A UNIT THEOREM FOR PRODUCTS OF GROUPS WITH SEVERAL ENDS AND APPLICATIONS

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Abstract. In his 1994 survey [25], Kleinert defined formally and formulated the problem to obtain unit theorems for unit groups of orders in a semisimple algebra A. If A is a group algebra FG, it boils down to classifying all finite groups G such that the unit groups of most orders in FG belong to a prescribed class $\mathcal{G}$ of infinite groups. We solve this problem for $\mathcal{G}$ consisting of the groups which are virtually a direct product of groups whose Cayley graph has more than one end. Subsequently, we obtain two types of applications. A first type being about the existence of torsion-free normal complements and a second about obtaining short and uniform proofs of some of the main results in [13, 18, 14, 15].

1. Unit theorems and the virtual structure problem

1.1. **Historical background of the problem.** The interest in structure theorems for unit groups of orders goes back to the dawn of ring theory via for example Dirichlet's Unit Theorem for ring of integers of number fields and to the pioneering work of Higman [9] in case of group algebras. Such results are called 'unit theorems' and a concrete meaning was formulated and asked for in Kleinert's 1994 survey [25]:

> "A unit theorem for a finite-dimensional semisimple rational algebra Λ consists of the definition, in purely group-theoretical terms, of a class of groups $C(\Lambda)$ such that almost all generic unit groups of Λ are members of $C(\Lambda)$."

Recall that a generic unit group of Λ is a subgroup of finite index in the group of elements of reduced norm 1 of an order[1] in Λ.

An important feature of orders in semisimple algebras is that their unit groups are commensurable [17, Lemma 4.6.9]. Therefore, in the case that the class $C(\Lambda)$ is closed under commensurability, a generic unit group will be in $C(\Lambda)$ if and only if the unit group of one specific order is in $C(\Lambda)$. For instance, in the case that $\Lambda = FG$ is a group algebra with F a number field, one could choose the order RG with R the ring of integers of F. In this setting, Kleinert's question boils down to the 'Virtual Structure Problem' which was formulated explicitly for the first time around 2000 in [16]:

Problem (Virtual Structure Problem)**.** Let $\mathcal{G}$ be a class of groups. Classify the finite groups G and the number fields F such that $\mathcal{U}(RG)$ has a subgroup of finite index which belongs to the class $\mathcal{G}$.

Up to our knowledge, until quite recently the only classes of groups $\mathcal{G}$ for which the virtual structure problem was known were the following:

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[1] In this article an order refers to a $\mathbb{Z}$-order, but the question also makes sense for more general orders.

- $\mathcal{G}_{fin} = \{$ finite groups $\}$, [17, Corollary 5.5.8]
- $\mathcal{G}_{ab} = \{$ abelian groups $\}$, [9]
- $\mathcal{G}_{solv} = \{$ solvable groups $\}$, [23, Theorem 2]
- $\mathcal{G}_{fbf} = \{$ direct product of free-by-free groups $\}$ and certain subclasses ([20, 16, 28, 21]),

The class $\mathcal{G}_{fbf}$ was the culmination of various works superseding each other. For instance, prior to $\mathcal{G}_{fbf}$ and building on [15, 20, 26, 19], Jespers and Del Rìo answered in [16] the virtual structure problem for

$$\mathcal{G}_{pab} = \Big\{ \prod_i A_{i,1} * \cdots * A_{i,t_i} \mid A_{i,j} \text{ are finitely generated abelian } \Big\},$$

where $t_i = 1$ is allowed (i.e. an abelian factor). It turns out that the classification coincides with that of the class of products of free groups (throughout, $\mathbb{Z}$ is considered to be free).

The answer for $\mathcal{G}_{fbf}$ already dates back to 2007 and skepticism grew whether there were other unit theorems. Recently, new classes were achieved through the point of view of geometric group theory. Concretely, the virtual structure problem was answered for classes of groups satisfying some fixed point property such as property (T) or HFA [2, 1].

Remarkably, in all these cases there is also a characterization in terms of the group algebra. For example, for $F = \mathbb{Q}$ and $\mathcal{G}_{fbf}$, one has that all generic unit groups of FG are virtually in $\mathcal{G}_{fbf}$ if and only if every simple factor of FG is either a field, a totally definite quaternion algebra or $\mathrm{M}_2(K)$, where K is either $\mathbb{Q}$, $\mathbb{Q}(i)$, $\mathbb{Q}(\sqrt{-2})$ or $\mathbb{Q}(\sqrt{-3})$.

1.2. **Contributions to the Virtual Structure Problem.** In this article, we will obtain new unit theorems. Foremost, we answer the virtual structure problem for the class

$$\mathcal{G}_\infty := \Big\{ \prod_i \Gamma_i \mid \Gamma_i \text{ has infinitely many ends } \Big\}.$$

Recall that given a finitely generated group Γ, the *number of ends* $e(\Gamma)$ is the infimum of the number of infinite connected components of $\mathrm{Cay}(\Gamma, S) \setminus F$, for F going through finite subsets of $\mathrm{Cay}(\Gamma, S)$. This number $e(\Gamma)$ does not depend on the choice of a generating set S for Γ.

It is well known that for a finitely generated group Γ, the number of ends $e(\Gamma)$ is either 0, 1, 2, or ∞. By Stallings' theorem [32, 31] a group has infinitely many ends if and only if it can be decomposed as an amalgamated product or an HNN extension, over a finite group. In fact, we will mainly work with this characterization. It is known that commensurable groups have equal number of ends, see [31, p. 38]. Hence resolving the virtual structure problem for $\mathcal{G}_\infty$ also yields a unit theorem in the sense of Kleinert.

Despite that the class $\mathcal{G}_\infty$ is much larger than the class

$$\Big\{ \prod_i \Gamma_i \mid \Gamma_i \text{ is a non-cyclic free group} \Big\},$$

we will prove that the virtual structure problem for them has the same answer. We will say that a group *virtually belongs to* $\mathcal{G}$ or *is virtually-*$\mathcal{G}$, if it has a subgroup of finite index belonging to $\mathcal{G}$. Also, recall that a group G is said to be a *cut group* if $\mathcal{Z}(\mathcal{U}(\mathbb{Z}G))$ is finite.

Theorem 1.1. *Let G be a finite group and F be a number field with ring of integers R. The following are equivalent.*

(i) Γ *is virtually-*$\mathcal{G}_\infty$ *for all orders $\mathcal{O}$ in FG and all finite-index subgroups $\Gamma \leq \mathcal{U}(\mathcal{O})$.*

(ii) $F = \mathbb{Q}$ *and* $\mathcal{U}(RG)$ *is virtually-*$\mathcal{G}_\infty$.

