THE PICKY CONJECTURE FOR GROUPS OF LIE TYPE

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Abstract. Recently, Moretó and Rizo proposed a conjecture, known as the Picky Conjecture, proposing new character correspondences extending the McKay Conjecture. We prove the Picky Conjecture for all quasi-simple groups of Lie type for non-defining primes. In favourable situations, we also obtain the stronger version postulating preservation of character values up to sign, and we show this stronger version holds in general when assuming certain natural properties of Lusztig's Jordan decomposition. Along the way, we complete the determination of semisimple picky elements in these groups by classifying picky 2- and 3-elements.

1. Introduction

The investigation of global to local character correspondences for finite groups has been a very fruitful area of research in the past several decades. Here we investigate a new type of correspondence that has recently been proposed by Alex Moretó and Noelia Rizo, coined the *Picky Conjecture*, and prove it for quasi-simple groups of Lie type in non-defining characteristic.

Let G be a finite group and ℓ a prime. We say that an ℓ-element $x \in G$ is *picky* if it lies in a unique Sylow ℓ-subgroup P of G. For any $x \in G$, we let $\mathrm{Irr}^x(G)$ denote the set of complex irreducible characters of G that take a non-zero value at x.

We are concerned with the following recent conjecture of Moretó and Rizo [MR25]:

Conjecture 1 (Picky Conjecture). *Let G be a finite group, ℓ a prime, $P \in \mathrm{Syl}_\ell(G)$ and $x \in P$ a picky ℓ-element. Then there exists a bijection*

$$f : \mathrm{Irr}^x(G) \to \mathrm{Irr}^x(\mathrm{N}_G(P))$$

such that for each $\chi \in \mathrm{Irr}^x(G)$,

(1) $\chi(1)_\ell = f(\chi)(1)_\ell$, *and*
(2) $\mathbb{Q}(\chi(x)) = \mathbb{Q}(f(\chi)(x))$.

We will say the Picky Conjecture *holds for (G, ℓ, x) if (1) and (2) hold.*

As observed in [MR25], the Picky Conjecture has strong connections with the recently-proved McKay Conjecture, as well as with its Galois equivariant version proposed by

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Navarro. In particular, if G has some picky ℓ-element, then the Picky Conjecture implies the McKay Conjecture for G at the prime ℓ.

We will work with two stronger versions of the Picky Conjecture.

Definition 2 (Picky+ Conjecture)**.** *We will say that the* Picky+ Conjecture *holds for* (G, ℓ, x) *if the Picky Conjecture holds for* (G, ℓ, x) *and the following further holds:*

(3) $\chi(x)_\ell = f(\chi)(x)_\ell$.

Definition 3 (Strong Picky Conjecture)**.** *We will say that the* Strong Picky Conjecture *holds for* (G, ℓ, x) *if the Picky Conjecture holds and the following strong version of (2) holds (which implies (2) and (3)):*

(2*) $\chi(x) = \pm f(\chi)(x)$.

We remark that the Strong Picky Conjecture does not hold in general. Indeed, as pointed out by Moretó and Rizo, there are counterexamples to (2*) for the groups with TI Sylow p-subgroups in [BM90, Thm 1.3(b)–(d), (f),(g)], which will be discussed further in [MR25]. However, Moretó–Navarro–Rizo recently showed the Strong Picky Conjecture for ℓ-solvable groups for odd primes ℓ in [MNR, Thm A], and Martínez Madrid recently showed in [Mar25, Thm A] that it holds for symmetric groups. Further, we find that the Strong Picky Conjecture holds for groups of Lie type in non-defining characteristic, assuming some standard conjectures on characters of groups of Lie type (that have been shown to hold in many cases; see Proposition 2.12 below).

Let $G = \mathbf{G}^F$ be a finite group of Lie type, where $\mathbf{G}$ is a connected reductive linear algebraic group in characteristic $p > 0$ and $F : \mathbf{G} \to \mathbf{G}$ is a Steinberg map. In [Ma25], the first author began the classification of picky elements for such G, as well as the larger study of subnormalisers of ℓ-elements and the corresponding more general version of Conjecture 1 for arbitrary ℓ-elements. It turns out that for $\ell = p$, there exist picky ℓ-elements even in highly non-abelian Sylow subgroups and in arbitrary rank; the character tables of Sylow normalisers in this case are not well-understood.

In contrast, if $p \neq \ell > 3$ and $\mathbf{G}$ is simple of simply connected type, by [Ma25, Thm 5.9] then G has no picky ℓ-elements unless P is abelian. Hence, our first focus in this paper is on the situation that $\ell \neq p$ and that P is abelian. In particular, we show that in this situation, the Picky Conjecture holds (see Proposition 2.9). Combining this with an investigation of the finitely many groups with picky 3-elements we obtain:

Theorem 4. *Let $G = \mathbf{G}^F$ be a finite group of Lie type, where $\mathbf{G}$ is simple and simply connected in characteristic p and F is a Steinberg morphism. Then the Picky+ Conjecture holds for G for any prime $\ell \geq 3$ different from p.*

In our proof of Theorem 4, in addition to the work in [Ma25], we also rely on the McKay bijection proved by the first author and Späth in [Ma07, Sp10].

We also classify the picky semisimple ℓ-elements in the case that $\ell = 2$, and we obtain the stronger version in this case:

Theorem 5. *Let $G = \mathbf{G}^F$ be a finite group of Lie type, where $\mathbf{G}$ is simple and simply connected in characteristic $p \neq 2$ and F is a Steinberg morphism. Then the Strong Picky Conjecture holds for G for the prime $\ell = 2$.*

With this we also eventually obtain the Picky Conjecture for quasi-simple groups of Lie type in non-defining characteristic:

Corollary 6. *The Picky Conjecture holds for any quasi-simple group H such that $H/\mathbf{Z}(H)$ is simple of Lie type in characteristic p for any $\ell \neq p$.*

The paper is organised as follows. In Section 2, we treat the generic case of finite reductive groups with abelian Sylow ℓ-subgroups, and also discuss the Suzuki and Ree groups. In Section 3, we determine quasi-simple groups of Lie type with semisimple picky 2-elements, and in the final section, Section 4, we complete the proofs of our main Theorems 4 and 5 and of Corollary 6 by first classifying and then dealing with picky 2- and 3-elements, as well as groups with exceptional covering groups.

2. The case of abelian Sylow subgroups

In this section, we prove Theorem 4 when the Sylow subgroups of G are abelian. First, let G be an arbitrary finite group with abelian Sylow subgroups. In this situation, the following reduces the study of the Picky Conjecture to a strengthening of bijections in the McKay Conjecture.

Lemma 2.1 (Moretó–Rizo)**.** *Let ℓ be a prime and suppose that G is a finite group with abelian Sylow ℓ-subgroups. Let $x \in G$ be a picky ℓ-element. Then $\mathrm{Irr}^x(G) = \mathrm{Irr}_{\ell'}(G)$.*

Remark 2.2. Note that in the situation of Lemma 2.1, (3) of Conjecture 1 follows by general properties of character values. Indeed, let $\chi \in \mathrm{Irr}_{\ell'}(G)$. Now $\chi(x)$ is a sum of $n := \chi(1)$ many ℓ-power order roots of unity, and if m denotes the length of its Galois orbit, then the norm $\mathcal{N}(\chi(x))$ is an integer that is a sum of n^m ℓ-power roots of unity. Now for $a \geq 2$ the sum over all $\ell^{a-1}(\ell-1)$ primitive ℓ^ath roots of unity is zero (inductively, since the sum over all ℓ^ath roots of unity vanishes), so we see that $\mathcal{N}(\chi(x))$ equals a sum of k ℓth roots of unity where $k \equiv n^m \pmod{\ell}$. Finally, the sum over the $\ell - 1$ primitive ℓth roots of unity equals -1, so $\mathcal{N}(\chi(x)) \equiv n^m \pmod{\ell}$ is prime to ℓ. This of course also holds for $\mathrm{N}_G(P)$ in place of G.

As the McKay Conjecture has been proven (see [CS24]), we see from Lemma 2.1 and Remark 2.2 that to prove the Picky+ Conjecture, resp. the Strong Picky Conjecture, for (G, ℓ, x) when $P \in \mathrm{Syl}_\ell(G)$ is abelian, it suffices to prove item (2), resp. (2*), of Conjecture 1 for some known McKay bijection $f\colon \mathrm{Irr}_{\ell'}(G) \to \mathrm{Irr}_{\ell'}(\mathrm{N}_G(P))$.

We also remark that the (Strong) Picky Conjecture is known for the case that P is cyclic [MR25]:

Lemma 2.3 (Moretó–Rizo)**.** *Let ℓ be a prime and suppose that G has cyclic Sylow ℓ-subgroups. Then the Strong Picky Conjecture holds for all picky ℓ-elements of G.*

2.1. **The Setting.** We consider the following setting, where we refer the reader to [MT11] and [GM20] for background and notation on linear algebraic groups and the character theory of finite groups of Lie type. As before, let $\mathbf{G}$ be simple of simply connected type in characteristic p with a Frobenius map $F : \mathbf{G} \to \mathbf{G}$ with respect to an $\mathbb{F}_q$-structure and $G := \mathbf{G}^F$. Let $(\mathbf{G}^*, F)$ be dual to $(\mathbf{G}, F)$, and write $G^* := (\mathbf{G}^*)^F$.

We suppose that $P \in \mathrm{Syl}_\ell(G)$ is abelian. This means that ℓ satisfies the condition $(*)$ of [Ma14, Sec. 2]. Further, this means that ℓ is an odd prime good for $\mathbf{G}$ which divides neither $[\mathbf{Z}(\mathbf{G})^F : \mathbf{Z}^\circ(\mathbf{G})^F]$ nor $[\mathbf{Z}(\mathbf{G}^*)^F : \mathbf{Z}^\circ(\mathbf{G}^*)^F]$, and that further $\ell \geq 5$ if G involves ${}^3\mathrm{D}_4(q)$. (See [Ma14, Lemma 2.1, Prop. 2.2].)

Let $d := d_\ell(q)$ be the order of q modulo ℓ and let $\mathbf{S}$ be a Sylow d-torus of $(\mathbf{G}, F)$ containing P, which exists by our assumptions on ℓ (see the proof of [Ma14, Prop. 2.3]). Let $\mathbf{L} := \mathrm{C}_{\mathbf{G}}(\mathbf{S})$ be the corresponding minimal d-split Levi subgroup of $\mathbf{G}$ and write $L := \mathbf{L}^F$. We further let $\mathbf{L}^* := \mathrm{C}_{\mathbf{G}^*}(\mathbf{S}^*)$, where $\mathbf{S}^*$ is a Sylow d-torus for $(\mathbf{G}^*, F)$ dual to $\mathbf{S}$ and (thus) $\mathbf{L}^*$ is dual to $\mathbf{L}$. (Such a choice exists by [Ma07, Lemma 3.3].) We then let $L^* := \mathbf{L}^{*F}$ and also write $N^* := \mathrm{N}_{G^*}(\mathbf{S}^*)$.

