Constrained Ramsey numbers for rainbow P_5

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Abstract

Given a graph H and a positive integer k, the *k-colored Ramsey number* $R_k(H)$ is the minimum integer n such that in every k-edge-coloring of the complete graph K_n, there is a monochromatic copy of H. Given two graphs H and G, the *constrained Ramsey number* (also called *rainbow Ramsey number*) $f(H,G)$ is defined as the minimum integer n such that, in every edge-coloring of K_n with any number of colors, there is either a monochromatic copy of H or a rainbow copy of G. Let P_t be the path on t vertices. Gyárfás, Lehel and Schelp proved that $f(H,P_5) = R_3(H)$ when H is a path or a cycle. Li, Besse, Magnant, Wang and Watts conjectured that $f(H,P_5) = R_3(H)$ for any graph H, and confirmed this for all connected graphs and all bipartite graphs. In this paper, we address this conjecture for multiple classes of disconnected graphs with chromatic number at least 3. Our newly established general results encompass all known results on this problem. We also obtain several results for a bipartite variation of the problem. In addition, we propose a series of questions concerning this problem from multiple distinct aspects for further research.

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1 Introduction

Ramsey theory investigates the thresholds at which specific patterns become unavoidable in arbitrarily edge-colored host graphs. Classical Ramsey problems, rooted in Ramsey's foundational 1930 theorem [39], investigate the inevitability of monochromatic subgraphs, where an edge-colored graph is called *monochromatic* if all edges are colored the same. In 1950, Erdős and Rado [11] proved the following *Canonical Ramsey Theorem*. Here an edge-colored graph is called *rainbow* if all edges are colored differently, and *lexical* if there is a total order of its vertices such that two edges have the same color if and only if they share the same larger endpoint.

Theorem 1.1 (Erdős-Rado Canonical Ramsey Theorem [11]). *For any positive integer p, there exists an integer n such that, in every edge-coloring of the complete graph K_n with any number of colors, there is a monochromatic, a rainbow, or a lexically colored K_p.*

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Motivated by the Canonical Ramsey Theorem, Eroh [12], Jamison, Jiang and Ling [23], and independently, Chen, Schelp and Wei [8], introduced the following notion. Given two graphs H and G, the *constrained Ramsey number* (also called *rainbow Ramsey number*) $f(H,G)$ is defined as the minimum integer n such that, in every edge-coloring of K_n with any number of colors, there is either a monochromatic copy of H or a rainbow copy of G. It follows from Theorem 1.1 that $f(H,G)$ exists if and only if either H is a star or G is acyclic (see also [12, 23]). Since its inception, this concept has spurred extensive research into analogous problems across diverse combinatorial structures, including bipartite graphs [4, 13, 43], hypergraphs [32, 34], ordered graphs [18], random graphs [5, 10], Euclidean space [14, 17] and Boolean lattices [7]. In addition, the study of rainbow colored graphs can be traced back to Euler's 1782 Latin square decomposition problem. In fact, many classical problems in combinatorics can be reformulated as the search for specific rainbow substructures within edge-colored graphs, such as Ringel's conjecture [37], the Caccetta-Häggkvist conjecture [1] and graph decomposition problems [20]. Moreover, rainbow generalizations of Ramsey-type problems [16, 27], Turán-type problems [30, 25] and saturation problems [29] has been extensively studied in the last two decades. For more results on this topic, we refer to [2, 3, 19, 24, 28, 35].

In 2003, Jamison, Jiang and Ling [23] initiated the study of $f(S,T)$, where both S and T are trees. They proved that $\Omega(st) \le f(S,T) \le O(st\cdot r(T))$, where $s = |E(S)|$, $t = |E(T)|$ and $r(T)$ is the radius of T. Let P_t be the path on t vertices. Jamison, Jiang and Ling asked whether $f(S,T)$ is maximized by $f(P_{s+1}, P_{t+1})$. In 2007, Gyárfás, Lehel and Schelp [21] showed that for $t+1 \in \{4,5\}$, the answer is negative. Moreover, Jamison, Jiang and Ling [23] conjectured that $f(S,T) = O(st)$. In 2009, Loh and Sudakov [35] proved that $f(S,P_t) = O(st\log t)$, which matches the lower bound up to a logarithmic factor. This result was improved by Gishboliner, Milojević, Sudakov and Wigderson [19] recently, who proved a nearly optimal upper bound for $f(S,P_t)$ which differs from the lower bound by a function of inverse-Ackermann type. Nowadays, $f(H,P_t)$ is still one of the most interesting cases of the constrained Ramsey problem.

For $t \in \{4,5\}$, the constrained Ramsey number $f(H,P_t)$ exhibits a surprising connection to the classical Ramsey number for H. For a graph F, we refer to a mapping $c : E(F) \to \{1,2,\ldots,k\}$ as a *k-edge-coloring* (not necessarily a proper edge-coloring) of F.[1] Given a graph H and a positive integer k, the *k-colored Ramsey number* $R_k(H)$ is the minimum integer n such that in every k-edge-coloring of the complete graph K_n, there is a monochromatic copy of H. Gyárfás, Lehel and Schelp [21] proved that $f(H,P_4) = R_2(H)$ for any graph H of order at least 5, and $f(H,P_5) = R_3(H)$ when H is a path or a cycle. Note that for any graph H, we have $f(H,P_t) \ge R_{t-2}(H)$ since a $(t-2)$-edge-colored graph trivially contains no rainbow P_t. In particular, we have

$$f(H,P_5) \ge R_3(H). \tag{1.1}$$

Motivated by these results, we propose the following conjecture.

Conjecture 1.2. *For any graph H, we have $f(H,P_5) = R_3(H)$.*

In fact, an equivalent conjecture was first posed by Li, Besse, Magnant, Wang and Watts [26] in the language of Gallai-Ramsey numbers. Given graphs G and H and a positive integer k, the *Gallai-Ramsey number $gr_k(G : H)$* is the minimum integer n such that every k-edge-coloring

[1] The mapping need not be surjective, so we do not require all the k colors to be used.

of K_n contains either a rainbow copy of G or a monochromatic copy of H.[2] In 2020, Li et al. conjectured that $gr_k(P_5 : H) = R_3(H)$ holds for any graph H and $k \geq 3$ (see [26, Conjecture 1]). Note that $f(H, P_5) = \max_{k \geq 1} gr_k(P_5 : H)$ and $gr_k(P_5 : H) \geq R_3(H)$ for $k \geq 3$. Therefore, the assertion that $f(H, P_5) = R_3(H)$ is equivalent to the assertion that $gr_k(P_5 : H) = R_3(H)$ holds for $k \geq 3$. Moreover, Li et al. obtained the following result (see [26, Lemmas 1 and 2]), generalizing the above mentioned result of Gyárfás, Lehel and Schelp [21].

Theorem 1.3. ([26]) *If H is a connected graph or a bipartite graph, then $f(H, P_5) = R_3(H)$.*

In this paper, we continue the study of constrained Ramsey number for rainbow P_5. According to Theorem 1.3, it suffices to study disconnected graphs with chromatic number at least 3. For such graphs H, Conjecture 1.2 was only confirmed in very few cases. In the following, we list all disconnected graphs H for which Conjecture 1.2 was confirmed (in the subsequent statements, the notation nG represents the disjoint union of n copies of a graph G):

- $H = tG$, where $t \geq 2$ and G is a 3-color-critical graph[3]; see [26, Theorem 5].
- $H = tK_r$, where $2 \leq t \leq r - 1$; see [26, Theorem 6].
- $H = 2G$, where G is a connected graph; see [26, Theorem 7].

In the remaining of this paper, we will study this problem along the following three natural directions:

(1) Given a graph H, find an integer k as small as possible such that $f(H, P_5) \leq R_k(H)$.

(2) Let H be the disjoint union of a connected graph and its subgraphs. Show that $f(H, P_5) = R_3(H)$.

(3) For graphs H of chromatic number 3, show that $f(H, P_5) = R_3(H)$.

We will present our results for each of these three directions in the following three subsections, respectively. We remark that all listed results above are special cases of our newly established general theorems, as shown below. In Section 6, we will also prove several results for a bipartite variation of the problem. Before presenting our main results, we first introduce some additional terminology and notation.

Notation. For a positive integer t, let $[t] := \{1, 2, \ldots, t\}$. A graph is called *nonempty* if it contains at least one edge. For two graphs G_1 and G_2, we use $G_1 \cup G_2$ to denote the disjoint union of G_1 and G_2, $G_1 \vee G_2$ to denote the join of G_1 and G_2, and $G_1 \subseteq G_2$ to denote that G_1 is a subgraph of G_2. For a graph G, let $\omega(G)$ be the *clique number* of G, that is, the maximum number of vertices in a complete subgraph of G. Let $\chi(G)$ be the *chromatic number* of G, that is, the minimum number of colors needed in a proper vertex-coloring of G. If $\chi(G) = k$, then G is said to be *k-chromatic*. For a k-chromatic graph G, let $\sigma(G)$ be the *chromatic surplus* of G, defined as the minimum number of vertices

[2] This definition does not require all k colors to be used. In the literature, a variant of the Gallai-Ramsey number requires that every color appears at least once (e.g., [31, 33, 44]). In the case $G = P_5$, these two definitions yield distinct values; see [33] for a concrete example.

[3] A graph is called *r-color-critical* if it has chromatic number r, and it has an edge (called a *critical edge*) whose removal reduces the chromatic number to $r - 1$.

in some color class under all proper k-vertex-colorings of G. Given an edge-colored graph G and an edge $e \in E(G)$, let $c(e)$ be the color assigned on edge e. For disjoint subsets $U, V \subset V(G)$, let $E(U,V) := \{uv \in E(G) \colon u \in U, v \in V\}$, $C(U,V) := \{c(e) \colon e \in E(U,V)\}$, and $C(U) := \{c(uv) \colon u, v \in U\}$. If $|C(U,V)| = 1$, then we use $c(U,V)$ to denote the unique color in $C(U,V)$. The subgraph of G induced by U is denoted by $G[U]$, and $G-U$ is shorthand for $G[V(G) \setminus U]$. If U consists of a single vertex u, then we simply write $E(\{u\},V)$, $C(\{u\},V)$, $c(\{u\},V)$ and $G-\{u\}$ as $E(u,V)$, $C(u,V)$, $c(u,V)$ and $G-u$, respectively. If $G' \subseteq G$, we use $G-G'$ to denote $G-V(G')$.

1.1 Integers k with $f(H,P_5) \leq R_k(H)$

Our first result is an upper bound on $f(H,P_5)$ in terms of the Ramsey number and chromatic number of H. As a corollary, we have $f(H,P_5) = R_3(H)$ for all bipartite graphs H.

Theorem 1.4. *For any nonempty graph H, we have $f(H,P_5) \leq R_{\chi(H)+1}(H)$.*

Let $G_1, G_2, \ldots, G_t$ be t graphs of the same order. We say that $G_1, G_2, \ldots, G_t$ are p-*homological* if $\max_{i\in[t]} \chi(G_i) = p$ and for every $i \in [t]$, G_i admits a proper p-vertex-coloring such that, for each $j \in [p]$, the number of vertices of color j in G_i is the same across all $i \in [t]$. In other words, the graphs $G_1, G_2, \ldots, G_t$ are p-homological if $\max_{i\in[t]} \chi(G_i) = p$ and there exist p positive integers $s_1, s_2, \ldots, s_p$ such that all of $G_1, G_2, \ldots, G_t$ are spanning subgraphs of $K_{s_1,\ldots,s_p}$. We can derive the following result for the disjoint union of homological graphs.

Theorem 1.5. *Let $G_1, G_2, \ldots, G_t$ be p-homological connected graphs. Then for $k = \max\{t,p,3\}$, we have $f(G_1 \cup G_2 \cup \cdots \cup G_t, P_5) \leq R_k(G_1 \cup G_2 \cup \cdots \cup G_t)$.*

Note that when $p \geq t$, Theorem 1.5 improves Theorem 1.4 in the case $H = G_1 \cup G_2 \cup \cdots \cup G_t$. Moreover, we have the following immediate consequence by Theorem 1.5 and Inequality (1.1).

Corollary 1.6. *For any connected graph G with $\chi(G) = 3$, we have $f(3G,P_5) = R_3(3G)$.*

1.2 Disjoint unions of a connected graph and its subgraphs

Our first result in this direction concerns the disjoint union of a connected graph and one of its connected subgraphs.

Theorem 1.7. *Let G_1, G_2 be two connected graphs with $G_1 \subseteq G_2$. Then $f(G_1 \cup G_2, P_5) = R_3(G_1 \cup G_2)$.*

As a corollary, we have $f(2G,P_5) = R_3(2G)$ for any connected graph G. For more graphs, we have the following result.

Theorem 1.8. *Let $G, G_1, \ldots, G_t$ be connected graphs with $t \leq \frac{(\chi(G)-2)(R_2(G)-1)+\sigma(G)-1}{\chi(G)|V(G)|}$ and $G_1, \ldots, G_t \subseteq G$. Then $f(G \cup G_1 \cup \cdots \cup G_t, P_5) = R_3(G \cup G_1 \cup \cdots \cup G_t)$.*

Note that $R_2(K_r) \geq r^2+1$ for all $r \geq 4$. Indeed, for $4 \leq r \leq 11$, we have $R_2(K_r) \geq r^2+1$ from the known lower bounds on $R_2(K_r)$ (see [38]); for $r \geq 12$, we have $R_2(K_r) \geq \binom{r-1}{3} \geq r^2+1$ from the well-known Nagy's construction. Hence, $\frac{(\chi(K_r)-2)(R_2(K_r)-1)+\sigma(K_r)-1}{\chi(K_r)|V(K_r)|} + 1 \geq \frac{(r-2)r^2}{r^2} + 1 = r-1$ when $r \geq 4$. This together with Theorems 1.7 and 1.8 implies that $f(tK_r, P_5) = R_3(tK_r)$ for $2 \leq t \leq r-1$. Moreover, we shall prove the following corollary of Theorem 1.8 in Section 4.

Corollary 1.9. *Let $G, G_1, \ldots, G_t$ be connected graphs with $\chi(G) \geq \frac{t+4+\sqrt{(t+4)^2-12}}{2}$ and $G_1, \ldots, G_t \subseteq G$. Then $f(G \cup G_1 \cup \cdots \cup G_t, P_5) = R_3(G \cup G_1 \cup \cdots \cup G_t)$.*

1.3 Graphs of chromatic number 3

In this subsection, we focus on graphs of chromatic number 3. The first result concerns disconnected graphs whose components are either 3-color-critical or bipartite.

Theorem 1.10. *For any graph H whose components are either 3-color-critical or bipartite, we have $f(H, P_5) = R_3(H)$.*

As an immediate consequence, we have $f(tG, P_5) = R_3(tG)$, where $t \geq 2$ and G is a 3-color-critical graph. Moreover, since odd cycles are 3-color-critical graph and even cycle are bipartite graphs, we have the following corollary.

Corollary 1.11. *Let H be any disjoint union of cycles. Then $f(H, P_5) = R_3(H)$.*

Recall that $\sigma(G)$ is the chromatic surplus of a graph G. Given a graph G with $\chi(G) \leq 3$, let $\sigma_3(G) := \sigma(G)$ if $\chi(G) = 3$, and $\sigma_3(G) := 0$ otherwise. In the following, we present two results related to $\sigma(G)$ or $\sigma_3(G)$.

Theorem 1.12. *For any $1 \leq s \leq t$, let $G_1, \dots, G_s$ be s connected graphs of chromatic number 3 and $G_{s+1}, \dots, G_t$ be $t - s$ connected bipartite graphs. Suppose that for each $i \in [t]$, G_i admits a proper 3-vertex-coloring with color classes of sizes $a_i \geq b_i \geq c_i = \sigma_3(G_i)$ such that*

(i) *$a_i - b_i \leq 1$ for each $i \in [s]$, $a_i = b_i$ for each $i \in \{s+1, \dots, t\}$, and*

(ii) *$b_i + c_i \geq \sum_{j \in [t] \setminus \{i\}} c_j$ for each $i \in [t]$.*

Then $f(G_1 \cup \cdots \cup G_t, P_5) = R_3(G_1 \cup \cdots \cup G_t)$.

Theorem 1.13. *Let H be a 3-chromatic graph satisfying*

(i) *H has exactly one component of chromatic number 3, and*

(ii) *every component of H has order at least $\sigma(H)$.*

Then $f(H, P_5) = R_3(H)$.

If H is a graph with $\chi(H) = 3$ and $\sigma(H) = 1$, then H has exactly one component of chromatic number 3. Thus by Theorem 1.13, we have the following corollary.

Corollary 1.14. *Let H be a graph with $\chi(H) = 3$ and $\sigma(H) = 1$. Then $f(H, P_5) = R_3(H)$.*

The remainder of this paper is organized as follows. In the next section, we provide some results that will be used in our proofs. In Section 3, we prove Theorems 1.4 and 1.5. In Section 4, we present our proofs of Theorems 1.7, 1.8 and Corollary 1.9. In Section 5, we prove Theorems 1.10, 1.12 and 1.13. Finally, in Section 6, we conclude this paper by presenting several open problems and making some remarks to illustrate the difficulty in solving Conjecture 1.2, and in particular, we present several new results for a bipartite variation of the problem.

2 Preliminaries

Firstly, we point out that in order to study Conjecture 1.2, it suffices to focus on graphs without isolated vertices. This follows from the following proposition.

Proposition 2.1. *Let H' be obtained from a graph H by adding an isolated vertex. If $f(H, P_5) = R_3(H)$, then $f(H', P_5) = R_3(H')$.*

Proof. By Inequality (1.1), it suffices to prove that $f(H', P_5) \leq R_3(H')$. Note that $R_3(H') \geq R_3(H)$ and $R_3(H') \geq |V(H')|$. Let F be an edge-colored $K_{R_3(H')}$. Since $R_3(H') \geq R_3(H) = f(H, P_5)$, F contains either a rainbow P_5 or a monochromatic H. If F contains a monochromatic H, then since $R_3(H') \geq |V(H')|$ and H' is obtained from H by adding an isolated vertex, F also contains a monochromatic H'. Hence, $f(H', P_5) \leq R_3(H')$. □

In the following, we introduce some Ramsey-type results and structural results that will be used in our proofs.

