

REPRESENTATIONS OF THE DRINFELD DOUBLES OF POINTED RANK ONE HOPF ALGEBRAS

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ABSTRACT. In this paper, we investigate the representations of the Drinfeld doubles $D(H_{\mathcal{D}})$ of pointed rank one Hopf algebras $H_{\mathcal{D}}$ over an algebraically closed field $\mathbb{k}$ of characteristic zero. We provide a complete classification of all finite-dimensional indecomposable $D(H_{\mathcal{D}})$ -modules up to isomorphism and explicitly describe the Auslander-Reiten sequences in the category of finite-dimensional $D(H_{\mathcal{D}})$ -modules. We show that $D(H_{\mathcal{D}})$ is of tame representation type.

1. INTRODUCTION

Finite-dimensional Hopf algebras play a central role in modern algebra and quantum group theory, particularly through their connections with tensor categories, knot and 3-manifold invariants, and noncommutative geometry. Among them, pointed Hopf algebras of rank one constitute one of the simplest nontrivial families. Despite their apparent simplicity, their structure and representation theory already exhibit many of the key phenomena appearing in the general theory.

The systematic study of pointed Hopf algebras began in the 1990s with the classification program of Andruskiewitsch and Schneider, who reduced the classification problem to the analysis of Nichols algebras. In the rank-one case, the corresponding Nichols algebra is one-dimensional, and the resulting Hopf algebras generalize the classical Taft algebras introduced by E. Taft [21], which provided the first explicit examples of finite-dimensional nonsemisimple, noncommutative Hopf algebras.

The first systematic investigation of finite-dimensional pointed Hopf algebras of rank one was carried out by Krop and Radford [13], who classified all such algebras over algebraically closed fields of characteristic zero. The classification in positive characteristic was later completed by Scherotzke [16]. These Hopf algebras can also be realized as quotients of Hopf-Ore extensions of group algebras, a framework developed by Wang, You, and Chen. The classification over arbitrary fields was subsequently obtained by Wang, You and Chen [24].

The representation theory of these Hopf algebras has been the subject of intensive study. In [9], Cibils investigated the indecomposable modules over Taft algebras and derived explicit decomposition formulas for tensor products of indecomposable modules. Building on Cibils's results, Chen, Van Oystaeyen, and Zhang determined

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the Green rings of all Taft algebras [8], while Li and Zhang computed the Green rings of generalized Taft algebras [14]. The representation theory of finite-dimensional pointed Hopf algebras of rank one was subsequently studied in full generality by Wang, Li, and Zhang [22, 23]. These works describe not only the categories of indecomposable modules but also the fusion rules governing their tensor product structures.

Another line of research concerns the Drinfeld doubles of pointed rank-one Hopf algebras, which are quasi-triangular and therefore play a fundamental role in the study of braided tensor categories. The representation theory of Drinfeld doubles of Taft algebras was studied in detail by Chen [2, 3, 4, 5], who described the simple and indecomposable modules, projective covers, and block structures. These results highlight the deep connections between pointed rank-one Hopf algebras, their doubles, and quantum groups at roots of unity. Erdmann et al. investigated the representations and stable Green rings of Drinfeld doubles of generalized Taft algebras in [10, 11], while Sun and Chen studied the representations of Drinfeld doubles of Radford Hopf algebras in [17]. The (generalized) Taft algebras are pointed rank-one Hopf algebras of nilpotent type, whereas the Radford algebras are of non-nilpotent type. Krop and Radford studied the representations of the Drinfeld double $D(H_{\mathcal{D}})$ of any finite-dimensional pointed rank one Hopf algebra $H_{\mathcal{D}}$ over an algebraically closed field of characteristic zero, classifying all simple and projective indecomposable $D(H_{\mathcal{D}})$ -modules in the case where the group $G(H_{\mathcal{D}})$ of group-like elements is abelian [13]. However, a complete classification of all finite-dimensional indecomposable $D(H_{\mathcal{D}})$ -modules remains open.

In this article, building on the work of Krop and Radford [13], we study the indecomposable representations of the Drinfeld doubles $D(H_{\mathcal{D}})$ of pointed rank-one Hopf algebras $H_{\mathcal{D}}$ over an algebraically closed field $\mathbb{k}$ of characteristic zero, assuming that $G(H_{\mathcal{D}})$ is abelian.

The paper is organized as follows. In Section 2, we recall the definition of a group datum, the structure of a pointed rank one Hopf algebra $H_{\mathcal{D}}$, and its Drinfeld double $D(H_{\mathcal{D}})$.

In Section 3, we review the simple modules and projective indecomposable modules over $D(H_{\mathcal{D}})$, showing that the Loewy length of $D(H_{\mathcal{D}})$ is three. We divide both the simple and non-simple projective indecomposable modules into the nilpotent and non-nilpotent cases, and provide standard $\mathbb{k}$ -bases for each of these modules.

In Section 4, we investigate the finite-dimensional indecomposable $D(H_{\mathcal{D}})$ -modules of Loewy length two. Using Auslander-Reiten theory, we describe all such modules and classify them up to isomorphism. Moreover, all Auslander-Reiten sequences in the category of finite-dimensional $D(H_{\mathcal{D}})$ -modules are explicitly presented. Finally, we show that $D(H_{\mathcal{D}})$ is of tame representation type. The main results of the paper are as follows:

Theorem 4.23 Assume $m > 1$. A complete set of representatives of isomorphism classes of finite dimensional indecomposable $D(H_{\mathcal{D}})$ -modules is given by

$$\left\{ \begin{array}{l} V(l', \lambda'), P(l, \lambda), \Omega^{\pm s} V(l, \lambda), \\ T_s(l, \lambda), \bar{T}_s(l, \lambda), M_s(l, \lambda, \eta) \end{array} \middle| \begin{array}{l} 1 \leq l \leq n-1, 1 \leq l' \leq n, s \geq 1, \\ \lambda \in I_l, \lambda' \in I_{l'}, \eta \in \mathbb{k}^{\times} \end{array} \right\}.$$

Theorem 4.30 Assume $m = 1$. A complete set of representatives of isomorphism classes of finite dimensional indecomposable $D(H_{\mathcal{D}})$ -modules is given by

$$\left\{ \begin{array}{l} V(l', \lambda'), P(l, \lambda), \\ \Omega^{\pm s} V(l, \lambda), W_s(l, \lambda, \eta) \end{array} \middle| \begin{array}{l} 1 \leq l \leq n-1, 1 \leq l' \leq n, s \geq 1, \\ \lambda \in I_l, \lambda' \in I_{l'}, \eta \in \mathbb{k} \end{array} \right\}.$$

Throughout, let $\mathbb{k}$ be an algebraically closed field with $\text{char } \mathbb{k} = 0$ and $\mathbb{k}^\times = \mathbb{k} \setminus \{0\}$. Unless otherwise stated, all algebras and Hopf algebras are defined over $\mathbb{k}$; all modules are finite dimensional and left modules; $\dim$ and $\otimes$ denote $\dim_{\mathbb{k}}$ and $\otimes_{\mathbb{k}}$, respectively. Let $\mathbb{Z}$ denote the set of all integers, $\mathbb{Z}_n := \mathbb{Z}/n\mathbb{Z}$ for an integer n , and let $\mathbb{N}$ denote all non-negative integers. The references [12, 15, 19] are basic references for the theory of Hopf algebras and quantum groups. The readers may refer to [1] for the representation theory of algebras.

2. POINTED HOPF ALGEBRAS OF RANK ONE AND THEIR DOUBLES

In this section, we recall the pointed Hopf algebras of rank one and their Drinfeld doubles.

Let $0 \neq q \in \mathbb{k}$. For any integer $n > 0$, set $(n)_q = 1 + q + \cdots + q^{n-1}$. Observe that $(n)_q = n$ when $q = 1$ and $(n)_q = \frac{q^n - 1}{q - 1}$ when $q \neq 1$. Define the q -factorial of n by $(0)!_q = 1$ and $(n)!_q = (n)_q(n-1)_q \cdots (1)_q$ for $n > 0$, see [12, p.74].

A quadruple $\mathcal{D} = (G, \chi, a, \alpha)$ is called a *group datum* if G is a finite group, χ is a $\mathbb{k}$ -valued character of G , a is a central element of G and $\alpha \in \mathbb{k}$ subject to $\chi^n = 1$ or $\alpha(a^n - 1) = 0$, where n is the order of $\chi(a)$. The group datum $\mathcal{D}$ is of *nilpotent type* if $\alpha(a^n - 1) = 0$, and it is of *non-nilpotent type* if $\alpha(a^n - 1) \neq 0$ and $\chi^n = 1$. For any group datum $\mathcal{D} = (G, \chi, a, \alpha)$, Krop and Radford constructed an associated finite dimensional pointed rank one Hopf algebra $H_{\mathcal{D}}$ and classified such Hopf algebras. They also described the Drinfeld doubles $D(H_{\mathcal{D}})$ of $H_{\mathcal{D}}$, see [13].

Let $\mathcal{D} = (G, \chi, a, \alpha)$ be a group datum. The Hopf algebra $H_{\mathcal{D}}$ is generated as an algebra by G and x subject to the group relations for G ,

$$x^n = \alpha(a^n - 1) \text{ and } xg = \chi(g)gx \text{ for all } g \in G.$$

The comultiplication Δ is given by

$$\Delta(x) = x \otimes a + 1 \otimes x \text{ and } \Delta(g) = g \otimes g, \quad g \in G.$$

$H_{\mathcal{D}}$ has a $\mathbb{k}$ -basis $\{gx^j | g \in G, 0 \leq j \leq n-1\}$. If $\mathcal{D}$ is of *non-nilpotent type*, then $H_{\mathcal{D}} \cong H_{\mathcal{D}'}$ as Hopf algebras, where $\mathcal{D}' = (G, \chi, a, 1)$. Therefore, we always assume that $\alpha = 1$ whenever $\mathcal{D}$ is of *non-nilpotent type*. For the details, one can refer to [13].

In the sequel, we let $\mathcal{D} = (G, \chi, a, \alpha)$ be a group datum, and assume that G is abelian. Let $\rho = \chi(a)$ and denote by n the order of ρ . Let $\Gamma = \text{Hom}(G, \mathbb{k}^\times)$ denote the group of $\mathbb{k}$ -valued characters. Note that $D(H_{\mathcal{D}}) = H_{\mathcal{D}}^{*\text{cop}} \bowtie H_{\mathcal{D}}$ is generated as an algebra by its two sub-Hopf algebra $H_{\mathcal{D}}^{*\text{cop}}$ and $H_{\mathcal{D}}$, where $H_{\mathcal{D}}^*$ is the dual Hopf algebra of $H_{\mathcal{D}}$. $H_{\mathcal{D}}^{*\text{cop}}$ is generated as an algebra by ξ and Γ subject to the following relations:

$$\xi^n = 0 \text{ and } \xi\gamma = \gamma(a)\gamma\xi \text{ for all } \gamma \in \Gamma.$$

The coalgebra structure of $H_{\mathcal{D}}^{*\text{cop}}$ is determined by $\Delta(\xi) = \xi \otimes \varepsilon + \chi \otimes \xi$, $\Delta(\gamma) = \gamma \otimes \gamma$ if $\mathcal{D}$ is of nilpotent type, and

$$\Delta(\gamma) = \gamma \otimes \gamma + (\gamma^n(a) - 1) \sum_{l+r=n} \frac{1}{(l)_{\rho}!(r)_{\rho}!} \gamma \chi^l \xi^r \otimes \gamma \xi^l$$

if $\mathcal{D}$ is of non-nilpotent type, where $\gamma \in \Gamma$.

Proposition 2.1. [13, Proposition 5] *The Drinfeld double $D(H_{\mathcal{D}})$ is generated as an algebra by G , x , Γ and ξ subject to the relations defining $H_{\mathcal{D}}$ and $H_{\mathcal{D}}^{*\text{cop}}$ and the following relations:*

- (a) $g\gamma = \gamma g$ for all $g \in G$ and $\gamma \in \Gamma$;
- (b) $\xi g = \chi^{-1}(g)g\xi$ for all $g \in G$;
- (c) $[x, \xi] = a - \chi$;
- (d) $\gamma(a)x\gamma = \gamma x$ if $\mathcal{D}$ is **nilpotent**;
- (e) $\gamma(a)x\gamma = \gamma x + \frac{\gamma^n(a)-1}{(n-1)_{\rho}!} \gamma(\rho a - \chi) \xi^{n-1}$ if $\mathcal{D}$ is **non-nilpotent**.

3. SIMPLE MODULES AND PROJECTIVE MODULES

Throughout this and the next sections, for any $D(H_{\mathcal{D}})$ -module V , let $P(V)$ and $I(V)$ denote the projective cover and the injective envelope of V , respectively. Let $l(V)$ and $\text{rl}(V)$ denote the length and the radical length (Loewy length) of V , respectively. Moreover, let sV denote the direct sum of s copies of V for any integer $s \geq 0$, where $sV = 0$ when $s = 0$. Let $V^x := \{v \in V | xv = 0\}$ and $V^{\xi} := \{v \in V | \xi v = 0\}$. For any $v_1, \dots, v_s \in V$, denote by $\langle v_1, \dots, v_s \rangle$ the submodule of V generated by $\{v_1, \dots, v_s\}$.

3.1. Simple modules. Krop and Radford classified the simple modules over $D(H_{\mathcal{D}})$ for any group datum $\mathcal{D} = (G, \chi, a, \alpha)$ with G being abelian in [13, Subsection 2.2]. In this subsection, we recall the simple $D(H_{\mathcal{D}})$ -modules.

Let $\widehat{H_{\mathcal{D}}^{*\text{cop}}}$ be the subalgebra of $D(H_{\mathcal{D}})$ generated by $H_{\mathcal{D}}^{*\text{cop}}$ and G . Then $\widehat{H_{\mathcal{D}}^{*\text{cop}}} = H_{\mathcal{D}}^{*\text{cop}} \# \mathbb{k}G$, a smash product. One can easily check that $\widehat{H_{\mathcal{D}}^{*\text{cop}}} \cong \mathbb{k}(G \times \Gamma) \oplus J(\widehat{H_{\mathcal{D}}^{*\text{cop}}})$, where $J(\widehat{H_{\mathcal{D}}^{*\text{cop}}})$ is the Jacobson radical of $\widehat{H_{\mathcal{D}}^{*\text{cop}}}$. Thus, all simple $\widehat{H_{\mathcal{D}}^{*\text{cop}}}$ -modules are one-dimensional. They are in 1-1 correspondence with characters of $G \times \Gamma$. For each character $\lambda : G \times \Gamma \rightarrow \mathbb{k}$, denote by $\mathbb{k}_{\lambda}$ the associated one dimensional $\widehat{H_{\mathcal{D}}^{*\text{cop}}}$ -module. Let 1_{λ} be a fixed non-zero element of $\mathbb{k}_{\lambda}$. Then $\mathbb{k}_{\lambda}$ is described by the relations $(g\gamma)1_{\lambda} = \lambda(g\gamma)1_{\lambda}$ and $J(\widehat{H_{\mathcal{D}}^{*\text{cop}}})1_{\lambda} = 0$, where we write $g\gamma$ for $(g, \gamma) \in G \times \Gamma$.

For a character λ , define a left $D(H_{\mathcal{D}})$ -module $Z(\lambda) := D(H_{\mathcal{D}}) \otimes_{\widehat{H_{\mathcal{D}}^{*\text{cop}}}} \mathbb{k}_{\lambda}$. For simplicity, the element $1 \otimes_{\widehat{H_{\mathcal{D}}^{*\text{cop}}}} 1_{\lambda} \in Z(\lambda)$ is still denoted by 1_{λ} . It is easy to see that $\{x^i 1_{\lambda} | 0 \leq i \leq n-1\}$ is a $\mathbb{k}$ -basis of $Z(\lambda)$. For an element $g \in G$, denote by $\hat{g}$ the character $\gamma \rightarrow \gamma(g)$ of Γ . Let ϕ be the character $\chi^{-1}\hat{a}$ of $G \times \Gamma$.

Proposition 3.1. [13, Proposition 6] *Let $Z(\lambda)$ be a $D(H_{\mathcal{D}})$ -module. Then the following statements hold:*

- (1) $Z(\lambda)$ is simple if and only if $\lambda(a\chi^{-1}) \notin \{1, \rho, \dots, \rho^{n-2}\}$.
- (2) If $\lambda(a\chi^{-1}) = \rho^s$ for some $0 \leq s \leq n-2$, then $Z(\lambda)$ contains a unique non-trivial submodule, which is generated by $x^{s+1}1_\lambda$.

Let $\text{rad}(Z(\lambda))$ denote the radical of $Z(\lambda)$. Then $\text{rad}(Z(\lambda))$ is the unique proper submodule of $Z(\lambda)$ if $\lambda(a\chi^{-1}) = \rho^s$ for some $0 \leq s \leq n-2$, and $\text{rad}(Z(\lambda))=0$ otherwise. Set $L(\lambda) := Z(\lambda)/\text{rad}Z(\lambda)$, and denote the element $1_\lambda + \text{rad}Z(\lambda) \in L(\lambda)$ still by 1_λ . Let $\Lambda := \widehat{G} \times \widehat{\Gamma}$ be the set of characters of $G \times \Gamma$. Define $\text{ev}_{a\chi^{-1}} : \Lambda \rightarrow \mathbb{K}^\times$ by $\text{ev}_{a\chi^{-1}}(\lambda) = \lambda(a\chi^{-1})$, $\lambda \in \Lambda$, and let $K := \text{Ker}(\text{ev}_{a\chi^{-1}})$.

Proposition 3.2. [13, Theorem 2] *Retain the above notation. The following statements hold:*

- (1) $L(\mu) \cong L(\lambda)$ if and only if $\mu = \lambda$. There are $|G|^2$ pairwise non-isomorphic simple $D(H_{\mathcal{D}})$ -modules.
- (2) For every d with $1 \leq d \leq n-1$, there are $|K|$ non-isomorphic simple $D(H_{\mathcal{D}})$ -modules of dimension d .
- (3) The number of n -dimensional simple $D(H_{\mathcal{D}})$ -modules is $|G|^2 - (n-1)|K|$ and is also greater than or equal to $|K|$.

Let $\widetilde{\Lambda}$ be the set of weights that map $a\chi^{-1}$ to a ρ^s for some $0 \leq s \leq n-2$. For a $\lambda \in \Lambda$, denote by $d(\lambda)$ the above integer s if $\lambda \in \widetilde{\Lambda}$, and set $d(\lambda) = -1$ otherwise. Define a mapping $\sigma : \Lambda \rightarrow \Lambda$ by $\sigma(\lambda) = \lambda\phi^{d(\lambda)+1}$. Let $\tau = \sigma^2$.

Lemma 3.3. [13, Lemma 5] *Retain the above notation and let $\lambda \in \widetilde{\Lambda}$. The following statements hold:*

- (1) $d(\sigma(\lambda)) = n - d(\lambda) - 2$.
- (2) $\sigma(\widetilde{\Lambda}) = \widetilde{\Lambda}$.
- (3) Let $m = \frac{1}{n}\text{ord}(a)$ if $\mathcal{D}$ is of non-nilpotent type, and $m = \frac{1}{n}\text{lcm}(\text{ord}(a), \text{ord}(\chi))$, otherwise. Then the mapping σ has order $2m$, and τ has order m .
- (4) $L(\tau^k\sigma(\lambda)) \not\cong L(\tau^{k'}(\lambda))$ for any $k, k' \in \mathbb{N}$.

