WILD GENUS-ZERO QUANTUM DE RHAM SPACES

by

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Abstract. — The wild de Rham spaces parameterize isomorphism classes of (stable) meromorphic connections, defined on principal bundles over wild Riemann surfaces. Working on the Riemann sphere, we will deformation-quantize the standard open part of de Rham spaces, which corresponds to the moduli of linear partial differential equations with meromorphic coefficients. We treat the general (typically nongeneric) untwisted/unramified case with semisimple formal residue, for any polar divisor and reductive structure group, without using parabolic/parahoric structures on the trivial bundle.

The main ingredients are: (i) constructing the quantum Hamiltonian reduction of a (tensor) product of quantized coadjoint orbits in dual truncated-current Lie algebras, involving the corresponding category-$\mathcal{O}$ Verma modules; and (ii) establishing sufficient conditions on the coadjoint orbits, so that generically all meromorphic connections are stable, and the (semiclassical) moment map for the gauge-group action is faithfully flat.

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1. Introduction, main results, layout

1.1. Main aim. — This is a text about the deformation quantization of complex symplectic (affine) varieties arising in meromorphic 2d gauge theory, parameterizing isomorphism classes of irregular-singular connections defined on principal bundles over wild Riemann surfaces: the wild *de Rham spaces* $\mathcal{M}_{\mathrm{dR}}$.

1.1.1. — In brief, the genus-zero examples contain open subspaces $\mathcal{M}^*_{\mathrm{dR}} \subseteq \mathcal{M}_{\mathrm{dR}}$, whose points naively correspond to linear meromorphic ODEs for an unknown holomorphic function $z \mapsto \psi(z) \in \mathbb{C}^m$ (for some rank $m \geqslant 1$), up to a global base-change in the target. Here we replace $\mathrm{GL}_m(\mathbb{C})$ by any connected complex reductive group G, and then (mostly) build on [23, 25, 40, 148, 42, 43, 67, 39] in order to:

1. recall that $\mathcal{M}^*_{\mathrm{dR}}$ is naturally identified with the G-Hamiltonian reduction of products of coadjoint orbits $\mathcal{O}'_a$ in dual *truncated-current Lie algebras* (= TCLAs);[(1)]
2. construct a (strong) *quantum* G-comoment map for the quantization of $\mathcal{O}'_a$;
3. study the flatness of the G-moment map defining $\mathcal{M}^*_{\mathrm{dR}}$;
4. and conclude that the *quantum* G-Hamiltonian reduction of products of 'quantum' orbits is generically a (flat) deformation quantization of $\mathcal{M}^*_{\mathrm{dR}}$.

This vindicates a remark in [39, § 17], and opens to several extensions—some of which are mentioned below. The rest of the introduction gathers more background/motivation (cf. § 1.2), before moving on to a statement of the main results (cf. § 1.3, to which experts might want to skip), and to a layout of the sections in the body/appendices of the article (cf. § 1.4).

1.2. More background/motivation. — Let Σ be a closed Riemann surface, and mark a finite set $\boldsymbol{a} \subseteq \Sigma$ of—unordered—points. (There is an analogous setup in the complex-algebraic category, working with nonsingular complex projective curves.) Let also $\mathfrak{g} := \mathrm{Lie}(G)$ be the Lie algebra of G.

(1) A.k.a. Takiff Lie algebras [134], cf. [70, 71].

1.2.1. — A *meromorphic connection on* $(\Sigma, \boldsymbol{a})$ is a pair $(\pi, \mathcal{A})$, consisting of: (i) a holomorphic principal G-bundle $\pi \colon \mathcal{E} \to \Sigma$; and (ii) a $\mathfrak{g}$-valued meromorphic 1-form $\mathcal{A}$ on the total space $\mathcal{E}$,[(2)] with poles along the divisor $\mathcal{E}_{\boldsymbol{a}} := \pi^{-1}(\boldsymbol{a}) \subseteq \mathcal{E}$, such that:

1. $R_g^*(\mathcal{A}) = \mathrm{Ad}_{g^{-1}}(\mathcal{A})$, where R_g is the structural right action of $g \in G$ on $\mathcal{E}$;
2. and $\langle \mathcal{A}, X^\sharp \rangle = X$, where $X^\sharp$ is the fundamental vector field of $X \in \mathfrak{g}$ on $\mathcal{E}$.

It follows that $\mathcal{A}$ is nonsingular along $\ker(d\pi) \subseteq T\mathcal{E}$, and taking $\boldsymbol{a} = \varnothing$ yields the (expected) notion of a holomorphic G-equivariant Ehresmann connection on π, cf. [4].

1.2.2. — For many different reasons, in complex Poisson/symplectic geometry, representation theory, and low-dimensional topology, one is interested in parameterizing isomorphism classes of meromorphic connections. Our main motivation involves the actions of generalized braid/mapping class groups, both before and after quantization, cf. §§ 1.2.6–1.2.8. (This relates with the mathematical formalization of conformal/topological field theories, cf. Rmk. 1.2.9 and [144].)

In any event, the corresponding moduli spaces can be deformed, in *isomonodromic* fashion, and quantized. But in the irregular-singular case one fixes more data than the base pointed surface $(\Sigma, \boldsymbol{a})$: following [23, 148], amongst many others, we also prescribe—the G-orbit of—nonresonant 'very good' formal normal forms at each pole (cf., e.g., the review [30]; recall that this is the starting point to study Stokes data, which we do *not* consider in this text.)

1.2.3. — Consider thus a marked point $a \in \boldsymbol{a}$, and a small (analytic) disc in Σ centred at a, with given coordinate z—vanishing at a. Upon trivializing the bundle thereon, pulling back the connection 1-form determines an element of the complex vector space $\mathfrak{g}\{\{z\}\} dz$, where $\mathfrak{g}\{\{z\}\} := \mathfrak{g} \otimes_{\mathbb{C}} \big(\mathbb{C}\{\{z\}\}\big)$, invoking the field of *convergent* Laurent series. (I.e., the fraction field of the ring of convergent power series in z.)

However, we are only interested in the 'formal' germ of $\mathcal{A}$ at a, forgetting about convergence. Precisely, consider the *completed* local ring $\mathscr{O}_a = \mathscr{O}_{\Sigma,a}$ of Σ at a, with maximal ideal $\mathfrak{m}_a$. (Hereafter, we will *not* consider the uncompleted versions.) Then the disc $\mathcal{D}_a := \mathrm{Spec}\, \mathscr{O}_a$—in the terminology of [68, § A.1.1]—is a formal neighbourhood of a, and z yields a *uniformizer* $\varpi_a = \varpi_{z,a} \in \mathfrak{m}_a$. With these choices, pulling back $\mathcal{A}$ even further determines an element

$$\widehat{\mathcal{A}}_a = \widehat{\mathcal{A}}_{z,a} \in \mathfrak{g}(\!(\varpi_a)\!)\, d\varpi_a, \qquad \mathfrak{g}(\!(\varpi_a)\!) := \mathfrak{g} \otimes_{\mathbb{C}} \big(\mathbb{C}(\!(\varpi_a)\!)\big), \tag{1}$$

now involving formal Laurent series. (One can phrase this as taking *formal* Laurent expansions at a, in the 'formal' coordinate ϖ_a, rather than ordinary Laurent expansions in z.)

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(2) We do *not* define a connection as a differential operator ∇ on an associated vector bundle, because fixing a (faithful) G-module tampers with the definition of 'genericity' in the theory of isomonodromic deformations, cf. [24, Lem. A.2]. On the other hand, using the category of linear representations of G is helpful in the Tannakian formalism, cf. [35, 36].

But (1) also depends on the choice of trivialization of a principal bundle over $\mathcal{D}_a$ (cf. § 2.1): now we act by the formal-holomorphic gauge group $G⟦\varpi_a⟧$, thereby changing trivialization in simply-transitive fashion, looking for a distinguished element in the corresponding (gauge) orbit $\widehat{\mathcal{O}}'_a$ of $\widehat{\mathcal{A}}$. Namely, we suppose that $\widehat{\mathcal{O}}'_a$ contains a (*formal*) *normal form*

$$\mathcal{A}'_a = \Lambda'_a \varpi_a^{-1} d\varpi_a + dQ'_a, \qquad Q'_a = \sum_{i=1}^{s_a-1} A'_{a,i} \frac{\varpi_a^{-i}}{-i}, \tag{2}$$

where: (i) $s_a \geqslant 1$ is the pole order of (1), assuming—w.l.o.g.—that $A'_{a,s_a-1} \neq 0$ if $s_a \geqslant 2$, else that $\Lambda'_a \neq 0$; (ii) the coefficients $\Lambda'_a, A'_{a,1}, \ldots, A'_{a,s_a-1} \in \mathfrak{g}$ are semisimple and commute with each other; and (iii) the (formal) residue $A'_{a,0} \coloneqq \Lambda'_a$ is *nonresonant*. (Cf. Def. 3.1.2.)

1.2.4. Remark. — In this text we deal with *untwisted/unramified* meromorphic connections, with a view towards quantization, and so in particular we do *not* use meromorphic/Puiseux gauge transformations—formal or convergent.

In any event, note that (2) is a (formal) meromorphic 1-form which coincides with its *principal part*. In the body of the article we will recall more terminology around them, using definitions which do *not* depend on uniformizers (cf. Rmk. 6.1.2). Moreover, the normal form prescribes the polar divisor of $\boldsymbol{\mathcal{A}}$, viz.,

$$D = D(\boldsymbol{a}, \widehat{\boldsymbol{\mathcal{O}}}') \coloneqq \sum_{\boldsymbol{a}} s_a [a]. \tag{3}$$ ◊

1.2.5. — Given a multiset of such gauge orbits $\widehat{\boldsymbol{\mathcal{O}}}' = \{ \widehat{\mathcal{O}}'_a \}_{\boldsymbol{a}}$, consider now the (unframed) *naive de Rham groupoid*

$$\mathcal{C}_{dR} = \mathcal{C}_{dR}(\widehat{\boldsymbol{\Sigma}}; G), \qquad \widehat{\boldsymbol{\Sigma}} \coloneqq (\Sigma, \boldsymbol{a}, \widehat{\boldsymbol{\mathcal{O}}}'), \tag{4}$$

which is the category of:

objects : meromorphic connections $(\boldsymbol{\pi}, \boldsymbol{\mathcal{A}})$ on $(\Sigma, \boldsymbol{a})$, such that $\widehat{\mathcal{A}}_a \in \widehat{\mathcal{O}}'_a$ for each $a \in \boldsymbol{a}$, in the notation of (1);

(iso)morphisms $(\boldsymbol{\pi}, \boldsymbol{\mathcal{A}}) \to (\widetilde{\boldsymbol{\pi}}, \widetilde{\boldsymbol{\mathcal{A}}})$ **:** (iso)morphisms $\Phi : \boldsymbol{\mathcal{E}} \xrightarrow{\simeq} \widetilde{\boldsymbol{\mathcal{E}}}$ of holomorphic principal G-bundles, covering the identity of Σ, such that $\Phi^*(\widetilde{\boldsymbol{\mathcal{A}}}) = \boldsymbol{\mathcal{A}}$.

In the nonresonant setting, one could equivalently impose that the principal part of (1) lies in the orbit of (2) for the action of a *truncated* gauge group G_{s_a} (cf. § 3). The latter Lie group integrates the TCLA

$$\mathfrak{g}_{s_a} = \mathfrak{g} \otimes_{\mathbb{C}} (\mathcal{O}_{s_a[a]}), \qquad \mathcal{O}_{s_a[a]} \coloneqq \mathcal{O}_a / \mathfrak{m}_a^{s_a}, \tag{5}$$

and the truncated gauge-orbit of the principal part is tantamount to a coadjoint orbit $\mathcal{O}'_a \subseteq \mathfrak{g}_{s_a}^{\vee}$: its (complex-algebraic) Kirillov–Kostant–Souriau symplectic structure (= KKS [91, 96, 130]) is what we concretely consider and quantize later on. Moreover, this truncation leads to the aforementioned finite-dimensional descriptions of (open parts of) the moduli spaces, in genus-zero, as complex-affine G-Hamiltonian reductions.[(3)] (Cf. § 1.2.11, and note that the orbits $\mathcal{O}'_a$ are *affine* varieties by Prop. 3.2.10.)