2.1 Ramsey-type results

For a set $\mathscr{H}$ of graphs, the Ramsey number $R_k(\mathscr{H})$ denotes the minimum integer n such that every k-edge-colored K_n contains a monochromatic copy of some graph in $\mathscr{H}$. For a disconnected graph H, let $\mathscr{C}(H)$ be the set of all connected graphs containing H as a subgraph (not necessarily spanning or induced). Li, Besse, Magnant, Wang and Watts [26] obtained the following result linking $f(H, P_5)$, $R_3(H)$ and $R_2(\mathscr{C}(H))$.

Lemma 2.2. ([26, Lemma 3]) *For any disconnected graph H, if $R_3(H) \geq R_2(\mathscr{C}(H))$, then $f(H, P_5) = R_3(H)$.*

For two graphs G_1 and G_2, the Ramsey number $R(G_1, G_2)$ denotes the minimum integer n such that every 2-edge-colored K_n contains either a monochromatic G_1 of color 1 or a monochromatic G_2 of color 2. We shall use the following results on 2-colored Ramsey numbers.

Lemma 2.3. *The following results have been established.*

(i) ([6]) *For any connected graph H, we have $R_2(H) \geq (\chi(H) - 1)(|V(H)| - 1) + \sigma(H)$.*

(ii) ([9]) *For any tree T_{t+1} on $t + 1$ vertices, we have $R(K_m, T_{t+1}) = t(m - 1) + 1$.*

(iii) ([15]) *For any graph H with no isolated vertices, independence number c and such that $K_c \cup K_{\lceil c/2 \rceil} \subseteq \overline{G}$, we have $R(mK_2, H) = \max\{|V(H)| + 2m - c - 1, |V(H)| + m - 1\}$.*

Applying Lemma 2.3 (iii), we can derive the following corollary.

Corollary 2.4. *For any graph H with no isolated vertices and with c components, if $m \leq c$, then $R(mK_2, H) = |V(H)| + m - 1$.*

Proof. We first show that $R(mK_2, H) > |V(H)| + m - 2$ by constructing a 2-edge-colored $K_{|V(H)|+m-2}$ as follows. We partition the vertex set into two subsets V_1 and V_2 with $|V_1| = m-1$ and $|V_2| = |V(H)| - 1$. We color all edges within V_1 and all edges between V_1 and V_2 with color 1, and all edges within V_2 with color 2. Then there is no monochromatic mK_2 of color 1 or monochromatic H of color 2.

We next show that $R(mK_2, H) \le |V(H)| + m - 1$. Let $H_1, \dots, H_c$ be the components of H. Since H contains no isolated vertices, we have $|V(H_i)| \ge 2$ for all $i \in [c]$. Let $K = K_{|V(H_1)|} \cup \cdots \cup K_{|V(H_c)|}$. Then $|V(H)| = |V(K)|$ and $R(mK_2, H) \le R(mK_2, K)$, so it suffices to show that $R(mK_2, K) \le |V(K)| + m - 1$. Note that the independence number of K is c and $2K_c \subseteq \overline{G}$. Since $m \le c$ and by Lemma 2.3 (iii), we have $R(mK_2, K) = \max\{|V(K)| + 2m - c - 1, |V(K)| + m - 1\} = |V(K)| + m - 1$. The proof is complete. $\square$

We next state and prove a lemma related to lower bounds on 3-colored Ramsey numbers.

Lemma 2.5. *For connected nonempty graphs $G_1, \dots, G_t$, the following statements hold.*

(i) $R_3(G_1 \cup \cdots \cup G_t) \ge \max_{1 \le i,j,\ell \le t} \left\{ (\chi(G_i) - 1)(R(G_j, G_\ell) - 1) + \sum_{k \in [t], \chi(G_k) = \chi(G_i)} \sigma(G_k) \right\}$.

(ii) $R_3(G_1 \cup \cdots \cup G_t) \ge \max_{1 \le i,j \le t} \left\{ \left(R_2\left(K_{\omega(G_i)}\right) - 1\right)\left(|V(G_j)| - 1\right) + 1 \right\}$.

(iii) *If $\chi(G_i) = 3$ for all $i \in [t]$, then $R_3(G_1 \cup \cdots \cup G_t) \ge 3\sum_{i \in [t]} |V(G_i)| - 2$.*

(iv) *If $\max_{i \in [t]} \chi(G_i) = 3$, then $R_3(G_1 \cup \cdots \cup G_t) \ge 2\sum_{i \in [t]} |V(G_i)| + \sum_{i \in [t]} \sigma_3(G_i) - 2$.*

Proof. We shall prove the results by constructing appropriate 3-edge-colored complete graphs without monochromatic copies of $G_1 \cup \cdots \cup G_t$.

(i) We choose i, j, ℓ with $1 \le i, j, \ell \le t$ arbitrarily. Let F be a 3-edge-colored K_n defined as follows, where $n = (\chi(G_i) - 1)(R(G_j, G_\ell) - 1) + \sum_{k \in I_i} \sigma(G_k) - 1$ and $I_i := \{k \in [t]\colon \chi(G_k) = \chi(G_i)\}$. We partition $V(F)$ into $\chi(G_i)$ subsets $V_1, V_2, \dots, V_{\chi(G_i)}$ such that $|V_m| = R(G_j, G_\ell) - 1$ for each $1 \le m \le \chi(G_i) - 1$, and $|V_{\chi(G_i)}| = \sum_{k \in I_i} \sigma(G_k) - 1$. For each $1 \le m \le \chi(G_i) - 1$, we color the edges within V_m using colors 1 and 2 so that it contains neither a monochromatic G_j of color 1 nor a monochromatic G_ℓ of color 2. We color all the remaining edges using color 3. Note that F contains neither a monochromatic G_j of color 1 nor a monochromatic G_ℓ of color 2. Thus F contains no monochromatic $G_1 \cup \cdots \cup G_t$ of color 1 or 2. For each $k \in I_i$, since $\chi(G_k) = \chi(G_i)$, every monochromatic G_k of color 3 must contains at least $\sigma(G_k)$ vertices in each of $V_1, V_2, \dots, V_{\chi(G_i)}$. But $|V_{\chi(G_i)}| = \sum_{k \in I_i} \sigma(G_k) - 1$, so F contains no monochromatic $G_1 \cup \cdots \cup G_t$ of color 3.

(ii) We choose i, j with $1 \le i, j \le t$ arbitrarily. Let F be a 3-edge-colored K_n defined as follows, where $n = \left(R_2\left(K_{\omega(G_i)}\right) - 1\right)\left(|V(G_j)| - 1\right)$. We partition $V(F)$ into $R_2\left(K_{\omega(G_i)}\right) - 1$ subsets $V_1, V_2, \dots, V_{R_2\left(K_{\omega(G_i)}\right)-1}$ each of size $|V(G_j)| - 1$. By the definition, there exists a 2-edge-colored $K_{R_2\left(K_{\omega(G_i)}\right)-1}$ using colors 1 and 2 without monochromatic copies of $K_{\omega(G_i)}$. Denoted such a 2-edge-colored $K_{R_2\left(K_{\omega(G_i)}\right)-1}$ by K, and let $V(K) = \left\{v_1, v_2, \dots, v_{R_2\left(K_{\omega(G_i)}\right)-1}\right\}$. We color the edges between the $R_2\left(K_{\omega(G_i)}\right) - 1$ parts of F so that it forms a blow-up of K, that is, for each $1 \le m_1 < m_2 \le R_2\left(K_{\omega(G_i)}\right) - 1$, we color all edges between V_{m_1} and V_{m_2} using the color of edge $v_{m_1}v_{m_2}$ in K. We color all the remaining edges in F using color 3. Note that F contains no monochromatic G_j of color 3, so it contains no monochromatic $G_1 \cup \cdots \cup G_t$ of color 3. Moreover, F contains no monochromatic $K_{\omega(G_i)}$ of color 1 or 2, so it contains no monochromatic G_i (and thus no monochromatic $G_1 \cup \cdots \cup G_t$) of color 1 or 2.

(iii) Let F be a 3-edge-colored K_n defined as follows, where $n = 3\sum_{i \in [t]} |V(G_i)| - 3$. We partition $V(F)$ into three subsets V_1, V_2, V_3 each of size $\sum_{i \in [t]} |V(G_i)| - 1$. We color all edges within V_1 and all edges between V_2 and V_3 with color 1, all edges within V_2 and all edges between V_3 and V_1 with color 2, and all edges within V_3 and all edges between V_1 and V_2 with color 3. Since $\chi(G_i) = 3$ for all $i \in [t]$, every monochromatic copy of G_i of color m ($m \in [3]$)

in F must be contained in $F[V_m]$. Since $|V_1| = |V_2| = |V_3| = \sum_{i\in[t]} |V(G_i)| - 1$, there is no monochromatic $G_1 \cup \cdots \cup G_t$ in F.

(iv) Let F be a 3-edge-colored K_n defined as follows, where $n = 2\sum_{i\in[t]} |V(G_i)| + \sum_{i\in[t]} \sigma_3(G_i) - 3$. We partition $V(F)$ into three subsets V_1, V_2, V_3 with $|V_1| = |V_2| = \sum_{i\in[t]} |V(G_i)| - 1$ and $|V_3| = \sum_{i\in[t]} \sigma_3(G_i) - 1$. We color all edges within V_1 with color 1, all edges within V_2 with color 2, and all the remaining edges with color 3. Note that F contains no monochromatic $G_1 \cup \cdots \cup G_t$ of color 1 or 2. Without loss of generality, we may assume that $G_1, \ldots, G_m$ are all the graphs of chromatic number 3 in $G_1, \ldots, G_t$. Then $|V_3| = \sum_{i\in[t]} \sigma_3(G_i) - 1 = \sum_{i\in[m]} \sigma(G_i) - 1$, and every monochromatic G_i ($i \in [m]$) of color 3 must contain at least $\sigma(G_i)$ vertices in V_3. Hence, there is no monochromatic $G_1 \cup \cdots \cup G_t$ of color 3 in F. $\square$

In the study of extremal graph theory problems, Simonovits [41] introduced the decomposition family of graphs. Given a graph H with $\chi(H) = p \geq 3$, the *decomposition family* $\mathcal{M}(H)$ of H is the set of minimal graphs M that satisfies the following: for each M, there exists an integer t such that $H \subseteq \left(M \cup \overline{K}_{t-|V(M)|}\right) \vee K_{(p-2)\times t}$, where $K_{(p-2)\times t}$ is the complete $(p-2)$-partite graph with t vertices in each partite set. In other words, a graph M is in $\mathcal{M}(H)$ if the graph obtained from embedding M (but not any of its proper subgraphs) into one partite set of a complete $(p-1)$-partite graph with large partite sets (e.g., each of size at least $|V(H)|$) contains H as a subgraph. We shall use the following Ramsey-type results related to the decomposition family of a graph.

Lemma 2.6. *For two connected graphs G and H, the following statements hold.*

(i) $R_2(H) \geq R(H, \mathcal{M}(H)) + (\chi(H) - 2)(|V(H)| - 1)$.

(ii) $R(G \cup H, \mathcal{M}(H)) \leq \max\{R(G, H), R(H, \mathcal{M}(H)) + |V(G)|\}$.

Proof. (i) Let F be a 2-edge-colored K_n defined as follows, where $n = R(H, \mathcal{M}(H)) + (\chi(H) - 2)(|V(H)| - 1) - 1$. We partition $V(F)$ into $\chi(H) - 1$ subsets $V_1, \ldots, V_{\chi(H)-1}$ with $|V_{\chi(H)-1}| = R(H, \mathcal{M}(H)) - 1$ and $|V_i| = |V(H)| - 1$ for $i \in \{1, \ldots, \chi(H) - 2\}$. We color the edges within $V_{\chi(H)-1}$ so that it contains neither a monochromatic H of color 1 nor a monochromatic graph in $\mathcal{M}(H)$ of color 2. We color all edges within V_i (for $i \in \{1, \ldots, \chi(H) - 2\}$) using color 1, and all the remaining edges using color 2. Since H is connected, there is no monochromatic H of color 1 in F. Moreover, since $F[V_{\chi(H)-1}]$ contains no monochromatic graph in $\mathcal{M}(H)$ of color 2, there is also no monochromatic H of color 2 in F.

(ii) Let $n = \max\{R(G, H), R(H, \mathcal{M}(H)) + |V(G)|\}$. For a contradiction, suppose that F is a 2-edge-colored K_n with no monochromatic $G \cup H$ of color 1 or monochromatic graph in $\mathcal{M}(H)$ of color 2. Note that for any graph $M \in \mathcal{M}(H)$, M is a subgraph of H. Then since $n \geq R(G, H)$, F must contains a monochromatic copy A of G of color 1. Let $F' = F - A$. Then $|V(F')| = n - |V(G)| \geq R(H, \mathcal{M}(H)) + |V(G)| - |V(G)| = R(H, \mathcal{M}(H))$. Thus F contains a monochromatic copy B of H of color 1. But then $A \cup B$ is a monochromatic $G \cup H$ of color 1, a contradiction. $\square$

Given a bipartite graph H and a positive integer k, the *k-colored bipartite Ramsey number* $BR_k(H)$ is the minimum integer n such that every k-edge-coloring of the complete bipartite graph $K_{n,n}$ contains a monochromatic copy of H. Note that given a bipartite graph H, if H is disconnected, then there possibly exist a variety of distinct proper 2-edge-colorings, and thus there are many distinct ways to define its partite sets. We define

$$s(H) := \min\{|S| \colon S \text{ and } T \text{ are the partite sets of } H \text{ with } |S| \leq |T|\},$$

$$t(H) := \max\{|T|\colon S \text{ and } T \text{ are the partite sets of } H \text{ with } |S| \le |T|\},$$
$$s^*(H) := \max\{|S|\colon S \text{ and } T \text{ are the partite sets of } H \text{ with } |S| \le |T|\},$$
$$t^*(H) := \min\{|T|\colon S \text{ and } T \text{ are the partite sets of } H \text{ with } |S| \le |T|\}.$$

Then $|V(H)| = s(H) + t(H) = s^*(H) + t^*(H)$, $s(H) \le s^*(H) \le \frac{1}{2}|V(H)| \le t^*(H) \le t(H)$. Moreover, if H is connected, then $s(H) = s^*(H)$ and $t(H) = t^*(H)$. We have the following lower bound on $BR_k(H)$ for connected bipartite graphs. Note that this lower bound is best possible since $BR_k(K_{1,n}) = k(n-1)+1$ (see [22]).

Lemma 2.7. *For any connected bipartite graph H, we have $BR_k(H) \ge k(t(H)-1)+1$.*

Proof. It is well-known that the complete bipartite graph $K_{k,k}$ can be decomposed into k edge-disjoint perfect matchings. Thus we can obtain a k-edge-coloring G of $K_{k,k}$ such that each color induces a perfect matching. We construct a k-edge-coloring F of $K_{k(t(H)-1),k(t(H)-1)}$ by substituting each vertex of G with $t(H)-1$ new vertices, i.e., F is a blow-up of G. Then every color induces a $kK_{t(H)-1,t(H)-1}$ in F. Since H is connected and $K_{t(H)-1,t(H)-1}$ contains no H, there exists no monochromatic copy of H in F. Hence, we have $BR_k(H) \ge k(t(H)-1)+1$. $\square$

2.2 Structural results

In 2007, Thomason and Wagner [42] characterized the structure of edge-colored complete graphs with no rainbow P_5 (the result is complicated, and we refer the interested readers to [42, Theorem 2] for more information). Using the result of Thomason and Wagner, Li, Besse, Magnant, Wang and Watts [26] derived the following technical lemma (see [26, Lemma 1 and Remark 1]).

Lemma 2.8. ([26]) *Let H be a nonempty graph and $n \ge R_3(H)$. Suppose that K_n is edge-colored using exactly k colors such that it contains neither a rainbow P_5 nor a monochromatic H. Then we can partition the vertex set into $k-1$ parts $V_2, V_3, \ldots, V_k$ satisfying (renumbering the colors if necessary)*[4]

(i) *$\{i\} \subseteq C(V_i) \subseteq \{1, i\}$ for every $i \in \{2, 3, \ldots, k\}$, and*

(ii) *$c(V_i, V_j) = 1$ for every $2 \le i < j \le k$.*

Applying Lemma 2.8, we can further derive the following result.

Lemma 2.9. *Let H be a nonempty graph with components $H_1, \ldots, H_t$, and let $n \ge R_3(H)$. Suppose that K_n is edge-colored with no rainbow P_5 or monochromatic H. Then we can partition the vertex set into $k-1$ parts $V_2, V_3, \ldots, V_k$ for some $k \ge 4$ satisfying (renumbering the colors if necessary)*

(i) *$\{i\} \subseteq C(V_i) \subseteq \{1, i\}$ for every $i \in \{2, 3, \ldots, k\}$,*

(ii) *$c(V_i, V_j) = 1$ for every $2 \le i < j \le k$, and*

(iii) *$|V_2| \ge |V_3| \ge \cdots \ge |V_k|$, $|V_3| \ge \max_{i\in[t]} |V(H_i)|$, $|V_4| \ge \min_{i\in[t]} |V(H_i)|$ and $|V_3 \cup \cdots \cup V_k| \ge |V(H)|$.*

[4] We remark that there is no V_1 in this partition.

