathcal{L}(\mathcal{B}(p/q, (c_0, c_1)))$.* Consider the subcases

(1) $\mathcal{L}(S(p/q) \cap I_1(c_0; c_1 - 1)) \geq \frac{\mathcal{L}(S(p/q))}{2}$: Split into another two further subcases

(a) $S(p/q) \cap I_1(c_0; c_1 - 1) \supseteq I_2(c_0; c_1 - 1, 1)$. In this case we take our collection of words to be $\{(c_0, c_1 - 1, 1)\}$. It is then straightforward to check that (4.37) and (4.38) are satisfied by this collection.

(b) $S(p/q) \cap I_1(c_0; c_1 - 1) \subset I_2(c_0; c_1 - 1, 1)$. Then take the collection of words

$$\{(c_0, c_1-1, 1, b)\}_{b=M}^{\left\lfloor \frac{1}{3q_{n+3}^2\Psi(q_{n+3})}\right\rfloor} \quad \text{with} \quad M := \min\{K : I_3(c_0; c_1 - 1, 1, K) \subseteq S(p/q)\}.$$

Note that $q_{n+2}(\mathbf{a}c_0c_1) = q_{n+3}(\mathbf{a}c_0(c_1 - 1)1)$. Therefore the set of digits that can appear in the last place of a word describing a fundamental interval in $B(p/q, (c_0, c_1))$ coincides with the set of digits that can appear in the last place of a word describing a fundamental interval in $B(p/q, (c_0, c_1 - 1, 1))$ (recall def. of $B(p/q, \mathbf{a})$ on pp. 20). Furthermore, $c_1 \geq 2$ since $c_1 = b_1 + 1$ and $b_1 \in \mathbb{N}$, and so we can apply Lemma 2.10 to see by pairwise comparison of the fundamental intervals appearing in $B(p/q, (c_0, c_1))$ and $B(p/q, (c_0, c_1 - 1, 1))$ that

$$\mathcal{L}\left(\mathcal{B}(p/q, (c_0, c_1))\right) > \mathcal{L}\left(\mathcal{B}(p/q, (c_0, c_1 - 1, 1))\right).$$

Therefore $\mathcal{L}(S(p/q)) \geq 20\mathcal{L}\left(\mathcal{B}(p/q,(c_0,c_1-1,1))\right)$. We can then use similar calculations to those appearing in case 2a of Lemma 4.5 to verify that our collection of words satisfies (4.37) and (4.38).

(2) $\mathcal{L}(S(p/q)\cap I_1(c_0;c_1)) \geq \frac{\mathcal{L}(S(p/q))}{2}$: Then take the collection of consecutive words

$$\{(c_0,c_1,b)\}_{b=M}^{\left\lfloor \frac{1}{3q_{n+2}^2\Psi(q_{n+2})}\right\rfloor} \quad \text{with} \quad M := \min\{K : I_2(c_0,c_1,K)\subseteq S(p/q)\}\,.$$

Using the assumption $\mathcal{L}(S(p/q) \geq 20\mathcal{L}(\mathcal{B}(p/q,(c_0,c_1))$ we see that if $S(p/q)$ intersects $\mathcal{B}(p/q,(c_0,c_1))$ the intersection can only remove a small amount. Repeating the argument appearing in case 3c of the proof of Lemma 4.5 we can deduce that the collection of consecutive words satisfies (4.37) and (4.38).

Case 3^*cii - $\mathcal{L}(S(p/q)) < 20\mathcal{L}(\mathcal{B}(p/q,(c_0,c_1))$. The argument is similar to that used in case 2^*bii above. Consider the intersection of $S(p/q)$ with $I_1(c_0;c_1)$. Since $S(p/q)\cap I_1(c_0;c_1-1)\neq\emptyset$ we have that $S(p/q)\cap\mathcal{B}(p/q,(c_0,c_1))\neq\emptyset$ as $\mathcal{B}(p/q,(c_0,c_1))$ appears on the right inside $I_1(c_0;c_1)$. Therefore

$$\bigcup_{b=c_2+1}^{\left\lfloor \frac{1}{3q_{n+2}^2\Psi(q_{n+2})}\right\rfloor} I_2(c_0;c_1,b) \subset S(p/q)\setminus\mathcal{B}(p/q,(c_0,c_1))\,,$$

By Lemma 4.11 with $j=1$ odd, and our case 3^*cii assumption, we have that

$$\mathcal{L}\left(\bigcup_{b=c_2+1}^{\left\lfloor \frac{1}{3q_{n+2}^2\Psi(q_{n+2})}\right\rfloor} I_2(c_0;c_1,b)\right) > \frac{1}{640}\mathcal{L}(\mathcal{B}(p/q,(c_0,c_1))) > \frac{1}{12800}\mathcal{L}(S(p/q))\,.$$

Thus (4.37) and (4.38) are satisfied by the collection of words $\{(c_0,c_1,b)\}_{b=c_2+1}^{\left\lfloor \frac{1}{3q_{n+2}^2\Psi(q_{n+2})}\right\rfloor}$.

Case 3^*d - $S(p/q)\cap\mathcal{B}(p/q,c_0)=\emptyset$ *and there exists* b_1 *such that* $S(p/q)\subset I_1(c_0;b_1)$. By Lemma 4.8 we know $b_1=c_1$, which satisfies (4.38). Just as in the proof of Lemma 4.5 consider the following four possible events

(1) $S(p/q)\cap\mathcal{B}(p/q,(c_0,c_1))\neq\emptyset$. We use the argument of either case 3^*ci or 3^*cii, depending on the size of $S(p/q)$ relative to $\mathcal{B}(p/q,(c_0,c_1))$, to see the set of words

$$\{(c_0,c_1,b)\}_{b=M}^{\left\lfloor \frac{1}{3q_{n+2}^2\Psi(q_{n+2})}\right\rfloor} \quad \text{with } M := \min\{K : I_2(c_0;c_1,K)\subset S(p/q)\}\,,$$

satisfies (4.37) and (4.38).

(2) $S(p/q)\cap\mathcal{B}(p/q,(c_0,c_1))=\emptyset$ *and there exists* b_2 *such that* $I_2(c_0;c_1,b_2)\subset S(p/q)$. We use the same argument as case 3^*b above. That is, take the collection of consecutive words $\{(c_0,c_1,b)\}_{b=M_1}^{M_2}$ where $M_1=\min\{K : I_2(c_0;c_1,K)\subseteq S(p/q)\}$ and $M_2=\max\{K : I_2(c_0;c_1,K)\subseteq S(p/q)\}$ which satisfies (4.37) and (4.38).

(3) *$S(p/q) \cap \mathcal{B}(p/q,(c_0,c_1)) = \emptyset$, there exists no b_2 such that $I_2(c_0;c_1,b_2) \subset S(p/q)$ and there exists b_2 such that $I_2(c_0;c_1,b_2) \cap S(p/q) \neq \emptyset$ and $I_2(c_0;c_1,b_2+1) \cap S(p/q) \neq \emptyset$.* By Lemma 4.8(*iii*) we have $b_2 = c_2$ which satisfies (4.38). We then use similar arguments to case 2^* to obtain a suitable collections of words.

(4) *$S(p/q) \cap \mathcal{B}(p/q,(c_0,c_1)) = \emptyset$ and there exists b_2 such that $S(p/q) \subset I_2(c_0;c_1,b_2)$.* By Lemma 4.8(*iii*) $b_2 = c_2$ and satisfies (4.38). If one of the following happens then we can duplicate the arguments as used in cases $3^*a - c$:
  (1') $S(p/q) \cap \mathcal{B}(p/q,(c_0,c_1,c_2)) \neq \emptyset$,
  (2') $S(p/q) \cap \mathcal{B}(p/q,(c_0,c_1,c_2)) = \emptyset$ and there exists $b_3 \in \mathbb{N}$ such that $S(p/q) \supset I_3(c_0;c_1,c_2,b_3)$,
  (3') $S(p/q) \cap \mathcal{B}(p/q,(c_0,c_1,c_2)) = \emptyset$, there exists no $b_3 \in \mathbb{N}$ such that $S(p/q) \supset I_3(c_0;c_1,c_2,b_3)$ and there exists $b_3 \in \mathbb{N}$ such that $S(p/q) \cap I_3(c_0;c_1,c_2,b_3) \neq \emptyset$ and $S(p/q) \cap I_3(c_0;c_1,c_2,b_3-1) \neq \emptyset$.

  This covers all cases except the possibility that there exists $b_3 \in \mathbb{N}$ such that $S(p/q) \subset I_3(c_0;c_1,c_2,b_3)$. In this case we must have $b_3 = c_3$. This puts us essentially back to the beginning of $3^*d(4)$ except now we also know that $b_3 = c_3$ which satisfies (4.38), by (4.36) and Lemma 4.8(*iii*). We can repeat this process, with the prefix word $(c_0,c_1,\ldots,c_i)$ being replaced by $(c_0,c_1,\ldots,c_i,c_{i+1})$ at each step where (1')-(3') do not occur. This step also verifies c_{i+1} satisfies (4.38). The process must eventually terminate. To see this assume that for $1 \leq i \leq m-1$ we have $S(p/q) \subset I_{i+1}(c_0;c_1,\ldots,c_i,b_{i+1})$ for some $b_{i+1} \in \mathbb{N}$. Consider the case where $i = m-1$ then $b_m = c_m$ and $S(p/q) \subset I_m(c_0;c_1,\ldots,c_m)$. The digits $c_1,\ldots,c_m$ each satisfy (4.38) by (4.36) and Lemma 4.8(*iii*). By Lemma 4.8 and Lemma 2.7 we know that the left endpoints of $S(p/q)$ and $I_m(c_0;c_1,\ldots,c_m)$ agree. Furthermore, m is even so $\mathcal{B}(p/q,(c_0,\ldots,c_m))$ appears on the left of $I_m(c_0;c_1,\ldots,c_m)$ thus $S(p/q) \cap \mathcal{B}(p/q,(c_0,\ldots,c_m)) \neq \emptyset$ and so we are in case (1'). At this point we use the bound $\mathcal{L}(S(p/q)) \geq 20\mathcal{L}(\mathcal{B}(p/q,(c_0,\ldots,c_m)))$ which follows from Lemma 4.10. We take our collection of words to be

$$\{(c_0,c_1,\ldots,c_m,b)\}_{b=M}^{\left\lfloor \frac{1}{3q_{n+m+1}^2\Psi(q_{n+m+1})}\right\rfloor} \quad \text{with} \quad M := \min\{K : I_{m+1}(c_0;c_1,\ldots,c_m,K) \subseteq S(p/q)\}.$$

  This collection satisfies (4.38) by the choice of c_i at each step, and (4.37) can be shown to hold by similar arguments to those appearing in cases $3^*ci(2)$ and 3^*d. Here we use our condition $\mathcal{L}(S(p/q)) \geq 20\mathcal{L}(\mathcal{B}(p/q,\mathbf{c}))$.