1.2.6. — On the subject of moduli spaces, recall that one way to start constructing (analytic) moduli stacks is to work in families. Namely, one considers deformations of the starting data of $\widehat{\boldsymbol{\Sigma}}$, parameterized by complex manifolds/analytic spaces $\mathbf{B}$, and defines a groupoid for each such $\mathbf{B}$: cf., e.g., [**57**, App. A]. (Or use $\mathbb{C}$-schemes in the complex-algebraic setup, working on the étale/fppf site, etc.)

In turn, there are *admissible deformations* [**29**, Def. 10.1 + Rmk. 10.6] of the wild Riemann surface underlying the decorated pointed surface $\widehat{\boldsymbol{\Sigma}}$ of (4), i.e., the triple

$$\boldsymbol{\Sigma} := (\Sigma, \boldsymbol{a}, \boldsymbol{\Theta}'),$$

where $\boldsymbol{\Theta}' := \{\Theta'_a\}_{\boldsymbol{a}}$ is the multiset of *irregular classes* obtained from $\widehat{\mathcal{O}}'$.[(4)] The main motivational statement is that these admissible deformations provide an intrinsic/topological view on the nonlinear PDEs which govern the isomonodromic deformations of $(\boldsymbol{\pi}, \boldsymbol{\mathcal{A}})$. This leads to algebraic Poisson/symplectic actions of 'wild' generalizations of the mapping class group of pointed surfaces, on *wild* character varieties [**29**, **33**], generalizing the much-studied representations of surface groups.

In brief, one may first consider a holomorphic family of n-pointed genus-g closed Riemann surfaces based at $(\Sigma, \boldsymbol{a})$, where $g, n \geqslant 0$ are the genus of Σ and the cardinality of $\boldsymbol{a}$, respectively. Then, in somewhat dual fashion, choose—w.l.o.g.—a maximal torus $T \subseteq G$, with associated Cartan subalgebra $\mathfrak{t} := \mathrm{Lie}(T) \subseteq \mathfrak{g}$, and introduce the Weyl group W of (G, T). Now the (universal) admissible deformations of the pole-order-bounded irregular class

$$\Theta'_a \in \mathfrak{t}\big(\mathfrak{m}_a^{-s_a}\mathscr{O}_a/\mathscr{O}_a\big)/W, \qquad \mathfrak{t}\big(\mathfrak{m}_a^{-s_a}\mathscr{O}_a/\mathscr{O}_a\big) := \mathfrak{t} \otimes_{\mathbb{C}} \big(\mathfrak{m}_a^{-s_a}\mathscr{O}_a/\mathscr{O}_a\big), \tag{6}$$

on the *fixed* pointed Riemann surface $(\Sigma, \boldsymbol{a})$, correspond to a quotient of the root-valuation stratum [**73**] of any lift of Θ'_a through the Weyl-group action. This lift is an *irregular type* Q'_a as in (2) (cf. the 'quotient' stratifications of [**39**].)

1.2.7. Remark. — The strata correspond to the 'fission' [**26**, **29**] of the structure group G, down to the Levi factor L of a parabolic subgroup $P \subseteq G$ (with $T \subseteq L$). This was pointed out earlier [**18**, Eqq. (2.2)–(2.3)] when $G = \mathrm{GL}_m(\mathbb{C})$ (cf. Rmk. 4.1.3). More recently, the explicit description of the admissible deformations of (6) has been taken on in [**56**, **55**, **32**, **58**] (cf. [**148**, § 4]), for any reductive structure group, including the general twisted/ramified setting of [**33**]. Moreover, in the framed case, where the action of W is broken at all marked points (and ordering them), the admissible

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(3) This might be viewed as (an irregular version of) a 'semiclassical' analogue of the well-known fact that the spaces of Wess–Zumino–Novikov–Witten conformal blocks (= WZNW [**140**, **111**, **142**, **143**]) admit descriptions in terms of (co)invariants of $\mathfrak{g}$-modules, while in principle they are defined from representations of the affinizations $\widehat{\mathfrak{g}} \twoheadrightarrow \mathfrak{g}((\varpi_a))$ (upon equipping $\mathfrak{g}$ with an Ad_G-invariant pairing, etc.).

(4) Cf. Def. 3.2.5. Here just note that $(\boldsymbol{\pi}, \boldsymbol{\mathcal{A}})$ determines the G-orbit of (2), and that the normal form per se can only be extracted by framing $\boldsymbol{\pi}$ at each marked point, cf. Rmk. 1.2.12.

deformations of 'labelled' wild Riemann surfaces lead to a fibration onto the moduli stack $\mathcal{M}_{g,n}$; and so there is a *wild* mapping class group extending $\Gamma_{g,n} \coloneqq \pi_1(\mathcal{M}_{g,n})$ [57]. (Please refer to the introductions of [56, 55, 32, 57, 58] for more details, and for references to the past work of many more people, besides the authors and their collaborators.) ♢

1.2.8. — Another important piece of motivation is that the isomonodromy equations can be locally expressed as the flow of an integrable (nonautonomous) Hamiltonian system: the latter can also be quantized, leading to 'quantum' actions of braid and mapping class groups. E.g., in the case of Fuchsian systems on the sphere, the quantization of the Schlesinger system [122, 123] yields the Knizhnik–Zamolodchikov connection (= KZ [94]); and adding one (generic) pole of order two at infinity leads to the connections of De Concini–Millson–Toledano Laredo/Felder–Markov–Tarasov–Varchenko (= DMT/FMTV [105, 66]), quantizing [88].[5] The monodromy of KZ is a representation of Artin's standard braid group [3], and features in the statement of the Drinfel'd–Kohno theorem [95, 60, 61] for the Jimbo–Drinfel'd quantum group [85, 59]; while the monodromy of DMT/FMTV adds on an action of Brieskorn–Deligne's 'generalized' G-braid groups [37, 51], and recovers Lusztig's action [102] (cf. [131, 93]), in a Drinfel'd–Kohno theorem for q-Weyl groups [136]. (The corresponding 'semiclassical' action is De Concini–Kac–Procesi's [50], as explained in [24].)

These 'quantum' flat connections were generalized in [117, 149], allowing for a (nongeneric) pole of order 3, in a quantization of the 'simply-laced' systems [28]. (This relates with the Okamoto diagrams for the Painlevé equations [113], cf. also [34, 54, 53]). In this text, instead, we treat meromorphic connections with *arbitrary* polar divisor, without relying on quiver-modularity results [28, 77].

Moreover, contrary to [69], fixing the formal gauge orbits $\widehat{\mathcal{O}}'$ selects a symplectic leaf within a Poisson moduli space, and so we are naturally led to quotients of the universal enveloping algebra of (5), cf. Rmk. 4.2.4. This is required, in order to relate the family of quantizations with bundles of conformal blocks for the category $\mathcal{O}$ of [42, 43], cf. [67, 39] (and [15, 16, 82] for the standard BGG category $\mathcal{O}$).

Finally, we will also use the 'confluence' of simple poles to establish the most difficult criteria for flatness (cf. the 'confluent' Hamiltonians of [108, 109]); however, our constructions rather go in the opposite direction, as we *unfold* [76] one irregular singularity into several regular ones, cf. Lem. 8.3.2 + Cor. 9.4.9. This makes it possible to embed wild de Rham spaces into 'residue manifolds' [80].

1.2.9. Remark. — There is quite some more literature around isomonodromy Hamiltonians, also in higher genus. In turn, the latter relates with (semiclassical limits of) the connections of Bernard/Tsuchiya–Ueno–Yamada (= KZB/TUY [14, 13, 137]).

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[5] Compared to KZ, DMT lives on the other side of the Howe duality [10, 136], and the latter can be construed as a 'quantum' version of the Fourier–Laplace transform [74, 146, 147], cf [118].

As a last bibliographical point, let us point out the construction of 'geometrized' versions of KZ(B)/TUY [**63**, **19**], related with 'parabolic' extensions of the Hitchin connection [**79**] (cf. [**6**])—in geometric quantization. They are constructed by algebro-geometric means (based on [**115**]), and expand on the famous relation [**11**, **65**, **100**] between nonabelian theta functions and WZNW conformal blocks in 2d conformal field theory [**12**]. Much of this can be generalized into the wild case. ♢

1.2.10. — We plan to pursue the above perspective on the 'time-dependent' quantization of the isomonodromy systems, and the (further) relation with WZNW conformal blocks, elsewhere. In this text we will rather work on a *fixed* wild Riemann surface, in order to prepare the construction of the 'quantum' phase-spaces.

Thus, in first approximation, one can consider the 'decategorification' of the groupoid $\mathcal{C}_{\mathrm{dR}}$, i.e., the set(6) of its isomorphism classes, denoted by

$$\mathcal{M}_{\mathrm{dR}} = \mathcal{M}_{\mathrm{dR}}\big(\widehat{\boldsymbol{\Sigma}}; \mathrm{G}\big).$$

Then there is also a different (standard) gauge-theoretic construction of the moduli spaces, which rephrases the moduli problem as a group action, matching up isomorphism classes with orbits in some large parameter space: this typically endows $\mathcal{M}_{\mathrm{dR}}$ with much more geometric structure, viewing it as the Hamiltonian reduction of an infinite-dimensional complex affine space. E.g., when on vector bundles, the main construction of [**18**] (cf. [**120**]) yields a noncompact hyperkähler manifold—whose metric is *complete* with the present hypotheses on the normal forms. The underlying holomorphic symplectic structure is a generalization of the Narasimhan/Atiyah–Bott/Goldmann form [**110**, **5**, **72**], first introduced in [**23**] (in the generic case) using Fréchet quotients, which also extends to more general reductive structure groups [**25**].

1.2.11. — And indeed, as in [**25**, **148**], we will *not* restrict to vector bundles. Rather, for any reductive structure group, we consider the genus-zero setup of the classical theory of isomonodromic deformations (i.e., again, linear systems of rational PDEs.)

Precisely, in addition to taking $\Sigma := \mathbb{C}P^1$, we consider meromorphic connections on the *trivial* holomorphic principal G-bundle $\mathcal{E}^* := \Sigma \times \mathrm{G} \xrightarrow{\pi^*} \Sigma$. This yields a (full) subgroupoid $\mathcal{C}^*_{\mathrm{dR}} \subseteq \mathcal{C}_{\mathrm{dR}}$, whose objects map bijectively(7) to the $\mathbb{C}$-points of a *finite-dimensional* irreducible complex affine variety M. The latter is a subvariety of the product of truncated gauge orbits $\boldsymbol{\mathcal{O}}' := \{\mathcal{O}'_a\}_a$, i.e., the multiset of coadjoint orbits passing through—the G-orbit of—the prescribed formal normal forms $\mathcal{A}'_a \in \mathfrak{g}^{\vee}_{s_a}$ (as in (2)); and two connections are isomorphic if and only if the corresponding points in M are related by the diagonal (coadjoint) action of G. (Cf. again [**23**, **25**, **148**]; this is recalled in § 6.) To get the corresponding moduli (sub)space $\mathcal{M}^*_{\mathrm{dR}} \subseteq \mathcal{M}_{\mathrm{dR}}$ we then choose sufficiently generic orbits $\boldsymbol{\mathcal{O}}'$, so that all connections are *naively* stable

(6) It is *not* a proper class, as the groupoid is small.