Proof. Let F be an edge-colored K_n using exactly k colors such that it contains neither a rainbow P_5 nor a monochromatic H. Then $k \geq 4$; otherwise F contains a monochromatic copy of H since $n \geq R_3(H)$. By Lemma 2.8, we can partition $V(F)$ into $k-1$ parts $V_2, V_3, \ldots, V_k$ satisfying (i) and (ii). Without loss of generality, we may assume that $|V_2| \geq |V_3| \geq \cdots \geq |V_k|$. Let F' be a 3-edge-coloring of K_n obtained from F by recoloring all edges of colors in $\{4, \ldots, k\}$ with color 3. Since $n \geq R_3(H)$, there is a monochromatic copy of H in F'. Denote such a monochromatic H by A. Since F contains no monochromatic copy of H, the color of A must be 3 and the components of A must be contained in at least two distinct parts of $V_3, V_4, \ldots, V_k$. Combining with $|V_2| \geq |V_3| \geq \cdots \geq |V_k|$, we have $|V_3| \geq \max_{i \in [t]} |V(H_i)|$, $|V_4| \geq \min_{i \in [t]} |V(H_i)|$ and $|V_3 \cup \cdots \cup V_k| \geq |V(H)|$. □

In 2019, Li, Wang and Liu [33] characterized the structures of edge-colored complete bipartite graphs without rainbow P_4 or P_5.[5]

Lemma 2.10. ([33]) *Let $K_{n,n}$ ($n \geq 2$) be edge-colored with exactly k ($k \geq 3$) colors such that it contains no rainbow P_4. Let U and V be the partite sets. Then U can be partitioned into k parts $U_1, U_2, \ldots, U_k$ such that $c(U_i, V) = i$ for every $i \in [k]$ (renumbering the colors or replacing U and V if necessary).*

Lemma 2.11. ([33]) *Let $K_{n,n}$ ($n \geq 3$) be edge-colored with exactly k ($k \geq 4$) colors such that it contains no rainbow P_5. Let U and V be the partite sets. Then one of the following holds (renumbering the colors or replacing U and V if necessary):*

(a) *U can be partitioned into two parts U_1, U_2 with $|U_1| \geq 1$ and $|U_2| \geq 0$, and V can be partitioned into k parts $V_1, V_2, \ldots, V_k$ with $|V_1| \geq 0$ and $|V_i| \geq 1$ for every $i \in \{2, 3, \ldots, k\}$, such that $c(U_2, V) = 1$ and $c(U_1, V_i) = i$ for every $i \in [k]$;*

(b) *U can be partitioned into $k-1$ parts $U_2, U_3, \ldots, U_k$, and V can be partitioned into $k-1$ parts $V_2, V_3, \ldots, V_k$, such that $\{i\} \subseteq C(U_i, V_i) \subseteq \{1, i\}$ for every $i \in \{2, 3, \ldots, k\}$, and $c(U_i, V_j) = 1$ for every $2 \leq i \neq j \leq k$;*

(c) *$n \in \{3, 4\}$, $k = 4$, and each color induces a matching.*

We shall also use the following structural result.

Observation 2.12. *Let G_1, G_2 be two graphs of chromatic number 3. Let s_1, s_2, s_3 (resp., t_1, t_2, t_3) be the sizes of color classes in a proper 3-vertex-coloring of G_1 (resp., G_2) with $s_1 \geq s_2 \geq s_3$ (resp., $t_1 \geq t_2 \geq t_3$). If one of the following two statements holds, then $K_{x,y,z}$ contains $G_1 \cup G_2$ as a subgraph:*

(i) *$x \geq \max\{s_1 + t_2, t_1 + s_2\}$, $y \geq \min\{s_1 + t_2, t_1 + s_2\}$ and $z \geq s_3 + t_3$, or*

(ii) *$x \geq s_1 + t_1$, $y \geq s_2 + t_2$ and $z \geq s_3 + t_3$.*

Proof. The proofs of (i) and (ii) follow from a similar argument. For brevity, we only present the proof of (i). Let F be a $K_{x,y,z}$ with partite sets X, Y and Z, where $|X| = x$, $|Y| = y$ and $|Z| = z$. Without loss of generality, we may assume that $s_1 + t_2 \geq t_1 + s_2$. Then $|X| = x \geq s_1 + t_2$ and $|Y| = y \geq t_1 + s_2$. We choose disjoint subsets $S_1, T_2 \subseteq X$, $S_2, T_1 \subseteq Y$ and $S_3, T_3 \subseteq Z$ with $|S_i| = s_i$ and $|T_i| = t_i$ for each $i \in [3]$. Then $F[S_1 \cup S_2 \cup S_3]$ contains G_1 as a subgraph, and $F[T_1 \cup T_2 \cup T_3]$ contains G_2 as a subgraph. Hence, F contains a $G_1 \cup G_2$. □

[5]The initial assertions made by Li, Wang and Liu are somewhat complicated (see [33, Theorems 1.4 and 1.5]). Lemmas 2.10 and 2.11 serve as refined reformulations of their results.

3 Proofs of results related to $f(H, P_5) \leq R_k(H)$

In this section, we present our proofs of Theorems 1.4 and 1.5.

Proof of Theorem 1.4. For a contradiction, suppose that F is an edge-colored K_n without rainbow P_5 or monochromatic H, where $n = R_{\chi(H)+1}(H)$. Let $[k]$ be the set of colors used on $E(F)$. Then $k \geq \chi(H) + 2$, since otherwise F contains a monochromatic copy of H. By Lemma 2.8, we can partition $V(F)$ into $k - 1$ parts $V_2, V_3, \ldots, V_k$ satisfying

(i) $\{i\} \subseteq C(V_i) \subseteq \{1, i\}$ for every $i \in \{2, 3, \ldots, k\}$, and

(ii) $c(V_i, V_j) = 1$ for every $2 \leq i < j \leq k$.

Choose a partition $I_1, \ldots, I_{\chi(H)}$ of $\{2, 3, \ldots, k\}$ and correspondingly define $A_1 = \bigcup_{i \in I_1} V_i, \ldots, A_{\chi(H)} = \bigcup_{i \in I_{\chi(H)}} V_i$ such that the following conditions hold:

(1) $|A_1| \geq \cdots \geq |A_{\chi(H)}|$,

(2) $|A_{\chi(H)}|$ is largest subject to (1),

(3) $|A_{\chi(H)-1}|$ is largest subject to (1) and (2),

$\vdots$

$(\chi(H))$ $|A_2|$ is largest subject to $(1), (2), \ldots, (\chi(H) - 1)$.

Claim 3.1. *If $|I_\ell| \geq 2$ for some $1 \leq \ell \leq \chi(H)$, then $|A_{\chi(H)}| \geq \frac{1}{2}|A_\ell|$.*

Proof. Suppose for a contradiction that there exists an ℓ with $|I_\ell| \geq 2$ and $|A_{\chi(H)}| < \frac{1}{2}|A_\ell|$. Let i_0 be the smallest index with $|A_{i_0}| = |A_{i_0+1}| = \cdots = |A_{\chi(H)}|$. Then $\ell < i_0 \leq \chi(H)$, $|A_{i_0-1}| > |A_{i_0}|$ and $|A_{i_0}| < \frac{1}{2}|A_\ell|$. Since $|I_\ell| \geq 2$, there exists a $j \in I_\ell$ with $|V_j| \leq \frac{1}{2}|A_\ell|$. Let

$$I'_i = \begin{cases} I_i & \text{for } i \in \{1, 2, \ldots, \chi(H)\} \setminus \{\ell, i_0\}, \\ I_i \setminus \{j\} & \text{for } i = \ell, \\ I_i \cup \{j\} & \text{for } i = i_0. \end{cases}$$

Let φ be a permutation of $1, 2, \ldots, \chi(H)$ such that $\left|\bigcup_{i \in I'_{\varphi(1)}} V_i\right| \geq \cdots \geq \left|\bigcup_{i \in I'_{\varphi(\chi(H))}} V_i\right|$.

If $|A_\ell \setminus V_j| \leq |A_{i_0} \cup V_j|$, then since $|A_{i_0}| < \frac{1}{2}|A_\ell|$ and $|V_j| \leq \frac{1}{2}|A_\ell|$, we have

$$|A_{i_0} \cup V_j| \geq |A_\ell \setminus V_j| = |A_\ell| - |V_j| \geq |A_\ell| - \frac{1}{2}|A_\ell| = \frac{1}{2}|A_\ell| > |A_{i_0}|.$$

If $|A_\ell \setminus V_j| > |A_{i_0} \cup V_j|$, then $|A_\ell \setminus V_j| > |A_{i_0} \cup V_j| > |A_{i_0}|$. In both cases, we have $\left|\bigcup_{i \in I'_{\varphi(i_0)}} V_i\right| > |A_{i_0}|$, and $|A_{i_0+1}| = \cdots = |A_{\chi(H)}| = \left|\bigcup_{i \in I'_{\varphi(\chi(H))}} V_i\right| = \cdots = \left|\bigcup_{i \in I'_{\varphi(i_0+1)}} V_i\right|$ (when $\chi(H) \geq i_0 + 1$). This contradicts the choice of the partition $I_1, \ldots, I_{\chi(H)}$ of $\{2, 3, \ldots, k\}$. $\square$

Let F' be a $(\chi(H) + 1)$-edge-coloring of K_n obtained from F by recoloring all edges of colors in I_i with color $i + 1$ for every $1 \leq i \leq \chi(H)$. Since $n = R_{\chi(H)+1}(H)$, there is a monochromatic copy of H in F', say of color m. Since F contains no monochromatic copy of H, we have $m \geq 2$ and $|I_{m-1}| \geq 2$. By Claim 3.1, we have $|A_{\chi(H)}| \geq \frac{1}{2}|A_{m-1}|$.

Hence, $|A_1| \geq \cdots \geq |A_{m-1}| \geq |V(H)|$ and $|A_m| \geq \cdots \geq |A_{\chi(H)}| \geq \frac{1}{2}|A_{m-1}| \geq \frac{1}{2}|V(H)|$. Let $s_1, s_2, \ldots, s_{\chi(H)}$ be the sizes of color classes in a proper $\chi(H)$-vertex-coloring of H, and assume that $s_1 \geq s_2 \geq \cdots \geq s_{\chi(H)}$. Then $|A_1| \geq |V(H)| \geq s_1$ and $|A_i| \geq \frac{1}{2}|V(H)| \geq s_2$ for each $2 \leq i \leq \chi(H)$. Thus the edges between the parts $A_1, A_2, \ldots, A_{\chi(H)}$ in F form a monochromatic graph of color 1 that contains $K_{s_1, s_2, \ldots, s_2}$ (and thus H) as a subgraph, a contradiction. This completes the proof of Theorem 1.4. $\square$

Proof of Theorem 1.5. Let $H = G_1 \cup G_2 \cup \cdots \cup G_t$. Since $G_1, G_2, \ldots, G_t$ is p-homological, we may assume that for every $i \in [t]$, G_i admits a proper p-vertex-coloring such that, for each $j \in [p]$, the number of vertices of color j in G_i is s_j. Then $G_1, G_2, \ldots, G_t$ are subgraphs of $K_{s_1, \ldots, s_p}$. Let $s = s_1 + s_2 + \cdots + s_p$, so $|V(G_1)| = |V(G_2)| = \cdots = |V(G_t)| = s$.

For a contradiction, suppose that F is an edge-colored K_n without rainbow P_5 or monochromatic H, where $n = R_k(H)$ and $k = \max\{t, p, 3\}$. Let $[k']$ be the set of colors used on $E(F)$. Then $k' \geq k + 1 = \max\{t, p, 3\} + 1$, since otherwise F contains a monochromatic copy of H. By Lemma 2.8, we can partition $V(F)$ into $k' - 1$ parts $V_2, V_3, \ldots, V_{k'}$ satisfying

(i) $\{i\} \subseteq C(V_i) \subseteq \{1, i\}$ for every $i \in \{2, 3, \ldots, k'\}$, and

(ii) $c(V_i, V_j) = 1$ for every $2 \leq i < j \leq k'$.

Without loss of generality, we may assume that $|V_2| \geq |V_3| \geq \cdots \geq |V_{k'}|$.

Let F' be a k-edge-coloring of K_n obtained from F by recoloring all edges of colors in $\{k+1, \ldots, k'\}$ with color k. Since $n = R_k(H)$, there is a monochromatic copy of H in F'. Denote such a monochromatic H by A. Since F contains no monochromatic copy of H, the color of A must be k and the components of A must appear in at least two distinct parts from $V_k, V_{k+1}, \ldots, V_{k'}$. Let i_0 be the largest index such that A has a component within V_{i_0}. Then $i_0 \geq k + 1$ and $|V_2| \geq |V_3| \geq \cdots \geq |V_{k+1}| \geq |V_{i_0}| \geq s$.

For each $i \in \{2, 3, \ldots, k+1\}$, let $V_{i,1}, \ldots, V_{i,p}$ be disjoint subsets of V_i with $|V_{i,j}| = s_j$ for each $j \in [p]$. Let[6]

$$
\begin{aligned}
W_1 &= V_{2,1} \cup V_{3,2} \cup \cdots \cup V_{p+1,p}, \\
W_2 &= V_{2,2} \cup V_{3,3} \cup \cdots \cup V_{p,p} \cup V_{k+1,1}, \\
W_3 &= V_{2,3} \cup V_{3,4} \cup \cdots \cup V_{p-1,p} \cup V_{k,1} \cup V_{k+1,2}, \\
&\vdots \\
W_p &= V_{2,p} \cup V_{k-p+3,1} \cup V_{k-p+4,2} \cup \cdots \cup V_{k+1,p-1}, \\
W_{p+1} &= V_{k-p+2,1} \cup V_{k-p+3,2} \cup \cdots \cup V_{k+1,p}, \\
W_{p+2} &= V_{k-p+1,1} \cup V_{k-p+2,2} \cup \cdots \cup V_{k,p}, \\
&\vdots \\
W_k &= V_{3,1} \cup V_{4,2} \cup \cdots \cup V_{p+2,p}.
\end{aligned}
$$

Then $W_1, W_2, \ldots, W_k$ are disjoint vertex subsets. Moreover, for each $i \in [k]$, $F[W_i]$ contains a monochromatic $K_{s_1, \ldots, s_p}$ of color 1. Since $k \geq t$, there is a monochromatic $G_1 \cup G_2 \cup \cdots \cup G_t$ in F. This contradiction completes the proof of Theorem 1.5. $\square$

[6]For clarity, we remark here that when $k = p$, only $W_1, W_2, \ldots, W_p (= W_k)$ exist, while all other W_i do not exist.

4 Proofs of results for the disjoint unions of a connected graph and its subgraphs

In this section, we present our proofs of Theorems 1.7, 1.8 and Corollary 1.9. We first prove Theorem 1.8 and Corollary 1.9. In our proof of Theorem 1.8, we shall apply Lemma 2.2. To this ends, we first prove the following result on $R_2(\mathscr{C}(G \cup G_1 \cup \cdots \cup G_t))$.

Lemma 4.1. *Let $G, G_1, \ldots, G_t$ be connected graphs with $t \leq \frac{(\chi(G)-2)(R_2(G)-1)+\sigma(G)-1}{\chi(G)|V(G)|}$ and $G_1, \ldots, G_t \subseteq G$. Then $R_2(\mathscr{C}(G \cup G_1 \cup \cdots \cup G_t)) \leq (\chi(G)-1)(R_2(G)-1)+\sigma(G)$.*

Proof. For a contradiction, suppose that F is a 2-edge-colored K_n without monochromatic copies of any graph in $\mathscr{C}(G \cup G_1 \cup \cdots \cup G_t)$, where $n = (\chi(G)-1)(R_2(G)-1)+\sigma(G)$. An old observation of Erdős and Rado states that every 2-edge-colored complete graph contains a monochromatic spanning tree. Hence, without loss of generality, we may assume that F contains a monochromatic spanning tree of color 1. Then F contains no monochromatic $G \cup G_1 \cup \cdots \cup G_t$ (and thus no monochromatic $(t+1)G$) of color 1. We choose a set $\{A_1, \ldots, A_k\}$ of disjoint monochromatic copies of G of color 1 in F such that k is maximal. Then $k \leq t$. Let $F' = F - (A_1 \cup \cdots \cup A_k)$. Then F' contains no monochromatic G of color 1.

Since $t \leq \frac{(\chi(G)-2)(R_2(G)-1)+\sigma(G)-1}{\chi(G)|V(G)|}$, we have

$$\begin{aligned}
|V(F')| = n - k|V(G)| \; &\geq n - t|V(G)| \\
&= (\chi(G)-1)(R_2(G)-1)+\sigma(G) - t|V(G)| \\
&= R_2(G) + (\chi(G)-2)(R_2(G)-1)+\sigma(G)-1-t|V(G)| \\
&\geq R_2(G) + t\chi(G)|V(G)| - t|V(G)| \\
&= R_2(G) + t(\chi(G)-1)|V(G)|.
\end{aligned}$$

Then F' contains $t(\chi(G)-1)+1$ disjoint monochromatic copies $B_1, \ldots, B_{t(\chi(G)-1)+1}$ of G. Since F' contains no monochromatic G of color 1, all of $B_1, \ldots, B_{t(\chi(G)-1)+1}$ are of color 2.

We define an auxiliary 2-edge-colored complete graph Γ as follows. Let $V(\Gamma) = \{v_1, \ldots, v_{t(\chi(G)-1)+1}\}$. For $1 \leq i < j \leq t(\chi(G)-1)+1$, the edge v_iv_j is of color 1 if all edges between $V(B_i)$ and $V(B_j)$ are of color 1 in F', and of color 2 if there exists an edge of color 2 between $V(B_i)$ and $V(B_j)$ in F'. By Lemma 2.3 (ii), Γ contains either a monochromatic $K_{\chi(G)}$ of color 1 or a monochromatic tree T_{t+1} on $t+1$ vertices of color 2. If Γ contains a monochromatic $K_{\chi(G)}$ of color 1, then F' contains a monochromatic G of color 1, a contradiction. If Γ contains a monochromatic T_{t+1} of color 2, then F' contains a monochromatic connected subgraph of color 2 that contains $(t+1)G$ (and thus $G \cup G_1 \cup \cdots \cup G_t$) as a subgraph, a contradiction. This completes the proof of Lemma 4.1. □

Now we have all ingredients to present our proof of Theorem 1.8.

