A Hall viscosity for skyrmion via magnon interaction

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We identify a Hall viscosity term directly from the Dzyaloshinskii-Moriya interaction (DMI), that breaks parity symmetry, in the skyrmion motion of insulating magnets by time-averaging the magnon contribution to all orders. The viscosity term is proportional to the skyrmion charge. Skyrmion Hall angle shows significant dependence on the skyrmion shape and size, the ratio of exchange over DMI parameters, while roughly independent of the Gilbert damping parameter. The Hall angles have the same magnitude for opposite skyrmion charges. We speculate a velocity-dependent Hall viscosity contribution to seek asymmetric Hall angles for the opposite charges.

I. INTRODUCTION

Hall viscosity [1] is a fundamental and universal transport coefficient in the absence of parity symmetry. Hydrodynamics has confirmed its existence [2, 3], and there have been a plethora of investigations in the context of quantum Hall systems [4–8]. Recently it is introduced to skyrmion physics using topological Ward identities in [9, 10] and various different ways to verify its existence have been proposed [11–13]. These topics of Hall viscosity in magnetic skyrmions have been also collected in a monograph [14]. Moreover, Hall effects haven been also observed in charge neutral spin objects such as skyrmions [15, 16] and magnons [17] and even for spin charge neutral phonons [18].

Here we investigate the Hall viscosity contributions in the skyrmion motion by including the skyrmion magnon interactions in the context of insulating magnets. We note there exist closely related works in literature on Landau-Lifshitz-Gilbert (LLG) equation in this context [19–24]. We generalize the results by including the magnon density to all orders. This work can be viewed as a part of bigger program of investigating Hall viscosity in skyrmion motion by extending the previous works on conducting magnets and also insulating magnets without magnon contributions [9, 10, 14]. We provide derivations fully in appendices.

At the outset, we clarify a few subtle technical details. To study the interaction between skyrmion and magnon, we use the decomposition of a magnetization vector as

$$\vec{m} = (1 - m_f^2)^{1/2}\vec{m}_s + \vec{m}_f \ . \tag{1}$$

with the slow mode $\vec{m}_s$ and the fast mode $\vec{m}_f$, which is typically much smaller than the slow mode $|\vec{m}_f| \ll |\vec{m}_s|$. The factor $(1 - m_f^2)^{1/2}$ is to ensure that both vectors $\vec{m}$ and $\vec{m}_s$ have the unit norms as $|\vec{m}^2| = 1$ and $|\vec{m}_s^2| = 1$. Here we consider the case when $\vec{m}_f$ has a periodic time dependence, which allows us to average over the fast mode in time (temporal coarse graining). This can be further justified as we are interested in steady-state motion of topological defects such as skyrmions and domain walls. Furthermore, we do not expand the square root factor aiming to get the full results without approximation, which is theoretically appealing.

We also note that the slow mode is also slowly varying in space compared to the fast mode, $|\vec{\nabla}_\alpha \vec{m}_s| \ll |\vec{\nabla}_\alpha \vec{m}_f|$ where $\alpha = 1, 2$ are the space indices. It is tempting to think that the terms combined with $\vec{m}_f$ and $\vec{\nabla}_\alpha \vec{m}_s$ are smaller than those combined with $\vec{m}_s$ and $\vec{\nabla}_\alpha \vec{m}_f$. For example, $(\vec{\nabla}_\alpha \vec{m}_s) \cdot \vec{m}_f \ll \vec{m}_s \cdot (\vec{\nabla}_\alpha \vec{m}_f)$. This is not the case as the two modes, $\vec{m}_s$ and $\vec{m}_f$, are perpendicular to each other and constrained to satisfy $\vec{m}_s \cdot \vec{m}_f = 0$. Thus, they satisfy

$$(\vec{\nabla}_\alpha \vec{m}_s) \cdot \vec{m}_f = -\vec{m}_s \cdot (\vec{\nabla}_\alpha \vec{m}_f) \ , \tag{2}$$

where $\cdot$ acts on the two magnetization vectors and $\vec{\nabla}_\alpha$ is gradient operator. Thus, we carefully keep all the terms when spatial derivatives are involved.

Also, we do not consider the spatial averaging for the fast modes as it is not clear how to take spatial averages consistently when involved with spatial derivatives. For example, we have the identity

$$\vec{m}_f (\vec{\nabla}_\alpha \vec{m}_s \cdot \vec{m}_f) = -\vec{m}_f (\vec{m}_s \cdot \vec{\nabla}_\alpha \vec{m}_f) \ . \tag{3}$$

The term in the left side gives a finite result while the right one vanishes if we average them over the spatial directions, say for the fast mode $\vec{m}_f \propto \sin(k_x x)\sin(k_y y)$ and the domains $0 \le x, y < 2\pi/k$.

In main sections, we construct the LLG equation and the Thiele equation for skyrmion by using the decomposition (1) and the temporal time averaging. Then we evaluate them with a continuous skyrmion model along with a circular symmetric magnons. The Hall viscosity term and the corresponding Hall angles are analyzed with opposite skyrmion charges. We seek to find a way for achieving asymmetric skyrmion angles before conclude.

II. TEMPORAL AVERAGING OF MAGNONS

There have been numerous results of separating the slow mode and the fast mode (1)for the Landau-Lifshitz-Gilbert (LLG) equation

$$\dot{\vec{m}} = -\gamma \vec{m} \times \vec{H}_{\rm eff} + \alpha \vec{m} \times \dot{\vec{m}} \ , \tag{4}$$

in the context of spins or magnetizations, e.g., [19–24]. Here $\vec{H}_{\rm eff} = -\delta\mathcal{H}/\delta\vec{m}$ with Hamiltonian $\mathcal{H}$ and only the precession term is captured by $\mathcal{H}$.

We study the time average of the fast mode with a periodic time dependence, after rewriting the magnetization with (1). Starting from the terms with the time derivatives, we get

$$\begin{aligned}\dot{\vec{m}} &= (1-m_f^2)^{1/2}\dot{\vec{m}}_s + \dot{\vec{m}}_f - \frac{(m_f^k \dot{m}_f^k)}{(1-m_f^2)^{1/2}}\vec{m}_s \\ &\to (1-\langle m_f^2\rangle_T)^{1/2}\dot{\vec{m}}_s\ , \qquad (5)\end{aligned}$$

where the second term, linear in $\vec{m}_f$, vanishes for the temporal averaging over a period. The third term, while quadratic, also vanishes as we will see later with a circular symmetric form of the fast mode. Similarly, the damping term, the last term in (4), after dropping the odd power of the fast mode, reduces

$$\vec{m}\times\dot{\vec{m}} \to (1-\langle m_f^2\rangle_T)\vec{m}_s\times\dot{\vec{m}}_s + \langle \vec{m}_f\times\dot{\vec{m}}_f\rangle_T\ . \qquad (6)$$

Note the last term $\langle \vec{m}_f\times\dot{\vec{m}}_f\rangle_T$ survives with the time dependence we choose and points to the direction of the slow mode [25].

The second term in (4) can be also evaluated with the Hamiltonian $\mathcal{H} = \int dV \left(\frac{J}{2}\,(\vec{\nabla}\vec{m})^2 + D\,\vec{m}\cdot(\vec{\nabla}\times\vec{m}) - \vec{H}\cdot\vec{m}\right)$. The form $\vec{H}_{\rm eff} = -\delta\mathcal{H}/\delta\vec{m}$ and detailed computations can be found in appendix §A. The resulting LLG equation after temporal average reads

$$\begin{aligned}\dot{\vec{m}}_s =& -\gamma\vec{m}_s\times\vec{H}^{0s}_{\rm eff} + (1-\langle m_f^2\rangle)^{1/2}\cdot\alpha\vec{m}_s\times\dot{\vec{m}}_s + \langle\vec{m}_f\times\dot{\vec{m}}_f\rangle_T \\ &- \frac{\gamma}{(1-\langle m_f^2\rangle)^{1/2}}\Big\{J\ \langle\vec{m}_f\times\nabla^2_\alpha\vec{m}_f\rangle_T - J\ \langle m_f^i\nabla_\alpha m_f^i\rangle_T[\vec{m}_s\times\nabla_\alpha\vec{m}_s] \\ &- 2D\ \langle\vec{m}_f\times(\nabla\times\vec{m}_f)\rangle_T + 2D\ \vec{m}_s\times[(\vec{\nabla}\langle m_f^2\rangle_T)\times\vec{m}_s\Big\}\ , \qquad (7)\end{aligned}$$

where $\vec{H}^{0s}_{\rm eff} = (1-\langle m_f^2\rangle)^{1/2}\{J(\nabla^2\vec{m}_s) - 2D(\vec{\nabla}\times\vec{m}_s)\} + \vec{H}$. Similar results were previously reported [19–24]. We include all the magnon contributions with only its temporal average. Detailed derivations can be found in §A.

