

Odd and Even Harder Problems on Cycle-Factors

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Abstract

For a graph (undirected, directed, or mixed), a *cycle-factor* is a collection of vertex-disjoint cycles covering the entire vertex set. Cycle-factors subject to parity constraints arise naturally in the study of structural graph theory and algorithmic complexity. In this work, we study four variants of the problem of finding a cycle-factor subject to parity constraints: (1) all cycles are odd, (2) all cycles are even, (3) at least one cycle is odd, and (4) at least one cycle is even. These variants are considered in the undirected, directed, and mixed settings. We show that all but the fourth problem are NP-complete in all settings, while the complexity of the fourth one remains open for the directed and undirected cases. We also show that in mixed graphs, even deciding the existence of any cycle factor is NP-complete.

Keywords: 2-factor; parity constraints; cycle partition; complexity; cycle decomposition

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1 Introduction

Congruency constraints have a well-established role in classical combinatorial optimization. Recent interest in congruency-constrained problems has been fueled by their deep connections with integer programming (e.g. [2]), and their relevance in classical problems. For example, the problem of minimum cuts in a given graph that satisfy certain parity and congruency constraints has been studied [7, 27, 28]. Further research focused on finding paths [21, 22, 24, 25, 33] and cycles [37, 39] with congruency constraints. In general, congruency-constrained variants of well-studied combinatorial optimization problems often require a fundamentally different analysis, as these constraints can alter the nature of the problem to the extent that classical techniques no longer apply. This is exemplified by the EXACT MATCHING problem introduced by Papadimitriou and Yannakakis in 1982 [30], for which the existence of a deterministic polynomial-time algorithm remains an open question.

We study parity-restricted cycle-factors in undirected, directed, and mixed graphs. Let G be a mixed graph with edge set $E(G)$ and arc set $A(G)$; we use the term “edge” in the undirected sense and “arc” in the directed sense. We say that $P = (s = v_0, v_1, v_2, \dots, v_{k+1} = t)$ is a *mixed (s, t) -path*¹ if $v_i v_{i+1} \in E(G)$ or $v_i v_{i+1} \in A(G)$ for every $i \in \{0, 1, \dots, k\}$; intuitively, it is possible to assign consistent directions to the undirected edges $v_i v_{i+1} \in E(G)$ in P . If the first and last vertices of a mixed path coincide, we call it a *mixed cycle*. We say that a mixed path or mixed cycle is *odd* (or *even*) if the total number of edges and arcs is odd (or even). An undirected graph or directed graph (or digraph) is a mixed graph with $A(G) = \emptyset$ or $E(G) = \emptyset$, respectively. If a path consists only of edges (or arcs) we call it an undirected path (or a directed path); similarly for cycles. We denote by $V(P)$, $E(P)$, and $A(P)$ the sets of vertices, of edges, and of arcs, respectively, that form P .

For a graph (undirected, directed, or mixed), a *cycle-factor* is a collection of vertex-disjoint cycles covering the vertex set. If F denotes a cycle-factor, $\mathcal{C}(F)$ is the set of cycles contained in F . A simple combinatorial problem for finding a cycle-factor in a (undirected, directed, or mixed) graph is stated below.

CYCLE-FACTOR

Input: A graph.

Goal: Find a cycle-factor.

Cycle-factors with parity constraints naturally appear in both structural graph theory and algorithmic complexity. We study twelve problems, namely the following four problems in each of the undirected, directed, and mixed cases.

∀ODD CYCLE-FACTOR

Input: A graph.

Goal: Find a cycle-factor such that it only contains odd cycles.

∀EVEN CYCLE-FACTOR

Input: A graph.

Goal: Find a cycle-factor such that it only contains even cycles.

∃ODD CYCLE-FACTOR

Input: A graph.

Goal: Find a cycle-factor such that it contains an odd cycle.

¹All paths are assumed to be simple; that is, all vertices must be distinct except for the case of $s = t$ when it is a cycle.

$\exists$ EVEN CYCLE-FACTOR

Input: A graph.

Goal: Find a cycle-factor such that it contains an even cycle.

1.1 Related Results

Cycle-factors. The existence of a cycle-factor in undirected graphs was shown by Petersen [31] for $2k$ -regular graphs for natural k . Belck [8] and Gallai [13] were the first to characterize a cycle-factor in undirected graphs, which in turn gave polynomial-time algorithms. Their proofs imply simple algorithms in which the problem is reduced to find a perfect matching (see e.g. [34]).

It is folklore that a directed graph has a cycle-factor if and only if the undirected bipartite graph constructed as follows has a perfect matching: split each vertex v into an in-copy v_{in} and an out-copy v_{out} and replace each arc uv by an edge between u_{out} and v_{in} . Bang-Jensen, Guo, and Yeo [5] established conditions characterizing when a directed graph admits a cycle-factor, as a consequence of their work on complementary cycles, i.e. two vertex-disjoint cycles that cover all the vertices of the digraph. See e.g. [6, Sec. 5.7, 6.10] for further results for specific directed graphs such as tournaments.

By contrast, to the best of our knowledge, cycle-factors in mixed graphs have not been studied. It should be noted that a careless reduction by replacing each edge with two arcs of opposite directions does not work, since each edge cannot be used twice originally but the two arcs form a directed cycle of length 2.

C_k -free 2-matchings. A cycle-factor in undirected graphs is also called a 2-factor. A simple 2-matching is a relaxation of a 2-factor, which is an edge subset such that each vertex has at most two incident edges; if a graph has a 2-factor, it is obviously a maximum simple 2-matching. A C_k -free 2-matching in undirected graphs is a simple 2-matching which does not contain cycles of length k or less. The complexity of finding a maximum C_k -free 2-matching depends on the input graph and on k . For instance, if k is at least half of the number of vertices, computing the maximum cardinality of C_k -free 2-matching is not easier than testing the existence of a Hamiltonian cycle, which is a well-known NP-complete problem. The problem is NP-hard when $k \geq 5$ in general [11], and is polynomial-time solvable when $k = 3$ [17, 19] or when $k = 4$ and the input graph is bipartite [4, 18, 29]; it is open when $k = 4$ and the input graph is general. There exists an extensive literature on different scenarios (see e.g. [3, 10, 35, 36]).

Odd cycles in directed and undirected graphs. It is easy to find an odd cycle in the directed and undirected case. More precisely, this can be done by DFS in linear time. Moreover, a shortest one can be found based on BFS in polynomial time; see e.g. [34].

Even cycles in undirected graphs. A connected undirected graph contains no even cycle if and only if it is a cactus (in which every edge is contained in at most one cycle) such that every cycle is odd. We can easily determine in polynomial time whether there exists an even cycle in undirected graphs based on this characterization. Arkin, Papadimitriou, and Yannakakis [1] gave a linear-time algorithm for a more general congruency constraint.

Even cycles in directed graphs Robertson, Seymour, and Thomas [32], and independently McCuaig [26], developed polynomial-time algorithms for detecting even dicycles by using the connection with Pfaffian orientations in bipartite graphs [38]. Important applications have emerged since then. Using the characterization of bipartite graphs that have a Pfaffian orientation, Guening and Thomas [16] gave an excluded-minor characterization of directed graphs such

Table 1: Summary of our results. NPC refers to NP-Complete.