(iii) *All simple components of* FG *are of the form* $\mathbb{Q}(\sqrt{-d})$ *with* $d \in \mathbb{N}$, $\left(\frac{-a,-b}{\mathbb{Q}}\right)$ *with* $a, b \in \mathbb{N} \setminus \{0\}$, *or* $\mathrm{M}_2(\mathbb{Q})$, *and the latter occurs at least once.*

If so, G *is a cut group and only* $(-1,-1)$ *and* $(-1,-3)$ *can occur for* $(-a,-b)$.

Using assertion (iii) of Theorem 1.1 in terms of simple components of FG and [16, Proposition 1.1], we see that the answer for the class $\mathcal{G}_\infty$ coincide with the answer for the class obtained in [26, Theorem 1].

Corollary 1.2. *Let* G *be a finite group and* R *the ring of integers of a number field. The following are equivalent.*

(i) $\mathcal{U}(RG)$ *is virtually in* $\mathcal{G}_\infty$.

(ii) $\mathcal{U}(RG)$ *is virtually in* $\left\{ \prod_i \Gamma_i \mid \Gamma_i \text{ is a non-cyclic free group} \right\}$.

In [20, 26] the finite groups satisfying assertion (iii) in Theorem 1.1 have been classified. Therefore Theorem 1.1 indeed answers the Virtual Structure problem for the class $\mathcal{G}_\infty$.

Remark. That the groups G such that $\mathcal{U}(RG)$ is virtually a direct product of non-abelian free groups have to be cut follows from the classification obtained in [26, Theorem 1], as all components mentioned in Theorem 1.1 have a maximal order with finite center. However the characterization in Corollary 1.2 via $\mathcal{G}_\infty$ gives a classification-free proof of that fact, since it is a general property of virtually-$\mathcal{G}_\infty$ groups to have finite center (see Lemma 2.1).

As mentioned earlier, the number of ends $e(\Gamma)$ is either 0, 1, 2, or ∞. In the remainder of the paper we describe when $e(\mathcal{U}(RG)) = 0, 2$ or ∞.

First, by definition, $e(\Gamma) = 0$ if and only if Γ is finite. Therefore, Higman's theorem [17, Theorem 1.5.6.] translates to:

$$e(\mathcal{U}(\mathbb{Z}G)) = 0 \text{ if and only if } \begin{cases} G \text{ is abelian with } \exp(G) \mid 4, 6 \\ \text{or} \\ G \cong Q_8 \times C_2^n \text{ for some } n \end{cases} \tag{1.1}$$

Using [11, Theorem 4.1] one can formulate the variant of (1.1) for twisted group rings $\mathcal{U}(R^\alpha G)$ over an arbitrary ring of integers R.

Next we describe the case that $e(\mathcal{U}(RG)) = \infty$. In the upcoming result we use the notation $D_{2n} = \langle a, b \mid a^n = 1 = b^2, a^b = a^{-1} \rangle$, $Dic_3 = \langle a, b \mid a^6, a^3 = b^2, a^b = a^{-1} \rangle$ and $C_4 \rtimes C_4 = \langle a, b \mid a^4 = b^4 = 1, a^b = a^{-1} \rangle$.

Theorem 1.3. *Let* G *be a finite group and* F *be a number field with ring of integers* R. *Then for a finite index subgroup* Γ *of* $\mathcal{U}(RG)$ *the following are equivalent:*

(i) $e(\Gamma) = \infty$.

(ii) $F = \mathbb{Q}$ *and* $\mathcal{U}(RG)$ *is virtually free.*

(iii) $F = \mathbb{Q}$ *and* G *is isomorphic to* D_6, D_8, Dic_3, *or* $C_4 \rtimes C_4$.

That $e(\Gamma) = \infty$ is equivalent with the unit group of the group ring being virtually free is readily proven using the methods used in the proof of Theorem 1.1. The most innovative part of the proof of Theorem 1.3 concerns the fact that $\mathcal{U}(\mathbb{Z}G)$ is virtually free for exactly the four groups cited. The latter equivalence is already known since [15, 16], however we will give a new and short proof of this fact using amalgamated product machinery. In particular, Theorem 1.3 should rather be viewed as an application.

Finally, recall that $e(\Gamma) = 2$ if and only if Γ has a subgroup of finite index isomorphic to $\mathbb{Z}$, see [10, 7] or [31, p. 38]. For $\Gamma = \mathcal{U}(RG)$ this happens for only few cases.

Proposition 1.4. *Let G be a finite group, F a number field and R its ring of integers. The following are equivalent:*

(i) *$e(\Gamma) = 2$ for some finite-index subgroup Γ in the unit group of an order $\mathcal{O}$ in FG.*

(ii) *$\mathcal{U}(RG)$ is $\mathbb{Z}$-by-finite and $F = \mathbb{Q}(\sqrt{-d})$ for $d \in \mathbb{N}$.*

(iii) *G is isomorphic to C_n with $n = 5, 8$ or 12 and $F = \mathbb{Q}(\zeta_d)$ with $d \nmid n$.*

Remark. In light of Theorem 1.1 and Proposition 1.4, it would be natural to consider the class

$$\mathcal{G}_{\neq 1} := \Big\{ \prod_i \Gamma_i \mid e(\Gamma_i) \neq 1 \Big\}.$$

In fact, with similar methods one can prove for a finite group G that

$$\mathcal{U}(\mathbb{Z}G) \text{ is virtually-}\mathcal{G}_{\neq 1} \iff \mathcal{U}(\mathbb{Z}G) \text{ is virtually-}\mathcal{G}_{pab}. \tag{1.2}$$

1.3. **Further applications to normal complements.** As mentioned earlier, Theorem 1.3 can be considered as a first application of our methods and the infinite number of ends property. Our next application concerns normal complements.

After a use of the equivalence (i) $\Leftrightarrow$ (ii) in Theorem 1.3, part (ii) of Theorem 1.5 below is in fact the main theorem of [15]. More precisely, the proof in [15] first classifies the finite groups G such that $V(\mathbb{Z}G)$ is virtually free (see (iii) in Theorem 1.3). Additionally, the papers [13, 18, 14] construct case-by-case a normal complement for the trivial units. We provide a new proof which is uniform, in the sense that it does not require to classify all possible G and in particular does not involve a case-by-case study. This is possible thanks to our characterization via $\mathcal{G}_\infty$.