□

Proof of Lemma 4.7. The proof is analogous to the proof of Lemma 4.1 with Lemma 4.12 replacing Lemma 4.5. We take our collection of words to be the set $\{\mathbf{b}_l\}_{l=1}^L$ provided by Lemma 4.12. Note that Lemma 4.2 and Lemma 4.3 are still applicable in the context of Lemma 4.7. Combining Lemma 4.2 with Lemma 4.12 implies that our set of words satisfies (4.24). (4.26) follows from (4.38), and (4.25) follows from the same calculations as in the proof of Lemma 4.1 with (4.37) replacing (4.14). □

5. Proof of Theorems 1.1–1.9

We begin with the proof of Theorem 1.1. We will then briefly sketch the differences in the argument required to prove the second part of Theorem 1.5. Recall that the first part of Theorem 1.5 is an immediate consequence of Theorem 1.6. We conclude this section with a proof of Theorem 1.9.

5.1. Proof of Theorem 1.1.

Proof. Let Ψ and ω satisfy the assumptions of our theorem. To prove our result we will construct an injective function $f:\{1,2\}^{\mathbb{N}}\to E(\Psi,\omega)$. Let us start by picking $N_0\in\mathbb{N}$ to be even and sufficiently large that if $\frac{p_{N_0}}{q_{N_0}}=[0;1^{N_0}]$ then q_{N_0} is sufficiently large that Lemma 4.1 can be applied. To simplify our notation we let $\mathbf{a}=(1^{N_0})$. Let us now pick $(\delta_i)_{i=1}^{\infty}\in\{1,2\}^{\mathbb{N}}$ and set out to define $f((\delta_i))$. We will construct f via an inductive argument.

Step 1. Applying Lemma 4.1 to p_{N_0}/q_{N_0} we can choose a finite word $\mathbf{b}_1=(b_{1,0},\dots,b_{1,m_1-1})$ in a way that depends only upon p_{N_0}/q_{N_0}, Ψ and ω such that

$$I(0;\mathbf{a}\mathbf{b}_1)\subset\left\{x:\Psi(q_{N_0})(1-\omega(q_{N_0}))\leq x-\frac{p_{N_0}}{q_{N_0}}\leq\Psi(q_{N_0})\right\},$$

and if $m_1\geq 2$ then

$$b_{1,i}\in\left\{1,2,\dots,\left\lfloor\frac{1}{3q_{N_0+i}^2\Psi(q_{N_0+i})}\right\rfloor\right\}\quad 1\leq i\leq m_1-1,$$

where $\frac{p_{N_0+i}}{q_{N_0+i}}=[0;\mathbf{a},b_{1,0},b_{1,1},\dots,b_{1,i-1}]$ for $1\leq i\leq m_1-1$. Let us now also pick $l_1\in\{1,2\}$ such that $N_0+m_1+l_1$ is even.

Inductive step. Let $k\geq 1$ and suppose that we have constructed k words $\mathbf{b}_1,\dots,\mathbf{b}_k$ of length $m_1,\dots,m_k$ respectively and $l_1,\dots,l_k\in\mathbb{N}$ such that the following properties hold:

(1) For each $1\leq j\leq k$ the sum $N_j:=N_0+\sum_{p=1}^{j}(m_p+l_p)$ is even.
(2) For each $1\leq j\leq k$ we have

$$I(0;\mathbf{a}\mathbf{b}_1\delta_1^{l_1}\mathbf{b}_2\delta_2^{l_2}\dots\delta_{j-1}^{l_{j-1}}\mathbf{b}_j)\subset\left\{x:\Psi(q_{N_{j-1}})(1-\omega(q_{N_{j-1}}))\leq x-\frac{p_{N_{j-1}}}{q_{N_{j-1}}}\leq\Psi(q_{N_{j-1}})\right\}.$$

(3) For each $1\leq j\leq k$ if $m_j\geq 2$ then

$$b_{j,i}\in\left\{1,2,\dots,\left\lfloor\frac{1}{3q_{N_{j-1}+i}^2\Psi(q_{N_{j-1}+i})}\right\rfloor\right\}\quad 1\leq i\leq m_j-1,$$

where $\frac{p_{N_{j-1}+i}}{q_{N_{j-1}+i}}=[0;\mathbf{a}\mathbf{b}_1\delta_1^{l_1}\mathbf{b}_2\delta_2^{l_2}\dots,\mathbf{b}_{j-1}\delta_{j-1}^{l_{j-1}}b_{j,0},b_{j,1},\dots,b_{j,i-1}]$.

We now apply Lemma 4.1 to $\frac{p_{N_k}}{q_{N_k}} = [0; \mathbf{a}\mathbf{b}_1\delta_1^{l_1} \dots \mathbf{b}_k\delta_k^{l_k}]$ and choose a word $\mathbf{b}_{k+1} = (b_{k+1,0}, \dots, b_{k+1,m_{k+1}-1})$ in a way depending only upon $\frac{p_{N_k}}{q_{N_k}}$, Ψ and ω so that

$$I(0; \mathbf{a}\mathbf{b}_1\delta_1^{l_1}\mathbf{b}_2\delta_2^{l_2} \dots \mathbf{b}_k\delta_k^{l_k}\mathbf{b}_{k+1}) \subset \left\{x : \Psi(q_{N_k})(1-\omega(q_{N_k})) \leq x - \frac{p_{N_k}}{q_{N_k}} \leq \Psi(q_{N_k})\right\},$$

and if $m_{k+1} \geq 2$ then

$$b_{k+1,i} \in \left\{1, 2, \dots, \left\lfloor \frac{1}{3q_{N_k+i}^2\Psi(q_{N_k+i})} \right\rfloor \right\} \quad 1 \leq i \leq m_{k+1} - 1,$$

where $\frac{p_{N_k+i}}{q_{N_k+i}} = [0; \mathbf{a}\mathbf{b}_1\delta_1^{l_1}\mathbf{b}_2\delta_2^{l_2} \dots, \mathbf{b}_k\delta_k^{l_k}b_{k+1,0}, b_{k+1,1}, \dots, b_{k+1,i-1}]$. We can apply Lemma 4.1 because N_k is even. We now choose $l_{k+1} \in \{1, 2\}$ so that $N_{k+1} = N_0 + \sum_{p=1}^{k+1}(m_p + l_p)$ is even. We observe that our new collection of words $\mathbf{b}_1, \dots, \mathbf{b}_{k+1}$ with lengths $m_1, \dots, m_{k+1}$ and the collection of integers $l_1, \dots, l_{k+1}$ satisfy properties $(1) - (3)$ listed above with k replaced with $k+1$. Thus we can repeat this step indefinitely.

Repeatedly applying our inductive step yields an infinite sequence of words $(\mathbf{b}_j)_{j=1}^{\infty}$ with corresponding lengths $(m_j)_{j=1}^{\infty}$ and the sequence of integers $(l_j)_{j=1}^{\infty}$. We define $f((\delta_i))$ to be

$$f((\delta_i)) = \bigcap_{j=1}^{\infty} I_{N_j}(0; \mathbf{a}\mathbf{b}_1\delta_1^{l_1} \dots \mathbf{b}_j\delta_j^{l_j}),$$

or equivalently $f((\delta_i))$ is the irrational number satisfying $f((\delta_i)) = [0; \mathbf{a}\mathbf{b}_1\delta_1^{l_1}\mathbf{b}_2\delta_2^{l_2} \dots]$. It remains to show that f is injective and maps into $E(\Psi, \omega)$.

f is injective. Let $(\delta_i), (\delta_i') \in \{1, 2\}^{\mathbb{N}}$ be distinct sequences and let $j \in \mathbb{N}$ be the minimal integer such that $\delta_j \neq \delta_j'$. Without loss of generality we assume $\delta_j = 1$ and $\delta_j' = 2$. Then it follows from our construction of f that there exists $\mathbf{b}_1 \dots, \mathbf{b}_j$ and $l_1, \dots, l_j$ such that

$$f((\delta_i)) \in I(0; \mathbf{a}\mathbf{b}_1\delta_1^{l_1}, \dots, \mathbf{b}_j, 1^{l_j})$$

and

$$f((\delta_i')) \in I(0; \mathbf{a}\mathbf{b}_1\delta_1^{l_1}, \dots, \mathbf{b}_j, 2^{l_j}).$$

Since the continued fraction expansion of an irrational number is unique, it follows that $f((\delta_i)) \neq f((\delta_i'))$ and the injectivity of f follows.

f maps into $E(\Psi, \omega)$. By construction, we know that

$$0 \leq f((\delta_i)) - \frac{p_{N_j}}{q_{N_j}} \leq \Psi(q_{N_j})$$

for all j. It remains to show that for any $(p, q) \in \mathbb{Z} \times \mathbb{N}$ with q sufficiently large we have

$$\Psi(q)(1 - \omega(q)) \leq \left| x - \frac{p}{q} \right|. \tag{5.1}$$

Since $\lim_{q\to\infty} q^2\Psi(q)=0$, by Lemma 2.3, to prove (5.1) holds for q sufficiently large, it suffices to show that

$$\Psi(q_n)(1-\omega(q_n)) \leq \left|x-\frac{p_n}{q_n}\right|. \tag{5.2}$$

for all n sufficiently large. If $n=N_j$ for some j then

$$\Psi(q_{N_j})(1-\omega(q_{N_j})) \leq x-\frac{p_{N_j}}{q_{N_j}}$$

by our construction. So (5.2) holds when $n=N_j$. Suppose $n\notin\{N_j\}$. We let (a_l) denote the sequence of partial quotients for $f((\delta_i))$. Then by our construction either $a_{n+1}\in\{1,2\}$ in which case

$$\left|x-\frac{p_n}{q_n}\right| \geq \frac{1}{6q_n^2} \tag{5.3}$$

by Lemma 2.2, or

$$a_{n+1}\in\left\{1,2,\ldots,\left\lfloor\frac{1}{3q_n^2\Psi(q_n)}\right\rfloor\right\}$$

where $\frac{p_n}{q_n}=[0;a_1,\ldots,a_n]$ and

$$\left|x-\frac{p_n}{q_n}\right| \geq \Psi(q) \tag{5.4}$$

holds by Lemma 2.2. Since $\lim_{q\to\infty} q^2\Psi(q)=0$, it follows from (5.3) and (5.4) that (5.2) holds for all n sufficiently large such that $n\notin\{N_j\}$. In summary, we have shown that (5.2) holds for all n sufficiently large. Therefore f maps into $E(\Psi,\omega)$ and our proof is complete. □

5.2. **Proof of Theorem 1.5.** Recall that the first part of Theorem 1.5 is a special case of Theorem 1.6. The second statement is proved below. The argument presented is similar in spirit to the proof of Theorem 1.1. The main difference being that we apply Lemma 4.6 in place of Lemma 4.1.

Proof of the second statement of Theorem 1.5. Let Ψ and ω satisfy the assumptions of the second statement of Theorem 1.5. To prove our result we will construct an injective function $f:\{2,3\}^{\mathbb{N}}\to E(\Psi,\omega)$[1]. Let us start by picking $N_0\in\mathbb{N}$ to be even and sufficiently large that if $\frac{p_{N_0}}{q_{N_0}}=[0;2^{N_0}]$ then q_{N_0} is sufficiently large that Lemma 4.6 can be applied. We let $\mathbf{a}_0=(2^{N_0})$. Let us now pick $(\delta_i)_{i=1}^{\infty}\in\{2,3\}^{\mathbb{N}}$ and set out to define $f((\delta_i))$. We will construct $f((\delta_i))$ via an inductive argument.