Thiele equation describes the linear motion of the slow mode center, $\vec{m}_s(r_\alpha(t))$, where $\alpha,\beta = 1,2$ for 2 dimensional space. Thus, $\dot{\vec{m}}_s = \dot{r}_\beta\nabla_\beta\vec{m}_s = v_\beta\nabla_\beta\vec{m}_s$. By contracting (7) with $(\vec{m}_s\times\nabla_\beta\vec{m}_s)\cdot$ ($\cdot$ acts on $\vec{m}$), we get

$$\begin{aligned}0 =& -\int dV\mathcal{G}_{\alpha\beta}v_\beta + \alpha\int dV(1-\langle m_f^2\rangle)^{1/2}\mathcal{D}_{\alpha\beta}v_\beta \\ &- \gamma\int dV\left\{(\nabla_\alpha\vec{m}_s)\cdot\vec{H} + (1-m_f^2)^{1/2}\left[(J)(\nabla_\alpha\vec{m}_s\cdot\nabla^2\vec{m}_s) - (2D)(\nabla_\alpha\vec{m}_s\cdot\vec{\nabla}\times\vec{m}_s)\right]\right\} \\ &- \gamma(J)\int dV\Big\{-\frac{[(\vec{m}_f\cdot\nabla_\beta\vec{m}_f)][(\nabla_\alpha\vec{m}_s)\cdot(\nabla_\beta\vec{m}_s)]}{(1-\langle m_f^2\rangle)^{1/2}} - \frac{[\vec{m}_s\cdot\nabla^2\vec{m}_f][\vec{m}_f\cdot\nabla_\alpha\vec{m}_s]}{(1-\langle m_f^2\rangle)^{1/2}}\Big\} \\ &+ \gamma(2D)\int dV\Big\{\frac{[\vec{m}_s\cdot(\vec{\nabla}\times\vec{m}_f)](\vec{m}_s\cdot\nabla_\alpha\vec{m}_f)}{(1-\langle m_f^2\rangle)^{1/2}} - \frac{(m_f^i\vec{\nabla}m_f^i)\cdot[\vec{m}_s\times\nabla_\alpha\vec{m}_s]}{(1-\langle m_f^2\rangle)^{1/2}}\Big\}\ , \qquad (8)\end{aligned}$$

where the integral is over a single skyrmion unit. Detailed derivation of (8) can be found in §B. Here, $\mathcal{G}_{\alpha\beta} = \vec{m}_s\cdot(\nabla_\alpha\vec{m}_s\times\nabla_\beta\vec{m}_s)$ is directly related to the skyrmion charge, while $\mathcal{D}_{\alpha\beta} = (\nabla_\alpha\vec{m}_s)\cdot(\nabla_\beta\vec{m}_s)$ is a drag term. While the last line contains 2 derivatives, one for slow mode and one fast, the derivative for the fast mode can be switched to the slow mode using (2). It turns out that the terms are proportional to the skyrmion charge. The integral contains the thickness of the ferromagnetic thin film, that is assumed to cancel out among the terms.

III. CONTINUOUS SKYRMION MODEL

We evaluate the Thiele equation (8) using the slow mode that satisfies $\vec{m}_s^2 = 1$ as

$$\vec{m}_s = \hat{e}_3 = \sin\theta\cos\phi\ \hat{x} + \sin\theta\sin\phi\ \hat{y} + \cos\theta\ \hat{z}\ , \qquad (9)$$

where θ and ϕ depends on the coordinates (ρ,φ) or $(x = \rho\cos\varphi, y = \rho\sin\varphi)$ as $\theta(\rho)$ and $\phi(\varphi)$. The fast mode, transverse to the slow mode, can be described as

$$\vec{m}_f = \psi(\rho,\varphi)e^{-i\omega t}\hat{e}_+ + \psi^*(\rho,\varphi)e^{i\omega t}\hat{e}_-\ , \qquad (10)$$

where $\psi(\vec{r}) = \psi_R(\vec{r}) + i\psi_I(\vec{r})$ and $\hat{e}_\pm = (\hat{e}_1 \pm i\hat{e}_2)/2$ with $\hat{e}_1 = \cos\theta\cos\phi\ \hat{x} + \cos\theta\sin\phi\ \hat{y} - \sin\theta\ \hat{z}$, $\hat{e}_2 = -\sin\phi\ \hat{x} + \cos\phi\ \hat{y}$. Here we focus on the 'circular symmetric' magnon with respect to slow mode $\vec{m}_s$. Thus, $\vec{m}_f = \psi_1\hat{e}_1 + \psi_2\hat{e}_2$ with $\psi_1 = \psi_R(\vec{r})\cos(\omega t) + \psi_I(\vec{r})\sin(\omega t)$ and $\psi_2 = \psi_R(\vec{r})\sin(\omega t) - \psi_I(\vec{r})\cos(\omega t)$. Then, the temporal time average gives $\langle\psi_1^2\rangle_T = \langle\psi_2^2\rangle_T = \langle\psi^2\rangle_T/2$. A temperature gradient is applied along x direction and thus $\nabla_x\langle\psi^2\rangle_T \neq 0$ (and assumed to be constant over the unit cell of the skyrmion) and $\nabla_y\langle\psi^2\rangle_T = 0$.

We consider a single skyrmion that has the angular profile with a finite volume, $0 \le \rho < 2P, 0 \le \varphi < 2\pi$.

$$\theta(\rho) = \begin{cases} \pi \ , & 0 < \rho < P - \frac{\omega_{\rm DW}}{2} \\ \frac{\pi}{2} - \frac{\rho - P}{\omega_{\rm DW}}\pi \ , & P - \frac{\omega_{\rm DW}}{2} \le \rho \le P + \frac{\omega_{\rm DW}}{2} \\ 0 \ , & P + \frac{\omega_{\rm DW}}{2} < \rho < 2P \end{cases} \tag{11}$$

and identify $\phi = \varphi$. The second line of (11) describes the domain wall (DW) in the middle range of the skyrmion. Then the Thiele equation reads

$$\begin{aligned} 0 &= 4\pi\epsilon_{\alpha\beta}v_\beta + \alpha(1 - \langle\psi^2\rangle)_T)^{1/2}\delta_{\alpha\beta}v_\beta \ \pi\Big[\frac{\pi^2 P}{\omega_{\rm DW}} + C_{\omega_{\rm DW}}\Big] \\ &- \hat{x}\ (J\gamma)\ \frac{\pi}{2}\Big[\log\big[\frac{2P}{\rho_{min}}\big] - C_{\omega_{\rm DW}}\Big]\cdot\frac{\nabla_x\langle\psi^2\rangle_T}{(1-\langle\psi^2\rangle_T)^{1/2}} \\ &+ \hat{y}\ (2D\gamma)\ \Big[\frac{\pi^2 P}{4}\Big]\cdot\frac{\nabla_x\langle\psi^2\rangle_T}{(1-\langle\psi^2\rangle_T)^{1/2}}\ . \end{aligned} \tag{12}$$

The first line of (12) contains the well known skyrmion Hall effect term, transverse to the direction of motion, and the longitudinal drag term, that depends on the details of the skyrmion structure and also the magnon density. The skyrmion with skinny DW region has bigger drag effect compared to the skyrmion with fat DW region. Here, $C_{\omega_{\rm DW}} = \int_{P-\omega_{\rm DW}/2}^{P+\omega_{\rm DW}/2}(\sin^2\theta/\rho)d\rho$ only depends on the ratio $P/\omega_{\rm DW}$ and is much smaller than the other contribution as we see below. Analytic result of $C_{\omega_{\rm DW}}$ is listed in (B5) in the appendix §B. We also note that the second line of (8) vanishes as they are circularly symmetric except the gradient ∇_α, which picks a direction along $\alpha = x, y$ with an integrating factor $\sin\varphi$ or $\cos\varphi$ for φ integral.