Proof of Theorem 1.8. By Lemma 4.1, we have $R_2(\mathscr{C}(G \cup G_1 \cup \cdots \cup G_t)) \leq (\chi(G)-1)(R_2(G)-1)+\sigma(G)$. By Lemma 2.5 (i), we have $R_3(G \cup G_1 \cup \cdots \cup G_t) \geq (\chi(G)-1)(R_2(G)-1)+\sigma(G)$. Thus $R_3(G \cup G_1 \cup \cdots \cup G_t) \geq R_2(\mathscr{C}(G \cup G_1 \cup \cdots \cup G_t))$. By Lemma 2.2, we have $f(G \cup G_1 \cup \cdots \cup G_t, P_5) = R_3(G \cup G_1 \cup \cdots \cup G_t)$. This completes the proof of Theorem 1.8. □

Proof of Corollary 1.9. By Lemma 2.3 (i), we have $R_2(G) \geq (\chi(G)-1)(|V(G)|-1)+\sigma(G)$. Thus

$$(\chi(G)-2)(R_2(G)-1)+\sigma(G)-1-t\chi(G)|V(G)|$$

$$
\begin{aligned}
&\geq (\chi(G)-2)((\chi(G)-1)(|V(G)|-1)+\sigma(G)-1)+\sigma(G)-1-t\chi(G)|V(G)| \\
&= ((\chi(G)-2)(\chi(G)-1)-t\chi(G))|V(G)|-(\chi(G)-2)(\chi(G)-1)+(\chi(G)-1)(\sigma(G)-1) \\
&= (\chi(G)^2-(t+3)\chi(G)+2)|V(G)|+(\chi(G)-1)(\sigma(G)-\chi(G)+1) \\
&= (\chi(G)^2-(t+4)\chi(G)+3)|V(G)|+(\chi(G)-1)(|V(G)|+\sigma(G)-\chi(G)+1) \\
&\geq (\chi(G)^2-(t+4)\chi(G)+3)|V(G)|. \qquad (4.1)
\end{aligned}
$$

Since $\chi(G) \geq \frac{t+4+\sqrt{(t+4)^2-12}}{2}$, we have $\chi(G)^2-(t+4)\chi(G)+3 \geq 0$. Combining with Inequality (4.1), we have $t \leq \frac{(\chi(G)-2)(R_2(G)-1)+\sigma(G)-1}{\chi(G)|V(G)|}$. By Theorem 1.8, the result follows. □

Next, we prove Theorem 1.7. We first prove the following result on $R_2(\mathscr{C}(G_1 \cup G_2))$.

Lemma 4.2. *Let G_1, G_2 be two connected graphs with $G_1 \subseteq G_2$ and $\chi(G_2) \geq 3$. Then the following statements hold.*

(i) $R_2(\mathscr{C}(G_1 \cup G_2)) \leq R_2(G_2)+\chi(G_2)|V(G_2)|$.

(ii) *If $\chi(G_1) < \chi(G_2)$, then* $R_2(\mathscr{C}(G_1 \cup G_2)) \leq R_2(G_2)+(\chi(G_2)-1)|V(G_2)|$.

Proof. (i) Let $n = R_2(G_2)+\chi(G_2)|V(G_2)|$. For a contradiction, suppose that F is a 2-edge-colored K_n containing no monochromatic copy of any graph in $\mathscr{C}(G_1 \cup G_2)$. Since $G_1 \subseteq G_2$, F contains no monochromatic copy of any graph in $\mathscr{C}(2G_2)$ either. Since $n = R_2(G_2)+\chi(G_2)|V(G_2)|$, F contains $\chi(G_2)+1$ disjoint monochromatic copies $A_1, \ldots, A_{\chi(G_2)+1}$ of G_2.

Claim 4.1. *Among the graphs $A_1, \ldots, A_{\chi(G_2)+1}$, at least $\chi(G_2)$ of them are of the same color.*

Proof. For a contradiction, suppose that for each color $i \in [2]$, at most $\chi(G_2)-1$ of $A_1, \ldots, A_{\chi(G_2)+1}$ are of color i. Since $\chi(G_2)+1 \geq 4$, there exist four distinct indices a, b, c, d such that A_a and A_b are of color 1, and A_c and A_d are of color 2. In order to avoid a monochromatic graph in $\mathscr{C}(2G_2)$, we have $c(V(A_a), V(A_b)) = 2$ and $c(V(A_c), V(A_d)) = 1$. Let $u \in V(A_a)$ and $v \in V(A_c)$. Without loss of generality, we may assume that $c(uv) = 1$. For avoiding a connected monochromatic graph of color 1 containing $2G_2$ as a subgraph, we can further deduce that $c(v, V(A_b)) = 2$ and $c(V(A_d), V(A_b)) = 2$. But then in $F[V(A_b) \cup V(A_c) \cup V(A_d)]$, there is a connected monochromatic graph of color 2 containing $2G_2$ as a subgraph, a contradiction. □

By Claim 4.1, we may assume that $A_1, \ldots, A_{\chi(G_2)}$ are of color 1 without loss of generality. For avoiding a connected monochromatic graph of color 1 containing $2G_2$, we have $c(V(A_i), V(A_j)) = 2$ for every $1 \leq i < j \leq \chi(G_2)$. Let $s_1, \ldots, s_{\chi(G_2)}$ be the sizes of color classes in a proper $\chi(G_2)$-vertex-coloring of G_2. For each $i \in [\chi(G_2)]$, we can partition $V(A_i)$ into subsets $V_{i,1}, \ldots, V_{i,\chi(G_2)}$ with $|V_{i,j}| = s_j$ for each $j \in [\chi(G_2)]$. Then each of $F\left[V_{1,1} \cup V_{2,2} \cup \cdots \cup V_{\chi(G_2),\chi(G_2)}\right]$ and $F\left[V_{1,2} \cup V_{2,3} \cup \cdots \cup V_{\chi(G_2)-1,\chi(G_2)} \cup V_{\chi(G_2),1}\right]$ contains a monochromatic G_2 of color 2. This implies that F contains a connected monochromatic graph of color 2 containing $2G_2$ as a subgraph, a contradiction.

(ii) The proof of (ii) is analogous to that of (i), so we only present the different parts. Suppose for a contradiction that F is a 2-edge-colored K_n containing no monochromatic copy of any graph in $\mathscr{C}(G_1 \cup G_2)$, where $n = R_2(G_2)+(\chi(G_2)-1)|V(G_2)|$. Then F contains $\chi(G_2)$ disjoint monochromatic copies $A_1, \ldots, A_{\chi(G_2)}$ of G_2. If $\chi(G_2) = 3$, then at least two of A_1, A_2, A_3 are of the same color by the pigeonhole principle; if $\chi(G_2) \geq 4$, the by an analogous argument as the proof of Claim 4.1, we can also derive that at least $\chi(G_2)-1$ of $A_1, \ldots, A_{\chi(G_2)}$ are of the same color. Next, we further show that all the $\chi(G_2)$ graphs are of the same color.

Claim 4.2. *All of $A_1, \dots, A_{\chi(G_2)}$ are of the same color.*

Proof. From the about argument, we know that at least $\chi(G_2) - 1$ of $A_1, \dots, A_{\chi(G_2)}$ are of the same color. Suppose the claim does not hold. Then without loss of generality, we may assume that $A_1, \dots, A_{\chi(G_2)-1}$ are of color 1 and $A_{\chi(G_2)}$ is of color 2. For avoiding a connected monochromatic graph of color 1 containing $2G_2$ as a subgraph, we have $c(V(A_i), V(A_j)) = 2$ for every $1 \le i < j \le \chi(G_2) - 1$. For the same reason, there exists an edge of color 2 between $V(A_{\chi(G_2)})$ and $V(A_1) \cup \cdots \cup V(A_{\chi(G_2)-1})$, say $c(uv) = 2$ with $u \in V(A_{\chi(G_2)})$ and $v \in V(A_1)$. Moreover, since $G_1 \subseteq G_2$ and $\chi(G_1) < \chi(G_2)$, there is a monochromatic G_1 of color 2 containing v in $F\left[V(A_1) \cup \cdots \cup V(A_{\chi(G_2)-1})\right]$. Then F contains a connected monochromatic graph of color 2 containing $G_1 \cup G_2$ as a subgraph, a contradiction. □

By Claim 4.2, all of $A_1, \dots, A_{\chi(G_2)}$ are of the same color. Then, by an argument analogous to that in the last paragraph of the proof of (i), we can also derive a contradiction. This completes the proof of (ii). □

Now we have all ingredients to present our proof of Theorem 1.7.

Proof of Theorem 1.7. Note that $\frac{1+4+\sqrt{(1+4)^2-12}}{2} < 5$. Thus the case $\chi(G_2) \ge 5$ follows from Corollary 1.9. If $\chi(G_2) \le 2$, then $G_1 \cup G_2$ is a bipartite graph, so the result follows from Theorem 1.3 or 1.4. Next, we consider the case $\chi(G_2) = 4$. By Lemma 2.3 (i), 2.5 (i), 4.2 (i) and since $|V(G_2)| \ge \chi(G_2) = 4$ and $\sigma(G_2) \ge 1$, we have

$$
\begin{aligned}
& R_3(G_1 \cup G_2) - R_2(\mathscr{C}(G_1 \cup G_2)) \\
\ge\ & (\chi(G_2) - 1)(R_2(G_2) - 1) + \sigma(G_2) - (R_2(G_2) + \chi(G_2)|V(G_2)|) \\
=\ & (\chi(G_2) - 2)R_2(G_2) - (\chi(G_2) - 1) + \sigma(G_2) - \chi(G_2)|V(G_2)| \\
\ge\ & (\chi(G_2) - 2)((\chi(G_2) - 1)(|V(G_2)| - 1) + \sigma(G_2)) - (\chi(G_2) - 1) + \sigma(G_2) - \chi(G_2)|V(G_2)| \\
\ge\ & 2(3(|V(G_2)| - 1) + 1) - 3 + 1 - 4|V(G_2)| \\
=\ & 2|V(G_2)| - 6 \ > 0.
\end{aligned}
$$

Thus $R_3(G_1 \cup G_2) > R_2(\mathscr{C}(G_1 \cup G_2))$. By Lemma 2.2, the result follows.

Now we consider the case $\chi(G_2) = 3$. If $\chi(G_1) \le 2$ and $\sigma(G_2) = 1$, then the result follows from Corollary 1.14. If $\chi(G_1) \le 2$ and $\sigma(G_2) \ge 2$, then by Lemma 2.3 (i), 2.5 (i) and 4.2 (ii), we have

$$
\begin{aligned}
& R_3(G_1 \cup G_2) - R_2(\mathscr{C}(G_1 \cup G_2)) \\
\ge\ & (\chi(G_2) - 1)(R_2(G_2) - 1) + \sigma(G_2) - (R_2(G_2) + (\chi(G_2) - 1)|V(G_2)|) \\
=\ & 2(R_2(G_2) - 1) + \sigma(G_2) - (R_2(G_2) + 2|V(G_2)|) \\
=\ & R_2(G_2) - 2 + \sigma(G_2) - 2|V(G_2)| \\
\ge\ & (2(|V(G_2)| - 1) + \sigma(G_2)) - 2 + \sigma(G_2) - 2|V(G_2)| \\
=\ & 2\sigma(G_2) - 4 \ \ge 0.
\end{aligned}
$$

Thus $R_3(G_1 \cup G_2) \ge R_2(\mathscr{C}(G_1 \cup G_2))$. By Lemma 2.2, the result follows.

The remainder of the proof is devoted to the case $\chi(G_1) = \chi(G_2) = 3$. Let s_1, s_2, s_3 (resp., t_1, t_2, t_3) be the sizes of color classes in a proper 3-vertex-coloring of G_1 (resp., G_2) with $s_1 \ge s_2 \ge s_3 = \sigma(G_1)$ (resp., $t_1 \ge t_2 \ge t_3 = \sigma(G_2)$). By Inequality (1.1), it suffices to show that $f(G_1 \cup G_2, P_5) \le R_3(G_1 \cup G_2)$. For a contradiction, suppose that F is an edge-colored

K_n without rainbow P_5 or monochromatic $G_1 \cup G_2$, where $n = R_3(G_1 \cup G_2)$. By Lemma 2.9, we can partition $V(F)$ into $k-1$ parts $V_2, V_3, \ldots, V_k$ for some $k \geq 4$ satisfying

(i) $\{i\} \subseteq C(V_i) \subseteq \{1, i\}$ for every $i \in \{2, 3, \ldots, k\}$,

(ii) $c(V_i, V_j) = 1$ for every $2 \leq i < j \leq k$,

and in addition, we have $|V_2| \geq |V_3| \geq \cdots \geq |V_k|$,

$$|V_2| \geq |V_3| \geq |V(G_2)| \tag{4.2}$$

and

$$|V_4 \cup \cdots \cup V_k| \geq |V_4| \geq |V(G_1)|. \tag{4.3}$$

Claim 4.3. $|V_4 \cup \cdots \cup V_k| \leq s_3 + t_3 - 1$.

Proof. Suppose $|V_4 \cup \cdots \cup V_k| \geq s_3 + t_3$. Let $x = \max\{|V_2|, |V_4 \cup \cdots \cup V_k|\}$, $y = |V_3|$ and $z = \min\{|V_2|, |V_4 \cup \cdots \cup V_k|\}$. By Inequality (4.2) and since $G_1 \subseteq G_2$, we have $y \geq |V(G_2)| \geq \frac{1}{2}|V(G_1)| + \frac{1}{2}|V(G_2)| \geq s_2 + t_2$ and $z \geq \min\{|V(G_2)|, s_3 + t_3\} \geq s_3 + t_3$. Note that $x + y + z = n = R_3(G_1 \cup G_2)$, $x \geq y$, $x \geq z$ and x is an integer. Combining with Lemma 2.5 (iii), we have

$$\begin{aligned} x \geq \left\lceil \frac{1}{3} R_3(G_1 \cup G_2) \right\rceil &\geq \left\lceil \frac{1}{3} \left(3(|V(G_1)| + |V(G_2)|) - 2\right) \right\rceil \\ = |V(G_1)| + |V(G_2)| &\geq s_1 + t_1. \end{aligned}$$

By Observation 2.12 (ii), there is a monochromatic $G_1 \cup G_2$ of color 1 using edges between V_2, V_3 and $V_4 \cup \cdots \cup V_k$, a contradiction. □

By Lemma 2.5 (i), Claim 4.3 and Inequalities (4.2) and (4.3), we have

$$\begin{aligned} |V_2| \geq \left\lceil \frac{1}{2}(n - |V_4 \cup \cdots \cup V_k|) \right\rceil &\geq \left\lceil \frac{1}{2}(R_3(G_1 \cup G_2) - (s_3 + t_3 - 1)) \right\rceil \\ &\geq \left\lceil \frac{1}{2}(2(R_2(G_2) - 1) + \sigma(G_1) + \sigma(G_2) - (s_3 + t_3 - 1)) \right\rceil \\ &= \left\lceil \frac{1}{2}(2R_2(G_2) - 1) \right\rceil = R_2(G_2) \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} |V(G_1)| = s_1 + s_2 + s_3 \leq \frac{3}{2}(s_1 + s_2) &= \frac{3}{2}(|V(G_1)| - s_3) \leq \frac{3}{2}(|V_4 \cup \cdots \cup V_k| - s_3) \\ \leq \frac{3}{2}(s_3 + t_3 - 1 - s_3) &= \frac{3}{2}(t_3 - 1) \leq \frac{3}{2}\left(\frac{1}{3}|V(G_2)| - 1\right) = \frac{1}{2}(|V(G_2)| - 3). \end{aligned}$$

In particular, we have $|V(G_1)| \leq |V(G_2)| - 1$. Combining with Lemma 2.3 (i) and Inequality (4.4), we have

$$|V_2| \geq R_2(G_2) \geq 2(|V(G_2)| - 1) + 1 \geq |V(G_1)| + |V(G_2)|. \tag{4.5}$$

Recall that $\mathcal{M}(H)$ is the decomposition family of a graph H.

Claim 4.4. $R_2(G_2) \geq R(G_1 \cup G_2, \mathcal{M}(G_2)) \geq R(G_1 \cup G_2, \mathcal{M}(G_1))$.

Proof. On the one hand, we have $R_2(G_2) \geq R(G_1, G_2)$ since $G_1 \subseteq G_2$. On the other hand, by Lemma 2.6 (i) and since $|V(G_1)| \leq |V(G_2)| - 1$, we have $R_2(G_2) \geq R(G_2, \mathcal{M}(G_2)) + |V(G_2)| - 1 \geq R(G_2, \mathcal{M}(G_2)) + |V(G_1)|$. Hence, combining with Lemma 2.6 (ii), we have $R_2(G_2) \geq \max\{R(G_1, G_2), R(G_2, \mathcal{M}(G_2)) + |V(G_1)|\} \geq R(G_1 \cup G_2, \mathcal{M}(G_2))$.

Note that if $M \in \mathcal{M}(G_2)$, then embedding a copy of M into one partite set of a complete bipartite graph of large partite sets creates a G_2 (and thus a G_1), so M or some of its subgraphs is a member of $\mathcal{M}(G_1)$. Therefore, $R(G_1 \cup G_2, \mathcal{M}(G_2)) \geq R(G_1 \cup G_2, \mathcal{M}(G_1))$. □

By Claim 4.4 and Inequality (4.4), we have $|V_2| \geq R_2(G_2) \geq R(G_1 \cup G_2, \mathcal{M}(G_2)) = R(\mathcal{M}(G_2), G_1 \cup G_2)$. Since F contains no monochromatic $G_1 \cup G_2$ of color 2, $F[V_2]$ contains a monochromatic copy M of some graph in $\mathcal{M}(G_2)$ of color 1. Let $U_1 \subseteq V_2$ with $V(M) \subseteq U_1$ and $|U_1| = |V(G_2)|$.

Claim 4.5. $|V_3| \leq |V(G_2)| - 3 + s_3$.