From the first author's work in [Ma14], we see:

Lemma 2.4. *In the above situation, we have*
(a) $\mathrm{N}_G(P) = \mathrm{N}_{\mathbf{G}}(\mathbf{L})^F$;
(b) $L = P \times \mathrm{O}^\ell(L)$;
(c) $\ell \nmid |\mathrm{W}_G(\mathbf{L})|$, *where* $\mathrm{W}_G(\mathbf{L}) := \mathrm{N}_{\mathbf{G}}(\mathbf{L})^F/L$;
(d) *any picky ℓ-element $x \in G$ lies in a unique G-conjugate of L; and*
(e) *if $x \in L$ is a picky ℓ-element, then $x^g \in L$ if and only if $g \in \mathrm{N}_G(P)$.*

Proof. The first three items are from [Ma14, Prop. 2.4]. The latter two items follow, since if $x \in P$ is picky and $x \in L^g$ for some $g \in G$, then $x \in P^g$ by (b). Hence $g \in \mathrm{N}_G(P) = \mathrm{N}_{\mathbf{G}}(\mathbf{L})^F$ by (a). Then $L^g = (\mathbf{L} \cap G)^g = \mathbf{L} \cap G = L$. □

Further, the picky elements as above were classified in [Ma25, Thm 5.2]:

Theorem 2.5. *In the above situation, $x \in P$ is picky if and only if $\mathrm{C}_{\mathbf{G}}(x) = \mathbf{L}$.*

Write $N := \mathrm{N}_G(P) = \mathrm{N}_{\mathbf{G}}(\mathbf{L})^F$. In what follows, we will consider a bijection

$$\Omega\colon \mathrm{Irr}_{\ell'}(G) \to \mathrm{Irr}_{\ell'}(N)$$

as constructed in [Ma07, Thm 7.8], which uses the results of [Sp09, Sp10] to obtain the local extensions that yield this bijection. Throughout, Ω will continue to denote such a bijection, and we will argue that any such Ω satisfies the properties for f in the Picky Conjecture.

We also note here that by [Ma07, Thm 7.8(c)], Ω has the additional property that χ and $\Omega(\chi)$ lie above the same character of $\mathbf{Z}(G)$ for any $\chi \in \mathrm{Irr}_{\ell'}(G)$. Now, if S is a simple group of Lie type in characteristic $p \neq \ell$ with non-exceptional Schur multiplier but not of Suzuki or Ree type, then we know that its universal covering group is of the form $G = \mathbf{G}^F$ above. Hence, together with the results of [Ma25], in such a case, to prove the Picky+ Conjecture for all quasi-simple groups H with $H/\mathbf{Z}(H) \cong S$ and primes $\ell > 3$ with $\ell \neq p$, it suffices to prove that $\mathbb{Q}(\chi(x)) = \mathbb{Q}(\Omega(\chi)(x))$ where G, x, Ω are as above. (Note that the cases $\ell \in \{2, 3\}$ or S has an exceptional Schur multiplier are treated in Section 4 below, and the Suzuki and Ree groups are discussed in Section 2.5 below.)

2.2. **Values on the local side.** Keep the setting from Section 2.1. We first establish the relevant values for characters in $\mathrm{Irr}_{\ell'}(N)$.

Let $\chi \in \mathrm{Irr}_{\ell'}(G)$. In this situation, the character $\Omega(\chi) \in \mathrm{Irr}_{\ell'}(N)$ is of the form

$$\Omega(\chi) = \mathrm{Ind}_{N_\theta}^N(\hat{\theta}\eta),$$

where $\hat{\theta} \in \mathrm{Irr}(N_\theta)$ is an extension of some character $\theta \in \mathrm{Irr}_{\ell'}(L)$ (whose existence is guaranteed by [Sp10, Thm 1.1]) and η is a character of $N_\theta/L = \mathrm{W}_G(\mathbf{L})_\theta$ viewed as a character of N_θ by inflation. We will say that $\Omega(\chi)$ corresponds to the pair (θ, η). Note here that since $P \lhd L$ is abelian, in fact $\mathrm{Irr}_{\ell'}(L) = \mathrm{Irr}(L)$ by Clifford theory.

Then since $x^g \in L \leq N_\theta$ for any $g \in N$ and η is trivial on L, we have

$$\Omega(\chi)(x) = \frac{1}{|N_\theta|} \sum_{g \in N; x^g \in N_\theta} \hat{\theta}(x^g)\eta(x^g) = \frac{1}{|N_\theta|} \sum_{g \in N} \theta(x^g)\eta(x^g) = \frac{\eta(1)}{|N_\theta|} \sum_{g \in N; x^g \in L} \theta(x^g).$$

Thus we see

$$\Omega(\chi)(x) = \eta(1)\frac{|L|}{|N_\theta|}\mathrm{Ind}_L^N(\theta)(x) \tag{2.1}$$

and hence in particular

$$\mathbb{Q}(\Omega(\chi)(x)) = \mathbb{Q}(\mathrm{Ind}_L^N(\theta)(x)). \tag{2.2}$$

2.3. **Values on the global side.** We continue to keep the setting from the previous sections and now investigate the relevant values for $\chi \in \mathrm{Irr}_{\ell'}(G)$. Recall that $\mathrm{C}^\circ_{\mathbf{G}}(x) = \mathrm{C}_{\mathbf{G}}(x) = \mathbf{L}$ by Theorem 2.5. By [GM20, Cor. 3.3.13] and the discussion after, we have:

$$\chi(x) = {}^*R_{\mathbf{L}}^{\mathbf{G}}(\chi)(x) = \sum_{\varphi \in \mathrm{Irr}(L)} \langle \chi, R_{\mathbf{L}}^{\mathbf{G}}(\varphi)\rangle \varphi(x), \tag{2.3}$$

where $R_{\mathbf{L}}^{\mathbf{G}}$ and ${}^*R_{\mathbf{L}}^{\mathbf{G}}$ denote Lusztig induction, resp. restriction.

Now, if χ lies in the rational Lusztig series $\mathcal{E}(G, s)$ of G corresponding to a semisimple element $s \in G^*$, then up to G^*-conjugacy, we may assume $s \in L^*$. (See [Ma07, Prop. 7.2].) From the work constructing the bijection Ω in [Ma07, Thm 7.5, 7.8 and Prop. 7.7], we see we may assume further that $\theta \in \mathcal{E}(L, s)$ if χ is such that $\Omega(\chi)$ corresponds to (θ, η) as in Section 2.2.

Remark 2.6. In the above situation, we note that $\mathrm{C}_{\mathbf{L}^*}(s)$ contains only components of classical type A, B, C, D. This is clear if $d = d_\ell(q)$ is a so-called *regular number* for $(\mathbf{G}, F)$, since then $\mathbf{L}^*$ is a torus (see [GM20, Ex. 3.5.7]). If d is not a regular number, the claim follows from [GM20, Tab. 3.3] and [BMM, Tab. 1].

In the following, we consider a regular embedding $\iota\colon \mathbf{G} \hookrightarrow \widetilde{\mathbf{G}} = \mathbf{G}\mathbf{Z}(\widetilde{\mathbf{G}})$ and its dual epimorphism $\iota^*\colon \widetilde{\mathbf{G}}^* \to \mathbf{G}^*$, as in [GM20, Prop. 1.7.5], and extend F naturally to $\widetilde{\mathbf{G}}$, resp. to $\widetilde{\mathbf{G}}^*$. We also recall that, thanks to Lemma 2.3, we are particularly interested in the case that P is not cyclic.

Proposition 2.7. *Keep the setting of Section 2.1. Thus, $G = \mathbf{G}^F$ where $\mathbf{G}$ is simple of simply connected type in characteristic $p \neq \ell$, with $F\colon \mathbf{G} \to \mathbf{G}$ a Frobenius map; $P \in \mathrm{Syl}_\ell(G)$ is abelian lying in a Sylow d-torus $\mathbf{S}$ with $d := d_\ell(q)$; and $L := \mathbf{L}^F$, where $\mathbf{L} := \mathrm{C}_{\mathbf{G}}(\mathbf{S})$. Further, write $\widetilde{\mathbf{L}} := \mathbf{L}\mathbf{Z}(\widetilde{\mathbf{G}})$ and $\widetilde{L} := \widetilde{\mathbf{L}}^F$.*

Let $\chi \in \mathrm{Irr}_{\ell'}(G) \cap \mathcal{E}(G, s)$ such that $\Omega(\chi)$ corresponds to (θ, η) as in Section 2.2, where $s \in L^*$ and $\theta \in \mathcal{E}(L, s)$. If

1. *$J_s^{\mathbf{G}} \circ R_{\mathbf{L}}^{\mathbf{G}} = R_{\mathrm{C}_{\mathbf{L}^*}(s)}^{\mathrm{C}_{\mathbf{G}^*}(s)} \circ J_s^{\mathbf{L}}$; or*
2. *P is non-cyclic,*

then $\langle \chi, R_{\mathbf{L}}^{\mathbf{G}}(\theta')\rangle \neq 0$ for some $\widetilde{L}$-conjugate θ' of θ. In Case (1), if $\mathrm{C}_{\mathbf{L}^*}(s)$ is further connected, then we may take $\theta' = \theta$.

Remark 2.8. We note that if d is a regular number for G, then (1) holds and $\mathrm{C}_{\mathbf{L}^*}(s)$ is connected, so that $\theta' = \theta$ in the situation of Proposition 2.7. Indeed, then $\mathrm{C}_{\mathbf{L}^*}(s) = \mathbf{L}^*$ is a torus, hence connected, and further the commutation holds by the defining properties of Jordan decomposition [GM20, Thm 2.6.22].

Proof of Proposition 2.7. From [Ma07, Prop. 7.7], θ corresponds under some Jordan decomposition $J_s^{\mathbf{L}}$ to a character $J_s^{\mathbf{L}}(\theta)$ of $\mathrm{C}_{L^*}(s)$ lying above some unipotent character $\lambda \in \mathrm{UCh}(\mathrm{C}^\circ_{\mathbf{L}^*}(s)^F) := \mathcal{E}(\mathrm{C}^\circ_{\mathbf{L}^*}(s)^F, 1)$ and furthermore

$$\mathrm{W}_{\mathrm{C}_{G^*}(s)}(\mathrm{C}^\circ_{\mathbf{L}^*}(s))_\lambda = \mathrm{W}_{N^*}(s, \lambda) \cong \mathrm{W}_G(\mathbf{L})_\theta \tag{2.4}$$

under duality, where we recall that $N^* := \mathrm{N}_{G^*}(\mathbf{S}^*)$ as in Section 2.1. Also, here $\mathrm{C}_{L^*}(s) = \mathrm{C}_{\mathrm{C}_{\mathbf{G}^*}(s)}(\mathbf{S})^F$ and $\mathrm{C}^\circ_{L^*}(s) := \mathrm{C}^\circ_{\mathbf{L}^*}(s)^F = \mathrm{C}_{\mathrm{C}^\circ_{\mathbf{G}^*}(s)}(\mathbf{S})^F$, and the latter is also a minimal d-split Levi subgroup of $(\mathrm{C}^\circ_{\mathbf{G}^*}(s), F)$ according to the proof of [Ma07, Cor. 6.6].