[1]The significance of taking $\{2,3\}^{\mathbb{N}}$ instead of $\{1,2\}^{\mathbb{N}}$, as in the proof of Theorem 1.1, is to ensure we satisfy the $a_1,a_n\geq 2$ assumption appearing in Lemma 4.6.

Step 1. We apply Lemma 4.6 to p_{N_0}/q_{N_0}. Therefore

$$I\left(0;\mathbf{a}_0,\frac{1}{q_{N_0}^2\Psi(q_{N_0})}-1,1,1,2^{N_0-2},1,1,\left\lfloor\frac{1}{3q_{*,1}^2\Psi(q_{*,1})}\right\rfloor\right)$$
$$\subset\left\{x:\Psi(q_{N_0})(1-\omega(q_{N_0}))\leq x-\frac{p_{N_0}}{q_{N_0}}\leq\Psi(q_{N_0})\right\}$$

where

$$\frac{p_{*,1}}{q_{*,1}}=\left[0;\mathbf{a}_0,\frac{1}{q_{N_0}^2\Psi(q_{N_0})}-1,1,1,2^{N_0-2},1,1\right].$$

We may assume that we chose N_0 to be sufficiently large that

$$\frac{1}{q_{N_0}^2\Psi(q_{N_0})}-1\geq 1\quad\text{and}\quad\left\lfloor\frac{1}{3q_{*,0}^2\Psi(q_{*,0})}\right\rfloor\geq 1. \tag{5.5}$$

We let

$$\mathbf{a}_1=\left(\mathbf{a}_0,\frac{1}{q_{N_0}^2\Psi(q_{N_0})}-1,1,1,2^{N_0-2},1,1,\left\lfloor\frac{1}{3q_{*,0}^2\Psi(q_{*,0})}\right\rfloor,\delta_1^{l_1}\right),$$

N_1 denote the length of $\mathbf{a}_1$, and

$$\frac{p_{N_1}}{q_{N_1}}=[0;\mathbf{a}_1].$$

Here $l_1\in\mathbb{N}$ is chosen so that N_1 is even and so that for all $q\geq q_{N_1}$ we have

$$\max_{1\leq i\leq N_1}a_{1,i}\leq\left\lfloor\frac{1}{3q^2\Psi(q)}\right\rfloor$$

where $\mathbf{a}_1=(a_{1,1},\dots a_{1,N_1})$. This is permissible as $\lim_{q\to\infty}q^2\Psi(q)=0$ and $\delta_1\in\{2,3\}$. We remark that the first digit and last digit of $\mathbf{a}_1$ are greater than or equal to two by construction and all other digits in $\mathbf{a}_1$ are strictly positive integers by (5.5).

Inductive step. Let $k\geq 1$ and suppose that we have constructed $k+1$ words $\{\mathbf{a}_j=(a_{j,1},\dots a_{j,N_j})\}_{j=0}^k$ such that N_j is even for each $0\leq j\leq k$ and $l_1,\dots,l_k\in\mathbb{N}$. For each $0\leq j\leq k$ let $\frac{p_{N_j}}{q_{N_j}}=[0;\mathbf{a}_j]$. Suppose that the following properties are satisfied:

(1) For each $0\leq j<k$ we have

$$\mathbf{a}_{j+1}=\left(\mathbf{a}_j,\frac{1}{q_{N_j}^2\Psi(q_{N_j})}-1,1,a_{j,N_j}-1,a_{j,N_j-1}\dots,a_{j,2},a_{j,1}-1,1,\left\lfloor\frac{1}{3q_{*,j}^2\Psi(q_{*,j+1})}\right\rfloor,\delta_j^{l_{j+1}}\right)$$

where

$$\frac{p_{*,j+1}}{q_{*,j+1}}=\left[0;\mathbf{a}_j,\frac{1}{q_{N_j}^2\Psi(q_{N_j})}-1,1,a_{j,N_j}-1,a_{j,N_j-1}\dots,a_{j,2},a_{j,1}-1,1\right]$$

(2) For each $0\leq j<k$ we have

$$I(0;\mathbf{a}_{j+1})\subset\left\{x:\Psi(q_{N_j})(1-\omega(q_{N_j}))\leq x-\frac{p_{N_j}}{q_{N_j}}\leq\Psi(q_{N_j})\right\}.$$

(3) For each $1 \leq j \leq k$, for any $q \geq q_{N_j}$ we have

$$\max_{1\leq i\leq N_j} a_{j,i} \leq \left\lfloor \frac{1}{3q^2\Psi(q)} \right\rfloor$$

(4) For each $0 \leq j \leq k$ the first and last digit of $\mathbf{a}_j$ is greater than or equal to two, and all other digits are strictly positive integers.

We now show how to define $\mathbf{a}_{k+1}$ and l_{k+1} so that the above properties are satisfied with $k+1$ in place of k. We apply Lemma 4.6 to $\frac{p_{N_k}}{q_{N_k}}$ so that

$$I\left(0; \mathbf{a}_k, \frac{1}{q_{N_k}^2\Psi(q_{N_k})} - 1, 1, a_{k,N_k} - 1, a_{k,N_k-1}\dots, a_{k,2}, a_{k,1} - 1, 1, \left\lfloor \frac{1}{3q_{*,k+1}^2\Psi(q_{*,k+1})} \right\rfloor\right)$$
$$\subset \left\{x : \Psi(q_{N_k})(1-\omega(q_{N_j})) \leq x - \frac{p_{N_k}}{q_{N_k}} \leq \Psi(q_{N_k})\right\}$$

where $\frac{p_{*,k+1}}{q_{*,k+1}} = \left[0; \mathbf{a}_k, \frac{1}{q_{N_k}^2\Psi(q_{N_k})} - 1, 1, a_{k,N_k} - 1, a_{k,N_k-1}\dots, a_{k,2}, a_{k,1} - 1, 1\right]$. Let $\mathbf{a}_{k+1}$ be given by

$$\left(\mathbf{a}_k, \frac{1}{q_{N_k}^2\Psi(q_{N_k})} - 1, 1, a_{k,N_k} - 1, a_{k,N_k-1}\dots, a_{k,2}, a_{k,1} - 1, 1, \left\lfloor \frac{1}{3q_{*,k+1}^2\Psi(q_{*,k+1})} \right\rfloor, \delta_{k+1}^{l_{k+1}}\right)$$

where l_{k+1} has been chosen to be sufficiently large that

$$\max_{1\leq i\leq N_{k+1}} a_{k+1,i} \leq \left\lfloor \frac{1}{3q^2\Psi(q)} \right\rfloor$$

for any $q \geq q_{N_{k+1}}$ where $\frac{p_{N_{k+1}}}{q_{N_{k+1}}} = [0; \mathbf{a}_{k+1}]$ and $\mathbf{a}_{k+1} = (a_{k+1,1}, \dots, a_{k+1,N_{k+1}})$. This is permissible as $\lim_{q\to\infty} q^2\Psi(q) = 0$ and $\delta_{k+1} \in \{2,3\}$. It is clear that for this choice of $\mathbf{a}_{k+1}$ and l_{k+1} properties (1), (2) and (3) all hold with k replaced by $k+1$. To see that the fourth property holds, observe that the first digit of $\mathbf{a}_{k+1}$ is the first digit of $\mathbf{a}_k$, so by assumption it is greater than or equal to 2. The last digit δ_{k+1} belongs to $\{2,3\}$, so the last digit also has the desired property. It remains to check that all other digits in $\mathbf{a}_{k+1}$ are strictly positive integers. If a digit belongs to $\{1,2,3\}$ then this is obviously true. If it equals one of $\mathbf{a}_k$'s digits, then by our assumption it is a strictly positive integer. The only digits that do not necessarily fall in to these categories are

$$\frac{1}{q_{N_k}^2\Psi(q_{N_k})} - 1, \quad a_{k,N_k} - 1 \quad \text{and} \quad \left\lfloor \frac{1}{3q_{*,k+1}^2\Psi(q_{*,k+1})} \right\rfloor.$$

Since $\lim_{q\to\infty} q^2\Psi(q) = 0$ we can assume that our initial choice of N_0 was sufficiently large that

$$\frac{1}{q_{N_k}^2\Psi(q_{N_k})} - 1 \geq 1 \quad \text{and} \quad \left\lfloor \frac{1}{3q_{*,k}^2\Psi(q_{*,k})} \right\rfloor \geq 1.$$

The remaining digit, $a_{k,N_k} - 1$, is the last term of $\mathbf{a}_k$ with one subtracted. By our assumption the last term of $\mathbf{a}_k$ is at least two Therefore $a_{k,N_k} - 1 \geq 1$. We may now conclude that property (4) is satisfied with k replaced by $k+1$.

Repeatedly applying our inductive step yields an infinite sequence of words $(\mathbf{a}_j)_{j=0}^{\infty}$ and an infinite sequence of integers $(l_j)_{j=1}^{\infty}$ such that properties (1) – (4) hold for any $k \geq 1$. We define $f((\delta_i))$ to be

$$f((\delta_i)) = \bigcap_{j=0}^{\infty} I(0; \mathbf{a}_j).$$

It follows from property (1) that the intervals appearing in this intersection are nested and consequently the intersection is non-empty. Moreover, by our construction this intersection must be a single point. Equivalently, $f((\delta_i))$ is the point whose sequence of partial quotients is equal to the componentwise limit of the sequence $(0\mathbf{a}_k 0^{\infty})_{k=0}^{\infty}$. Here we are using property (4) which guarantees that for each k each digit in $\mathbf{a}_k$ is a strictly positive integer. This guarantees that $(0\mathbf{a}_k 0^{\infty})_{k=0}^{\infty}$ will converge to the continued fraction expansion of some real number.

It remains to show that f is injective and maps into $E(\Psi, \omega)$. Injectivity follows by an analogous argument to that given in the proof of Theorem 1.1. As such we omit the argument and focus on proving that f maps into $E(\Psi, \omega)$.

f **maps into** $E(\Psi, \omega)$. By property (2) we know that

$$\Psi(q_{N_j})(1 - \omega(q_{N_j}) \leq f((\delta_i)) - \frac{p_{N_j}}{q_{N_j}} \leq \Psi(q_{N_j})$$

for all $j \in \mathbb{N}$. Therefore, to show that $f((\delta_i)) \in E(\Psi, \omega)$, it remains to show that there are at most finitely many $(p, q) \in \mathbb{Z} \times \mathbb{N}$ satisfying

$$\left| x - \frac{p}{q} \right| < \Psi(q)(1 - \omega(q)). \tag{5.6}$$

Since $\lim_{q \to \infty} q^2 \Psi(q) = 0$ we know that $\Psi(q) \leq \frac{1}{2q^2}$ for all q sufficiently large. Therefore, by Lemma 2.3, to show that (5.6) has finitely many solutions it suffices to show that for all n sufficiently large we have

$$\left| x - \frac{p_n}{q_n} \right| \geq \Psi(q_n)(1 - \omega(q_n)). \tag{5.7}$$

By property (2) (5.7) is satisfied whenever $n \in \{N_j\}_{j=0}^{\infty}$. It remains to consider the remaining case when $n \notin \{N_j\}_{j=0}^{\infty}$. Let us fix such an n and let k be such that $N_k < n < N_{k+1}$. Upon inspection three cases naturally arise from our construction. Either $a_{n+1} \in \{1, 2, 3\}$, $a_{n+1} \in \{a_{k,1} - 1, a_{k,2}, \ldots, a_{k,N_k - 1}, a_{k,N_k} - 1\}$ or $q_n = q_{*,k+1}$. We treat each case below individually.