Problem	Directed	Undirected	Mixed
CYCLE-FACTOR	P (Folklore)	P [8, 13]	NPC (Thm. 5.1)
$\forall$ ODD CYCLE-FACTOR	NPC (Thm. 3.1)	NPC (Cor. 4.2)	NPC (Thm. 3.1)
$\forall$ EVEN CYCLE-FACTOR	NPC (Thm. 3.4)	NPC (Cor. 4.3)	NPC (Thm. 3.4)
$\exists$ ODD CYCLE-FACTOR	NPC (Thm. 3.5)	NPC (Cor. 4.4)	NPC (Thm. 3.5)
$\exists$ EVEN CYCLE-FACTOR	Open	Open	NPC (Thm. 5.1)

that in any subgraph the maximum number of disjoint dicycles coincides with the minimum size of a transversal of dicycles (a vertex set whose removal makes the graph acyclic). Gorsky, Kawarabayashi, Kreutzer, and Wiederrecht [15] and Gorsky [14] recently proved relaxed versions of the so-called Erdős–Pósa property for even dicycles, which established a number of further implications. Björklund, Husfeldt, and Kaski [9] proposed a randomized polynomial-time algorithm for computing a shortest even dicycle in digraphs via advanced algebraic methods.

1.2 Our Results

Our results are summarized in Table 1. We show that three out of the four versions of the problem are NP-complete in the directed case, while the complexity of $\exists$ EVEN CYCLE-FACTOR remains open for both the directed and undirected settings. By observing that there exists a reduction from directed to undirected graphs that preserves the number of even and odd cycles, we also establish the NP-completeness results for the undirected case. Finally, we investigate the mixed setting, where the problem is already NP-complete even when no parity constraint is imposed.

The rest of the paper is organized as follows. We prepare basic terminology and notation in Section 2. We prove hardness for directed graphs in Section 3, and extend these results to undirected graphs in Section 4. Our main result for mixed graphs is shown in Section 5. We conclude the paper in Section 6.

2 Preliminaries

For an integer number $n \geq 1$, we use $[n] = \{1, 2, \dots, n\}$.

Undirected graphs. Let G be an undirected graph. For a set $F \subseteq E(G)$, we denote the set of vertices that are incident to at least one edge of F by $V(F)$. We denote the degree of a vertex by $\deg_G(v)$ or simply by $\deg(v)$. Given $F \subseteq E(G)$ and $v \in V(G)$, we use $\deg_F(v)$ for $\deg_H(v)$, where H is defined by $V(H) = V(G)$ and $E(H) = F$. We say G is *cubic* if $\deg_G(v) = 3$ for all $v \in V(G)$.

For $F \subseteq E(G)$, an orientation of F is a set of arcs $\vec{F}$ such that for each $uv \in F$, precisely one of the two arcs uv and vu belongs to $\vec{F}$. A *k-edge-coloring* is an assignment of numbers in $[k]$ to the edges of a graph such that any two adjacent edges receive different colors.

Digraphs. Let $D = (V, A)$ be a directed graph. We denote the set of vertices of $F \subseteq A$ by $V(F)$. We denote the in-degree of a vertex v by $\deg^-(v) = |\{uv \in A \mid u \in V\}|$ and out-degree of v by $\deg^+(v) = |\{vu \in A \mid u \in V\}|$. For $s, t \in V(D)$, a *Hamiltonian (s, t)-path* is an (s, t) -path that visits all the vertices once.

3 Directed Graphs

In a digraph, a cycle-factor corresponds to a spanning subgraph in which each vertex has in-degree 1 and out-degree 1. Throughout this section, we refer to such a cycle-factor as a $(1,1)$ -factor, where the first coordinate indicates the in-degree and the second indicates the out-degree.

3.1 All Odd $(1,1)$ -Factor

First, we consider the problem of finding a $(1,1)$ -factor whose directed cycles have odd length.

Theorem 3.1. $\forall \text{ODD } (1,1)\text{-FACTOR is NP-complete.}$

Proof. Clearly, the problem is in NP. Next, we make a reduction from the following NP-complete problem (Karp [23]).

HAMILTONIAN (s,t) -PATH

Input: A directed graph H .

Goal: Decide whether H contains a directed Hamiltonian (s,t) -path or not.

Let (H, s, t) be an instance of HAMILTONIAN (s,t) -PATH. We now construct a digraph D in the following way. We let $V(D)$ consist of $V(H)$ and three extra vertices x_1^a, x_2^a , and x_3^a for every $a \in A(H)$. Next, we let $A(D)$ consist of the sets of five arcs, $S_a = \{ux_1^a, x_1^ax_2^a, x_2^ax_3^a, x_3^ax_1^a, x_3^av\}$, for all $a = uv \in A(D)$ and the arc ts . This finishes the description of D . It is not difficult to see that D can be constructed from H in polynomial time. For an illustration, see Figure 1.

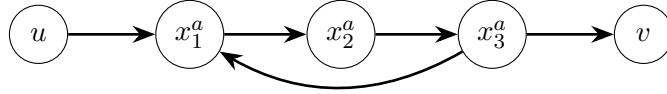


Figure 1: The gadget for an arc $a = uv \in A(H)$ in the reduction of Theorem 3.1.

We confirm that (H, s, t) is a Yes-instance of HAMILTONIAN (s,t) -PATH if and only if D is a Yes-instance of $\forall \text{ODD } (1,1)\text{-FACTOR}$.

First suppose that (H, s, t) is a Yes-instance of HAMILTONIAN (s,t) -PATH, so there exists a directed Hamiltonian (s,t) -path P in H . We now let F consist of the four arcs $ux_1^a, x_1^ax_2^a, x_2^ax_3^a, x_3^av$ for all $a = uv \in A(P)$, the arc ts , and the three arcs $x_1^ax_2^a, x_2^ax_3^a, x_3^ax_1^a$ for all $a \in A(H) \setminus A(P)$. It follows directly by construction that F is a $(1,1)$ -factor of D . We still need to show that C is odd for all $C \in \mathcal{C}(F)$. First observe that for all $a \in A(D) \setminus A(P)$ the unique cycle $C \in \mathcal{C}(F)$ with $V(C) = \{x_1^a, x_2^a, x_3^a\}$ contains three vertices and hence is clearly odd. The only directed cycle $C \in \mathcal{C}(F)$ that is not of that form satisfies $|V(C)| = |V(P)| + 3|A(P)| = 4|A(P)| + 1$, which is also odd. Hence D is a Yes-instance of $\forall \text{ODD } (1,1)\text{-FACTOR}$.

Now suppose that D is a Yes-instance of $\forall \text{ODD } (1,1)\text{-FACTOR}$, so there exists a $(1,1)$ -factor F of D such that C is odd for all $C \in \mathcal{C}(F)$.

Claim 3.2. *For every $a = uv \in A(H)$, exactly one of the following holds:*

- $S_a \cap F = \{ux_1^a, x_1^ax_2^a, x_2^ax_3^a, x_3^av\}$,
- $S_a \cap F = \{x_1^ax_2^a, x_2^ax_3^a, x_3^ax_1^a\}$.