Proof. Suppose that $|V_3| \geq |V(G_2)| - 2 + s_3$. Then we can choose disjoint subsets $W_1, W_2 \subseteq V_3$ with $|W_1| = |V(G_2)| - 2$ and $|W_2| = s_3$. Then $F[U_1 \cup W_1]$ contains a monochromatic G_2 of color 1. By Inequalities (4.3) and (4.5), we have $|V_4| \geq |V(G_1)|$ and $|V_2 \setminus U_1| \geq |V(G_1)|$. Thus $F[(V_2 \setminus U_1) \cup W_2 \cup V_4]$ contains a monochromatic G_1 of color 1. Then F contains a monochromatic $G_1 \cup G_2$ of color 1, a contradiction. □

Since $G_1 \subseteq G_2$, we have $s_3 = \sigma(G_1) \leq \sigma(G_2) = t_3$. By Lemma 2.5 (i), Lemma 2.3 (i) and Claims 4.3 and 4.5, we have

$$\begin{aligned}
|V_2| &= R_3(G_1 \cup G_2) - |V_3| - |V_4 \cup \cdots \cup V_k| \\
&\geq 2(R_2(G_2) - 1) + \sigma(G_1) + \sigma(G_2) - (|V(G_2)| - 3 + s_3) - (s_3 + t_3 - 1) \\
&= 2R_2(G_2) + 2 - |V(G_2)| - s_3 \\
&\geq R_2(G_2) + (2(|V(G_2)| - 1) + \sigma(G_2)) + 2 - |V(G_2)| - s_3 \\
&= R_2(G_2) + |V(G_2)| + \sigma(G_2) - s_3 \ \geq R_2(G_2) + |V(G_2)|.
\end{aligned}$$

Thus $|V_2 \setminus U_1| \geq R_2(G_2)$. By Claim 4.4, we further have $|V_2 \setminus U_1| \geq R(G_1 \cup G_2, \mathcal{M}(G_1)) = R(\mathcal{M}(G_1), G_1 \cup G_2)$. Since F contains no monochromatic $G_1 \cup G_2$ of color 2, $F[V_2 \setminus U_1]$ contains a monochromatic copy M' of some graph in $\mathcal{M}(G_1)$ of color 1. By Inequality (4.5), we can choose $U_2 \subseteq V_2 \setminus U_1$ with $V(M') \subseteq U_2$ and $|U_2| = |V(G_1)|$. Recall that $|V_3| \geq |V(G_2)|$ and $|V_4| \geq |V(G_1)|$. Thus $F[U_1 \cup V_3]$ (resp., $F[U_2 \cup V_4]$) contains a monochromatic G_2 (resp., G_1) of color 1. Then F contains a monochromatic $G_1 \cup G_2$ of color 1, a contradiction. This completes the proof of Theorem 1.7. □

5 Proofs of results for graphs of chromatic number 3

In this section, we present our proofs of Theorems 1.10, 1.12 and 1.13.

Proof of Theorem 1.10. By Theorem 1.3 or 1.4, it suffices to assume that $\chi(H) = 3$, that is, H contains at least one 3-color-critical component. Let $H = G_1 \cup \cdots \cup G_t$, where $G_1, \ldots, G_s$ are s connected 3-color-critical graphs and $G_{s+1}, \ldots, G_t$ are $t - s$ connected bipartite graphs for some $1 \leq s \leq t$. Note that for $i \in [s]$, the graph G_i can be obtained from a bipartite graph by adding one edge within one partite set. Let a_i and b_i be the sizes of the two partite sets, and assume that the critical edge is contained in the partite set of size a_i. For $i \in [t] \setminus [s]$, we can also assume that a_i and b_i are the sizes of the two partite sets of G_i. By Proposition 2.1, it suffices to assume that $|V(G_i)| \geq 2$ for all $i \in [t]$. Hence, we have $a_i \geq 2$ for $i \in [s]$, $a_i \geq 1$ for $i \in [t] \setminus [s]$ and $b_i \geq 1$ for $i \in [t]$. Moreover, by Lemma 2.5 (iv), we have

$$R_3(H) \geq 2|V(H)| + s - 2. \tag{5.1}$$

By Inequality (1.1), it suffices to show that $f(H, P_5) \leq R_3(H)$. For a contradiction, suppose that there exists an edge-coloring F of $K_{R_3(H)}$ without rainbow P_5 or monochromatic H. By Lemma 2.9, we can partition $V(F)$ into $k-1$ parts $V_2, V_3, \ldots, V_k$ for some $k \geq 4$ satisfying

(i) $\{i\} \subseteq C(V_i) \subseteq \{1, i\}$ for every $i \in \{2, 3, \ldots, k\}$,

(ii) $c(V_i, V_j) = 1$ for every $2 \leq i < j \leq k$, and

(iii) $|V_2| \geq |V_3| \geq \cdots \geq |V_k|$, $|V_3| \geq \max_{i\in[t]} |V(G_i)|$, $|V_4| \geq \min_{i\in[t]} |V(G_i)|$ and $|V_3 \cup \cdots \cup V_k| \geq |V(H)| = \sum_{i\in[t]}(a_i + b_i)$.

We first prove a claim related to the upper bound on the size of V_2.

Claim 5.1. $|V_2| \leq |V(H)| + s - 2$.

Proof. Suppose that $|V_2| \geq |V(H)| + s - 1$. Then by Corollary 2.4, we have $|V_2| \geq R(sK_2, H)$. Since F contains no monochromatic H of color 2, there must be a monochromatic matching $M = \{u_1v_1, \ldots, u_sv_s\}$ of color 1 within V_2. Since $|V_2 \setminus V(M)| \geq |V(H)| - s - 1 > \sum_{i\in[s]}(a_i - 2) + \sum_{i\in[t]\setminus[s]} a_i$ and $|V_3 \cup \cdots \cup V_k| \geq |V(H)| = \sum_{i\in[t]}(a_i + b_i)$, we can choose disjoint subsets $U_1, \ldots, U_t \subseteq V_2 \setminus V(M)$ and $W_1, \ldots, W_t \subseteq V_3 \cup \cdots \cup V_k$ such that $|U_i| = a_i - 2$ for $i \in [s]$, $|U_i| = a_i$ for $i \in [t] \setminus [s]$ and $|W_i| = b_i$ for $i \in [t]$. Then for each $i \in [s]$ (resp., $i \in [t] \setminus [s]$), $F[U_i \cup W_i \cup \{u_i, v_i\}]$ (resp., $F[U_i \cup W_i]$) contains a monochromatic G_i of color 1. Hence, F contains a monochromatic H of color 1, a contradiction. □

We divide the rest of the proof into two cases based on the size of V_2.

Case 1. $|V_2| \geq |V(H)|$.

By Claim 5.1, we may assume that $|V_2| = |V(H)| + m - 1$ for some $1 \leq m \leq s - 1$. Then by Corollary 2.4, we have $|V_2| \geq R(mK_2, H)$. Since F contains no monochromatic H of color 2, there must be a monochromatic matching $M = \{u_1v_1, \ldots, u_mv_m\}$ of color 1 within V_2. Since

$$\begin{aligned} |V_2 \setminus V(M)| \geq |V(H)| - m - 1 &\geq \sum\nolimits_{i\in[m]}(a_i - 2) + \sum\nolimits_{i\in[t]\setminus[m]} a_i + \sum\nolimits_{i\in[t]} b_i \\ &> \sum\nolimits_{i\in[m]}(a_i - 2) + \sum\nolimits_{i\in[t]\setminus[s]} a_i, \end{aligned} \tag{5.2}$$

we can choose disjoint subsets $U_1, \ldots, U_m, U_{s+1}, \ldots, U_t \subseteq V_2 \setminus V(M)$ with $|U_i| = a_i - 2$ for $i \in [m]$ and $|U_i| = a_i$ for $i \in [t] \setminus [s]$. Let $V_2' := V_2 \setminus (V(M) \cup U_1 \cup \cdots \cup U_m \cup U_{s+1} \cup \cdots \cup U_t)$.

We next state and prove three claims related to the size of V_3.

Claim 5.2. *We have that either* $|V_3| \leq \sum_{i\in[t]} b_i - 1$ *or* $|V_3| \geq |V(H)|$.

Proof. Suppose that $\sum_{i\in[t]} b_i \leq |V_3| \leq |V(H)| - 1$. By Inequality (5.1), we have

$$|V_4 \cup \cdots \cup V_k| = R_3(H) - |V_2| - |V_3| \geq 2|V(H)| + s - 2 - (|V(H)| + m - 1) - (|V(H)| - 1) = s - m.$$

Let $u_{m+1}, \dots, u_s$ be $s-m$ distinct vertices in $V_4 \cup \cdots \cup V_k$. By Inequality (5.2), we can choose $s-m$ disjoint subsets $U_{m+1}, \dots, U_s$ of V_2' with $|U_i| = a_i - 1$ for each $i \in [s] \setminus [m]$. Moreover, since $|V_3| \geq \sum_{i \in [t]} b_i$, we can choose t disjoint subsets $W_1, \dots, W_t$ of V_3 with $|W_i| = b_i$ for each $i \in [t]$. Let $F_i := F[U_i \cup W_i \cup \{u_i, v_i\}]$ for $i \in [m]$, $F_i := F[U_i \cup W_i \cup \{u_i\}]$ for $i \in [s] \setminus [m]$, and $F_i := F[U_i \cup W_i]$ for $i \in [t] \setminus [s]$. Then for each $i \in [t]$, F_i contains a monochromatic G_i of color 1. Hence, F contains a monochromatic H of color 1, a contradiction. □

Claim 5.3. $|V_3| \leq \sum_{i \in [t]} b_i - 1$.

Proof. Suppose for a contradiction that $|V_3| \geq \sum_{i \in [t]} b_i$, so $|V_3| \geq |V(H)|$ by Claim 5.2. Let $n := |V_3| + 1 - |V(H)|$ if $|V_3| \leq |V(H)| + (s-m) - 2$, and $n := s - m$ if $|V_3| \geq |V(H)| + (s-m) - 1$. Then by Corollary 2.4, we have $|V_3| \geq R(nK_2, H)$. Since F contains no monochromatic H of color 3, there must be a monochromatic matching $M' = \{u_{m+1}v_{m+1}, \dots, u_{m+n}v_{m+n}\}$ of color 1 within V_3. By Inequality (5.1), if $|V_3| \leq |V(H)| + (s-m) - 2$, then

$$\begin{aligned} |V_4 \cup \cdots \cup V_k| &= R_3(H) - |V_2| - |V_3| \\ &\geq 2|V(H)| + s - 2 - (|V(H)| + m - 1) - (|V(H)| + n - 1) \\ &= s - m - n; \end{aligned}$$

if $|V_3| \geq |V(H)| + (s-m) - 1$, then

$$|V_4 \cup \cdots \cup V_k| > 0 = s - m - n.$$

Thus there eixst $s-m-n$ distinct vertices $u_{m+n+1}, \dots, u_s$ in $V_4 \cup \cdots \cup V_k$. By Inequality (5.2), we can choose $s-m$ disjoint subsets $U_{m+1}, \dots, U_s$ of V_2' with $|U_i| = b_i$ for $i \in [m+n] \setminus [m]$ and $|U_i| = a_i - 1$ for $i \in [s] \setminus [m+n]$. Moreover, since $|V_3| \geq |V(H)|$, we can choose t disjoint subsets $W_1, \dots, W_t \subseteq V_3 \setminus V(M')$ with $|W_i| = a_i - 2$ for $i \in \{m+1, \dots, m+n\}$ and $|W_i| = b_i$ for $i \in [t] \setminus \{m+1, \dots, m+n\}$. Let $F_i := F[U_i \cup W_i \cup \{u_i, v_i\}]$ for $i \in [m+n]$, $F_i := F[U_i \cup W_i \cup \{u_i\}]$ for $i \in [s] \setminus [m+n]$, and $F_i := F[U_i \cup W_i]$ for $i \in [t] \setminus [s]$. Then for each $i \in [t]$, F_i contains a monochromatic G_i of color 1. Hence, F contains a monochromatic H of color 1, a contradiction. □

Claim 5.4. $|V_3| \leq s - m - 1$.

Proof. Suppose that $|V_3| \geq s - m$. By Inequality (5.1) and Claim 5.3, we have

$$\begin{aligned} |V_4 \cup \cdots \cup V_k| &= R_3(H) - |V_2| - |V_3| \\ &\geq 2|V(H)| + s - 2 - (|V(H)| + m - 1) - \left(\sum\nolimits_{i \in [t]} b_i - 1\right) \\ &> s - m. \end{aligned}$$

Thus we can choose $s-m$ distinct vertices $u_{m+1}, \dots, u_s \in V_3$ and $s-m$ distinct vertices $v_{m+1}, \dots, v_s \in V_4 \cup \cdots \cup V_k$ such that $M' := \{u_{m+1}v_{m+1}, \dots, u_s v_s\}$ is a monochromatic matching of color 1. Moreover, since

$$|V_3 \cup \cdots \cup V_k| = R_3(H) - |V_2| \geq 2|V(H)| + s - 2 - (|V(H)| + m - 1) \geq |V(H)|,$$

we can choose disjoint subsets $W_1, \dots, W_t \subseteq (V_3 \cup \cdots \cup V_k) \setminus V(M')$ such that $|W_i| = a_i - 2$ for $i \in \{m+1, \dots, s\}$ and $|W_i| = b_i$ for each $i \in [t] \setminus \{m+1, \dots, s\}$. By Inequality (5.2), we can choose $s-m$ disjoint subsets $U_{m+1}, \dots, U_s$ of V_2' with $|U_i| = b_i$ for each $i \in \{m+1, \dots, s\}$. Then for each $i \in [s]$ (resp., $i \in [t] \setminus [s]$), $F[U_i \cup W_i \cup \{u_i, v_i\}]$ (resp., $F[U_i \cup W_i]$) contains a monochromatic G_i of color 1. Hence, F contains a monochromatic H of color 1, a contradiction. □

By Claim 5.3, we have $|V_3| \le \sum_{i\in[t]} b_i - 1$. Recall that $|V_3\cup\cdots\cup V_k| \ge |V(H)| = \sum_{i\in[t]}(a_i + b_i)$. We now choose $W \subseteq V_3\cup\cdots\cup V_k$ with $|W| = \sum_{i\in[t]} b_i$ starting with V_3, and all the vertices of V_i are selected before taking vertices from V_{i+1} for $i \ge 3$. Assume that for some $j > 3$ we have $|V_3\cup\cdots\cup V_{j-1}| \le \sum_{i\in[t]} b_i - 1$ and $|V_3\cup\cdots\cup V_j| \ge \sum_{i\in[t]} b_i$. Recall that $|V_2| \ge |V_3| \ge \cdots \ge |V_k|$. Thus by Claim 5.4, we have $|V_3\cup\cdots\cup V_j| \le |V_3\cup\cdots\cup V_{j-1}| + |V_3| \le \sum_{i\in[t]} b_i - 1 + s - m - 1$. Then combining with Inequality (5.1), we have

$$\begin{aligned}|V_{j+1}\cup\cdots\cup V_k| &= R_3(H) - |V_2| - |V_3\cup\cdots\cup V_j|\\ &\ge 2|V(H)| + s - 2 - (|V(H)| + m - 1) - \left(\sum\nolimits_{i\in[t]} b_i - 1 + s - m - 1\right)\\ &= |V(H)| - \sum\nolimits_{i\in[t]} b_i + 1 \ > \sum\nolimits_{i\in[t]} a_i - s.\end{aligned}$$

Now we can choose distinct vertex $u_1,\ldots,u_s \in V_2$ and disjoint subsets $W_1,\ldots,W_t \subseteq W$ and $U_1,\ldots,U_t \subseteq V_{j+1}\cup\cdots\cup V_k$ such that $|W_i| = b_i$ for $i\in[t]$, $|U_i| = a_i - 1$ for $i\in[s]$, and $|U_i| = a_i$ for $i \in [t]\setminus[s]$. Then for each $i\in[s]$ (resp., $i\in[t]\setminus[s]$), $F[U_i\cup W_i\cup\{u_i\}]$ (resp., $F[U_i\cup W_i]$) contains a monochromatic G_i of color 1. Hence, F contains a monochromatic H of color 1, a contradiction. This completes the proof of Case 1.

Case 2. $|V_2| \le |V(H)| - 1$.

Let $x := \max\left\{\sum_{i\in[t]} a_i - s, \sum_{i\in[t]} b_i\right\}$ and $y := \min\left\{\sum_{i\in[t]} a_i - s, \sum_{i\in[t]} b_i\right\}$. Then $x \ge y \ge t \ge s$ and

$$2y \le x + y = \sum\nolimits_{i\in[t]} a_i - s + \sum\nolimits_{i\in[t]} b_i = |V(H)| - s. \tag{5.3}$$

We next state and prove three claims related to the sizes of V_2 and V_3.

Claim 5.5. $|V_2| \le x - 1$.

Proof. Suppose that $|V_2| \ge x$. Let j be the minimum index such that $|V_3\cup\cdots\cup V_j| \ge y$. Note that $|V_3\cup\cdots\cup V_j| \le |V(H)| - 1$. To see this, if $j = 3$, then $|V_3\cup\cdots\cup V_j| = |V_3| \le |V_2| \le |V(H)| - 1$; if $j \ge 4$, then $|V_j| \le |V_3| \le |V_3\cup\cdots\cup V_{j-1}| \le y - 1$, so $|V_3\cup\cdots\cup V_j| \le 2(y-1) \le |V(H)| - 1$ by Inequality (5.3). Combining with Inequality (5.1), we have

$$\begin{aligned}|V_{j+1}\cup\cdots\cup V_k| &= R_3(H) - |V_2| - |V_3\cup\cdots\cup V_j|\\ &\ge 2|V(H)| + s - 2 - (|V(H)| - 1) - (|V(H)| - 1) \ = s.\end{aligned}$$

Then there is a monochromatic $K_{x,y,s}$ (and thus a monochromatic H) of color 1 between V_2, $V_3\cup\cdots\cup V_j$ and $V_{j+1}\cup\cdots\cup V_k$, a contradiction. □

Claim 5.6. $|V_2| + |V_3| \le x + s - 1$.

Proof. Suppose that $|V_2| + |V_3| \ge x + s \ge \max\left\{\sum_{i\in[t]} a_i, \sum_{i\in[t]} b_i + s\right\}$. Then by Claim 5.5, we have $|V_3| \ge x + s - (x-1) \ge s + 1$. Let $u_1,\ldots,u_s \in V_3$ be s distinct vertices.

We claim that $|V_4\cup\cdots\cup V_k| \le \sum_{i\in[t]} b_i - 1$, $x = \sum_{i\in[t]} b_i$ and $y = \sum_{i\in[t]} a_i - s$. To see this, we first suppose that $|V_4\cup\cdots\cup V_k| \ge \sum_{i\in[t]} b_i$. Then we can choose disjoint subsets $W_1,\ldots,W_t \subseteq V_4\cup\cdots\cup V_k$ with $|W_i| = b_i$ for each $i\in[t]$. Let $v_1,\ldots,v_s \in V_2$ be s distinct vertices, and let $U_1,\ldots,U_t \subseteq (V_2\cup V_3)\setminus\{u_1,\ldots,u_s,v_1,\ldots,v_s\}$ be t disjoint subsets with $|U_i| = a_i - 2$ for $i\in[s]$ and $|U_i| = a_i$ for $i\in[t]\setminus[s]$. Then for each $i\in[s]$ (resp., $i\in[t]\setminus[s]$), $F[U_i\cup W_i\cup\{u_i, v_i\}]$ (resp., $F[U_i\cup W_i]$) contains a monochromatic G_i of color 1. This implies that

F contains a monochromatic H of color 1, a contradiction. Hence, $|V_4 \cup \cdots \cup V_k| \le \sum_{i\in[t]} b_i - 1$. On the other hand, by Inequalities (5.1), (5.3) and Claim 5.5, we have

$$|V_4 \cup \cdots \cup V_k| \ge R_3(H) - 2|V_2| \ge 2|V(H)| + s - 2 - 2(x-1) > y.$$

Thus $y < |V_4 \cup \cdots \cup V_k| \le \sum_{i\in[t]} b_i - 1$, so $\sum_{i\in[t]} b_i = x$ and $\sum_{i\in[t]} a_i - s = y < |V_4 \cup \cdots \cup V_k|$.