Throughout the following, let m be the positive integer defined in Lemma 3.3(3).

Remark 3.4. The character $\phi = \chi^{-1}\hat{a}$ has order mn . If $\mathcal{D}$ is of non-nilpotent type, then $m > 1$. In fact, if $\mathcal{D}$ is of non-nilpotent type and $m = 1$, then $\text{ord}(a) = n$ by Lemma 3.3(3), and hence $x^n = \alpha(a^n - 1) = 0$, a contradiction.

Now we describe the simple modules $L(\lambda)$ more explicitly in terms of their dimensions. For any $1 \leq l \leq n-1$, let

$$I_l = \{\lambda \in \widetilde{\Lambda} \mid \lambda(a\chi^{-1}) = \rho^{l-1}\}, \quad I_n := \Lambda \setminus \widetilde{\Lambda} = \{\lambda \in \Lambda \mid \lambda \notin \widetilde{\Lambda}\}, \text{ and}$$

$$I'_n = \{\lambda \in I_n \mid \lambda(a\chi^{-1}) \neq \rho^k, 0 \leq k \leq n-1\}, \quad I''_n = \{\lambda \in I_n \mid \lambda(a\chi^{-1}) = \rho^{n-1}\}.$$

Then $I_n = I'_n \cup I''_n$, $\cup_{l=1}^{n-1} I_l = \widetilde{\Lambda}$ and $\cup_{l=1}^n I_l = \Lambda$. For any $1 \leq l \leq n$ and $\lambda \in I_l$, denote by $V(l, \lambda)$ the l -dimensional simple $D(H_{\mathcal{D}})$ -module $L(\lambda)$.

Corollary 3.5. Let $1 \leq l \leq n-1$ and $\lambda \in I_l$. Then $\sigma(\lambda), \sigma^{-1}(\lambda) \in I_{n-l}$. Moreover, $\sigma^{-1}(\lambda) = \lambda\phi^{d(\lambda)+1-n} = \lambda\phi^{l-n}$.

Proof. It follows from Lemma 3.3 and a straightforward verification. $\square$

In the sequel, we describe $\mathbb{k}$ -bases of simple modules $V(l, \lambda)$. For the convenience in classifying the non-simple indecomposable $D(H_{\mathcal{D}})$, we will need two types of bases for each simple $D(H_{\mathcal{D}})$ -module $V(l, \lambda)$. For any $1 \leq l \leq n$ and $\lambda \in I_l$, define

$$\alpha_i(l, \lambda) := (i)_\rho(\lambda(\chi) - \lambda(a)\rho^{1-i}) \in \mathbb{k}, \quad i \geq 1.$$

Since $\sigma^2(\lambda) = \lambda\phi^n$, we have $\alpha_i(l, \tau(\lambda)) = \alpha_i(l, \lambda)$. We define $\beta(l, \lambda) \in \mathbb{k}$ by $\beta(1, \lambda) := 1$ if $l = 1$ and $\beta(l, \lambda) := \alpha_1(l, \lambda)\alpha_2(l, \lambda) \cdots \alpha_{l-1}(l, \lambda)$ if $1 < l \leq n$. Note that $\alpha_i(l, \lambda) \neq 0$ for all $1 \leq i \leq l-1$, and hence $\beta(l, \lambda) \neq 0$.

Let $1 \leq l \leq n$ and $\lambda \in I_l$. Define $v_i, m_i \in V(l, \lambda)$, $0 \leq i \leq l-1$, as follows:

$$v_i := x^i 1_\lambda; \quad m_i := \begin{cases} \alpha_{i+1}(l, \lambda)\alpha_{i+2}(l, \lambda) \cdots \alpha_{l-1}(l, \lambda)x^i 1_\lambda, & \text{if } 0 \leq i \leq l-2, \\ x^{l-1} 1_\lambda, & \text{if } i = l-1. \end{cases}$$

Then both $\{v_i | 0 \leq i \leq l-1\}$ and $\{m_i | 0 \leq i \leq l-1\}$ form $\mathbb{k}$ -bases of $V(l, \lambda)$. The former is called the *natural basis* of $V(l, \lambda)$ and the latter is called the *standard basis* of $V(l, \lambda)$.

We now present the $D(H_{\mathcal{D}})$ -module actions on both bases. For $1 \leq l \leq n-1$, the $D(H_{\mathcal{D}})$ -module action on the natural basis of $V(l, \lambda)$ is determined by

$$(g\gamma)v_i = (\phi^i \lambda)(g\gamma)v_i, \quad g \in G, \gamma \in \Gamma, \quad 0 \leq i \leq l-1,$$

$$xv_i = \begin{cases} v_{i+1}, & 0 \leq i \leq l-2, \\ 0, & i = l-1, \end{cases} \quad \xi v_i = \begin{cases} 0, & i = 0, \\ \alpha_i(l, \lambda)v_{i-1}, & 1 \leq i \leq l-1, \end{cases}$$

while the $D(H_{\mathcal{D}})$ -module action on the standard basis of $V(l, \lambda)$ is given by

$$(g\gamma)m_i = (\phi^i \lambda)(g\gamma)m_i, \quad g \in G, \gamma \in \Gamma, \quad 0 \leq i \leq l-1,$$

$$xm_i = \begin{cases} \alpha_{i+1}(l, \lambda)m_{i+1}, & 0 \leq i \leq l-2, \\ 0, & i = l-1, \end{cases} \quad \xi m_i = \begin{cases} 0, & i = 0, \\ m_{i-1}, & 1 \leq i \leq l-1. \end{cases}$$

It is clear that $V(l, \lambda)^x = \mathbb{k}v_{l-1} = \mathbb{k}m_{l-1}$ and $V(l, \lambda)^\xi = \mathbb{k}v_0 = \mathbb{k}m_0$.

Next we consider the case that $l = n$ with $\lambda \in I_n$. In this situation, $V(n, \lambda)$ is projective. To describe the $D(H_{\mathcal{D}})$ -module structure on $V(n, \lambda)$, it is necessary to distinguish between two cases for $\mathcal{D}$: the nilpotent case and the non-nilpotent case.

If $\mathcal{D}$ is of **nilpotent type**, the $D(H_{\mathcal{D}})$ -module action on the natural basis of $V(n, \lambda)$ is given by

$$(g\gamma)v_i = (\phi^i \lambda)(g\gamma)v_i, \quad g \in G, \gamma \in \Gamma, \quad 0 \leq i \leq n-1,$$

$$xv_i = \begin{cases} v_{i+1}, & 0 \leq i \leq n-2, \\ 0, & i = n-1, \end{cases} \quad \xi v_i = \begin{cases} 0, & i = 0, \\ \alpha_i(l, \lambda)v_{i-1}, & 1 \leq i \leq n-1, \end{cases}$$

while the $D(H_{\mathcal{D}})$ -module action on the standard basis of $V(n, \lambda)$ is given by

$$(g\gamma)m_i = (\phi^i \lambda)(g\gamma)m_i, \quad g \in G, \gamma \in \Gamma, \quad 0 \leq i \leq n-1,$$

$$xm_i = \begin{cases} \alpha_{i+1}(n, \lambda)m_{i+1}, & 0 \leq i \leq n-2, \\ 0, & i = n-1, \end{cases} \quad \xi m_i = \begin{cases} 0, & i = 0, \\ m_{i-1}, & 1 \leq i \leq n-1. \end{cases}$$

It is clear that $V(n, \lambda)^x = \mathbb{k}v_{n-1} = \mathbb{k}m_{n-1}$ and $V(n, \lambda)^\xi = \mathbb{k}v_0 = \mathbb{k}m_0$.

If $\mathcal{D}$ is of **non-nilpotent type**, then the $D(H_{\mathcal{D}})$ -module actions on the two bases of $V(n, \lambda)$ are given respectively by

$$(g\gamma)v_i = (\phi^i\lambda)(g\gamma)v_i, \quad g \in G, \gamma \in \Gamma, 0 \leq i \leq n-1,$$

$$xv_i = \begin{cases} v_{i+1}, & 0 \leq i \leq n-2, \\ (\lambda^n(a) - 1)v_0, & i = n-1, \end{cases} \quad \xi v_i = \begin{cases} 0, & i = 0, \\ \alpha_i(l, \lambda)v_{i-1}, & 1 \leq i \leq n-1. \end{cases}$$

and

$$(g\gamma)m_i = (\phi^i\lambda)(g\gamma)m_i, \quad g \in G, \gamma \in \Gamma, 0 \leq i \leq n-1,$$

$$xm_i = \begin{cases} \alpha_{i+1}(n, \lambda)m_{i+1}, & 0 \leq k \leq n-2, \\ \frac{\lambda^n(a)-1}{\beta(n, \lambda)}m_0, & i = n-1, \end{cases} \quad \xi m_i = \begin{cases} 0, & i = 0, \\ m_{i-1}, & 1 \leq i \leq n-1. \end{cases}$$

Clearly, $V(n, \lambda)^\xi = \mathbb{k}v_0 = \mathbb{k}m_0$.

By the above discussion, we have following explicit description of the simple $D(H_{\mathcal{D}})$ -modules.

Proposition 3.6. *The following set*

$$\{V(l, \lambda) | 1 \leq l \leq n, \lambda \in I_l\}$$

gives a complete set of representatives of isomorphism classes of simple $D(H_{\mathcal{D}})$ -modules. Moreover, $V(n, \lambda)$ is a projective $D(H_{\mathcal{D}})$ -module for any $\lambda \in I_n$.

Corollary 3.7. *Let M be a finite dimensional semisimple $D(H_{\mathcal{D}})$ -module. Then $l(M) = \dim(M^\xi)$. Furthermore, if $\mathcal{D}$ is of nilpotent type, then $l(M) = \dim(M^x)$.*

3.2. Indecomposable projective modules. Krop and Radford described all projective indecomposable $D(H_{\mathcal{D}})$ -modules for any group datum $\mathcal{D} = (G, \chi, a, \mu)$ with G abelian in [13, Subsection 2.3], including their radical series and composition factors.

Since $D(H_{\mathcal{D}})$ is a symmetric algebra, we have $P(V) \cong I(V)$ and

$$P(V)/\text{rad}(P(V)) \cong \text{soc}(P(V)) \cong V$$

for any simple $D(H_{\mathcal{D}})$ -module V . Let J denote the Jacobson radical of $D(H_{\mathcal{D}})$, and let $P(\lambda)$ be the projective cover of $L(\lambda)$. Denote by $J^i(\lambda)$, $i = 0, 1, \dots$, the terms of radical series of $P(\lambda)$. Then, by [13, Subsection 2.3], we have the following proposition and corollary.

Proposition 3.8. [13, Theorem 3] *Let $\lambda \in \tilde{\Lambda}$. The radical series of $P(\lambda)$ is given by*

$$P(\lambda) \supset J(\lambda) \supset L(\lambda) \supset 0$$

with

$$J(\lambda)/L(\lambda) \cong L(\sigma(\lambda)) \oplus L(\sigma^{-1}(\lambda)).$$

Corollary 3.9. *Let $\lambda \in \Lambda$. Then:*

- (1) *If $\lambda \in I_n$, the simple module $Z(\lambda)$ is projective.*
- (2) *If $\lambda \in \tilde{\Lambda}$, the radical length of $P(\lambda)$ is $\text{rl}(P(\lambda)) = 3$ and its length is $l(P(\lambda)) = 4$.*

(3) The radical length of $D(H_{\mathcal{D}})$ is $\text{rl}(D(H_{\mathcal{D}})) = 3$; in particular, $J^3 = 0$.

For a more explicit description of the structure of indecomposable projective modules, we consider separately the two cases in which $\mathcal{D}$ is nilpotent or non-nilpotent.

Case 1: $\mathcal{D}$ is of nilpotent type.

Let $1 \leq l \leq n-1$, $\lambda \in I_l$, and let $P(l, \lambda)$ be a $2n$ -dimensional vector space with a $\mathbb{k}$ -basis $\{v_0, v_1, \dots, v_{n-1}, u_0, u_1, \dots, u_{n-1}\}$. A straightforward verification shows that $P(l, \lambda)$ becomes a $D(H_{\mathcal{D}})$ -module under the action defined by

$$\begin{aligned} (g\gamma)v_i &= (\phi^i\lambda)(g\gamma)v_i, & g \in G, \gamma \in \Gamma, 0 \leq i \leq n-1, \\ (g\gamma)u_i &= (\phi^{i-n+l}\lambda)(g\gamma)u_i, & g \in G, \gamma \in \Gamma, 0 \leq i \leq n-1, \\ xv_i &= \begin{cases} v_{i+1}, & 0 \leq i \leq n-2, \\ 0, & i = n-1, \end{cases} & xu_i = \begin{cases} u_{i+1}, & 0 \leq i \leq n-2, \\ 0, & i = n-1, \end{cases} \end{aligned}$$

$$\begin{aligned} \xi v_i &= \begin{cases} u_{n-l-1}, & i = 0, \\ \alpha_i(l, \lambda)v_{i-1} + u_{n-l+i-1}, & 1 \leq i \leq l-1, \\ u_{n-1}, & i = l, \\ \alpha_{i-l}(n-l, \sigma(\lambda))v_{i-1}, & l+1 \leq i \leq n-1, \end{cases} \\ \xi u_i &= \begin{cases} 0, & i = 0, \\ \alpha_i(n-l, \sigma^{-1}(\lambda))u_{i-1}, & 1 \leq i \leq n-l-1, \\ 0, & i = n-l, \\ \alpha_{i-n+l}(l, \lambda)u_{i-1}, & n-l+1 \leq i \leq n-1. \end{cases} \end{aligned}$$

The above basis is called the *standard basis* of $P(l, \lambda)$. Clearly, $P(l, \lambda)^x = \mathbb{k}v_{n-1} + \mathbb{k}u_{n-1}$ and $P(l, \lambda)^\xi = \mathbb{k}u_0 + \mathbb{k}u_{n-l}$.

Case 2: $\mathcal{D}$ is of non-nilpotent type. Let $1 \leq l \leq n-1$, and $\lambda \in I_l$, and consider a $2n$ -dimensional vector space $P(l, \lambda)$ with a $\mathbb{k}$ -basis $\{v_0, v_1, \dots, v_{n-1}, u_0, u_1, \dots, u_{n-1}\}$. It can be easily checked that $P(l, \lambda)$ carries a structure of a $D(H_{\mathcal{D}})$ -module with the action defined by

$$\begin{aligned} (g\gamma)v_i &= (\phi^{i-n+l}\lambda)(g\gamma)v_i, & g \in G, \gamma \in \Gamma, 0 \leq i \leq n-1, \\ (g\gamma)u_i &= (\phi^i\lambda)(g\gamma)u_i, & g \in G, \gamma \in \Gamma, 0 \leq i \leq n-1, \\ xv_i &= \begin{cases} \alpha_{i+1}(n-l, \sigma^{-1}(\lambda))v_{i+1}, & 0 \leq i \leq n-l-2, \\ u_0, & i = n-l-1, \\ \alpha_{i+1-n+l}(l, \lambda)v_{i+1} + u_{i+1-n+l}, & n-l \leq i \leq n-2, \\ y_{l,\lambda}v_0 + u_l, & i = n-1, \end{cases} \\ xu_i &= \begin{cases} \alpha_{i+1}(l, \lambda)u_{i+1}, & 0 \leq i \leq l-2, \\ 0, & i = l-1, \\ \alpha_{i+1-l}(n-l, \sigma(\lambda))u_{i+1}, & l \leq i \leq n-2, \\ z_{l,\lambda}u_0, & i = n-1, \end{cases} \end{aligned}$$

$$\xi v_i = \begin{cases} 0, & i = 0, \\ v_{i-1}, & 1 \leq i \leq n-1, \end{cases} \quad \xi u_i = \begin{cases} 0, & i = 0, \\ u_{i-1}, & 1 \leq i \leq n-1, \end{cases}$$

where $y_{l,\lambda} = \frac{\rho^{1-l}\lambda(a)-\rho^l\lambda(\chi)}{(n-1)!_\rho}$ and $z_{l,\lambda} = \frac{\rho\lambda(a)-\lambda(\chi)}{(n-1)!_\rho}$. Moreover, $y_{l,\lambda} + z_{l,\lambda} = 0$ and $z_{n-l,\sigma^{-1}(\lambda)} = y_{l,\lambda}$ by $\lambda(\chi) = \lambda(a)\rho^{1-l}$.

Such a basis is referred to as the *standard basis* of $P(l, \lambda)$. Clearly, $P(l, \lambda)^x = \mathbb{k}u_{l-1} + \mathbb{k}(u_{n-1} - z_{l,\lambda}v_{n-l-1})$ and $P(l, \lambda)^\xi = \mathbb{k}v_0 + \mathbb{k}u_0$.

Proposition 3.10. *Let $1 \leq l \leq n-1$ and $\lambda \in I_l$. Then $P(l, \lambda) \cong P(\lambda)$.*

Proof. Let $\{v_0, v_1, \dots, v_{n-1}, u_0, u_1, \dots, u_{n-1}\}$ be the standard basis of $P(l, \lambda)$ as stated above.

Case 1: $\mathcal{D}$ is of nilpotent type. Let $M := \langle u_{n-1} \rangle$. Then it is easy to verify that $M = \text{span}\{u_{n-l}, u_{n-l+1}, \dots, u_{n-1}\}$. Using the natural basis of $V(l, \lambda)$, one sees that $M \cong V(l, \lambda)$. Let N be a simple submodule of $P(l, \lambda)$. By Corollary 3.7, we have $\dim(N^x) = 1$. Choose $0 \neq z \in N^x$. Since $N^x \subseteq P(l, \lambda)^x$, we can write

$$z = \beta_1 v_{n-1} + \beta_2 u_{n-1}, \text{ for some } \beta_1, \beta_2 \in \mathbb{k}.$$

If $\beta_1 \neq 0$, then

$$az - \rho^{1-l}\lambda(a)z = (\rho - \rho^{1-l})\lambda(a)\beta_1 v_{n-1}.$$

Hence $v_{n-1} \in N$, and therefore $\xi^{n-l}v_{n-1} = \beta(n-l, \sigma(\lambda))u_{n-1} \in N$. Thus, $v_{n-1}, u_{n-1} \in N$, implying $\dim(N^x) = 2$, a contradiction. Therefore, $\beta_1 = 0$ and $\beta_2 \neq 0$, so $u_{n-1} = \beta_2^{-1}z \in N$. This shows $M = \langle u_{n-1} \rangle \subseteq N$, and since M and N are both simple, we have $N = M$. Thus, $\text{soc}P(l, \lambda) = M \cong V(l, \lambda)$. Since $D(H_{\mathcal{D}})$ is a symmetric algebra, it follows that $P(l, \lambda)$ is isomorphic to a submodule of $P(\lambda)$.