Theorem 1.5. *Let G be a finite group, F a number field and R its ring of integers. Then the following hold:*

(i) *If $\mathcal{U}(RG)$ is virtually-$\mathcal{G}_\infty$, then for each $e \in \mathrm{PCI}(FG)$ such that FGe is not a division algebra, there is a free normal subgroup N_e of $\mathcal{U}(RG)e$ such that*

$$V(RG)e \cong N_e \rtimes Ge.$$

(ii) *If $e(\mathcal{U}(RG)) = \infty$, then there exists a normal free subgroup F_m in $\mathcal{U}(RG)$ such that*

$$V(RG) \cong F_m \rtimes G.$$

(iii) *If $\mathcal{U}(RG)$ is virtually-$\mathcal{G}_\infty$ and H a finite subgroup of $V(RG)$, then $(He)^{\alpha_e} \leq Ge$ for some $\alpha_e \in \mathcal{U}(RGe)$.*

Part (iii) of Theorem 1.5 is a third Zassenhaus conjecture type of statement. In the upcoming work [3], more applications of Theorem 1.1 to the "blockwise Zassenhaus conjecture" will be investigated, that is, to the question whether He is conjugated inside $\mathcal{U}(RGe)$ to a subgroup of Ge, for H a finite subgroup of $V(RG)$ and e a primitive central idempotent of FG.

Remark 1.6. It follows from the proof of Theorem 1.5 and [11, Corollary 10.14] that the subgroup N_e in (i) is generated by the H-units recently introduced in [11].

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2. Proofs of the statements

2.1. **Proof of VSP for products of groups with infinitely many ends.** The remainder of the paper is dedicated to the proof of Theorem 1.1 We first need to record some general facts on the number of ends and to be virtually-$\mathcal{G}_\infty$.

Lemma 2.1. *Let Γ_1, Γ_2 and Γ be finitely generated groups.*

1. *If Γ_1 and Γ_2 are commensurable, then $e(\Gamma_1) = e(\Gamma_2)$. Furthermore, Γ_1 is virtually-$\mathcal{G}_\infty$ if and only if Γ_2 is virtually-$\mathcal{G}_\infty$.*
2. *If $e(\Gamma) = \infty$ and Γ is commensurable with $\Gamma_1 \times \Gamma_2$, then Γ_1 or Γ_2 is finite.*
3. *If Γ is virtually-$\mathcal{G}_\infty$, then $\mathcal{Z}(\Gamma)$ is finite.*

The first part in particular applies to $\Gamma_i = \mathcal{U}(\mathcal{O}_i)$ for two orders $\mathcal{O}_i$ in a finite dimensional semisimple algebra A.

With " Γ is commensurable with $\Gamma_1 \times \Gamma_2$" we mean the following: Γ_1 and Γ_2 are normal in Γ, the intersection $\Gamma_1 \cap \Gamma_2$ is finite and $\Gamma_1\Gamma_2$ is of finite index in Γ.

Proof. That the number of ends is constant on commensurability classes is classical and can be found in [31, p. 38]. Now we prove that being virtually-$\mathcal{G}_\infty$ is also a property shared by commensurable groups. By definition, $\Gamma_1 \cap \Gamma_2$ is a common subgroup of finite index and hence it is enough to obtain the desired statement for the case that Γ_2 is a subgroup of finite index in Γ_1. In that case the $\mathcal{G}_\infty$ subgroup of finite index in Γ_2 is also of finite index in Γ_1. Thus assume that Γ_1 is virtually-$\mathcal{G}_\infty$. Denote by $H = \prod_{j=1}^m H_j$ a subgroup of finite index in Γ_1 with $e(H_j) = \infty$ for all $1 \leq j \leq m$. Then $H \cap \Gamma_2 = \prod_{j=1}^m (H_j \cap \Gamma_2)$ is of finite index in Γ_2. Moreover, $H_j \cap \Gamma_2$ is of finite index in H_j and hence by the first part $e(H_j) = e(H_j \cap \Gamma_2)$, which shows that Γ_2 is virtually-$\mathcal{G}_\infty$ via $H \cap \Gamma_2$.

For the second statement note that the combination of the first part and the assumption yields that $e(\Gamma_1 \times \Gamma_2) = \infty$. However, the Cayley graph of a direct product is the cartesian product of the Cayley graphs. Using this one can see that the number of ends of a direct product of infinite finitely generated groups is always one, a contradiction.

For the third part, consider a subgroup $H = \prod_{j=1}^m H_j$ of finite index in Γ with $e(H_j) = \infty$ for $1 \leq j \leq m$. First remark that $\mathcal{Z}(H_j)$ is finite for each j. Indeed by Stallings' theorem [32, 31] H_j can be decomposed as a non-trivial amalgamated product or HNN extension over a finite group C. In such decomposition one necessarily has that $\mathcal{Z}(H_j) \leq C$. Therefore also $\mathcal{Z}(H_j)$ is finite, as claimed. Consequently, $\mathcal{Z}(H)$ is finite. Now, since $H \cap \mathcal{Z}(\Gamma) \leq \mathcal{Z}(H)$ is of finite index in $\mathcal{Z}(\Gamma)$ we obtain that $\mathcal{Z}(\Gamma$, as desired.

Finally, unit groups of orders are commensurable by [17, Lemma 4.6.9] and hence indeed the first statement applies to them. □

Next we need the following lemma, which is a generalization of [16, Proposition 4.5].

Lemma 2.2. *Let G be a finite group, D be a finite dimensional division algebra over F with $\mathrm{char}(F) = 0$, different[2] of $\left(\frac{-2,-5}{\mathbb{Q}}\right)$, and suppose $\mathrm{M}_n(D)$ with $n \geq 2$ is a simple component of FG. If $\mathcal{O}$ is an order in $\mathrm{M}_n(D)$, then $e(\mathcal{U}(\mathcal{O})) = \infty$ if and only if $n = 2$ and $D = F = \mathbb{Q}$.*

[2]This condition is not necessary, i.e the number of ends of $GL_2(\mathcal{O})$ for $\mathcal{O}$ an order in $\left(\frac{-2,-5}{\mathbb{Q}}\right)$ is not infinite. However including this case would make the proof unnecessarily lengthy.

Proof. Suppose $e(\mathcal{U}(\mathcal{O})) = \infty$. Lemma 2.1 allows to assume without lose of generality that $\mathcal{O}$ is a maximal order in $\mathrm{M}_n(D)$. It is well known that in that case $\mathcal{O} \cong \mathrm{M}_n(\mathcal{O}_{max})$ with $\mathcal{O}_{max}$ a maximal order in D.

Next recall that any group with infinitely many ends has finite center (as central elements need to be in the subgroup over which the amalgam and HNN are constructed, which is finite in this case). Therefore, $\mathrm{SL}_n(\mathcal{O}_{max})$ has finite index in $\mathrm{GL}_n(\mathcal{O}_{max})$ and hence $\mathrm{SL}_n(\mathcal{O}_{max})$ also has infinitely many ends. This implies that $\mathrm{SL}_n(\mathcal{O}_{max})$ has S-rank 1, with S the set of infinite places, as otherwise it has hereditarily Serre's property FA (even property (T) [27, 6]) and hence can not have a non-trivial amalgam or HNN splitting.