(7) Cf. (40), and note that nonresonance is required to ensure surjectivity; otherwise, cf. Rmk. 6.1.4.

(cf. [22, Rmk. 2.37] and [28, Rmk. 9.7] in the vector-bundle case). I.e., without fixing parabolic/parahoric structures on the bundle, we provide a sufficient condition so that *all* points of M are stable for the action of the projectivized group

$$\mathbf{P}(G) \coloneqq G/Z(G),$$

cf. § 10. (None of this relies on a construction of the partial compactification $\mathcal{M}_{dR}$.)

Before moving on to a statement of main results, we make three remarks.

***1.2.12. Remark.* —** The normal form $\mathcal{A}'_a$ at the marked point $a \in \mathbf{a}$ is determined upon framing the bundle there. Else, it is only the G-*orbit* of $\mathcal{A}'_a$—the 'normal orbit' of Def. 3.2.5 (1.)—which is canonical. Equivalently, choosing a maximal torus $T \subseteq G$, one finds a Weyl orbit which 'extends' the irregular class (6) down to the residue.

With this caveat, note that the stability condition of Prop. 10.1.1 does not break the Weyl symmetry, and it is only the *unmarked* orbit $\mathcal{O}'_a \subseteq \mathfrak{g}^{\vee}_{s_a}$ which enters into the definition of M. Moreover, while we do choose a marking of $\mathcal{O}'_a$ to construct a $*$-product thereon, we also prove (in Lem. 4.4.2) that the deformation quantization does *not* depend upon this choice. In conclusion, all our 'semiclassical' constructions are completely canonical, while the 'quantum' ones only rely on the choice of a polarization of $\mathcal{O}'_a$—as it typically happens, cf. § 4.4. ♢

***1.2.13. Remark.* —** Recall that the stable locus $M^s \subseteq M$ might be empty: the (unramified, irregular) *additive* [98, 99] Deligne–Simpson problem [126] is to characterize the choices of orbits $\boldsymbol{\mathcal{O}}'$ such that there exist stable connections on the trivial bundle with such local data. E.g., in the logarithmic vector-bundle case, a necessary/sufficient condition [49] relies on the quiver-theoretic description of the moduli spaces [48]; this was extended to the irregular-singular case in [21] (cf. [28, § 10]).

In our situation, the level set M is in particular *nonempty* if the G-moment map (42) is faithfully flat, which follows from the assumptions of Thm. 1.3.4; the condition (88) then insures that $M^s = M \neq \varnothing$. (We do *not* investigate necessary nonemptyness conditions.) ♢

***1.2.14. Remark.* —** A finer notion of stability involves parabolic degrees/slopes, cf. again [18]; and in principle *parahoric* structures are required for a general structure group G, even in the tame case [27]. We will *not* discuss this in this text, particularly because we focus on the de Rham side of nonabelian Hodge theory—and so we are not concerned with the rotation of the weights, etc.

Briefly, recall that $\mathcal{M}_{dR}$ is but one avatar of the 'wild' first cohomology of $\boldsymbol{\Sigma}$ [20]. It is complemented by the Dolbeault space $\mathcal{M}_{Dol}$ (parameterizing meromorphic G-Higgs-bundles), and by the Betti space $\mathcal{M}_B$ (parameterizing Stokes filtered/graded G-local systems, i.e., monodromy/Stokes data of meromorphic G-connections [31]). In the nonsingular case—where $\mathbf{a} = \varnothing$—, this triple of mutually nonisomorphic complex-algebraic varieties is the upshot of the nonabelian Hodge correspondence [78, 52, 47, 127], cf. [128, 129] (and [125] for the regular-singular case, where $s_a = 1$ for $a \in \mathbf{a}$.)

The quantization of $\mathcal{M}_{\mathrm{Dol}}$ and $\mathcal{M}_{\mathrm{B}}$ have also been widely studied: let us *not* attempt a review here. In any event, the former leads to (autonomous) quantized Hitchin systems, related with *isospectral* deformations, which are not immediately equivalent to the isomonodromic setting underlying this text. And the quantum (wild) character varieties are also different, as we are not directly quantizing the de Rham/Betti *holomorphic* symplectic structure; but rather an *algebraic* one, on (an open subspace of) the domain of the transcendental map which takes the monodromy/Stokes data—a.k.a. the Riemann–Hilbert–Birkhoff correspondence. ♢

1.3. Main results. — In the end, the affine GIT quotient of $M = M^s$ by G yields a genuine orbit space, which is our model for $\mathcal{M}^*_{\mathrm{dR}}$. Thus, we can—and will—work with a mildly-singular (irreducible) symplectic affine variety, defined over $\mathbb{C}$ (cf. § 6.1; taking the stacky quotient is also possible, but the exact relation with symplectic structures and quantization is more sophisticated.)

1.3.1. — First, recall that [39] set up a representation-theoretic quantization of the coadjoint orbits $\mathcal{O}'_a \subseteq \mathfrak{g}^\vee_{s_a}$ through the principal parts $\mathcal{A}'_a$ of (2), based on [2]. It relies on certain induced modules $M^\pm_a = M^\pm_{\mathcal{A}'_a}$ for TCLAs to be generically simple/irreducible (cf. Def. 4.2.2). These are the (parabolic) Verma modules of the category $\mathcal{O}$ of [42, 43], generalizing the standard (parabolic) Verma modules for $U(\mathfrak{g})$—and referred to as 'singularity modules' in [39], cf. [141, 67] in the regular/generic case (and cf. Rmk. 4.2.4 in general).

Concretely, let $\mathcal{R}_{a,0} := \mathbb{C}[\mathcal{O}'_a]$ be the commutative Poisson ring of regular functions on the (affine) coadjoint orbit $\mathcal{O}'_a$ through an untwisted semisimple principal part $\mathcal{A}'_a$, and let also $\hbar$ be a formal deformation parameter. In this text we complement the deformation quantization of $\mathcal{R}_{a,0}$, by proving the following:

***1.3.2. Theorem* (cf. Prop. 4.2.6 + Rmk. 4.3.3 + Thm. 4.3.5). —** *Choose two 'opposite' parabolic sequences in* G, *as in* (20), *such that the associated polarizations of* $T\mathcal{O}'_a$ *are 'balanced'. Then:*

1. *the* $*$*-product* $\mathcal{R}_{a,0} \otimes_{\mathbb{C}} \mathcal{R}_{a,0} \to \mathcal{R}_{a,0}[\![\hbar]\!]$ *of* [39] *admits a* strong *quantum comoment map, generating the natural 'quantum'* G_{s_a}*-action;*
2. *and if* $\mathcal{A}'_a$ *is nonresonant, then the* $U(\mathfrak{g}_{s_a})$*-modules* $M^\pm_a$, *are* simple.

1.3.3. — Second, we construct the quantum Hamiltonian reduction of the corresponding (completed (45)) product of 'quantum' orbits $\{\widehat{\mathcal{R}}_{a,\hbar}\}_a$, modulo the diagonal G-action, in the standard way [64] (cf. [118]). This yields in principle a deformation quantization of wild de Rham spaces: the caveat is that one must prove that 'quantization and reduction commute', viz., that the corresponding deformation of the coordinate ring is *flat*. A uniform way to ensure it lies in the geometry of the G-moment map for the 'semiclassical' action on the orbit product, cf. Lem. 6.2.4.

At this final stage, it is cleaner to assume that G is *semisimple*, cf. Lem.-Def. 7.1.3. (As it typically happens, this is not a serious restriction: any symplectomorphism class of de Rham spaces has a representative with semisimple structure group.) Then we give an explicit 'global' criterion for the faithful flatness of the moment map. Namely, for each marked point $a \in \mathbf{a}$ denote by $\nu_a \geqslant 0$ the number of nested Ad_G-stabilizers of the coefficients of (2), starting from the top and including the residue, which are equal to a maximal torus of G. This involves a 'fission' sequence of reductive subgroups of G [**26, 29**], cf. again Rmk. 4.1.3.[8] Importantly, the multiset $\{\nu_a\}_{\mathbf{a}}$ is *intrinsic*, viz., independent of the choice of uniformizers at the marked points (cf. Lem. 5.1.2). Moreover, it only depends on the G-orbit of (2), whence it is uniquely determined by the choice of wild Riemann sphere $\boldsymbol{\Sigma}$, together with a residue orbit at each marked point. (Thereby selecting a symplectic leaf within a Poisson moduli space.)

In particular, one can associate such 'moduli' numbers $\nu_a \geqslant 0$ to the decorated surface $\widehat{\boldsymbol{\Sigma}}$ of (4), which controls the de Rham groupoid/space; or, equivalently, to the triple $(\Sigma, \mathbf{a}, \mathcal{O}')$. Finally, one can prove the following (main):

***1.3.4. Theorem* (cf. Lem. 6.2.4 + Prop. 8.1.1 + Lem. 8.1.4 + Corr. 9.3.9 + 9.4.11)**
Suppose that G *is semisimple, and that*

$$\chi(\widehat{\boldsymbol{\Sigma}}) < 0, \qquad \text{where} \qquad \chi(\widehat{\boldsymbol{\Sigma}}) := 2 - \sum_{\mathbf{a}} \nu_a \in \mathbb{Z}. \tag{7}$$

Then:

1. *the semiclassical 'diagonal'* G*-moment map on the orbit product is (faithfully)* flat*;*
2. *and the (separated) level-zero quantum* G*-Hamiltonian reduction of the tensor product of quantum orbits is a* flat *deformation quantization of* $\mathcal{M}^*_{\mathrm{dR}}$.

1.3.5. — As an important particular case, assume that the irregular singularity at the point a is *generic*, i.e., such that the leading coefficient of the formal normal form is regular semisimple. (This is A'_{a,s_a-1} if $s_a > 1$; else it is the residue.) Then the nested centralizers are equal to one and the same maximal torus of G, so that $\nu_a = s_a$.

Again this is an intrinsic notion, and one readily extracts the following:

***1.3.6. Corollary*. —** *Suppose that* G *is semisimple, that all the poles are generic, and that*

$$2 - |D| < 0, \qquad \text{where} \qquad |D| := \sum_{\mathbf{a}} s_a,$$

in the notation of (3)*. Then the conclusions of Thm. 1.3.4 hold true.*

1.3.7. — Thus, generically, the criterion for flatness only depends on the pole orders.

More importantly, Cor. 1.3.6 includes the G-bundle generalization of the examples considered in the seminal work of Jimbo–Miwa–Ueno (= JMU [**89, 86, 87**]). E.g., under this genericity assumption one thus quantizes (an Hamiltonian reduction of) the symplectic phase-spaces for the following Lax pairs/representations:

- several simple poles, i.e., the Schlesinger system [**122, 123**];

[8] W.l.o.g., these are the nested stabilizers which are equal to a fixed maximal torus $T \subseteq G$, taking $\Lambda'_a, A'_{a,1}, \ldots, A'_{a,s_a-1} \in \mathfrak{t}$, in the notation of § 1.2.6.

- as a particular case, four simple poles for $G = SL_2(\mathbb{C})$, i.e., (one Hamiltonian formulation of) PVI;
- one pole of order 2, and several simple poles, i.e., the system of Jimbo–Miwa–Môri–Sato (= JMMS [88]);
- as a particular case, one pole of order 2 and one simple pole, i.e., the 'dual' Schlesinger system [74];
- as a particular particular case, for $G = SL_3(\mathbb{C})$, the 'dual' reading of PVI;
- the standard rank-2 representations of PV (one pole of order 2 and two simple poles) and PIV (one pole of order 3 and one simple pole);
- and the other standard untwisted/unramified (non-simply-laced) representations of PIII (two poles of order 2) and PII (one pole of order 4).