On the local side, we have seen that $\Omega(\chi)$ corresponds to the pair (θ, η), or the triple (s, λ, η). On the global side, χ also corresponds to the triple (s, λ, η), which in the context of [Ma07, Sec. 6,7] means that χ corresponds via some Jordan decomposition $J_s^{\mathbf{G}}$ to $J_s^{\mathbf{G}}(\chi) \in \mathrm{UCh}(\mathrm{C}_{G^*}(s))$ lying in the d-Harish-Chandra series of $(\mathrm{C}^\circ_{L^*}(s), \lambda)$ as defined in [Ma07, Thm 4.6]. (Here $J_s^{\mathbf{G}}(\chi)$, in turn, corresponds to η under the bijection of [Ma07, Thm 4.6(b)].) Then in particular, $J_s^{\mathbf{G}}(\chi)$ is a constituent of

$$\mathrm{Ind}^{\mathrm{C}_{G^*}(s)}_{\mathrm{C}^\circ_{G^*}(s)}(R^{\mathrm{C}^\circ_{\mathbf{G}^*}(s)}_{\mathrm{C}^\circ_{\mathbf{L}^*}(s)}(\lambda)) = R^{\mathrm{C}_{\mathbf{G}^*}(s)}_{\mathrm{C}_{\mathbf{L}^*}(s)}(\mathrm{Ind}^{\mathrm{C}_{L^*}(s)}_{\mathrm{C}^\circ_{L^*}(s)}(\lambda)), \tag{2.5}$$

where $R^{\mathrm{C}_{\mathbf{G}^*}(s)}_{\mathrm{C}_{\mathbf{L}^*}(s)}$ is as defined in [GM20, Def. 4.8.8] and the equality follows from [DM94, Cor. 2.4(ii)] by adjunction. Note further that by [GM20, 3.3.20], the constituents of $R^{\mathbf{G}}_{\mathbf{L}}(\theta)$ lie in $\mathcal{E}(G, s)$.

Assume first that (1) holds. If further $\mathrm{C}_{\mathbf{L}^*}(s)$ is connected, then χ lies in the d-Harish-Chandra series of $(\mathbf{L}, \theta)$. Indeed, we then have $J_s^{\mathbf{L}}(\theta) = \lambda$ and $J_s^{\mathbf{G}}(R^{\mathbf{G}}_{\mathbf{L}}(\theta)) = R^{\mathrm{C}_{\mathbf{G}^*}(s)}_{\mathrm{C}_{\mathbf{L}^*}(s)}(\lambda)$ contains $J_s^{\mathbf{G}}(\chi)$. Since $J_s^{\mathbf{G}}$ is a bijection from $\mathcal{E}(G, s)$ to $\mathrm{UCh}(\mathrm{C}_{G^*}(s))$, this means that $\langle \chi, R^{\mathbf{G}}_{\mathbf{L}}(\theta)\rangle \neq 0$. If $\mathrm{C}_{\mathbf{L}^*}(s)$ is disconnected, $J_s^{\mathbf{G}}(\chi)$ is a constituent of $R^{\mathrm{C}_{\mathbf{G}^*}(s)}_{\mathrm{C}_{\mathbf{L}^*}(s)}(\hat\lambda)$, for some $\hat\lambda$ lying above λ. Let $\theta' \in \mathcal{E}(L, s)$ be such that $J_s^{\mathbf{L}}(\theta') = \hat\lambda$. Then since $J_s^{\mathbf{L}}(\theta')$ and $J_s^{\mathbf{L}}(\theta)$ lie above the same unipotent character of $\mathrm{C}^\circ_{L^*}(s)$, it follows from the analysis in [GM20, Sec. 2.6] that θ and θ' are $\widetilde{L}$-conjugate. Here we have $J_s^{\mathbf{G}}(\chi)$ is a constituent of $J_s^{\mathbf{G}}(R^{\mathbf{G}}_{\mathbf{L}}(\theta'))$, and again we see $\langle \chi, R^{\mathbf{G}}_{\mathbf{L}}(\theta')\rangle \neq 0$.

In Remark 2.8 we see that if d is regular, we have (1) and $\mathrm{C}_{\mathbf{L}^*}(s)$ is connected, so χ lies in the d-Harish-Chandra series of $(\mathbf{L}, \theta)$. In particular, we may assume now that d is not a regular number.

Now assume that (2) holds. If G is of exceptional type, since d is not regular and P is not cyclic by (2), we conclude that $G = \mathrm{E}_7(q)$ and $d = 4$. Hence, we may assume that either G is of classical type or that $G = \mathrm{E}_7(q)$ with $d = 4$, and recall from Remark 2.6 that $\mathrm{C}_{\mathbf{L}^*}(s)$ contains only components of classical type in either situation. Now, since $\mathbf{Z}(\widetilde{\mathbf{G}})$ is connected, the Mackey formula holds for $\widetilde{\mathbf{G}}$ by [GM20, Thm 3.3.7(5)]. Then further $J_{\tilde s}^{\widetilde{\mathbf{G}}} \circ R^{\widetilde{\mathbf{G}}}_{\widetilde{\mathbf{L}}} = R^{\mathrm{C}_{\widetilde{\mathbf{G}}^*}(\tilde s)}_{\mathrm{C}_{\widetilde{\mathbf{L}}^*}(\tilde s)} \circ J_{\tilde s}^{\widetilde{\mathbf{L}}}$ for $\tilde s \in \widetilde{\mathbf{G}}^{*F}$ with $\iota^*(\tilde s) = s$, using [GM20, Thm 4.7.2] when G is of classical type and using the reasoning in the second two paragraphs of the proof of [GM20, Thm 4.7.5] if $G = \mathrm{E}_7(q)$.

Note here that $\widetilde{\mathbf{L}}$ is a minimal d-split Levi subgroup of $\widetilde{\mathbf{G}}$. Further, since $\ell \nmid |\widetilde{G}/G\mathbf{Z}(\widetilde{G})|$ by our assumptions, we see that $\mathrm{Irr}_{\ell'}(\widetilde{G})$ is exactly the set of characters lying above those in $\mathrm{Irr}_{\ell'}(G)$. In particular, considerations similar to above, now using the parametrization of $\mathrm{Irr}_{\ell'}(\widetilde{G})$ from [CS13, Sec. 4], yield that we may find a character $\widetilde{\chi} \in \mathcal{E}(\widetilde{G}, \widetilde{s}) \cap \mathrm{Irr}_{\ell'}(\widetilde{G})$ lying above χ such that $\langle \widetilde{\chi}, R_{\widetilde{\mathbf{L}}}^{\widetilde{\mathbf{G}}}(\widetilde{\theta})\rangle \neq 0$, where $\widetilde{\theta} \in \mathcal{E}(\widetilde{L}, \widetilde{s})$ lies above θ. (Note that the parametrisation in [CS13] requires an extension map $\widetilde{\Lambda}$ with respect to $\widetilde{L} \lhd \widetilde{N}$. Such a map $\widetilde{\Lambda}$ exists for types A, B, C, and D according to [CS17a, Cor. 5.14], [CS19, Sec. 5.5], [CS17b, Thm 6.1], and [CS24, Thms 4.1, 6.9], respectively. See also [MS16, Prop. 3.20] for the case $d \in \{1, 2\}$. A map $\widetilde{\Lambda}$ is shown to exist for type E_7 in the course of checking the conditions for [CS19, Thm 4.2] in the proof of [CS19, Thm A].) Then by [GM20, Cor. 3.3.25], we have $\langle \chi, R_{\mathbf{L}}^{\mathbf{G}}(\theta')\rangle \neq 0$ for some $\widetilde{L}$-conjugate θ' of θ. ☐

The following proves Theorem 4 for $\ell > 3$, thanks to [Ma25, Thm 5.9].

Proposition 2.9. *Keep the setting in Section 2.1. Then Conjecture 1(2) holds. That is, the Picky+ Conjecture holds for (G, ℓ, x) for all picky ℓ-elements $x \in G = \mathbf{G}^F$ when ℓ is an odd prime such that the Sylow ℓ-subgroups of G are abelian.*

Proof. Recall from Remark 2.2 that the second statement follows from the first. By Lemma 2.3 we may assume that $P \in \mathrm{Syl}_\ell(G)$ is non-cyclic. Hence we are as in the hypotheses in Proposition 2.7, so $\langle \chi, R_{\mathbf{L}}^{\mathbf{G}}(\theta')\rangle \neq 0$ for some $\widetilde{L}$-conjugate θ' of θ, where we recall that $\chi \in \mathrm{Irr}_{\ell'}(G) \cap \mathcal{E}(G, s)$ with $s \in L^*$ and $\theta \in \mathcal{E}(L, s)$.

Also, recall that, without loss, $x \in P \leq \mathbf{S}$. Note that since $x \in \mathbf{S}$ is centralised by any $g \in \widetilde{L}$, we see (2.2) holds with θ replaced with θ'. Then for the purposes of proving that $\mathbb{Q}(\chi(x)) = \mathbb{Q}(\Omega(\chi)(x))$, we may assume that $\langle \chi, R_{\mathbf{L}}^{\mathbf{G}}(\theta)\rangle \neq 0$.

To prove that $\mathbb{Q}(\chi(x)) = \mathbb{Q}(\Omega(\chi)(x))$, (2.2) and (2.3), together with Lemma 2.4(e), yield that it now suffices to show that

$$\sum_{\varphi \in \mathrm{Irr}(L)} \langle \chi, R_{\mathbf{L}}^{\mathbf{G}}(\varphi)\rangle \varphi(x) = r \sum_{g \in N/N_\theta} \theta^g(x) \quad \text{for some } r \in \mathbb{Q}^\times.$$

Thanks to [Ma07, Lemma 3.3, Prop. 7.7], we have that $N/N_\theta \cong N^*/L^* \mathrm{C}_{N^*}(s)_\lambda$ where χ corresponds to the triple (s, λ, η) as before. Using [CE99, Prop. 1.9], we see that for $g \in N/N_\theta$, we have $\mathcal{E}(L, s)^g = \mathcal{E}(L, s^{g^*})$ where $g \mapsto g^*$ denotes some such isomorphism. Further, by [Ma07, Lemma 2.2], each character in $\mathcal{E}(L, s)^g$ lies above the same character of $\mathbf{Z}(L)$, so that $\varphi(x)/\varphi(1) = \theta^g(x)/\theta(1)$ is constant for $\varphi \in \mathcal{E}(L, s)^g$ since $\theta \in \mathcal{E}(L, s)$ and $x \in P \leq \mathbf{Z}(L)$.