Now we have

$$\begin{aligned} |V_2| + |V_3| &= R_3(H) - |V_4 \cup \cdots \cup V_k| \ \ge 2|V(H)| + s - 2 - \left(\sum\nolimits_{i\in[t]} b_i - 1\right) \\ &= 2\sum\nolimits_{i\in[t]} a_i + \sum\nolimits_{i\in[t]} b_i + s - 1. \end{aligned} \tag{5.4}$$

We next show that $|V_2| \le \sum_{i\in[s]} b_i - 1$. Suppose for a contradiction that $|V_2| \ge \sum_{i\in[s]} b_i$. Recall that $|V_2| + |V_3| \ge x + s \ge \sum_{i\in[t]} b_i + s$. Thus we can choose disjoint subsets $W_1, \ldots, W_s \subseteq V_2$ and $W_{s+1}, \ldots, W_t \subseteq (V_2 \cup V_3) \setminus (W_1 \cup \cdots \cup W_s \cup \{u_1, \ldots, u_s\})$ with $|W_i| = b_i$ for $i \in [t]$. Since $|V_4 \cup \cdots \cup V_k| > \sum_{i\in[t]} a_i - s$, we can choose s distinct vertices $v_1, \ldots, v_s \in V_4 \cup \cdots \cup V_k$ and t disjoint subsets $U_1, \ldots, U_t \subseteq (V_4 \cup \cdots \cup V_k) \setminus \{v_1, \ldots, v_s\}$ with $|U_i| = a_i - 2$ for $i \in [s]$ and $|U_i| = a_i$ for $i \in [t] \setminus [s]$. Then for each $i \in [s]$ (resp., $i \in [t] \setminus [s]$), $F[U_i \cup W_i \cup \{u_i, v_i\}]$ (resp., $F[U_i \cup W_i]$) contains a monochromatic G_i of color 1. This implies that F contains a monochromatic H of color 1, a contradiction.

Now by Lemma 2.5 (iii), we have

$$\begin{aligned} |V_4 \cup \cdots \cup V_k| &= R_3(H) - |V_2| - |V_3| \ \ge R_3(G_1 \cup \cdots \cup G_s) - 2|V_2| \\ &\ge 3\sum\nolimits_{i\in[s]} (a_i + b_i) - 2 - 2\left(\sum\nolimits_{i\in[s]} b_i - 1\right) \\ &= 3\sum\nolimits_{i\in[s]} a_i + \sum\nolimits_{i\in[s]} b_i. \end{aligned}$$

Moreover, by Inequality (5.4), we have $|V_2| \ge \frac{1}{2}(|V_2| + |V_3|) > \sum_{i\in[t]} a_i$. Recall that $|V_3 \cup \cdots \cup V_k| \ge |V(H)| = \sum_{i\in[t]} (a_i + b_i)$. Thus we can choose s distinct vertices $v_1, \ldots, v_s \in V_2$, t disjoint subsets $U_1, \ldots, U_t \subseteq V_2 \setminus \{v_1, \ldots, v_s\}$, s disjoint subsets $W_1, \ldots, W_s \subseteq V_4 \cup \cdots \cup V_k$ and $t - s$ subsets $W_{s+1}, \ldots, W_t \subseteq (V_3 \cup \cdots \cup V_k) \setminus (W_1 \cup \cdots \cup W_s \cup \{u_1, \ldots, u_s\})$ with $|U_i| = a_i - 2$ for $i \in [s]$, $|U_i| = a_i$ for $i \in [t] \setminus [s]$ and $|W_i| = b_i$ for $i \in [t]$. Then for each $i \in [s]$ (resp., $i \in [t] \setminus [s]$), $F[U_i \cup W_i \cup \{u_i, v_i\}]$ (resp., $F[U_i \cup W_i]$) contains a monochromatic G_i of color 1. This implies that F contains a monochromatic H of color 1. This contradiction completes the proof of Claim 5.6. □

Claim 5.7. $|V_2| \le s - 1$.

Proof. Suppose that $|V_2| \ge s$. We first show that $|V_2| \le \sum_{i\in[t]} a_i - s - 1$. To see this, if $|V_2| \ge \sum_{i\in[t]} a_i - s$, then let j be the minimum index with $|V_3 \cup \cdots \cup V_j| \ge \sum_{i\in[t]} b_i$. Note that if $j \ge 4$, then $|V_3 \cup \cdots \cup V_{j-1}| \le \sum_{i\in[t]} b_i - 1$. By Claim 5.6, we have

$$\begin{aligned} |V_2 \cup \cdots \cup V_j| &\le |V_2| + \max\left\{|V_3|, \sum\nolimits_{i\in[t]} b_i - 1 + |V_j|\right\} \\ &\le |V_2| + \sum\nolimits_{i\in[t]} b_i - 1 + |V_3| \ \le x + s + \sum\nolimits_{i\in[t]} b_i - 2. \end{aligned}$$

Then

$$|V_{j+1} \cup \cdots \cup V_k| = R_3(H) - |V_2 \cup \cdots \cup V_j|$$

$$\geq 2|V(H)| + s - 2 - \left(x + s + \sum\nolimits_{i\in[t]} b_i - 2\right) > s.$$

Now there is a monochromatic $K_{x,y,s}$ (and thus a monochromatic H) of color 1 between V_2, $V_3 \cup \cdots \cup V_j$ and $V_{j+1} \cup \cdots \cup V_k$, a contradiction. Hence, $|V_2| \leq \sum_{i\in[t]} a_i - s - 1$.

Let ℓ be the minimum index with $|V_2 \cup \cdots \cup V_\ell| \geq \sum_{i\in[t]} a_i$. Then $\ell \geq 3$, $|V_3 \cup \cdots \cup V_\ell| \geq s+1$, $|V_2 \cup \cdots \cup V_{\ell-1}| \leq \sum_{i\in[t]} a_i - 1$. Combining with Claim 5.5, we have

$$\begin{aligned} |V_{\ell+1} \cup \cdots \cup V_k| &= R_3(H) - |V_2 \cup \cdots \cup V_{\ell-1}| - |V_\ell| \geq R_3(H) - |V_2 \cup \cdots \cup V_{\ell-1}| - |V_2| \\ &\geq 2|V(H)| + s - 2 - \left(\sum\nolimits_{i\in[t]} a_i - 1\right) - (x-1) > \sum\nolimits_{i\in[t]} b_i. \end{aligned}$$

Let $M = \{u_1v_1, \ldots, u_sv_s\}$ be a monochromatic matching with $u_1, \ldots u_s \in V_2$ and $v_1, \ldots, v_s \in V_3 \cup \cdots \cup V_\ell$. Let $U_1, \ldots, U_t \subseteq (V_2 \cup \cdots \cup V_\ell) \setminus V(M)$ and $W_1, \ldots, W_t \subseteq V_{\ell+1} \cup \cdots \cup V_k$ be disjoint subsets with $|U_i| = a_i - 2$ for $i \in [s]$, $|U_i| = a_i$ for $i \in [t] \setminus [s]$ and $|W_i| = b_i$ for $i \in [t]$. Then for each $i \in [s]$ (resp., $i \in [t] \setminus [s]$), $F[U_i \cup W_i \cup \{u_i, v_i\}]$ (resp., $F[U_i \cup W_i]$) contains a monochromatic G_i of color 1. This implies that F contains a monochromatic H of color 1. This contradiction completes the proof of Claim 5.7. □

Let j_1 be the minimum index with $|V_2 \cup \cdots \cup V_{j_1}| \geq x$. Combining with Claim 5.7, we have $j_1 \geq 3$, $|V_2 \cup \cdots \cup V_{j_1-1}| \leq x - 1$ and $|V_{j_1}| \leq |V_2| \leq s - 1$. Let j_2 be the minimum index with $|V_{j_1+1} \cup \cdots \cup V_{j_2}| \geq y$. Combining with Claim 5.7, we have $j_2 \geq j_1+2$, $|V_{j_1+1} \cup \cdots \cup V_{j_2-1}| \leq y-1$ and $|V_{j_2}| \leq |V_2| \leq s - 1$. Then by Inequalities (5.1) and (5.3), we have

$$\begin{aligned} |V_{j_2+1} \cup \cdots \cup V_k| &= R_3(H) - |V_2 \cup \cdots \cup V_{j_1-1}| - |V_{j_1}| - |V_{j_1+1} \cup \cdots \cup V_{j_2-1}| - |V_{j_2}| \\ &\geq 2|V(H)| + s - 2 - (x-1) - (s-1) - (y-1) - (s-1) \geq s. \end{aligned}$$

Then there is a monochromatic $K_{x,y,s}$ (and thus a monochromatic H) of color 1 between $V_2 \cup \cdots \cup V_{j_1}$, $V_{j_1+1} \cup \cdots \cup V_{j_2}$ and $V_{j_2+1} \cup \cdots \cup V_k$, a contradiction. This completes the proof of Theorem 1.10. □

Proof of Theorem 1.12. Let $H := G_1 \cup G_2 \cup \cdots \cup G_t$. By Inequality (1.1), it suffices to show that $f(H, P_5) \leq R_3(H)$. For a contradiction, suppose that F is an edge-colored $K_{R_3(H)}$ using exactly k colors such that it contains no rainbow P_5 or monochromatic H. By Lemma 2.8, we can partition $V(F)$ into $k-1$ parts $V_2, V_3, \ldots, V_k$ satisfying

(i) $\{i\} \subseteq C(V_i) \subseteq \{1, i\}$ for every $i \in \{2, 3, \ldots, k\}$, and

(ii) $c(V_i, V_j) = 1$ for every $2 \leq i < j \leq k$.

Without loss of generality, we may assume that $|V_2| = \max_{2\leq i\leq k} |V_i|$.

Let F' be a 3-edge-coloring of K_n obtained from F by recoloring all edges of colors in $[k] \setminus [3]$ with color 3. Since $n = R_3(H)$, there is a monochromatic copy of H in F'. Denote such a monochromatic H by A. Since F contains no monochromatic copy of H, the color of A must be 3 and the components of A must be contained in at least two distinct parts of $V_3, V_4, \ldots, V_k$. Without loss of generality, let $V_3, \ldots, V_\ell$ be the parts containing components of A for some $4 \leq \ell \leq k$, and assume that $|V_3| \geq \cdots \geq |V_\ell|$. For each $i \in \{3, \ldots, \ell\}$, let x_i be the number of components of A contained in V_i, and let $G_{i_1}, \ldots, G_{i_{x_i}}$ be these components. Then $\sum_{i=3}^{\ell} x_i = t$ and $|V_2| \geq |V_i| \geq \sum_{j=1}^{x_i} |V(G_{i_j})| = \sum_{j=1}^{x_i} (a_{i_j} + b_{i_j} + c_{i_j})$ for each $i \in \{3, \ldots, \ell\}$.

By the assumptions of the theorem, we have $a_i \geq b_i$ and $b_i + c_i \geq \sum_{j\in[t]\setminus\{i\}} c_j$ for all $i \in [t]$. Then

$$\begin{aligned}|V_2| \geq |V_3| \ &\geq \sum\nolimits_{j=1}^{x_3}(a_{3_j} + b_{3_j} + c_{3_j}) \ \geq a_{3_1} + c_{3_1} + \sum\nolimits_{j=1}^{x_3} b_{3_j} \ \geq b_{3_1} + c_{3_1} + \sum\nolimits_{j=1}^{x_3} b_{3_j} \\ &\geq \ \sum\nolimits_{j\in[t]\setminus\{3_1\}} c_j + \sum\nolimits_{j=1}^{x_3} b_{3_j} \ \geq \sum\nolimits_{i=4}^{\ell} \sum\nolimits_{j=1}^{x_i} c_{i_j} + \sum\nolimits_{j=1}^{x_3} b_{3_j}.\end{aligned}$$

Thus we can choose disjoint subsets $V_{2,2}, U_4, \ldots, U_\ell$ of V_2 with $|V_{2,2}| = \sum_{j=1}^{x_3} b_{3_j}$ and $|U_i| = \sum_{j=1}^{x_i} c_{i_j}$ for each $i \in \{4, \ldots, \ell\}$. Moreover, by the assumptions of the theorem, we have $b_i \leq a_i \leq \frac{1}{2}(a_i + b_i + c_i) = \frac{1}{2}|V(G_i)|$ for all $i \in [t]$. Combining with $|V_3| \geq \cdots \geq |V_\ell|$, we can deduce that for $i \in \{3, \ldots, \ell - 1\}$,

$$\begin{aligned}\sum\nolimits_{j=1}^{x_i} a_{i_j} + \sum\nolimits_{j=1}^{x_{i+1}} b_{(i+1)_j} &\leq \frac{1}{2}\sum\nolimits_{j=1}^{x_i} |V(G_{i_j})| + \frac{1}{2}\sum\nolimits_{j=1}^{x_{i+1}} |V(G_{(i+1)_j})| \\ &\leq \frac{1}{2}|V_i| + \frac{1}{2}|V_{i+1}| \ \leq \frac{1}{2}|V_i| + \frac{1}{2}|V_i| \ = |V_i|.\end{aligned}$$

Thus for each $i \in \{3, \ldots, \ell - 1\}$, we can choose disjoint subsets $V_{i,1}, V_{i,2}$ of V_i with $|V_{i,1}| = \sum_{j=1}^{x_i} a_{i_j}$ and $|V_{i,2}| = \sum_{j=1}^{x_{i+1}} b_{(i+1)_j}$. Furthermore, since $b_i + c_i \geq \sum_{j\in[t]\setminus\{i\}} c_j$ for all $i \in [t]$, we have

$$\begin{aligned}|V_\ell| \geq \ &\sum\nolimits_{j=1}^{x_\ell}(a_{\ell_j} + b_{\ell_j} + c_{\ell_j}) \ \geq \left(\sum\nolimits_{j=1}^{x_\ell} a_{\ell_j}\right) + (b_{\ell_1} + c_{\ell_1}) \\ \geq \ &\sum\nolimits_{j=1}^{x_\ell} a_{\ell_j} + \sum\nolimits_{j\in[t]\setminus\{\ell_1\}} c_j \ \geq \sum\nolimits_{j=1}^{x_\ell} a_{\ell_j} + \sum\nolimits_{j=1}^{x_3} c_{3_j}.\end{aligned}$$

Thus we can choose disjoint subsets $V_{\ell,1}, U_3$ of V_ℓ with $|V_{\ell,1}| = \sum_{j=1}^{x_\ell} a_{\ell_j}$ and $|U_3| = \sum_{j=1}^{x_3} c_{3_j}$. Now for each $i \in \{3, \ldots, \ell\}$, $F[V_{i,1} \cup V_{i-1,2} \cup U_i]$ contains a monochromatic $G_{i_1} \cup \cdots \cup G_{i_{x_i}}$ of color 1. Thus F contains a monochromatic copy of H, a contradiction. This completes the proof of Theorem 1.12. $\square$

Proof of Theorem 1.13. Let $H_1, H_2, \ldots, H_t$ be the components of H with $\chi(H_1) = 3$, $\chi(H_2) = \cdots = \chi(H_t) = 2$ and $\min_{i\in[t]} |V(H_i)| \geq \sigma(H) = \sigma(H_1)$. Let s_1, s_2, s_3 (resp., t_1, t_2) be the sizes of color classes in a proper 3-vertex-coloring of H_1 (resp., a proper 2-vertex-coloring of $H_2 \cup \cdots \cup H_t$) with $s_1 \geq s_2 \geq s_3 = \sigma(H_1)$ (resp., $t_1 \geq t_2$).

By Inequality (1.1), it suffices to show that $f(H, P_5) \leq R_3(H)$. Suppose, for the sake of contradiction, that there exists an edge-coloring F of $K_{R_3(H)}$ without rainbow P_5 or monochromatic H. By Lemma 2.9, we can partition $V(F)$ into $k-1$ parts $V_2, V_3, \ldots, V_k$ for some $k \geq 4$ satisfying

(i) $\{i\} \subseteq C(V_i) \subseteq \{1, i\}$ for every $i \in \{2, 3, \ldots, k\}$,

(ii) $c(V_i, V_j) = 1$ for every $2 \leq i < j \leq k$, and

(iii) $|V_2| \geq |V_3| \geq \cdots \geq |V_k|$, $|V_3| \geq \max_{i\in[t]} |V(H_i)| \geq s_1 + s_2 + s_3$, $|V_4| \geq \min_{i\in[t]} |V(H_i)| \geq \sigma(H_1) = s_3$ and $|V_3 \cup \cdots \cup V_k| \geq |V(H)|$.

Let $X \subseteq V_2$, $Y \subseteq V_3$ and $Z \subseteq V_4 \cup \cdots \cup V_k$ with $|X| = s_2$, $|Y| = s_1$ and $|Z| = s_3$. Then the edges between X, Y, Z form a monochromatic K_{s_1,s_2,s_3} of color 1 that contains H_1 as a subgraph. Let $F' = F - (X \cup Y \cup Z)$. If F' contains a monochromatic K_{t_1,t_2} of color 1, then F contains a monochromatic H, a contradiction. Hence, we may assume that the following statement holds.

Fact 5.1. *F' contains no monochromatic K_{t_1,t_2} of color 1.*

We next state and prove two claims related to the sizes of V_2 and V_3.

Claim 5.8. $|V_2| \leq s_2 + t_2 - 1$.

Proof. Suppose that $|V_2| \geq s_2 + t_2$. Then we can choose $U \subseteq V_2 \setminus X$ with $|U| = t_2$. Since $|V_3 \cup \cdots \cup V_k| - |Y \cup Z| \geq |V(H)| - s_1 - s_3 > t_1$, the edges between U and $(V_3 \cup \cdots \cup V_k) \setminus (Y \cup Z)$ form a monochromatic subgraph of color 1 that contains K_{t_1,t_2} as a subgraph. This contradicts Fact 5.1. $\square$

Claim 5.9. $|V_2| + |V_3| \leq s_1 + s_2 + t_2 - 1$.