But Thm. 1.3.4 also includes nongeneric cases, such as the simply-laced isomonodromy systems [28] (one pole of order 3 with nongeneric leading term, and several simple poles), which encompass all the previous examples. In particular, the 'higher' Painlevé equations fit this setup; cf. § 11.4 of op. cit, as well as [17].

The main point, however, is that the main result covers a plethora of unlisted cases with arbitrary polar divisor and nongeneric irregular singularities [148].

Before moving on to a layout of the article, we make two concluding remarks.

1.3.8. Remark. — In general, flatness need *not* hold: e.g., if $\mathbf{a} = \{ a \}$ is a singleton, and $s_a = 1$, then the G-moment map is the (closed) embedding of a semisimple orbit in $\mathfrak{g}^\vee$. Conversely, a uniform *necessary* condition will depend on the exact relation of the sequences of Levi factors of parabolic subgroups of G at each pole, and we do *not* look for one.(9) Nonetheless, the 'wild' Euler–Poincaré characteristic of (7) is intimately related with the Deligne–Mumford condition for the stacks $\mathcal{WM}_{0,n,\bullet} \twoheadrightarrow \mathcal{M}_{0,n}$ of wild Riemann spheres [57], where again $n := |\mathbf{a}|$. (The general condition for flatness should also involve subtracting twice the genus of the base surface.)

More precisely, in the tame/logarithmic case, flatness generically holds if $2-n < 0$, which is precisely the condition for $\mathcal{M}_{0,n}$ to be Deligne–Mumford; in the wild setting, however, there are also flat examples with just $n \in \{1,2\}$ marked points, and in turn $\mathcal{WM}_{0,n,\bullet}$ can be Deligne–Mumford if the pole orders there are high enough, coherently with Thm. 1.3.4. (The latter simply fails for the stacky quotients

$$\mathcal{M}_{0,2} \simeq \left[\mathrm{pt} / \mathbb{C}^\times\right], \qquad \mathcal{M}_{0,1} \simeq \left[\mathrm{pt} / (\mathbb{C}^\times \ltimes \mathbb{C})\right].) \qquad \diamondsuit$$

1.3.9. Remark. — The stability condition of Prop. 10.1.1 also implies that 'quantization and reduction commute'. Namely, recall [41, § 23.2] that the moment map $M \to \mathfrak{g}^\vee$ is a submersion if the action on M is *locally free*, and so a fortiori if the

(9) If $G = PGL_m(\mathbb{C})$ for some integer $m \geqslant 1$, then the 'Birkhoff' orbits (47) are isomorphic to cotangent bundles of representation space of quivers, and the exact (nontrivial!) conditions to ensure flatness are as in [48, Thm. 1.1]; e.g., there are flat examples involving five nongeneric simple poles with $m = 3$.

stabilizers are finite. Therefore, the proof of [101, Prop. 2 (a)] extends to the present situation (and condition (b) of loc. cit. automatically holds when G is reductive.) ♢

1.4. Layout. — The layout of this article is as follows.

1.4.1. — In § 2 we set up the basic terminology for (formal) connections on (formal) discs, and the (formal) gauge action thereon.

In § 3 we explain that the moduli of nonresonant untwisted semisimple connections on discs are (G-orbits through) distinguished spaces of principal parts, in dual TCLAs.

In § 4 we recall the definition of a $*$-product on certain coadjoint orbits in dual TCLAs, and prove Thm. 1.3.2.

In § 5 we show that the previous constructions are independent of the choice of uniformizers, up to isomorphism; i.e., that one can work canonically and 'globally' on—principal bundles over—pointed Riemann surfaces.

In § 6 we recall the standard finite-dimensional presentation of genus-zero de Rham varieties, for trivial G-bundles, and define their putative deformation quantization via quantum G-Hamiltonian reduction.

In § 7 we prepare the basic terminology of the subsequent §§ 8–9, where we establish sufficient criteria for the flatness of the G-moment map—proving Thm. 1.3.4.

Finally, in § 10 we provide a sufficient condition ensuring that all points in the zero-level set of the G-moment map are stable for the projectification of G.

1.4.2. — The appendix § A gathers few standard complements about the (based/framed) formal classification of connections on discs.

The appendix § B describes the group of Lie-algebra automorphisms of TCLAs.

The appendix § C collects proofs which have been postponed—aiming to ease the reading flow.

1.5. Conventions, terminology, etc. — Henceforth, in this text, we tacitly use the following notations/conventions/facts (unless otherwise specified):

- tensor products are $\mathbb{C}$-multilinear;
- a $\mathbb{C}$-*variety* is a separated reduced scheme S, of finite type, defined over (Spec)$\mathbb{C}$;
- if V is a finite-dimensional complex vector space, we identify it with the (closed) $\mathbb{C}$-points of the affine variety $\operatorname{Spec}\operatorname{Sym}(V^{\vee})$;
- the latter is also—abusively—denoted by V;
- if S is a nonsingular $\mathbb{C}$-variety, an algebraic map $\mu \colon S \to V$ (of $\mathbb{C}$-varieties) induces a holomorphic map $\mu^{\mathrm{an}} \colon S(\mathbb{C}) \to V$ (of complex manifolds);
- the notation might *not* distinguish between μ and its analytification μ^{an}, and, at times, neither between S and $S(\mathbb{C}) = \operatorname{Hom}_{\mathrm{Sch}}(\operatorname{Spec}\mathbb{C}, S)$;
- in the previous situation, μ is surjective, with equidimensional (scheme-theoretic) fibres, if and only if this holds for μ^{an};

- on the same token, if $\mathbf{G}$ is a complex algebraic group, an algebraic (left) action $\mathbf{G} \times S \to S$ induces a holomorphic action $G \times S(\mathbb{C}) \to S(\mathbb{C})$, where $G \coloneqq \mathbf{G}(\mathbb{C})$ is the complex Lie group of $\mathbb{C}$-points of $\mathbf{G}$;
- the notation might *not* distinguish between these two actions;
- a *Hamiltonian* $\mathbf{G}$*-variety* is a Poisson $\mathbb{C}$-variety $\big(S, \{\cdot,\cdot\}\big)$, equipped with a $\mathbf{G}$-action by (algebraic) Poisson automorphisms, such that the infinitesimal action of $\mathfrak{g} \coloneqq \mathrm{Lie}(\mathbf{G})$ is given by (the Hamiltonian vector fields associated with) a $\mathbf{G}$-equivariant *moment map* $\mu \colon S \to \mathfrak{g}^\vee$;
- if G is a complex reductive Lie group, and $T \subseteq G$ a maximal torus, the *Weyl group of* (G, T) is denoted by $W = W(G,T) \coloneqq N_G(T)/T$;
- in the same situation, if $\mathfrak{t} \coloneqq \mathrm{Lie}(T) \subseteq \mathfrak{g}$ is the corresponding Cartan subalgebra, then the *root system of* $(\mathfrak{g}, \mathfrak{t})$ is denoted by $\Phi = \Phi(\mathfrak{g}, \mathfrak{t}) \subseteq \mathfrak{t}^\vee$;
- we let W act on $\mathfrak{t}$ and $\mathfrak{t}^\vee$ in the standard way, permuting the roots and the coroots $\alpha^\vee \in \Phi^\vee \subseteq \mathfrak{t}$, invoking the *dual/inverse* root system of Φ.

The end of remarks/examples is signalled by a '$\diamondsuit$', inspired by [92].

2. Formal connections

2.1. Principal bundles on DVRs. — We first consider 'formal germs' of meromorphic connections on principal bundles over Riemann surfaces, in an abstract setting: cf., e.g., [7]; and [83], [57, § 2.3.1], for a modern view. (Again, we only consider the untwisted/unramified case.)

Experts might want to skip to the main classification statement of § 3; or even directly to § 4, for the 'quantum' version.

2.1.1. — Let $\mathbf{G}$ be a connected reductive algebraic group defined over $\mathbb{C}$, with Lie algebra $\mathfrak{g} \coloneqq \mathrm{Lie}(\mathbf{G})$. Let also $\mathscr{O}$ be a *complete* DVR (= discrete valuation ring), with maximal ideal $\mathfrak{m} \subseteq \mathscr{O}$ and residue field isomorphic to $\mathbb{C}$: we fix once and for all an identification $\mathscr{O}/\mathfrak{m} \simeq \mathbb{C}$, as well as a ring embedding $\mathbb{C} \hookrightarrow \mathscr{O}$ which splits the canonical projection onto the residue field.[10] Denote by $\mathscr{O} \hookrightarrow \mathscr{K} \coloneqq \mathrm{Frac}(\mathscr{O})$ the fraction field, and consider the $\mathscr{O}$-submodules

$$\mathfrak{m}^{-s}\mathscr{O} \coloneqq \{ f \in \mathscr{K} \mid \mathfrak{m}^s f \subseteq \mathscr{O} \}, \qquad s \in \mathbb{Z}_{>0}. \tag{8}$$

(Equivalently, using the discrete valuation $\nu \colon \mathscr{K} \to \mathbb{Z} \cup \{+\infty\}$, look at elements such that $\nu(f) \geqslant -s$.)

The ($\mathscr{O}$-)*disc* is the affine $\mathbb{C}$-scheme $\mathcal{D} \coloneqq \mathrm{Spec}\,\mathscr{O}$, with *origin* $0 \coloneqq \mathfrak{m}$. (It is *not* of finite type over $\mathbb{C}$, and so not a variety.)

(10) Thus we choose a *coefficient field*, as per the structure theorem of complete Noetherian local rings [44]. This is w.l.o.g., since we only care about completed local rings of Riemann surfaces/complex projective curves.

2.1.2. — Now consider a principal $\mathbf{G}$-bundle $\pi : \mathcal{E} \to \mathcal{D}$, i.e., a (right) $\mathbf{G}$-*torsor*. By definition, π is a faithfully-flat affine morphism from a $\mathbb{C}$-scheme $\mathcal{E}$, equipped with a right $\mathbf{G}$-action $\rho : \mathcal{E} \times \mathbf{G} \to \mathcal{E}$, such that: (i) $\pi \circ \rho = \pi \circ p_1$, where $p_1 : \mathcal{E} \times \mathbf{G} \to \mathcal{E}$ is the canonical projection; and (ii) the shear map $\mathcal{E} \times \mathbf{G} \to \mathcal{E} \times_{\mathcal{D}} \mathcal{E}$—induced by (p_1, ρ)—is an isomorphism. Moreover, we require that π be locally trivial in the flat topology (a.k.a. *fppf = fidèlement plate de présentation finie*); however, over local Henselian rings this the same as being étale-locally-trivial [1, Prop. 8.1]. In turn, the first cohomology $H^1_{\text{ét}}(\mathcal{D}, \mathbf{G})$ is trivial, and so π is actually *globally* trivializable. (This does *not* require that $\mathbf{G}$ is reductive.)

Hereafter, fix a trivialization $\mathcal{E} \simeq \mathcal{D} \times \mathbf{G} \to \mathcal{D}$, and regard π as the canonical projection.

2.1.3. — Changing global trivialization amounts to acting by the (formal) *gauge group of* π, i.e., the group of automorphisms of π covering the identity on the base. As customary, we identify it with the $\mathscr{O}$-points of $\mathbf{G}$:

$$\mathbf{G}(\mathscr{O}) := \mathrm{Hom}_{\mathrm{Sch}}(\mathcal{D}, \mathbf{G}). \tag{9}$$

A *frame of* π, at the origin, is the choice of a $\mathbb{C}$-point in the fibre over 0, i.e., equivalently, an element $g \in G := \mathbf{G}(\mathbb{C})$.(11) The triple $(\pi, 0, g)$ is a *framed* principal $\mathbf{G}$-bundle over the pointed disc $(\mathcal{D}, 0)$.