The S-rank being one means that $n = 2$ and D is either $\mathbb{Q}(\sqrt{-d})$, with $d \geq 0$ or $\left(\frac{-a,-b}{\mathbb{Q}}\right)$ with a, b strictly positive integers (see for instance [2, Theorem 2.10.]). Furthermore it was proven in [5] that the condition that $\mathrm{M}_2(D)$ is a component of a group algebra yields that $d \in \{0, -1, -2, -3\}$ and $(a, b) \in \{(1, 1), (1, 3), (2, 5)\}$. All these division algebras are (right norm) Euclidean and due to this have a unique maximal order (see [2, remark 3.13]), which we denote $\mathcal{O}_D$. By assumption $(a, b) = (2, 5)$ does not occur. Now, following [2, Theorem 5.1] $\mathrm{GL}_2(\mathcal{O}_D)$ has property FA except if $D = \mathbb{Q}$ or $\mathbb{Q}(\sqrt{-2})$. In case of $D = \mathbb{Q}(\sqrt{-2})$ one can use the amalgam decomposition of $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$ given in [8, Theorem 2.1] to see that the group does not admit a splitting over a finite group. Finally, $\mathrm{GL}_2(\mathbb{Z}) = D_8 *_{C_2 \times C_2} D_{12}$ and hence $e(\mathrm{GL}_2(\mathbb{Z})) = \infty$, finishing the proof. □

We now proceed with the proof of the main result.

Proof of Theorem 1.1. First observe that Lemma 2.1 entails that if $\Gamma \leq \mathcal{U}(\mathcal{O})$ is virtually-$\mathcal{G}_\infty$ for some order $\mathcal{O}$ in FG, then also all other finite index subgroups of the unit group of any other order in FG is virtually-$\mathcal{G}_\infty$. In particular we obtained:

Claim 0: Statement (i) holds if and only if $\mathcal{U}(RG)$, with R the ring of integers in F, is virtually-$\mathcal{G}_\infty$.

The above claim yields the implication (ii) ⇒ (i) and the converse follows once we prove that virtually-$\mathcal{G}_\infty$ forces F to be the field of rational numbers.

Next, for the remaining of the proof fix a Wedderburn-Artin decomposition $FG = \bigoplus_{i=1}^{q} \mathrm{M}_{n_i}(D_i)$. Furthermore, fix a maximal order $\mathcal{O}_i$ in D_i for each i such that $\mathcal{U}(RG)$ is a subgroup of $\prod_{i=1}^{q} \mathrm{GL}_{n_i}(\mathcal{O}_i)$.

By our starting observation one has that $\mathcal{U}(RG)$ is virtually-$\mathcal{G}_\infty$ if and only if $\prod_{i=1}^{q} \mathrm{GL}_{n_i}(\mathcal{O}_i)$ is. Now if $\mathrm{M}_{n_i}(D_i)$ is either a field or a totally definite quaternion algebra, then $\mathrm{SL}_1(\mathcal{O}_i)$ is finite by [17, Proposition 5.5.6]. Also recall that F is in the center of every simple component of FG. Hence, if $\mathrm{M}_2(\mathbb{Q})$ is a simple component, then $F = \mathbb{Q}$. We now obtain the implication (iii) ⇒ (ii) by combining all the previous with Lemma 2.2.

Now suppose that (i) holds. Equivalently, by claim 0, suppose that $\mathcal{U}(RG)$ is virtually-$\mathcal{G}_\infty$ and therefore also $\prod_{i=1}^{q} \mathrm{GL}_{n_i}(\mathcal{O}_i)$.

Claim 1: $\mathcal{Z}(\mathcal{U}(RG))$ and $\mathcal{Z}(\mathcal{O}_i)$ are finite for all $1 \leq i \leq q$.

This follows from Lemma 2.1 and the virtually-$\mathcal{G}_\infty$ property.

Claim 1 implies that $\mathrm{SL}_{n_j}(\mathcal{O}_j)$ is of finite index in $\mathrm{GL}_{n_j}(\mathcal{O}_j)$ for all $1 \leq j \leq q$. In particular, $e(\mathrm{GL}_{n_j}(\mathcal{O}_j)) = e(\mathrm{SL}_{n_j}(\mathcal{O}_j))$.

Claim 2: $e(\mathrm{SL}_{n_j}(\mathcal{O}_j)) = \infty$ for all j such that $M_{n_j}(D_j)$ is different of a field or totally definite quaternion algebra (e.g. all j for which $n_j \geq 2$).

Let $H = \prod_{j=1}^{m} H_m$ be a subgroup of finite index in $\mathcal{U}(RG)$ such that $e(H_j) = \infty$ for each $1 \leq j \leq m$. Denote $S_j := \mathrm{SL}_{n_j}(\mathcal{O}_j) \cap H$ which is of finite index in $\mathrm{SL}_{n_j}(\mathcal{O}_j)$,

hence it is enough to proof that $e(S_j) = \infty$. Let p_k be the projection of H on H_k. Fix some j as in the statement of claim 2. The condition is equivalent with saying that $\mathrm{SL}_{n_j}(\mathcal{O}_j)$ is infinite [23]. In particular there exists some k such that $p_k(S_j)$ is infinite[3]. For such k we will now prove that $|p_k(\prod_{i\neq j} S_i)| < \infty$. For this consider $S := S_j \times \prod_{i\neq j} S_i$ which by the first claim is of finite index in H. Therefore $p_k(S)$ is of finite index in H_k and hence $e(p_k(S)) = \infty$. Now note that $p_k(S)$ is commensurable with $p_k(S_j) \times p_k(\prod_{i\neq j} S_i)$. Indeed, the two subgroups clearly commute, are normal in $p_k(S)$ and $p_k(S_j) \cap p_k(\prod_{i\neq j} S_i) \subseteq \mathcal{Z}(p_k(S))$ which is finite since $p_k(S)$ has infinitely many ends. Thus we may apply[4] Lemma 2.1 to conclude that $|p_k(\prod_{i\neq j} S_i)| < \infty$.

Now consider the set $\mathcal{I}_j := \{k \mid |p_k(S_j)| < \infty\}$. From the previous it follows that if $k \in \{1, \dots, q\} \setminus \mathcal{I}_j$, then $p_k(S_j)$ is of finite index in H_k. Hence $S_j/(S_j \cap \prod_{i\in\mathcal{I}_j} H_i)$ is a subgroup of finite index in $\prod_{k\notin\mathcal{I}_j} H_k$. As the quotient was with a finite subgroup, we obtain that S_j is virtually-$\mathcal{G}_\infty$ and hence $\mathrm{SL}_{n_j}(\mathcal{O}_j)$ also. However, under the conditions stated in claim 2, SL_1 is virtual indecomposable [24, Theorem 1]. Therefore $\mathrm{SL}_{n_j}(\mathcal{O}_j)$ in fact is even virtually a group with infinitely many ends and so in fact $e(\mathrm{SL}_{n_j}(\mathcal{O}_j)) = \infty$, finishing the proof of claim 2.