Proof. Suppose that $|V_2| + |V_3| \geq s_1 + s_2 + t_2$. Then we can choose $U \subseteq (V_2 \cup V_3) \setminus (X \cup Y)$ with $|U| = t_2$. By Lemma 2.5 (iv) and Claim 5.8, we have

$$\begin{aligned} |V_4 \cup \cdots \cup V_k| - |Z| &= R_3(H) - |V_2| - |V_3| - s_3 \ \geq R_3(H) - 2|V_2| - s_3 \\ &\geq 2|V(H)| + \sigma(H) - 2 - 2(s_2 + t_2 - 1) - s_3 \\ &= 2(s_1 + s_2 + s_3 + t_1 + t_2) + s_3 - 2 - 2(s_2 + t_2 - 1) - s_3 \\ &= 2(s_1 + s_3 + t_1) \ > t_1. \end{aligned}$$

Then the edges between U and $(V_4 \cup \cdots \cup V_k) \setminus Z$ form a monochromatic subgraph of color 1 that contains K_{t_1,t_2} as a subgraph. This contradicts Fact 5.1. $\square$

By Lemma 2.5 (iv) and Claim 5.9, we have

$$\begin{aligned} |V_4 \cup \cdots \cup V_k| &= R_3(H) - |V_2| - |V_3| \\ &\geq 2|V(H)| + \sigma(H) - 2 - (s_1 + s_2 + t_2 - 1) \\ &= 2(s_1 + s_2 + s_3 + t_1 + t_2) + s_3 - 2 - (s_1 + s_2 + t_2 - 1) \\ &= s_1 + s_2 + 3s_3 + 2t_1 + t_2 - 1. \end{aligned} \tag{5.5}$$

In particular, we have $|V_4 \cup \cdots \cup V_k| > |V_2|$ by Claim 5.8. Thus $k \geq 5$ since $|V_2| \geq |V_3| \geq \cdots \geq |V_k|$. Choose a subset $I \subseteq \{4, 5, \ldots, k\}$ and correspondingly define $A_1 = \bigcup_{i \in I} V_i$ and $A_2 = (V_4 \cup \cdots \cup V_k) \setminus A_1$ such that the following conditions hold:

(1) $|A_1| \geq |A_2|$,

(2) $|A_1| - |A_2|$ is smallest subject to (1).

By Inequality (5.5) and Condition (1), we have

$$|A_1| \geq \frac{1}{2}|V_4 \cup \cdots \cup V_k| \geq \frac{1}{2}(s_1 + s_2 + 3s_3 + 2t_1 + t_2 - 1).$$

In particular, we have $|A_1| > s_3 + t_1 > s_3$. Hence, we may assume that the subset Z of $V_4 \cup \cdots \cup V_k$ is contained in A_1. Let $U \subset A_1 \setminus Z$ with $|U| = t_1$.

If $|A_2| \geq t_2$, then the edges between U and A_2 form a monochromatic subgraph of color 1 that contains K_{t_1,t_2} as a subgraph. This contradicts Fact 5.1. Therefore, we may assume that $|A_2| < t_2$. Combining with $t_2 \leq t_1$, Inequality (5.5) and Claim 5.8, we have

$$|A_2| < t_2 < \frac{1}{3}|V_4 \cup \cdots \cup V_k| = \frac{1}{3}(|A_1| + |A_2|)$$

and

$$|A_1| = |V_4 \cup \cdots \cup V_k| - |A_2| > (s_1 + s_2 + 3s_3 + 2t_1 + t_2 - 1) - t_2 > s_2 + t_2 > |V_2|,$$

so $|A_2| < \frac{1}{2}|A_1|$ and $|I| \geq 2$. Let $j \in I$ with $|V_j| = \min_{i \in I} |V_i|$, so $|V_j| \leq \frac{1}{2}|A_1|$. If $|A_1 \setminus V_j| \geq |A_2 \cup V_j|$, then $I' = I \setminus \{j\}$ is a better choice than I by Condition (2). Thus $|A_1 \setminus V_j| < |A_2 \cup V_j|$. Let $I^* = (\{4, 5, \ldots, k\} \setminus I) \cup \{j\}$, $A_1^* = \bigcup_{i \in I^*} V_i$ and $A_2^* = (V_4 \cup \cdots \cup V_k) \setminus A_1^*$. Then $|A_1^*| = |A_2 \cup V_j| > |A_1 \setminus V_j| = |A_2^*|$ and

$$\begin{aligned}
&(|A_1| - |A_2|) - (|A_1^*| - |A_2^*|) \\
=\ &(|A_1| - |A_2|) - (|A_2| + |V_j| - (|A_1| - |V_j|)) \\
=\ &2(|A_1| - |A_2| - |V_j|)\ > 2\left(|A_1| - \frac{1}{2}|A_1| - \frac{1}{2}|A_1|\right)\ = 0,
\end{aligned}$$

contradicting the choice of I. This completes the proof of Theorem 1.13. □

6 Concluding remarks

In this paper, we address Conjecture 1.2 for multiple classes of disconnected graphs with chromatic number at least 3. Our newly established general results encompass all known results on this problem as shown in Section 1. In the following, we make some remarks to illustrate the challenges in solving Conjecture 1.2 completely and present several related results and open problems. In particular, we will prove several results for a bipartite variation of the constrained Ramsey number.

(1) We first illustrate the challenges in solving Conjecture 1.2. Firstly, in our proofs of Theorems 1.7 and 1.8, we utilized Lemma 2.2 which states that if $R_3(H) \geq R_2(\mathscr{C}(H))$, then $f(H, P_5) = R_3(H)$. This lemma provides a powerful sufficient condition to assure $f(H, P_5) = R_3(H)$. However, there exist graphs H such that $R_3(H) < R_2(\mathscr{C}(H))$. For instance, Lorimer and Segedin [36] proved that $R_3(nK_r) \leq 3(n-1)r + R_3(K_r)$, while Roberts [40] proved that $R_2(\mathscr{C}(nK_r)) = (r^2 - r + 1)n - r + 1$ for $n \geq R_2(K_r)$. Thus $R_3(nK_r) < R_2(\mathscr{C}(nK_r))$ when $r \geq 4$ and $n \geq \max\left\{R_2(K_r), \frac{R_3(K_r) - 2r}{r^2 - 4r + 1}\right\}$. Therefore, if Conjecture 1.2 is true, then the condition $R_3(H) \geq R_2(\mathscr{C}(H))$ cannot be a necessary condition for $f(H, P_5) = R_3(H)$. In order to address Conjecture 1.2 for graphs H with $R_3(H) < R_2(\mathscr{C}(H))$, some novel ideas are called for.

Secondly, in our proofs of Theorems 1.4, 1.5, 1.10, 1.12 and 1.13, we applied Lemma 2.8 or 2.9 which characterize the possible structure of an edge-colored $K_{R_3(H)}$ without a rainbow P_5 or a monochromatic H. As in our proofs, upon employing this structural result, we frequently utilize lower bounds of the Ramsey number $R_3(H)$ to derive an upper or lower bound on the sizes of $V_2, V_3, \ldots, V_k$. The main challenge in this procedure lies in the fact that we currently lack an efficient lower bound on the Ramsey number $R_3(H)$. As research on $R_3(H)$ advances, we believe that more results related to Conjecture 1.2 will become attainable.

Thirdly, for graphs H with $\chi(H) \geq 4$, the problem becomes more difficult. When we employ the partition $V_2, V_3, \ldots, V_k$ guaranteed by Lemma 2.8 or 2.9, we commonly find a monochromatic copy of H of color 1 consisting of certain edges between the parts. If $k = 4$, it is impossible to find a copy of H (if $\chi(H) \geq 4$) using edges between V_2, V_3 and V_4. To acquire a monochromatic copy of H, we must conduct an in-depth study of the structure within each part. However, this needs a good understanding of the relationship between $R_3(H)$

and 2-colored Ramsey numbers $R(H, H')$ for H and its subgraphs H'. We believe that the relationship between 3-colored Ramsey numbers and 2-colored Ramsey numbers will very fertile for future research. Further research in this area could be fruitful.

(2) As the first step towards solving Conjecture 1.2, we propose the following foundational problem.

Problem 6.1. *Given two connected graphs G_1, G_2 with $\chi(G_1) \leq \chi(G_2) = 3$, determine whether $f(G_1 \cup G_2, P_5) = R_3(G_1 \cup G_2)$.*

We can prove the following partial result; however, solving the problem in general remains challenging.

Proposition 6.2. *Let G_1, G_2 be two connected graphs with $\chi(G_1) \leq \chi(G_2) = 3$ and $\min\{|V(G_1)|, |V(G_2)|\} \geq \sigma_3(G_1) + \sigma_3(G_2)$. Then $f(G_1 \cup G_2, P_5) = R_3(G_1 \cup G_2)$*

Proof. If $\chi(G_1) = 2$, then $\sigma_3(G_1) = 0$, so $\min\{|V(G_1)|, |V(G_2)|\} \geq \sigma_3(G_1) + \sigma_3(G_2)$ is equivalent to $|V(G_1)| \geq \sigma(G_1 \cup G_2)$. Then the proposition is true by Theorem 1.13. Assume now $\chi(G_1) = \chi(G_2) = 3$ and $\min\{|V(G_1)|, |V(G_2)|\} \geq \sigma_3(G_1) + \sigma_3(G_2)$. Let s_1, s_2, s_3 (resp., t_1, t_2, t_3) be the sizes of color classes in a proper 3-vertex-coloring of G_1 (resp., G_2) with $s_1 \geq s_2 \geq s_3 = \sigma(G_1)$ (resp., $t_1 \geq t_2 \geq t_3 = \sigma(G_2)$).

By Inequality (1.1), it suffices to show that $f(H, P_5) \leq R_3(H)$. Suppose, for the sake of contradiction, that there exists an edge-coloring F of $K_{R_3(G_1 \cup G_2)}$ without rainbow P_5 or monochromatic $G_1 \cup G_2$. By Lemma 2.9, we can partition $V(F)$ into $k-1$ parts $V_2, V_3, \ldots, V_k$ for some $k \geq 4$ satisfying

(i) $\{i\} \subseteq C(V_i) \subseteq \{1, i\}$ for every $i \in \{2, 3, \ldots, k\}$,

(ii) $c(V_i, V_j) = 1$ for every $2 \leq i < j \leq k$,

and in addition, we have $|V_2| \geq |V_3| \geq \cdots \geq |V_k|$,

$$|V_3| \geq \max\{|V(G_1)|, |V(G_2)|\} \geq \frac{1}{2}|V(G_1)| + \frac{1}{2}|V(G_2)| \geq s_2 + t_2$$

and

$$\min\{|V_2|, |V_4 \cup \cdots \cup V_k|\} \geq |V_4| \geq \min\{|V(G_1)|, |V(G_2)|\} \geq \sigma_3(G_1) + \sigma_3(G_2) = s_3 + t_3.$$

Moreover, by Lemma 2.5 (iii) and since $|V_2| \geq |V_3|$ and $|V_2| + |V_3| + |V_4 \cup \cdots \cup V_k| = R_3(G_1 \cup G_2)$, we have

$$\begin{aligned} \max\{|V_2|, |V_4 \cup \cdots \cup V_k|\} \geq \left\lceil \frac{1}{3} R_3(G_1 \cup G_2) \right\rceil &\geq \left\lceil \frac{1}{3} \left(3(|V(G_1)| + |V(G_2)|) - 2\right) \right\rceil \\ &= |V(G_1)| + |V(G_2)| \geq s_1 + t_1. \end{aligned}$$

By Observation 2.12 (ii), there is a monochromatic $G_1 \cup G_2$ of color 1 using edges between V_2, V_3 and $V_4 \cup \cdots \cup V_k$, a contradiction. This completes the proof of Proposition 6.2. $\square$

(3) For all graphs H for which we have proved so far that $f(H, P_5) = R_3(H)$, the corresponding extremal constructions are 3-edge-colored $K_{R_3(H)-1}$ without monochromatic copies of H. This is in fact not related to the condition of rainbow P_5, since a 3-edge-colored graph certainly contains no rainbow P_5. We do not know whether there exists an extremal construction using at least four colors. We suspect that the problem may behave very differently if we restrict our attention to edge-colorings with at least four colors. Therefore, we suggest the following problem for further research.

Problem 6.3. *Let H be a graph.*

(i) *Determine the minimum integer n such that in every edge-coloring of K_n **with at least four colors**, there is either a monochromatic copy of H or a rainbow copy of P_5.*

(ii) *For any integer $k \geq 4$, determine the minimum integer n such that in every edge-coloring of K_n **with exactly k colors**, there is either a monochromatic copy of H or a rainbow copy of P_5.*

Regarding Problem 6.3 (ii), we can provide the following lower bound construction when $k \leq \chi(H)$. Recall that $\mathcal{M}(H)$ is the decomposition family of H. For $2 \leq i \leq \chi(H) - 1$, let $\mathcal{M}_i(H)$ be the set of minimal graphs M that satisfies the following: for each M, there exists an integer t such that $H \subseteq \big(M \cup \overline{K}_{t-|V(M)|}\big) \vee K_{(\chi(H)-i)\times t}$. Note that $\mathcal{M}_2(H) = \mathcal{M}(H)$. Then for $4 \leq k \leq \chi(H)$, a lower bound on the minimum integer n in Problem 6.3 (ii) is $(k-2)(|V(H)|-1) + R(\mathcal{M}_{\chi(H)-k+2}(H), H)$. To see this, we defined a k-edge-colored complete graph on $(k-2)(|V(H)|-1) + R(\mathcal{M}_{\chi(H)-k+2}(H), H) - 1$ vertices as follows. We partition the vertex set into $k-1$ parts $V_2, V_3, \ldots, V_k$ with $|V_2| = |V_3| = \cdots = |V_{k-1}| = |V(H)| - 1$ and $|V_k| = R(\mathcal{M}_{\chi(H)-k+2}(H), H) - 1$. For each $i \in \{2, 3, \ldots, k-1\}$, we color all the edges within V_i using color i. We color the edges within V_k using colors 1 and k such that it contains neither a monochromatic graph in $\mathcal{M}_{\chi(H)-k+2}(H)$ of color 1 nor a monochromatic H of color 2. We color all the remaining edges with color 1. Then there is no monochromatic H or rainbow P_5 in this k-edge-colored complete graph, and thus the lower bound holds. This lower bound is sharp when H is a clique K_p. Indeed, it was shown by the authors and Wang [33] that for $p \geq 5$ and $k \in \{p-1, p\}$, the answer to Problem 6.3 (ii) is $(k-2)(p-1) + R(K_{p-k+2}, K_p)$. For $4 \leq k \leq p-2$, the first author and Su showed in an unpublished manuscript that the answer to Problem 6.3 (ii) is $\max_{(a_2,\ldots,a_k)\in X} \big(\sum_{i=2}^{k}(R(K_{a_i+1}, K_p) - 1)\big) + 1$, where $X = \big\{(a_2, \ldots, a_k) \in \{1, 2, \ldots, p-k+2\}^{k-1} \colon \sum_{i=2}^{k} a_i = p-1\big\}$. Note that $\max_{(a_2,\ldots,a_k)\in X} \big(\sum_{i=2}^{k}(R(K_{a_i+1}, K_p) - 1)\big) + 1 = (k-2)(p-1) + R(K_{p-k+2}, K_p)$ if the inequality $R(K_a, K_p) + R(K_b, K_p) \leq R(K_{a-1}, K_p) + R(K_{b+1}, K_p)$ holds for all $3 \leq a \leq b \leq p-1$. For general graphs, especially disconnected graphs, Problem 6.3 remains open.

(4) For general path P_t with $t \geq 6$, we have $f(H, P_t) \geq R_{t-2}(H)$ since a $(t-2)$-edge-colored graph trivially contains no rainbow P_t. However, we have no convincing evidence to suggest that $f(H, P_t) \leq R_{t-2}(H)$ holds true. Hence, we pose the following problem for further research.

Problem 6.4. *Given a graph H and integer $t \geq 6$, determine an upper bound on $f(H, P_t)$.*

(5) Recently, the first author and Wang [32] studied a hypergraph version of the problem. There are three 3-uniform paths of length 3: the tight path $\mathcal{T} = \{v_1v_2v_3, v_2v_3v_4, v_3v_4v_5\}$, the messy path $\mathcal{M} = \{v_1v_2v_3, v_2v_3v_4, v_4v_5v_6\}$ and the loose path $\mathcal{L} = \{v_1v_2v_3, v_3v_4v_5, v_5v_6v_7\}$. The authors in [32] characterized the structures of edge-colored $K_n^{(3)}$ without rainbow $\mathcal{T}$, $\mathcal{M}$ and $\mathcal{L}$, respectively. As applications, they showed that $f(H, G) = R_2(H)$ for $G \in \{\mathcal{T}, \mathcal{M}, \mathcal{L}\}$ and infinitely many 3-uniform hypergraphs H. For higher uniformity or higher length, the situation becomes more complicated. We propose the following related problem for further research.

Problem 6.5. *For the 3-uniform tight path or loose path P of length 4 and a 3-uniform hypergraph H, determine if $f(H,P) = R_3(H)$.*

(6) In 2004, Eroh and Oellermann [13] introduced a bipartite variation of the constrained Ramsey number $f(H,G)$. Given two bipartite graphs H and G, the *bipartite constrained Ramsey number* (also called *bipartite rainbow Ramsey number*) $h(H,G)$ is defined as the minimum integer n such that, in every edge-coloring of $K_{n,n}$ with any number of colors, there is either a monochromatic copy of H or a rainbow copy of G. Eroh and Oellermann [13] showed that $h(H,G)$ exists if and only if either H is a star or G is a star forest. Therefore, if G is a path of length at least 3, then $h(H,G)$ exists if and only if H is a star. Eroh and Oellermann [13] proved that for integers $m, n \geq 2$ with $m \leq 2n-3$, we have $h(K_{1,n}, P_{m+1}) = (m-1)(n-1)+1 = BR_{m-1}(K_{1,n})$. In particular, it follows that $h(K_{1,n}, P_4) = BR_2(K_{1,n})$ for $n \geq 3$, and $h(K_{1,n}, P_5) = BR_3(K_{1,n})$ for $n \geq 4$. One can also derive that $h(K_{1,2}, P_4) = 3 = BR_2(K_{1,2})$, $h(K_{1,2}, P_5) = 5 = BR_3(K_{1,2}) + 1$ and $h(K_{1,3}, P_5) = 7 = BR_3(K_{1,3})$ from Lemmas 2.10 and 2.11. Thus $h(K_{1,n}, P_5) = BR_3(K_{1,n})$ only holds for $n \geq 3$.