Case 1 - $a_{n+1} \in \{1, 2, 3\}$**.** In this case it follows from Lemma 2.2 that

$$\left| x - \frac{p_n}{q_n} \right| \geq \frac{1}{9q_n^2}.$$

Since $\lim_{q \to \infty} q^2 \Psi(q) = 0$ there are at most finitely many values of n where $a_{n+1} \in \{1, 2, 3\}$ satisfying (5.7).

Case 2 - $a_{n+1} \in \{a_{k,1}-1, a_{k,2}, \ldots, a_{k,N_k-1}, a_{k,N_k}-1\}$. In this case we have the following bound for a_{n+1} that follows from property (3):

$$a_{n+1} \leq \left\lfloor \frac{1}{3q_n^2\Psi(q_n)} \right\rfloor .$$

It now follows from an application of Lemma 2.2 that such an n cannot contribute any solutions to (5.7).

Case 3 - $q_n = q_{*,k+1}$. In this case, we know by our construction that

$$a_{n+1} = \left\lfloor \frac{1}{3q_n^2\Psi(q_n)} \right\rfloor .$$

Again by an application of Lemma 2.2 no such n can contribute any solutions to (5.7).

In summary, we have shown that there are finitely many values of $n \notin \{N_j\}_{j=0}^{\infty}$ that satisfy (5.7). Combined with our earlier observation, we may conclude that (5.7) has finitely many solutions and $f((\delta_i)) \in E(\Psi, \omega)$. This completes our proof. □

5.3. **Proof of Theorem 1.9.**

Proof of Theorem 1.9. Let $\omega : (0,\infty) \to (0,\infty)$ and $\tau > 2$ be arbitrary. Moreover let $\psi : (0,\infty) \to (0,\infty)$ be given by $\psi(q) = \frac{1}{q^2\lceil q^{\tau-2}\rceil}$. Clearly ψ satisfies the square divisibility condition and $\lim_{q\to\infty} \frac{-\log\psi(q)}{\log q} = \tau$. We will construct the desired Ψ by perturbing ψ. These perturbations will ensure $E(\Psi,\omega) \neq \emptyset$ but will be sufficiently small to leave the lower limit at infinity unchanged.

Let $N_0 \in \mathbb{N}$ and set $\frac{p_{N_0}}{q_{N_0}} = [0; 1^{N_0}]$. We can choose N_0 sufficiently large so that

$$q^2\psi(q) \leq \tfrac{1}{2} \tag{5.8}$$

for all $q \geq q_{N_0}$. Define the function Ψ at q_{N_0} so that

$$\frac{1}{q_{N_0}^2\Psi(q_{N_0})} - \frac{q_{N_0-1}}{q_{N_0}} := \left[\frac{1}{q_{N_0}^2\psi(q_{N_0})} - 1; 1, 1, 1, \ldots\right] .$$

That is, writing $\varphi = [0; 2, 1, 1 \ldots]$, we set

$$\Psi(q_{N_0}) = \frac{1}{q_{N_0}^2\left(\frac{1}{q_{N_0}^2\psi(q_{N_0})} - \varphi + \frac{q_{N_0-1}}{q_{N_0}}\right)} . \tag{5.9}$$

Here we are using the formula $-[0; b_1, b_2, \ldots,] = [-1; 1, b_1 - 1, b_2, b_3, \ldots]$. Which holds for any sequence of positive integers $(b_i)_{i=1}^{\infty}$. For a proof see [36]. It is a consequence of (5.8) and the square divisibility condition that

$$\frac{1}{q_{N_0}^2\psi(q_{N_0})} - 1 \geq 1$$

and is an integer. It follows from (5.9) that if N_0 is chosen sufficiently large then

$$\frac{1}{2}\psi(q_{N_0}) \leq \Psi(q_{N_0}) \leq 2\psi(q_{N_0}) . \tag{5.10}$$

We now want to pick a fundamental interval contained inside $S(\Psi,\omega,p_{N_0}/q_{N_0})$. We write it in this format to emphasis $S(\Psi,\omega,p_{N_0}/q_{N_0})$ is the set appearing in Proposition 2.5(ii) for Ψ, not ψ. Choose $R_1 \in \mathbb{N}$ to be an even number and large enough (depending on ω, ψ and N_0) so that

$$\sum_{i=1}^{2} \mathcal{L}\left(I_{R_1}\left(\frac{1}{q_{N_0}^2\psi(q_{N_0})} - 1; 1^{R_1-1}, i\right)\right) < \frac{\omega(q_{N_0})}{q_{N_0}^2\Psi(q_{N_0})}. \tag{5.11}$$

Equation (5.11) must eventually hold since the terms on the left tend to zero as R_1 increases and the term on the right is positive and does not depend upon R_1. We remark that the left endpoint of $S(\Psi,\omega,p_{N_0}/q_{N_0})$ is contained in

$$I_{R_1}\left(\frac{1}{q_{N_0}^2\psi(q_{N_0})} - 1; 1^{R_1}\right),$$

and to the immediate right of this fundamental interval we have

$$I_{R_1}\left(\frac{1}{q_{N_0}^2\psi(q_{N_0})} - 1; 1^{R_1-1}, 2\right)$$

since R_1 is even. It follows from these observations, (5.11) and the lower bound

$$\mathcal{L}(S(\Psi,\omega,p_{N_0}/q_{N_0})) > \frac{\omega(q_{N_0})}{q_{N_0}^2\Psi(q_{N_0})}$$

that

$$I_{R_1}\left(\frac{1}{q_{N_0}^2\psi(q_{N_0})} - 1; 1, \dots, 1, 2\right) \subset S(\Psi,\omega,p_{N_0}/q_{N_0}). \tag{5.12}$$

Define

$$\mathbf{c}_1 := \left(1^{N_0}, \frac{1}{q_{N_0}^2\psi(q_{N_0})} - 1, 1^{R_1-1}, 2\right)$$

Set $N_1 = N_0 + R_1 + 1$, define $\frac{p_{N_1}}{q_{N_1}} := [0; \mathbf{c}_1]$, and choose R_1 large enough so that

$$2\psi(q_{N_1}) < \frac{1}{2}\psi(q_{N_0}). \tag{5.13}$$

We now repeat the above step with $\frac{p_{N_0}}{q_{N_0}}$ replaced by $\frac{p_{N_1}}{q_{N_1}}$. That is, define Ψ at q_{N_1} by

$$\Psi(q_{N_1}) := \frac{1}{q_{N_1}^2\left(\frac{1}{q_{N_1}^2\psi(q_{N_1})} - \varphi + \frac{q_{N_1-1}}{q_{N_1}}\right)}.$$

Note that the inequalities

$$\frac{1}{2}\psi(q_{N_1}) \le \Psi(q_{N_1}) \le 2\psi(q_{N_1}) \tag{5.14}$$

can be guaranteed by our initial choice of N_0. Again, we can choose some $R_2 \in \mathbb{N}$ a large even integer so that

$$I_{R_2}\left(\frac{1}{q_{N_1}^2\psi(q_{N_1})} - 1; 1^{R_2-1}, 2\right) \subset S(\Psi,\omega,p_{N_1}/q_{N_1}). \tag{5.15}$$

It is a consequence of (5.8) and that ψ satisfies the square divisibility condition that

$$\frac{1}{q_{N_1}^2\psi(q_{N_1})}-1\geq 1$$

and the left hand side of this inequality is an integer. Define

$$\mathbf{c}_2:=\left(\mathbf{c}_1,\frac{1}{q_{N_1}^2\psi(q_{N_1})}-1,1^{R_2-1},2\right).$$

Set $N_2=N_1+R_2+1$ and $\frac{p_{N_2}}{q_{N_2}}=[0;\mathbf{c}_2]$. Again we may assume that R_2 is chosen sufficiently large so that

$$2\psi(q_{N_2})<\frac{1}{2}\psi(q_{N_1}). \tag{5.16}$$

Repeating the above arguments indefinitely yields two sequences of integers $(N_i)_{i=0}^{\infty}$, $(R_i)_{i=1}^{\infty}$, a sequence of rationals $(p_{N_i}/q_{N_i})_{i=0}^{\infty}$, and a sequence of words $(\mathbf{c}_i)_{i=1}^{\infty}$ whose digits are strictly positive integers. Moreover $\frac{p_{N_i}}{q_{N_i}}=[0;\mathbf{c}_i]$ and $\mathbf{c}_i$ has length N_i for all $i\geq 1$. We define Ψ at q_{N_i} according to the rule

$$\Psi(q_{N_i}):=\frac{1}{q_{N_i}^2\left(\frac{1}{q_{N_i}^2\psi(q_{N_i})}-\varphi+\frac{q_{N_i-1}}{q_{N_i}}\right)}.$$

The following properties are a consequence of our construction:

(1) For all $i\geq 1$ we have
$$\mathbf{c}_{i+1}=\left(\mathbf{c}_i,\frac{1}{q_{N_i}^2\Psi(q_{N_i})}-1,1^{R_{i+1}-1},2\right).$$

(2) For all $i\geq 0$ we have
$$I_{R_{i+1}}\left(\frac{1}{q_{N_i}^2\psi(q_{N_i})}-1;1^{R_{i+1}-1},2\right)\subset S(\Psi,\omega,p_{N_i}/q_{N_i}).$$

(3) Equations (5.10) and (5.14) hold for all i, i.e. for all $i\geq 0$ we have
$$\frac{1}{2}\psi(q_{N_i})\leq\Psi(q_{N_i})\leq 2\psi(q_{N_i})$$

(4) Equations (5.13) and (5.16) holds for all $i\geq 0$, i.e. for all $i\geq 0$ we have
$$2\psi(q_{N_{i+1}})<\frac{1}{2}\psi(q_{N_i}).$$

By property (1) we know that

$$x=\lim_{i\to\infty}[0;\mathbf{c}_i]:=[0;a_1(x),a_2(x),\ldots]$$

is well defined and its sequence of partial quotients is the component wise limit of the sequence $(0\mathbf{c}_i0^{\infty})_{i=1}^{\infty}$. Moreover this x has the following properties:

(A) for every N_i we have
$$[a_{N_i+1}(x);a_{N_i+2}(x),\ldots]\in S(\Psi,\omega,p_{N_i}(x)/q_{N_i}(x)),$$

(B) for every $j\notin\{N_i+1\}_{i\in\mathbb{N}\cup\{0\}}$ we have $a_j(x)\leq 2$.

Property (A) is a consequence of Property (2) above. Property (B) is a consequence of the formula stated in property (1).

Define Ψ to be the non-increasing function given by

$$\Psi(q) := \begin{cases} \Psi(q_{N_i}) & q = q_{N_i} \\ \min\{\Psi(q_{N_i}), \max\{\psi(q), \Psi(q_{N_{i+1}})\}\} & q_{N_i} < q < q_{N_{i+1}}\,. \end{cases}$$

It is a consequence of properties (3) and (4) that Ψ is non-increasing and $\lim_{q\to\infty}\frac{-\log\Psi(q)}{\log q} = \tau$. It follows from the equality $\lim_{q\to\infty}\frac{-\log\Psi(q)}{\log q} = \tau$ that $q^2\Psi(q) < 1/2$ for all sufficiently large q. Hence we can apply Proposition 2.5 together with properties (A) and (B) to conclude that $x \in E(\Psi, \omega)$ so the set is non-empty. $\square$

6. Proof of Theorem 1.10 and Theorem 1.11

We begin our proofs with some discussion on the lower order of the function $q \mapsto \Psi(q)\omega(q)$. As we will see, two divided by the lower order coincides with the natural upper bound for the Hausdorff dimension of $E(\Psi, \omega)$ provided by (1.1). The benefit of the lower order is that when doing dimension calculations it is easier to work with than the natural upper bound.