Proof. First observe that $\{x_1^a x_2^a, x_2^a x_3^a\} \subseteq F$ as F is a $(1, 1)$ -factor in D and these are the only arcs incident to x_2^a in D . Next, if $x_3^a x_1^a \in F$, we obtain that $\{ux_1^a, x_3^a v\} \cap F = \emptyset$ as F is a $(1, 1)$ -factor in D . Finally, if $x_3^a x_1^a$ is not contained in F , we obtain that $\{ux_1^a, x_3^a v\} \subseteq F$ as F is a $(1, 1)$ -factor in D . $\square$

Claim 3.3. *Let $C \in \mathcal{C}(F)$ be a directed cycle with $V(C) \cap V(H) \neq \emptyset$. Then, $ts \in A(C)$.*

Proof. Suppose for the sake of a contradiction that there exists a directed cycle $C \in \mathcal{C}(F)$ with $V(C) \cap V(H) \neq \emptyset$ and such that $A(C)$ does not contain the arc ts . Let $A_0 \subseteq A(H)$ consist of the arcs $a \in A(H)$ such that $A(C) \cap S_a \neq \emptyset$. As C is a directed cycle and $V(C) \cap V(H) \neq \emptyset$, we obtain from Claim 3.2 that $|A(C) \cap S_a| = 4$ for all $a \in A_0$. Hence, as $\{S_a \mid a \in A(H)\}$ is a partition of $A(D) \setminus \{ts\}$ and $A(C)$ does not contain ts , we obtain that $|V(C)| = 4|A_0|$. In particular, we obtain that C is even, contradicting the assumption on F . $\square$

As F is a $(1, 1)$ -factor, there exists a cycle C in $\mathcal{C}(F)$ with $V(H) \cap V(C) \neq \emptyset$, which is unique as $ts \in A(C)$ by Claim 3.3. This means $V(H) \subseteq V(C)$. Now let P be the subgraph of H such that $A(P)$ contains all $a = uv \in A(H)$ with $S_a \cap F = \{ux_1^a, x_1^a x_2^a, x_2^a x_3^a, x_3^a v\}$. It follows by Claim 3.2 and the fact that C is a cycle in D that P is a directed path in H . As $V(P) = V(C) \cap V(H) = V(H)$, we obtain that P is a Hamiltonian (s, t) -path in H , so (H, s, t) is Yes-instance of HAMILTONIAN (s, t) -PATH. $\square$

3.2 All Even $(1, 1)$ -Factor

The next problem asks for a $(1, 1)$ -factor whose directed cycles are even.

Theorem 3.4. $\forall\text{EVEN } (1, 1)\text{-FACTOR}$ is NP-complete.

Proof. Clearly, the problem is in NP. Next, consider the following problem that was shown to be NP-complete by Holyer [20].²

3-EDGE-COLORING

Input: A cubic undirected graph H .

Goal: Decide whether H is 3-edge-colorable or not.

We construct a digraph D by replacing each edge of H by a gadget: the vertex set is $V(D) = V(H) \cup \{w'_{uv}, w''_{uv} \mid uv \in A(H)\}$ and the arc set $A(D)$ is the union of $A_{uv} = \{uw'_{uv}, vw'_{uv}, w''_{uv}u, w''_{uv}v, w'_{uv}w''_{uv}, w''_{uv}w'_{uv}\}$ for all edges $uv \in A(H)$. Note that for every edge $uv \in A(H)$, the gadget contains unique Hamiltonian paths P_{uv} from u to v and P_{vu} from v to u . This finishes the description of D . It is not difficult to see that D can be computed from H in polynomial time. An illustration of the gadget can be found in Figure 2.

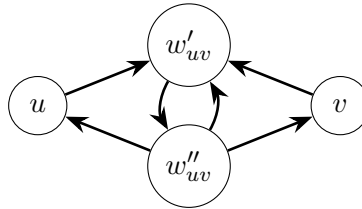


Figure 2: The gadget for an edge $uv \in A(H)$ in the reduction of Theorem 3.4.

²As observed in the proof, if we consider $\forall\text{EVEN } 2\text{-FACTOR}$ in cubic undirected graphs, the problem can be seen as almost equivalent.

We confirm that H is a Yes-instance of 3-EDGE-COLORING if and only if D is a Yes-instance of $\forall\text{EVEN } (1,1)\text{-FACTOR}$.

First suppose that H is a Yes-instance of 3-EDGE-COLORING and let ϕ be a 3-edge-coloring of H . As H is cubic, every vertex is incident to exactly three edges. As ϕ is a proper edge coloring, every edge has a different color. Every color defines a perfect matching, and so two colors define a series of even cycles covering every vertex in H , that is, two colors define an even cycle-cover. Thus, by fixing two colors and orientations of the even cycles and by replacing the resulting arcs uv with the corresponding paths P_{uv} of length 3 in D , we obtain a collection of vertex-disjoint even dicycles in D which covers all the vertices in $V(H)$. By adding the dicycle of length 2 of form $(w'_{uv}, w''_{uv}, w'_{uv})$ for each edge uv of the remaining color, we obtain a $(1,1)$ -factor of D in which all the cycles are even.

Now suppose that D is a Yes-instance of $\forall\text{EVEN } (1,1)\text{-FACTOR}$. Let $\vec{F}$ be a $(1,1)$ -factor of D in which all the cycles are even. Since H is cubic, each $u \in V(H)$ has exactly three neighbors $v_1, v_2, v_3 \in V(H)$ in H . By the construction of D and since $\vec{F}$ is a $(1,1)$ -factor in which all the directed cycles are even, we obtain that there exists an ordering (i, j, k) of $[3]$ such that the following holds:

- $\vec{F} \cap A_{uv_i} = A(P_{uv_i})$,
- $\vec{F} \cap A_{uv_j} = A(P_{v_j u})$,
- $\vec{F} \cap A_{uv_k} = \{w'_{uv_k} w''_{uv_k}, w''_{uv_k} w'_{uv_k}\}$.

Now let $F_1 \subseteq E(H)$ contain all edges uv such that $\vec{F} \cap A_{uv} = \{w'_{uv} w''_{uv}, w''_{uv} w'_{uv}\}$. It follows directly from the above observation that F_1 is a perfect matching in H . We further set $F_2 = E(H) \setminus F_1$ and observe that every $v \in V(H)$ is incident to exactly two edges in F_2 . It follows that F_2 forms a collection $\mathcal{C}$ of cycles in H . Next, every $C \in \mathcal{C}$ that is of length k for some positive integer k corresponds to a directed cycle in D formed by $\vec{F}$ of length $3k$. By the assumption on $\vec{F}$, we obtain that all cycles in $\mathcal{C}$ are even. Hence for every $C \in \mathcal{C}$, there exists a proper 2-edge-coloring of $E(C)$. Using these colorings and assigning the edges in F_1 a third color, we obtain a proper 3-edge-coloring of H . $\square$

3.3 Exists Odd $(1,1)$ -Factor

The next problem is when a $(1,1)$ -factor should contain at least one odd cycle.

Theorem 3.5. $\exists\text{ODD } (1,1)\text{-FACTOR}$ is NP-complete.

Proof. Clearly, the problem is in NP. Next, we make a reduction from the following NP-complete problem [12].

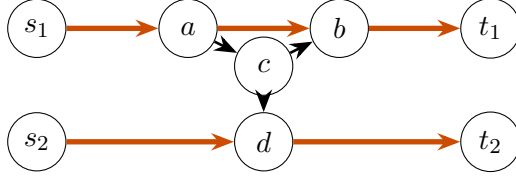
2-VERTEX-DISJOINT PATHS (2VDP)

Input: A directed graph H , and 2 vertex pairs (s_1, t_1) and (s_2, t_2) of $V(H)$.