Furthermore, $\dim((P(l, \lambda)/\text{soc}P(l, \lambda))^x) = 2$, and thus, by Corollary 3.7, $l(\text{soc}(P(l, \lambda)/\text{soc}P(l, \lambda))) \leq 2$. Let $\bar{v}$ denote the image of $v \in P(l, \lambda)$ under the canonical epimorphism $P(l, \lambda) \rightarrow P(l, \lambda)/\text{soc}P(l, \lambda)$. Then one can check that

$$M_1 := \text{span}\{\bar{v}_l, \bar{v}_{l+1}, \dots, \bar{v}_{n-1}\}, \quad M_2 := \text{span}\{\bar{u}_0, \bar{u}_1, \dots, \bar{u}_{n-l-1}\}$$

are submodules of $P(l, \lambda)/\text{soc}P(l, \lambda)$. By Corollary 3.5, we have

$$M_1 \cong V(n-l, \sigma(\lambda)), \text{ and } M_2 \cong V(n-l, \sigma^{-1}(\lambda)).$$

It follows that $\text{soc}(P(l, \lambda)/\text{soc}P(l, \lambda)) = M_1 \oplus M_2$, and consequently,

$$\text{soc}^2 P(l, \lambda) = \text{span}\{v_l, \dots, v_{n-1}, u_0, \dots, u_{n-1}\}.$$

Now it is straightforward to see that

$$P(l, \lambda)/\text{soc}^2 P(l, \lambda) \cong V(l, \lambda).$$

Therefore, the length of $P(l, \lambda)$ is $l(P(l, \lambda)) = 4$, and by Corollary 3.9(2), $P(l, \lambda) \cong P(\lambda)$.

Case 2: $\mathcal{D}$ is of non-nilpotent type. Let $M := \langle u_0 \rangle$. It is straightforward to verify that $M = \text{span}\{u_0, u_1, \dots, u_{l-1}\}$. Using the standard basis of $V(l, \lambda)$, one sees that $M \cong V(l, \lambda)$.

Let N be a simple submodule of $P(l, \lambda)$. By Corollary 3.7, we have $\dim(N^\xi) = 1$.

Choose $0 \neq z \in N^\xi$. Since $N^\xi \subseteq P(l, \lambda)^\xi$, we can write $z = \beta_1 v_0 + \beta_2 u_0$, for some $\beta_1, \beta_2 \in \mathbb{k}$. If $\beta_1 \neq 0$, then

$$az - \lambda(a)z = (\rho^{n-l} - 1)\lambda(a)\beta_1 v_0.$$

Hence $v_0 \in N$, and therefore $x^{n-l}v_0 = \beta(n-l, \sigma^{-1}(\lambda))u_0 \in N$. Thus, $v_0, u_0 \in N$, which implies $\dim(N^\xi) = 2$, a contradiction. Therefore, $\beta_1 = 0$ and $\beta_2 \neq 0$, and so $u_0 = \beta_2^{-1}z \in N$. This shows $M = \langle u_0 \rangle \subseteq N$, and since both M and N are simple, we conclude $N = M$. Hence, $\text{soc}P(l, \lambda) = M \cong V(l, \lambda)$. Because $D(H_{\mathcal{D}})$ is a symmetric algebra, it follows that $P(l, \lambda)$ is isomorphic to a submodule of $P(\lambda)$. Clearly,

$$\dim((P(l, \lambda)/\text{soc}P(l, \lambda))^\xi) = 2.$$

Therefore, by Corollary 3.7, we have

$$l(\text{soc}(P(l, \lambda)/\text{soc}P(l, \lambda))) \leq 2.$$

Let $\bar{v}$ denote the image of $v \in P(l, \lambda)$ under the canonical epimorphism $P(l, \lambda) \rightarrow P(l, \lambda)/\text{soc}P(l, \lambda)$. Then one can verify that

$$M_1 := \text{span}\{\bar{v}_0, \bar{v}_1, \dots, \bar{v}_{n-l-1}\}, \quad M_2 := \text{span}\{\bar{u}_l, \bar{u}_{l+1}, \dots, \bar{u}_{n-1}\}$$

are submodules of $P(l, \lambda)/\text{soc}P(l, \lambda)$, satisfying

$$M_1 \cong V(n-l, \sigma^{-1}(\lambda)) \text{ and } M_2 \cong V(n-l, \sigma(\lambda)).$$

It follows that

$$\text{soc}(P(l, \lambda)/\text{soc}P(l, \lambda)) = M_1 \oplus M_2,$$

and consequently,

$$\text{soc}^2 P(l, \lambda) = \text{span}\{v_0, v_1, \dots, v_{n-l-1}, u_0, u_1, \dots, u_{n-1}\}.$$

Now one can easily check that $P(l, \lambda)/\text{soc}^2 P(l, \lambda) \cong V(l, \lambda)$. Therefore, the length of $P(l, \lambda)$ is $l(P(l, \lambda)) = 4$, and by Corollary 3.9(2), $P(l, \lambda) \cong P(\lambda)$. $\square$

Lemma 3.11. *Let $1 \leq l \leq n-1$ and $\lambda \in I_l$. Then:*

- (1) $\text{soc}P(l, \lambda) = \text{rad}^2 P(l, \lambda) \cong V(l, \lambda)$ and $\text{soc}^2 P(l, \lambda) = \text{rad}P(l, \lambda)$.
- (2) $\text{soc}^2 P(l, \lambda)/\text{soc}P(l, \lambda) = V(n-l, \sigma(\lambda)) \oplus V(n-l, \sigma^{-1}(\lambda))$.

Proof. It follows from Proposition 3.10 and its proof. $\square$

Corollary 3.12. *A complete set of representatives for the isomorphism classes of indecomposable projective $D(H_{\mathcal{D}})$ -modules is given by*

$$\{P(l, \lambda), V(n, \mu) | 1 \leq l \leq n-1, \lambda \in I_l, \mu \in I_n\}.$$

Proof. It follows from Propositions 3.6 and 3.10. $\square$

Corollary 3.13. *If P is a non-simple indecomposable projective $D(H_{\mathcal{D}})$ -module, then $\text{rad}P = \text{soc}^2 P$ and $\text{rad}^2 P = \text{soc}P$.*

Proof. It follows from Proposition 3.6, Lemma 3.11 and Corollary 3.12. $\square$

Lemma 3.14. *Let M be an indecomposable $D(H_{\mathcal{D}})$ -module. If $\text{rl}(M)=1$, then $M \cong V(l, \lambda)$ for some $1 \leq l \leq n$ and $\lambda \in I_l$. If $\text{rl}(M) = 3$, then $M \cong P(l, \lambda)$ for some $1 \leq l \leq n-1$ and $\lambda \in I_l$.*

Proof. It follows from Proposition 3.6, Corollary 3.12 and [3, Lemma 3.5]. $\square$

Proposition 3.15. [13, Theorem 4] *Each block of $D(H_{\mathcal{D}})$ is either isomorphic to $M_n(\mathbb{k})$, or isomorphic as a left $D(H_{\mathcal{D}})$ -module to*

$$(\oplus_{k=0}^{m-1} lP(l, \tau^k(\lambda))) \oplus (\oplus_{k=0}^{m-1} (n-l)P(n-l, \tau^k(\sigma(\lambda)))),$$

where $1 \leq l \leq n-1$ and $\lambda \in I_l$.

4. INDECOMPOSABLE MODULES WITH LOEWY LENGTH TWO

In this section, we investigate the non-simple, non-projective indecomposable $D(H_{\mathcal{D}})$ -modules. By Lemma 3.14, such indecomposable modules have Loewy length 2.

Proposition 4.1. *Let M be an indecomposable $D(H_{\mathcal{D}})$ -module with $\text{rl}(M) = 2$. Then there exist a $1 \leq l \leq n-1$ and a $\lambda \in I_l$ such that $I(M) \cong \oplus_{k=0}^{m-1} s_k P(l, \tau^k(\lambda))$ for some $s_k \in \mathbb{N}$, $0 \leq k \leq m-1$.*

Proof. Clearly, M is an indecomposable module over a non-simple block of $D(H_{\mathcal{D}})$, and its injective hull $I(M)$ is projective. By Proposition 3.15, we can write $I(M) = P_1 \oplus P_2$, where

$$P_1 \cong \oplus_{k=0}^{m-1} s_k P(l, \tau^k(\lambda)) \text{ and } P_2 \cong \oplus_{k=0}^{m-1} t_k P(n-l, \tau^k(\sigma(\lambda)))$$

for some character $\lambda \in I_l$, $1 \leq l \leq n-1$ and integers $s_k, t_k \in \mathbb{N}$, $0 \leq k \leq m-1$. We may view M as a submodule of $I(M)$ and set $M_1 = M \cap P_1$ and $M_2 = M \cap P_2$. Then $\text{soc} M = \text{soc} P_1 \oplus \text{soc} P_2$ and so $\text{soc} M \subset M_1 \oplus M_2$. Since $\text{rl}(M) = 2$, the quotient $M/\text{soc} M$ is semisimple. Hence

$$M/\text{soc} M \subseteq \text{soc}^2 I(M)/\text{soc} I(M),$$

and consequently $M \subseteq \text{rad} I(M) = \text{soc}^2 I(M)$ by Corollary 3.13.

Let $\pi : I(M) \rightarrow I(M)/\text{soc} I(M)$ be the canonical epimorphism. Then

$$M/\text{soc} M \subseteq \text{rad} I(M)/\text{rad}^2 I(M) = \pi(\text{rad} P_1) \oplus \pi(\text{rad} P_2).$$

By Lemma 3.11, we have

$$\begin{aligned} \pi(\text{rad} P_1) &\cong \text{rad} P_1 / \text{rad}^2 P_1 \cong \oplus_{k=0}^{m-1} (s_k + s_{k+1}) V(n-l, \tau^k(\sigma(\lambda))), \\ \pi(\text{rad} P_2) &\cong \text{rad} P_2 / \text{rad}^2 P_2 \cong \oplus_{k=0}^{m-1} (t_{k-1} + t_k) V(l, \tau^k(\lambda)), \end{aligned}$$

where $s_m = s_0$ and $t_{-1} = t_{m-1}$. By Lemma 3.3(4), if V is a simple submodule of $M/\text{soc} M$, then either $V \subseteq \pi(\text{rad} P_1)$ or $V \subseteq \pi(\text{rad} P_2)$. In the former case,

$$\pi^{-1}(V) \subseteq M \cap (\text{rad} P_1 + \text{rad}^2 I(M)) = M \cap (\text{rad} P_1 \oplus \text{rad}^2 P_2) \subseteq M_1 \oplus M_2.$$

Similarly, if $V \subseteq \pi(\text{rad} P_2)$, then $\pi^{-1}(V) \subseteq M_1 \oplus M_2$. Therefore, $M \subseteq M_1 \oplus M_2$, and hence $M = M_1 \oplus M_2$.

Since M is indecomposable, it follows that $M_1 = 0$ or $M_2 = 0$, and consequently $P_1 = 0$ or $P_2 = 0$. Thus, we conclude either

$$I(M) \cong \oplus_{k=0}^{m-1} s_k P(l, \tau^k(\lambda)), \text{ or } I(M) \cong \oplus_{k=0}^{m-1} t_k P(n-l, \tau^k(\sigma(\lambda))).$$

$\square$

Let M be an indecomposable $D(H_{\mathcal{D}})$ -module with $\text{rl}(M) = 2$. By [3, Lemma 3.7], we have $\text{soc}M = \text{rad}M$. If $l(\text{soc}M) = t$ and $l(\text{head}M) = s$, then we say that M is of (s, t) -type (cf. [3]).

Corollary 4.2. *Let M be of (s, t) -type such that*

$$\text{soc}M \cong \oplus_{k=0}^{m-1} t_k V(l, \tau^k(\lambda))$$

for some $1 \leq l \leq n-1$ and $\lambda \in I_l$. Then:

- (1) $\text{head}M \cong \oplus_{k=0}^{m-1} s_k V(n-l, \tau^k(\sigma(\lambda)))$ for some integers $s_k \geq 0$.
- (2) $\text{Hom}_{D(H_{\mathcal{D}})}(M, \text{soc}M) = \text{Hom}_{D(H_{\mathcal{D}})}(\text{head}M, \text{soc}M) = 0$.

Proof. (1) follows from the proof of Proposition 4.1. Since $\text{rad}M = \text{soc}M$ and $\text{soc}M$ is semisimple, it follows from (1) and together with Lemma 3.3(4) that

$$\begin{aligned} \text{Hom}_{D(H_{\mathcal{D}})}(M, \text{soc}M) &= \text{Hom}_{D(H_{\mathcal{D}})}(\text{head}M, \text{soc}M) \\ &\cong \text{Hom}_{D(H_{\mathcal{D}})}(\oplus_{k=0}^{m-1} s_k V(n-l, \tau^k(\sigma(\lambda))), \oplus_{k=0}^{m-1} t_k V(l, \tau^k(\lambda))) = 0. \end{aligned}$$

This proves (2). $\square$

Let M be an $D(H_{\mathcal{D}})$ -module. Fix a projective cover $f : P(M) \rightarrow M$ and define the syzygy ΩM of M to be $\text{Ker}f$. Dually, fix an injective envelope $f : M \rightarrow I(M)$ and define the cosyzygy $\Omega^{-1}M$ of M to be $\text{Coker}f$. By [1, p.126], one obtains a family of indecomposable $D(H_{\mathcal{D}})$ -modules

$$\Omega^{\pm s}V(l, \lambda), \text{ where } s \geq 1, 1 \leq l \leq n-1, \lambda \in I_l.$$

Lemma 4.3. *Let M be of (s, t) -type.*

- (1) $t \leq 2s$, moreover, if $s \neq 1$ then $t < 2s$ and ΩM is of $(2s-t, s)$ -type.
- (2) $s \leq 2t$, moreover, if $t \neq 1$ then $s < 2t$ and $\Omega^{-1}M$ is of $(t, 2t-s)$ -type.
- (3) If $s = t$, then both ΩM and $\Omega^{-1}M$ are of (t, t) -type.

Proof. (1) By the proof of Proposition 4.1, we have

$$\text{head}M \cong \oplus_{k=0}^{m-1} s_k V(l, \tau^k(\lambda)) \text{ and } P(M) \cong \oplus_{k=0}^{m-1} s_k P(l, \tau^k(\lambda))$$

for some $s_k \in \mathbb{N}$, $0 \leq k \leq m-1$ with $\sum_{k=0}^{m-1} s_k = s$. Let $f : P(M) \rightarrow M$ be an epimorphism. Then the restriction $f|_{\text{rad}P(M)} : \text{rad}P(M) \rightarrow \text{rad}M$ remains an epimorphism. Since $\text{rad}M = \text{soc}M$ is semisimple, we have $\text{rad}^2P(M) \subseteq \text{Ker}(f|_{\text{rad}P(M)})$. Hence,

$$l(\text{soc}M) \leq l(\text{rad}P(M)/\text{rad}^2P(M)) = 2s,$$

and therefore $t \leq 2s$.

Now assume $s \neq 1$. Then $\text{soc}P(M) = \text{rad}^2P(M) \subseteq \text{Ker}f$. Thus, f induces an epimorphism

$$\bar{f} : P(M)/\text{rad}^2P(M) \rightarrow M.$$

If $t = 2s$, then $l(M) = 3s = l(P(M)/\text{soc}P(M))$, so $\bar{f}$ would be an isomorphism. However, $P(M)/\text{rad}^2P(M)$ is not indecomposable when $s > 1$, a contradiction. Therefore, $t < 2s$. Meanwhile, since $\Omega M \cong \text{Ker}f \subseteq \text{rad}P(M)$ and $\text{rad}^2P(M) \neq \text{Ker}f$, it follows that $\text{rl}(\Omega M) = 2$ and $l(\text{soc}(\Omega M)) = l(\text{soc}P(M)) = s$. Moreover,

$l(\Omega M) = l(P(M)) - l(M) = 3s - t$, so $l(\Omega M/\text{soc}(\Omega M)) = 2s - t$. That is, ΩM is of $(2s - t, s)$ -type.

(2) The statement is dual to (1).

(3) The result is clear. $\square$

Proposition 4.4. *Let M be an indecomposable $D(H_{\mathcal{D}})$ -module of (s, t) -type.*

- (1) *If $s < t$, then $t = s + 1$ and $M \cong \Omega^{-s}V(l, \lambda)$ for some $1 \leq l \leq n - 1$ and $\lambda \in I_l$.*
- (2) *If $s > t$, then $s = t + 1$ and $M \cong \Omega^tV(l, \lambda)$ for some $1 \leq l \leq n - 1$ and $\lambda \in I_l$.*

Proof. (1) We first consider the case $1 = s < t$. Then $t = 2$ by Lemma 4.3(1), and hence $\text{head}M$ is simple. Let $f : P(M) \rightarrow M$ be an epimorphism. Since $l(M) = 3$ and $l(P(M)) = 4$, it follows that $l(\text{Ker}f) = 1$. Thus, $\text{Ker}f = \text{soc}P(M) \cong V(l, \lambda)$ for some $1 \leq l \leq n - 1$ and $\lambda \in I_l$. Consequently,

$$M \cong P(M)/\text{Ker}f = P(M)/\text{soc}P(M) \cong \Omega^{-1}V(l, \lambda).$$

Now assume $1 < s < t$. Then $t < 2s$ by Lemma 4.3. Set $r = t - s$, so $1 \leq r < s$. There exists a unique positive integer l' such that

$$l'r < s \leq (l' + 1)r.$$

By Lemma 4.3(1), one can show by the induction on i that $\Omega^i M$ is of $(s - ir, s - (i - 1)r)$ -type for all $1 \leq i \leq l'$.

In particular, $\Omega^{l'} M$ is of $(s - l'r, s - (l' - 1)r)$ -type.

If $s - l'r > 1$, then by Lemma 4.3(1) we have

$$s - (l' - 1)r < 2(s - l'r),$$

which implies $(l' + 1)r < s$, a contradiction. Hence $s - l'r = 1$ and therefore $r = 1$. It follows that $t = s + 1$, $s = l' + 1$, and $\Omega^{l'} M$ is of $(1, 2)$ -type. Thus, $\Omega^{s-1} M \cong \Omega^{-1}V(l, \lambda)$ for some $1 \leq l \leq n - 1$ and $\lambda \in I_l$, and so $M \cong \Omega^{-s}V(l, \lambda)$.

(2) It is similar to (1) or dual to (1). $\square$

Lemma 4.5. *For any $1 \leq l \leq n - 1$, $\lambda \in I_l$, we have the following Auslander-Reiten sequences in $\text{mod}D(H_{\mathcal{D}})$:*

- (1) $0 \rightarrow \Omega V(l, \lambda) \rightarrow V(n - l, \sigma(\lambda)) \oplus V(n - l, \sigma^{-1}(\lambda)) \oplus P(l, \lambda) \rightarrow \Omega^{-1}V(l, \lambda) \rightarrow 0;$
- (2) $0 \rightarrow \Omega^{t+2}V(l, \lambda) \rightarrow \Omega^{t+1}V(n - l, \sigma(\lambda)) \oplus \Omega^{t+1}V(n - l, \sigma^{-1}(\lambda)) \rightarrow \Omega^tV(l, \lambda) \rightarrow 0;$
- (3) $0 \rightarrow \Omega^{-t}V(l, \lambda) \rightarrow \Omega^{-(t+1)}V(n - l, \sigma(\lambda)) \oplus \Omega^{-(t+1)}V(n - l, \sigma^{-1}(\lambda)) \rightarrow \Omega^{-(t+2)}V(l, \lambda) \rightarrow 0,$

where $t \geq 0$ and $\Omega^0V(l, \lambda) = V(l, \lambda)$.