Altogether: Claim 1 says that $\mathcal{Z}(\mathcal{O}_i)$ is finite and consequently $\mathrm{SL}_{n_j}(\mathcal{O}_j)$ is of finite index in $\mathrm{GL}_{n_j}(\mathcal{O}_j)$ for all j. In particular, $e(\mathrm{GL}_{n_j}(\mathcal{O}_j)) = \infty$ if $n_j \geq 2$. Now Lemma 2.2 implies that $n_j = 2$, i.e. no higher matrix algebras appear in the decomposition of FG. In such a case no $\left(\frac{-2,-5}{\mathbb{Q}}\right)$ component arises. Indeed, following [2, table appendix] such a component can only arise if $F = \mathbb{Q}$ and G maps onto one of the groups with SmallGroupID [40,3], [240,89] or [240,90]. But a direct verification, e.g. via the Wedderga package in GAP, shows that these groups all have matrix components of reduced degree at least 3.

Consequently, Lemma 2.2 says that all matrix components of FG must be isomorphic to $\mathrm{M}_2(\mathbb{Q})$ and in particular $F = \mathbb{Q}$ (as F is contained in the center of every simple component). As pointed out earlier, together with claim 0 this finishes the proof the equivalence (i) $\Leftrightarrow$ (ii)

Furthermore, by [2, Theorem 2.10. & Proposition 6.11.], if $\mathbb{Q}Ge$ is a division algebra D for some primitive central idempotent e of $\mathbb{Q}G$ then D is $\mathbb{Q}(\sqrt{-d})$ with $d \in \mathbb{Z}_{\geq 0}$ or it is a totally definite quaternion algebra with center $\mathbb{Q}$. In summary, we obtained that all components of $\mathbb{Q}G$ are of the desired form, yielding the remaining implication (i) $\Rightarrow$ (iii).

Finally, that only the parameters $(-1,-1)$ and $(-1,-3)$ appear is due to [33, Theorem 11.5.14] saying that else $\mathcal{U}(\mathcal{O})$ is cyclic for any order $\mathcal{O}$ in $\left(\frac{-a,-b}{\mathbb{Q}}\right)$. In those cases $Ge \leq \mathcal{U}(\mathbb{Z}Ge)$ would have an abelian $\mathbb{Q}$-span and thus $\mathbb{Q}Ge \neq \left(\frac{-a,-b}{\mathbb{Q}}\right)$, a contradiction. □

2.2. **Proofs VSP for a given amount of ends.** Finally, we characterize the case that $\mathcal{U}(RG)$ is virtually a group with infinitely many ends, and not only a product of such.

Proof of Theorem 1.3. Let $FG \cong \prod_{i=1}^{q} \mathrm{M}_{n_i}(D_i)$ be the Wedderburn-Artin decomposition of FG. Further consider maximal orders $\mathcal{O}_i$ in D_i. Thanks to the commensurability of unit groups of orders and of the number of ends, one has that $\mathcal{U}(RG)$ is a

[3]Otherwise S_j would be finite and hence also the overgroup of finite index $\mathrm{SL}_{n_j}(\mathcal{O}_j)$.

[4]Instead of that statement one could have used the well known [31, 4.A.6.3.] saying that infinite finitely generated normal subgroups of a group with infinitely many ends need to have finite index.

subgroup of finite index in $\prod_{i=1}^{q} \mathrm{GL}_{n_i}(\mathcal{O}_i)$ and $e(\Gamma) = e(\mathcal{U}(RG)) = e(\prod_{i=1}^{q} \mathrm{GL}_{n_i}(\mathcal{O}_i))$. However, the Cayley graph of a direct product is the cartesian product of the Cayley graphs. Using this we see that $e(Q \times P) = 1$ for any finitely generated infinite groups P, Q. Therefore $e(\mathcal{U}(RG)) = \infty$ if and only if $e(\mathrm{GL}_{n_{i_0}}(\mathcal{O}_{i_0})) = \infty$ for exactly one i_0 and the other factors are finite. In light of Lemma 2.2 and [2, Theorem 2.10.] this happens exactly when FG has exactly one $\mathrm{M}_2(\mathbb{Q})$ component and all the others are $\mathbb{Q}, \mathbb{Q}(\sqrt{-d})$ or $\left(\frac{-a,-b}{\mathbb{Q}}\right)$. In particular, F must be $\mathbb{Q}$. Furthermore, since $\mathrm{GL}_2(\mathbb{Z})$ is virtually free we see that in those cases $\mathcal{U}(RG)$ is indeed virtually free. This concludes the equivalence (i) $\Leftrightarrow$ (ii) and moreover gives a description in terms of the isomorphism type of the simple factors of FG.

Suppose that G is isomorphic to D_6, D_8, Dic_3 or $C_4 \rtimes C_4$. It easily verified that they have the following Wedderburn-Artin decomposition:

$$\begin{array}{lcl} \mathbb{Q}[D_6] & \cong & \mathbb{Q}^{\times 2} \times \mathrm{M}_2(\mathbb{Q}) \\ \mathbb{Q}[D_8] & \cong & \mathbb{Q}^{\times 4} \times \mathrm{M}_2(\mathbb{Q}) \\ \mathbb{Q}[Dic_3] & \cong & \mathbb{Q}^{\times 2} \times \mathbb{Q}(i) \times \left(\frac{-1,-3}{\mathbb{Q}}\right) \times \mathrm{M}_2(\mathbb{Q}) \\ \mathbb{Q}[C_4 \rtimes C_5] & \cong & \mathbb{Q}^{\times 4} \times \mathbb{Q}(i)^{\times 2} \times \left(\frac{-1,-1}{\mathbb{Q}}\right) \times \mathrm{M}_2(\mathbb{Q}) \end{array}$$

By the description obtained earlier via simple component of FG, we obtain that for all the groups above $e(\mathcal{U}(\mathbb{Z}G)) = \infty$, yielding (iii) $\Rightarrow$ (ii). It remains to prove that these groups are the only one where simple components of $\mathbb{Q}G$ satisfies the necessary restrictions