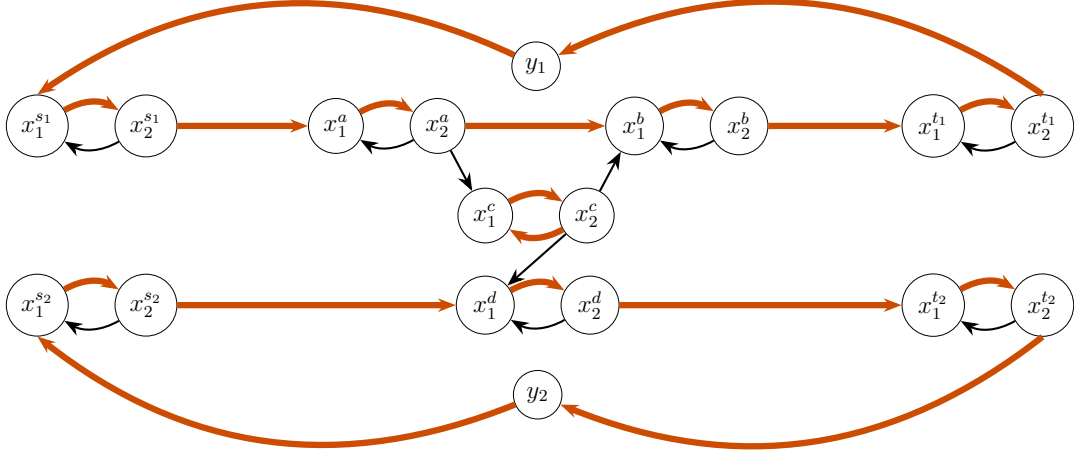
Goal: Decide if H contains vertex-disjoint (s_i, t_i) -paths P_i , $(i = 1, 2)$ or not.

Let $(H, (s_1, t_1), (s_2, t_2))$ be an instance of 2VDP. We now construct a digraph D as follows. We let $V(D)$ consist of two vertices x_1^v, x_2^v for every $v \in V(H)$ and two more vertices y_1 and y_2 . Further, let $A(D)$ be the union of $S_v = \{x_1^v x_2^v, x_2^v x_1^v\}$ for all $v \in V(H)$, plus two more arcs $x_2^{t_i} y_i$ and $y_i x_1^{s_i}$ for $i \in [2]$. This finishes the description of D . It is not difficult to see that D can be computed from H in polynomial time. An illustration can be found in Figure 3.

We confirm that $(H, (s_1, t_1), (s_2, t_2))$ is a Yes-instance of 2VDP if and only if D is a Yes-instance of $\exists\text{ODD } (1,1)\text{-FACTOR}$.



(a) Example of 2VDP instance $(H, (s_1, t_1), (s_2, t_2))$; solution in thick orange.



(b) $\exists$ ODD $(1, 1)$ -FACTOR instance D obtained from (a). Solution shown in thick orange.

Figure 3: Illustration of the reduction of Theorem 3.5.

First suppose that $(H, (s_1, t_1), (s_2, t_2))$ is a Yes-instance of 2VDP, so there exist a directed (s_1, t_1) -path P_1 and a directed (s_2, t_2) -path P_2 in H such that $V(P_1) \cap V(P_2) = \emptyset$. For $i \in [2]$, we now define a directed cycle C_i by $A(C_i) = \bigcup_{v \in V(P_i)} \{x_1^v x_2^v\} \cup \bigcup_{uv \in A(P_i)} \{x_2^u x_1^v\} \cup \{x_2^{t_i} y_i, y_i x_1^{s_i}\}$. It is not difficult to see that C_i is a directed cycle for $i \in [2]$ indeed. Further, for all $v \in V(H) \setminus (V(P_1) \cup V(P_2))$, we let C_v be the directed cycle defined by $A(C_v) = S_v$. We let F be defined by $F = A(C_1) \cup A(C_2) \cup \bigcup_{v \in V(H) \setminus (V(P_1) \cup V(P_2))} A(C_v)$. It follows by construction that F is a $(1, 1)$ -factor in D , where $C_1 \in \mathcal{C}(F)$ is odd as $|V(C_1)| = 2|V(P_1)| + 1$. This yields that D is a Yes-instance of $\exists$ Odd $(1, 1)$ -Factor.

Now suppose that D is a Yes-instance of $\exists$ Odd $(1, 1)$ -Factor, so there exists a $(1, 1)$ -factor F of D such that $\mathcal{C}(F)$ contains at least one odd cycle.

Claim 3.6. *Let $C \in \mathcal{C}(F)$. Then C is odd if and only if $|V(C) \cap \{y_1, y_2\}| = 1$.*

Proof. Let V_0 be the set of vertices v in $V(H)$ such that $S_v \cap A(C) \neq \emptyset$. If there exists some $v \in V_0$ such that $S_v \subseteq A(C)$, then we have $A(C) = S_v$, from which the statement clearly follows. We may hence suppose by construction that $S_v \cap A(C) = \{x_1^v x_2^v\}$ for all $v \in V_0$. Further, let V_1 be the set of vertices in $\{y_1, y_2\} \cap V(C)$. We obtain by the above observation that $|V(C)| = 2|V_0| + |V_1|$, from which the statement follows directly. $\square$

By Claim 3.6 and as $\mathcal{C}(F)$ contains an odd cycle by assumption, there exists a directed cycle $C_1 \in \mathcal{C}(F)$ with $|V(C_1) \cap \{y_1, y_2\}| = 1$. By symmetry, we may assume $y_1 \in V(C_1)$ and $y_2 \notin V(C_1)$. As F is a $(1, 1)$ -factor, there also exists a directed cycle $C_2 \in \mathcal{C}(F)$ with $y_2 \in V(C_2)$. Now for $i \in [2]$, let P_i be the directed path such that $A(P_i) = \{uv \mid x_2^u x_1^v \in A(C_i)\}$. It is easy to see that P_i is a directed (s_i, t_i) -path. Also, as C_1 and C_2 are vertex-disjoint, we get that P_1 and P_2 are vertex-disjoint. Thus $(H, (s_1, t_1), (s_2, t_2))$ is a Yes-instance of 2VDP. $\square$

3.4 Exists Even (1,1)-Factor

In this section, we give remarks on $\exists\text{EVEN (1,1)-FACTOR}$, whose complexity is open. We first show that the problem of deciding whether a digraph has an even cycle is polynomially reducible to this problem.

EVEN DICYCLE

Input: A directed graph H .

Goal: Find an even directed cycle.

Given an instance H of **EVEN DICYCLE** we construct in linear time an instance D of $\exists\text{EVEN (1,1)-FACTOR}$ as follows. We let $V(D)$ consist of six vertices $v, v^1, v^2, v^3, v^4, v^5$ for all vertices $v \in V(H)$. Let $A(D) = A(H) \cup \bigcup_{v \in V(H)} \{vv^1, v^1v^2, v^2v^3, v^3v^4, v^4v^5, v^5v^3, v^2v^3, v^5v^1\}$. An illustration can be found in Figure 4.

Since each v is a cut vertex in D and $\{v, v^1, v^2, v^3, v^4, v^5\}$ contains only odd cycles, if D has a $(1,1)$ -factor F that contains an even directed cycle C , then C is an even directed cycle of H . On the other hand, if H has an even directed cycle C , then a $(1,1)$ -factor in D consists of C , the directed cycles $(v^1, v^2, v^3, v^4, v^5, v^1)$ for $v \in V(C)$, and the directed cycles (v, v^1, v^2, v) and (v^3, v^4, v^5, v^3) for $v \in V(H) \setminus V(C)$.