Proof. It is similar to [3, Theorem 3.17]. $\square$

In what follows, we investigate the indecomposable $D(H_{\mathcal{D}})$ -modules of (t, t) -type. For any $D(H_{\mathcal{D}})$ -module M and $\lambda \in \Lambda$, define

$$M_{\lambda} := \{v \in M \mid (g\gamma)v = \lambda(g\gamma)v, g\gamma \in G \times \Gamma\}.$$

Then M_{λ} is a subspace of M .

Lemma 4.6. *Let M be a $D(H_{\mathcal{D}})$ -module.*

- (1) $M = \bigoplus_{\lambda \in \Lambda} M_{\lambda}$ as vector spaces.
- (2) If $\mathcal{D}$ is of nilpotent type, then $xM_{\lambda} \subseteq M_{\lambda\phi}$ for any $\lambda \in \Lambda$, and hence $M^x = \bigoplus_{\lambda \in \Lambda} (M^x \cap M_{\lambda})$.
- (3) If $\mathcal{D}$ is of non-nilpotent type, then $\xi M_{\lambda} \subseteq M_{\lambda\phi^{-1}}$ for any $\lambda \in \Lambda$, and hence $M^{\xi} = \bigoplus_{\lambda \in \Lambda} (M^{\xi} \cap M_{\lambda})$.
- (4) If K and N are submodules of M such that $M = K \oplus N$, then $M_{\lambda} = K_{\lambda} \oplus N_{\lambda}$ for all $\lambda \in \Lambda$.
- (5) If $f : M \rightarrow N$ is a $D(H_{\mathcal{D}})$ -module map, then $f(M_{\lambda}) \subseteq N_{\lambda}$ for all $\lambda \in \Lambda$. Furthermore, if f is surjective, then $f(M_{\lambda}) = N_{\lambda}$ for all $\lambda \in \Lambda$.

Proof. It follows from a straightforward verification. $\square$

Now let m be the integer defined in Lemma 3.3(3). To classify all indecomposable modules, we consider the two cases $m = 1$ and $m > 1$.

4.1. The case of $m > 1$.

Throughout this subsection, we assume that $m > 1$.

Let $1 \leq l \leq n-1$ and $\lambda \in I_l$. Let $\{v_0, v_1, \dots, v_{n-1}, u_0, u_1, \dots, u_{n-1}\}$ be a standard basis of $P(l, \lambda)$ as defined previously. Define two subspaces $T_1(l, \lambda), \bar{T}_1(l, \lambda) \subset P(l, \lambda)$ as follows:

- If $\mathcal{D}$ is of nilpotent type:

$$T_1(l, \lambda) = \text{span}\{v_l, \dots, v_{n-1}, u_{n-l}, \dots, u_{n-1}\}, \quad \bar{T}_1(l, \lambda) = \text{span}\{u_0, u_1, \dots, u_{n-1}\}$$

- If $\mathcal{D}$ is of non-nilpotent type:

$$T_1(l, \lambda) = \text{span}\{u_0, u_1, \dots, u_{n-1}\}, \quad \bar{T}_1(l, \lambda) = \text{span}\{v_0, \dots, v_{n-l-1}, u_0, \dots, u_{l-1}\}.$$

By the proof of Proposition 3.10, both $T_1(l, \lambda)$ and $\bar{T}_1(l, \lambda)$ are submodules of $P(l, \lambda)$ of $(1, 1)$ -type. Moreover, if T is any submodules of $P(l, \lambda)$ of $(1, 1)$ -type, then $T = T_1(l, \lambda)$ or $T = \bar{T}_1(l, \lambda)$. Furthermore, the socle and head of these submodules are given by

$$\text{soc}T_1(l, \lambda) \cong \text{soc}\bar{T}_1(l, \lambda) \cong V(l, \lambda),$$

$$\text{head}T_1(l, \lambda) \cong V(n-l, \sigma(\lambda)),$$

$$\text{head}\bar{T}_1(l, \lambda) \cong V(n-l, \sigma^{-1}(\lambda)).$$

The above observation immediately yield the following lemma.

Lemma 4.7. *Let $1 \leq l, l' \leq n-1$ and $\lambda \in I_l, \mu \in I_{l'}$.*

- (1) $T_1(l, \lambda) \not\cong \bar{T}_1(l', \mu)$.
- (2) $T_1(l, \lambda) \cong T_1(l', \mu)$ if and only if $l = l'$ and $\lambda = \mu$.

(3) $\bar{T}_1(l, \lambda) \cong \bar{T}_1(l', \mu)$ if and only $l = l'$ and $\lambda = \mu$.

Remark 4.8. Let $1 \leq l \leq n-1$ and $\lambda \in I_l$. The following isomorphisms hold:

$$\begin{aligned} \Omega^{-1}T_1(l, \lambda) &\cong T_1(n-l, \sigma^{-1}(\lambda)), & \Omega T_1(l, \lambda) &\cong T_1(n-l, \sigma(\lambda)), \\ \Omega^{-1}\bar{T}_1(l, \lambda) &\cong \bar{T}_1(n-l, \sigma(\lambda)), & \Omega \bar{T}_1(l, \lambda) &\cong \bar{T}_1(n-l, \sigma^{-1}(\lambda)). \end{aligned}$$

Consequently,

$$\Omega^2 T_1(l, \lambda) \cong T_1(l, \tau(\lambda)) \text{ and } \Omega^2 \bar{T}_1(l, \lambda) \cong \bar{T}_1(l, \tau^{-1}(\lambda)).$$

Moreover, $\text{End}_{D(H_D)}(T_1(l, \lambda)) \cong \mathbb{k}$, $\text{End}_{D(H_D)}(\bar{T}_1(l, \lambda)) \cong \mathbb{k}$, and for all $k \in \mathbb{Z}$ with $m \nmid k$,

$$\text{Hom}_{D(H_D)}(T_1(l, \lambda), T_1(l, \tau^k(\lambda))) = 0, \quad \text{Hom}_{D(H_D)}(\bar{T}_1(l, \lambda), \bar{T}_1(l, \tau^k(\lambda))) = 0.$$

Similarly to [17, Theorems 4.18], one can establish the following two propositions.

Proposition 4.9. Let $1 \leq l \leq n-1$, $\lambda \in I_l$, and $t \in \mathbb{Z}$ with $t \geq 1$. Then there exists an indecomposable $D(H_D)$ -module $T_t(l, \lambda)$ of (t, t) -type. This module satisfies the following properties:

- (1) $\Omega^2 T_t(l, \lambda) \cong T_t(l, \tau(\lambda))$, and $\text{soc} T_t(l, \lambda) \cong \bigoplus_{k=0}^{t-1} V(l, \tau^{-k}(\lambda))$, $\text{head} T_t(l, \lambda) \cong \bigoplus_{k=0}^{t-1} V(n-l, \tau^{-k}(\sigma(\lambda)))$.
- (2) For every $1 \leq j < t$, the module $T_t(l, \lambda)$ contains a unique submodule of (j, j) -type, which is isomorphic to $T_j(l, \lambda)$. Moreover, the corresponding quotient module satisfies

$$T_t(l, \lambda)/T_j(l, \lambda) \cong T_{t-j}(l, \tau^{-j}(\lambda)).$$

- (3) For every $1 \leq j < t$, the unique submodule of (j, j) -type of $T_t(l, \lambda)$ is contained in the unique submodule of $(j+1, j+1)$ -type.
- (4) There exist AR-sequences:

$$\begin{aligned} 0 \rightarrow T_1(l, \lambda) &\xrightarrow{f_1} T_2(l, \lambda) \xrightarrow{g_1} T_1(l, \tau^{-1}(\lambda)) \rightarrow 0, \\ 0 \rightarrow T_t(l, \lambda) &\xrightarrow{\begin{pmatrix} g_{t-1} \\ f_t \end{pmatrix}} T_{t-1}(l, \tau^{-1}(\lambda)) \oplus T_{t+1}(l, \lambda) \xrightarrow{(f'_{t-1}, g_t)} T_t(l, \tau^{-1}(\lambda)) \rightarrow 0, \end{aligned}$$

where $t \geq 2$.

Proposition 4.10. For any $1 \leq l \leq n-1$, $\lambda \in I_l$ and $t \in \mathbb{Z}$ with $t \geq 1$, there is an indecomposable $D(H_D)$ -module $\bar{T}_t(l, \lambda)$ of (t, t) -type. We have the following properties:

- (1) $\Omega^2 \bar{T}_t(l, \lambda) \cong \bar{T}_t(l, \tau^{-1}(\lambda))$, $\text{soc} \bar{T}_t(l, \lambda) \cong \bigoplus_{k=0}^{t-1} V(l, \tau^k(\lambda))$ and $\text{head} \bar{T}_t(l, \lambda) \cong \bigoplus_{k=0}^{t-1} V(n-l, \tau^{k-1}(\sigma(\lambda)))$.
- (2) For every $1 \leq j < t$, $\bar{T}_t(l, \lambda)$ contains a unique submodule of (j, j) -type, which is isomorphic to $\bar{T}_j(l, \lambda)$ and the corresponding quotient module satisfies

$$\bar{T}_t(l, \lambda)/\bar{T}_j(l, \lambda) \cong \bar{T}_{t-j}(l, \tau^j(\lambda)).$$

- (3) For every $1 \leq j < t$, the unique submodule of (j, j) -type of $\bar{T}_t(l, \lambda)$ is contained in the unique submodule of $(j+1, j+1)$ -type.

(4) *There exist AR-sequences:*

$$\begin{aligned}
0 \rightarrow \bar{T}_1(l, \lambda) &\xrightarrow{f_1} \bar{T}_2(l, \lambda) \xrightarrow{g_1} \bar{T}_1(l, \tau(\lambda)) \rightarrow 0, \\
0 \rightarrow \bar{T}_t(l, \lambda) &\xrightarrow{\begin{pmatrix} g_{t-1} \\ f_t \end{pmatrix}} \bar{T}_{t-1}(l, \tau(\lambda)) \oplus \bar{T}_{t+1}(l, \lambda) \xrightarrow{(f'_{t-1}, g_t)} \bar{T}_t(l, \tau(\lambda)) \rightarrow 0,
\end{aligned}$$

where $t \geq 2$.

Corollary 4.11. *Let $1 \leq l, l' \leq n-1$, $\lambda \in I_l$, $\mu \in I_{l'}$ and $t, t' \in \mathbb{Z}$ with $t \geq 1$ and $t' \geq 1$. Then the following isomorphisms hold:*

- (1) $T_t(l, \lambda) \not\cong \bar{T}_{t'}(l', \mu)$.
- (2) $T_t(l, \lambda) \cong T_{t'}(l', \mu)$ if and only if $t = t'$, $l = l'$ and $\lambda = \mu$.
- (3) $\bar{T}_t(l, \lambda) \cong \bar{T}_{t'}(l', \mu)$ if and only if $t = t'$, $l = l'$ and $\lambda = \mu$.

Proof. It follows from Lemma 4.7, Propositions 4.9(2) and 4.10(2) that

$$l(T_t(l, \lambda)) = 2t \text{ and } l(\bar{T}_{t'}(l', \mu)) = 2t'.$$

□

Proposition 4.12. *Let M be an indecomposable $D(H_{\mathcal{D}})$ -module of (t, t) -type. If M contains a submodule of $(1, 1)$ -type, then $M \cong T_t(l, \lambda)$ or $M \cong \bar{T}_t(l, \lambda)$ for some $1 \leq l \leq n-1$ and $\lambda \in I_l$.*

Proof. If $t = 1$, then M is of $(1, 1)$ -type. Hence there exist $1 \leq l \leq n-1$ and $\lambda \in I_l$ such that $\text{soc} M \cong V(l, \lambda)$. Consequently, $P(l, \lambda)$ serves as an injective envelope of M . Thus, M is isomorphic to a submodule of $P(l, \lambda)$. It follows that $M \cong T_1(l, \lambda)$ or $M \cong \bar{T}_1(l, \lambda)$.

Now let $t > 1$ and assume that M contains a submodule N of $(1, 1)$ -type. Then as shown above, $N \cong T_1(l, \lambda)$ or $N \cong \bar{T}_1(l, \lambda)$ for some $1 \leq l \leq n-1$ and $\lambda \in I_l$. By an argument similar to the proof of [3, Theorem 4.16] we have $M \cong T_t(l, \lambda)$ or $M \cong \bar{T}_t(l, \lambda)$. □

Let $1 \leq l \leq n-1$, $\lambda \in I_l$ and $P = \bigoplus_{k=0}^{m-1} P(l, \tau^k(\lambda))$. For any $0 \leq k \leq m-1$, let $\{v_0^k, v_1^k, \dots, v_{n-1}^k, u_0^k, u_1^k, \dots, u_{n-1}^k\}$ be a standard basis of $P(l, \tau^k(\lambda))$. Then $\{v_j^k, u_j^k | 0 \leq j \leq n-1, 0 \leq k \leq m-1\}$ forms a basis of P .

Assume that $\mathcal{D}$ is of **nilpotent type**. Let $\eta \in \mathbb{k}^\times$. For all $0 \leq j \leq n-1$ and $0 \leq k \leq m-1$, define $x_j^k \in P$ by $x_j^k = u_j^k$ if $n-l \leq j \leq n-1$; $x_j^k = v_{j+l}^k + u_j^{k+1}$ if $0 \leq j \leq n-l-1$ and $0 \leq k < m-1$; $x_j^{m-1} = v_{j+l}^{m-1} + \eta u_j^0$ if $0 \leq j \leq n-l-1$ and $k = m-1$. A straightforward verification shows that

$$(g\gamma)x_j^k = \begin{cases} (\phi^{j+l}\tau^k(\lambda))(g\gamma)x_j^k, & 0 \leq j \leq n-l-1, 0 \leq k \leq m-1, \\ (\phi^{j-n+l}\tau^k(\lambda))(g\gamma)x_j^k, & n-l \leq j \leq n-1, 0 \leq k \leq m-1, \end{cases}$$

$$\begin{aligned}
 xx_j^k &= \begin{cases} x_{j+1}^k, & 0 \leq j \leq n-l-2, 0 \leq k \leq m-1, \\ x_{n-l}^{k+1}, & j = n-l-1, 0 \leq k < m-1, \\ \eta x_{n-l}^0, & j = n-l-1, k = m-1, \\ x_{j+1}^k, & n-l \leq j \leq n-2, 0 \leq k \leq m-1, \\ 0, & j = n-1, 0 \leq k \leq m-1, \end{cases} \\
 \xi x_j^k &= \begin{cases} x_{n-1}^k, & j = 0, 0 \leq k \leq m-1, \\ \alpha_j(n-l, \sigma(\lambda))x_{j-1}^k, & 1 \leq j \leq n-l-1, 0 \leq k \leq m-1, \\ 0, & j = n-l, 0 \leq k \leq m-1, \\ \alpha_{j-n+l}(l, \lambda)x_{j-1}^k, & n-l+1 \leq j \leq n-1, 0 \leq k \leq m-1, \end{cases}
 \end{aligned}$$

where $g\gamma \in G \times \Gamma$. It is easy to see that $\text{span}\{x_j^k | 0 \leq j \leq n-1, 0 \leq k \leq m-1\}$ is a submodule of $\text{rad}P$, denoted by $M_1(l, \lambda, \eta)$. Clearly, $\{x_j^k | 0 \leq j \leq n-1, 0 \leq k \leq m-1\}$ is a $\mathbb{k}$ -basis of $M_1(l, \lambda, \eta)$. Such a basis is called a *standard basis* of $M_1(l, \lambda, \eta)$. Note that $\alpha_j(n-l, \sigma(\tau^k(\lambda))) = \alpha_j(n-l, \sigma(\lambda))$ and $\alpha_j(l, \tau^k(\lambda)) = \alpha_j(l, \lambda)$, $j \geq 1$.

Assume that $\mathcal{D}$ is of **non-nilpotent type**. Let $\eta \in \mathbb{k}^\times$. For any $0 \leq j \leq n-1$ and $0 \leq k \leq m-1$, define $x_j^k \in P$ by $x_j^k = u_j^k$ if $0 \leq j \leq l-1$; $x_j^k = u_j^k + v_{j-l}^{k+1}$ if $l \leq j \leq n-1$ and $0 \leq k < m-1$; $x_j^{m-1} = u_j^{m-1} + \eta v_{j-l}^0$ if $l \leq j \leq n-1$ and $k = m-1$. A straightforward verification shows that

$$(g\gamma)x_j^k = (\phi^j \tau^k(\lambda))(g\gamma)x_j^k, \quad g\gamma \in G \times \Gamma, 0 \leq j \leq n-1, 0 \leq k \leq m-1,$$

$$\begin{aligned}
 xx_j^k &= \begin{cases} \alpha_{j+1}(l, \lambda)x_{j+1}^k, & 0 \leq j \leq l-2, 0 \leq k \leq m-1, \\ 0, & j = l-1, 0 \leq k \leq m-1, \\ \alpha_{j+1-l}(n-l, \sigma(\lambda))x_{j+1}^k, & l \leq j \leq n-2, 0 \leq k \leq m-1, \\ z_{l,\lambda}x_0^k + x_0^{k+1}, & j = n-1, 0 \leq k < m-1, \\ z_{l,\lambda}x_0^{m-1} + \eta x_0^0, & j = n-1, k = m-1, \end{cases} \\
 \xi x_j^k &= \begin{cases} 0, & j = 0, 0 \leq k \leq m-1, \\ x_{j-1}^k, & 1 \leq j \leq n-1, 0 \leq k \leq m-1. \end{cases}
 \end{aligned}$$

This implies that $\text{span}\{x_j^k | 0 \leq j \leq n-1, 0 \leq k \leq m-1\}$ is a submodule of P , denoted by $M_1(l, \lambda, \eta)$. Clearly, $\{x_j^k | 0 \leq j \leq n-1, 0 \leq k \leq m-1\}$ is a $\mathbb{k}$ -basis of $M_1(l, \lambda, \eta)$. Such a basis is a *standard basis* of $M_1(l, \lambda, \eta)$. Note that $z_{l,\tau^k(\lambda)} = z_{l,\lambda}$ for any integer k .

Lemma 4.13. *Let $1 \leq l \leq n-1$, $\lambda \in I_l$, and $\eta \in \mathbb{k}^\times$. We have the following:*

- (1) $\text{rl}(M_1(l, \lambda, \eta)) = 2$, $\text{soc}M_1(l, \lambda, \eta) \cong \oplus_{k=0}^{m-1} V(l, \tau^k(\lambda))$ and $\text{head}M_1(l, \lambda, \eta) \cong \oplus_{k=0}^{m-1} V(n-l, \tau^k(\sigma(\lambda)))$.
- (2) $M_1(l, \lambda, \eta)$ is an indecomposable module of (m, m) -type.
- (3) $M_1(l, \lambda, \eta)$ does not contain any submodule of $(1, 1)$ -type.
- (4) $M_1(l, \lambda, \eta) \not\cong T_m(l', \mu)$ and $M_1(l, \lambda, \eta) \not\cong \bar{T}_m(l', \mu)$ for any $1 \leq l' \leq n-1$ and $\mu \in I_{l'}$.