The *based* gauge group of $(\pi, 0, g)$ consists of the subgroup of (9) preserving the frame: this is the kernel of the group morphism $\mathbf{G}(\mathscr{O}) \twoheadrightarrow G$, suggestively denoted by $\mathbf{G}(\mathfrak{m})$; while G acts on frames in simply-transitive fashion. The corresponding group sequence splits, and the semidirect product $\mathbf{G}(\mathscr{O}) \simeq G \ltimes \mathbf{G}(\mathfrak{m})$ is implicit in what follows. (This corresponds to the exact $\mathscr{O}$-linear sequence $0 \to \mathfrak{m} \to \mathscr{O} \to \mathbb{C} \to 0$, whose splitting relies on the above choice of coefficient field.)

There is a corresponding split short exact sequence of complex Lie algebras:

$$0 \longrightarrow \mathfrak{g}(\mathfrak{m}) \longrightarrow \mathfrak{g}(\mathscr{O}) \longrightarrow \mathfrak{g} \longrightarrow 0,$$

where $\mathfrak{g}(\mathfrak{m}) := \mathfrak{g} \otimes \mathfrak{m}$ and $\mathfrak{g}(\mathscr{O}) := \mathfrak{g} \otimes \mathscr{O}$ inherit a Lie bracket from $\mathfrak{g}$ via

$$\left[X \otimes f, Y \otimes f'\right] := [X, Y] \otimes (ff'), \qquad X, Y \in \mathfrak{g}, \quad f, f' \in \mathscr{O}.$$

(Recall that $\mathfrak{g}(\mathscr{O})$ is a.k.a. the *current* Lie algebra.) More generally, for any $\mathscr{O}$-module V we consider the complex vector space $\mathfrak{g}(V) := \mathfrak{g} \otimes V$, and if $W \subseteq V$ is a submodule then the $\mathbb{C}$-linear isomorphism $\mathfrak{g}(V/W) \simeq \mathfrak{g}(V)\big/\mathfrak{g}(W)$ is tacit in what follows. As above, if—moreover—$\mathcal{R}$ is a (possibly nonunital) commutative $\mathbb{C}$-algebra then we regard $\mathfrak{g}(\mathcal{R})$ as a complex Lie algebra; e.g., the (formal) *loop* algebra of $(\mathfrak{g}, \mathscr{K})$ is $\mathfrak{g}(\mathscr{K})$.

(11) As in § 1, we view G as a (connected) complex Lie group with (reductive) Lie algebra $\mathfrak{g}$. Then the identity component of $Z(G)$ is isomorphic to a finite power of $\mathbb{C}^\times$, cf. [46, Prop. D.2.1 + Exmp. D.3.3].

2.2. Connections. — Now consider connections on π, possibly singular at the origin. As in the nonformal setting (cf. § 1.2), one can define them in terms of $\mathfrak{g}$-valued 1-forms on the total space of the principal bundle; but here we will just work on the base—in the given trivialization.

2.2.1. — Namely, consider the module of *continuous* Kähler differentials of $\mathscr{K}/\mathbb{C}$, denoted by

$$\mathscr{K} \xrightarrow{\mathrm{d}} \Omega^1_{\mathscr{K}} = \Omega^1_{\mathscr{K}/\mathbb{C}}. \tag{10}$$

It is the $\mathfrak{m}$-adic completion of the module of ordinary Kähler differentials. (Again, we will *not* use the uncompleted version.) Then a *connection on* π, possibly singular at 0, can be just defined as an element $\widehat{\mathcal{A}} \in \mathfrak{g}(\Omega^1_{\mathscr{K}})$. We will also use the (completed) $\mathscr{O}$-submodule

$$\Omega^1_{\mathscr{O}} = \Omega^1_{\mathscr{O}/\mathbb{C}} \hookrightarrow \Omega^1_{\mathscr{K}},$$

so that $\mathfrak{g}(\Omega^1_{\mathscr{O}}) \subseteq \mathfrak{g}(\Omega^1_{\mathscr{K}})$ is the vector subspace of *nonsingular connections*. There is also an $\mathscr{O}$-linear identification $\Omega^1_{\mathscr{K}} \simeq \Omega^1_{\mathscr{O}} \otimes_{\mathscr{O}} \mathscr{K}$ (extending scalars), and analogously to (8) we consider the $\mathscr{O}$-submodule

$$\mathfrak{m}^{-s}\Omega^1_{\mathscr{O}} := \Omega^1_{\mathscr{O}} \otimes_{\mathscr{O}} \big(\mathfrak{m}^{-s}\mathscr{O}\big) \simeq \Big\{ \alpha \in \Omega^1_{\mathscr{K}} \;\Big|\; \mathfrak{m}^s \alpha \in \Omega^1_{\mathscr{O}} \Big\}, \qquad s \in \mathbb{Z}_{>0}.$$

Then $\mathfrak{g}(\mathfrak{m}^{-s}\Omega^1_{\mathscr{O}})$ is the subspace of connections *with pole order bounded by* s. Finally, given a connection $\widehat{\mathcal{A}}$, its *principal part* $\mathcal{A}$ is the equivalence class of $\widehat{\mathcal{A}}$ in the quotient $\mathfrak{g}\big(\Omega^1_{\mathscr{K}}/\Omega^1_{\mathscr{O}}\big)$. (Strictly speaking, completions are not required here.) Finally, a principal part (also) has pole order bounded by s if it lies in the finite-dimensional subspace $\mathfrak{g}\big(\mathfrak{m}^{-s}\Omega^1_{\mathscr{O}}/\Omega^1_{\mathscr{O}}\big)$.

2.2.2. — Fix now also an Ad_G-invariant nondegenerate symmetric bilinear form $(\cdot \mid \cdot) : \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$. Using (a slight modification of) the usual 2-cocycle of $\mathfrak{g}(\mathscr{K})$, which involves the *residue map* $\mathrm{Res} : \Omega^1_{\mathscr{K}} \to \mathbb{C}$—extended $\mathbb{C}$-bilinearly to $\mathfrak{g}(\Omega^1_{\mathscr{K}})$—, we follow [23] and pair the space of pole-order-bounded principal parts with the TCLA

$$\mathfrak{g}_s = \mathfrak{g}(\mathscr{O}_s), \qquad \mathscr{O}_s := \mathscr{O}/\mathfrak{m}^s, \tag{11}$$

analogously to (5). (The marked point is still $0 \in \mathcal{D}$.) This remains nondegenerate, and so hereafter we tacitly consider the principal part $\mathcal{A}$, of a pole-order-bounded connection $\widehat{\mathcal{A}}$, as an element of $\mathfrak{g}_s^{\vee}$.

In addition, we denote by $\Lambda := \mathrm{Res}(\widehat{\mathcal{A}}) \in \mathfrak{g}$ the residue of (the principal part of) a connection $\widehat{\mathcal{A}}$. Somewhat conversely, the *irregular part of* $\widehat{\mathcal{A}}$ is the equivalence class of $\widehat{\mathcal{A}}$ in $\mathfrak{g}\big(\mathfrak{m}^{-s}\Omega^1_{\mathscr{O}}/\mathfrak{m}^{-1}\Omega^1_{\mathscr{O}}\big)$. In view of the exactness of the $\mathbb{C}$-linear sequence

$$\mathfrak{g}(\mathscr{K}) \xrightarrow{\mathrm{d}} \mathfrak{g}(\Omega^1_{\mathscr{K}}) \xrightarrow{\mathrm{Res}} \mathfrak{g} \to 0,$$

there exists $Q \in \mathfrak{g}(\mathfrak{m}^{1-s}\mathscr{O})$ such that the irregular part is represented by the residue-free element dQ; the equality $d(\mathscr{O}) = \Omega^1_{\mathscr{O}} \subseteq \Omega^1_{\mathscr{K}}$ implies that it is only the class of Q modulo $\mathfrak{g}(\mathscr{O})$ which matters, and the latter is actually uniquely determined by the irregular class. (Up to tensoring with $\mathfrak{g}$, this is a truncated version of the fact that the structural derivation in (10) induces a $\mathbb{C}$-linear isomorphism $\mathscr{K}/\mathscr{O} \simeq \Omega^1_{\mathscr{K}}/\mathfrak{m}^{-1}\Omega^1_{\mathscr{O}}$.)

2.2.3. — Note that (11) is also the complex Lie algebra of the (nonreductive, finite-dimensional, connected) complex Lie group $G_s := \mathbf{G}(\mathscr{O}_s)$, i.e., the *truncated-current Lie group* (= TCLG). Then the canonical projection $\mathscr{O}_s \twoheadrightarrow \mathscr{O}_s/(\mathfrak{m}/\mathfrak{m}^s) \simeq \mathbb{C}$, yields a short exact group sequence:

$$1 \longrightarrow \mathrm{Bir}_s \longrightarrow G_s \longrightarrow G \longrightarrow 1, \tag{12}$$

whose kernel is sometimes referred to as the *Birkhoff* subgroup. (The notation avoids conflicts with Borel subgroups of G.) Moreover, the given splitting $\mathbb{C} \hookrightarrow \mathscr{O}$ induces an analogous one $\mathbb{C} \hookrightarrow \mathscr{O}_s$,(12) whence a semidirect factorization $G_s \simeq G \ltimes \mathrm{Bir}_s$—which is tacit in what follows. Infinitesimally, this yields a Lie-algebra decomposition

$$\mathfrak{g}_s \simeq \mathfrak{g} \ltimes \mathfrak{bir}_s, \qquad \mathfrak{bir}_s := \mathfrak{g}(\mathfrak{m}/\mathfrak{m}^s) \simeq \mathrm{Lie}(\mathrm{Bir}_s).$$

Then the dual vector space splits as $\mathfrak{g}_s^\vee \simeq \mathfrak{g}^\vee \oplus \mathfrak{bir}_s^\vee$: the former direct summand corresponds to the residue, and the latter to the irregular part (cf. also Rmk. 2.3.4).

2.2.4. Remark. — By hypothesis, the Lie algebra $\mathfrak{g} \simeq \mathfrak{g}_s/\mathfrak{bir}_s$ is reductive, and $\mathfrak{bir}_s \subseteq \mathfrak{g}_s$ is a nilpotent ideal: hence, it is the nilradical. Analogously, $\mathrm{Bir}_s \subseteq G_s$ is the unipotent radical.

It follows that $\exp : \mathfrak{bir}_s \to \mathrm{Bir}_s$ is bijective, and that the product of the latter is controlled by the Lie bracket of the former via—the inverse of—the truncated Baker–Campbell–Hausdorff formula (= BCH [62]). In particular, geometrically, one can view Bir_s as a complex affine space. ♢

2.3. Formal gauge action. — The gauge group (9) acts by pullback on the space of connections, on the *right*, in affine fashion. One can express this intrinsically, as usual, correcting the (inverse) Adjoint action by the formal logarithmic derivative—viz., the completed pullback of the left-invariant Maurer–Cartan form (cf. [83, § 2.1]).

Nonetheless, we will conclude this section of intrinsic definitions by giving the explicit formula that one concretely uses: this relies on one further choice, which is tacitly used until § 5 (where we prove an invariance statement, to get rid of it.)

2.3.1. — Let ϖ be a *uniformizer for* $\mathscr{O}$, i.e., a generator of $\mathfrak{m}$. Then there are isomorphisms of $\mathbb{C}$-algebras

$$\mathscr{O} \simeq \mathbb{C}[\![\varpi]\!], \qquad \mathscr{K} \simeq \mathbb{C}(\!(\varpi)\!).$$

(12) Making $\mathscr{O}_s$ into a *finite-dimensional* $\mathbb{C}$-algebra, so that G_s can also be regarded as a complex (affine) algebraic group; the notation will not distinguish the two structures.