Given $\Psi : (0, \infty) \to (0, \infty)$ and $\omega : (0, \infty) \to (0, 1)$ we define the lower order of (Ψ, ω) to be

$$\lambda(\Psi, \omega) = \liminf_{q\to\infty} \frac{-\log\Psi(q)\omega(q)}{\log q}.$$

The following simple proposition connects the lower order of (Ψ, ω) with the upper bound appearing in (1.1). The proof is standard but we include it for completeness.

Proposition 6.1. *Let $\Psi : (0, \infty) \to (0, \infty)$ and $\omega : (0, \infty) \to (0, 1)$ be such that $q \to q^2\Psi(q)\omega(q)$ is non-increasing. Then*

$$\inf\left\{s \geq 0 : \sum_{q=1}^{\infty} q(\Psi(q)\omega(q))^s < \infty\right\} = \frac{2}{\lambda(\Psi, \omega)}.$$

Proof. Let $\lambda < \lambda(\Psi, \omega)$ be arbitrary. Then for all q sufficiently large we have $\Psi(q)\omega(q) < q^{-\lambda}$. Taking $s > 2/\lambda$, it now follows that

$$\sum_{q=1}^{\infty} q(\Psi(q)\omega(q))^s \ll \sum_{q=1}^{\infty} q^{1-s\lambda} < \infty.$$

Since s and λ were arbitrary we have

$$\inf\left\{s \geq 0 : \sum_{q=1}^{\infty} q(\Psi(q)\omega(q))^s < \infty\right\} \leq \frac{2}{\lambda(\Psi, \omega)}. \tag{6.1}$$

Let us now establish the reverse inequality. Let us assume that $\lambda(\Psi, \omega) < \infty$. Otherwise our desired equality follows from (6.1), as in this case both sides equal zero. Let

$\lambda > \lambda(\Psi,\omega)$ be arbitrary. Then for infinitely many $q \in \mathbb{R}$ we have $\Psi(q)\omega(q) > q^{-\lambda}$. Let $s < 2/\lambda$ be arbitrary. We observe the following:

$$\sum_{q=1}^{\infty} q(\Psi(q)\omega(q))^s = \sum_{n=1}^{\infty}\sum_{2^n<q\leq 2^{n+1}} q(\Psi(q)\omega(q))^s = \sum_{n=1}^{\infty}\sum_{2^n<q\leq 2^{n+1}} q^{1-2s}(q^2\Psi(q)\omega(q))^s$$
$$\gg \sum_{n=1}^{\infty} 2^{n(1-2s)} \sum_{2^n<q\leq 2^{n+1}} (q^2\Psi(q)\omega(q))^s. \tag{6.2}$$

Suppose that $q' \in \mathbb{R}$ is such that $\Psi(q')\omega(q') > (q')^{-\lambda}$ and n is such that $2^{n+1} < q' \leq 2^{n+2}$, then since $q \to q^2\Psi(q)\omega(q)$ is decreasing, for such an n we have

$$\sum_{2^n<q\leq 2^{n+1}} (q^2\Psi(q)\omega(q))^s \geq \sum_{2^n<q\leq 2^{n+1}} ((q')^2\Psi(q')\omega(q'))^s \gg 2^{n(1+2s-s\lambda)}. \tag{6.3}$$

$$2^{n(1-2s)} \sum_{2^n<q\leq 2^{n+1}} (q^2\Psi(q)\omega(q))^s \geq 2^{n(1-2s)}\cdot 2^{n(1+2s-s\lambda)} = 2^{n(2-\lambda s)} \geq 1. \tag{6.4}$$

In the final inequality we used that $2-\lambda s > 0$. This inequality is a consequence of our choice of s. Combining (6.2) with the fact (6.4) holds for infinitely many n, we see that $\sum_{q=1}^{\infty} q(\Psi(q)\omega(q))^s = \infty$. Since s and λ were arbitrary, it follows that

$$\frac{2}{\lambda(\Psi,\omega)} \leq \inf\left\{ s\geq 0 : \sum_{q=1}^{\infty} q(\Psi(q)\omega(q))^s < \infty \right\}. \tag{6.5}$$

Combining (6.1) and (6.5) our result now follows. □

Observe that the conclusion of Proposition 6.1 can be made under the assumptions of Theorems 1.10 and 1.11. Hence to prove these statements it remains to show that

$$\dim_H(E(\Psi,\omega)) \geq \frac{2}{\lambda(\Psi,\omega)} \tag{6.6}$$

within each of the relevant settings. Our proof is an adaptation of an argument of Bugeaud [8]. We will prove (6.6) by constructing a sufficiently large Cantor set

$$\mathcal{C} = \bigcap_{k=1}^{\infty} E_k \subseteq E(\Psi,\omega)$$

and then obtain (6.6) by applying Proposition 2.12 to $\mathcal{C}$.

The proofs of Theorems 1.10 and 1.11 follow the same strategy. For the purpose of brevity, we only prove Theorem 1.11 in full. The proof of Theorem 1.11 contains additional technicalities that are unnecessary in the proof of Theorem 1.10. For the rest of this section we fix Ψ and ω satisfying the assumptions of Theorem 1.11. We will highlight where differences arise in our proof when compared to the proof of Theorem 1.10. This will often simply involve using Lemma 4.7 in place of Lemma 4.1.

6.1. **Construction of a Cantor subset by induction.** In our construction of $\mathcal{C}$ we adopt the following notational convention. Let $p/q \in \mathbb{Q} \cap (0,1)$ be such that q satisfies the relevant partial quotient bound and is large enough so that Lemma 4.7 can be applied. We also assume that q is sufficiently large so that Lemma 4.3 can be applied. Given words $\{\mathbf{b}_l\}_{l=1}^{L}$ satisfying the conclusion of Lemma 4.7 for p/q, we let

$$U\left(\frac{p}{q}\right) := \bigcup_{l=1}^{L} I(0; \mathbf{a}\mathbf{b}_l)$$

where $\mathbf{a} = (a_1, \dots, a_n)$ is the unique word of even length such that $p/q = [0; a_1, \dots, a_n]$[2]. Such a word exists by Lemma 2.1. We will build $\mathcal{C}$ inductively by starting with the base case and then repeatedly applying Lemma 6.2 below.

6.1.1. *Base level construction.* We start by defining the set E_1. Consider the fundamental interval $I_{N_0}(0; 2^{N_0})$ corresponding to the digit 2 repeated N_0 times. We can assume that $N_0 \in \mathbb{N}$ has been chosen sufficiently large so that for all $q \geq q_{N_0}(2^{N_0})$:

$$\Psi(q) \leq \frac{1}{1000q^2}, \tag{6.7}$$

and

$$\prod_{j=1}^{\infty} \left(1 - 21 \cdot 2^{j-1} q^2 \Psi(2^{(j-1)/2} q)\right) \geq \frac{1}{2}. \tag{6.8}$$

The former assumption can be shown to be possible by the condition $\lim_{q\to\infty} q^2\Psi(q) = 0$, while the latter assumption follows from the condition $\sum_{q=1}^{\infty} q\Psi(q) < \infty$, Lemma 2.11, and well-known properties of infinite series and infinite products.

Define the subset of the fundamental interval

$$\begin{aligned} J_0 := &\left\{ x \in I_{N_0}(0; \mathbf{2}^{N_0}) : a_{N_0+j}(x) \leq \frac{1}{7 \cdot 2^{j-1} q_{N_0}(2^{N_0})^2 \Psi(2^{(j-1)/2} q_{N_0}(2^{N_0}))}, \quad j \geq 1 \right\} \\ &\cup \bigcup_{m=1}^{\infty} \left\{ [0; 2^{N_0}, a_1, \dots, a_m] : a_j \leq \frac{1}{7 \cdot 2^{j-1} q_{N_0}(2^{N_0})^2 \Psi(2^{(j-1)/2} q_{N_0}(2^{N_0}))}, \quad 1 \leq j \leq m \right\}. \end{aligned}$$

Observe that all irrational points in J_0 satisfy Proposition 2.5 (i). This is due to the non-increasing property of $q \mapsto q^2\Psi(q)$ and noting that

$$2^{(j-1)/2} q_{N_0}(2^{N_0}) \leq q_{N_0+j}(2^{N_0}\mathbf{a}) \tag{6.9}$$

[2]In our proof of Theorem 1.10 the parameter q is large enough so that Lemma 4.1 and Lemma 4.3 can be applied. We then use the words $\{\mathbf{b}_l\}_{l=1}^{L}$ satisfying the conclusion of Lemma 4.1.

for any word $\mathbf{a} \in \mathbb{N}^j$. This is a consequence of (2.1). Therefore by Lemma 2.8, (6.8) and (6.9) we have

$$\begin{aligned}\mathcal{L}(J_0) &\geq \prod_{j=1}^{\infty}\left(1 - \frac{3}{\frac{1}{7\cdot 2^{j-1} q_{N_0}(2^{N_0})^2 \Psi(2^{(j-1)/2} q_{N_0}(2^{N_0}))}}\right)\mathcal{L}\left(I_{N_0}(0;2^{N_0})\right)\\ &\geq \prod_{j=1}^{\infty}\left(1 - 21\cdot 2^{j-1} q_{N_0}(2^{N_0})^2 \Psi\left(2^{(j-1)/2} q_{N_0}(2^{N_0})\right)\right)\mathcal{L}\left(I_{N_0}(0;2^{N_0})\right)\\ &= \frac{\mathcal{L}\left(I_{N_0}(0;2^{N_0})\right)}{2}. \end{aligned} \tag{6.10}$$

Hence J_0 retains a positive proportion of the measure of $I_{N_0}(0;2^{N_0})$.