This analytic result (12) also contains the force density contributions that is parallel and perpendicular to the gradient of magnon density. The latter is the same as the direction of applied temperature gradient. The second line of (8) is the combination of two terms that are involved with J with $\log[2P/\rho_{min}] - C_{\omega_{\rm DW}} > 0$. Detailed computations are provided in §B. Note that the $-\nabla_x\langle\psi^2\rangle$ tells that the direction of force is opposite to the gradient of magnon density, the direction of magnon motion. We also note that it depends on the details of skyrmion structure, the area $2\pi P\omega_{\rm DW}$ of domain wall region of the skyrmion.

The last term in (12) is the sought contribution that is transverse to the direction of the temperature gradient. The magnitude is directly proportional to $2\pi P$, the circumference of the middle of circular skyrmion, independent of the thickness of the DW inside the skyrmion. We advertise that this is an explicit example of Hall viscosity contribution that is originated from the term $-D\vec{m}\cdot(\vec{\nabla}\times\vec{m})$ in Hamiltonian which breaks the parity symmetry. Note that the term is generated through the interaction between the skyrmion and magnon that is assumed to have effects along the direction of temperature gradient, $\nabla_x\langle\psi^2\rangle$.

Now solving (12) gives

$$\begin{aligned} v_x &= \frac{\tilde{\alpha}A + 4\pi B}{(4\pi)^2 + \tilde{\alpha}^2}\frac{\nabla_x\langle\psi^2\rangle_T}{(1-\langle\psi^2\rangle_T)^{1/2}}\ , \\ v_y &= \frac{4\pi A - \tilde{\alpha}B}{(4\pi)^2 + \tilde{\alpha}^2}\ \frac{\nabla_x\langle\psi^2\rangle_T}{(1-\langle\psi^2\rangle_T)^{1/2}}\ , \end{aligned} \tag{13}$$

where $\tilde{\alpha} = \alpha(1-\langle\psi^2\rangle)_T)^{1/2}\pi\cdot C_\alpha$ with $C_\alpha = \frac{\pi^2 P}{\omega_{\rm DW}} + C_{\omega_{\rm DW}}$, $A = (\pi/2)\gamma J\cdot C_A$ with $C_A = \log\big[\frac{2P}{\rho_{min}}\big] - C_{\omega_{\rm DW}}$ and $B = (\pi^2/2)\gamma DP$. C_α and C_A only depends on the details of the skyrmion profile. We can see that the terms $\tilde{\alpha}, A, B$ are sensitive to the details of the skyrmion structure. In particular $\tilde{\alpha}$ only depends on the ratio P/ω_{WD}. On the other hand, A depends on the size and the cutoff of lower radial integral domain of the skyrmion, while B only depends on the size. From the analytic form of the first equation in (13), the longitudinal velocity along the temperature gradient behaves $v_x \propto \nabla_x\langle\psi^2\rangle \propto \nabla_x T$ as its coefficient is positive and magnon density is proportional to temperature. Thus the skyrmion [20–22] and Domain Wall [24] move toward the hotter region .

IV. HALL ANGLES

The Hall angle can be obtained from (13) as $\tan\theta_H = v_y/v_x = (4\pi A - \tilde{\alpha}B)/(\tilde{\alpha}A + 4\pi B)$, where the magnon contributions and details of driving force cancel out. Here we choose $\langle\psi^2\rangle_T \ll 1$. Instead of estimating the Hall angle with a set of particular parameter values, we rewrite the Hall angle in terms of the parameters J/D, α and the parameters depending on skyrmion structure, P, C_α, C_A.

$$\tan\theta_H = \frac{4C_A(J/D) - \pi\alpha C_\alpha P}{\alpha C_\alpha C_A(J/D) + 4\pi P}\ . \tag{14}$$

We note that the Hall angle depends on J and D only through the ratio J/D. For a reasonable range of parameters, $C_\alpha \sim \mathcal{O}(10)$ and $C_A \sim \mathcal{O}(1)$. For example, $C_\alpha = 23.71$, $C_A = 4.39$ for $P = 50\ nm, \omega_{WD} = 21\ nm$ and $\rho_{min} = 1\ nm$.

We consider the Hall viscosity dependence on the three parameters with ranges as $10^{-12}m < J/D < 10^{-6}m, P \sim 10^{-8}m, 10^{-5} < \alpha < 10^{-2}$ for a continuum skyrmion model. Note that the denominator in (14) is roughly order of 10^{-7}–$10^{-6}\ m$ due to the fixed value $4\pi P \sim 10^{-7}\ m$ [30]. Thus, the Hall angle is large for a large portion of the parameter space (note that the plot has a log scale). It changes steeply when $J/D \sim P$ and $\tan\theta_H \sim C_A(J/D)/(\pi P)$. The Hall angle (in degree) is depicted in Fig. 1 with the values of J/D and α in log-log scales with base 10, e.g. $Log[\alpha] = -2$ for $\alpha = 10^{-2}$. Note that the Hall angle strongly depends on the combination J/D, increasing steeply around $J/D \sim 10^{-7}\ m$.

The result tells that the skyrmion size has a significant role in Hall angle, especially when $\alpha\cdot(J/D)$ is smaller than the skyrmion size P, while the angle is almost independent of α. With a smaller size skyrmion,

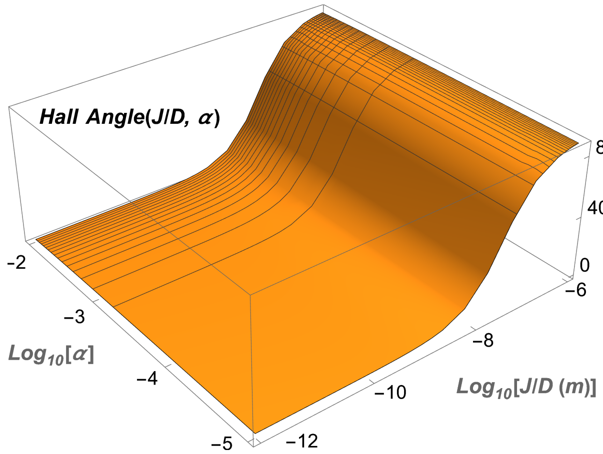

FIG. 1. Hall angle with Log-Log scales for $(J/D, \alpha)$

$P = 10\ nm, \omega_{WD} = 5\ nm$ and $\rho_{min} = 1\ nm$, the Hall angle increases steeply around $J/D \sim 10^{-8}\ m$, an order of smaller value of J/D.

When the skyrmion has an opposite topological charge, the Thiele equation changes the signs of the transverse terms, the first term with $\epsilon_{\alpha\beta}$ and the last term that is proportional to D. Thus the velocities (13) are the same with a relative sign for v_y, and the corresponding Hall angle (14) also changes sign. This is expected as both the terms, the Magnus force and the DM term, are related to the magnetic properties, meaning to be involved with a charged transport rather than neutral one.

Now, it is interesting to see whether the broken parity symmetry in a charged sector also breaks the parity in a neutral sector. Rephrase differently, is there a neutral Hall viscosity effect or neutral momentum transport (in addition to the momentum driven by charged dynamics) even in this case at hand, either directly or through some interactions between the charged and neutral sectors? We discuss this in the following section.

V. ASYMMETRIC SKYRMION HALL ANGLE?

We saw that the skyrmions with positive and negative charges have the same Hall angles (of course with opposite directions). It is interesting to achieve asymmetric skyrmion Hall angles. Some available experimental data appears to show discrepancies between these Hall angles [15, 27]. The asymmetric Hall angles provide direct ways to verify the existence of Hall viscosity as their values are different from expected ones [11] as well as they meet in different point compared to an expected on in the parameter space [12][13]. Moreover, it will shed a new light on how to achieve the skyrmion motion without transverse motion. See some ways achieving vanishing Hall angle by considering antiferromagnetically exchange-coupled bilayer system [28], by using two different DM interactions [29], or at angular momentum compensation point [27].