Now, let $\varphi \in \mathrm{Irr}(L)$ with $\langle \chi, R_{\mathbf{L}}^{\mathbf{G}}(\varphi)\rangle \neq 0$. Then $\varphi \in \mathcal{E}(L, t)$ with $t \in L^*$ a G^*-conjugate of s, by [GM20, Prop. 3.3.20]. This in turn implies that t is N^*-conjugate to s, since N^* controls fusion in L^* by [Ma07, Prop. 5.11]. That is, such a character φ lies in $\mathcal{E}(L, s^{g^*})$ for some $g^* \in N^*/L^* \mathrm{C}_{N^*}(s)$. It follows that $\varphi(x) = \theta^g(x)\varphi(1)/\theta(1)$. Then we have

$$\sum_{\varphi \in \mathrm{Irr}(L)} \langle \chi, R_{\mathbf{L}}^{\mathbf{G}}(\varphi)\rangle \varphi(x) = \sum_{g \in N/N_\theta} \sum_{\varphi \in \mathcal{E}(L,s)^g} \langle \chi, R_{\mathbf{L}}^{\mathbf{G}}(\varphi)\rangle \varphi(x) = \frac{1}{\theta(1)} \sum_{g \in N/N_\theta} \theta^g(x)\ r_g$$

where $r_g := \sum_{\varphi\in\mathcal{E}(L,s)^g}\langle\chi, R_{\mathbf{L}}^{\mathbf{G}}(\varphi)\rangle\varphi(1)$ for $g \in N/N_\theta$. It now suffices to see that $r_g = r_1$ for each $g \in N/N_\theta$. But this follows since for $\varphi_1 \in \mathcal{E}(L,s)$, we have $\varphi_1^g \in \mathcal{E}(L,s)^g$ and $\langle\chi, R_{\mathbf{L}}^{\mathbf{G}}(\varphi_1^g)\rangle\varphi_1^g(1) = \langle\chi, R_{\mathbf{L}}^{\mathbf{G}}(\varphi_1)\rangle\varphi_1(1)$. □

Remark 2.10. Note that since $\mathbf{L}$ is a minimal d-split Levi subgroup of $\mathbf{G}$, each $\varphi \in \mathrm{Irr}(L)$ contributing to the sum in (2.3) is d-cuspidal. A conjecture of Cabanes–Enguehard (see the discussion after [CE99, Not. 1.11]) posits that the d-cuspidal pairs under a given character of G must be G-conjugate. If this conjecture holds, we would have that the $\varphi \in \mathrm{Irr}(L)$ contributing non-trivially to the sum in (2.3) are always of the form $\varphi = \theta^g$ for some $g \in G$. In the literature, it is sometimes said that *generalised d-Harish-Chandra theory* holds for $\mathcal{E}(G,s)$ if moreover there is a bijection with the characters of relative Weyl groups (see [KM13, Def. 2.9]). We note that by [En13, Prop. 2.2.2] and [CE99, Thm 4.1], the conjecture of Cabanes–Enguehard holds (given the properties of ℓ discussed in Section 2.1) whenever s is an ℓ'-element. Further, when d is a regular number and thus $\mathbf{L}$ is an F-stable maximal torus, we have $\langle\chi, R_{\mathbf{L}}^{\mathbf{G}}(\varphi)\rangle \neq 0$ if and only if $(\mathbf{L},\varphi)$ is G-conjugate to $(\mathbf{L},\theta)$, if and only if $(\mathbf{L},\varphi) = (\mathbf{L},\theta^g)$ for some $g \in N = \mathrm{N}_G(\mathbf{L})$, using [GM20, Prop. 2.5.18 and Defs 2.5.14, 2.5.17]. Further, in this case, $R_{\mathbf{L}}^{\mathbf{G}}(\varphi) = R_{\mathbf{L}}^{\mathbf{G}}(\theta)$, by [GM20, Cor. 2.2.10].

In particular, if we know that Cabanes–Enguehard's conjecture holds for $\mathcal{E}(G,s)$ (or, the more modest assumption that every $\varphi \in \mathrm{Irr}(L)$ contributing non-trivially to the sum in (2.3) is of the form $\varphi = \theta^g$ for $g \in G$), then we obtain an alternate proof for Proposition 2.9 since then

$$\chi(x) = {}^*R_{\mathbf{L}}^{\mathbf{G}}(\chi)(x) = r\sum_{g\in N/N_\theta}\theta^g(x)$$

for some integer $r \neq 0$, using (2.3) and Proposition 2.7. Hence $\mathbb{Q}(\chi(x)) = \mathbb{Q}(\Omega(\chi)(x))$ when combined with (2.2).

2.4. **On the Strong Picky Conjecture.** Keep the situation of Setting 2.1, with $\chi \in \mathrm{Irr}_{\ell'}(G)\cap\mathcal{E}(G,s)$ for $s \in G^*$ semisimple and $\Omega(\chi) \in \mathrm{Irr}_{\ell'}(N)$ corresponding to (θ,η) as in Sections 2.2 and 2.3.

In some situations, such as when $\mathrm{C}_{G^*}(s) \le L^*$ or when L is a torus and $N_\theta = L$ (i.e., θ is in *general position*), we have $\pm R_{\mathbf{L}}^{\mathbf{G}}(\theta) \in \mathrm{Irr}(G)$ (see [GM20, Thm 3.3.22, Cor. 2.2.9]). In this situation, we obtain the strong form of the conjecture:

Proposition 2.11. *Keep the setting of Section 2.1. If $\chi = \pm R_{\mathbf{L}}^{\mathbf{G}}(\theta) \in \mathrm{Irr}_{\ell'}(G)$, then Conjecture 1(2*) holds for χ at the prime ℓ.*

That is, the Strong Picky Conjecture holds for the subset $\mathfrak{G}$ of characters of the form $\pm R_{\mathbf{L}}^{\mathbf{G}}(\theta) \in \mathrm{Irr}_{\ell'}(G)$ for a minimal d-split Levi subgroup $\mathbf{L}$ and the restriction of Ω to $\mathfrak{G}$.

Proof. Let $x \in G$ be a picky ℓ-element. Replacing x by a suitable conjugate, by Lemma 2.4 we may assume that $x \in L$. Now we have $R_{\mathbf{L}}^{\mathbf{G}}(\theta)\otimes\mathrm{St}_{\mathbf{G}} = \pm\mathrm{Ind}_L^G(\theta\otimes\mathrm{St}_{\mathbf{L}})$ by [GM20, Cor. 3.4.11]. Since x is an ℓ-element and $\ell \nmid |\mathbf{Z}(\mathbf{G})/\mathbf{Z}^\circ(\mathbf{G})|$, its centraliser $\mathrm{C}_{\mathbf{G}}(x)$ is connected (see, e.g. [MT11, Exer. 20.16]). Recall from Theorem 2.5 that the picky element x satisfies $\mathrm{C}_G(x) = L$. Then from [GM20, Prop. 3.4.10], we have

$$\pm\,\mathrm{St}_{\mathbf{G}}(x) = |\,\mathrm{C}_G(x)|_p = |L|_p.$$

From this, we see

$$\pm|L|_p \cdot R_{\mathbf{L}}^{\mathbf{G}}(\theta)(x) = \mathrm{Ind}_L^G(\theta \otimes \mathrm{St}_{\mathbf{L}})(x) = \frac{1}{|L|}\sum_{g\in G; x^g \in L} \theta(x^g)\,\mathrm{St}_{\mathbf{L}}(x^g) = \frac{1}{|L|}\sum_{g\in N}\theta(x^g)\,\mathrm{St}_{\mathbf{L}}(x^g),$$

where the last equality is from Lemma 2.4(e).

Again using [GM20, Prop. 3.4.10], Theorem 2.5, and that x is picky, for each $g \in N$, we have $\mathrm{St}_{\mathbf{L}}(x^g) = \epsilon_{\mathbf{L}}\epsilon_{\mathrm{C}^\circ_{\mathbf{L}}(x^g)}|\,\mathrm{C}^\circ_{\mathbf{L}}(x^g)^F|_p = \epsilon_{\mathbf{L}}\epsilon_{\mathbf{L}}|\,\mathrm{C}_L(x^g)|_p = |L|_p$. (Here the values $\epsilon_{\mathbf{G}} \in \{\pm 1\}$ are as in [GM20, Def. 2.2.11].) Then $R_{\mathbf{L}}^{\mathbf{G}}(\theta)(x) = \frac{\pm 1}{|L|}\sum_{g\in N}\theta(x^g) = \frac{\pm 1}{|L|}\sum_{g\in G; x^g\in L}\theta(x^g)$.

That is, we see

$$R_{\mathbf{L}}^{\mathbf{G}}(\theta)(x) = \pm\mathrm{Ind}_L^G(\theta)(x) = \pm\mathrm{Ind}_L^N(\theta)(x). \tag{2.6}$$

Recall that Conjecture 1(1) holds for G by Lemma 2.1 combined with [Ma07, Thm 7.8] and [Sp10, Thm 1.1]. The claim then follows from Equations (2.1) and (2.6), once we know that $\mathrm{W}_G(\mathbf{L})_\theta$ (hence η) is trivial.

Now, the latter holds when the Mackey formula holds, as in [DM20, Cor. 9.3.1]. This leaves only the cases $G = {}^2\mathrm{E}_6(2)$ and $G = \mathrm{E}_8(2)$ when $\mathbf{L}$ is not a torus (hence d is not a regular number) to consider, by [GM20, Thm 3.3.7]. In these cases, the Sylow ℓ-subgroups are cyclic of prime order, and hence by Lemma 2.3, the result still holds. □

We next obtain the Strong Picky Conjecture under suitable assumptions.

Proposition 2.12. *Keep the notation from Section 2.1, and assume that (1) of Proposition 2.7 holds and that Cabanes–Enguehard's conjecture holds under χ. (See Remark 2.10.) Then*

$$\chi(x) = \pm\Omega(\chi)(x).$$

In particular, if d is a regular number, then the Strong Picky Conjecture holds for (G, ℓ, x) for all picky ℓ-elements x.