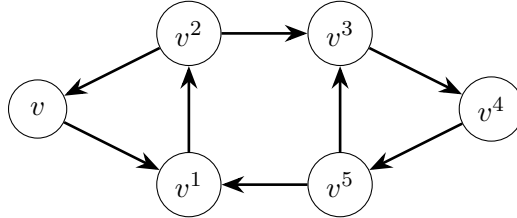


Figure 4: The gadget for a vertex $v \in V(H)$ in the reduction from **EVEN DICYCLE**.

Note that, **EVEN DICYCLE** is polynomially solvable as mentioned in Section 1.1, hence the above reduction does not imply the **NP**-hardness of the problem. However, it had been a long-standing open problem until Robertson, Seymour, and Thomas [32] and independently McCuaig [26] solved it. The above reduction implies that, if the problem is in **P**, a polynomial-time algorithm for $\exists\text{EVEN (1,1)-FACTOR}$ might either use **EVEN DICYCLE** as a subroutine or extend the rather complex ideas from [26, 32]. We pose this as an open problem.

4 Undirected Graphs

In an undirected graph, a cycle-factor corresponds to a subgraph in which each vertex has degree 2. Throughout this section, we refer to such a cycle-factor as a 2 -factor. We will show that there exists a reduction from directed to undirected graphs which preserves the number of even cycles and the number of odd cycles. This key property shows that finding a 2 -factor with parity constraints is not easier than finding a $(1,1)$ -factor with the same parity constraints.

Given an undirected graph G and $a, b \in \mathbb{Z}_{\geq 0}$, an $(a, b, 2)$ -factor is a 2 -factor F in G such that $\mathcal{C}(F)$ consists of a even cycles and b odd cycles. Similarly, given a directed graph D and $a, b \in \mathbb{Z}_{\geq 0}$, an $(a, b, 1, 1)$ -factor is a $(1, 1)$ -factor $\vec{F}$ in D such that $\mathcal{C}(\vec{F})$ consists of a even directed cycles and b odd directed cycles.

Lemma 4.1. *Given a digraph D , in polynomial time, we can compute a graph G such that for any $a, b \in \mathbb{Z}_{\geq 0}$, we have that G contains an $(a, b, 2)$ -factor if and only if D contains an $(a, b, 1, 1)$ -factor.*

Proof. Let D be a digraph. We now construct a graph G in the following way. We let $V(G)$ consist of three vertices a_v, b_v, c_v for all $v \in V(D)$. Next, for all $v \in V(D)$, we let $E(G)$ contain two edges $a_v b_v$ and $b_v c_v$. Finally, for every $uv \in A(D)$, we let $E(G)$ contain the edge $c_u a_v$. This completes the description of G . It is easy to see that G can be built from D in polynomial time. An illustration appears in Figure 5.

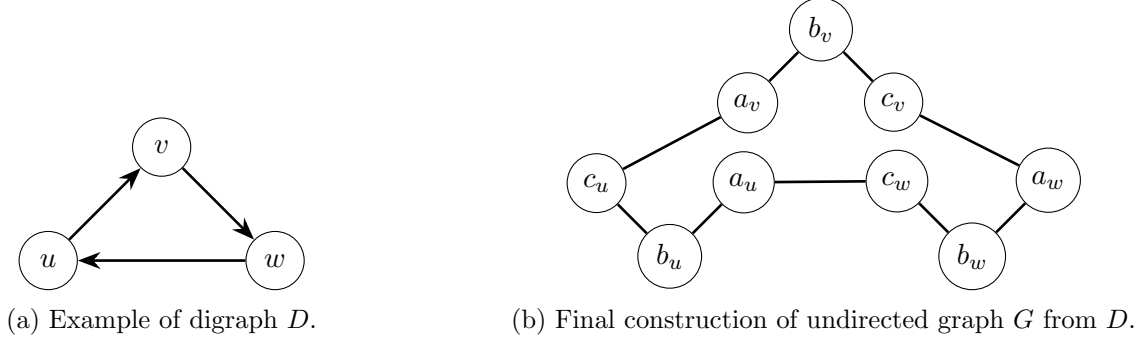


Figure 5: Illustration for Lemma 4.1.

We confirm that D contains an $(a, b, 1, 1)$ -factor if and only if G contains an $(a, b, 2)$ -factor.

First suppose that D contains an $(a, b, 1, 1)$ -factor $\vec{F}$. Let $F \subseteq E(G)$ be the set consisting of $a_v b_v$ and $b_v c_v$ for all $v \in V(D)$ and the edges $c_u a_v$ for all $u, v \in V(D)$ with $uv \in \vec{F}$. We will show that F is an $(a, b, 2)$ -factor of G . First observe that for every $v \in V(D)$, we have $\deg_F(a_v) = \deg_{\vec{F}}^-(v) + 1 = 2$, $\deg_F(b_v) = 2$, and $\deg_F(c_v) = \deg_{\vec{F}}^+(v) + 1 = 2$, so F is a 2-factor of G . Next, we describe a bijection $f: \mathcal{C}(\vec{F}) \rightarrow \mathcal{C}(F)$. Namely, let $\vec{C} \in \mathcal{C}(\vec{F})$. Then, we let $f(\vec{C})$ be defined by $E(f(\vec{C})) = \bigcup_{v \in V(D)} \{a_v b_v, b_v c_v\} \cup \bigcup_{uv \in A(\vec{C})} \{c_u a_v\}$. It is not difficult to see that f is a bijection from $\mathcal{C}(\vec{F})$ to $\mathcal{C}(F)$ indeed. Moreover, for every $\vec{C} \in \mathcal{C}(\vec{F})$, we have $|E(f(\vec{C}))| = 3|A(\vec{C})|$. In particular, we obtain that $|E(f(\vec{C}))|$ and $|A(\vec{C})|$ have the same parity. It follows that F is an $(a, b, 2)$ -factor of G .

Now suppose that G contains an $(a, b, 2)$ -factor F . We define a set $\vec{F} \subseteq A(D)$. Namely, we let $\vec{F}$ contain the arcs $uv \in A(D)$ for all pairs of $u, v \in V(G)$ with $c_u a_v \in F$. Observe that $\vec{F}$ is well-defined by the construction of G . We show that $\vec{F}$ is an $(a, b, 1, 1)$ -factor of D . Note that for every $v \in V(D)$, as F is a 2-factor of G and by the construction of G , we have that $\{a_v b_v, b_v c_v\} \subseteq F$. As F is a 2-factor of G , for every $v \in V(D)$, there exists exactly one edge in $(\{a_v u \mid u \in V(G)\} \setminus \{a_v b_v\}) \cap F$. It follows by construction that this edge is of the form $c_u a_v$ for some $u \in V(D)$ with $uv \in A(D)$. By the definition of $\vec{F}$, we get that $\deg_{\vec{F}}^-(v) = 1$. A similar argument shows $\deg_{\vec{F}}^+(v) = 1$. It follows that $\vec{F}$ is a $(1, 1)$ -factor of D . Next, we describe a bijection $g: \mathcal{C}(F) \rightarrow \mathcal{C}(\vec{F})$. Namely, let $C \in \mathcal{C}(F)$. Then, we let $g(C)$ be defined by $A(g(C)) = \{uv \in A(D) \mid c_u a_v \in E(C)\}$. It is easy to see that g is a bijection from $\mathcal{C}(F)$ to $\mathcal{C}(\vec{F})$ indeed. Also, for every $C \in \mathcal{C}(F)$, we have $3|A(g(C))| = |E(C)|$. In particular, we get that $|A(g(C))|$ and $|E(C)|$ have the same parity. Hence $\vec{F}$ is an $(a, b, 1, 1)$ -factor of D . $\square$

We obtain the following results for the undirected case.