Proof. (1) Since $M_1(l, \lambda, \eta) \subseteq \text{rad}P$, $\text{rl}(M_1(l, \lambda, \eta)) \leq 2$. However, $\text{soc}M_1(l, \lambda, \eta) = \text{soc}P \cong \oplus_{k=0}^{m-1} V(l, \tau^k(\lambda))$, hence $\text{rl}(M_1(l, \lambda, \eta)) \neq 1$. Therefore, $\text{rl}(M_1(l, \lambda, \eta)) = 2$. Moreover, $\text{head}M_1(l, \lambda, \eta) \cong \oplus_{k=0}^{m-1} V(n-l, \tau^k(\sigma(\lambda)))$.

(2) **Case 1: $\mathcal{D}$ is of nilpotent type.** Assume that $M_1(l, \lambda, \eta) = N \oplus K$ for some submodules N and K of $M_1(l, \lambda, \eta)$. Then for any $\mu \in \Lambda$, we have

$$M_1(l, \lambda, \eta)_\mu = N_\mu \oplus K_\mu.$$

By the $D(H_{\mathcal{D}})$ -module action, one can see that $x_j^k \in M_1(l, \lambda, \eta)_{\lambda\phi^j+l+kn}$ for $0 \leq j \leq n-l-1$ and $0 \leq k \leq m-1$, $x_j^k \in M_1(l, \lambda, \eta)_{\lambda\phi^j-n+l+kn}$ for $n-l \leq j \leq n-1$ and $0 \leq k \leq m-1$.

Since $\text{ord}(\phi) = mn$, $\dim M_1(l, \lambda, \eta)_{\lambda\phi^j+kn} = 1$ for $0 \leq j \leq n-1$ and $0 \leq k \leq m-1$. Therefore, each x_j^k belongs either to N or to K . In particular, $x_0^0 \in N$ or $x_0^0 \in K$. Without loss of generality, we may assume $x_0^0 \in N$. Then $x^{n-1}x_0^0 = x_{n-1}^1 \in N$.

If $x_0^1 \in K$, then $\xi x_0^1 = x_{n-1}^1 \in K$, which is a contradiction. Therefore $x_0^1 \in N$. Similarly, one can verify that $x_0^2, \dots, x_0^{m-1} \in N$. However, $M_1(l, \lambda, \eta) = \langle x_0^0, x_0^1, \dots, x_0^{m-1} \rangle$, which implies that $N = M_1(l, \lambda, \eta)$. Therefore, $M_1(l, \lambda, \eta)$ is indecomposable. By the result in (1), it follows that $M_1(l, \lambda, \eta)$ is of (m, m) -type.

Case 2: $\mathcal{D}$ is of non-nilpotent type. Suppose $M_1(l, \lambda, \eta) = N \oplus K$ for some submodules N and K of $M_1(l, \lambda, \eta)$. Then for any $\mu \in \Lambda$, we have

$$M_1(l, \lambda, \eta)_\mu = N_\mu \oplus K_\mu.$$

By the module action, one sees that $x_j^k \in M_1(l, \lambda, \eta)_{\lambda\phi^j+nk}$ for $0 \leq j \leq n-1$ and $0 \leq k \leq m-1$. Moreover, $\dim M_1(l, \lambda, \eta)_{\lambda\phi^j+nk} = 1$ for $0 \leq j \leq n-1$, $0 \leq k \leq m-1$. Therefore, each x_j^k belongs either to N or to K . In particular, $x_{n-1}^0 \in N$ or $x_{n-1}^0 \in K$. Without loss of generality, we may assume $x_{n-1}^0 \in N$. Then $xx_{n-1}^0 = z_{l,\lambda}x_0^0 + x_0^1 \in N$, however $\xi^{n-1}x_{n-1}^0 = x_0^0 \in N$. Consequently, $x_0^1 \in N$. If $x_{n-1}^1 \in K$ then $\xi^{n-1}x_{n-1}^1 = x_0^1 \in K$, a contradiction. Therefore, $x_{n-1}^1 \in N$. Similarly, one can verify that $x_{n-1}^2, x_{n-1}^3, \dots, x_{n-1}^{m-1} \in N$.

However, $M_1(l, \lambda, \eta) = \langle x_{n-1}^0, x_{n-1}^1, \dots, x_{n-1}^{m-1} \rangle$, which implies that $N = M_1(l, \lambda, \eta)$. Hence, $M_1(l, \lambda, \eta)$ is indecomposable. By the result in (1), it follows that $M_1(l, \lambda, \eta)$ is of (m, m) -type.

(3) Let $\pi : M_1(l, \lambda, \eta) \rightarrow \text{head}M_1(l, \lambda, \eta)$ be the canonical epimorphism.

Case 1: Suppose that $\mathcal{D}$ is of **nilpotent type**. Assume, for contradiction, that $M_1(l, \lambda, \eta)$ contains a submodule N of $(1, 1)$ -type. By (1), the image $\pi(N)$ is a simple submodule of $\text{head}M_1(l, \lambda, \eta)$, so $\pi(N) \cong V(n-l, \tau^k(\sigma(\lambda)))$ for some $0 \leq k \leq m-1$. Consequently,

$$\pi(N) = \bigoplus_{j=l}^{n-1} \pi(N)_{\lambda\phi^{kn+j}} \text{ and } \dim(\pi(N)_{\lambda\phi^{kn+j}}) = 1, \text{ for all } l \leq j \leq n-1.$$

By Lemma 4.6(5), $\pi(N)_{\lambda\phi^{kn+j}} = \pi(N_{\lambda\phi^{kn+j}})$, implying $N_{\lambda\phi^{kn+j}} \neq 0$. From the argument in (2), we have

$$N_{\lambda\phi^{kn+j}} = M_1(l, \lambda, \eta)_{\lambda\phi^{kn+j}} = \mathbb{k}x_{j-l}^k, \quad l \leq j \leq n-1,$$

so in particular, $x_0^k \in N$, and thus $\langle x_0^k \rangle \subseteq N$. But $\text{soc}\langle x_0^k \rangle \cong V(l, \tau^k(\lambda)) \oplus V(l, \tau^{k+1}(\lambda))$, while $l(\text{soc}N) = 1$, a contradiction. Therefore, $M_1(l, \lambda, \eta)$ contains no submodule of $(1, 1)$ -type.

Case 2: Suppose that $\mathcal{D}$ is of **non-nilpotent type**. Assume, for contradiction, that $M_1(l, \lambda, \eta)$ contains a submodule N of $(1, 1)$ -type. By (1), the image

$\pi(N)$ under the canonical projection is a simple submodule of $\text{head}M_1(l, \lambda, \eta)$, so $\pi(N) \cong V(n-l, \tau^k(\sigma(\lambda)))$ for some $0 \leq k \leq m-1$. It follows that

$$\pi(N) = \oplus_{j=l}^{n-1} \pi(N)_{\lambda\phi^{kn+j}} \text{ and } \dim(\pi(N)_{\lambda\phi^{kn+j}}) = 1, \text{ for all } l \leq j \leq n-1.$$

By Lemma 4.6(5), $\pi(N)_{\lambda\phi^{kn+j}} = \pi(N_{\lambda\phi^{kn+j}})$, and hence $N_{\lambda\phi^{kn+j}} \neq 0$. By the proof of (2), we have

$$N_{\lambda\phi^{kn+j}} = M_1(l, \lambda, \eta)_{\lambda\phi^{kn+j}} = \mathbb{K}x_j^k \text{ for } l \leq j \leq n-1,$$

so in particular, $x_{n-1}^k \in N$, and therefore $\langle x_{n-1}^k \rangle \subseteq N$. It is easy to see that $\text{soc}\langle x_{n-1}^k \rangle \cong V(l, \tau^k(\lambda)) \oplus V(l, \tau^{k+1}(\lambda))$, whereas $l(\text{soc}N) = 1$, a contradiction. Therefore, $M_1(l, \lambda, \eta)$ does not contain any submodule of $(1, 1)$ -type.

(4) This follows from (3) and Propositions 4.9(2) and 4.10(2). $\square$

Lemma 4.14. *Let $1 \leq l \leq n-1$, $\lambda \in I_l$ and $\eta \in \mathbb{K}^\times$. Then:*

- (1) $\Omega^{-1}M_1(l, \lambda, \eta) \cong M_1(n-l, \sigma^{-1}(\lambda), (-1)^m\eta)$.
- (2) $\Omega^{-2}M_1(l, \lambda, \eta) \cong M_1(l, \tau^{-1}(\lambda), \eta)$ and $\Omega^2M_1(l, \lambda, \eta) \cong M_1(l, \tau(\lambda), \eta)$.

Proof. With the notations above, let $P = \oplus_{k=0}^{m-1} P(l, \tau^k(\lambda))$ denote the injective envelope of $M_1(l, \lambda, \eta)$, so that $M_1(l, \lambda, \eta) \subseteq P$. For any $y \in P$, let $\bar{y}$ denote its image under the canonical epimorphism $P \rightarrow P/M_1(l, \lambda, \eta)$.

Case 1: Suppose that $\mathcal{D}$ is of **nilpotent type**. First, we have $\overline{v_{j+l}^k} = -\overline{u_j^{k+1}}$ for $0 \leq j \leq n-l-1$ and $0 \leq k < m-1$, and $\overline{v_{j+l}^{m-1}} = -\eta\overline{u_j^0}$ for $0 \leq j \leq n-l-1$. Define elements $y_j^k \in P/M_1(l, \lambda, \eta)$ for $0 \leq j \leq n-1$ and $0 \leq k \leq m-1$ by

$$y_j^k := \begin{cases} (-1)^k \overline{v_j^k}, & \text{if } 0 \leq j \leq l-1, \\ (-1)^k \overline{u_{j-l}^k}, & \text{if } l \leq j \leq n-1 \end{cases}$$

It is easy to see that $\{y_j^k | 0 \leq j \leq n-1, 0 \leq k \leq m-1\}$ forms a basis of $P/M_1(l, \lambda, \eta)$. A straightforward verification shows that for any $g\gamma \in G \times \Gamma$,

$$(g\gamma)y_j^k = \begin{cases} (\phi^{j+n-l}\tau^k(\sigma^{-1}(\lambda)))(g\gamma)y_j^k, & 0 \leq j \leq l-1, 0 \leq k \leq m-1, \\ (\phi^{j-l}\tau^k(\sigma^{-1}(\lambda)))(g\gamma)y_j^k, & l \leq j \leq n-1, 0 \leq k \leq m-1, \end{cases}$$

$$xy_j^k = \begin{cases} y_{j+1}^k, & 0 \leq j \leq l-2, 0 \leq k \leq m-1, \\ y_l^{k+1}, & j = l-1, 0 \leq k < m-1, \\ (-1)^m \eta y_l^0, & j = l-1, k = m-1, \\ y_{j+1}^k, & l \leq j \leq n-2, 0 \leq k \leq m-1, \\ 0, & j = n-1, 0 \leq k \leq m-1, \end{cases}$$

$$\xi y_j^k = \begin{cases} y_{n-1}^k, & j = 0, 0 \leq k \leq m-1, \\ \alpha_j(l, \lambda)y_{j-1}^k, & 1 \leq j \leq l-1, 0 \leq k \leq m-1, \\ 0, & j = l, 0 \leq k \leq m-1, \\ \alpha_{j-l}(n-l, \sigma^{-1}(\lambda))y_{j-1}^k, & l+1 \leq j \leq n-1, 0 \leq k \leq m-1. \end{cases}$$

It follows that $\Omega^{-1}M_1(l, \lambda, \eta) \cong P/M_1(l, \lambda, \eta) \cong M_1(n-l, \sigma^{-1}(\lambda), (-1)^m\eta)$.

Case 2: $\mathcal{D}$ is of non-nilpotent type. In $P/M_1(l, \lambda, \eta)$, we have $\overline{u_j^k} = -\overline{v_{j-l}^{k+1}}$ for $l \leq j \leq n-1$ and $0 \leq k < m-1$, and $\overline{u_j^{m-1}} = -\overline{\eta v_{j-l}^0}$ for $l \leq j \leq n-1$. Now let $y_j^k := (-1)^k \overline{v_j^k}$ for all $0 \leq j \leq n-1$ and $0 \leq k \leq m-1$. Then $\{y_j^k | 0 \leq j \leq n-1, 0 \leq k \leq m-1\}$ is a basis of $P/M_1(l, i, \eta)$. A straightforward verification shows that

$$(g\gamma)y_j^k = (\phi^{j-n+l}\tau^k(\lambda))(g\gamma)y_j^k, \quad 0 \leq j \leq n-1, 0 \leq k \leq m-1,$$

$$xy_j^k = \begin{cases} \alpha_{j+1}(n-l, \sigma^{-1}(\lambda))y_{j+1}^k, & 0 \leq j \leq n-l-2, 0 \leq k \leq m-1, \\ 0, & j = n-l-1, 0 \leq k \leq m-1, \\ \alpha_{j+1-n+l}(l, \lambda)y_{j+1}^k, & n-l \leq j \leq n-2, 0 \leq k \leq m-1, \\ y_{l,\lambda}y_0^k + y_0^{k+1}, & j = n-1, 0 \leq k < m-1, \\ y_{l,\lambda}y_0^{m-1} + (-1)^m \eta y_0^0, & j = n-1, k = m-1, \end{cases}$$

$$\xi y_j^k = \begin{cases} 0, & j = 0, 0 \leq k \leq m-1, \\ y_{j-1}^k, & 1 \leq j \leq n-1, 0 \leq k \leq m-1. \end{cases}$$

Hence $\Omega^{-1}M_1(l, \lambda, \eta) \cong P/M_1(l, \lambda, \eta) \cong M_1(n-l, \sigma^{-1}(\lambda), (-1)^m \eta)$ by $y_{l,\lambda} = z_{n-l, \sigma^{-1}(\lambda)}$ for $0 \leq k \leq m-1$. This shows (1). (2) follows from (1). $\square$

Lemma 4.15. *Let $1 \leq l \leq n-1$, $\lambda \in I_l$ and $\eta \in \mathbb{k}^\times$. Then*

$$M_1(l, \lambda, \eta) \cong M_1(l, \tau(\lambda), \eta).$$

Proof. Let $\{x_j^k | 0 \leq j \leq n-1, 0 \leq k \leq m-1\}$ and $\{y_j^k | 0 \leq j \leq n-1, 0 \leq k \leq m-1\}$ be the standard bases of $M_1(l, \lambda, \eta)$ and $M_1(l, \tau(\lambda), \eta)$, respectively. Define a linear map $f : M_1(l, \tau(\lambda), \eta) \rightarrow M_1(l, \lambda, \eta)$ by

$$f(y_j^k) = \begin{cases} x_j^{k+1}, & 0 \leq j \leq n-1, 0 \leq k < m-1, \\ \eta x_j^0, & 0 \leq j \leq n-1, k = m-1. \end{cases}$$

Then f is a linear isomorphism. By a straightforward verification, one can check that for any $g \in G$, $\gamma \in \Gamma$, $0 \leq j \leq n-1$ and $0 \leq k \leq m-1$,

$$f((g\gamma)y_j^k) = (g\gamma)f(y_j^k), f(xy_j^k) = xf(y_j^k), f(\xi y_j^k) = \xi f(y_j^k).$$

Hence f is a $D(H_{\mathcal{D}})$ -module isomorphism. This completes the proof. $\square$

Corollary 4.16. *Let $1 \leq l \leq n-1$, $\lambda \in I_l$ and $\eta \in \mathbb{k}^\times$. Then*

$$\Omega^2 M_1(l, \lambda, \eta) \cong M_1(l, \lambda, \eta) \cong \Omega^{-2} M_1(l, \lambda, \eta).$$

Proof. It follows from Lemma 4.14(2) and Lemma 4.15. $\square$

Proposition 4.17. *Let $1 \leq l, l' \leq n-1$, $\lambda \in I_l, \lambda' \in I_{l'}$ and $\eta, \eta' \in \mathbb{k}^\times$. Then*

$$M_1(l, \lambda, \eta) \cong M_1(l', \lambda', \eta') \Leftrightarrow l = l', \eta = \eta' \text{ and } \lambda' = \tau^k(\lambda) \text{ for some } k \in \mathbb{N}.$$

Proof. If $l = l'$, $\eta = \eta'$ and $\lambda' = \tau^k(\lambda)$ for some $k \in \mathbb{N}$, then by Lemma 4.15, we have

$$M_1(l, \lambda, \eta) \cong M_1(l', \lambda', \eta').$$

Conversely, assume that $M_1(l, \lambda, \eta) \cong M_1(l', \lambda', \eta')$. Then

$$\text{soc} M_1(l, \lambda, \eta) \cong \text{soc} M_1(l', \lambda', \eta').$$

By Lemma 4.13(1), we have $V(l, \tau^k(\lambda)) \cong V(l', \lambda')$ for some $0 \leq k \leq m-1$. It follows from Proposition 3.2(1) or Proposition 3.6 that $l = l'$ and $\lambda' = \tau^k(\lambda)$. Hence, by Lemma 4.15, $M_1(l', \lambda', \eta') \cong M_1(l, \lambda, \eta')$, and consequently $M_1(l, \lambda, \eta) \cong M_1(l, \lambda, \eta')$.

Let $f : M_1(l, \lambda, \eta) \rightarrow M_1(l, \lambda, \eta')$ be a module isomorphism. Denote by

$$\{x_j^k | 0 \leq j \leq n-1, 0 \leq k \leq m-1\} \text{ and } \{y_j^k | 0 \leq j \leq n-1, 0 \leq k \leq m-1\}$$

the standard bases of $M_1(l, \lambda, \eta)$ and $M_1(l, \lambda, \eta')$, respectively.

If $\mathcal{D}$ is of **nilpotent type**, then by the proof of Lemma 4.13(2),

$$M_1(l, \lambda, \eta)_{\lambda\phi^j+l+nk} = \mathbb{K}x_j^k, \text{ and } M_1(l, \lambda, \eta')_{\lambda\phi^j+l+nk} = \mathbb{K}y_j^k$$

for $0 \leq j \leq n-l-1$ and $0 \leq k \leq m-1$, and

$$M_1(l, \lambda, \eta)_{\lambda\phi^j-n+l+nk} = \mathbb{K}x_j^k \text{ and } M_1(l, \lambda, \eta')_{\lambda\phi^j-n+l+nk} = \mathbb{K}y_j^k$$

for $n-l \leq j \leq n-1$ and $0 \leq k \leq m-1$. Hence by Lemma 4.6(5), there exist scalars $\beta_0, \beta_1, \dots, \beta_{m-1} \in \mathbb{K}^\times$ such that

$$f(x_0^k) = \beta_k y_0^k \text{ for all } 0 \leq k \leq m-1.$$

For any $1 \leq j \leq n-l-1$ and $0 \leq k \leq m-1$, we have

$$f(x_j^k) = f(x^j x_0^k) = x^j f(x_0^k) = x^j (\beta_k y_0^k) = \beta_k y_j^k.$$

In particular, $f(x_{n-l-1}^k) = \beta_k y_{n-l-1}^k$. From $f(xx_{n-l-1}^k) = xf(x_{n-l-1}^k)$, it follows that $f(x_{n-l}^k) = \beta_{k-1} y_{n-l}^k$ for all $1 \leq k \leq m-1$ and $f(x_{n-l}^0) = \beta_{m-1} \eta^{-1} \eta' y_{n-l}^0$. Therefore, for any $n-l \leq j \leq n-1$ and $0 \leq k \leq m-1$,

$$f(x_j^k) = f(x^{j-n+l} x_{n-l}^k) = x^{j-n+l} f(x_{n-l}^k) = \begin{cases} \beta_{k-1} y_j^k, & 1 \leq k \leq m-1, \\ \beta_{m-1} \eta^{-1} \eta' y_j^0, & k = 0. \end{cases}$$

Finally, from $f(\xi x_0^k) = \xi f(x_0^k)$, we obtain $\beta_k = \beta_{k-1}$ for all $1 \leq k \leq m-1$ and $\beta_0 = \beta_{m-1} \eta^{-1} \eta'$. Consequently, $\eta = \eta'$.