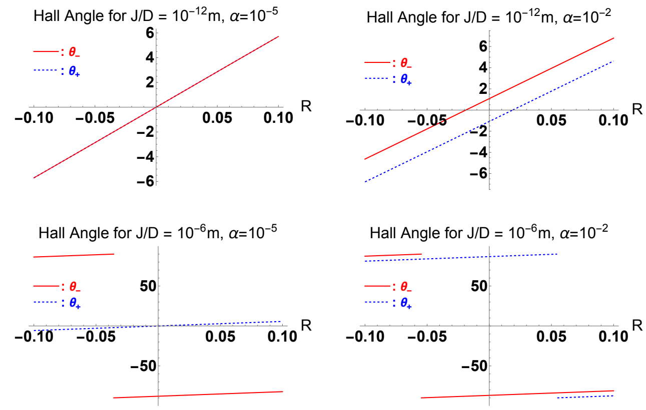

FIG. 2. Hall angles $\theta_\pm(R)$ for $-0.1 < R < 0.1$

Any moving object has a momentum regardless of its charge and is under the influence of broken parity once it is broken either at the fundamental level or through some interactions. Here we proceed to provide a way to describe the Hall viscosity that directly depends on skyrmion velocity and is independent of the skyrmion charge. This can be done by generalizing the skyrmion center collective coordinates $r_\alpha(t)$ for $\vec{m}_s(r_\alpha(t))$ as

$$r_\alpha = v_\alpha t + R\epsilon_{\alpha\beta} v_\beta t \ , \tag{15}$$

which gives $\dot{\vec{m}}_s = (v_\alpha + R\epsilon_{\alpha\beta}v_\beta)\nabla_\alpha \vec{m}_s$ instead of $\dot{\vec{m}}_s = v_\beta \nabla_\beta \vec{m}_s$. Then, the Thiele equation (12) is changed with two more terms by replacing v_α to $v_\alpha + R\epsilon_{\alpha\beta}v_\beta$. Solving the modified Thiele equation gives

$$v_x = \frac{(\tilde{\alpha} \mp 4\pi R)A + (4\pi \pm \tilde{\alpha} R)B}{(16\pi^2 + \tilde{\alpha}^2)(1+R^2)} \cdot \frac{\nabla_x \langle \psi^2 \rangle_T}{(1 - \langle \psi^2 \rangle_T)^{1/2}} \ ,$$
$$v_y = \frac{(\pm 4\pi + \tilde{\alpha} R)A + (\mp\tilde{\alpha} + 4\pi R)B}{(16\pi^2 + \tilde{\alpha}^2)(1+R^2)} \cdot \frac{\nabla_x \langle \psi^2 \rangle_T}{(1 - \langle \psi^2 \rangle_T)^{1/2}} \ .$$

Here we include both the cases with the positive $+1$ and negative -1 skyrmion charges. This equation reduces to (13) for positive charge $+1$ when $R = 0$.

The generalized Hall angle, in terms of $J/D, \alpha, P$, is

$$\tan\theta_\pm = \frac{[\pm 4 + \alpha C_\alpha R] C_A(J/D) + \pi[\mp \alpha C_\alpha + 4R]P}{[\alpha C_\alpha \mp 4R] C_A(J/D) + \pi[4 \pm \alpha C_\alpha R]P} \ . \tag{16}$$

Previous estimates for R is order of 10^{-2} [11]. Moreover, R in (15) is dimensionless and is expected to be small. To see its effect, we study the Hall angle for the range, $-0.1 < R < 0.1$. The result (16) shows that the parameter R only mixes with the combination of number 4 and αC_α. Thus the effects of R enhances when α increases. This can be checked in the top two plots in Fig. 2, where the two Hall angles $\theta_\pm$ almost coincide in the top left inset with $\alpha = 10^{-5}$ compared to the right one with $\alpha = 10^{-2}$.

We also notice that there can be a jump of the Hall angle from $\theta = 90^o$ to -90^o when $\alpha C_\alpha < 4R$ and the denominator of (16) flips sign passing through 0. It is surprising that this can happen with a relatively small value of α. This can be checked in the bottom plots in Fig. 2. In the bottom left inset with $J/D = 10^{-6}m, \alpha = 10^{-5}$, this jump happens for $\alpha = -0.037$ for θ_+ while there is no jump for θ_- for the studied range. In the bottom right inset with $J/D = 10^{-6}m, \alpha = 10^{-2}$, this jumps happen for $\alpha = -0.055$ for θ_- and for $\alpha = 0.055$ for θ_+.

VI. OUTLOOK

A Hall viscosity term, transverse to the driving force, e.g. temperature gradient, is identified for the skyrmion motion. It depends on the strength of DM interaction and size of the skyrmion, but is independent of skyrmion velocity. It is nothing but the parity-symmetry breaking DM interaction term, after time averaging the circular symmetric magnon contribution. In real materials with temperature gradient, megnons cannot be perfectly circular symmetric. Thus in practice, the strength of the Hall viscosity term is expected to be bigger than our estimates. In this sense, this Hall viscosity term is universal and plays role as far as DM interaction and magnon contributions are involved.

We showed that this DM Hall viscosity term depends on the skyrmion charge. Thus all the transverse terms, Magnus force and the viscosity, in the Thiele equation depend on the skyrmion charge. It is expected as the two parity breaking sources, magnetic field and DM term, are related to magnetic properties. The Hall viscosity related to this charged dynamics is quite interesting. On the other hand, it is also tempting to look into the possibility for the Hall viscosity effect for neutral dynamics. Is there neutral Hall viscosity effect when the parity symmetry is broken by magnetic field and DM interaction? This is an interesting question along with much recent investigation related to charge neutral Hall viscosity.

We introduced the neutral Hall viscosity effect, that is proportional to the magnitude of skyrmion velocity (with transverse direction), by the parameter R. We propose that the neutral Hall viscosity is independent of the skyrmion charge and thus provides an asymmetry for the skyrmon Hall effect. We checked that the bigger the Gilbert damping parameter α, the bigger the neutral Hall viscosity effect. For the parameter range where the skyrmion Hall is larger ~ 80, there can be a surprising signature for the Hall angle: the Hall angles can jump from $\theta = 90^o$ to $\theta = -90^o$ as R varies. This can be also achieved by changing other parameters including J/D and α, which is clear in (16). It will be interesting to verify this behavior experimentally.

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Appendix A: Evaluating LLG equation

Let us compute the second term in the LLG equation (4). The variation of Hamiltonian with a vector $\vec{m}$ cam be done in terms of index notation. For example, the DM interaction term in Hamiltonian can be evaluated as

$$H_{\text{eff},i} = -\frac{\delta(D\epsilon_{jlm} m_j \nabla_l m_m)}{\delta m_i} = -D\epsilon_{ilm}\nabla_l m_m + D\epsilon_{jli}\nabla_l m_j = -2D\epsilon_{ilm}\nabla_l m_m \ , \tag{A1}$$

up to a total derivative term. Here the index i is a vector index, and we used the epsilon tensor for the cross product $(\vec{\nabla}_\alpha \times \vec{m})_i = \epsilon_{ilm}\nabla_l m_m$. We write the expansion of the effective field $\vec{H}_{\text{eff}}$

$$\begin{aligned}
\vec{H}_{\text{eff}} &= J(\nabla_\alpha^2 \vec{m}) - 2D\vec{\nabla}_\alpha \times \vec{m} + \vec{H} \\
&= J\Big\{(1-m_f^2)^{1/2}\nabla_\alpha^2 \vec{m}_s - \frac{(m_f^i \nabla_\alpha m_f^i)}{(1-m_f^2)^{1/2}}\nabla_\alpha \vec{m}_s - \nabla_\alpha\big(\frac{(m_f^i \nabla_\alpha m_f^i)}{(1-m_f^2)^{1/2}}\big)\vec{m}_s + \nabla_\alpha^2 \vec{m}_f\Big\} \\
&\quad - 2D\Big\{(1-m_f^2)^{1/2}\vec{\nabla}_\alpha \times \vec{m}_s + \vec{\nabla}_\alpha \times \vec{m}_f - \frac{(m_f^i \vec{\nabla}_\alpha m_f^i)}{(1-m_f^2)^{1/2}} \times \vec{m}_s\Big\} + \vec{H} \ .
\end{aligned} \tag{A2}$$