Proof. From (2.1) and the line before, we have $\Omega(\chi)(x) = \eta(1)\sum_{g\in N/N_\theta}\theta^g(x)$. Since (1) of Proposition 2.7 holds, there is an $\widetilde{L}$-conjugate θ' of θ such that $\langle\chi, R_{\mathbf{L}}^{\mathbf{G}}(\theta')\rangle \neq 0$. Arguing as in the beginning of the proof of Proposition 2.9, we may assume instead that $\langle\chi, R_{\mathbf{L}}^{\mathbf{G}}(\theta)\rangle \neq 0$. On the other hand, from (2.3) and the proof of Proposition 2.9, we know then that

$$\chi(x) = \langle\chi, R_{\mathbf{L}}^{\mathbf{G}}(\theta)\rangle \sum_{g\in N/N_\theta}\theta^g(x),$$

since we have assumed that every $\varphi \in \mathrm{Irr}(L)$ contributing to the sum in (2.3) is conjugate to θ.

We make the identification $\mathrm{W}_{\mathrm{C}_{G^*}(s)}(\mathrm{C}^\circ_{\mathbf{L}^*}(s))_\lambda \cong \mathrm{W}_G(\mathbf{L})_\theta$ as in (2.4). By combining [Ma07, Thm 4.6(b)], Broué–Malle–Michel's Comparison Theorem [GM20, 4.6.21], and (2.5), we have

$$\begin{aligned}\pm\eta(1) = \pm\langle\eta, \mathrm{Ind}_1^{\mathrm{W}_G(\mathbf{L})_\theta}(1)\rangle &= \langle J_s^{\mathbf{G}}(\chi), \mathrm{Ind}_{\mathrm{C}^\circ_{G^*}(s)}^{\mathrm{C}_{G^*}(s)}(R_{\mathrm{C}^\circ_{\mathbf{L}^*}(s)}^{\mathrm{C}^\circ_{\mathbf{G}^*}(s)}(\lambda))\rangle \\ &= \langle J_s^{\mathbf{G}}(\chi), R_{\mathrm{C}_{\mathbf{L}^*}(s)}^{\mathrm{C}_{\mathbf{G}^*}(s)}(\mathrm{Ind}_{\mathrm{C}^\circ_{L^*}(s)}^{\mathrm{C}_{L^*}(s)}(\lambda))\rangle.\end{aligned}$$

Now, recall from the proof of Proposition 2.7 that if $\langle J_s^{\mathbf{G}}(\chi), R_{\mathrm{C}_{\mathbf{L}^*}(s)}^{\mathrm{C}_{\mathbf{G}^*}(s)}(\hat\lambda)\rangle \neq 0$ for some $\hat\lambda \neq J_s^{\mathbf{L}}(\theta)$ lying above λ, then $\hat\lambda = J_s^{\mathbf{L}}(\theta')$ for some $\widetilde{L}$-conjugate θ' of θ. But since (1) of

Proposition 2.7 holds, we then have $\langle \chi, R_{\mathbf{L}}^{\mathbf{G}}(\theta')\rangle \neq 0$. From our assumption that Cabanes–Enguehard's conjecture holds under χ, this means θ' is further N-conjugate to θ. That is, $\theta' = \theta^g = \theta^n$ for $g \in \widetilde{L}$ and $n \in \mathrm{N}_G(\mathbf{L})$. Then note that s^{n^*} is L-conjugate to s (where $n^* \in N^*$ is associated to n as before), as θ and θ^n lie under the same character $\widetilde{\theta}$ of $\widetilde{L}$. Hence $n^* \in \mathrm{C}_{N^*}(s)L^*$. Applying the equivariance of Jordan decomposition from [CS13, Thm 3.1] to $\widetilde{L}$, we see that since $\widetilde{\theta}^n$ corresponds to λ^{n^*} but also to λ (since $J_s^{\mathbf{L}}(\theta^n)$ lies over λ), it follows that in fact $n^* \in \mathrm{C}_{N^*}(s)_\lambda L^*$. Hence by the isomorphism (2.4), we have $\theta^n = \theta$. That is, $\theta' = \theta$ and $\hat{\lambda} = J_s^{\mathbf{L}}(\theta)$. Together with the computation above, this yields

$$\pm\eta(1) = \langle J_s^{\mathbf{G}}(\chi), R_{\mathrm{C}_{\mathbf{L}^*}(s)}^{\mathrm{C}_{\mathbf{G}^*}(s)}(J_s^{\mathbf{L}}(\theta))\rangle = \langle \chi, R_{\mathbf{L}}^{\mathbf{G}}(\theta)\rangle,$$

where we again have used that (1) of Proposition 2.7 holds.

All together, this gives

$$\chi(x) = \pm\eta(1) \sum_{g\in N/N_\theta} \theta^g(x) = \pm\Omega(\chi)(x),$$

as stated.

For the last statement, recall that, as noted in Remark 2.8, (1) of Proposition 2.7 holds when d is a regular number (i.e., when $\mathbf{L}$ is a torus). Further, in that case generalised d-Harish-Chandra theory holds by properties of Deligne–Lusztig characters (see [GM20, Cor. 2.2.10]). $\square$

2.5. **Suzuki and Ree groups.** We now assume that F is a Steinberg, but not Frobenius, morphism. That is, G is a Suzuki or Ree group.

Proposition 2.13. *The Picky+ Conjecture holds for any quasi-simple group H such that $S = H/\mathbf{Z}(H)$ is a Suzuki or Ree group ($^2\mathrm{B}_2(2^{2f+1})$ with $f \geq 1$, $^2\mathrm{G}_2(3^{2f+1})'$ with $f \geq 0$ or $^2\mathrm{F}_4(2^{2f+1})'$ with $f \geq 0$) for all primes ℓ distinct from the defining prime. (That is, $\ell \neq 2$, 3, resp. 2.)*

Proof. For any S as in the statement, the Schur multiplier of S is trivial unless $S = {}^2\mathrm{B}_2(8)$, in which case it is a Klein four group. In the case $S = {}^2\mathrm{B}_2(8)$, the Sylow ℓ-subgroups of S, and hence of H, are cyclic so this case is covered by Lemma 2.3. So, we may assume that S is its own universal covering group.

Further, we note that the case $S = {}^2\mathrm{G}_2(3)' \cong \mathrm{PSL}_2(8)$ has been handled above in Proposition 2.9, so we may assume S is not this group. If $S = {}^2\mathrm{F}_4(2)'$, then $S \lhd G := {}^2\mathrm{F}_4(2)$ with index 2. In particular, S and G have the same picky ℓ-elements. (See [Ma25, Lemma 2.3].) From here, the statement can be completed by checking with GAP. Hence, we may assume further that $S \neq {}^2\mathrm{F}_4(2)'$.

Then from now on, we may assume S is of the form $G = \mathbf{G}^F$ with $\mathbf{G}$ simple of simply connected type and F a Steinberg morphism. By [Ma25, Thm 5.11], here an ℓ-element $x \in G$ is picky if and only if $\ell \geq 5$ and x is a regular ℓ-element. For $^2\mathrm{B}_2(q^2)$ and $^2\mathrm{G}_2(q^2)$, the Sylow ℓ-subgroups are then cyclic, so we are again done by Lemma 2.3.

Hence, we are left to consider the group $G = {}^2\mathrm{F}_4(q^2)$, where $q^2 = 2^{2f+1}$. Here the Sylow ℓ-subgroups are again abelian, so Lemma 2.1 and Remark 2.2 apply. As noted in the proof of [Ma14, Thm 5.1], the conclusion of Lemma 2.4 still holds here, taking $\mathbf{L}$ to be a minimal Φ-split Levi subgroup as defined in loc. cit. The structure of $L = \mathbf{L}^F$ (which is abelian) and $N = \mathrm{N}_G(P) = \mathrm{N}_G(\mathbf{L})$ is given in [Ma14, Tab. 2], and a bijection analogous to Ω above is given in [Ma07, Thm 8.5]. Note that again $\mathrm{C}_{\mathbf{G}}(x) = \mathbf{L}$. From here, the same considerations as in Sections 2.1–2.3 yield the statement for $G = {}^2\mathrm{F}_4(q^2)$, where $q^2 = 2^{2f+1}$ with $f \geq 1$. □

3. Picky 2-Elements

In this section, we complete the determination of quasi-simple groups of Lie type with picky semisimple ℓ-elements started in [Ma25] by considering the remaining open case $\ell = 2$. Throughout, $\mathbf{G}$ is a simple linear algebraic group of simply connected type over an algebraically closed field of odd characteristic $p > 2$, $F : \mathbf{G} \to \mathbf{G}$ is a Frobenius map with respect to an $\mathbb{F}_q$-structure and $G := \mathbf{G}^F$. In order to determine picky 2-elements, we make use of the fact that Sylow 2-subgroups of G are self-normalising most of the time (see [Ko85, Cor.]):

Proposition 3.1. *Let G be as above. Then a Sylow 2-subgroup P of G is self-normalising except when one of:*
(1) *$G = \mathrm{SL}_n(\epsilon q)$ with $\epsilon \in \{\pm 1\}$, $n \geq 3$, where $\mathrm{N}_G(P)/P \cong C_{(q-\epsilon)_{2'}}^{t-1}$;*
(2) *$G = \mathrm{Sp}_{2n}(q)$ with $n \geq 1$ and $q \equiv 3, 5 \pmod 8$, where $\mathrm{N}_G(P)/P \cong C_3^t$; or*
(3) *$G = \mathrm{E}_6(\epsilon q)$ with $\epsilon \in \{\pm 1\}$, where $\mathrm{N}_G(P)/P \cong C_{(q-\epsilon)_{2'}}$.*
Here, t in (1) and (2) denotes the number of non-zero digits in the 2-adic expansion of n.

Lemma 3.2. *Let G be as above and $x \in G$ be a picky 2-element. Then x lies in an F-stable maximal torus $\mathbf{T}$ of $\mathbf{G}$ with $|\mathbf{T}^F| = (q-1)^a(q+1)^b$ for some $a, b \geq 0$. Moreover, x lies in a unique F-stable maximal torus of $\mathbf{G}$ except possibly when $G = \mathrm{Sp}_{2n}(3)$ for some $n \geq 1$.*

Proof. Since x is semisimple, it is contained in some F-stable maximal torus $\mathbf{T}$ of $\mathbf{G}$. The order polynomial of $\mathbf{T}$ is a product of cyclotomic polynomials. Assume it is divisible by Φ_d for some $d > 2$. Since q is odd, by Zsigmondy's theorem there exists a primitive prime divisor $r > d > 2$ of $\Phi_d(q)$ and hence of $|\mathbf{T}^F|$, so of $|\,\mathrm{C}_G(x)|$. But by Proposition 3.1 the order of a Sylow 2-normaliser is only divisible by prime divisors of $6(q^2-1)$, so we obtain a contradiction to [Ma25, Lemma 2.1].