Now assume that $\mathcal{D}$ is of **non-nilpotent type**. By the proof of Lemma 4.13(2), we have

$$M_1(l, \lambda, \eta)_{\lambda\phi^j+nk} = \mathbb{K}x_j^k \text{ and } M_1(l, \lambda, \eta')_{\lambda\phi^j+nk} = \mathbb{K}y_j^k$$

for all $0 \leq j \leq n-1$ and $0 \leq k \leq m-1$. Hence, by Lemma 4.6(5), there exist scalars $\beta_0, \beta_1, \dots, \beta_{m-1} \in \mathbb{K}^\times$ such that $f(x_{n-1}^k) = \beta_k y_{n-1}^k$ for all $0 \leq k \leq m-1$. For any $0 \leq j \leq n-1$ and $0 \leq k \leq m-1$, we have

$$f(x_j^k) = f(\xi^{n-1-j} x_{n-1}^k) = \xi^{n-1-j} f(x_{n-1}^k) = \xi^{n-1-j} (\beta_k y_{n-1}^k) = \beta_k y_j^k.$$

Moreover, from $f(xx_{n-1}^k) = xf(x_{n-1}^k)$ for all $0 \leq k \leq m-1$, it follows that $\beta_0 = \beta_1 = \dots = \beta_{m-1}$ and $\eta = \eta'$. $\square$

Corollary 4.18. *Let $1 \leq l < n$, $\lambda \in I_l$ and $\eta \in \mathbb{K}^\times$. Then*

$$\text{End}_{D(H_D)}(M_1(l, \lambda, \eta)) \cong \mathbb{K}.$$

Proof. It follows from the proof of Proposition 4.17. $\square$

Lemma 4.19. *Let M be an indecomposable $D(H_{\mathcal{D}})$ -module of (t, t) -type. Assume that M contains no submodule of $(1, 1)$ -type. Then M contains a submodule isomorphic to $M_1(l, \lambda, \eta)$ for some $1 \leq l < n$, $\lambda \in I_l$ and $\eta \in \mathbb{k}^\times$.*

Proof. Clearly, $t > 1$. By Proposition 4.1, we have

$$P = \bigoplus_{k=0}^{m-1} t_k P(l, \tau^k(\lambda))$$

as an injective envelope of M for some $1 \leq l \leq n-1$, $\lambda \in I_l$ and $t_k \in \mathbb{N}$ satisfying $\sum_{k=0}^{m-1} t_k = t$. Hence we may assume that

$$M \subseteq \text{soc}^2 P = \text{rad} P = \bigoplus_{k=0}^{m-1} t_k \text{rad} P(l, \tau^k(\lambda)).$$

It follows that $\text{soc} M = \text{soc} P = \bigoplus_{k=0}^{m-1} t_k \text{soc} P(l, \tau^k(\lambda))$. For each $1 \leq s \leq t_k$, let

$$\{v_0^{k,s}, \dots, v_{n-1}^{k,s}, u_0^{k,s}, \dots, u_{n-1}^{k,s}\}$$

be a standard basis of the s -th copy $P(l, \tau^k(\lambda))$ in the direct sum $t_k P(l, \tau^k(\lambda))$.

Case 1: $\mathcal{D}$ is of nilpotent type. In this case, $\text{rad} P$ has a $\mathbb{k}$ -basis

$$\{v_l^{k,s}, \dots, v_{n-1}^{k,s}, u_0^{k,s}, \dots, u_{n-1}^{k,s} \mid 0 \leq k \leq m-1, 1 \leq s \leq t_k\}$$

and

$$\text{soc} M = \text{soc} P = \text{span}\{u_{n-l}^{k,s}, \dots, u_{n-1}^{k,s} \mid 0 \leq k \leq m-1, 1 \leq s \leq t_k\},$$

see the proof of Proposition 3.10. Let

$$V = \text{span}\{v_l^{k,s}, \dots, v_{n-1}^{k,s}, u_0^{k,s}, \dots, u_{n-l-1}^{k,s} \mid 0 \leq k \leq m-1, 1 \leq s \leq t_k\}$$

and $U = M \cap V$. Then as vector spaces,

$$\text{rad} P = V \oplus \text{soc} P \text{ and } M = U \oplus \text{soc} M.$$

For any $0 \leq j \leq n-l-1$ and $0 \leq k \leq m-1$,

$$(\text{rad} P)_{\lambda \phi^{kn-n+l+j}} = \text{span}\{u_j^{k,s}, v_{j+l}^{k-1,s'} \mid 1 \leq s \leq t_k, 1 \leq s' \leq t_{k-1}\}.$$

For any $n-l \leq j \leq n-1$ and $0 \leq k \leq m-1$,

$$(\text{rad} P)_{\lambda \phi^{kn-n+l+j}} = \text{span}\{u_j^{k,s} \mid 1 \leq s \leq t_k\},$$

where we adopt the conventions $t_{-1} = t_{m-1}$ and $v_{j+l}^{-1,s'} = v_{j+l}^{m-1,s'}$. Note that

$$\text{soc} P = \bigoplus_{j=n-l}^{n-1} \bigoplus_{k=0}^{m-1} (\text{rad} P)_{\lambda \phi^{kn-n+l+j}} \text{ and } V = \bigoplus_{j=0}^{n-l-1} \bigoplus_{k=0}^{m-1} (\text{rad} P)_{\lambda \phi^{kn-n+l+j}}.$$

Hence

$$\text{soc} M = \bigoplus_{j=n-l}^{n-1} \bigoplus_{k=0}^{m-1} M_{\lambda \phi^{kn-n+l+j}} \text{ and } U = \bigoplus_{j=0}^{n-l-1} \bigoplus_{k=0}^{m-1} M_{\lambda \phi^{kn-n+l+j}}.$$

Moreover, $M_{\lambda \phi^{kn-n+l+j}} = (\text{rad} P)_{\lambda \phi^{kn-n+l+j}}$ for all $n-l \leq j \leq n-1$ and $0 \leq k \leq m-1$. For any $0 \leq j \leq n-1$, define $P_{[j]} = \bigoplus_{k=0}^{m-1} (\text{rad} P)_{\lambda \phi^{kn-n+l+j}}$ and $M_{[j]} = M \cap P_{[j]}$. Then

$$\text{rad} P = \bigoplus_{j=0}^{n-1} P_{[j]}, \quad M_{[j]} = \bigoplus_{k=0}^{m-1} M_{\lambda \phi^{kn-n+l+j}} \text{ and } M = \bigoplus_{j=0}^{n-1} M_{[j]}.$$

Moreover,

$$M_{[j]} = P_{[j]} \text{ for all } n-l \leq j \leq n-1, \text{soc} M = \bigoplus_{j=n-l}^{n-1} M_{[j]} \text{ and } U = \bigoplus_{j=0}^{n-l-1} M_{[j]}.$$

By the structure of $\text{rad} P$, the maps

$$M_{[j]} \rightarrow M_{[j+1]}, v \mapsto xv \text{ and } M_{[j+1]} \rightarrow M_{[j]}, v \mapsto \xi v$$

are both bijective for any $0 \leq j < n-1$ with $j \neq n-l-1$. It follows that $\dim M_{[j]} = t$ for all $0 \leq j \leq n-1$ since M is of (t, t) -type.

If $M^x \cap M_{[n-l-1]} \neq 0$, then by Lemma 4.6(2) and the equality $M_{[n-l-1]} = \bigoplus_{k=0}^{m-1} M_{\lambda\phi^{kn-1}}$ we have $M^x \cap M_{\lambda\phi^{kn-1}} \neq 0$ for some $0 \leq k \leq m-1$. Since $M_{\lambda\phi^{kn-1}} \subseteq (\text{rad} P)_{\lambda\phi^{kn-1}}$, it follows from the action of x on $\text{rad} P$ that

$$M^x \cap M_{\lambda\phi^{kn-1}} \subseteq \text{span}\{v_{n-1}^{k-1,s} | 1 \leq s \leq t_{k-1}\}.$$

Let $0 \neq w \in M^x \cap M_{\lambda\phi^{kn-1}}$. Then

$$w = \sum_{s=1}^{t_{k-1}} \beta_s v_{n-1}^{k-1,s} \text{ for some } \beta_1, \dots, \beta_{t_{k-1}} \in \mathbb{k}.$$

A direct verification shows that the submodule $\langle w \rangle$ is isomorphic to $T_1(l, \tau^{k-1}(\lambda))$. Hence, M contains a submodule of $(1, 1)$ -type, a contradiction. Therefore, $M^x \cap M_{[n-l-1]} = 0$.

For any $0 \leq k \leq m-1$, set

$$V_k^0 = \text{span}\{v_{n-1}^{k-1,s} | 1 \leq s \leq t_{k-1}\} \text{ and } V_k^1 = \text{span}\{u_{n-l-1}^{k,s} | 1 \leq s \leq t_k\}.$$

Then

$$(\text{rad} P)_{\lambda\phi^{kn-1}} = V_k^0 \oplus V_k^1 \text{ and } M_{\lambda\phi^{kn-1}} \cap V_k^0 = 0.$$

Note that $V_k^0 = 0$ whenever $t_{k-1} = 0$. Hence $V_k^0 \oplus M_{\lambda\phi^{kn-1}} \subseteq V_k^0 \oplus V_k^1$ and hence $\dim M_{\lambda\phi^{kn-1}} \leq \dim V_k^1 = t_k$. However, since

$$\sum_{k=0}^{m-1} \dim M_{\lambda\phi^{kn-1}} = \dim M_{[n-l-1]} = t = \sum_{k=0}^{m-1} t_k,$$

it follows that $\dim M_{\lambda\phi^{kn-1}} = \dim V_k^1 = t_k$ and hence $(\text{rad} P)_{\lambda\phi^{kn-1}} = V_k^0 \oplus V_k^1 = V_k^0 \oplus M_{\lambda\phi^{kn-1}}$ for all $0 \leq k \leq m-1$.

If one of $t_0, t_1, \dots, t_{m-1}$ is zero, we may assume $t_{k-1} = 0$ but $t_k \neq 0$ for some $0 \leq k \leq m-1$. Then

$$M_{\lambda\phi^{kn-1}} = V_k^1 = \text{span}\{u_{n-l-1}^{k,s} | 1 \leq s \leq t_k\}.$$

In this case, one can check that the submodule $\langle u_{n-l-1}^{k,1} \rangle$ of M is isomorphic to $\overline{T}_1(l, \tau^k(\lambda))$, a contradiction. Therefore, $t_0, t_1, \dots, t_{m-1}$ are all nonzero.

For each $0 \leq k \leq m-1$, there exists a basis $\{x_{k,s} | 1 \leq s \leq t_k\}$ of $M_{\lambda\phi^{kn-1}}$ such that

$$x_{k,s} - u_{n-l-1}^{k,s} \in V_k^0 \text{ for all } 1 \leq s \leq t_k.$$

We claim that the elements

$$x_{k,1} - u_{n-l-1}^{k,1}, \dots, x_{k,t_k} - u_{n-l-1}^{k,t_k}$$

are linearly independent over $\mathbb{k}$ for all $0 \leq k \leq m-1$.

Suppose, to the contrary, that $x_{k,1} - u_{n-l-1}^{k,1}, \dots, x_{k,t_k} - u_{n-l-1}^{k,t_k}$ are linearly dependent over $\mathbb{k}$ for some $0 \leq k \leq m-1$. If $t_k = 1$, then $x_{k,1} - u_{n-l-1}^{k,1} = 0$, which implies $u_{n-l-1}^{k,1} = x_{k,1} \in M$. In this case, M contains a submodule $\langle u_{n-l-1}^{k,1} \rangle$ of $(1, 1)$ -type, contradicting our assumption. Hence $t_k > 1$.

Without loss of generality, we may assume that

$$x_{k,t_k} - u_{n-l-1}^{k,t_k} = \sum_{j=1}^{t_k-1} \beta_j (x_{k,j} - u_{n-l-1}^{k,j}) \text{ for some } \beta_j \in \mathbb{k}.$$

Then

$$u_{n-l-1}^{k,t_k} - \sum_{j=1}^{t_k-1} \beta_j u_{n-l-1}^{k,j} = x_{k,t_k} - \sum_{j=1}^{t_k-1} \beta_j x_{k,j} \in M.$$

Thus, M contains a submodule

$$\langle u_{n-l-1}^{k,t_k} - \sum_{j=1}^{t_k-1} \beta_j u_{n-l-1}^{k,j} \rangle$$

of $(1,1)$ -type as above, again a contradiction. This proves the claim. Consequently, $t_{k-1} = \dim V_k^0 \geq t_k$ for all $0 \leq k \leq m-1$, and hence

$$t_0 = t_1 = \cdots = t_{m-1}.$$

Therefore, for each $0 \leq k \leq m-1$, there exists an invertible matrix $X_k \in M_{t_0}(\mathbb{k})$ such that

$$(x_{k,1} - u_{n-l-1}^{k,1}, x_{k,2} - u_{n-l-1}^{k,2}, \dots, x_{k,t_0} - u_{n-l-1}^{k,t_0}) = (v_{n-1}^{k-1,1}, v_{n-1}^{k-1,2}, \dots, v_{n-1}^{k-1,t_0}) X_k.$$

It follows that $(x_{k,1}, \dots, x_{k,t_0}) X_k^{-1} = (u_{n-l-1}^{k,1}, \dots, u_{n-l-1}^{k,t_0}) X_k^{-1} + (v_{n-1}^{k-1,1}, \dots, v_{n-1}^{k-1,t_0})$.

Let $X = X_{m-1}^{-1} \cdots X_1^{-1} X_0^{-1}$. Then X is an invertible matrix in $M_{t_0}(\mathbb{k})$. Since $\mathbb{k}$ is an algebraically closed field, there exists a nonzero vector $B = (\beta_1, \dots, \beta_{t_0})^T \in \mathbb{k}^{t_0}$ and a nonzero scalar $\eta \in \mathbb{k}$ such that

$$XB = \eta B,$$

where $(\beta_1, \dots, \beta_{t_0})^T$ denotes the transposition of $(\beta_1, \dots, \beta_{t_0})$. For $0 \leq k \leq m-1$, define $y_{n-l-1}^k \in M_{\lambda\phi^{kn-1}}$ by

$$y_{n-l-1}^k := (x_{k,1}, x_{k,2}, \dots, x_{k,t_0}) X_k^{-1} \cdots X_1^{-1} X_0^{-1} B.$$

Then we have

$$\begin{aligned} y_{n-l-1}^0 &= (u_{n-l-1}^{0,1}, \dots, u_{n-l-1}^{0,t_0}) X_0^{-1} B + (v_{n-1}^{m-1,1}, \dots, v_{n-1}^{m-1,t_0}) B, \\ y_{n-l-1}^k &= (u_{n-l-1}^{k,1}, \dots, u_{n-l-1}^{k,t_0}) X_k^{-1} \cdots X_1^{-1} X_0^{-1} B \\ &\quad + (v_{n-1}^{k-1,1}, \dots, v_{n-1}^{k-1,t_0}) X_{k-1}^{-1} \cdots X_0^{-1} B, \quad 1 \leq k \leq m-1. \end{aligned}$$

Note that the map $P_{[0]} \rightarrow P_{[n-l-1]}, v \mapsto x^{n-l-1}v$ is a bijection and its restriction gives rise to a bijection from $M_{[0]}$ onto $M_{[n-l-1]}$. Hence if $v \in P_{[0]}$ satisfies $x^{n-l-1}v \in M_{[n-l-1]}$, then $v \in M_{[0]}$. Define $y_0^k \in P_{[0]}$, $0 \leq k \leq m-1$, by

$$\begin{aligned} y_0^0 &= (u_0^{0,1}, \dots, u_0^{0,t_0}) X_0^{-1} B + (v_l^{m-1,1}, \dots, v_l^{m-1,t_0}) B, \\ y_0^k &= (u_0^{k,1}, \dots, u_0^{k,t_0}) X_k^{-1} \cdots X_1^{-1} X_0^{-1} B \\ &\quad + (v_l^{k-1,1}, \dots, v_l^{k-1,t_0}) X_{k-1}^{-1} \cdots X_0^{-1} B, \quad 1 \leq k \leq m-1. \end{aligned}$$

Let $0 \leq k \leq m-1$. It is easy to see that $x^{n-l-1}y_0^k = y_{n-l-1}^k$, and hence $y_0^k \in M_{[0]}$. Let $y_j^k = x^j y_0^k$ for $1 \leq j \leq n-l-1$ and $0 \leq k \leq m-1$. Then $y_j^k \in M_{\lambda\phi^{kn-n+l+j}}$ for

all $0 \leq j \leq n-l-1$. Moreover, $xy_j^k = y_{j+1}^k$ for all $0 \leq j < n-l-1$. Furthermore, for all $0 \leq j \leq n-l-1$, we have

$$\begin{aligned} y_j^0 &= (u_j^{0,1}, \dots, u_j^{0,t_0})X_0^{-1}B + (v_{l+j}^{m-1,1}, \dots, v_{l+j}^{m-1,t_0})B, \\ y_j^k &= (u_j^{k,1}, \dots, u_j^{k,t_0})X_k^{-1} \cdots X_1^{-1}X_0^{-1}B \\ &\quad + (v_{l+j}^{k-1,1}, \dots, v_{l+j}^{k-1,t_0})X_{k-1}^{-1} \cdots X_0^{-1}B, \quad 1 \leq k \leq m-1. \end{aligned}$$

In particular, for all $0 \leq j \leq n-l-1$, we have

$$y_j^{m-1} = (u_j^{m-1,1}, \dots, u_j^{m-1,t_0})\eta B + (v_{l+j}^{m-2,1}, \dots, v_{l+j}^{m-2,t_0})X_{m-2}^{-1} \cdots X_0^{-1}B.$$

For all $n-l \leq j \leq n-1$ and $0 \leq k \leq m-1$, define $y_j^k \in \text{rad}P$ by

$$y_j^k = \begin{cases} (u_j^{m-1,1}, \dots, u_j^{m-1,t_0})B, & \text{if } k = 0, \\ (u_j^{k-1,1}, \dots, u_j^{k-1,t_0})X_{k-1}^{-1} \cdots X_0^{-1}B, & \text{if } 1 \leq k \leq m-1. \end{cases}$$

Clearly, the set $\{y_j^k | 0 \leq j \leq n-1, 0 \leq k \leq m-1\}$ is linearly independent. A straightforward verification shows that for any $g\gamma \in G \times \Gamma$,

$$(g\gamma)y_j^k = \begin{cases} (\phi^{l+j}\tau^{k-1}(\lambda))(g\gamma)y_j^k, & 0 \leq j \leq n-l-1, 0 \leq k \leq m-1, \\ (\phi^{j-n+l}\tau^{k-1}(\lambda))(g\gamma)y_j^k, & n-l \leq j \leq n-1, 0 \leq k \leq m-1, \end{cases}$$

$$xy_j^k = \begin{cases} y_{j+1}^k, & 0 \leq j \leq n-l-2, 0 \leq k \leq m-1, \\ y_{n-l}^{k+1}, & j = n-l-1, 0 \leq k < m-1 \\ \eta y_{n-l}^0, & j = n-l-1, k = m-1 \\ y_{j+1}^k, & n-l \leq j \leq n-2, 0 \leq k \leq m-1, \\ 0, & j = n-1, 0 \leq k \leq m-1 \end{cases}$$

$$\xi y_j^k = \begin{cases} y_{n-1}^k, & j = 0, 0 \leq k \leq m-1 \\ \alpha_j(n-l, \sigma(\lambda))y_{j-1}^k, & 1 \leq j \leq n-l-1, 0 \leq k \leq m-1 \\ 0, & j = n-l, 0 \leq k \leq m-1 \\ \alpha_{j-n+l}(l, \lambda)y_{j-1}^k, & n-l+1 \leq j \leq n-1, 0 \leq k \leq m-1 \end{cases}$$

It follows that $\{y_j^k | 1 \leq j \leq n, 0 \leq k \leq m-1\}$ is a basis of the submodule $\langle y_0^0, y_0^1, \dots, y_0^{m-1} \rangle$ of M and $\langle y_0^0, y_0^1, \dots, y_0^{m-1} \rangle \cong M_1(l, \tau^{-1}(\lambda), \eta)$.