The term $\vec{m} \times \vec{H}_{\text{eff}}$ after the temporal averaging becomes

$$\begin{aligned}
&\vec{m} \times \vec{H}_{\text{eff}} \approx (1 - \langle m_f^2\rangle_T)^{1/2}\vec{m}_s \times \vec{H} \\
&+ J\Big\{(1-\langle m_f^2\rangle_T)\ \vec{m}_s \times \nabla^2 \vec{m}_s - (1/2)(\nabla_\alpha \langle m_f^2\rangle_T)(\vec{m}_s \times \nabla_\alpha \vec{m}_s) + \langle \vec{m}_f \times \nabla_\alpha^2 \vec{m}_f\rangle_T\Big\} \\
&- 2D\Big\{(1-\langle m_f^2\rangle_T)\ \vec{m}_s \times (\vec{\nabla}\times\vec{m}_s) - (1/2)\vec{m}_s \times (\vec{\nabla}\langle m_f^2\rangle_T \times \vec{m}_s) + \langle \vec{m}_f \times (\vec{\nabla}\times\vec{m}_f)\rangle_T\Big\} \ .
\end{aligned} \tag{A3}$$

Here several terms involved with two derivatives on $\vec{m}_f$ drop out as $\vec{m}_s \times \vec{m}_s = 0$, and the terms with odd powers of $\vec{m}_f$ also drop out with temporal averaging. We note that the result is valid for all orders of $\vec{m}_f$ after the temporal averaging with the magnon profile (10). We also note that the term proportional to $(-1/2)(\nabla_\alpha\langle \vec{m}_f^2\rangle_T)(\vec{m}_s \times \nabla_\alpha \vec{m}_s)$ in (A3) is canceled out by a contribution that is from the other term $\langle \vec{m}_f \times \nabla_\alpha^2 \vec{m}_f\rangle_T$ for circular symmetric magnons. Of course, these two terms cannot be canceled out for realistic situation such as the magnos under temperature gradient.

For the rest of this section, we look into this by introducing some magnon currents that are discussed in literature, e.g., [19–21, 24]. The last term in the second line of (A3) can be rewritten as $J\ \langle \vec{m}_f \times \nabla_\alpha^2 \vec{m}_f\rangle_T = J\ \nabla_\alpha \vec{j}_\alpha^f$, where $\vec{j}_\alpha^f = \vec{m}_f \times \nabla_\alpha \vec{m}_f$ is perpendicular to $\vec{m}_f$ and is a tensor with spin index and also space index α. This spin current $\vec{j}_\alpha^f = \vec{m}_f \times \nabla_\alpha \vec{m}_f$ can be decomposed into components parallel and perpendicular to $\vec{m}_s$ as $\vec{j}_\alpha^f = \vec{m}_s[\vec{j}_\alpha^f \cdot \vec{m}_s] + (\vec{m}_s \times \vec{m}_f)[\vec{j}_\alpha^f \cdot (\vec{m}_s \times \vec{m}_f)]/m_f^2$. After a little of algebra, we get

$$\vec{m}_f \times \nabla_\alpha \vec{m}_f = \vec{m}_s[(\vec{m}_f \times \nabla_\alpha \vec{m}_f)\cdot \vec{m}_s] + \vec{m}_s \times [\vec{m}_f(\nabla_\alpha \vec{m}_s \cdot \vec{m}_f)] \ . \tag{A4}$$

We also use $\nabla_\alpha(\vec{m}_s \cdot \vec{m}_f) = \nabla_\alpha \vec{m}_s \cdot \vec{m}_f + \nabla_\alpha \vec{m}_f \cdot \vec{m}_s = 0$. Note that one cannot discard $(\nabla_\alpha \vec{m}_s)\cdot \vec{m}_f$ while keeping $(\nabla_\alpha \vec{m}_f)\cdot \vec{m}_s$ by arguing that $\vec{m}_s$ is slowing varying compared to the fast mode $\vec{m}_f$ as discussed in the introduction.

We evaluate the second term in (A4) for the circular magnons with the wave function (10). We use $\vec{\nabla}\vec{m}_s = [\hat{\rho}\partial_\rho\theta + (\hat{\varphi}/\rho)\partial_\varphi\theta]\ \hat{e}_1 + [\hat{\rho}\partial_\rho\phi + (\hat{\varphi}/\rho)\partial_\varphi\phi]\sin\theta\ \hat{e}_2 = \hat{\rho}\partial_\rho\theta\ \hat{e}_1 + (\hat{\varphi}/\rho)\partial_\varphi\phi\sin\theta\ \hat{e}_2$. Then,

$$\begin{aligned}\langle\vec{m}_f(\vec{\nabla}_\alpha\vec{m}_s\cdot\vec{m}_f)\rangle_T &= \langle\psi_1^2\rangle_T\hat{\rho}\partial_\rho\theta\ \hat{e}_1 + \langle\psi_2^2\rangle_T(\hat{\varphi}/\rho)\partial_\varphi\phi\sin\theta\ \hat{e}_2\ , \\ &= \frac{1}{2}\langle\vec{m}_f^2\rangle_T\ \vec{\nabla}_\alpha\vec{m}_s\ ,\end{aligned} \tag{A5}$$

where we use $\langle\psi_1^2\rangle_T = \langle\psi_2^2\rangle_T = \langle\psi^2\rangle_T/2 = \langle\vec{m}_f^2\rangle_T/2$. In the right hand side of the first line, ∂_ρ and ∂_φ are parts of the vector $\vec{\nabla}_\alpha$. Then, $\langle\vec{m}_f\times\nabla_\alpha\vec{m}_f\rangle_T = \vec{m}_s(\langle\vec{m}_s\cdot(\vec{m}_f\times\nabla_\alpha\vec{m}_f)\rangle_T) + (\langle\vec{m}_f^2\rangle_T/2)\ (\vec{m}_s\times\ \vec{\nabla}\vec{m}_s)$. Thus,

$$\begin{aligned}J\ \langle\vec{m}_f\times\nabla_\alpha^2\vec{m}_f\rangle_T &= J\ \nabla_\alpha\langle\vec{m}_f\times\nabla_\alpha\vec{m}_f\rangle_T \\ &= J\ j_\alpha(\nabla_\alpha\vec{m}_s) + J\ \langle\vec{m}_f\cdot\nabla_\alpha\vec{m}_f\rangle_T\ (\vec{m}_s\times\vec{\nabla}_\alpha\vec{m}_s) + J\ (\langle\vec{m}_f^2\rangle_T/2)\ (\vec{m}_s\times\vec{\nabla}_\alpha^2\vec{m}_s)\ ,\end{aligned} \tag{A6}$$

where we use $\nabla_\alpha j_\alpha = 0$ and define the magnon current as

$$j_\alpha = \langle\vec{m}_s\cdot(\vec{m}_f\times\nabla_\alpha\vec{m}_f)\rangle_T\ . \tag{A7}$$

The last term in (A6) can be added to $\vec{H}^{0s}_{\text{eff}}$ that is defined below (7). The LLG equation for the slow mode reads

$$\begin{aligned}\dot{\vec{m}}_s = &-\gamma\vec{m}_s\times\vec{H}^s_{\text{eff}} + (1-\langle m_f^2\rangle)^{1/2}\cdot\alpha\vec{m}_s\times\dot{\vec{m}}_s \\ &-\frac{\gamma}{(1-\langle m_f^2\rangle)^{1/2}}\Big\{-J\ \langle\vec{m}_f\cdot\nabla_\alpha\vec{m}_f\rangle_T\ (\vec{m}_s\times\vec{\nabla}_\alpha\vec{m}_s) + J\ j_\alpha(\nabla_\alpha\vec{m}_s) + J\ \langle\vec{m}_f\cdot\nabla_\alpha\vec{m}_f\rangle_T\ (\vec{m}_s\times\vec{\nabla}_\alpha\vec{m}_s) \\ &\qquad - 2D\ \langle\vec{m}_f\times(\nabla\times\vec{m}_f)\rangle_T + 2D\ \vec{m}_s\times[(\vec{\nabla}\langle m_f^2\rangle_T)\times\vec{m}_s\Big\}\ .\end{aligned} \tag{A8}$$

Here, $\vec{H}^s_{\text{eff}}$ contains the terms with derivatives only on the slow mode, and slightly modified as $\vec{H}^s_{\text{eff}} = \vec{H} + J\frac{1-\langle m_f^2\rangle/2}{(1-\langle m_f^2\rangle)^{1/2}}\ (\nabla^2\vec{m}_s) - 2D(1-\langle m_f^2\rangle)^{1/2}(\vec{\nabla}\times\vec{m}_s)$. Note that the first and last terms in the second line cancel each other. Thus the Thiele equation simplifies for the circular symmetric magnons. For more realistic magnons in the presence of temperature gradient, the magnon profile cannot be completely circular symmetric as it transfers energy from the hot region to the cold region. This means $\langle\psi_1^2\rangle_T \neq \langle\psi_2^2\rangle_T$, and thus (A5) does not holds.