Moreover, there are $\mathbb{C}$-linear isomorphisms

$$\mathfrak{m}^s \simeq \varpi^s \mathbb{C}[\![\varpi]\!], \qquad \mathfrak{m}^{-s}\mathscr{O} \simeq \varpi^{-s}\mathbb{C}[\![\varpi]\!],$$

as well as

$$\Omega^1_{\mathscr{O}} \simeq \mathbb{C}[\![\varpi]\!]\, \mathrm{d}\varpi, \qquad \Omega^1_{\mathscr{K}} \simeq \mathbb{C}(\!(\varpi)\!)\, \mathrm{d}\varpi, \qquad \mathfrak{m}^{-s}\Omega^1_{\mathscr{O}} \simeq \varpi^{-s}\mathbb{C}[\![\varpi]\!]\, \mathrm{d}\varpi,$$

which intertwine the structures of $\mathscr{O}$- and $\mathbb{C}[\![\varpi]\!]$-modules.

As customary, we then also write $\mathbf{G}(\mathscr{O}) = G[\![\varpi]\!]$, and $\mathfrak{g}(\mathscr{O}) = \mathfrak{g}[\![\varpi]\!]$, etc. In particular, connections become elements $\widehat{\mathcal{A}} \in \mathfrak{g}(\!(\varpi)\!)\, \mathrm{d}\varpi$. Those of pole order bounded by $s \geqslant 1$ can be uniquely written as $\widehat{\mathcal{A}} = \mathcal{A} + \mathcal{B}$, with

$$\mathcal{A} = \sum_{i=0}^{s-1} A_i \varpi^{-i-1}\, \mathrm{d}\varpi, \qquad \mathcal{B} \in \mathfrak{g}[\![\varpi]\!]\, \mathrm{d}\varpi, \tag{13}$$

for coefficients $A_0, \dots, A_{s-1} \in \mathfrak{g}$. Assume—w.l.o.g.—that $A_{s-1} \neq 0$, whence it is the *leading coefficient*, and the pole order is *equal to* s. (The latter notion is intrinsic.) We will also say that $\mathcal{B}$ is the *nonsingular part of* $\widehat{\mathcal{A}}$, and decompose the principal part as

$$\mathcal{A} = \mathcal{A}_0 + \mathrm{d}Q, \qquad \mathcal{A}_0 = \Lambda \varpi^{-1}\, \mathrm{d}\varpi,$$

where in turn

$$\Lambda := A_0 = \mathrm{Res}(\mathcal{A}) = \mathrm{Res}(\widehat{\mathcal{A}}), \qquad Q := \sum_{i=1}^{s-1} A_i \frac{\varpi^{-i}}{-i}, \tag{14}$$

invoking the residue and the irregular part. In this setting there is a canonical representative for the latter, because of the vector-space splitting

$$\mathfrak{g}(\!(\varpi)\!)\, \mathrm{d}\varpi = \varpi^{-1}\mathfrak{g}[\varpi^{-1}]\, \mathrm{d}\varpi \oplus \mathfrak{g}[\![\varpi]\!]\, \mathrm{d}\varpi.$$

(But the coefficients different from the residue are *not* intrinsically determined by $\widehat{\mathcal{A}}$.)

Finally, in the ring isomorphism $\mathscr{O}_s \simeq \mathbb{C}[\![\varpi]\!]/\varpi^s\mathbb{C}[\![\varpi]\!]$, we will also—abusively—denote by ϖ the (nilpotent) class of the uniformizer. In particular, there are vector-space decompositions

$$\bigoplus_{i=1}^{s-1} \mathfrak{g} \otimes (\mathbb{C} \cdot \varpi^i) = \mathfrak{bir}_s \subseteq \mathfrak{g}_s = \bigoplus_{i=0}^{s-1} \mathfrak{g} \otimes (\mathbb{C} \cdot \varpi^i).$$

2.3.2. — Now the 'constant' gauge transformations act in ϖ-graded fashion:

$$\widehat{\mathcal{A}}.g = \mathrm{Ad}_{g^{-1}}\big(\widehat{\mathcal{A}}\big), \qquad g \in G. \tag{15}$$

On the other hand, given $\mathbf{X} \in \varpi\mathfrak{g}[\![\varpi]\!]$, the based gauge transformation $\mathbf{h} := e^{\mathbf{X}}$ reads

$$\widehat{\mathcal{A}}.\mathbf{h} = e^{-\mathrm{ad}_{\mathbf{X}}}\big(\widehat{\mathcal{A}}\big) + \widetilde{e}^{\,\mathrm{ad}_{\mathbf{X}}}(\mathrm{d}\mathbf{X}), \tag{16}$$

setting

$$\mathrm{ad}_{\mathbf{X}}(\mathbf{Y}\, \mathrm{d}\varpi) := \mathrm{ad}_{\mathbf{X}}(\mathbf{Y})\, \mathrm{d}\varpi, \qquad \mathbf{X}, \mathbf{Y} \in \mathfrak{g}(\!(\varpi)\!),$$

as well as

$$\widetilde{e}^{\mathrm{ad}_X} = \frac{1-e^{-\mathrm{ad}_X}}{\mathrm{ad}_X} := \sum_{i\geqslant 0} \frac{(-1)^i}{(i+1)!}\,\mathrm{ad}_X^i \in \mathrm{End}_{\mathbb{C}}\big(\mathfrak{g}[\![\varpi]\!]\,d\varpi\big).$$

(The point is that one has the identity $d(e^X) = e^X \cdot \widetilde{e}^{\mathrm{ad}_X}(dX)$.)

2.3.3. *Remark.* — When a uniformizer is given, denote the based gauge group by $G_1[\![\varpi]\!] \subseteq G[\![\varpi]\!]$, and set $\mathfrak{g}_1[\![\varpi]\!] := \varpi\mathfrak{g}[\![\varpi]\!] = \mathrm{Lie}\big(G_1[\![\varpi]\!]\big)$. Taking limits in Rmk. 2.2.4, as $s \to \infty$, it is well-known that the exponential map $\mathfrak{g}_1[\![\varpi]\!] \to G_1[\![\varpi]\!]$ is bijective, as every element of the based gauge group can be *uniquely* written as an infinite (ordered) product of elementary transformations of the form $e^{X\varpi^i}$, for $X \in \mathfrak{g}$ and for an integer $i > 0$ (cf. [7, § 1.5]).

Moreover, one can equip $\mathfrak{g}(\!(\varpi)\!)$ with the standard complete ϖ-adic (ultra)metric, so that $\varpi^i\mathfrak{g}[\![\varpi]\!]$ is a closed subspace for any $i \in \mathbb{Z}$. Then, upgrading the exponential map to a homeomorphism, we can (and will) regard the based gauge group as a topological group.(13) Transferring the ϖ-adic topology onto $\mathfrak{g}(\!(\varpi)\!)\,d\varpi$, in $\mathbb{C}$-linear fashion, the action (16) is then *continuous*. (There is a textbook proof using uniformly-convergent sequences, which we omit.) ♢

2.3.4. *Remark.* — In principle, the duality of § 2.2.2 matches up an element of $\mathfrak{g}$ with a residue 1-form $\mathcal{A}_0 = \Lambda\varpi^{-1}\,d\varpi$—as in (13). Nonetheless, the residue Λ is independent of ϖ, and we will (also) just view it as an element of $\mathfrak{g}^\vee$ using the nondegenerate pairing.

Again, the above does *not* work for the coefficients A_i of the irregular part, without tensoring by $\varpi^{-i-1}\,d\varpi$, for $i \in \{1,\dots,s-1\}$. On the contrary, if ϖ is given, then after truncation the action (15) matches up with the inverse Ad_G-action on the coefficients of a principal part $\mathcal{A}$: the latter is (thus) identified with the restriction $\mathfrak{g}_s^\vee \times G \to \mathfrak{g}_s^\vee$ of the inverse $\mathrm{Ad}^\vee_{G_s}$-action—in the given group embedding $G \hookrightarrow G_s$. In particular, the $\mathrm{Ad}^\vee_G$-stabilizer of $\mathcal{A}$ (which is intrinsic) is the intersection of the Ad_G-stabilizer of the coefficients of $\mathcal{A}$; and the $\mathrm{Ad}^\vee_G$-stabilizer of an irregular part dQ is the intersection of the Ad_G-stabilizer of the coefficients of Q.

(Henceforth, all these identifications are tacit.) ♢

3. Normal forms/orbits

3.1. Based gauge invariants. — Here we recall one main statement around the classification of untwisted/unramified connections on a framed (trivial) principal G-bundle over the disc. This amounts to looking at certain $G_1[\![\varpi]\!]$-orbits inside $\mathfrak{g}(\!(\varpi)\!)\,d\varpi$; cf. again [7], as well as [8], amongst many others.

(13) We will *not* use proalgebraic structures; the main point is that we consider $G[\![\varpi]\!]$-orbits which are controlled by their (algebraic) truncated analogues, and work with the latter.

3.1.1. — Analogously to Rmk. 2.3.4, note that (16) preserves the subspace $\mathfrak{g}[\![\varpi]\!]\,\mathrm{d}\varpi$, and that it acts *linearly* on principal parts—upon removing the nonsingular part of the output. Moreover, it preserves the spaces of principal parts of pole order bounded by $s \geqslant 1$, on which the subgroup $\exp\bigl(\varpi^s \mathfrak{g}[\![\varpi]\!]\bigr) \subseteq G_1[\![\varpi]\!]$ acts trivially. Hence, truncating the gauge action,(14) a $G_1[\![\varpi]\!]$-orbit yields an $\mathrm{Ad}^{\vee}_{\mathrm{Bir}_s}$-orbit in $\mathfrak{g}^{\vee}_s$, in the duality of § 2.2.2 (and in the notation of (11)–(12)): in this text we consider connections whose based gauge orbit is—conversely—determined by the Birkhoff orbit of their principal part, as follows. (We view this terminology as standard.)

3.1.2. Definition. — Let $\widehat{\mathcal{A}}$ be a connection on the framed principal G-bundle $(\pi, 0, g)$. Then:

1. $\widehat{\mathcal{A}}$ is *untwisted* (a.k.a. *unramified*) if the based gauge orbit $G_1[\![\varpi]\!].\widehat{\mathcal{A}}$ contains an element $\widehat{\mathcal{A}}' = (\mathcal{A}'_0 + \mathrm{d}Q') + \mathcal{B}'$ such that the coefficients of Q' commute with each other and are semisimple, in the notation of (13) and (14);
2. if this holds, $\widehat{\mathcal{A}}$ is *semisimple* if moreover $\Lambda' = A'_0 := \mathrm{Res}\bigl(\widehat{\mathcal{A}}'\bigr)$ is semisimple and commutes with (the coefficients of) Q';
3. if both hold, $\widehat{\mathcal{A}}$ is *nonresonant* if moreover $\mathrm{ad}_{\Lambda'}$ has no nonzero integer eigenvalues upon restriction to the ad-stabilizer of (the coefficients of) Q'.

3.1.3. Definition. — For short, framed UnTwisted Semisimple connections (resp., Nonresonant such) are said to be *UTS* (resp., *NUTS*). Moreover, if $\widehat{\mathcal{A}}$ is NUTS:

1. the principal part $\mathcal{A}' = \mathcal{A}'_0 + \mathrm{d}Q'$ is the *normal form of* $\widehat{\mathcal{A}}$, and—conversely—such principal parts are also said to be *NUTS*;
2. Q' is the *irregular type of* $\widehat{\mathcal{A}}$;
3. and $\Lambda' \in \mathfrak{g}$ is the *normal residue of* $\widehat{\mathcal{A}}$.