Case 2: $\mathcal{D}$ is of **non-nilpotent type**. In this case, $\text{rad}P$ has a $\mathbb{k}$ -basis

$$\{v_0^{k,s}, \dots, v_{n-l-1}^{k,s}, u_0^{k,s}, \dots, u_{n-1}^{k,s} | 0 \leq k \leq m-1, 1 \leq s \leq t_k\}$$

and $\text{soc}M = \text{soc}P = \text{span}\{u_0^{k,s}, \dots, u_{l-1}^{k,s} | 0 \leq k \leq m-1, 1 \leq s \leq t_k\}$. Let $V = \text{span}\{v_0^{k,s}, \dots, v_{n-l-1}^{k,s}, u_l^{k,s}, \dots, u_{n-1}^{k,s} | 0 \leq k \leq m-1, 1 \leq s \leq t_k\}$ and $U = M \cap V$. As vector spaces, we have

$$\text{rad}P = V \oplus \text{soc}P \text{ and } M = U \oplus \text{soc}M.$$

For any $0 \leq j \leq l-1$ and $0 \leq k \leq m-1$,

$$(\text{rad}P)_{\lambda\phi^{kn+j}} = \text{span}\{u_j^{k,s} | 1 \leq s \leq t_k\}.$$

For any $l \leq j \leq n-1$ and $0 \leq k \leq m-1$,

$$(\text{rad}P)_{\lambda\phi^{kn+j}} = \text{span}\{u_j^{k,s}, v_{j-l}^{k+1,s'} | 1 \leq s \leq t_k, 1 \leq s' \leq t_{k+1}\},$$

where we set $t_m = t_0$ and $v_{j-l}^{m,s'} = v_{j-l}^{0,s'}$. Note that

$$\text{soc}P = \bigoplus_{j=0}^{l-1} \bigoplus_{k=0}^{m-1} (\text{rad}P)_{\lambda\phi^{kn+j}} \text{ and } V = \bigoplus_{j=l}^{n-1} \bigoplus_{k=0}^{m-1} (\text{rad}P)_{\lambda\phi^{kn+j}}.$$

Hence

$$\text{soc}M = \bigoplus_{j=0}^{l-1} \bigoplus_{k=0}^{m-1} M_{\lambda\phi^{kn+j}} \text{ and } U = \bigoplus_{j=l}^{n-1} \bigoplus_{k=0}^{m-1} M_{\lambda\phi^{kn+j}}.$$

Moreover, $M_{\lambda\phi^{kn+j}} = (\text{rad}P)_{\lambda\phi^{kn+j}}$ for all $0 \leq j \leq l-1$ and $0 \leq k \leq m-1$.

For any $0 \leq j \leq n-1$, let $P_{[j]} = \bigoplus_{k=0}^{m-1} (\text{rad}P)_{\lambda\phi^{kn+j}}$ and $M_{[j]} = M \cap P_{[j]}$. Then

$$\text{rad}P = \bigoplus_{j=0}^{n-1} P_{[j]}, \quad M_{[j]} = \bigoplus_{k=0}^{m-1} M_{\lambda\phi^{kn+j}} \text{ and } M = \bigoplus_{j=0}^{n-1} M_{[j]}.$$

Moreover, $M_{[j]} = P_{[j]}$ for all $0 \leq j \leq l-1$, $\text{soc}M = \bigoplus_{j=0}^{l-1} M_{[j]}$ and $U = \bigoplus_{j=l}^{n-1} M_{[j]}$. By the structure of $\text{rad}P$, the maps

$$M_{[j]} \rightarrow M_{[j+1]}, v \mapsto xv \text{ and } M_{[j+1]} \rightarrow M_{[j]}, v \mapsto \xi v$$

are both bijective for any $0 \leq j < n-1$ with $j \neq l-1$. It follows that $\dim M_{[j]} = t$ for all $0 \leq j \leq n-1$ since M is of (t, t) -type.

Suppose that $M^\xi \cap M_{[l]} \neq 0$. Then by Lemma 4.6(3) and the decomposition $M_{[l]} = \bigoplus_{k=0}^{m-1} M_{\lambda\phi^{kn+l}}$, there exists some $0 \leq k \leq m-1$ such that

$$M^\xi \cap M_{\lambda\phi^{kn+l}} \neq 0.$$

Since $M_{\lambda\phi^{kn+l}} \subseteq (\text{rad}P)_{\lambda\phi^{kn+l}}$, the action of ξ on $\text{rad}P$ implies that

$$M^\xi \cap M_{\lambda\phi^{kn+l}} \subseteq \text{span}\{v_0^{k+1,s} | 1 \leq s \leq t_{k+1}\}.$$

Let $0 \neq w \in M^\xi \cap M_{\lambda\phi^{kn+l}}$. Then we can write $w = \sum_{s=1}^{t_{k+1}} \beta_s v_0^{k+1,s}$ for some $\beta_1, \dots, \beta_{t_{k+1}} \in \mathbb{k}$. A straightforward verification shows that the submodule $\langle w \rangle$ is isomorphic to $\bar{T}_1(l, \tau^{k+1}(\lambda))$, which is a contradiction. Therefore, we must have $M^\xi \cap M_{[l]} = 0$.

For each $0 \leq k \leq m-1$, set

$$V_k^0 = \text{span}\{v_0^{k+1,s} | 1 \leq s \leq t_{k+1}\} \text{ and } V_k^1 = \text{span}\{u_l^{k,s} | 1 \leq s \leq t_k\}.$$

Then

$$(\text{rad}P)_{\lambda\phi^{kn+l}} = V_k^0 \oplus V_k^1 \text{ and } M_{\lambda\phi^{kn+l}} \cap V_k^0 = 0.$$

Note that $V_k^0 = 0$ if $t_{k+1} = 0$. Hence we have

$$V_k^0 \oplus M_{\lambda\phi^{kn+l}} \subseteq V_k^0 \oplus V_k^1,$$

which implies $\dim M_{\lambda\phi^{kn+l}} \leq \dim V_k^1 = t_k$. However,

$$\sum_{k=0}^{m-1} \dim M_{\lambda\phi^{kn+l}} = \dim M_{[l]} = t = \sum_{k=0}^{m-1} t_k.$$

It follows that $\dim M_{\lambda\phi^{kn+l}} = \dim V_k^1 = t_k$. Consequently, we have

$$(\text{rad}P)_{\lambda\phi^{kn+l}} = V_k^0 \oplus V_k^1 = V_k^0 \oplus M_{\lambda\phi^{kn+l}}$$

for all $0 \leq k \leq m-1$.

If one of $t_0, t_1, \dots, t_{m-1}$ is zero, we may assume $t_{k+1} = 0$ but $t_k \neq 0$ for some $0 \leq k \leq m-1$. In this case,

$$M_{\lambda\phi^{kn+l}} = V_k^1 = \text{span}\{u_l^{k,s} | 1 \leq s \leq t_k\},$$

and one can check that the submodule $\langle u_l^{k,1} \rangle$ of M is isomorphic to $T_1(l, \tau^k(\lambda))$, a contradiction. Hence $t_0, t_1, \dots, t_{m-1}$ are all nonzero.

Then, for each $0 \leq k \leq m-1$, there exists a basis $\{x_{k,s} | 1 \leq s \leq t_k\}$ of $M_{\lambda\phi^{kn+l}}$ such that $x_{k,s} - u_l^{k,s} \in V_k^0$ for all $1 \leq s \leq t_k$. An argument similar to Case 1 shows that the vectors

$$x_{k,1} - u_l^{k,1}, \dots, x_{k,t_k} - u_l^{k,t_k}$$

are linearly independent over $\mathbb{k}$ for all $0 \leq k \leq m-1$. Consequently,

$$t_{k+1} = \dim V_k^0 \geq t_k \text{ for all } 0 \leq k \leq m-1,$$

which implies $t_0 = t_1 = \dots = t_{m-1}$. Therefore, for each $0 \leq k \leq m-1$, there exists an invertible matrix $X_k \in M_{t_0}(\mathbb{k})$ such that

$$(x_{k,1} - u_l^{k,1}, x_{k,2} - u_l^{k,2}, \dots, x_{k,t_0} - u_l^{k,t_0}) = (v_0^{k+1,1}, v_0^{k+1,2}, \dots, v_0^{k+1,t_0})X_k.$$

Then

$$(x_{k,1}, x_{k,2}, \dots, x_{k,t_0}) = (u_l^{k,1}, u_l^{k,2}, \dots, u_l^{k,t_0}) + (v_0^{k+1,1}, v_0^{k+1,2}, \dots, v_0^{k+1,t_0})X_k.$$

Let $X := X_{m-1} \cdots X_1 X_0$. Then X is an invertible matrix in $M_{t_0}(\mathbb{k})$. Since $\mathbb{k}$ is an algebraically closed field, there is exists a nonzero vector $B = (\beta_1, \dots, \beta_{t_0})^T \in \mathbb{k}^{t_0}$ and a nonzero scalar $\eta \in \mathbb{k}$ such that

$$XB = \eta B,$$

For $0 \leq k \leq m-1$, define $y_l^k \in M_{\lambda\phi^{kn+l}}$ by

$$y_l^k := \begin{cases} (x_{0,1}, \dots, x_{0,t_0})B, & \text{if } k = 0, \\ (x_{k,1}, \dots, x_{k,t_0})X_{k-1} \cdots X_1 X_0 B, & \text{if } 1 \leq k \leq m-1. \end{cases}$$

Then, we have

$$\begin{aligned} y_l^0 &= (u_l^{0,1}, \dots, u_l^{0,t_0})B + (v_0^{1,1}, \dots, v_0^{1,t_0})X_0 B, \\ y_l^k &= (u_l^{k,1}, \dots, u_l^{k,t_0})X_{k-1} \cdots X_1 X_0 B \\ &\quad + (v_0^{k+1,1}, \dots, v_0^{k+1,t_0})X_k \cdots X_1 X_0 B, \quad 1 \leq k \leq m-1. \end{aligned}$$

Note that the map $P_{[n-1]} \rightarrow P_{[l]}, y \mapsto \xi^{n-l-1}y$ is a bijection and its restriction gives rise to a bijection from $M_{[n-1]}$ onto $M_{[l]}$. Hence, if $y \in P_{[n-1]}$ satisfies $\xi^{n-l-1}y \in M_{[l]}$, then $y \in M_{[n-1]}$. Define $y_{n-1}^k \in P_{[n-1]}$, $0 \leq k \leq m-1$, by

$$\begin{aligned} y_{n-1}^0 &= (u_{n-1}^{0,1}, \dots, u_{n-1}^{0,t_0})B + (v_{n-l-1}^{1,1}, \dots, v_{n-l-1}^{1,t_0})X_0 B, \\ y_{n-1}^k &= (u_{n-1}^{k,1}, \dots, u_{n-1}^{k,t_0})X_{k-1} \cdots X_1 X_0 B \\ &\quad + (v_{n-l-1}^{k+1,1}, \dots, v_{n-l-1}^{k+1,t_0})X_k \cdots X_1 X_0 B, \quad 1 \leq k \leq m-1. \end{aligned}$$

Let $0 \leq k \leq m-1$. It is easy to see that $\xi^{n-l-1}y_{n-1}^k = y_l^k$, and hence $y_{n-1}^k \in M_{[n-1]}$. Let $y_j^k = \xi^{n-1-j}y_{n-1}^k$ for $0 \leq j < n-1$. Then $y_j^k \in M_{\lambda\phi^{kn+j}}$ for all $0 \leq j \leq n-1$. Moreover, we have

$$y_j^k = \begin{cases} (u_j^{0,1}, \dots, u_j^{0,t_0})B, & 0 \leq j \leq l-1, k = 0, \\ (u_j^{k,1}, \dots, u_j^{k,t_0})X_{k-1} \cdots X_1 X_0 B, & 0 \leq j \leq l-1, 1 \leq k \leq m-1, \\ (u_j^{0,1}, \dots, u_j^{0,t_0})B + (v_{j-l}^{1,1}, \dots, v_{j-l}^{1,t_0})X_0 B, & l \leq j \leq n-1, k = 0, \\ (u_j^{k,1}, \dots, u_j^{k,t_0})X_{k-1} \cdots X_1 X_0 B \\ \quad + (v_{j-l}^{k+1,1}, \dots, v_{j-l}^{k+1,t_0})X_k \cdots X_1 X_0 B, & l \leq j \leq n-1, 1 \leq k \leq m-1. \end{cases}$$

In particular, for any $l \leq j \leq n-1$, we have

$$y_j^{m-1} = (u_j^{m-1,1}, \dots, u_j^{m-1,t_0})X_{m-2} \cdots X_0 B + \eta(v_{j-l}^{0,1}, \dots, v_{j-l}^{0,t_0})B.$$

Clearly, the set $\{y_j^k | 0 \leq j \leq n-1, 0 \leq k \leq m-1\}$ is linearly independent. By a straightforward verification, one can check that

$$(g\gamma)y_j^k = (\phi^j \tau^k(\lambda))(g\gamma)y_j^k, \quad g\gamma \in G \times \Gamma, 0 \leq j \leq n-1, 0 \leq k \leq m-1,$$

$$xy_j^k = \begin{cases} \alpha_{j+1}(l, \lambda)y_{j+1}^k, & 0 \leq j \leq l-2, 0 \leq k \leq m-1, \\ 0, & j = l-1, 0 \leq k \leq m-1, \\ \alpha_{j+1-l}(n-l, \sigma(\lambda))y_{j+1}^k, & l \leq j \leq n-2, 0 \leq k \leq m-1, \\ z_{l,\lambda}y_0^k + y_0^{k+1}, & j = n-1, 0 \leq k < m-1, \\ z_{l,\lambda}y_0^{m-1} + \eta y_0^0, & j = n-1, k = m-1, \end{cases}$$

$$\xi y_j^k = \begin{cases} 0, & j = 0, 0 \leq k \leq m-1, \\ y_{j-1}^k, & 1 \leq j \leq n-1, 0 \leq k \leq m-1. \end{cases}$$

It follows that $\{y_j^k | 0 \leq j \leq n-1, 0 \leq k \leq m-1\}$ is a basis of the submodule $\langle y_{n-1}^0, y_{n-1}^1, \dots, y_{n-1}^{m-1} \rangle$ of M and $\langle y_{n-1}^0, y_{n-1}^1, \dots, y_{n-1}^{m-1} \rangle \cong M_1(l, \lambda, \eta)$. $\square$

Similarly to [17, Theorem 4.37], one can prove the following proposition.

Proposition 4.20. *For any $\eta \in \mathbb{k}^\times$, $l, t \in \mathbb{Z}$ with $1 \leq l \leq n-1$, $t \geq 1$, and $\lambda \in I_l$ there exists an indecomposable $D(H_D)$ -module $M_t(l, \lambda, \eta)$ of (tm, tm) -type. Moreover,*

- (1) $\Omega^2 M_t(l, \lambda, \eta) \cong M_t(l, \lambda, \eta)$, $\text{soc} M_t(l, \lambda, \eta) \cong \bigoplus_{k=0}^{m-1} tV(l, \tau^k(\lambda))$ and $\text{head} M_t(l, \lambda, \eta) \cong \bigoplus_{k=0}^{m-1} tV(n-l, \tau^k(\sigma(\lambda)))$.
- (2) If $M_t(l, \lambda, \eta)$ contains a submodule of (s, s) -type, then $m|s$. Moreover, for any $1 \leq j < t$, $M_t(l, \lambda, \eta)$ contains a unique submodule of (jm, jm) -type, which is isomorphic to $M_j(l, \lambda, \eta)$ and the quotient module of $M_t(l, \lambda, \eta)$ modulo the submodule of (jm, jm) -type is isomorphic to $M_{t-j}(l, \lambda, \eta)$.
- (3) For any $1 \leq j < t$, the unique submodule of (jm, jm) -type of $M_t(l, \lambda, \eta)$ is contained in that of $((j+1)m, (j+1)m)$ -type.
- (4) $M_t(l, \lambda, \eta)$ is not isomorphic to $T_{tm}(l', \lambda')$ or $\overline{T}_{tm}(l', \lambda')$ for any $1 \leq l' \leq n-1$ and $\lambda' \in I_{l'}$.
- (5) There exist Auslander-Reiten sequences:

$$0 \rightarrow M_1(l, \lambda, \eta) \xrightarrow{f_1} M_2(l, \lambda, \eta) \xrightarrow{g_1} M_1(l, \lambda, \eta) \rightarrow 0,$$

$$0 \rightarrow M_t(l, \lambda, \eta) \xrightarrow{\begin{pmatrix} g_{t-1} \\ f_t \end{pmatrix}} M_{t-1}(l, \lambda, \eta) \oplus M_{t+1}(l, \lambda, \eta) \xrightarrow{(f'_{t-1}, g_t)} M_t(l, \lambda, \eta) \rightarrow 0 \quad (t \geq 2).$$

Corollary 4.21. *Let $\eta, \eta' \in \mathbb{k}^\times$ and $l, l', t, t' \in \mathbb{Z}$ with $1 \leq l, l' \leq n-1$, $t, t' \geq 1$ and $\lambda \in I_l, \lambda' \in I_{l'}$. Then $M_t(l, \lambda, \eta) \cong M_{t'}(l', \lambda', \eta') \Leftrightarrow t = t', l = l', \eta = \eta'$ and $\lambda' = \tau^k(\lambda)$ for some integer k*

Proof. It follows from Proposition 4.17, Proposition 4.20(2,5), and the facts that $l(M_t(l, \lambda, \eta)) = 2tm$ and $l(M_{t'}(l', \lambda', \eta')) = 2t'm$. $\square$

Proposition 4.22. *Let M be an indecomposable $D(H_{\mathcal{D}})$ -module of (t, t) -type. If M contains a submodule isomorphic to $M_1(l, \lambda, \eta)$ for some $1 \leq l \leq n-1$, $\lambda \in I_l$ and $\eta \in \mathbb{k}^\times$, then $t = sm$ and $M \cong M_s(l, \lambda, \eta)$ for some $s \geq 1$.*

Proof. Assume that M contains a submodule isomorphic to $M_1(l, \lambda, \eta)$ for some $1 \leq l \leq n-1$, $\lambda \in I_l$ and $\eta \in \mathbb{k}^\times$. Then $t \geq m$. By Proposition 4.12, Lemma 4.13(4), and Propositions 4.9(2,3) and 4.10(2,3), one can verify that M does not contain any submodule of $(1, 1)$ -type. Hence, by the proof of Lemma 4.19, m divides t , so that $t = sm$ for some $s \geq 1$. An argument similar to that in [3, Theorem 4.16] then shows that $M \cong M_t(l, \lambda, \eta)$. $\square$

Summarizing the discussions in the last section and this section, we obtain the classification of finite dimensional indecomposable $D(H_{\mathcal{D}})$ -modules as follows.