If we intend to study the problem for other monochromatic subgraphs, we must take into account an additional restriction on the number of colors to ensure the study is well-defined. Given two bipartite graphs H, G and a positive integer k, let $h_k(H,G)$ be the minimum integer n such that in any edge-coloring of $K_{n,n}$ with at most k colors, there is either a monochromatic copy of H or a rainbow copy of G. We can prove the following two results related to rainbow P_4 and P_5. Recall that $s(H), t(H), s^*(H), t^*(H)$ are the parameters defined in Section 2.1.

Proposition 6.6. *Let H be a bipartite graph and $k \geq 3$ be an integer. If H is connected or $k \geq \frac{t(H)-1}{s(H)-1}$, then $h_k(H, P_4) = \max\{BR_2(H), k(s(H)-1)+1\}$.*

Proof. For the lower bound, since a 2-edge-colored graph contains no rainbow P_4, we have $h_k(H,P_4) \geq BR_2(H)$. In order to prove $h_k(H,P_4) \geq k(s(H)-1)+1$, we defined an edge-colored $K_{k(s(H)-1),k(s(H)-1)}$ with partite sets X and Y as follows. We partition X into k parts $X_1, X_2, \ldots, X_k$ with $|X_1| = |X_2| = \cdots = |X_k| = s(H)-1$, and color the edges such that $c(X_i, Y) = i$ for every $i \in [k]$. Note that there is no rainbow P_4 or monochromatic H is this edge-colored $K_{k(s(H)-1),k(s(H)-1)}$, so $h_k(H,P_4) \geq k(s(H)-1)+1$.

We now show that if H is connected or $k \geq \frac{t(H)-1}{s(H)-1}$, then $h_k(H,P_4) \leq \max\{BR_2(H), k(s(H)-1)+1\}$. We shall prove a slightly stronger statement: if $s(H) = s^*(H)$ or $t(H) \leq \max\{BR_2(H), k(s(H)-1)+1\}$, then $h_k(H,P_4) \leq \max\{BR_2(H), k(s(H)-1)+1\}$. For a contradiction, suppose that there exists an edge-coloring F of $K_{n,n}$ with at most k colors such that there is no rainbow P_4 or monochromatic H in F, where $n = \max\{BR_2(H), k(s(H)-1)+1\}$. Let k' be the number of colors used on $E(F)$, and let U, V be the partite sets of F. Then $3 \leq k' \leq k$, and by Lemma 2.10, we may assume that U can be partitioned into k' parts $U_1, U_2, \ldots, U_{k'}$ such that $c(U_i, V) = i$ for every $i \in [k']$. Note that $\max_{i\in[k']} |U_i| \geq \lceil \frac{n}{k'} \rceil \geq s(H)$ and $|V| = n \geq BR_2(H) \geq t^*(H)$. In the case that $s(H) = s^*(H)$, we have $\max_{i\in[k']} |U_i| \geq s(H) = s^*(H)$, so there is a monochromatic $K_{s^*(H),t^*(H)}$ (and thus a monochromatic H) in F, a contradiction. In the case that $t(H) \leq \max\{BR_2(H), k(s(H)-1)+1\} = n = |V|$, there is a monochromatic $K_{s(H),t(H)}$ (and thus a monochromatic H) in F, a contradiction. This completes the proof of Proposition 6.6. $\square$

Proposition 6.7. *Let H be a bipartite graph with $V(H) \geq 4$ and $k \geq 4$ be an integer. If H is connected or $BR_2(H) + t(H) - 1 \leq \max\{BR_3(H), k(s(H)-1)+1\}$, then $h_k(H,P_5) = \max\{BR_3(H), k(s(H)-1)+1\}$.*

Proof. For the lower bound, since a 3-edge-colored graph contains no rainbow P_5, we have $h_k(H, P_5) \geq BR_3(H)$. Moreover, since $P_4 \subseteq P_5$, we have $h_k(H, P_5) \geq h_k(H, P_4) \geq k(s(H) - 1) + 1$ by the proof of Proposition 6.6.

For the upper bound, suppose for a contradiction that there exists an edge-coloring F of $K_{n,n}$ with at most k colors such that there is no rainbow P_5 or monochromatic H in F, where $n = \max\{BR_3(H), k(s(H) - 1) + 1\}$. By the above mentioned result on $h(K_{1,n}, P_5)$ and since $|V(H)| \geq 4$, we may assume that H is not a star. Then $s(H) \geq 2$ and $n \geq k(s(H) - 1) + 1 \geq k + 1 \geq 5$. Let k' be the number of colors used on $E(F)$, and let U, V be the partite sets of F. Then $4 \leq k' \leq k$, and by Lemma 2.11, one of Lemma 2.11 (a) and (b) holds.

We first assume that Lemma 2.11 (a) holds. Note that $\max_{i\in[k']} |V_i| \geq \lceil \frac{n}{k'} \rceil \geq s(H)$. If $|U_1| \geq t(H)$ or $|U_2| \geq s^*(H)$, then F contains a monochromatic $K_{s(H),t(H)}$ or $K_{s^*(H),n}$ (and thus a monochromatic H), a contradiction. Therefore, $n = |U_1|+|U_2| \leq t(H)-1+s^*(H)-1 < BR_2(H)+t(H)-1$. Thus we only need to consider the case that H is connected. By Lemma 2.7, we have $\max\{|U_1|, |U_2|\} \geq \frac{1}{2}n \geq \frac{1}{2}BR_3(H) \geq \frac{1}{2}(3(t(H) - 1) + 1) \geq t(H)$. Thus F contains a monochromatic $K_{s(H),t(H)}$ (and thus a monochromatic H), a contradiction.

Now we assume that Lemma 2.11 (b) holds. In the case that H is connected, we can define a 3-edge-coloring F' of $K_{n,n}$ by recoloring all edges of colors in $\{4, \ldots, k'\}$ with color 3. Note that F' contains a monochromatic H since $n \geq BR_3(H)$. Since H is connected, such an H is also monochromatic in F (although its color in F is possibly different from that in F'), a contradiction. For the case $BR_2(H) + t(H) - 1 \leq \max\{BR_3(H), k(s(H) - 1) + 1\}$, we may assume that $|U_2| = \max_{2\leq i\leq k'} |U_i|$ without loss of generality. Thus $|U_2| \geq \lceil \frac{n}{k'-1} \rceil \geq s(H)$. Then $|V_3 \cup \cdots \cup V_{k'}| \leq t(H)-1$, since otherwise there is a monochromatic H of color 1 between U_2 and $V_3 \cup \cdots \cup V_{k'}$. Thus $|V_2| = |V| - |V_3 \cup \cdots \cup V_{k'}| \geq n - (t(H) - 1) \geq BR_2(H) \geq s(H)$. Then $|U_3 \cup \cdots \cup U_{k'}| \leq t(H)-1$, since otherwise there is a monochromatic H of color 1 between V_2 and $U_3 \cup \cdots \cup U_{k'}$. Thus $|U_2| = |U| - |U_3 \cup \cdots \cup U_{k'}| \geq n - (t(H) - 1) \geq BR_2(H)$. But then since $|U_2| \geq BR_2(H)$, $|V_2| \geq BR_2(H)$ and $C(U_2, V_2) \subseteq \{1, 2\}$, there is a monochromatic H between U_2 and V_2, a contradiction. This completes the proof of Proposition 6.7. □

Regarding general disconnected bipartite graphs, the problem under consideration seems difficult to resolve. Therefore, we formulate the following problem for subsequent research.

Problem 6.8. *Given a disconnected bipartite graph H and integers $k \geq t - 1$ with $t \in \{4, 5\}$, determine whether $h_k(H, P_t) = \max\{BR_{t-2}(H), k(s(H) - 1) + 1\}$.*

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References

[1] R. Aharoni, M. DeVos, and R. Holzman, Rainbow triangles and the Caccetta-Häggkvist conjecture, *J. Graph Theory* **92**(4) (2019), 347–360. `https://doi.org/10.1002/jgt.22457`

[2] N. Alon, T. Jiang, Z. Miller, and D. Pritikin, Properly colored subgraphs and rainbow subgraphs in edge-colorings with local constraints, *Random Structures Algorithms* **23**(4) (2003), 409–433. https://doi.org/10.1002/rsa.10102

[3] M. Axenovich and H. Lefmann, Canonical theorems for colored integers with respect to some linear combinations, *SIAM J. Discrete Math.* **38**(1) (2024), 609–628. https://doi.org/10.1137/21M1454195

[4] P.N. Balister, A. Gyárfás, J. Lehel, and R.H. Schelp, Mono-multi bipartite Ramsey numbers, designs, and matrices, *J. Combin. Theory Ser. A* **113**(1) (2006), 101–112. https://doi.org/10.1016/j.jcta.2005.07.003

[5] N. Behague, R. Hancock, J. Hyde, S. Letzter, and N. Morrison, Thresholds for constrained Ramsey and anti-Ramsey problems, *European J. Combin.* **129** (2025), Paper No. 104159. https://doi.org/10.1016/j.ejc.2025.104159

[6] S.A. Burr, Ramsey numbers involving graphs with long suspended paths, *J. London Math. Soc. (2)* **24**(3) (1981), 405–413. https://doi.org/10.1112/jlms/s2-24.3.405

[7] H.-B. Chen, Y.-J. Cheng, W.-T. Li, and C.-A. Liu, The Boolean rainbow Ramsey number of antichains, Boolean posets and chains, *Electron. J. Combin.* **27**(4) (2020), Paper No. 4.38, 12 pp. https://doi.org/10.37236/9034

[8] G.T. Chen, R.H. Schelp, and B. Wei, Monochromatic-rainbow Ramsey numbers, *14th Cumberland Conference Abstracts*, (2001).

[9] V. Chvátal, Tree-complete graph Ramsey numbers, *J. Graph Theory* **1**(1) (1977), 93. https://doi.org/10.1002/jgt.3190010118

[10] M. Collares, Y. Kohayakawa, C.G. Moreira, and G.O. Mota, The threshold for the constrained Ramsey property, *arXiv:2207.05201v1*, (2022), 15 pp. https://doi.org/10.48550/arXiv.2207.05201

[11] P. Erdős and R. Rado, A combinatorial theorem, *J. London Math. Soc.* **s1-25**(4) (1950), 249–255. https://doi.org/10.1112/jlms/s1-25.4.249

[12] L. Eroh, Rainbow Ramsey numbers, *PhD thesis, Western Michigan University*, (2000). http://scholarworks.wmich.edu/dissertations/1448

[13] L. Eroh and O.R. Oellermann, Bipartite rainbow Ramsey numbers, *Discrete Math.* **277**(1-3) (2004), 57–72. https://doi.org/10.1016/S0012-365X(03)00154-7

[14] Y.J. Fang, G.N. Ge, Y. Shu, Q. Xu, Z.X. Xu, and D.L. Yang, Canonical Ramsey: triangles, rectangles and beyond, *arXiv:2510.11638v2*, (2025), 27 pp. https://doi.org/10.48550/arXiv.2510.11638

[15] R.J. Faudree, R.H. Schelp, and J. Sheehan, Ramsey numbers for matchings, *Discrete Math.* **32**(2) (1980), 105–123. https://doi.org/10.1016/0012-365X(80)90049-7

[16] J. Fox, A. Grinshpun, and J. Pach, The Erdős-Hajnal conjecture for rainbow triangles, *J. Combin. Theory, Ser. B* **111** (2015), 75–125. https://doi.org/10.1016/j.jctb.2014.09.005

[17] P. Gehér, A. Sagdeev, and G. Tóth, Canonical theorems in geometric Ramsey theory, *arXiv:2404.11454v2*, (2025), 9 pp, *Comb. Theory*, to appear. https://doi.org/10.48550/arXiv.2404.11454

[18] A. Girão, F. Illingworth, L. Michel, M. Savery, and A. Scott, Flashes and rainbows in tournaments, *Combinatorica* **44**(3) (2024), 675–690. https://doi.org/10.1007/s00493-024-00090-7

[19] L. Gishboliner, A. Milojević, B. Sudakov, and Y. Wigderson, Canonical Ramsey numbers of sparse graphs, *SIAM J. Discrete Math.* **39**(3) (2025), 1491–1519. https://doi.org/10.1137/24M1714964

[20] S. Glock, D. Kühn, R. Montgomery, and D. Osthus, Decompositions into isomorphic rainbow spanning trees, *J. Combin. Theory Ser. B* **146** (2021), 439–484. https://doi.org/10.1016/j.jctb.2020.03.002

[21] A. Gyárfás, J. Lehel, and R.H. Schelp, Finding a monochromatic subgraph or a rainbow path, *J. Graph Theory* **54**(1) (2007), 1–12. https://doi.org/10.1002/jgt.20179

[22] J.H. Hattingh and M.A. Henning, Bipartite Ramsey theory, *Util. Math.* **53** (1998), 217–230. https://www.scopus.com/pages/publications/0032222120

[23] R.E. Jamison, T. Jiang, and A.C.H. Ling, Constrained Ramsey numbers of graphs, *J. Graph Theory* **42**(1) (2003), 1–16. https://doi.org/10.1002/jgt.10072

[24] Z.M. Jin, A heuristic algorithm for rainbow matchings and its application in rainbow Ramsey number for matchings, *Discrete Appl. Math.* **368** (2025), 153–161. https://doi.org/10.1016/j.dam.2025.02.040

[25] P. Keevash, D. Mubayi, B. Sudakov, and J. Verstraëte, Rainbow Turán problems, *Combin. Probab. Comput.* **16**(1) (2007), 109–126. https://doi.org/10.1017/S0963548306007760

[26] X.H. Li, P. Besse, C. Magnant, L.G. Wang, and N. Watts, Gallai-Ramsey numbers for rainbow paths, *Graphs Combin.* **36** (2020), 1163–1175. https://doi.org/10.1007/s00373-020-02175-8

[27] X.H. Li, H. Broersma, and L.G. Wang, The Erdős-Gyárfás function with respect to Gallai-colorings, *J. Graph Theory* **101**(2) (2022), 242–264. https://doi.org/10.1002/jgt.22822

[28] R.N. Li, R.H. Lu, X.L. Su, and S.G. Zhang, The absence of monochromatic triangle implies various properly colored spanning trees, *arXiv:2403.09082v3*, (2024), 15 pp. https://doi.org/10.48550/arXiv.2403.09082

[29] X.H. Li, J. Ma, and T.Y. Xie, Weak rainbow saturation numbers of graphs, *J. Graph Theory* **109**(1) (2025), 35–42. https://doi.org/10.1002/jgt.23211

[30] X.H. Li, J. Ma, and Z.H. Zheng, On the multicolor Turán conjecture for color-critical graphs, *Canad. J. Math.*, (2025), online, 31 pp. https://doi.org/10.4153/S0008414X25101521

[31] X.L. Li and Y. Si, Gallai-Ramsey multiplicity for rainbow small trees, *Graphs Combin.* **40**(4) (2024), Paper No. 91, 30 pp. https://doi.org/10.1007/s00373-024-02819-z

[32] X.H. Li and R.S. Wang, Edge-colored 3-uniform hypergraphs without rainbow paths of length 3 and its applications to Ramsey theory, *arXiv:2510.18246v1*, (2025), 27 pp. https://arxiv.org/abs/2510.18246v1

[33] X.H. Li, L.G. Wang, and X.X. Liu, Complete graphs and complete bipartite graphs without rainbow path, *Discrete Math.* **342**(7) (2019), 2116–2126. https://doi.org/10.1016/j.disc.2019.04.010

[34] X.X. Liu, Constrained Ramsey numbers for the loose path, cycle and star, *Discrete Math.* **347**(1) (2024), Paper No. 113697, 11 pp. https://doi.org/10.1016/j.disc.2023.113697

[35] P.-S. Loh and B. Sudakov, Constrained Ramsey numbers, *Combin. Probab. Comput.* **18**(1-2) (2009), 247–258. https://doi.org/10.1017/S0963548307008875

[36] P.J. Lorimer and R.J. Segedin, Ramsey numbers for multiple copies of complete graphs, *J. Graph Theory* **2**(1) (1978), 89–91. https://doi.org/10.1002/jgt.3190020112

[37] R. Montgomery, A. Pokrovskiy, and B. Sudakov, A proof of Ringel's conjecture, *Geom. Funct. Anal.* **31**(3) (2021), 663–720. https://doi.org/10.1007/s00039-021-00576-2

[38] S.P. Radziszowski, Small Ramsey numbers, *Electron J. Combin.* (2024), Dynamic Survey 1, 133 pp. https://doi.org/10.37236/21

[39] F.P. Ramsey, On a problem of formal logic, *Proc. Lond. Math. Soc. (2)* **30**(1) (1930), 264–286. https://doi.org/10.1112/plms/s2-30.1.264

[40] B. Roberts, Ramsey numbers of connected clique matchings, *Electron. J. Combin.* **24**(1) (2017), Paper No. 1.36, 7 pp. https://doi.org/10.37236/6284

[41] M. Simonovits, Extremal graph problems and graph products, *Studies in pure mathematics*, 669–680, *Birkhäuser Verlag, Basel*, 1983.

[42] A. Thomason and P. Wagner, Complete graphs with no rainbow path, *J. Graph Theory* **54**(3) (2007), 261–266. https://doi.org/10.1002/jgt.20207

[43] Y. Wang and Y.S. Li, Bounds for bipartite rainbow Ramsey numbers, *Graphs Combin.* **33**(4) (2017), 1065–1079. https://doi.org/10.1007/s00373-017-1794-3

[44] M.Q. Wei, Y.P. Mao, I. Schiermeyer, and Z. Wang, Ramsey and Gallai-Ramsey numbers of cycles and books, *Acta Math. Appl. Sin. Engl. Ser.* **41**(2) (2025), 425–440. https://doi.org/10.1007/s10255-025-0009-6