Corollary 4.2. $\forall \text{ODD 2-FACTOR}$ is NP-complete.

Proof. Clearly, the problem is in NP. Next, we make a reduction from $\forall \text{ODD (1, 1)-FACTOR}$ which is NP-complete by Theorem 3.1. Let D be an instance of $\forall \text{ODD (1, 1)-FACTOR}$. We now use Lemma 4.1 to compute, in polynomial time, a graph G such that for all nonnegative integers a and b , we have that G contains an $(a, b, 2)$ -factor if and only if D contains an $(a, b, 1, 1)$ -factor.

In particular, we have indeed that G contains a $(0, b, 2)$ -factor for some $b \geq 0$ if and only if D contains a $(0, b, 1, 1)$ -factor, as required. $\square$

Similar arguments yield the following two corollaries using Theorems 3.4 and 3.5 instead of Theorem 3.1, respectively.

Corollary 4.3. $\forall \text{EVEN 2-FACTOR}$ is NP-complete.

Corollary 4.4. $\exists \text{ODD 2-FACTOR}$ is NP-complete.

5 Mixed Graphs

In a mixed graph, a cycle-factor corresponds to mixed cycles in which all the edges can be oriented to obtain a directed cycle. In this section, we refer to cycle-factors on mixed graphs as *mixed cycle-factors*, and the problem CYCLE-FACTOR on mixed graphs as MCF.

Theorem 5.1. *MCF is NP-complete.*

We prove Theorem 5.1 through the hardness of two intermediate problems. In the following, given a ground set E and a collection $\mathcal{P}$ of 2-element subsets of E , we say that a set $F \subseteq E$ is $\mathcal{P}$ -respecting if $|F \cap p| \leq 1$ for all $p \in \mathcal{P}$. We first consider the following problem:

PAIR-RESTRICTED CYCLE-FACTOR (PRCF)

Input: An undirected graph H and a collection $\mathcal{P}$ of 2-element subsets of $E(H)$.

Goal: Decide whether there exists a $\mathcal{P}$ -respecting 2-factor of H or not.

Lemma 5.2. *PRCF is NP-complete.*

Proof. Clearly, the problem is in NP. Next, we make a reduction from the following NP-complete problem (Karp [23]).

3-DIMENSIONAL MATCHING (3DM)

Input: Three sets X, Y, Z with $|X| = |Y| = |Z|$ and a set of 3-element tuples $T \subseteq X \times Y \times Z$.

Goal: Decide whether there exists a perfect matching $M \subseteq T$ (i.e., for every $v \in X \cup Y \cup Z$ exactly one tuple in M contains v) or not.

Let (X, Y, Z, T) be an instance of 3DM. We now define an instance $(H, \mathcal{P})$ of PRCF in the following way. First, we let $V(H) = X \cup Y \cup Z$. Next, we let $E(H)$ consist of three edges xy, yz, zx for all tuples $t = (x, y, z) \in T$; we allow parallel edges between the same pair of vertices in $V(H)$, and we denote the edges by $(xy)_t, (yz)_t, (zx)_t$ to emphasize that they come from t . Finally, we define $\mathcal{P}$ as follows: for each pair of distinct tuples $t_1 = (x_1, y_1, z_1)$ and $t_2 = (x_2, y_2, z_2)$ in T ,

- if $x_1 = x_2 = x$, then $\mathcal{P}$ contains four pairs $\{(xy_1)_{t_1}, (xy_2)_{t_2}\}, \{(xy_1)_{t_1}, (xz_2)_{t_2}\}, \{(xz_1)_{t_1}, (xy_2)_{t_2}\}$, and $\{(xz_1)_{t_1}, (xz_2)_{t_2}\}$,
- if $y_1 = y_2 = y$, then $\mathcal{P}$ contains four pairs $\{(yx_1)_{t_1}, (yx_2)_{t_2}\}, \{(yx_1)_{t_1}, (yz_2)_{t_2}\}, \{(yz_1)_{t_1}, (yx_2)_{t_2}\}$, and $\{(yz_1)_{t_1}, (yz_2)_{t_2}\}$, and
- if $z_1 = z_2 = z$, then $\mathcal{P}$ contains four pairs $\{(zx_1)_{t_1}, (zx_2)_{t_2}\}, \{(zx_1)_{t_1}, (zy_2)_{t_2}\}, \{(zy_1)_{t_1}, (zx_2)_{t_2}\}$, and $\{(zy_1)_{t_1}, (zy_2)_{t_2}\}$.

This finishes the description of $(H, \mathcal{P})$. It is easy to see that $(H, \mathcal{P})$ can be computed from (X, Y, Z, F) in polynomial time.

We confirm that (X, Y, Z, T) is a Yes-instance of 3DM if and only if $(H, \mathcal{P})$ is a Yes-instance of PRCF.

First suppose that (X, Y, Z, T) is a Yes-instance of 3DM, so there exists a perfect matching, i.e., a subset $M \subseteq T$ such that for every $v \in X \cup Y \cup Z$ there exists exactly one tuple $(x, y, z) \in M$ with $v \in \{x, y, z\}$. We let $F \subseteq E(H)$ consist of the three edges $(xy)_t, (yz)_t, (zx)_t$ for all $t = (x, y, z) \in M$. As M is a perfect matching and $V(H) = X \cup Y \cup Z$, F is a 2-factor of H by definition. In addition, since every $p = \{e, f\} \in \mathcal{P}$ comes from a pair of distinct tuples $t_1, t_2 \in T$ that share at least one element in $V(H) = X \cup Y \cup Z$ (i.e., e is of form $(\cdot)_{t_1}$ and f is of form $(\cdot)_{t_2}$), clearly $|F \cap p| \leq 1$ holds (as M is a perfect matching, again). It means that F is $\mathcal{P}$ -respecting, and $(H, \mathcal{P})$ is a Yes-instance of PRCF.

Now suppose that $(H, \mathcal{P})$ is a Yes-instance of PRCF, so there exists a $\mathcal{P}$ -respecting 2-factor F of H . Let $v \in V(H)$. Suppose that $v \in X$ and let $e, f \in F$ be the two edges incident to v . Then, by definition of $E(H)$ and $\mathcal{P}$, there exists a tuple $t = (v, y, z) \in T$ such that $\{e, f\} = \{(vy)_t, (vz)_t\}$ as F is $\mathcal{P}$ -respecting. The same is true when $v \in Y$ and $v \in Z$, which means that for any $C \in \mathcal{C}(F)$, there exists a tuple $t_C = (x, y, z) \in T$ such that $E(C) = \{(xy)_t, (yz)_t, (zx)_t\}$. We let $M \subseteq T$ consist of the tuples t_C for all $C \in \mathcal{C}$. Then M is a perfect matching as F is a 2-factor of H , and (X, Y, Z, T) is a Yes-instance of 3DM. $\square$

In the second intermediate problem, we consider mixed graphs but omit the condition that mixed cycles need to cover the entire vertex set. Given a mixed graph G , a *Z-mixed cycle-factor* of G is a set $F \subseteq E(G) \cup A(G)$ forming a collection of vertex-disjoint mixed cycles such that every $z \in Z$ is incident to exactly two elements of F . Formally, we consider the following problem:

STEINER MIXED CYCLE-FACTOR (SMCF)

Input: A mixed graph G and a set $Z \subseteq V(G)$.