Now assume x lies in at least two F-stable maximal tori of $\mathbf{G}$. Then $\mathrm{C}_{\mathbf{G}}(x)$ is not a torus, so has positive semisimple rank. In particular $\mathrm{C}_{\mathbf{G}}(x)$ contains unipotent elements, and then so does $\mathrm{C}_G(x)$. But again by Proposition 3.1 the order of a Sylow 2-normaliser is prime to p unless $\mathbf{G} = \mathrm{Sp}_{2n}$ and $p = 3$. So assume $G = \mathrm{Sp}_{2n}(q)$ with $q = 3^f$. As $\mathrm{C}_{\mathbf{G}}(x)$ has semisimple rank at least 1, then $\mathrm{C}_G(x)$ contains an A_1-type subgroup, of order divisible by $(q^2-1)/2$. Now note that $(q^2-1)/2$ is prime to 3, whence again using Proposition 3.1, [Ma25, Lemma 2.1] forces $q^2 - 1$ to be a 2-power, which only holds when $q = 3$. □

Proposition 3.3. *Assume that G is one of $\mathrm{SL}_n(\epsilon q)$ with $n \geq 6$, $\mathrm{Sp}_{2n}(q)$ with $n \geq 6$, $\mathrm{E}_6(\epsilon q)$, $\mathrm{E}_7(q)$, or $\mathrm{E}_8(q)$. Then G possesses no picky 2-element.*

Proof. For each group in the hypothesis, we will exhibit a subgroup $H \leq G$ such that
(1) H contains representatives for all classes of maximal tori of G of order $(q-1)^a(q+1)^b$;
(2) $\mathrm{N}_G(H)$ contains a Sylow 2-subgroup of G; and
(3) the Sylow 2-subgroups of $\mathrm{N}_G(H)/H$ are not normal.

If such H exists, then by Lemma 3.2 and (1) any class of picky 2-elements has a representative in H; but by (2) and (3) any such lies in at least two Sylow 2-subgroups of G, whence no picky 2-element can exist.

To ensure (1), we let $H = \mathbf{H}^F$ for a suitably chosen F-stable connected reductive subgroup $\mathbf{H} \le \mathbf{G}$ containing a maximal torus of $\mathbf{G}$ such that the Weyl group of $\mathbf{H}$ contains representatives of all F-classes of elements of order at most 2 in the Weyl group W of $\mathbf{G}$. Then, by [MT11, Prop. 25.3], H will contain the F-fixed points of representatives for all classes of F-stable maximal tori of $\mathbf{G}$ of orders $(q-1)^a(q+1)^b$. Condition (2) will be checked from the order formulas, and (3) by inspection.

We consider the various types in turn. First assume $G = \mathrm{SL}_n(\epsilon q)$ with $n \ge 6$, and let $m := \lfloor n/2 \rfloor$. Then $W \cong \mathfrak{S}_n$ contains $m+1$ conjugacy classes of elements of order at most 2, all of which possess representatives in the natural Young subgroup $\mathfrak{S}_2^m$, the Weyl group of the standard Levi subgroup H of type $\mathrm{GL}_2(\epsilon q)^m$ of G. Thus H satisfies (1). Now $N := \mathrm{N}_G(H)$ contains a Sylow 2-subgroup of G, and $N/H \cong \mathfrak{S}_m$, see [GLS98, Thm 4.10.6 and Tab. 4.10.6]. Since the Sylow 2-subgroups of $\mathfrak{S}_m$ are not normal for $m \ge 3$, we are done.

For $G = \mathrm{Sp}_{2n}(q)$ write $n = 2m + c$ with $c \in \{0, 1\}$ and let H be a natural subgroup $\mathrm{Sp}_4(q)^m \times \mathrm{Sp}_2(q)^c$. Observe that the Weyl group of H contains representatives of all involution classes in W; furthermore $\mathrm{N}_G(H) \ge \mathrm{Sp}_4(q) \wr \mathfrak{S}_m \times \mathrm{Sp}_2(q)^c$ satisfies (2). For $m \ge 3$, so $n \ge 6$, $\mathfrak{S}_m$ has non-normal Sylow 2-subgroups.

For $G = \mathrm{E}_6(\epsilon q)$ let $H = (q-\epsilon)^2.\mathrm{A}_1(q)^4$, the centraliser of a maximal commuting product of fundamental SL_2-subgroups (see [GLS98, Tab. 4.10.6]). Then H contains representatives of all five classes of maximal tori of G of order $(q-\epsilon)^a(q+\epsilon)^b$, $2 \le a \le 6$, and (2) is satisfied as $\mathrm{N}_G(H)/H \cong \mathfrak{S}_4$. This also shows (3).

For $G = \mathrm{E}_7(q)$ we choose a subgroup $H = \mathrm{A}_1(q)^3.\mathrm{D}_4(q)$, with $\mathrm{N}_G(H)/H \cong \mathfrak{S}_3$ and for $G = \mathrm{E}_8(q)$ a subgroup $H = \mathrm{D}_4(q)^2$, with $\mathrm{N}_G(H)/H \cong \mathfrak{S}_3 \times 2$ (see [MT11, Tab. p. 168]). By direct computation in Chevie, these satisfy (1), and then (2) and (3) are also seen to hold. □

Proposition 3.4. *Assume G contains a picky 2-element. Then one of the following holds:*
(1) $q = 3$;
(2) $G = \mathrm{SL}_n(\epsilon q)$ *for* $n \in \{3, 5\}$ *with* $q \equiv -\epsilon \pmod 4$;
(3) $G = \mathrm{Sp}_{2n}(5)$ *with* $n \le 5$; *or*
(4) $G = \mathrm{SL}_2(q)$ *or* $\mathrm{Sp}_4(q)$.

Proof. Let $x \in G$ be a picky 2-element. By Proposition 3.3 we may assume G is not $\mathrm{SL}_n(\epsilon q)$ with $n \ge 6$ or $\mathrm{E}_6(\epsilon q)$. By Lemma 3.2, x lies in some maximal torus T of G of order $(q-1)^a(q+1)^b$. We first assume $q \equiv 1 \pmod 4$. Then $m := (q+1)/2 \ge 3$ is odd and prime to $q-1$. By Proposition 3.1 the order of the normaliser of a Sylow 2-subgroup of G is not divisible by m unless $G = \mathrm{SU}_n(q)$ with n not a 2-power, or $\mathrm{Sp}_{2n}(5)$. So except for those latter cases we conclude $|T| = (q-1)^n$ using [Ma25, Lemma 2.1]. If on the other hand $q \equiv -1 \pmod 4$ and $q > 3$ then $m := (q-1)/2 \ge 3$ is odd and prime to $q+1$, and so unless $G = \mathrm{SL}_n(q)$ with n not a 2-power we infer that $|T| = (q+1)^n$.

Thus, unless we are in one of the exceptions in (1)–(3), for $\mathbf{T} \le \mathbf{G}$ an F-stable maximal torus with $\mathbf{T}^F = T$, by [Ma07, Prop. 5.20] the normaliser $N := \mathrm{N}_G(\mathbf{T})$ contains a Sylow 2-subgroup of G. We have thus shown that $x \in T \trianglelefteq N$ with $N/T = W$ the relative Weyl group of a Sylow d-torus of $\mathbf{G}$, where $d = 1$ if $q \equiv 1 \pmod 4$ and $d = 2$ otherwise. If W has non-normal Sylow 2-subgroups then x cannot be picky in N and hence neither in G. If W has normal Sylow 2-subgroups, it is of type A_1 or B_2 by [Ma25, Prop. 5.5], so G is $\mathrm{SL}_2(q)$ or $\mathrm{Sp}_4(q)$ as in (4). □

We now discuss the cases left open in the previous result.

Remark 3.5. The picky ℓ-elements of a given finite permutation group or matrix group over a finite field can be determined effectively using the criterion in [Ma25, Cor. 2.10], which we have implemented in the GAP system [GAP]. This was used to deal with some of the small cases below.

Lemma 3.6. *Let $G = \mathrm{SL}_3(\epsilon q)$ with $q \equiv -\epsilon \pmod 4$. Then G contains picky 2-elements if and only if q is a Fermat or Mersenne prime, or $q = 9$.*

Proof. By Lemma 3.2 any picky 2-element $x \in G$ lies in a unique maximal torus T of G, of order $(q-1)^a(q+1)^b$. If $|T| = (q-\epsilon)^2$ then $|\operatorname{C}_G(x)|$ is divisible by $(q-\epsilon)_{2'}^2$, which is only possible for $q = 3$ by Proposition 3.1. Otherwise, $|T| = q^2 - 1$, which is divisible by an odd prime not dividing $q - \epsilon$ unless $q + \epsilon$ is a 2-power, so q is a Mersenne or Fermat prime, or $q = 9$. In the latter case, the normaliser of T contains a Sylow 2-subgroup P of G and $\mathrm{N}_G(P) = \mathrm{N}_G(T) = T.2$ by Proposition 3.1. Now let $x \in T$ be a generator of the cyclic Sylow 2-subgroup of T. Then $\langle x\rangle$ has index 2 in P, so is normal in any Sylow 2-subgroup containing it. In particular, any such Sylow 2-subgroup is contained in $\mathrm{N}_G(T_2) = \mathrm{N}_G(T) = \mathrm{N}_G(P)$ and so equals P. Hence x is picky. □

The next assertion is also shown in [MMM, Thm 6.1]; we provide a short proof for completeness.

Lemma 3.7. *Let $G = \mathrm{SL}_2(q)$ with $q \geq 5$ odd. Then G contains a picky 2-element if and only if q is a Fermat or Mersenne prime or $q = 9$.*

Proof. Any 2-element of G is contained in a maximal torus of G of order $q-\epsilon$ for $\epsilon \in \{\pm 1\}$ with $q \equiv \epsilon \pmod 4$. First assume $q \equiv \epsilon \pmod 8$. Then the Sylow 2-normalisers in G are dihedral of order $2(q-\epsilon)_2$. If q is neither a Fermat or a Mersenne prime nor $q = 9$, then $(q-\epsilon)_2 < q - \epsilon$ and thus there are no picky 2-elements by [Ma25, Lemma 2.1]. Now assume $q \equiv 4 + \epsilon \pmod 8$. In this case, the Sylow 2-normaliser is isomorphic to $\mathrm{SL}_2(3)$. Since the largest element order in that group is 6, we conclude $q - \epsilon \leq 6$ if G has a picky 2-element. This forces $q = 5$, a Fermat prime.