Recall that the unit group of the maximal orders of $\left(\frac{-1,-1}{\mathbb{Q}}\right)$ and $\left(\frac{-1,-3}{\mathbb{Q}}\right)$ are respectively $\mathrm{SL}(2,3) \cong Q_8 \rtimes C_3$ and Dic_3. As $\mathbb{Q}[\mathrm{SL}_2(\mathbb{F}_3)]$ has a component $\mathrm{M}_3(\mathbb{Q})$, we have that G can not map onto $\mathrm{SL}_2(\mathbb{F}_3)$. Since $\mathrm{SL}_2(\mathbb{F}_3)$ has a unique isomorphism type of non-abelian subgroup, namely Q_8, the group G must map onto Q_8 if $\left(\frac{-1,-1}{\mathbb{Q}}\right)$ is a simple factor of FG. Now recall that $\mathcal{U}(\mathbb{Z}G)$ is contained in $\prod_{i=1}^{q} \mathrm{GL}_{n_i}(\mathcal{O}_i)$ with $\mathrm{GL}_{n_i}(\mathcal{O}_i)$ isomorphic to $\mathrm{GL}_2(\mathbb{Z}) \cong D_{12} *_{C_2 \times C_2} D_8$ or a maximal order in a quaternion algebra or field mentioned earlier. Then, altogether we have that

$$\mathcal{U}(\mathbb{Z}G) \text{ is a subgroup of finite index in } (D_8 \times U) *_{C_2 \times C_2 \times U} (D_{12} \times U) \tag{2.1}$$

where $U = A \times Q_8^s \times Dic_3^t$ for some s, t and with A abelian with $\exp(A) \mid 4, 6$. Using the description of torsion subgroups in amalgamated products we know that, up to conjugation, G is a subgroup of $C_2 \times C_2 \times U$ or it contains transversal elements in the factor D_8 or D_{12}.

First suppose that G is conjugated to a subgroup of the amalgamated part $C_2 \times C_2 \times U$. Recall that all subgroups of Dic_3 and Q_8 are cyclic. From this it is readily concluded that the only way to have exactly one matrix component $\mathrm{M}_2(\mathbb{Q})$ is for G to be Dic_3.

Next, suppose that G is not conjugated to a subgroup of the amalgamated part. Then $\mathcal{U}(\mathbb{Z}G)$ contains a non-trivial amalgamated product $G *_{G \cap (C_2 \times C_2 \times U)} (D_m \times U)$ with $m = 8$ or 12. Now [12, Proposition 2.7] yields an irreducible representation ρ of G of reduced degree at least 2 such that $\ker(\rho)$ is contained in $G \cap (C_2 \times C_2 \times U)$. By assumption on the components of the group algebra, the representation co-restricts to a morphism $G \to \mathrm{GL}_2(\mathbb{Z})$. Note that the image $\rho(G)$ needs to be D_6 or D_8 as D_{12} has two matrix components. From all this we can infer that in order for $\mathbb{Q}G$ to not have more than one matrix component, the intersection $G \cap (C_2 \times C_2 \times U)$ needs to be a central subgroup of order 2. Thus G is a central extension of D_6 or D_8 with a C_2. A look at the classification of groups of order 12 and 16 shows that G is isomorphic to either Dic_3 or $C_4 \times C_4$, finishing the proof. $\square$

Next we prove the characterization of when a unit group has exactly two ends.

Proof of Proposition 1.4. As recalled in the proof of Lemma 2.2, $e(\Gamma_1) = e(\Gamma_2)$ if Γ_1 and Γ_2 are commensurable (see [31, p. 38]). Moreover, the unit group of two orders are commensurable [17, lemma 4.6.9]. Therefore, $e(\Gamma) = 2$ if and only if $e(\mathcal{U}(RG)) = 2$ for R the ring of integers of F. In particular, (ii) implies (i). Furthermore, since for general finitely generated groups Γ one has that $e(\Gamma) = 2$ if and only if Γ is $\mathbb{Z}$-by-finite, we obtain that statement (i) implies that $\mathcal{U}(RG)$ is $\mathbb{Z}$-by-finite.

We will now prove simultaneously that $\mathcal{U}(RG)$ to be $\mathbb{Z}$-by-finite implies the restrictions on F and G mentionned in (ii) and (iii), which would finish the proof. To do so, recall a result by Kleinert [23] (or [17, Corollary 5.5.7]) saying that $\mathcal{U}(RG)$ is abelian-by-finite if and only if all the simple components of FG are either fields or totally definite quaternion algebras. In particular FG has no non-trivial nilpotent elements in which case [30, Theorem 2.6] tells that G is either abelian or $G \cong Q_8 \times C_2^m \times A$ with $m \geq 0$ and A an abelian group of odd order. We first consider the case that $G \cong Q_8 \times C_2^m \times A$. Then

$$FG \cong F[Q_8 \times C_2^m] \otimes_F FA \cong (4m\, F \oplus m \left(\frac{-1,-1}{F}\right)) \otimes_F FA.$$

We now see that in order to obtain a *single* copy of $\mathbb{Z}$ in $\mathcal{U}(RG)$ that this will have to come from a component of FA. However this component will appear at least 4 times and hence such groups are never $\mathbb{Z}$-by-finite.

Now suppose that G is abelian. By the theorem of Perlis-Walker [29, Theorem 3.5.4]

$$FG \cong \bigoplus_{d||G|} a_d \frac{[\mathbb{Q}(\zeta_d) : \mathbb{Q}]}{[F(\zeta_d) : F]} F(\zeta_d) \tag{2.2}$$

with a_d the number of different cyclic subgroups of order d. Now denote by $R_{F,d}$ the ring of integers of $F(\zeta_d)$ and recall that by Dirichlet Unit theorem [17, Theorem 5.2.4] the rank of the finitely generated abelian group $\mathcal{U}(R_{F,d})$ is $n_1 + n_2 - 1$ with n_1 the number of real embeddings of $F(\zeta_d)$ and r_2 the number of pairs of complex embeddings. Note that the rank of $\mathcal{U}(R_{F,d})$ is at least the one of the unit group of the ring of integers of the cyclotomic field $\mathbb{Q}(\zeta_d)$. The latter rank is well-known to be $\frac{\varphi(d)}{2} - 1$. A direct computation yields that $\varphi(d) \leq 4$ if and only if $d \in \{2, 3, 4, 5, 6 8, 10, 12\}$ with equality only for $\{5, 8, 10, 12\}$. This combined with (2.2) we see that we have *exactly one* copy of $\mathbb{Z}$ if and only if G is isomorphic to C_n with $n = 5, 8$ or 12, $\operatorname{rank} \mathcal{U}(R_{F,n}) = \operatorname{rank} \mathcal{U}(R_{\mathbb{Q},n}) = 1$ and $\operatorname{rank} \mathcal{U}(R) = 0$. The latter means that F is either $\mathbb{Q}$ or an imaginary quadratic extension of $\mathbb{Q}$ and hence yields the implication (i) $\Rightarrow$ (ii). For the restriction on F written in (iii) note that due to the values of n the field $\mathbb{Q}(\zeta_d)$ with $d \nmid n$ is either $\mathbb{Q}$ or an imaginary quadratic extension and hence $\operatorname{rank} \mathcal{U}(R) = 0$. Furthermore, $F(\zeta_n) = \mathbb{Q}(\zeta_n)$ and hence $\operatorname{rank} \mathcal{U}(R_{F,n}) = \operatorname{rank} \mathcal{U}(R_{\mathbb{Q},n})$ as desired. □

2.3. **Proof of Theorem 1.5.** For efficiency reasons we will first prove the Zassenhaus conjecture type of property for the projections to the simple components of FG.