Goal: Decide whether there exists a Z -mixed cycle-factor in G or not.

Lemma 5.3. *SMCF is NP-complete*

Proof. Clearly, the problem is in NP. Next, we make a reduction from PRCF, which is NP-complete by Lemma 5.2. Let $(H, \mathcal{P})$ be an instance of RSCF. Further, let $v_1, \dots, v_n$ be an arbitrary enumeration of $V(H)$ and let $e_1, \dots, e_m$ be an arbitrary enumeration of $E(H)$. We now construct an instance (G, Z) of SMCF. We first let $V(G)$ consist of $V(H)$, a vertex $w_{e,f}$ for all $(e, f) \in E(H)^2$, and two vertices $z_{e,f}$ and $z_{f,e}$ for every $\{e, f\} \in \mathcal{P}$. Next, for every $e = v_i v_j \in E(H)$ with $i < j$, let us let $E(G)$ contain $S_e = \{v_i w_{e,e_1}, w_{e,e_m} v_j\} \cup \{w_{e,e_k} w_{e,e_{k+1}} \mid k \in [m-1]\}$. Next, we let $E(G)$ contain the edge $z_{e,f} z_{f,e}$ for all $\{e, f\} \in \mathcal{P}$. Further, for every $\{e, f\} \in \mathcal{P}$, we let $A(G)$ contain four arcs $w_{e,f} z_{e,f}$, $z_{f,e} w_{e,f}$, $w_{f,e} z_{f,e}$, and $z_{e,f} w_{f,e}$. We finally set $Z = \bigcup_{\{e,f\} \in \mathcal{P}} \{z_{e,f}, z_{f,e}\}$. This finishes the description of (G, Z) . It is easy to see that (G, Z) can be computed from $(H, \mathcal{P})$ in polynomial time. An illustration can be found in Figure 6.

We confirm that $(H, \mathcal{P})$ is a Yes-instance of PRCF if and only if (G, Z) is a Yes-instance of SMCF.

First suppose that $(H, \mathcal{P})$ is a Yes-instance of PRCF, so there exists a $\mathcal{P}$ -respecting 2-factor F of H . We now define a set $F' \subseteq E(G) \cup A(G)$. First, we let F'_0 be defined as $\bigcup_{e \in F} S_e$. Now consider $p = \{e, f\} \in \mathcal{P}$. As F is $\mathcal{P}$ -respecting, at least one of e and f does not belong to F . By symmetry, we assume that $e \notin F$. Then we let F'_p consist of the two arcs $w_{e,f} z_{e,f}$ and $z_{f,e} w_{f,e}$ and the edge $z_{e,f} z_{f,e}$. Let $F' = F'_0 \cup \bigcup_{p \in \mathcal{P}} F'_p$. By construction, F'_p forms a mixed cycle of length 3 for every $p \in \mathcal{P}$. It also follows from the fact that F is a 2-factor in H and by construction

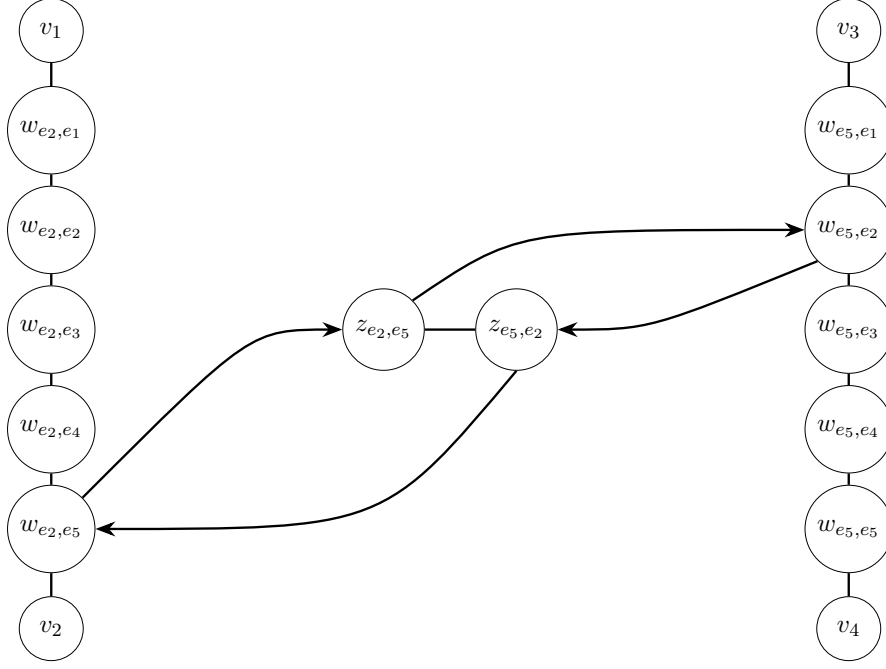


Figure 6: Illustration of the reduction of Lemma 5.3. There exist five edges e_i ($i \in [5]$) in H , and the illustrated part corresponds to a pair $\{e_2, e_5\} \in \mathcal{P}$ such that $e_2 = v_1v_2$ and $e_5 = v_3v_4$. Then, z_{e_2,e_5} and z_{e_5,e_2} are in Z , which can be covered only by mixed cycles whose vertex sets are included in $\{z_{e_2,e_5}, z_{e_5,e_2}, w_{e_2,e_5}, w_{e_5,e_2}\}$.

that F'_0 is the edge set of a collection of disjoint undirected cycles in G . Moreover, as e is chosen so that $e \in p \setminus F$ for each $p = \{e, f\} \in \mathcal{P}$, the cycles formed by F'_p and by F'_0 are disjoint. Hence F' is a mixed cycle-factor in G . In addition, we have $Z \subseteq \bigcup_{p \in \mathcal{P}} V(F'_p) \subseteq V(F')$, so F' is a Z -mixed cycle-factor in G . We obtain that (G, Z) is a Yes-instance of SMCF.

Now suppose that (G, Z) is a Yes-instance of SMCF, so there exists a Z -mixed cycle-factor F' in G .

Claim 5.4. *Let $e \in E(H)$. Then either $S_e \subseteq F'$ or $S_e \cap F' = \emptyset$.*

Proof. Suppose otherwise, so there exists some $f \in E(H)$ such that F' contains exactly one of the two edges in S_e incident to $w_{e,f}$. Let $C \in \mathcal{C}(F')$ be the mixed cycle that contains this edge. As C is a mixed cycle, we get that $A(C)$ contains exactly one of the two arcs $w_{e,f}z_{e,f}$ and $z_{f,e}w_{e,f}$. By symmetry, we may suppose that $A(C)$ contains $w_{e,f}z_{e,f}$ and does not contain $z_{f,e}w_{e,f}$. As F' is a mixed cycle-factor, neither F' does not contain the arc $z_{f,e}w_{e,f}$. Thus, as F' is a Z -mixed cycle-factor and $z_{f,e} \in Z$, we get that F' contains the arc $w_{f,e}z_{f,e}$ and the edge $z_{e,f}z_{f,e}$, which belong to $E(C) \cup A(C)$. This contradicts that C is a mixed cycle since either orientation of the edge $z_{e,f}z_{f,e}$ is inconsistent. $\square$