Conversely, arguing similarly to Lemma 3.6, it is easy to see that for $q \geq 5$ a Fermat or Mersenne prime or $q = 9$, any 2-element of G of order at least 4 lies in a unique Sylow 2-subgroup. □

Lemma 3.8. *Let $G = \mathrm{Sp}_4(q)$ with q odd. Then G contains picky 2-elements if and only if $q \neq 5, 7$ is a Fermat or Mersenne prime.*

Proof. Since $\mathrm{Sp}_4(3) \cong 2.\mathrm{SU}_4(2)$ the claim in this case follows from [Ma25, Prop. 3.2], while $\mathrm{Sp}_4(5)$, $\mathrm{Sp}_4(7)$ and $\mathrm{Sp}_4(9)$ have no picky 2-element by explicit computation. So now assume $q \geq 11$ and $q \equiv \epsilon \pmod 4$ with $\epsilon \in \{\pm 1\}$. The maximal tori of $G = \mathrm{Sp}_4(q)$ have orders $(q \pm \epsilon)^2$, $q^2 - 1$ and $q^2 + 1$, so using Proposition 3.1 and [Ma25, Lemma 2.1] we check that any picky 2-element $x \in G$ must be contained in a torus T of order $(q-\epsilon)^2$ and $q-\epsilon$ necessarily is a 2-power.

Thus $q - \epsilon = 2^f$ with $f \geq 4$. The normaliser $N := \mathrm{N}_G(T) = T.W(\mathrm{B}_2)$ of T is the normaliser of a Sylow 2-subgroup of G. Now by explicit computation all elements of N lying in one of the seven non-trivial cosets of T either have some fourth or eighth root of unity as eigenvalue, or their set of eigenvalues is closed under negation. Thus, for $x \in T$ an element whose set of eigenvalues consists of 16th roots of unity but is not closed under negation, the only G-conjugates of x in N are its eight N-conjugates in T, and thus x is picky by [Ma25, Lemma 2.9]. □

We thus reach the following classification of groups with a picky 2-element:

Theorem 3.9. *Let G be as above. Then G contains a picky 2-element if and only if G is one of*

(1) $\mathrm{SL}_2(q)$ *where q is a Fermat or Mersenne prime or $q = 9$;*
(2) $\mathrm{SL}_3(q)$ *with $q \equiv 3 \pmod 4$ where q is a Mersenne prime;*
(3) $\mathrm{SU}_3(q)$ *with $q \equiv 1 \pmod 4$ where q is a Fermat prime or $q = 9$;*
(4) $\mathrm{Sp}_4(q)$ *where $q \neq 5, 7$ is a Fermat or Mersenne prime;*
(5) $\mathrm{SL}_4(3), \mathrm{SU}_3(3), \mathrm{SU}_4(3), \mathrm{Spin}_7(3), \mathrm{Sp}_6(3), \mathrm{Spin}_8^{\pm}(3), \mathrm{G}_2(3)$.

Proof. We go though the cases in Proposition 3.4. If $q = 3$ first assume $G = \mathrm{Sp}_{2n}(3)$ with $n \geq 3$. Let $x \in G$ be a picky 2-element. If x has order at least 16 then its centraliser contains a cyclic subgroup $\mathrm{GL}_1(80)$ and thus has order divisible by 5, contradicting Proposition 3.1. Furthermore, if the eigenspace of x for some 8th root of unity has dimension at least 2 the centraliser contains a subgroup $\mathrm{SL}_2(9)$, again of order divisible by 5. Now let V be the subspace of the natural module for G generated by the eigenspaces of x for eigenvalues of order at most 4 and H the (symplectic) group induced by G on V. Since $\mathrm{Sp}_4(3)$ has no picky elements of order 2 or 4, we conclude that G has no picky 2-elements if $2n \geq 8$. The same reasoning applies for $\mathrm{Spin}_{2n+1}(3)$ and $\mathrm{Spin}_{2n}^{\pm}(3)$, using that none of $\mathrm{SO}_5(3)$ and $\mathrm{GO}_6^{\pm}(3)$ possesses picky 2- or 4-elements, to conclude that $2n+1 \leq 7$, respectively $2n \leq 8$. Using Proposition 3.3 for the remaining types we then arrive at the groups listed in (5), $\mathrm{SL}_3(3)$ and $\mathrm{Sp}_4(3)$ in (2) and (4), and the groups

$$\mathrm{SL}_5(3), \mathrm{SU}_5(3), {}^3\mathrm{D}_4(3), \mathrm{F}_4(3).$$

By calculations in GAP [GAP], none of the latter different from $\mathrm{F}_4(3)$ possesses picky 2-elements. For $G = \mathrm{F}_4(3)$ one observes that none of its maximal tori of order $(q-1)^a(q+1)^b$ does contain regular elements (this can also be checked from the data at [Lue]). Thus the centraliser of any 2-element $x \in G$ contains unipotent elements, while the Sylow 2-subgroups are self-normalising by Proposition 3.1, and so x cannot be picky.

Next let $G = \mathrm{SL}_5(\epsilon q)$ with $q \equiv -\epsilon \pmod 4$. By Lemma 3.2 any picky 2-element $x \in G$ lies in a unique maximal torus T of G, of order $(q-1)^a(q+1)^b$. Now $|T|$ is divisible by $(q-\epsilon)^2$, so $|\,\mathrm{C}_G(x)|$ is divisible by $(q-\epsilon)_{2'}^2$ which is only possible for $q = 3$ by Proposition 3.1, a case we already discussed. The case of $\mathrm{SL}_3(\epsilon q)$ was dealt with in Lemma 3.6, leading to (2) and (3).

The argument for $\mathrm{Sp}_{2n}(5)$, $n \geq 3$, is completely analogous to the one for $\mathrm{Sp}_{2n}(3)$ above, showing that only $n > 3$ is not possible. Again by explicit computation, $\mathrm{Sp}_6(5)$ has no picky 2-element. The case of $\mathrm{SL}_2(q)$ is discussed in Lemma 3.7. Finally, $G = \mathrm{Sp}_4(q)$ leads to (4) by Lemma 3.8. □

The question of whether there are infinitely many groups G as in the previous statement hence hinges on the number theoretic question of existence or non-existence of infinitely many Fermat or Mersenne primes.

4. The Picky Conjecture for $\ell \leq 3$ and for exceptional covering groups

We now complete the proofs of Theorems 4 and 5 and derive Corollary 6.

4.1. **The case $\ell = 2$.** To prove Theorem 5, we first explicitly classify all picky semisimple 2-elements:

Lemma 4.1. *Let G be the full covering group of one of the groups in Theorem 3.9. Then the picky 2-elements $x \in G$ are as follows:*
(1) *in $\mathrm{SL}_2(q)$ with $3 < q = 2^n \pm 1$: any 2-element of order at least 4;*
(2) *in $\mathrm{SL}_3(q)$ with $q = 2^n - 1$: any 2-element of order at least 4;*
(3) *in $\mathrm{SU}_3(q)$ with $q = 2^n + 1$: any 2-element of order at least 4;*
(4) *in $\mathrm{Sp}_4(q)$ with $5, 7 \neq q = 2^n \pm 1$: certain regular 2-elements of order at least 8; or*
(5) *$G, o(x)$ and $\mathrm{C}_G(x)$ are as in Table 1.*

Table 1. Picky 2-elements

G	$o(x)$	$\lvert \mathrm{C}_G(x)\rvert$	$\lvert \mathrm{Irr}^x_{2^i}(G)\rvert$
$\mathrm{SL}_4(3)$	8	16 (2×)	12
$\mathrm{SU}_3(3)$	8	8 (2×)	8
$3_i.\mathrm{SU}_4(3)$	8	96 (4×)	$24, 6, 24$
$6.\Omega_7(3)$	8	96 (2×)	$48, 0, 0, 12$
	8	48	48
$\mathrm{Sp}_6(3)$	8	192 (2×)	$32, 24$
	8	32	32
$\Omega_8^+(3)$	8	64 (2×)	$16, 0, 4, 8$
	8	32	$16, 0, 4$
$2.\Omega_8^-(3)$	8	64 (2×)	$32, 0, 0, 8$
$3.\mathrm{G}_2(3)$	8	24 (2×)	24

In the column headed $\lvert \mathrm{Irr}^x_{2^i}(G)\rvert$ we give the number of characters in $\mathrm{Irr}^x(G)$ with 2-part of the degree equal to 2^i, for $i = 0, 1, 2, \ldots$

Proof. This is obtained by explicit computation in these matrix groups in GAP. □

Lemma 4.2. *The Strong Picky Conjecture holds for all picky 2-elements of the groups listed in Table 1.*

Proof. This can be checked in GAP. The relevant numbers of non-vanishing characters and the 2-parts of their degrees are displayed in Table 1. □

Lemma 4.3. *The Strong Picky Conjecture holds for $G = \mathrm{SL}_2(q)$, q odd, for any picky 2-element $x \in G$.*

Proof. From Lemma 4.1 and the proof of Lemma 3.7, if G has a picky 2-element $x \in P$ with $P \in \mathrm{Syl}_2(G)$, then either $G = \mathrm{SL}_2(5)$ and $\mathrm{N}_G(P) \cong \mathrm{SL}_2(3)$ or $q = 2^n + \epsilon$ with $n \geq 3$ and $\epsilon \in \{\pm 1\}$, and $\mathrm{N}_G(P) \cong \mathrm{Dih}_{2^{n+1}}$. In either case, recall that any 2-element with $|x| \geq 4$ is picky.

If $q = 5$, we see directly from GAP that there is a unique class of picky elements (of order 4) and both G and $\mathrm{N}_G(P)$ have four characters that do not vanish on this class, all having odd degree and taking values in $\{\pm 1\}$ on x for both groups.

Now let $q = 2^n + \epsilon$ with $n \geq 3$, and let $x \in P$ be a 2-element of order larger than 2. From the well-known character tables of G and $\mathrm{Dih}_{2^{n+1}}$, we see that both groups have four characters (all of odd degree) that take values ± 1 on x and a family χ_k of characters with $1 \leq k < (q-\epsilon)/2$ that take values $\pm(\zeta^k + \zeta^{-k})$ with ζ a root of unity satisfying $|x| = |\zeta|$. The latter family has degree 2 in $\mathrm{N}_G(P)$ and degree $q+\epsilon$ in G, so that $\chi_k(1)_2 = 2$ in both groups. Hence we see that (1) and (2*) hold. □

Lemma 4.4. *The Strong Picky Conjecture holds for any picky 2-element $x \in G = \mathrm{SL}_3(\epsilon q)$ with $\epsilon \in \{\pm 1\}$ and q odd.*

Proof. From Lemma 4.1, we see that if picky 2 elements exist, then $q = 2^n - \epsilon$ is a prime or $G = \mathrm{SU}_3(9)$, and then any 2-element of order larger than 2 is picky.