Pick a large integer $Q_1 \in \mathbb{N}$ such that

$$Q_1 > \max\left\{-\frac{1000}{\mathcal{L}\left(I_{N_0}(0;2^{N_0})\right)}\log\left(\mathcal{L}\left(I_{N_0}(0;2^{N_0})\right)\right)\right\}. \tag{6.11}$$

Note that this lower bound for Q_1 depends only upon N_0. We will impose other lower bounds on the size of Q_1 later. By Dirichlet's Theorem in Diophantine approximation, for any $x \in I_{N_0}(0;2^{N_0})$ there exists $(p,q) \in \mathbb{Z}\times\mathbb{N}$ satisfying

$$\left|x - \frac{p}{q}\right| < \frac{1}{qQ_1}, \quad 1 \leq q \leq Q_1.$$

Let

$$A = \left\{x \in I_{N_0}(0;2^{N_0}) : \left|x - \frac{p}{q}\right| < \frac{1}{qQ_1}, \quad 1 \leq q \leq \frac{Q_1}{10}\right\}.$$

By (6.11) and a standard measure calculation we have that

$$\begin{aligned}\mathcal{L}(A) &\leq \sum_{1\leq q\leq \frac{Q_1}{10}} \frac{2}{qQ_1}\left(q\mathcal{L}(I_{N_0}(0;2^{N_0})) + 1\right) \leq \frac{2}{10}\mathcal{L}\left(I_{N_0}(0;2^{N_0})\right) + \frac{2}{Q_1}\sum_{1\leq q\leq \frac{Q_1}{10}} q^{-1}\\ &\leq \frac{1}{4}\mathcal{L}\left(I_{N_0}(0;2^{N_0})\right). \end{aligned} \tag{6.12}$$

Let $\tilde{T} = \{\frac{r_j}{s_j}\}_{1\leq j\leq \tilde{t}}$ be a maximal $(Q_1)^{-2}$-separated family of rational points $\frac{r}{s} \in \mathbb{Q}\cap J_0$ with $\frac{Q_1}{10} \leq s \leq Q_1$. We claim that

$$J_0 \setminus A \subseteq \bigcup_{j=1}^{\tilde{t}}\left[\frac{r_j}{s_j} - \frac{11}{Q_1^2}, \frac{r_j}{s_j} + \frac{11}{Q_1^2}\right]. \tag{6.13}$$

This follows because if $x \in J_0 \setminus A$, then by Dirichlet's Theorem there exists $(u,v) \in \mathbb{Z}\times\mathbb{N}$ with $\frac{Q_1}{10} \leq v \leq Q_1$ such that

$$\left|x - \frac{u}{v}\right| < \frac{1}{vQ_1} \leq \frac{1}{v^2}.$$

By Lemma 2.4 this $(u,v) \in \mathbb{Z}\times\mathbb{N}$ must satisfy

$$\frac{u}{v} \in \{[0;a_1(x),\ldots,a_l(x)], [0;a_1(x),\ldots,a_l(x),1], [0;a_1(x),\ldots,a_l(x),a_{l+1}(x)-1]\} \tag{6.14}$$

for some $l \in \mathbb{N}$. Note that l can be made arbitrarily large by choosing Q_1 suitably large in a way that does not depend upon x. Here we are using the fact that the size of the partial quotients is bounded at each level for elements of J_0. Hence we can choose Q_1 large enough so that $l > N_0$ thus guaranteeing $\frac{u}{v} \in J_0$. Consequently

$$J_0 \setminus A \subseteq \bigcup_{\frac{u}{v} \in J_0 : \frac{Q_1}{10} \leq v \leq Q_1} \left[\frac{u}{v} - \frac{1}{vQ_1}, \frac{u}{v} + \frac{1}{vQ_1} \right] . \tag{6.15}$$

Equation (6.13) now follows from (6.15) and an application of the triangle inequality. Namely, if $\frac{u}{v} \notin \tilde{T}$ then by the maximality of $\tilde{T}$ there exists some $\frac{r}{s} \in \tilde{T}$ such that $|r/s - u/v| \leq Q_1^{-2}$. Then, by the triangle inequality

$$\left| x - \frac{r}{s} \right| \leq \left| x - \frac{u}{v} \right| + \left| \frac{r}{s} - \frac{u}{v} \right| \leq \frac{1}{vQ_1} + \frac{1}{Q_1^2} \leq \frac{10}{Q_1^2} + \frac{1}{Q_1^2} .$$

giving us (6.13). We now use our measure bounds for J_0 and A to determine a lower bound on the cardinality of $\tilde{T}$. By (6.10), (6.12), and (6.13) we have that

$$\tilde{t} \cdot \frac{22}{Q_1^2} \geq \mathcal{L} \left(\bigcup_{j=1}^{\tilde{t}} \left[\frac{r_j}{s_j} - \frac{11}{Q_1^2}, \frac{r_j}{s_j} + \frac{11}{Q_1^2} \right] \right) \geq \mathcal{L}(J_0 \setminus A) \geq \frac{1}{4} \mathcal{L}\left(I_{N_0}(0; 2^{N_0}) \right)$$

Thus we can rearrange to obtain

$$\tilde{t} \geq \frac{Q_1^2}{88} \mathcal{L}\left(I_{N_0}(0; 2^{N_0}) \right) .$$

Lastly note that since each $\frac{r}{s} \in \tilde{T}$ belongs to J_0 we can write $\frac{r}{s} = [0; a_1, \dots, a_n]$ for some $n = N_0 + j$ for some $j \in \mathbb{N}$ where the partial quotients $a_{N_0+1}, \dots, a_n$ satisfy the upper bounds imposed by the definition of J_0. We would like to apply Lemma 4.7 to $\frac{r}{s}$, however it is possible that n is odd. If n is odd we write $\frac{r}{s} = [0; a_1, \dots, a_{n-1}, a_n - 1, 1]$ if $a_n \geq 2$ and $\frac{r}{s} = [0; a_1, \dots, a_{n-1} + 1]$ if $a_n = 1$. Since $a_i = 2$ for $1 \leq i \leq N_0$, $q \to q^2\Psi(q)$ is non-increasing, and (6.9) we have

$$a_{N_0+i} + 1 \leq \frac{1}{7 \cdot 2^{j-1} q_{N_0}(2^{N_0})^2 \Psi(2^{(j-1)/2} q_{N_0}(2^{N_0}))} + 1 < \frac{1}{6s^2\Psi(s)} , \qquad 1 \leq i \leq j .$$

Thus, regardless of the parity of n we can find a continued fraction expansion of $\frac{r}{s}$ of even length so that the bound

$$\max_{1 \leq i \leq n} a_i < \frac{1}{6s^2\Psi(s)}$$

is satisfied. Therefore Lemma 4.7 can be applied to each $\frac{r}{s} \in \tilde{T}$. We define

$$E_1 := \bigcup_{j=1}^{\tilde{t}} U\left(\frac{r_j}{s_j} \right) .$$

In this final line we have assumed that Q_1 is sufficiently large so that Lemma 4.7 can be applied to each element in $\tilde{T}$.

6.1.2. *Inductive step.* The following lemma allows us to apply an inductive argument.

Lemma 6.2. *Let $Q \geq Q_1$. Then for all $Q' \geq Q$ sufficiently large in a way that only depends upon Q, Ψ and ω we have the following: Suppose $\frac{p}{q}$ is such that*

1) $\frac{Q}{10} \leq q \leq Q$.
2) For any $x \in U(\frac{p}{q})$ there are no solutions to

$$\left|x - \frac{r}{s}\right| < (1 - \omega(s))\Psi(s) \quad Q_1 \leq s \leq Q\,.$$

3) Writing $\frac{p}{q} = [0; c_1, \ldots, c_n]$ for n even, we have

$$c_i \leq \frac{1}{6q^2\Psi(q)}\,, \quad 1 \leq i \leq n\,.$$

Then we can pick a collection of rational points $T = \{\frac{r_j}{s_j}\}_{1 \leq j \leq t}$ with

$$t = \left\lfloor \frac{(Q'^2)\Psi(Q)\omega(Q)}{10^{10}} \right\rfloor$$

such that

a) $\frac{r_j}{s_j} \in U(\frac{p}{q})$ *and* $\frac{Q'}{10} \leq s_j \leq Q'$ *for each* $1 \leq j \leq t$,
b) $|\frac{r_j}{s_j} - \frac{r_i}{s_i}| \geq (Q')^{-2}$ *for every* $1 \leq j < i \leq t$,
c) Writing $\frac{r_j}{s_j} = [0; a_1, \ldots, a_l]$ for some l even we have that

$$a_i \leq \frac{1}{6s_j^2\Psi(s_j)}\,, \quad 1 \leq i \leq l\,.$$

d) $U(\frac{r_j}{s_j}) \subset U(\frac{p}{q})$ *for every* $1 \leq j \leq t$,
e) For every $x \in U(\frac{r_j}{s_j})$
i)

$$\left|x - \frac{r}{s}\right| < \Psi(s), \quad Q < s \leq Q'$$

has a solution $(r, s) \in \mathbb{Z} \times \mathbb{N}$.
ii)

$$\left|x - \frac{r}{s}\right| < (1 - \omega(s))\Psi(s), \quad Q_1 \leq s \leq Q'$$

has no solutions $(r, s) \in \mathbb{Z} \times \mathbb{N}$.

Remark 6.3. Note the sets $U(\frac{r_j}{s_j})$ appearing in $d), e)$ are well-defined since Lemma 4.7 is applicable to $\frac{r_j}{s_j} \in T$ by $c)$.

Remark 6.4. Lemma 6.2 can be adapted to prove Theorem 1.10. Under the assumptions of Theorem 1.10 its statement can be simplified. In particular, to apply Lemma 4.1 (rather than Lemma 4.7) we do not need assumption 3). Moreover, we do not require the resulting set of rationals T to satisfy c). Furthermore, the constant appearing in the denominator in the formula for t can be improved, but this has no effect on the later dimension calculations.

Proof. Let us begin by fixing $Q \geq Q_1$. We will now show that for all Q' sufficiently large in a way that depends upon Q, Ψ and ω the desired conclusion holds. Let $\frac{p}{q}$ be as in the statement of our lemma. Write $\frac{p}{q} = [0; \mathbf{c}]$ where $\mathbf{c} = (c_1, \ldots, c_n)$ with n even (this is possible by Lemma 2.1). By assumption 3) we can apply Lemma 4.7 to p/q[3] and let $\{\mathbf{b}_l\}_{l=1}^{L}$ denote the corresponding set of words. We let m denote their common length. For each l we let $\frac{p(\mathbf{cb}_l)}{q(\mathbf{cb}_l)} = [0; \mathbf{cb}_l]$ and

$$J(\mathbf{b}_l) := \left\{ x \in I(0; \mathbf{cb}_l) : a_{n+m+j}(x) \leq \frac{1}{7 \cdot 2^{j-1} q(\mathbf{cb}_l)^2 \Psi\left(2^{(j-1)/2} q(\mathbf{cb}_l)\right)}, \quad j \geq 1 \right\}$$
$$\cup \bigcup_{m=1}^{\infty} \left\{ [0; \mathbf{cb}_l, a_1, \ldots, a_m] : a_j \leq \frac{1}{7 \cdot 2^{j-1} q_{N_0}(\mathbf{cb}_l)^2 \Psi(2^{(j-1)/2} q_{N_0}(\mathbf{cb}_l))}, \quad 1 \leq j \leq m \right\}.$$

and $J = \bigcup_{l=1}^{L} J(\mathbf{b}_l)$.