We now define $F \subseteq E(H)$ to be the set consisting of all $e \in E(H)$ with $S_e \subseteq F'$. It follows directly from Claim 5.4 and the fact that F' is a cycle-factor that F is a 2-factor in H . It remains to prove that F is $\mathcal{P}$ -respecting. Suppose for the sake of a contradiction that there exists some $p = \{e, f\} \in \mathcal{P}$ with $p \not\subseteq F$. We obtain by Claim 5.4 that $S_e \cup S_f \subseteq F'$. As F' is a mixed cycle-factor, we obtain that F' contains none of the four arcs $w_{e,f}z_{e,f}$, $z_{f,e}w_{e,f}$, $w_{f,e}z_{f,e}$, and $z_{e,f}w_{f,e}$. This contradicts that F' is a Z -mixed cycle-factor since $z_{e,f} \in Z$ has only one remaining incident edge $z_{e,f}z_{f,e}$ in G . It follows that F is $\mathcal{P}$ -respecting, so $(H, \mathcal{P})$ is a Yes-instance of PRCF. $\square$

We are finally ready to prove Theorem 5.1.

Proof of Theorem 5.1. Clearly, the problem is in NP. We make a reduction from SMCF, which is NP-complete by Lemma 5.3. Let (H, Z) be an instance of SMCF. We now create a mixed graph G . First, we let $V(G)$ consist of $V(H)$ and the set $U_v = \{u_v^1, u_v^2, u_v^3\}$ of three vertices for every $v \in V(H) \setminus Z$. We further let $E(G)$ consist of $E(H)$ and the set of five edges $S_v = \{vu_v^1, u_v^1u_v^2, u_v^2u_v^3, u_v^3u_v^1, u_v^3v\}$ for every $v \in V(H) \setminus Z$. This finishes the description of G . It is easy to see that G can be computed from (H, Z) in polynomial time. An illustration is shown in Figure 7.

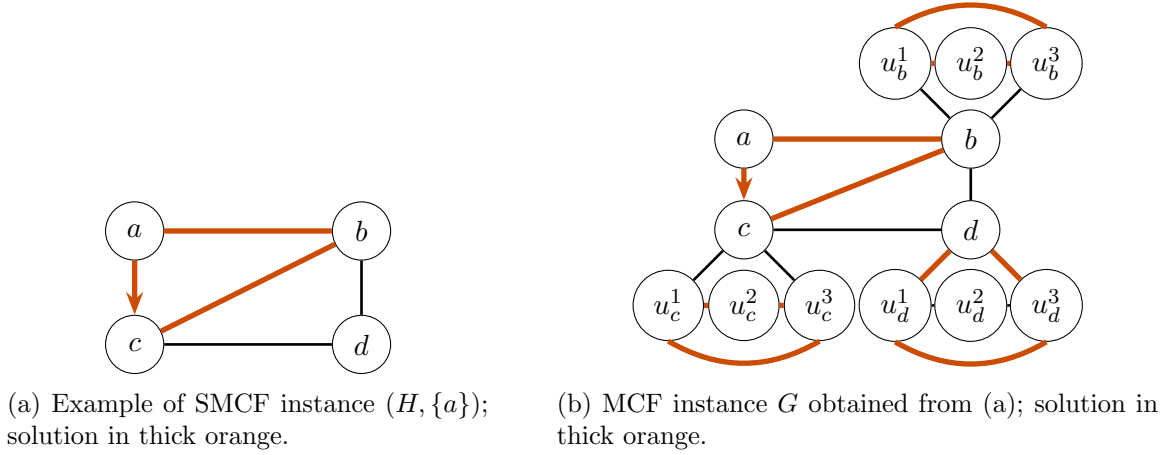


Figure 7: Illustration of reduction of Theorem 5.1.

We confirm the equivalence of the instance (H, Z) of SMCF and the instance G of MCF.

First suppose that (H, Z) is a Yes-instance of SMCF, so there exists a Z -mixed cycle-factor F in H . For every $v \in V(H) \setminus V(F) \subseteq V(H) \setminus Z$, we set $F'_v = \{vu_v^1, u_v^1u_v^2, u_v^2u_v^3, u_v^3u_v^1\} \subseteq S_v$. For every $v \in (V(H) \setminus Z) \cap V(F)$, we set $F'_v = \{u_v^1u_v^2, u_v^2u_v^3, u_v^3u_v^1\} \subseteq S_v$. We now let $F' = F \cup \bigcup_{v \in V(H) \setminus Z} F'_v$. It follows from the fact that F is a Z -mixed cycle-factor in H and by construction that F' is a mixed cycle-factor in G . Hence G is a Yes-instance of MCF.

Next suppose that G is a Yes-instance of MCF, so there exists a mixed cycle-factor F' in G .

Claim 5.5. *Let $C \in \mathcal{C}(F')$. Then either $E(C) \cup A(C) \subseteq E(H) \cup A(H)$ or $E(C) \cup A(C) \subseteq S_v$ for some $v \in V(H)$.*

Proof. For each $v \in V(H) \setminus Z$, the two edges $u_v^1u_v^2$ and $u_v^2u_v^3$ must be contained in F' as it is a mixed cycle-factor. If C contains at least one of vu_v^1 and vu_v^3 for some $v \in V(H) \setminus Z$, then $E(C) = \{vu_v^1, u_v^1u_v^2, u_v^2u_v^3, u_v^3u_v^1\} \subseteq S_v$ and $A(C) = \emptyset$ as C is a mixed cycle; this satisfies the statement. Otherwise, C contains none of vu_v^1 and vu_v^3 for any $v \in V(H) \setminus Z$, which means $E(C) \cup A(C) \subseteq E(H) \cup A(H)$; this also satisfies the statement. $\square$

Now let $F = F' \cap (E(H) \cup A(H))$. It follows directly from Claim 5.5 and by construction that F is a Z -mixed cycle-factor in H . Hence (G, Z) is a Yes-instance of SMCF. $\square$

We remark here that Theorem 5.1 also directly implies the hardness of $\exists$ EVEN CYCLE-FACTOR in mixed graphs. Indeed, given an instance G of MCF, we can add another component consisting of two vertices s and t and two arcs st and ts . It is easy to see that this new graph admits a mixed cycle-factor with an even cycle if and only if G admits a mixed cycle-factor.

On the other hand, as directed graphs are (a special case of) mixed graphs, the NP-completeness of $\forall$ ODD CYCLE-FACTOR, $\forall$ EVEN CYCLE-FACTOR, and $\exists$ ODD CYCLE-FACTOR in mixed graphs is a consequence of Theorems 3.1–3.5.

6 Discussion

Finding a cycle-factor in an undirected or directed graph is a well-studied problem in combinatorial optimization. The analogous problem for mixed graphs, however, was open; we show that it is already NP-complete. We also study natural parity-constrained variants in both settings and observe that all of these variants are NP-complete, with the sole exception of the decision problem that asks whether a given undirected or directed graph admits a cycle-factor that contains at least one even-length cycle. Techniques that combine algebraic methods with graph-theoretic arguments appear to be promising for resolving the remaining open complexity questions as mentioned in Section 3.4, particularly in light of the connection between the undirected and directed cases established in Section 4. Another interesting direction is to characterize classes of graphs or parameters for which the parity-constrained cycle-factor problems become tractable or trivial (always admit or always do not admit such a cycle-factor).

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