More generally, if $\widehat{\mathcal{A}}$ is UTS then $\mathcal{A}'$ is said to be its (UTS) *normal principal part*, and we use the same terminology for its *irregular type* and *normal residue*.

3.1.4. Remark. — Again, Deff. 3.1.2–3.1.3 are far from general, with a view towards quantization. E.g., one can define nonresonance also when the normal residue has a nilpotent part—but it is still only the semisimple part which plays any role. And one can enter into the twisted/ramified setting, where the normal forms involve formal Puiseux gauge transformations, i.e., negative powers of a root of the formal variable ϖ (again, cf. [**33**, **32**, **58**] and references therein, e.g., [**9**, **103**, **104**], etc.).

On the contrary, Deff. 3.1.2–3.1.3 include *generic* connections, whose based gauge orbit contain an element with regular semisimple leading coefficient, which used to be the classical setup in the irregular-singular case (particularly JMU [**89**], when on vector bundles). More generally, they include the case of 'complete fission' [**26**], i.e., when the Ad_G-stabilizer of (the coefficients of) Q' is a maximal torus of G. ♢

(14) And inverting it, for convenience, turning it into a *left* action.

3.1.5. Remark. — If the condition of Def. 3.1.2 (1.) holds, then up to based gauge one can assume that Λ' commutes with Q': the condition (2.) is precisely that the projection of the residue on the $\mathrm{ad}_{\mathfrak{g}}$-stabilizer of Q' be semisimple.

Moreover, the former is equivalent to the following: there exists a maximal torus $T \subseteq G$, with (Cartan) Lie subalgebra $\mathfrak{t} := \mathrm{Lie}(T) \subseteq \mathfrak{g}$, such that $Q' \in \mathfrak{t}(\!(\varpi)\!)/\mathfrak{t}[\![\varpi]\!]$. Neglecting parabolic/parahoric structures, this is the condition for 'very good' connections typically found in the literature, and used, e.g., in [148, 56, 57].(15) ♢

3.1.6. — The main point here is that the normal form is a *complete* invariant of NUTS based gauge orbits. To state it precisely, for any element Q as in (14) denote by $\mathfrak{l}_Q \subseteq \mathfrak{g}$ the intersection of the $\mathrm{ad}_{\mathfrak{g}}$-stabilizer subalgebras

$$\mathfrak{g}^{A_i} := \ker(\mathrm{ad}_{A_i}), \qquad i \in \{1, \dots, s-1\}.$$

Then:

3.1.7. Proposition-Definition. — *Let $X''_s \subseteq \varpi^{-s}\mathfrak{g}[\![\varpi]\!]\,d\varpi$ be the topological subspace of NUTS connections of pole order bounded by s, and denote by $\tau''_s : X''_s \to \mathfrak{g}^{\vee}_s$ the map $\widehat{\mathcal{A}} \mapsto \mathcal{A}'$ (taking the normal form). Then τ''_s is a (globally) trivializable topological principal $G_1[\![\varpi]\!]$-bundle over its image, viz., over the subspace*

$$\mathfrak{g}''_s := \Big\{ \Lambda'\varpi^{-1}d\varpi + dQ' \in \mathfrak{g}'_s \;\Big|\; \mathrm{coker}\big(\mathrm{ad}_{\Lambda'}(Q',\kappa)\big) = 0 \text{ for } \kappa \in \mathbb{Z}\setminus\{0\} \Big\}, \tag{17}$$

where, in turn:

1. *we let*

$$\mathfrak{g}'_s := \Big\{ \Lambda'\varpi^{-1}d\varpi + dQ' \in \mathfrak{g}^{\vee}_s \;\Big|\; \Lambda', A'_1, \dots, A'_{s-1} \text{ are semisimple and commute} \Big\}; \tag{18}$$

2. *we set*

$$\mathrm{ad}_{\Lambda'}(Q',\kappa) := \big(\mathrm{ad}_{\Lambda'}\big|_{\mathfrak{l}_{Q'}} - \kappa\cdot\mathrm{Id}_{\mathfrak{l}_{Q'}}\big) \in \mathfrak{gl}_{\mathbb{C}}(\mathfrak{l}_{Q'}), \qquad \kappa \in \mathbb{Z};$$

3. *and we identify $\mathfrak{g}^{\vee}_s$ with the topological quotient $\varpi^{-s}\mathfrak{g}[\![\varpi]\!]\,d\varpi/\mathfrak{g}[\![\varpi]\!]\,d\varpi$.*

Scheme of a proof. — Besides the fact that all the based gauge orbits through NUTS connections meet (17) in one point, and only one, the statement also summarizes the well-known fact that the $G_1[\![\varpi]\!]$-stabilizers of NUTS connections are trivial.

As per the topologies and the global trivialization, one can show that τ''_s is continuous, and that the (bijective) restriction $\mathfrak{g}''_s \times G_1[\![\varpi]\!] \to X''_s$ of the based-gauge-action map is a homeomorphism. By construction, it intertwines the canonical projection onto $\mathfrak{g}''_s$ with τ''_s. (Cf. § A for few standard complements.) □

(15) Again, beware that this description is coordinate-dependent, and that the intrinsic definition is rather as an element of $\mathfrak{t}(\mathscr{K})/\mathfrak{t}(\mathscr{O})$ (cf. [29, Def. 7.1]); if one insists in *not* fixing a maximal torus from the start, then one should instead consider $\mathbb{C}[\![\varpi]\!]^{\times}$-orbits of elements Q' having semisimple commuting coefficients (cf. § 5).

***3.1.8. Remark.* —** In (17), it would be equivalent to ask that

$$\operatorname{coker}\big(\mathrm{ad}_{\Lambda'}(Q',\kappa)\big) = 0, \qquad \kappa \in \mathbb{Z}_{>0},$$

because the nonzero eigenvalues of the adjoint action arise in opposite pairs. (And strictly speaking, to put a connection in normal form only the positive integer eigenvalues play any role; cf. again § A, as well as Rmk. 4.2.7.) ♢

***3.1.9. Example.* —** Suppose that $s = 1$ (a.k.a. the *tame/logarithmic* case).

A connection $\widehat{\mathcal{A}} = \mathcal{A}_0 + \mathcal{B} \in \varpi^{-1}\mathfrak{g}[\![\varpi]\!]\,d\varpi$ is UTS (resp., NUTS) if and only if it has semisimple residue $\Lambda \in \mathfrak{g}$ (resp., also nonresonant)—and it is vacuously untwisted/unramified. Then $\mathfrak{g}'_1$ can be identified with the semisimple locus in $\mathfrak{g}$, and $\mathfrak{g}''_1$ with the nonresonant part therein. Now τ''_1 simply deletes the nonsingular part, as the $G_1[\![\varpi]\!]$-action cannot modify the leading term, which here coincides with the (normal) residue $\Lambda = \Lambda'$. ♢

3.2. About full gauge invariants. — The upshot of § 3.1 is that the based gauge orbits of NUTS connections are determined by the coadjoint Birkhoff orbit of their principal part, and that the latter meets (17) in exactly one point (cf. the first part of [39]).

3.2.1. — Hereafter, we thus focus on $\mathfrak{g}_s^{\vee}$, which we now rather view as a complex affine Poisson *variety*—in Zariski topology. To get to the underlying (unframed) principal G-bundle π, one must act by the 'constant' part (15) of the full gauge action, which can be regarded as an algebraic **G**-action; and to this extent:

***3.2.2. Lemma.* —**

1. *The subspaces* $\mathfrak{g}''_s \subseteq \mathfrak{g}'_s \subseteq \mathfrak{g}^{\vee}_s$ *of* (17)–(18) *are* G*-invariant.*
2. *And if an* $\mathrm{Ad}^{\vee}_{G_s}$*-orbit* $\mathcal{O}' \subseteq \mathfrak{g}^{\vee}_s$ *intersects either of the subspaces* (17)–(18)*, then the intersection is precisely a* G*-orbit.*

Proof postponed to C.1. — □

3.2.3. — By Lem. 3.2.2, in particular, the normal form is *not* a well-defined invariant for the full gauge action, and one can use the following (well-posed) variations of Deff. 3.1.2–3.1.3:

***3.2.4. Definition.* —** A connection $\widehat{\mathcal{A}}$ on the principal G-bundle π is *nonresonant/untwisted/semisimple* if either condition holds upon framing π at 0.

***3.2.5. Definition.* —** For short, UnTwisted Semisimple connections (resp., Nonresonant such) are said to be *UTS* (resp., *NUTS*). Moreover, if $\widehat{\mathcal{A}}$ is NUTS:

1. the *normal orbit of* $\widehat{\mathcal{A}}$ is the G-orbit of the normal principal part $\mathcal{A}'$, in any frame (of π, at 0);
2. the *irregular class* $\Theta' = \Theta(Q')$ *of* $\widehat{\mathcal{A}}$ is the G-orbit of the irregular type Q', in any frame;
3. and the *normal residue orbit of* $\widehat{\mathcal{A}}$ is the G-orbit of the normal residue Λ', in any frame.

More generally, if $\widehat{\mathcal{A}}$ is UTS then the G-orbit of $\mathcal{A}'$ is said to be its *normal principal orbit*, and we use the same terminology for its *irregular class* and *normal residue orbit*.

3.2.6. Remark. — We will not use the normal residue G-orbit in any meaningful way. Nonetheless, when a frame is given, the orbit of the normal residue for the action of the Ad-stabilizer of the irregular type will play a role (cf., e.g., the proof of Prop. 3.2.10). ♢

3.2.7. Remark. — If we choose a starting maximal torus $T \subseteq G$, as in Rmk. 3.1.5, and impose that Q' takes coefficients in its Lie algebra $\mathfrak{t} \subseteq \mathfrak{g}$, then we can break the action of G down to the Weyl group W. In particular, we find the usual notion that untwisted/unramified irregular classes are W-orbits in the quotient $\mathfrak{t}(\!(\varpi)\!)/\mathfrak{t}[\![\varpi]\!]$ [**29**, Rmk. 10.6] (cf. [**55**] and (6)).(16) ♢

3.2.8. — Neglecting topologies, a corollary of Prop.-Def. 3.1.7 is a canonical bijection $X''_s/G[\![\varpi]\!] \xrightarrow{\simeq} \mathfrak{g}''_s/G$—taking the normal orbit. What matters here is that choosing an $\mathrm{Ad}^\vee_{G_s}$-orbit $\mathcal{O}' \subseteq \mathfrak{g}^\vee_s$ which intersects $\mathfrak{g}''_s \subseteq \mathfrak{g}^\vee_s$ is the same as prescribing a full gauge orbit $\widehat{\mathcal{O}}' \subseteq \varpi^{-s}\mathfrak{g}[\![\varpi]\!]\, d\varpi$. As mentioned in the introduction § 1, these coadjoint orbits can (thus) be viewed as the local pieces of wild de Rham spaces.