Proof of (iii) *of Theorem 1.5.* Supppose that $\mathcal{U}(RG)$ is virtually-$\mathcal{G}_\infty$. Let H be a finite subgroup of $V(RG)$ and $e \in \mathrm{PCI}(FG)$ and $H \leq V(RG)$. Observe that the theorem of Cohn-Livingstone [4, Corollary 2.7] implies that

$$|He| \mid |Ge| \tag{2.3}$$

Indeed, denote by $\Phi : \mathbb{Z}[G] \to \mathbb{Z}[Ge]$ the ring epimorphism associated to e which satisfied $\Phi(V(\mathbb{Z}[G])) \subseteq V(\mathbb{Z}[Ge])$. Thus $|\Phi(H)| \mid |Ge|$. To prove that $|He| \mid |\Phi(H)|$

consider the ring epimorphism $\pi\colon \mathbb{Z}[G] \to \mathbb{Z}[G]e : x \mapsto xe$. One has that $\omega(G,N) \subseteq \ker(\pi)$. It follows that π factorises through $\mathbb{Z}[Ge]$ i.e. there exists a unique morphism $\sigma\colon \mathbb{Z}[Ge] \to \mathbb{Z}[G]e$ such that $\pi = \sigma \circ \Phi$. In particular, the order of $He = \pi(H) = \sigma(\Phi(H))$ divides $|\Phi(H)|$, as desired.

Now we consider the possible isomorphism types for FGe separately. First suppose that FGe is a division algebra, i.e. $\mathbb{Q}(\sqrt{-d})$ or $\left(\frac{-a,-b}{\mathbb{Q}}\right)$ by Theorem 1.1. Then the unit group of its unique maximal order is finite [23] and in particular also $\mathcal{U}(RGe)$ is finite. Hence if FGe is a field, then the latter unit group is cyclic and thus $He \leq Ge$ due to (2.3). If $FGe \cong \left(\frac{-a,-b}{\mathbb{Q}}\right)$, then by [33, Theorem 11.5.14] $\mathcal{U}(RGe)$ is cyclic except if (1) $(a,b) = (-1,-1)$ and RGe the Lipschitz or Hurwitz order (which is maximal) or (2) $(a,b) = (-1,-3)$ and RGe is the maximal order[5] of $\left(\frac{-1,-3}{\mathbb{Q}}\right)$. In case of the maximal orders, one must have that Ge is their unit group and hence $He \leq Ge$. If RGe is the Lipschitz order, then $Ge \cong Q_8$ and He is some subgroup of the unit group of the Hurwitz quaternions, i.e. a subgroup of $\mathrm{SL}_2(\mathbb{F}_3) \cong Q_8 \rtimes C_3$. Via (2.3) we see that in fact $He \leq Ge$, as claimed.

It remains to consider the components where $FGe \cong \mathrm{M}_2(\mathbb{Q})$. In that case, $\mathcal{U}(RG)e$ is a subgroup of finite index in $\mathrm{GL}_2(\mathbb{Z}) \cong D_8 *_{C_2\times C_2} D_{12}$. Therefore there exists some $\alpha_e \in \mathcal{U}(RGe)$ such that $\alpha_e^{-1}Ge\alpha_e$ is a subgroup of D_8 or D_{12}. Since the $\mathbb{Q}$-span of Ge is $\mathrm{M}_2(\mathbb{Q})$ we get that $\alpha_e^{-1}Ge\alpha_e = D_6, D_{12}$ or D_8. In particular $|Ge|$ determines uniquely its isomorphism type. The same holds for He if He is non-abelian. Therefore, by (2.3), we have the desired statement in that case. Suppose now that $|He| = 4$. If $He \cong C_4$, then He is up to conjugation a subgroup of D_8 and due to the dividing of the orders, again $He \leq Ge$ after conjugation. If it is an elementary abelian 2-group, then it is uniquely defined in both D_{12} and D_8. As this subgroup is amalgamated we are done. □

Proof of (i) *and* (ii) *of Theorem 1.5.* Suppose that $U(RG)$ is virtually-$\mathcal{G}_\infty$. For each $e \in \mathrm{PCI}(FG)$ consider a maximal order $\mathcal{O}_e$ of FGe containing RGe. One has that $V(RG)$ is of finite index in $\prod_{e\in\mathrm{PCI}(FG)}\mathcal{U}(\mathcal{O}_e)$. Therefore, $V(RG)e$ is of finite index in $\mathcal{U}(\mathcal{O}_e)$.

Now suppose that FGe is not a division algebra and hence $FGe \cong \mathrm{M}_2(\mathbb{Q})$ by Theorem 1.1. Recall that $\mathrm{M}_2(\mathbb{Z})$ is the unique maximal order of $\mathrm{M}_2(\mathbb{Q})$. Therefore $V(RG)e$ is of finite index in $\mathrm{GL}_2(\mathbb{Z}) \cong D_8 *_{C_2\times C_2} D_{12}$. This entails that Ge is conjugated to $D_{2\ell}$ for $\ell = 3, 4$ or 6. Moreover, by part (iii) all torsion subgroups in $V(RG)e$ are conjugated to a subgroup of Ge. Combining these facts with [11, Proposition 8.2] and Karrass-Solitar's description [22, Theorem 5] of subgroups of an amalgam one can deduce that $A *_C B$, it follows that $V(RG)e \cong \langle F_{m_e}, C_2 \times C_2\rangle *_{C_2\times C_2} Ge$ for some free subgroup F_{m_e} with m_e given in [22, Corollary pg 247]. From this follows that $V(RG)e \cong \langle F_{m_e}^G\rangle \rtimes G$, as desired.