From the considerations in the proof of Lemma 3.6, we have $\mathrm{N}_G(P) = T.2 \cong C_{q^2-1}.2$. Here the order-2 action of $\mathrm{N}_G(P)/T$ on the cyclic group T is given by $a \mapsto a^{\epsilon q}$. Let x have order $|\zeta_2^a|$, where ζ_2 is a primitive (q^2-1)th root of unity. Now, from the character table in Chevie [Chv], we see the characters that are non-vanishing on x are $1_G, \mathrm{St}_G$, which have odd degrees and take values ± 1 on x; two families of characters of odd degree indexed by $1 \leq k < \frac{q-\epsilon}{2}$, which take values $\pm\zeta_2^{ak(q+\epsilon)}$ on x; and one family of characters indexed by $1 \leq k < q^2 - 1$ with $(q+\epsilon) \nmid k$ whose degree is divisible by 2 exactly once, which take values $\pm(\zeta_2^{ak} + \zeta_2^{-ak})$ on x. Then we see using Clifford theory that we have a map satisfying (1) and (2*) by mapping the two extensions of $1_{C_{q^2-1}}$ to $\{1_G, \mathrm{St}_G\}$; the extensions of the characters that are stable under $\mathrm{N}_G(P)/T$ to the two other families of odd-degree characters in $\mathrm{Irr}^x(G)$; and the characters of $\mathrm{N}_G(P)$ that do not restrict irreducibly to the remaining family in $\mathrm{Irr}^x(G)$. □

Lemma 4.5. *The Strong Picky Conjecture holds for $G = \mathrm{Sp}_4(q)$, q odd, for any picky 2-element $x \in G$.*

Proof. From Lemma 4.1 and the considerations in the proof of Lemma 3.8, we have $q = 2^n + \epsilon$ with $\epsilon \in \{\pm 1\}$ and $n \geq 4$ and x is a regular 2-element lying in $T \cong C_{q-\epsilon}^2$. We further have $P = \mathrm{N}_G(P) = T.W(\mathrm{B}_2) \cong (C_{q-\epsilon}.2) \wr C_2$. Here the action of $C_{q-\epsilon}.2 \setminus C_{q-\epsilon}$ is by inversion. From here, using the character table for G constructed by Srinivasan [Sri68] and (somewhat tedious) Clifford theory arguments for P, we again obtain the result. □

Lemmas 4.2–4.5 complete the proof of Theorem 5, when combined with Theorem 3.9 and Lemma 4.1.

4.2. **The case $\ell = 3$.** The simply connected quasi-simple groups of Lie type with non-abelian Sylow 3-subgroups containing picky semisimple 3-elements were determined in

[Ma25, Thm 5.10]. By explicit computation in these groups in GAP, one obtains the following:

Lemma 4.6. *Let G be the full covering group of one of the quasi-simple groups in* [Ma25, Thm 5.10]. *Then the picky 3-elements $x \in G$, their orders and centralisers are as in Table 2.*

TABLE 2. Picky 3-elements

G	$o(x)$	$\lvert \mathrm{C}_G(x)\rvert$	$\lvert \mathrm{Irr}^x_{3^i}(G)\rvert$
$4_i.\mathrm{SL}_3(4)$	3	36	24
$\mathrm{SU}_3(8)$	9	81 (9×)	6, 21
$2.\mathrm{SU}_4(2)$	9	18 (2×)	18
$\mathrm{SU}_5(2)$	9	54 (2×), 27 (2×)	27
$6.\mathrm{SU}_6(2)$	9	324 (6×)	36, 72
	9	54	36
$\mathrm{SU}_7(2)$	9	81	54
$\mathrm{SU}_8(2)$	9	486, 243	162
$2.\mathrm{Sp}_6(2)$	9	18	18
$\mathrm{Sp}_8(2)$	9	54, 27	27
$\mathrm{Sp}_{10}(2)$	9	162 (2×)	81
$2.\Omega_8^+(2)$	9	54 (3×)	36
$\Omega_{10}^+(2)$	9	27	27
$\Omega_8^-(2)$	9	9	9
$\Omega_{10}^-(2)$	9	324, 162 (2×), 81	54
$\Omega_{12}^-(2)$	9	162	81
$\mathrm{G}_2(8)$	9	81 (3×)	9, 15
${}^3\mathrm{D}_4(2)$	9	54 (3×)	9, 6
$2.\mathrm{F}_4(2)$	9	108 (2×)	54

Again, in the column headed $\lvert \mathrm{Irr}^x_{3^i}(G)\rvert$ we give the number of characters in $\mathrm{Irr}^x(G)$ with 3-part of the degree equal to 3^i, for $i = 0, 1, 2, \ldots$. Note that we do not consider the non-perfect group $\mathrm{G}_2(2) \cong \mathrm{SU}_3(3).2$.

Lemma 4.7. *The Strong Picky Conjecture holds for all picky 3-elements of the groups listed in Table 2.*

Proof. All relevant character tables are contained in or can be computed by GAP, except for the tables of $\mathrm{SU}_8(2)$ and of $G_2(8)$. The generic character table of $G_2(q)$ can be found in [Chv]. From this, by specialising q to 8, one finds nine irreducible $3'$-characters and 15 characters χ with $\chi(1)_3 = 3$ for $G = G_2(8)$, as well as for $\mathrm{N}_G(P)$, satisfying the assertions of Conjecture 1^+. Finally, for $G = \mathrm{SU}_8(2)$ we use Lusztig's description of the irreducible characters. The picky 3-elements lie in a maximal torus T of order $\Phi_2^3\Phi_6^2$. Any irreducible character of G taking non-zero value on one of those elements has to lie in a Lusztig series $\mathcal{E}(G, s)$ for which the parametrising semisimple element $s \in G^* = \mathrm{PgU}_8(2) \cong G$ contains a torus T^* dual to T, so lies in T^*. Using the 8-dimensional matrix representation of G we determine all such elements s of order at most 3 up to G-conjugation. The values of the characters in $\mathcal{E}(G,s)$ on the two classes of picky 3-elements can then be determined by the character formula. The values, up to sign, are collected in Table 3. Here, the second column gives the number of elements with the given centraliser, the next two columns show the multiplicities of values (up to sign) on representatives x_1, x_2 of the two classes of picky 3-elements. Since the squares of these values add up to the centraliser order of x_i in G, all other irreducible characters of G vanish on both x_i. By direct computation, we obtain the same values up to sign (with multiplicities) as for the normaliser of a Sylow 3-subgroup of G.

The relevant numbers of non-vanishing characters for all groups in question are displayed in Table 2. $\square$

Table 3. Non-vanishing characters of $\mathrm{SU}_8(2)$

$C_G(s)$	#	$\lvert\chi(x_1)\rvert$	$\lvert\chi(x_2)\rvert$
${}^2A_7(q)$	1	$1\,(\times 6), 2\,(\times 9), 4\,(\times 3)$	$1\,(\times 12), 2\,(\times 6)$
$\Phi_2.{}^2A_6(q)$	2	$1\,(\times 6), 2\,(\times 3)$	$1\,(\times 6), 2\,(\times 3)$
$\Phi_2.A_1(q).{}^2A_5(q)$	2	$1\,(\times 6), 2\,(\times 9), 4\,(\times 3)$	$1\,(\times 12), 2\,(\times 6)$
$\Phi_2^2.{}^2A_5(q)$	1	$1\,(\times 6), 2\,(\times 3)$	$1\,(\times 6), 2\,(\times 3)$
$\Phi_2.{}^2A_2(q).{}^2A_4(q)$	2	$1\,(\times 9), 2\,(\times 9)$	$1\,(\times 18)$
$\Phi_2.{}^2A_3(q)^2$	1	$1\,(\times 9)$	$1\,(\times 9)$
$\Phi_2^2.{}^2A_2(q).{}^2A_3(q)$	2	$1\,(\times 9)$	$1\,(\times 9)$
$\Phi_2^2.A_1(q).{}^2A_2(q)^2$	1	$1\,(\times 9), 2\,(\times 9)$	$1\,(\times 18)$

Lemma 4.7 completes the proof of Theorem 4, when combined with Propositions 2.9 and 2.13.

4.3. **Exceptional covering groups.** We now wish to prove the following:

Proposition 4.8. *Let ℓ be a prime and let S be a simple group of Lie type defined in characteristic $p \neq \ell$, such that S has an exceptional Schur multiplier. Then the Picky Conjecture holds for any quasi-simple group H such that $H/\mathbf{Z}(H) = S$.*

Proof. Let H be the universal covering group of S. There is some simple simply connected linear algebraic group $\mathbf{G}$ and Steinberg map F such that $S = G/\mathbf{Z}(G)$ for $G := \mathbf{G}^F$, and we can write $G = H/Z$ for some $Z \le \mathbf{Z}(H)$. By [Ma25, Lemma 2.4], the picky ℓ-elements of H are preimages of picky ℓ-elements of G under the natural map.

First assume that $\ell \ge 5$. By [MT11, Tab. 24.3] we note that $|Z|$ is not divisible by ℓ so we know again using [Ma25, Thm 5.10] that $P \in \mathrm{Syl}_\ell(H)$ is abelian if H has picky ℓ-elements. By Lemma 2.3 we may also assume that Sylow ℓ-subgroups of H and hence of S are non-cyclic. This only leaves the possibilities that $\ell = 5$ and $S = \Omega_8^+(2), G_2(4), F_4(2), {}^2E_6(2)$, or $\ell = 7$ and $S = F_4(2), {}^2E_6(2)$. Now, by the known character tables, in any of these groups the centralisers of ℓ-elements are strictly bigger than the centraliser of an ambient Sylow d-torus, so by [Ma25, Thm 5.3] none of them has picky ℓ-elements, and so neither has H.

For $\ell = 2, 3$ the validity of the Picky Conjecture was checked in Lemmas 4.2 and 4.7 for all groups in question apart from central quotients of $2.\mathrm{SL}_4(2)$ and $6.{}^2E_6(2)$, for $\ell = 3$. For $2.\mathrm{SL}_4(2)$ verifying the conjecture is an easy exercise using GAP, while ${}^2E_6(2)$ and hence $6.{}^2E_6(2)$ does not possess picky 3-elements by inspection of the character table. $\square$

Finally, we obtain Corollary 6:

Proof of Corollary 6. From Propositions 2.13 and 4.8, we may assume that S is not a Suzuki or Ree group and that S has a non-exceptional Schur multiplier. In this case, the universal covering group is of the form $G = \mathbf{G}^F$ with $\mathbf{G}$ simple of simply connected type and F a Frobenius morphism. If $\ell = 3$ and the Sylow ℓ-subgroups of G are non-abelian, G and all its central quotients satisfies the Picky+ Conjecture by Lemma 4.7.

Otherwise, the Sylow ℓ-subgroups of G are abelian, and the statement follows from the proof of Proposition 2.9, since Ω preserved the characters on restriction to the centre. □

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