Appendix B: Evaluating Thiele equation

In this appendix, we show explicit computations for contracting $(\vec{m}_s\times\nabla_\beta\vec{m}_s)\cdot$ on the LLG equation (7), with a dot product acting on the spin vector index, to get (8), followed by the computation leading to (12) using the local coordinates listed in (9) and (10). We write again the LLG equation (7) for the slow mode so that the appendix can be read in self-contained manner.

$$\begin{aligned}0 = &-\dot{\vec{m}}_s + (1-\langle m_f^2\rangle)^{1/2}\cdot\alpha\vec{m}_s\times\dot{\vec{m}}_s \\ &-\gamma\vec{m}_s\times\Big\{\vec{H} + J(1-\langle m_f^2\rangle)^{1/2}\ (\nabla^2\vec{m}_s) - 2D(1-\langle m_f^2\rangle)^{1/2}(\vec{\nabla}\times\vec{m}_s)\Big\} \\ &-\frac{\gamma}{(1-\langle m_f^2\rangle)^{1/2}}\Big\{J\ \langle\vec{m}_f\times\nabla_\alpha^2\vec{m}_f\rangle_T - J\ \langle m_f^i\nabla_\alpha m_f^i\rangle_T[\vec{m}_s\times\nabla_\alpha\vec{m}_s] \\ &\qquad - 2D\ \langle\vec{m}_f\times(\nabla\times\vec{m}_f)\rangle_T + 2D\ \vec{m}_s\times[(\vec{\nabla}\langle m_f^2\rangle_T)\times\vec{m}_s\Big\}\ .\end{aligned} \tag{B1}$$

The first line of (B1) contains the terms with $\dot{\vec{m}}_s$, a time derivative of the slow mode. Then

$$(\vec{m}_s\times\nabla_\alpha\vec{m}_s)\cdot\dot{\vec{m}}_s = \epsilon_{ijk}(m_s^j\nabla_\alpha m_s^k)\dot{m}_s^i = \vec{m}_s\cdot(\nabla_\alpha\vec{m}_s\times\nabla_\beta\vec{m}_s)v_\beta \equiv \mathcal{G}_{\alpha\beta}v_\beta\ , \tag{B2}$$

where ϵ_{ijk} is the totally antisymmetric tensor with $\epsilon_{123} = 1$. $\dot{\vec{m}}_s = v_\alpha\nabla_\alpha\vec{m}_s$ with a space index $\alpha,\beta = x,y$ or $1,2$.

$$\begin{aligned}(\vec{m}_s\times\nabla_\alpha\vec{m}_s)\cdot(\vec{m}_s\times\dot{\vec{m}}_s) &= \epsilon_{ijk}\epsilon_{ilm}(m_s^j\nabla_\alpha m_s^k)(m_s^l\nabla_\beta m_s^m)\ v_\beta \\ &= m_s^2\ (\nabla_\alpha\vec{m}_s\cdot\nabla_\beta\vec{m}_s)v_\beta - (\vec{m}_s\cdot\nabla_\alpha\vec{m}_s)(\vec{m}_s\cdot\nabla_\beta\vec{m}_s)v_\beta = (\nabla_\alpha\vec{m}_s\cdot\nabla_\beta\vec{m}_s)\ v_\beta \equiv \mathcal{D}_{\alpha\beta}v_\beta\ ,\end{aligned}$$

where we use $m_s^2 = 1$, $\vec{m}_s \cdot \nabla_\alpha \vec{m}_s = \nabla_\alpha m_s^2/2 = 0$, and the identity $\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$. These two terms are in the first line of (8). The second line of (B1) contains the usual terms without the magnon contribution with modification with the magnitude of magnons $(1 - \langle m_f^2 \rangle)^{1/2}$. Using $(\vec{m}_s \times \nabla_\alpha \vec{m}_s) \cdot (\vec{m}_s \times \vec{A}) = (\vec{A} \cdot \nabla_\alpha m_s) - (\vec{m}_s \cdot \vec{A})(\vec{m}_s \cdot \nabla_\alpha \vec{m}_s) = (\vec{A} \cdot \nabla_\alpha m_s)$, we arrive the second line of (8).

At this point, we evaluate the first and second lines of (8) by integrating over a unit skyrmion size. We use the local coordinates (9), (10) and the skyrmion profile (11). $\mathcal{G}_{\alpha\beta}$ is anti-symmetric ($\mathcal{G}_{xx} = \mathcal{G}_{yy} = 0$) and we only evaluate $\mathcal{G}_{xy}$.

$$\begin{aligned}\int dV \mathcal{G}_{xy} v_y &= \int_0^{2P} \rho d\rho \int_0^{2\pi} d\varphi \ \hat{e}_3 \cdot (\nabla_x \hat{e}_3 \times \nabla_y \hat{e}_3) v_y = \int_0^{2P} \rho d\rho \int_0^{2\pi} d\varphi \ \frac{1}{\rho} (\partial_\rho \theta(\rho))(\partial_\varphi \phi(\varphi)) \sin\theta(\rho) v_y \\ &= -2\pi \int_0^{2P} d\rho (\partial_\rho \cos\theta(\rho) v_y = -2\pi \cos\theta(\rho)\Big|_0^{2P} v_y = 4\pi v_y \ , \end{aligned} \tag{B3}$$

where we use

$$\begin{aligned} \nabla_x \hat{e}_3 &= (\partial_x \rho)(\partial_\rho \theta)\hat{e}_1 + (\partial_x \varphi)(\partial_\varphi \phi) \sin\theta \hat{e}_2 \ , \\ \nabla_y \hat{e}_3 &= (\partial_y \rho)(\partial_\rho \theta)\hat{e}_1 + (\partial_y \varphi)(\partial_\varphi \phi) \sin\theta \hat{e}_2 \ , \\ \mathcal{G}_{xy} &= \hat{e}_3 \cdot (\nabla_x \hat{e}_3 \times \nabla_y \hat{e}_3) = \{(\partial_x \rho)(\partial_y \varphi) - (\partial_x \varphi)(\partial_y \rho)\}(\partial_\rho \theta)(\partial_\varphi \phi) \sin\theta = \frac{1}{\rho}(\partial_\rho \theta)(\partial_\varphi \phi) \sin\theta \ , \end{aligned}$$

and $\theta = \theta(\rho), \phi(\varphi) = \varphi$, $x = \rho\cos\varphi, y = \rho \sin\varphi$.