Now suppose that $e(\mathcal{U}(RG)) = \infty$. By Theorem 1.3 this means that there is exactly one non-division component of the form $\mathrm{M}_2(\mathbb{Q})$ and the other components are of the form $\mathbb{Q}(\sqrt{-d})$ with $d \in \mathbb{N}$ or $\left(\frac{-a,-b}{\mathbb{Q}}\right)$. As proven in the proof of Theorem 1.3, see (2.1), we have that $V(RG)$ is a subgroup of finite index in $(D_8 \times U) *_{C_2\times C_2\times U} (D_{12} \times U)$ with $U = A \times Q_8^s \times Dic_3^t$ for some s, t and with A abelian with $\exp(A) \mid 4, 6$. An analogue reasoning as for (i) now yields the desired conclusion. □

[5] A short concrete summary of the above orders can also be found before and after Theorem 3.14. of [2].

References

[1] Andreas Bächle, Geoffrey Janssens, Eric Jespers, Ann Kiefer, and Doryan Temmerman. A dichotomy for integral group rings via higher modular groups as amalgamated products. *J. Algebra*, 604:185–223, 2022. 2

[2] Andreas Bächle, Geoffrey Janssens, Eric Jespers, Ann Kiefer, and Doryan Temmerman. Abelianization and fixed point properties of units in integral group rings. *Math. Nachr.*, 296(1):8–56, 2023. 2, 6, 7, 8, 10

[3] Robynn Corveleyn, Geoffrey Janssens, and Doryan Temmerman. The virtual structure problem for higher modular groups, 2025. In preparation. 4

[4] Ángel del Río. Finite groups in integral group rings. *Lecture Notes*, page 24pg, 2018. https://arxiv.org/abs/1805.06996. 9

[5] Florian Eisele, Ann Kiefer, and Inneke Van Gelder. Describing units of integral group rings up to commensurability. *J. Pure Appl. Algebra*, 219(7):2901–2916, 2015. 6

[6] Mikhail Ershov and Andrei Jaikin-Zapirain. Property (T) for noncommutative universal lattices. *Invent. Math.*, 179(2):303–347, 2010. 6

[7] Hans Freudenthal. Über die Enden diskreter Räume und Gruppen. *Comment. Math. Helv.*, 17:1–38, 1945. 4

[8] Charles Frohman and Benjamin Fine. Some amalgam structures for Bianchi groups. *Proc. Amer. Math. Soc.*, 102(2):221–229, 1988. 6

[9] Graham. Higman. The units of group-rings. *Proc. London Math. Soc. (2)*, 46:231–248, 1940. 1, 2

[10] Heinz Hopf. Enden offener Räume und unendliche diskontinuierliche Gruppen. *Comment. Math. Helv.*, 16:81–100, 1944. 4

[11] Geoffrey Janssens, Eric Jespers, and Ofir Schnabel. Units of twisted group rings and their correlations to classical group rings. *Adv. Math.*, 458(part B):Paper No. 109983, 81, 2024. 3, 4, 10

[12] Geoffrey Janssens, Doryan Temmerman, and Francois Thilmany. Simultaneous ping-pong for finite subgroups of reductive groups. page 55pg, 2025. to appear. 8

[13] E. Jespers and G. Leal. Generators of large subgroups of the unit group of integral group rings. *Manuscripta Math.*, 78(3):303–315, 1993. 1, 4

[14] E. Jespers and M. M. Parmenter. Bicyclic units in $\mathbf{Z}S_3$. *Bull. Soc. Math. Belg. Sér. B*, 44(2):141–146, 1992. 1, 4

[15] Eric Jespers. Free normal complements and the unit group of integral group rings. *Proc. Amer. Math. Soc.*, 122(1):59–66, 1994. 1, 2, 3, 4

[16] Eric Jespers and Angel del Río. A structure theorem for the unit group of the integral group ring of some finite groups. *J. Reine Angew. Math.*, 521:99–117, 2000. 1, 2, 3, 5

[17] Eric Jespers and Ángel del Río. *Group ring groups. Vol. 1. Orders and generic constructions of units*. De Gruyter Graduate. De Gruyter, Berlin, 2016. 1, 2, 3, 5, 6, 9

[18] Eric Jespers and Guilherme Leal. Describing units of integral group rings of some 2-groups. *Comm. Algebra*, 19(6):1809–1827, 1991. 1, 4

[19] Eric Jespers and Guilherme Leal. Free products of abelian groups in the unit group of integral group rings. *Proc. Amer. Math. Soc.*, 126(5):1257–1265, 1998. 2

[20] Eric Jespers, Guilherme Leal, and Angel del Río. Products of free groups in the unit group of integral group rings. *J. Algebra*, 180(1):22–40, 1996. 2, 3

[21] Eric Jespers, Antonio Pita, Ángel del Río, Manuel Ruiz, and Pavel Zalesskii. Groups of units of integral group rings commensurable with direct products of free-by-free groups. *Adv. Math.*, 212(2):692–722, 2007. 2

[22] A. Karrass and D. Solitar. The subgroups of a free product of two groups with an amalgamated subgroup. *Trans. Am. Math. Soc.*, 150:227–255, 1970. 10

[23] E. Kleinert. Two theorems on units of orders. *Abh. Math. Sem. Univ. Hamburg*, 70:355–358, 2000. 2, 7, 9, 10

[24] E. Kleinert and Á. del Río. On the indecomposability of unit groups. *Abh. Math. Sem. Univ. Hamburg*, 71:291–295, 2001. 7

[25] Ernst Kleinert. Units of classical orders: a survey. *Enseign. Math. (2)*, 40(3-4):205–248, 1994. 1

[26] Guilherme Leal and Angel del Río. Products of free groups in the unit group of integral group rings. II. *J. Algebra*, 191(1):240–251, 1997. 2, 3

[27] G. A. Margulis. *Discrete subgroups of semisimple Lie groups*, volume 17 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1991. 6

[28] Antonio Pita, Ángel Del Río, and Manuel Ruiz. Groups of units of integral group rings of Kleinian type. *Trans. Amer. Math. Soc.*, 357(8):3215–3237, 2005. 2

[29] César Polcino Milies and Sudarshan K. Sehgal. *An introduction to group rings*, volume 1 of *Algebra and Applications*. Kluwer Academic Publishers, Dordrecht, 2002. 9
[30] Sudarshan K. Sehgal. Nilpotent elements in group rings. *Manuscripta Math.*, 15:65–80, 1975. 9
[31] John Stallings. *Group theory and three-dimensional manifolds*, volume 4 of *Yale Mathematical Monographs*. Yale University Press, New Haven, Conn.-London, 1971. A James K. Whittemore Lecture in Mathematics given at Yale University, 1969. 2, 4, 5, 7, 9
[32] John R. Stallings. On torsion-free groups with infinitely many ends. *Ann. of Math. (2)*, 88:312–334, 1968. 2, 5
[33] John Voight. *Quaternion algebras*, volume 288 of *Graduate Texts in Mathematics*. Springer, Cham, [2021] ©2021. 7, 10

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