Theorem 4.23. *Assume $m > 1$. A complete set of representatives of isomorphism classes of finite dimensional indecomposable $D(H_{\mathcal{D}})$ -modules is given by*

$$\left\{ \begin{array}{l} V(l', \lambda'), P(l, \lambda), \Omega^{\pm s} V(l, \lambda), \\ T_s(l, \lambda), \bar{T}_s(l, \lambda), M_s(l, \lambda, \eta) \end{array} \middle| \begin{array}{l} 1 \leq l \leq n-1, 1 \leq l' \leq n, s \geq 1, \\ \lambda \in I_l, \lambda' \in I_{l'}, \eta \in \mathbb{k}^\times \end{array} \right\}.$$

4.2. The case of $m = 1$.

Throughout this subsection, unless otherwise stated, assume $m = 1$. In this case, $\mathcal{D}$ is of nilpotent type by Remark 3.4. Let ∞ be a symbol with $\infty \notin \mathbb{k}$ and set $\bar{\mathbb{k}} := \mathbb{k} \cup \{\infty\}$.

Let $1 \leq l \leq n-1$ and $\lambda \in I_l$. Let $\{v_0, v_1, \dots, v_{n-1}, u_0, u_1, \dots, u_{n-1}\}$ be a standard basis of $P(l, \lambda)$ as given previously. For any $\eta \in \bar{\mathbb{k}}$, define a subspace $W_1(l, \lambda, \eta) \subset P(l, \lambda)$ by

$$W_1(l, \lambda, \eta) := \begin{cases} \text{span}\{u_0 + \eta v_l, \dots, u_{n-l-1} + \eta v_{n-1}, u_{n-l}, \dots, u_{n-1}\}, & \eta \in \mathbb{k}, \\ \text{span}\{v_l, \dots, v_{n-1}, u_{n-l}, \dots, u_{n-1}\}, & \eta = \infty. \end{cases}$$

Since $m = 1$, we have $\sigma^{-1} = \sigma$ and $\text{ord}(\phi) = n$. One can easily check that $W_1(l, \lambda, \eta)$ is a submodule of $P(l, \lambda)$, with

$$\text{soc} W_1(l, \lambda, \eta) \cong V(l, \lambda) \text{ and } W_1(l, \lambda, \eta) / \text{soc} W_1(l, \lambda, \eta) \cong V(n-l, \sigma(\lambda))$$

for any $\eta \in \bar{\mathbb{k}}$. Hence, $W_1(l, \lambda, \eta)$ is an indecomposable module of $(1, 1)$ -type.

Lemma 4.24. *Let $1 \leq l \leq n-1$, $\lambda \in I_l$ and $\eta \in \mathbb{k}$. Then $W_1(l, \lambda, \infty) \not\cong W_1(l, \lambda, \eta)$.*

Proof. Using the above notations, we have $P(l, \lambda)^x = \mathbb{k}v_{n-1} + \mathbb{k}u_{n-1}$. Hence

$$W_1(l, \lambda, \eta)^x = \mathbb{k}u_{n-1} \text{ and } W_1(l, \lambda, \infty)^x = \mathbb{k}v_{n-1} + \mathbb{k}u_{n-1}.$$

This shows that $\dim W_1(l, \lambda, \eta)^x = 1$ and $\dim W_1(l, \lambda, \infty)^x = 2$, and therefore $W_1(l, \lambda, \infty) \not\cong W_1(l, \lambda, \eta)$. $\square$

Lemma 4.25. *Let M be an indecomposable $D(H_{\mathcal{D}})$ -module of $(1, 1)$ -type. Then M is isomorphic to $W_1(l, \lambda, \eta)$ for some $1 \leq l \leq n-1$, $\lambda \in I_l$ and $\eta \in \bar{\mathbb{k}}$.*

Proof. Since M is an indecomposable module of $(1, 1)$ -type, its socle $\text{soc}M$ is simple but not projective. Hence, $\text{soc}M \cong V(l, \lambda)$ for some $1 \leq l \leq n-1$ and $\lambda \in I_l$, and $P(l, \lambda)$ is an injective envelope of M . We may assume that M is a submodule of $P(l, \lambda)$. Then $\text{soc}M = \text{soc}P(l, \lambda)$, $M/\text{soc}M \subset \text{soc}^2P(l, \lambda)/\text{soc}P(l, \lambda)$ and $M \subset \text{soc}^2P(l, \lambda)$.

By the proof of Proposition 3.10, $\text{soc}^2P(l, \lambda)/\text{soc}P(l, \lambda) \cong 2V(n-l, \sigma(\lambda))$. Hence, $\text{head}M \cong V(n-l, \sigma(\lambda))$, and so

$$\text{head}M = \bigoplus_{j=0}^{n-l-1} (\text{head}M)_{\sigma(\lambda)\phi^j} = \bigoplus_{j=l}^{n-1} (\text{head}M)_{\lambda\phi^j}.$$

Moreover, $\dim((\text{head}M)_{\lambda\phi^j}) = 1$ for all $l \leq j \leq n-1$. By Lemma 4.6(5), $(\text{head}M)_{\lambda\phi^l} = \pi(M_{\lambda\phi^l})$ and $(\text{soc}^2P(l, \lambda)/\text{soc}P(l, \lambda))_{\lambda\phi^l} = \pi((\text{soc}^2P(l, \lambda))_{\lambda\phi^l})$, where $\pi : P(l, \lambda) \rightarrow P(l, \lambda)/\text{soc}P(l, \lambda)$ is the canonical epimorphism.

Let $\{v_0, v_2, \dots, v_{n-1}, u_0, u_1, \dots, u_{n-1}\}$ be the standard basis of $P(l, \lambda)$. Then by the proof of Proposition 3.10, we have

$$\text{soc}^2P(l, \lambda) = \text{span}\{v_l, \dots, v_{n-1}, u_0, u_1, \dots, u_{n-1}\}.$$

Hence $(\text{soc}^2P(l, \lambda))_{\lambda\phi^l} = \mathbb{k}v_l + \mathbb{k}u_0$. Since $0 \neq M_{\lambda\phi^l} \subseteq (\text{soc}^2P(l, \lambda))_{\lambda\phi^l}$, there exist scales $\alpha, \beta \in \mathbb{k}$ such that

$$0 \neq \alpha u_0 + \beta v_l \in M_{\lambda\phi^l}.$$

If $\alpha = 0$, then $v_l \in M$, and hence $\langle v_l \rangle \subseteq M$. However, $\langle v_l \rangle = \text{span}\{v_l, \dots, v_{n-1}, u_{n-l}, \dots, u_{n-1}\}$ is a module of $(1, 1)$ -type. Thus, $\langle v_l \rangle = M$, and so $M \cong W_1(l, \lambda, \infty)$.

If $\alpha \neq 0$, then $u_0 + \eta v_l \in M$, where $\eta = \alpha^{-1}\beta \in \mathbb{k}$. In this case, a similar argument shows that $M \cong W_1(l, \lambda, \eta)$. $\square$

Proposition 4.26. *Let $1 \leq l, l' \leq n-1$, $\lambda \in I_l$, $\lambda' \in I_{l'}$ and $\eta, \eta' \in \bar{\mathbb{k}}$. Then $W_1(l, \lambda, \eta) \cong W_1(l', \lambda', \eta')$ if and only if $l = l'$, $\lambda = \lambda'$ and $\eta = \eta'$.*

Proof. If $l = l'$, $\lambda = \lambda'$ and $\eta = \eta'$, then clearly, $W_1(l, \lambda, \eta) \cong W_1(l', \lambda', \eta')$.

Conversely, assume $W_1(l, \lambda, \eta) \cong W_1(l', \lambda', \eta')$. Then $\text{soc}W_1(l, \lambda, \eta) \cong \text{soc}W_1(l', \lambda', \eta')$, and hence $V(l, \lambda) \cong V(l', \lambda')$. By Proposition 3.2(1), $l = l'$ and $\lambda = \lambda'$, and so

$$W_1(l, \lambda, \eta) \cong W_1(l, \lambda, \eta').$$

By Lemma 4.24, it is enough to consider the case where η and η' are contained in $\mathbb{k}$. Let $f : W_1(l, \lambda, \eta) \rightarrow W_1(l, \lambda, \eta')$ be a module isomorphism. Then

$$f(W_1(l, \lambda, \eta)_{\lambda\phi^j}) = W_1(l, \lambda, \eta')_{\lambda\phi^j} \text{ for all } 0 \leq j \leq n-1.$$

Using the bases of $W_1(l, \lambda, \eta)$ and $W_1(l, \lambda, \eta')$ given before, we have

$$W_1(l, \lambda, \eta)_{\lambda\phi^l} = \mathbb{k}(u_0 + \eta v_l), \quad W_1(l, \lambda, \eta')_{\lambda\phi^l} = \mathbb{k}(u_0 + \eta' v_l).$$

Hence, there exists some $\alpha \in \mathbb{k}^\times$ such that

$$f(u_0 + \eta v_l) = \alpha(u_0 + \eta' v_l).$$

Applying x^{n-1} , we obtain

$$f(x^{n-1}(u_0 + \eta v_l)) = x^{n-1}f(u_0 + \eta v_l),$$

which implies that $f(u_{n-1}) = \alpha u_{n-1}$.

Similarly, applyinh ξ , we obtain

$$f(\xi(u_0 + \eta v_l)) = \xi f(u_0 + \eta v_l),$$

which yields $\eta \alpha u_{n-1} = \eta' \alpha u_{n-1}$. This implies $\eta = \eta'$. $\square$

Remark 4.27. Let $1 \leq l \leq n-1$, $\lambda \in I_l$ and $\eta \in \bar{\mathbb{K}}$. Then we have

$$\Omega^{-1}W_1(l, \lambda, \eta) \cong W_1(n-l, \sigma(\lambda), -\eta), \quad \Omega W_1(l, \lambda, \eta) \cong W_1(n-l, \sigma(\lambda), -\eta),$$

where we regard $-\infty = \infty$. Hence $\Omega^2 W_1(l, \lambda, \eta) \cong W_1(l, \lambda, \eta)$. Moreover, one can check that $\text{End}_{D(H_{\mathcal{D}})}(W_1(l, \lambda, \eta)) \cong \mathbb{K}$ and $\text{Hom}_{D(H_{\mathcal{D}})}(W_1(l, \lambda, \eta), W_1(l, \lambda, \eta')) = 0$ for any $\eta' \in \bar{\mathbb{K}}$ with $\eta' \neq \eta$.

Similarly to [17, Theorem 4.18], one can prove the following proposition.

Proposition 4.28. For any $1 \leq l \leq n-1$, $\lambda \in I_l$, $\eta \in \bar{\mathbb{K}}$ and $t \in \mathbb{Z}$ with $t \geq 1$, there is an indecomposable $D(H_{\mathcal{D}})$ -module $W_t(l, \lambda, \eta)$ of (t, t) -type. We have the following properties:

- (1) $\Omega^2 W_t(l, \lambda, \eta) \cong W_t(l, \lambda, \eta)$, $\text{soc} W_t(l, \lambda, \eta) \cong tV(l, \lambda)$ and $\text{head} W_t(l, \lambda, \eta) \cong tV(n-l, \sigma(\lambda))$.
- (2) For any $1 \leq j < t$, $W_t(l, \lambda, \eta)$ contains a unique submodule of (j, j) -type, which is isomorphic to $W_j(l, \lambda, \eta)$ and the quotient module of $W_t(l, \lambda, \eta)$ modulo the submodule of (j, j) -type is isomorphic to $W_{t-j}(l, \lambda, \eta)$.
- (3) For any $1 \leq j < t$, the unique submodule of (j, j) -type of $W_t(l, \lambda, \eta)$ is contained in that of $(j+1, j+1)$ -type.
- (4) There are AR-sequences

$$\begin{aligned} 0 \rightarrow W_1(l, \lambda, \eta) &\xrightarrow{f_1} W_2(l, \lambda, \eta) \xrightarrow{g_1} W_1(l, \lambda, \eta) \rightarrow 0, \\ 0 \rightarrow W_t(l, \lambda, \eta) &\xrightarrow{\begin{pmatrix} g_{t-1} \\ f_t \end{pmatrix}} W_{t-1}(l, \lambda, \eta) \oplus W_{t+1}(l, \lambda, \eta) \xrightarrow{(f'_{t-1}, g_t)} W_t(l, \lambda, \eta) \rightarrow 0 \\ &\text{for } t \geq 2. \end{aligned}$$

Proposition 4.29. Let M be an indecomposable left $D(H_{\mathcal{D}})$ -module of (t, t) -type. Then M contains a submodule of $(1, 1)$ -type, and consequently $M \cong W_t(l, \lambda, \eta)$ for some $1 \leq l \leq n-1$, $\lambda \in I_l$ and $\eta \in \bar{\mathbb{K}}$.

Proof. First, by an argument similar to, but much easier than, the proof of Lemma 4.19, one can check that M contain a submodule isomorphic to $W_1(l, \lambda, \eta)$ for some $1 \leq l \leq n-1$, $\lambda \in I_l$ and $\eta \in \bar{\mathbb{K}}$. Then, similarly to [3, Theorem 4.16], one can show that $M \cong W_t(l, \lambda, \eta)$. $\square$

Summarizing the above discussion, we obtain the classification of finite dimensional indecomposable modules over $D(H_{\mathcal{D}})$ as follows.

Theorem 4.30. Assume $m = 1$. A complete set of representatives of isomorphism classes of finite dimensional indecomposable $D(H_{\mathcal{D}})$ -modules is given by

$$\left\{ \begin{array}{l} V(l', \lambda'), P(l, \lambda), \\ \Omega^{\pm s} V(l, \lambda), W_s(l, \lambda, \eta) \end{array} \middle| \begin{array}{l} 1 \leq l \leq n-1, 1 \leq l' \leq n, s \geq 1, \\ \lambda \in I_l, \lambda' \in I_{l'}, \eta \in \bar{\mathbb{K}}. \end{array} \right\}.$$

By Theorem 4.23 and Theorem 4.30, we immediately obtain the following corollary.

Corollary 4.31. *In all cases, where $m > 1$ or $m = 1$, the Drinfeld double $D(H_{\mathcal{D}})$ is of tame representation type.*

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REFERENCES

- [1] M. Auslander, I. Reiten and S. O. Smalø, Representation Theory of Artin Algebras, Cambridge Univ. Press, Cambridge, 1995.
- [2] H. X. Chen, Irreducible representations of a class of quantum doubles, J. Algebra 225 (2000), 391-409.
- [3] H. X. Chen, Finite-dimensional representations of a quantum double, J. Algebra 251 (2002), 751-789.
- [4] H. X. Chen, Representations of a class of Drinfeld's doubles, Comm. Algebra 33 (2005), 2809-2825.
- [5] H. X. Chen, The Green ring of Drinfeld double $D(H_4)$, Algebra and Representation Theory 17 (2014), 1457-1483.
- [6] H. X. Chen, H. S. E. Mohammed and H. Sun, Indecomposable decomposition of tensor products of modules over Drinfeld doubles of Taft algebras, J. Pure Appl. Algebra 221 (2017), 2752-2790.
- [7] H. X. Chen, H. S. E. Mohammed, W. Lin and H. Sun, The projective class rings of a family of pointed Hopf algebras of rank two, Bull. Belg. Math. Soc. Simon Stevin 23 (2016), 693-711.
- [8] H. X. Chen, F. Van Oystaeyen and Y. H. Zhang, The Green rings of Taft algebras, Proc. Amer. Math. Soc. 142 (2014), 765-775.
- [9] C. Cibils, A quiver quantum group, Comm. Math. Phys. 157 (1993), 459-477.
- [10] K. Erdmann, E. L. Green, N. Snashall and R. Taillefer, Representation theory of the Drinfeld doubles of a family of Hopf algebra, J. Pure Appl. Algebra 204 (2006), 413-454.
- [11] K. Erdmann, E. L. Green, N. Snashall and R. Taillefer, Stable Green ring of the Drinfeld doubles of the generalized Taft algebras (corrections and new results), Algebr. Represent. Theor. 22 (2019), 757-783.
- [12] C. Kassel, Quantum groups, Springer-Verlag, New York, 1995.
- [13] L. Krop, D. E. Radford, Finite-dimensional Hopf algebras of rank one in characteristic zero, J. Algebra 302 (2006) 214-230.
- [14] L. Li, Y. H. Zhang, The Green rings of the generalized Taft algebras, Contemporary Math. 585 (2013) 275-288.
- [15] S. Montgomery, Hopf Algebras and their actions on rings, CBMS Series in Math., Vol. 82, Am. Math. Soc., Providence, 1993.
- [16] S. Scherotzke, Classification of pointed rank one Hopf algebras, J. Algebra 319 (2008), 2889-2912.
- [17] H. Sun and H. X. Chen, Representations of Drinfeld doubles of Radford Hopf algebras, Sci. China Math. 68 (2025), 761-784.
- [18] H. Sun, H. S. E. Mahammed, W. Lin and H. X. Chen, Green rings of Drinfeld doubles of Taft algebras, Comm. Algebra 48 (2020), 3933-3947.
- [19] M. E. Sweedler, Hopf Algebras, Benjamin, New York, 1969.
- [20] R. Suter, Modules for $\mathcal{U}_q(\mathfrak{sl}_2)$, Comm. Math. phys. 163 (1994), 359-393.
- [21] E. J. Taft, The order of the antipode of a finite-dimensional Hopf algebra, Proc. Nat. Acad. Sci. USA 68 (1971), 2631-2633.
- [22] Z. H. Wang, L. B. Li and Y. H. Zhang, Green rings of pointed rank one Hopf algebras of nilpotent type, Algebr. Represent. Theory 17 (2014), 1901-1924.

- [23] Z. H. Wang, L. B. Li and Y. H. Zhang, Green rings of pointed rank one Hopf algebras of non-nilpotent type, J. Algebra 449 (2016), 108-137.
- [24] Z. Wang, L. You and H. X. Chen, Representations of Hopf-Ore extensions of group algebras and pointed Hopf algebras of rank one, Algebr Represent Theor 18, 801-830 (2015).

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