By Lemma 2.8, (6.8), and the definition of $U\left(\frac{p}{q}\right)$ we have

$$\begin{aligned} \mathcal{L}(J) = \sum_{l=1}^{L} \mathcal{L}(J(\mathbf{b}_l)) &\geq \sum_{l=1}^{L} \prod_{j=1}^{\infty} \left(1 - \frac{3}{\frac{1}{7 \cdot 2^{j-1} q(\mathbf{cb}_l)^2 \Psi(2^{(j-1)/2} q(\mathbf{cb}_l))}}\right) \mathcal{L}\left(I(0; \mathbf{cb}_l)\right) \\ &\geq \frac{1}{2} \sum_{l=1}^{L} \mathcal{L}\left(I(0; \mathbf{cb}_l)\right) \\ &= \frac{1}{2} \mathcal{L}\left(U\left(\frac{p}{q}\right)\right). \end{aligned} \tag{6.16}$$

Let us assume that Q' has been chosen sufficiently large so that we have:

$$Q' \geq \max_{p'/q' \in \mathbb{Q} \cap (0,1) : Q/10 \leq q' \leq Q} 1000 \left| U\left(\frac{p'}{q'}\right) \right|^{-1} \log\left(\left| U\left(\frac{p'}{q'}\right) \right|^{-1} \right), \tag{6.17}$$

$$\max_{\substack{p'/q' \in \mathbb{Q} \cap (0,1) \\ Q/10 \leq q' \leq Q}} \max_{\mathbf{b}_l \text{ provided by Lemma 4.7}} \frac{1}{2} q(\mathbf{cb}_l)^2 \Psi\left(q(\mathbf{cb}_l)\right) > \left(\frac{Q'}{10}\right)^2 \Psi\left(\frac{Q'}{10}\right), \tag{6.18}$$

$$Q' > \max_{\substack{p'/q' \in \mathbb{Q} \cap (0,1) \\ Q/10 \leq q' \leq Q}} \max_{\mathbf{b}_l \text{ provided by Lemma 4.7}} 10 q(\mathbf{cb}_l), \tag{6.19}$$

$$Q^2 \Psi(Q) > 12 \left(\frac{Q'}{10}\right)^2 \Psi\left(\frac{Q'}{10}\right) \tag{6.20}$$

Note that (6.17), (6.19) and (6.20) are possible since one side is fixed (depending on Q, Ψ and ω), and that (6.18) is possible since $\lim_{q \to \infty} q^2 \Psi(q) = 0$. By applying Dirichlet's Theorem for Q' and an arbitrary choice of $x \in U(\frac{p}{q})$, we see there exists a solution $(u, v) \in \mathbb{Z} \times \mathbb{N}$ satisfying

$$\left| x - \frac{u}{v} \right| < \frac{1}{vQ'}, \quad 1 \leq v \leq Q'.$$

[3] In the proof of Theorem 1.10 we would apply Lemma 4.1 at this point.

Letting

$$A=\left\{x\in U\left(\frac{p}{q}\right):\left|x-\frac{u}{v}\right|<\frac{1}{vQ'}\,,\ 1\leq v\leq \frac{Q'}{10}\right\}$$

we see as before that

$$\mathcal{L}(A)\leq \sum_{1\leq v\leq \frac{Q'}{10}}\frac{2}{vQ'}\left(v\mathcal{L}\left(U\left(\frac{p}{q}\right)\right)+1\right)\leq \mathcal{L}\left(U\left(\frac{p}{q}\right)\right)\frac{2}{10}+\frac{2}{Q'}\sum_{1\leq v\leq \frac{Q'}{10}}\frac{1}{v}$$

$$\leq \frac{1}{4}\mathcal{L}\left(U\left(\frac{p}{q}\right)\right), \tag{6.21}$$

where in the final line we have used (6.17). Let $\tilde{T}=\{\frac{r_j}{s_j}\}_{1\leq j\leq \tilde{t}}$ be a maximal $(Q')^{-2}$-separated family of rational points $\frac{r}{s}\in\mathbb{Q}\cap J$ with $\frac{Q'}{10}\leq s\leq Q'$. We claim that

$$J\setminus A\subseteq \bigcup_{j=1}^{\tilde{t}}\left[\frac{r_j}{s_j}-\frac{11}{(Q')^2},\frac{r_j}{s_j}+\frac{11}{(Q')^2}\right]. \tag{6.22}$$

This follows because if $x\in J\setminus A$, then by Dirichlet's Theorem there exists $(u,v)\in\mathbb{Z}\times\mathbb{N}$ with $\frac{Q'}{10}\leq v\leq Q'$ such that

$$\left|x-\frac{u}{v}\right|<\frac{1}{vQ'}\leq\frac{1}{v^2}\,.$$

By Lemma 2.4 this $(u,v)\in\mathbb{Z}\times\mathbb{N}$ must satisfy

$$\frac{u}{v}\in\{[0;a_1(x),\ldots,a_l(x)],[0;a_1(x),\ldots,a_l(x),1],[0;a_1(x),\ldots,a_l(x),a_{l+1}(x)-1]\} \tag{6.23}$$

for some $l\in\mathbb{N}$. Note that (6.19) implies that $v\geq\frac{Q'}{10}>q(\mathbf{c}\mathbf{b}_l)$ for any $\mathbf{c}$ and $\mathbf{b}_l$. Therefore l must be strictly greater than the length of $\mathbf{c}\mathbf{b}_l$ for any $\mathbf{c}$ and $\mathbf{b}_l$. Using this fact we see that (6.23) implies that $\frac{u}{v}\in J$. Consequently

$$J\setminus A\subseteq \bigcup_{\frac{u}{v}\in J:Q'/10\leq v\leq Q'}\left[\frac{u}{v}-\frac{1}{vQ'},\frac{u}{v}+\frac{1}{vQ'}\right] \tag{6.24}$$

(6.22) now follows from (6.24) and an application of the triangle inequality.

Hence, by (6.16), (6.21) and (6.22), we have that

$$\mathcal{L}\left(\bigcup_{j=1}^{\tilde{t}}\left[\frac{r_j}{s_j}-\frac{11}{(Q')^2},\frac{r_j}{s_j}+\frac{11}{(Q')^2}\right]\right)\geq\frac{1}{4}\mathcal{L}\left(U\left(\frac{p}{q}\right)\right),$$

and so we have the lower bound

$$\tilde{t}\geq\frac{(Q')^2\mathcal{L}\left(U\left(\frac{p}{q}\right)\right)}{88}. \tag{6.25}$$

By Lemma 4.7 and using the fact $q \to \Psi(q)\omega(q)$ is non-increasing, the above implies the following lower bound[4]

$$\tilde{t} \geq \frac{(Q')^2\Psi(Q)\omega(Q)}{88 \times 10^8}.$$

We now remove the rightmost rational from $\tilde{T}$. Thus we are left with at least

$$\frac{(Q')^2\Psi(Q)\omega(Q)}{88 \times 10^8} - 1$$

many rationals. We may assume that Q' has been chosen sufficiently large that from those rationals that remain, there exists a subset of cardinality

$$\left\lfloor \frac{(Q')^2\Psi(Q)\omega(Q)}{10^{10}} \right\rfloor.$$

We let this set be our set T. This collection satisfies the required properties:

- Conditions $a)$ and $b)$ are satisfied immediately because T is a subset of $\tilde{T}$.
- To see that condition $c)$ holds let $\frac{r_j}{s_j} \in T$ and $(a_1, \dots, a_l)$ be such that $\frac{r_j}{s_j} = [0; a_1, \dots, a_l]$ and $(a_1, \dots, a_l)$ satisfies the restrictions imposed by the set J, i.e. $(a_1, \dots, a_l)$ begins with $\mathbf{cb}_l$ for some $\mathbf{b}_l$ and satisfies the partial quotient upper bound. The integer l is not necessarily even. We will explain how to overcome this technicality at the end of this discussion. We see that $c)$ holds by the following case analysis:
  - For $1 \leq i \leq n$ condition $c)$ holds by our assumption 3) combined with the non-increasing property of $q \mapsto q^2\Psi(q)$.
  - For $i = n+1$ Lemma 4.3 implies that $a_{n+1} \leq \frac{2}{q^2\Psi(q)}$. Using this inequality and the non-increasing property of $q \mapsto q^2\Psi(q)$ and (6.20) we have that
  $$a_{n+1} \leq \frac{2}{q^2\Psi(q)} \leq \frac{2}{Q^2\Psi(Q)} < \frac{1}{6\left(\frac{Q'}{10}\right)^2\Psi\left(\frac{Q'}{10}\right)} \leq \frac{1}{6s^2\Psi(s)}.$$
  - Let $\mathbf{b}_l$ be such that $\frac{r_j}{s_j} \in J(\mathbf{b}_l)$. Then for $n+2 \leq i \leq n+m$ condition $c)$ holds by the non-increasing property of $q \mapsto q^2\Psi(q)$, Lemma 4.7 and (6.18). In particular, for each $1 \leq i \leq m$ we have $a_{n+i} = b_{i-1}$ and so
  $$a_{n+i} \leq \left\lfloor \frac{1}{3q_{n+i-1}^2\Psi(q_{n+i-1})} \right\rfloor \leq \frac{1}{3q(\mathbf{cb}_l)^2\Psi(q(\mathbf{cb}_l))} \leq \frac{1}{6\left(\frac{Q'}{10}\right)^2\Psi\left(\frac{Q'}{10}\right)} \leq \frac{1}{6s^2\Psi(s)}.$$
  - For $\mathbf{b}_l$ as above and $n+m+1 \leq i \leq l$, we have by construction of the set J and the non-increasing condition on $q \mapsto q^2\Psi(q)$, that the following holds assuming we chose our original Q_1 to be sufficiently large
  $$a_i + 1 \leq \frac{1}{7 \cdot 2^{i-n-m-1}q(\mathbf{cb}_l)^2\Psi(2^{(i-n-m-1)/2}q(\mathbf{cb}_l))} + 1 < \frac{1}{6s^2\Psi(s)}.$$

[4]It is here where a change in constant appears in the proof of Theorem 1.10, since Lemmas 4.1 and 4.7 give different bounds on the size of $U(p/q)$. Namely, the constant in the denominator would be 563200 rather than 88×10^8. This has a knock-on effect in later constants.

Here we have used the fact that $s \geq 2^{(i-n-m-1)/2} q(\mathbf{cb}_l)$, which is a consequence of (2.1). It follows that

$$\max_{1 \leq i \leq l} a_i < \frac{1}{6s^2 \Psi(s)}.$$

So $c)$ is satisfied by $(a_1, \dots, a_l)$ which is not necessarily of even length. Because of the additional $+1$ term appearing in the above equation, if we use the substitutions $[0; a_1 \dots, a_l] = [0; a_1, \dots, a_l - 1, 1]$ if $a_l \geq 2$ and $[0; a_1 \dots, a_l] = [0; a_1, \dots, a_{l-1} + 1]$ if $a_l = 1$ to ensure our word is of even length, then $c)$ is satisfied by the resulting even length word. This means that Lemma 4.7 is applicable to each $\frac{r_j}{s_j} \in T$ and so the sets $U(\frac{r_j}{s_j})$ are well-defined.