The symmetric term with $\mathcal{D}_{\alpha\beta} = \nabla_\alpha \vec{m}_s \cdot \nabla_\beta \vec{m}_s$ can be evaluated similarly. For example the term with $\mathcal{D}_{xx}$ is

$$\begin{aligned} &\alpha \int dV (1 - \langle m_f^2 \rangle)^{1/2} (\nabla_x \vec{m}_s \cdot \nabla_x \vec{m}_s) \ v_x \\ &= \alpha v_x \ (1 - \langle \psi^2 \rangle)^{1/2} \int_0^{2P} \rho d\rho \int_0^{2\pi} d\varphi \left\{ \cos^2\varphi (\partial_\rho \theta)^2 + \sin^2\varphi \frac{\sin^2\theta}{\rho^2} \right\} \\ &= \alpha v_x \ (1 - \langle \psi^2 \rangle)^{1/2} \times \pi \left\{ \int_{P-\omega_{\rm DW}/2}^{P+\omega_{\rm DW}/2} \rho d\rho \left(\frac{\pi}{\omega_{\rm DW}} \right)^2 + \int_{P-\omega_{\rm DW}/2}^{P+\omega_{\rm DW}/2} d\rho \frac{\sin^2\theta}{\rho} \right\} \\ &= \alpha v_x \ (1 - \langle \psi^2 \rangle)^{1/2} \times \pi \left\{ \frac{\pi^2 P}{\omega_{\rm DW}} + C_{\omega_{\rm DW}} \right\} \ , \end{aligned} \tag{B4}$$

where we assume that the magnon profile does not change much over the unit skyrmion size, so the factor $(1-\langle\psi^2\rangle)^{1/2}$ can come out of the integral. We also use $\partial_\rho \hat{e}_3 = \partial_\rho \theta(\rho)\hat{e}_1$ and $\partial_\varphi \hat{e}_3 = \sin\theta(\rho)\hat{e}_2$ as $\partial_\varphi \phi = 1$ with the identification $\phi = \varphi$. Then, $\mathcal{D}_{xx} = \nabla_x \hat{e}_3 \cdot \nabla_x \hat{e}_3 = ((\nabla_x \rho)(\partial_\rho \theta) \ \hat{e}_1 + (\nabla_x \varphi)(\partial_\varphi \phi) \sin\theta \ \hat{e}_2)^2 = \cos^2\varphi (\partial_\rho \theta)^2 + \frac{\sin^2\varphi}{\rho^2} \sin^2\theta$.

$$\begin{aligned} C_{\omega_{\rm DW}} = \int_{P-\omega_{\rm DW}/2}^{P+\omega_{\rm DW}/2} d\rho \frac{\sin^2\theta}{\rho} &= \frac{1}{2} \log \frac{(2P + \omega_{\rm DW})}{(2P - \omega_{\rm DW})} + \frac{1}{2} \cos\left(\frac{2\pi P}{\omega_{\rm DW}}\right) \left\{ {\rm Ci}\left(\frac{2\pi P}{\omega_{\rm DW}} + \pi\right) - {\rm Ci}\left(\frac{2\pi P}{D} - \pi\right) \right\} \\ &+ \frac{1}{2} \sin\left(\frac{2\pi P}{\omega_{\rm DW}}\right) \left\{ {\rm Si}\left(\frac{2\pi P}{\omega_{\rm DW}} + \pi\right) - {\rm Si}\left(\frac{2\pi P}{\omega_{\rm DW}} - \pi\right) \right\} . \end{aligned}$$

Here the integral range is reduced to the domain wall inside the skyrmion as $\sin 0 = \sin\pi = 0$. This $C_{\omega_{\rm DW}}$ is typically much smaller compared to the first term $\pi^2 P/\omega_{\rm DW}$. It depends only on the combination $P/\omega_{\rm DW}$ and decreases as the skyrmion DW region gets skinnier (meaning increasing $P/\omega_{\rm DW}$).

The three terms in the second line of (8) vanish when evaluating with the local coordinates for $\alpha = x$ or y.

$$\begin{aligned} \gamma \int dV \ (\nabla_x \vec{m}_s) \cdot \vec{H} &= \gamma \int dV \ \big((\partial_x \rho)(\partial_\rho \theta)\hat{e}_1 + (\partial_x \varphi)(\partial_\varphi \phi) \sin\theta \hat{e}_2 \big) \cdot \hat{z} H \\ &= -\gamma H \int dV \ (\partial_x \rho)(\partial_\rho \theta) \sin\theta \propto \int_0^{2\pi} d\varphi \sin\varphi = 0 \ , \end{aligned} \tag{B5}$$

$$-\gamma J (1 - m_f^2)^{1/2} \int dV (\nabla_x \vec{m}_s \cdot \nabla^2 \vec{m}_s) = \int dV \ (\partial_x \rho)(\cdots) = (\text{terms independent of } \varphi) \times \int_0^{2\pi} d\varphi \sin\varphi = 0 \ , \tag{B6}$$

$$2D\gamma (1 - m_f^2)^{1/2} \int dV (\nabla_x \vec{m}_s \cdot \vec{\nabla} \times \vec{m}_s) = \int dV \ (\partial_x \varphi)(\cdots) = (\text{terms independent of } \varphi) \times \int_0^{2\pi} d\varphi \sin\varphi = 0 \ . \tag{B7}$$

Here the first term is evaluated for constant magnetic field along $\hat{z}$ direction $\vec{H} = \hat{z}H$ along with $\hat{e}_1 \cdot \hat{z} = \sin\theta, \partial_x \rho = \cos\varphi$. The second term also vanishes similarly as $\nabla^2 \vec{m}_s = \{(\partial_\rho \theta)/\rho + \partial_\rho^2 \theta - \sin\theta\cos\theta/\rho^2\}\hat{e}_1 - \{(\partial_\rho\theta)^2 + \sin^2\theta/\rho^2\}\hat{e}_3$, and thus $\nabla_x \vec{m}_s \cdot \nabla^2 \vec{m}_s = (\partial_x \rho)(\partial_\rho \theta)\{(\partial_\rho\theta)/\rho + \partial_\rho^2\theta - \sin\theta\cos\theta/\rho^2\} = \cos\varphi \times$ (terms independent of φ). The third term is evaluated using

$$\begin{aligned}\vec{\nabla} \times \vec{m}_s &= (\hat{\rho}\partial_\rho + \frac{\hat{\varphi}}{\rho}\partial_\varphi) \times \hat{e}_3 = (\partial_\rho\theta)\hat{\rho} \times \hat{e}_1 + \frac{\sin\theta(\partial_\varphi\phi)}{\rho}\hat{\varphi} \times \hat{e}_2 \\ &= (\partial_\rho\theta)(\cos\theta\hat{e}_1 + \sin\theta\hat{e}_3) \times \hat{e}_1 + \frac{\sin\theta(\partial_\varphi\phi)}{\rho}\hat{e}_2 \times \hat{e}_2 = (\partial_\rho\theta)\sin\theta\ \hat{e}_2\ ,\end{aligned} \tag{B8}$$

and thus $(\nabla_x \vec{m}_s \cdot \vec{\nabla} \times \vec{m}_s) = (\partial_x\varphi)(\partial_\varphi\phi)(\partial_\rho\theta)\sin^2\theta =$ (terms independent of φ) $\times (-\sin\varphi)$. Here we used $d\varphi = (\cos\varphi dy - \sin\varphi dx)/\rho$ by taking a derivative on both sides of $\tan\varphi = y/x$.

Until this point, we show that the first two lines of the Thiele equation (8) collapsed into the first line of (12). We turn to consider the terms that are proportional to J in (7).

Next we study the term, $\frac{\gamma(J)}{(1-\langle m_f^2\rangle)^{1/2}}\ \langle m_f^i \nabla_\alpha m_f^i\rangle_T[\vec{m}_s \times \nabla_\alpha \vec{m}_s]$, in LLG equation (7) to arrive the corresponding term in (12). By contracting $(\vec{m}_s \times \nabla_\beta \vec{m}_s)$ with a dot product acting on the spin vector $[\vec{m}_s \times \nabla_\alpha \vec{m}_s]$

$$(\vec{m}_s \times \nabla_\beta \vec{m}_s) \cdot [\vec{m}_s \times \nabla_\alpha \vec{m}_s] = (\nabla_\beta \vec{m}_s \cdot \nabla_\alpha \vec{m}_s) - (\vec{m}_s \cdot \nabla_\beta \vec{m}_s)(\vec{m}_s \cdot \nabla_\alpha \vec{m}_s]\ .$$

The last term vanishes as $\nabla_\alpha m_s^2 = 0$ and thus $\vec{m}_s \cdot \nabla_\alpha \vec{m}_s = 0$.

Now we integrate the term over a unit volume of the skyrmion $\gamma(J)\int dV \frac{(\vec{m}_f\cdot\nabla_\beta\vec{m}_f)(\nabla_\beta\vec{m}_s\cdot\nabla_\alpha\vec{m}_s)}{(1-\langle m_f^2\rangle)^{1/2}}$ with $\mathcal{D}_{\beta\alpha} = \nabla_\beta \vec{m}_s \cdot \nabla_\alpha \vec{m}_s$. We consider the magnon profile has non-zero contribution along x direction, for example with temperature gradient along x direction, so that $\nabla_x\langle\psi^2\rangle \neq 0$, while $\nabla_y\langle\psi^2\rangle = 0$ in the 2 dimensional plane with coordinates (x, y). Then

$$\langle \vec{m}_f \cdot \nabla_\beta \vec{m}_f\rangle_T = (1/2)\nabla_x\langle\psi^2\rangle\ .$$

We also can check that $\mathcal{D}_{\alpha\beta} \propto \delta_{\alpha\beta}$. And

$$\begin{aligned}\mathcal{D}_{xx} &= \nabla_x\hat{e}_3 \cdot \nabla_x\hat{e}_3 = ((\nabla_x\rho)(\partial_\rho\theta)\ \hat{e}_1 + (\nabla_x\varphi)(\partial_\varphi\phi)\sin\theta\ \hat{e}_2)^2 \\ &= \cos^2\varphi(\partial_\rho\theta)^2 + \frac{\sin^2\varphi}{\rho^2}\sin^2\theta\ ,\end{aligned} \tag{B9}$$

where we use $\partial_\rho\hat{e}_3 = \partial_\rho\theta(\rho)\hat{e}_1$ and $\partial_\varphi\hat{e}_3 = \sin\theta(\rho)\hat{e}_2$ as $\partial_\varphi\phi = 1$ with the identification $\phi = \varphi$.

$$\begin{aligned}&\gamma(J)\int dV \frac{(\vec{m}_f \cdot \nabla_\beta \vec{m}_f)(\nabla_\beta \vec{m}_s \cdot \nabla_\alpha \vec{m}_s)}{(1-\langle m_f^2\rangle)^{1/2}} \\ &= \hat{x}\ \gamma(J)\ \frac{\nabla_x\langle\psi^2\rangle/2}{(1-\langle\psi^2\rangle)^{1/2}}\int_0^{2P}\rho d\rho\int_0^{2\pi}d\varphi\left\{\cos^2\varphi(\partial_\rho\theta)^2 + \frac{\sin^2\varphi}{\rho^2}\sin^2\theta\right\} \\ &= \hat{x}\ \gamma(J)\ \frac{\nabla_x\langle\psi^2\rangle}{(1-\langle\psi^2\rangle)^{1/2}} \times \frac{\pi}{2}\left\{\pi^2\frac{P}{\omega_{\rm DW}} + C_{\omega_{\rm DW}}\right\}\ ,\end{aligned}$$

where

$$\begin{aligned}C_{\omega_{\rm DW}} = \int_{P-\omega_{\rm DW}/2}^{P+\omega_{\rm DW}/2} d\rho\frac{\sin^2\theta}{\rho} &= \frac{1}{2}\log\frac{(2P+\omega_{\rm DW})}{(2P-\omega_{\rm DW})} + \frac{1}{2}\cos\left(\frac{2\pi P}{\omega_{\rm DW}}\right)\left\{{\rm Ci}\left(\frac{2\pi P}{\omega_{\rm DW}} + \pi\right) - {\rm Ci}\left(\frac{2\pi P}{D} - \pi\right)\right\} \\ &+ \frac{1}{2}\sin\left(\frac{2\pi P}{\omega_{\rm DW}}\right)\left\{{\rm Si}\left(\frac{2\pi P}{\omega_{\rm DW}} + \pi\right) - {\rm Si}\left(\frac{2\pi P}{\omega_{\rm DW}} - \pi\right)\right\}\ .\end{aligned}$$

Here we evaluate the slow and fast modes independently by assuming $\nabla_x\langle\psi^2\rangle_T$ is constant over the unit skyrmion volume.

Similarly, the other term $\gamma(J)\int dV\frac{[\vec{m}_s\cdot\nabla^2\vec{m}_f][\vec{m}_f\cdot\nabla_\alpha\vec{m}_s]}{(1-\langle m_f^2\rangle)^{1/2}}$ that contributes to x direction can be evaluated. Using

$$\vec{m}_s \cdot \nabla^2\vec{m}_f = -\frac{1}{\rho}\Big\{2\rho(\partial_\rho\psi_1)(\partial_\rho\theta) - (\partial_\rho\theta)\psi_1 + \rho(\partial_\rho^2\theta)\Big\} + \frac{1}{\rho^2}\Big\{(\partial_\varphi\phi)\sin\theta\cos\theta\psi_1 + 2\sin\theta(\partial_\rho\psi_2)\Big\}\ , \tag{B10}$$

$$\vec{m}_f \cdot \nabla_\beta\vec{m}_s = (\delta_{\beta\rho})\hat{\rho}(\partial_\rho\theta)\psi_1 + (\delta_{\beta\varphi})(\hat{\varphi}/\rho)\sin\theta\psi_2\ .,\tag{B11}$$

and after taking the temporal coarse graining, we get

$$\langle[\vec{m}_s \cdot \nabla^2 \vec{m}_f][\vec{m}_f \cdot \nabla_\alpha \vec{m}_s]\rangle_T = -(\delta_{\alpha\rho})\hat{\rho}(\partial_\rho\theta)^2\Big\{\partial_\rho\langle\psi_1^2\rangle_T + \frac{\psi_1^2}{\rho} + \frac{\sin\theta\cos\theta}{\rho^2}\psi_1^2\Big\} - (\delta_{\alpha\varphi})(\hat{\varphi})\frac{\sin\theta}{\rho}\partial_\varphi\langle\psi_2^2\rangle_T$$
$$= -\hat{x}\Big\{(\partial_\rho\theta)^2\nabla_x\langle\psi_1^2\rangle\cos^2\varphi + \nabla_x\langle\psi_2^2\rangle\frac{\sin^2\varphi}{\rho^2}\Big\}\,. \tag{B12}$$

This can be decomposed into x and y coordinates. By keeping the contributions for $\nabla_x\langle\psi_1^2\rangle = \nabla_x\langle\psi_2^2\rangle = \nabla_x\langle\psi^2\rangle/2$, we get

$$\begin{aligned}
&\gamma(J)\int dV\,\frac{[\vec{m}_s \cdot \nabla^2 \vec{m}_f][\vec{m}_f \cdot \nabla_\alpha \vec{m}_s]}{(1-\langle m_f^2\rangle)^{1/2}}\\
&= -\hat{x}\;\gamma(J)\;\frac{\nabla_x\langle\psi^2\rangle/2}{(1-\langle\psi^2\rangle)^{1/2}}\int_0^{2P}\rho d\rho\int_0^{2\pi}d\varphi\left\{\cos^2\varphi(\partial_\rho\theta)^2 + \frac{\sin^2\varphi}{\rho^2}\right\}\\
&= -\hat{x}\;\gamma(J)\;\frac{\nabla_x\langle\psi^2\rangle}{(1-\langle\psi^2\rangle)^{1/2}}\times\frac{\pi}{2}\left\{\pi^2\frac{P}{\omega_{\rm DW}} + \log\Big(\frac{2P}{\rho_{min}}\Big)\right\}\;,
\end{aligned}$$

Here $4\pi P^2 = \int_0^{2P}\rho d\rho\int_0^{2\pi}d\varphi$ is the skyrmion unit volume.

By combining these two contributions, we get

$$\hat{x}\;\frac{\gamma(J)\;\nabla_x\langle\psi^2\rangle_T}{(1-\langle\psi^2\rangle)_T}\Big\{\frac{\pi}{2}[\frac{\pi^2P}{\omega_{\rm DW}} + C_{\omega_{\rm DW}}] - \frac{\pi}{2}[\frac{\pi^2P}{\omega_{\rm DW}} + \log\Big(\frac{2P}{\rho_{min}}\Big)]\Big\}$$
$$= -\hat{x}\;\frac{\gamma(J)\;\nabla_x\langle\psi^2\rangle_T}{(1-\langle\psi^2\rangle)_T}\frac{\pi}{2}\Big\{\log\Big(\frac{2P}{\rho_{min}}\Big) - C_{\omega_{\rm DW}}\Big\}\,. \tag{B13}$$

From this analytic result, we see that the longitudinal force on skyrmion due to the magnons depends on the details of skyrmion structure, in particular the area of domain wall region of the skyrmion, and is opposite direction to the magnon density change.