- Let $\frac{r_j}{s_j} \in T$. Since $s_j \geq Q_1$ and $s_j \geq \frac{Q'}{10}$, it follows from (6.7) that $\Psi(s_j) \leq (Q')^{-2}$. Recall that we removed the rightmost rational from $\tilde{T}$ to define T. Since $\tilde{T}$ is a $(Q')^{-2}$ separated subset of $U\left(\frac{p}{q}\right)$ and $\Psi(s_j) \leq (Q')^{-2}$, it follows therefore that

$$U\left(\frac{r_j}{s_j}\right) \subseteq \left[\frac{r_j}{s_j}, \frac{r_j}{s_j} + \Psi(s_j)\right] \subseteq \left[\frac{r_j}{s_j}, \frac{r_j}{s_j} + (Q')^{-2}\right] \subseteq U\left(\frac{p}{q}\right).$$

So condition $d)$ holds.
- Let $\frac{r_j}{s_j} \in T$ and $x \in U\left(\frac{r_j}{s_j}\right)$. It follows from Lemma 4.7 that $|x - \frac{r_j}{s_j}| \leq \Psi(s_j)$[5]. We can choose Q' large enough so that $\frac{Q'}{10} > Q$. Since $\frac{Q'}{10} \leq s_j \leq Q'$ we have $Q < s_j \leq Q'$ and so condition $e)i)$ holds.
- Let $\frac{r_j}{s_j} \in T$ and $x \in U\left(\frac{r_j}{s_j}\right)$. By (6.7) and Lemma 2.3, if $(u, v) \in \mathbb{Z} \times \mathbb{N}$ with $v \geq Q_1$ satisfies

$$\left|x - \frac{u}{v}\right| < (1 - \omega(v))\Psi(v)$$

then $\frac{u}{v}$ is a convergent of x. Thus, to verify condition $e)ii)$, it suffices to consider convergents $\frac{p_k}{q_k}$ to x satisfying $Q_1 \leq q_k \leq Q'$. Therefore, to show that condition $e)ii)$ holds, it suffices to show that there exists no $k \in \mathbb{N}$ such that

$$\left|x - \frac{p_k}{q_k}\right| < (1 - \omega(q_k))\Psi(q_k) \quad \text{and} \quad Q_1 \leq q_k \leq Q'. \tag{6.26}$$

Let $(a_k)_{k=1}^{\infty}$ be sequence of partial quotients for x. Since $x \in U\left(\frac{p}{q}\right)$ we have

$$\frac{p}{q} = [0; a_1, \dots, a_n].$$

Moreover, since $x \in U\left(\frac{r_j}{s_j}\right)$ there exists $n_1 > n$ such that

$$\frac{r_j}{s_j} = [0; a_1, \dots, a_{n_1}].$$

Since $(q_k)_{k=1}^{\infty}$ is an increasing sequence, $q = q_n \leq Q$, and $x \in U\left(\frac{p}{q}\right)$, it follows from statement 2) in our assumptions that (6.26) has no solutions for $k \leq n$. If

[5] In the proof of Theorem 1.10 this would follow from Lemma 4.1.

$n < k \le n+m-1$, then since $[0; a_1, \dots, a_{n_1}]$ is the continued fraction expansion of $\frac{r_j}{s_j}$ and $\frac{r_j}{s_j} \in J$, it follows that a_{k+1} is a digit coming from a word belonging to $\{\mathbf{b}_l\}_{l=1}^{L}$, in which case

$$a_{k+1} \le \left\lfloor \frac{1}{3q_k^2 \Psi(q_k)} \right\rfloor \tag{6.27}$$

by Lemma 4.7[6]. Combining Lemma 2.2 and (6.27) we have

$$\left| x - \frac{p_k}{q_k} \right| \ge \frac{1}{3a_{k+1}q_k^2} \ge \Psi(q_k).$$

So there are no solutions to (6.26) for $n < k \le n+m-1$. If $n+m \le k < n_1$ then since $[0; a_1, \dots, a_{n_1}]$ is the continued fraction expansion of $\frac{r_j}{s_j}$ and $\frac{r_j}{s_j} \in J$, there exists $\mathbf{b}_l$ such that we have

$$a_{k+1} \le \frac{1}{7 \cdot 2^{k-m-n} q(\mathbf{c}\mathbf{b}_l)^2 \Psi(2^{(k-m-n)/2} q(\mathbf{c}\mathbf{b}_l))}.$$

Appealing to (2.1) we can show that $q_k \ge 2^{(k-m-n)/2} q(\mathbf{c}\mathbf{b}_l)$. Using our assumption that the function $q \mapsto q^2\Psi(q)$ is non-increasing, the following holds

$$q_k^2(x)\Psi(q_k(x)) \le 2^{k-m-n} q(\mathbf{c}\mathbf{b}_l)^2 \Psi(2^{(k-m-n)/2} q(\mathbf{c}\mathbf{b}_l)). \tag{6.28}$$

Hence, by Lemma 2.2 and (6.28) we have

$$\begin{aligned}\left| x - \frac{p_k(x)}{q_k(x)} \right| > \frac{1}{3a_{k+1}(x)q_k(x)^2} &\ge \frac{1}{3\frac{1}{7 \cdot 2^{k-m-n} q(\mathbf{c}\mathbf{b}_l)^2 \Psi(2^{(k-m-n)/2} q(\mathbf{c}\mathbf{b}_l))} q_k(x)^2} \\ &\ge \Psi(q_k(x)),\end{aligned}$$

So (6.26) has no solutions for $n+m \le k < n_1$. Taking $k = n_1$, then since $x \in U\left(\frac{r_j}{s_j}\right)$ it follows from the definition of $U\left(\frac{r_j}{s_j}\right)$ that

$$\left| x - \frac{p_{n_1}}{q_{n_1}} \right| = \left| x - \frac{r_j}{s_j} \right| \ge (1 - \omega(s_j))\Psi(s_j).$$

So (6.26) has no solution when $k = n_1$. Finally, if $k > n_1$ then by applying Lemma 4.3 to $\frac{r_j}{s_j}$[7] written with an even number of partial quotients we have that

$$a_{n_1+1} \ge \frac{1}{2q_{n_1}^2 \Psi(q_{n_1})}.$$

So, combined with (2.1), (6.7) and the bound $\frac{Q'}{10} \le s_j$ we have

$$q_k \ge a_{n_1+1} q_{n_1} \ge \frac{q_{n_1}}{2q_{n_1}^2 \Psi(q_{n_1})} = \frac{s_j}{2s_j^2 \Psi(s_j)} \ge 500 s_j > Q'.$$

Thus any convergent $\frac{p_k}{q_k}$ of $x \in U(\frac{r_j}{s_j})$ with $k > n_1$ cannot contribute a solution to (6.26). Summarising, we have shown that (6.26) has no solutions for all $k \in \mathbb{N}$. This proves condition $e)ii)$ holds.

[6] Or by Lemma 4.1 in the proof of Theorem 1.10

[7] which is satisfied under the assumptions of both Theorem 1.10 and Theorem 1.11

□

Let E_1 and Q_1 be as above. Now let $(Q_k)_{k=1}^{\infty}$ be a rapidly increasing sequence of integers satisfying the following properties:

- For each $k \in \mathbb{N}$, Q_{k+1} is sufficiently large that the conclusion of Lemma 6.2 holds when we have taken $Q = Q_k$ in the statement of this lemma.
- We have

$$\lim_{k\to\infty} \frac{-\log \Psi(Q_k)\omega(Q_k)}{\log Q_k} = \lambda(\Psi,\omega). \tag{6.29}$$

- We have

$$\lim_{k\to\infty} \frac{\log \Psi(Q_k)\omega(Q_k)}{\log \Psi(Q_{k+1})\omega(Q_{k+1})} = 0. \tag{6.30}$$

Repeatedly applying Lemma 6.2 yields a sequence of sets $(E_k)_{k=1}^{\infty}$ satisfying the following properties:

- For each E_k there exists a set of rationals $S_k = \{\frac{r_{j,k}}{s_{j,k}}\}_{j\in J_k}$ such that

$$E_k = \bigcup_{j\in J_k} U\left(\frac{r_{j,k}}{s_{j,k}}\right)$$

  is a disjoint union and $\frac{Q_k}{10} \leq s_{j,k} \leq Q_k$ for all $j \in J_k$.
- For all $k \geq 1$ we have $E_{k+1} \subset E_k$.
- For all $x \in E_k$ there exists at least k distinct choices of $(r,s) \in \mathbb{Z}\times\mathbb{N}$ satisfying

$$\left|x - \frac{r}{s}\right| < \Psi(s), \quad Q_1 \leq s \leq Q_k.$$

- For all $x \in E_k$ there are no $(r,s) \in \mathbb{Z}\times\mathbb{N}$ satisfying

$$\left|x - \frac{r}{s}\right| < (1-\omega(s)\Psi(s), \quad Q_1 \leq s \leq Q_k.$$

6.1.3. *Hausdorff dimension of* $\mathcal{C}$. Taking

$$\mathcal{C} = \bigcap_{k=1}^{\infty} E_k,$$

then it is clear from the above that $\mathcal{C} \subset E(\Psi,\omega)$. It also follows from Lemma 6.2 that we can assume that our sets $\{E_k\}$ satisfy the following properties:

- For each $k \in \mathbb{N}$ and $\frac{r_{j,k}}{s_{j,k}} \in S_k$, there exists

$$\left\lfloor \frac{(Q_{k+1})^2\Psi(Q_k)\omega(Q_k)}{10^{10}} \right\rfloor$$

  choices of $j' \in J_{k+1}$ such that

$$U\left(\frac{r_{j',k+1}}{s_{j',k+1}}\right) \subset U\left(\frac{r_{j,k}}{s_{j,k}}\right).$$

- For each $k \in \mathbb{N}$ and $\frac{r_{j,k}}{s_{j,k}} \in S_k$, we have
$$\mathcal{L}\left(U\left(\frac{r_{j,k}}{s_{j,k}}\right)\right) \geq \frac{\Psi(Q_k)\omega(Q_k)}{10^8}.$$
This follows from Lemma 4.7 [8] and the fact $q \mapsto \Psi(q)\omega(q)$ is a decreasing function.
- For each $k \in \mathbb{N}$ and $\frac{r_{j,k}}{s_{j,k}}, \frac{r_{j',k}}{s_{j',k}} \in S_k$, there is a gap of size at least $\frac{1}{2Q_k^2}$ between $U\left(\frac{r_{j,k}}{s_{j,k}}\right)$ and $U\left(\frac{r_{j',k}}{s_{j',k}}\right)$. This follows from (6.7) and the inequality $\left|\frac{r_{j,k}}{s_{j,k}} - \frac{r_{j',k}}{s_{j',k}}\right| \geq \frac{1}{Q_k^2}$ which is a consequence of Lemma 6.2.

Using the properties listed above and Proposition 2.12 we have
$$\begin{aligned}\dim_H(E(\Psi,\omega)) \geq \dim_H(\mathcal{C}) &\geq \liminf_{k\to\infty} \frac{\log\lfloor \frac{Q_k^2\Psi(Q_{k-1})\omega(Q_{k-1})}{10^{10}}\rfloor}{-\log\frac{\Psi(Q_k)\omega(Q_k)}{10^8}}\\ &\geq \liminf_{k\to\infty} \frac{\log Q_k^2\Psi(Q_{k-1})\omega(Q_{k-1})}{-\log\Psi(Q_k)\omega(Q_k)}\\ &= \liminf_{k\to\infty} \frac{\log Q_k^2}{-\log\Psi(Q_k)\omega(Q_k)}\\ &= \frac{2}{\lambda(\Psi,\omega)}.\end{aligned}$$

In the penultimate line we used (6.30). In the final line we used (6.29). Summarising, we have shown that (6.6) holds, and so the proof of Theorem 1.11 is complete. Note that the changes in constants that appear in the proof of Theorem 1.10 make no difference to the final dimension calculation. Thus this theorem follows similarly.

Acknowledgments. The first author was supported by an EPSRC New Investigators Award (EP/W003880/1). The second author is supported by a Leverhulme Trust Early Career Research Fellowship ECF-2024-401.

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[8] Or from Lemma 4.1 in the proof of Theorem 1.10 with a slightly better constant

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