3.2.9. — Hereafter, the $\mathrm{Ad}^\vee_{G_s}$-orbits $\mathcal{O}' \subseteq \mathfrak{g}^\vee_s$ are regarded as (nonsingular, irreducible) complex algebraic varieties; but recall that we want to quantize symplectic *affine* varieties. To this end, we use the following generalization of the well-known fact that a standard $\mathrm{Ad}^\vee_G$-orbit $\mathcal{O}' \subseteq \mathfrak{g}^\vee$ is affine if—and only if—it is semisimple:

3.2.10. Proposition. — *The $\mathrm{Ad}^\vee_{G_s}$-orbits through* (18)*, i.e., the UTS coadjoint orbits, are Zariski-closed affine subvarieties of $\mathfrak{g}^\vee_s$. (And so the same holds a fortiori for the orbits intersecting* (17)*, i.e., the NUTS coadjoint orbits.)*

Proof. — Choose an element $\mathcal{A}' \in \mathfrak{g}'_s$, and let $\mathcal{O}' := G_s.\mathcal{A}'$ be its $\mathrm{Ad}^\vee_{G_s}$-orbit. First, recall [**77**, Prop. 2.12] that $\mathcal{O}'$ is (algebraically) symplectomorphic to the Hamiltonian reduction of an 'extended' orbit $\widetilde{\mathcal{O}}' \subseteq G \times \mathfrak{g}^\vee_s$ (cf. (48)), with respect to the free action of the $\mathrm{Ad}^\vee_G$-stabilizer $L' := L_{Q'} \subseteq G$ of the irregular type Q', along the $\mathrm{Ad}^\vee_{L'}$-orbit of $\Lambda' \in \mathfrak{g} \simeq \mathfrak{g}^\vee$ (cf. (53)). In turn, $\widetilde{\mathcal{O}}'$ is (algebraically) symplectomorphic to the direct product of T^*G with the $\mathrm{Ad}^\vee_{\mathrm{Bir}_s}$-orbit of the irregular part dQ' [**148**, Cor. 3.2] (cf. (49)), and so it is affine. Finally, by hypothesis, Λ' is semisimple, whence the residue orbit is also affine. Altogether, it follows that $\mathcal{O}'$ is the quotient of an affine variety by a free L'-action. Now [**116**, Thm. 4.2] implies that this quotient is a geometric quotient: it must be the *affine* one—by the uniqueness of the categorical quotient [**106**].

Furthermore, let $\overline{\mathcal{O}'} \subseteq \mathfrak{g}^\vee_s$ be the Zariski-closure of the orbit, and suppose by contradiction that the complement $Z := \overline{\mathcal{O}'} \setminus \mathcal{O}'$ is nonempty. First, since $\mathcal{O}'$ is affine and

(16)Therefore, a notion of unramified irregular classes which is both coordinate- and torus-free can be obtained by looking at the orbits of elements Q' (with semisimple commuting coefficients) under the group $G \times \mathbb{C}[\![\varpi]\!]^\times$, noting that the actions of the two factors commute (cf. Fn. 15, and again § 5).

open in $\overline{\mathcal{O}'}$, it is well-known that Z has pure codimension 1. On the other hand, Z is a finite union of $\mathrm{Ad}^{\vee}_{G_s}$-orbits, and so it has codimension greater than 1. □

3.2.11. Remark. — We omit the proof of the following (partial) converse: the $\mathrm{Ad}^{\vee}_{G_s}$-orbit through an untwisted principal part $\mathcal{A}$ is Zariski-closed *only if* $\mathcal{A}$ is UTS. ♢

4. Quantization of coadjoint orbits in dual TCLAs

4.1. Stabilizers and Levi/parabolic filtrations. — Here we recall and complement the construction of a deformation quantization of $\mathrm{Ad}^{\vee}_{G_s}$-orbits through UTS principal parts. (We still use a uniformizer ϖ.)

4.1.1. — We follow [39]. Let $\mathcal{O}' \subseteq \mathfrak{g}_s^{\vee}$ be an $\mathrm{Ad}^{\vee}_{G_s}$-orbit such that $\mathcal{O}' \cap \mathfrak{g}'_s \neq \varnothing$: under a further technical hypothesis (cf. Rmk. 4.3.3), the second part of op. cit. considers 'parabolically' induced $U(\mathfrak{g}_s)$-modules, which ultimately lead to a deformation quantization of the (nondegenerate) restriction of the Lie–Poisson bracket of $\mathfrak{g}_s^{\vee}$ to the symplectic leaf $\mathcal{O}'$—i.e., the inverse of the KKS symplectic structure. Please refer to [39, § 16] for details, and note that op. cit. is phrased in the complex-analytic category; here we work in the complex-algebraic one.

4.1.2. — Choose a UTS principal part $\mathcal{A}' \in \mathfrak{g}'_s$, so that Prop. 3.2.10 ensures that the orbit $\mathcal{O}' = G_s.\mathcal{A}'$ is affine. Then the $\mathrm{Ad}^{\vee}_{G_s}$-stabilizer $\mathbf{L} := G_s^{\mathcal{A}'} \subseteq G_s$—of $\mathcal{A}'$—decomposes as:

$$\mathbf{L} = L_s \ltimes \prod_{i=1}^{s-1} \exp(\mathfrak{l}_{s-i} \cdot \varpi^i), \tag{19}$$

where, in turn: (i) $L_s = L_{\mathcal{A}'} \subseteq G$ is the Ad_G-stabilizer of (the coefficients of) $\mathcal{A}'$, including the normal residue $\Lambda' = A'_0$; and (ii)

$$\mathfrak{l}_i := \mathfrak{g}^{A'_{s-1}} \cap \dots \cap \mathfrak{g}^{A'_{s-i}} \subseteq \mathfrak{g}, \qquad i \in \{1, \dots, s-1\}.$$

(In particular $\mathfrak{l}_{s-1} = \mathfrak{l}_{Q'}$, in the notation of § 3.1.) Set also $\mathfrak{l}_s := \mathrm{Lie}(L_s) \subseteq \mathfrak{g}$: then $\mathfrak{l}_i$ is a reductive Lie subalgebra, integrating to a well-defined connected reductive (Zariski-)closed subgroup $L_i \subseteq G$, for $i \in \{1, \dots, s\}$.

Finally, denote by $\boldsymbol{\mathfrak{l}} := \mathfrak{g}_s^{\mathcal{A}'} \subseteq \mathfrak{g}_s$ the $\mathrm{ad}^{\vee}_{\mathfrak{g}_s}$-stabilizer of $\mathcal{A}'$, i.e., the Lie algebra of (19).

4.1.3. Remark. — Equivalently, consider the decreasing sequence of reductive subgroups of G obtained from the nested Ad_G-stabilizers of the coefficients of $\mathcal{A}'$:

$$L_0 := G, \qquad L_{i+1} = L_i^{A'_{s-i}} := \left\{ g \in L_i \mid \mathrm{Ad}_g(A'_{s-i}) = A'_{s-i} \right\},$$

for $i \in \{1, \dots, s\}$. (And then take their Lie algebras.)

This is a.k.a. the *fission sequence* (of G) determined by $\mathcal{A}'$. Beware that [26, 29] uses this terminology for the subsequence $L_{s-1} \subseteq \dots \subseteq L_1 \subseteq G$ determined by the irregular type Q', but in this text the residue centralizers also play an important role in light of the main Thm. 1.3.4. ♢

4.1.4. — Now choose also (noncanonically) two decreasing sequences

$$G =: P_0^{\pm} \supseteq P_1^{\pm} \supseteq \cdots \supseteq P_s^{\pm}, \tag{20}$$

of parabolic subgroups of G, such that $P_i^+ \cap P_i^- = L_i$, so that L_i is a (common) Levi factor of $P_i^{\pm}$. The corresponding sequence $\mathfrak{p}_i^{\pm} := \mathrm{Lie}(P_i^{\pm}) \subseteq \mathfrak{g}$ of parabolic subalgebras determines two *opposite polarizations* on $\mathcal{O}'$, i.e., a G_s-invariant (algebraic) Lagrangian splitting of the tangent bundle

$$T\mathcal{O}' \simeq G_s \times^{\boldsymbol{L}} (\mathfrak{g}_s/\boldsymbol{\mathfrak{l}}) \longrightarrow \mathcal{O}',$$

in the usual identification $\mathcal{O}' \simeq G_s/\boldsymbol{L}$—as homogeneous G_s-varieties. (Beware that in §§ 8–9 we will also consider a *different* parabolic sequence.)

The actual polarizations are defined from the nested nilradicals of the parabolic subalgebras, which must satisfy a further condition (cf. again Rmk. 4.3.3). At this stage, just observe that this is an entry point into a representation-theoretic quantization of $\mathcal{O}'$.

4.2. Representation-theoretic setup. — Namely, we use the parabolic filtrations $\mathfrak{p}_s^{\pm} \subseteq \cdots \subseteq \mathfrak{p}_1^{\pm} \subseteq \mathfrak{g}$ to define parabolically-induced Verma modules for TCLAs.

4.2.1. — By construction, the coefficient A'_{s-i} (of ϖ^{s-i-1}) in $\mathcal{A}'$ lies in the Lie-centre $\mathfrak{Z}(\mathfrak{l}_i) \subseteq \mathfrak{l}_i$, for $i \in \{0, \ldots, s-1\}$. In view of the given dualities, it corresponds to a linear map on $\mathfrak{Z}(\mathfrak{l}_i)$, and the latter extends (by zero) to a character $\chi_i^{\pm} : \mathfrak{p}_i^{\pm} \to \mathbb{C}$. We assemble them into a character $\boldsymbol{\chi}^{\pm}$ of the Lie subalgebra $\boldsymbol{\mathfrak{p}}^{\pm} := \bigoplus_{i=0}^{s-1} (\mathfrak{p}_{s-i}^{\pm} \cdot \varpi^i) \subseteq \mathfrak{g}_s$, in ϖ-graded fashion:

$$\boldsymbol{\chi}^{\pm} : \sum_{i=0}^{s-1} X_i \varpi^i \longmapsto \sum_{i=0}^{s-1} \chi_{s-i}^{\pm}(X_i), \qquad X_i \in \mathfrak{p}_{s-i}^{\pm}.\text{[17]}$$

4.2.2. Definition **(Cf. [43, 39]).** — The balanced tensor products

$$M^{\pm} = M^{(s)}_{\boldsymbol{\mathfrak{p}}^{\pm}, \boldsymbol{\chi}^{\pm}} := \mathrm{Ind}_{U(\boldsymbol{\mathfrak{p}}^{\pm})}^{U(\mathfrak{g}_s)}(\boldsymbol{\chi}^{\pm}) = U(\mathfrak{g}_s) \otimes_{U(\boldsymbol{\mathfrak{p}}^{\pm})} (\boldsymbol{\chi}^{\pm}), \tag{21}$$

are the *parabolic Verma modules of highest/lowest weight* $\mathcal{A}'$, *parabolic filtration* $\boldsymbol{\mathfrak{p}}^{\pm}$, and *truncation index* s.

4.2.3. Example. — Again, the generic examples correspond to the case where $\mathfrak{l}_1 \subseteq \mathfrak{g}$ is a Cartan subalgebra, i.e., where the leading term A'_{s-1} is regular semisimple. In this case $\mathfrak{p}_s^{\pm} = \cdots = \mathfrak{p}_1^{\pm}$ are two opposite Borel subalgebras, which is the setup of [67] (cf. [141]).

Somewhat conversely, if $s = 1$—with $\mathfrak{l}_1$ arbitrary—then $M^{\pm}$ are two 'opposite' standard parabolic Verma modules for $U(\mathfrak{g})$, cf. [82, Chp. 9]. ♢

[17] This relies on the identity $[\boldsymbol{\mathfrak{p}}^{\pm}, \boldsymbol{\mathfrak{p}}^{\pm}] = \bigoplus_{i=0}^{s-1} \big([\mathfrak{p}_{s-i}^{\pm}, \mathfrak{p}_{s-i}^{\pm}] \cdot \varpi^i \big) \subseteq \mathfrak{g}_s$.

4.2.4. Remark. — It follows that $M^{\pm}$ are cyclically generated—over $U(\mathfrak{g}_s)$—by the vectors

$$v^{\pm} = v^{\pm}_{\mathfrak{p}^{\pm},\chi^{\pm}} := 1 \otimes_{U\mathfrak{p}^{\pm}} 1, \tag{22}$$

respectively, on which $U\mathfrak{p}^{\pm}$ act by (the universal enveloping version of) $\chi^{\pm}$. This yields an alternative presentation of (21), as quotients of $U(\mathfrak{g}_s)$ by the left submodules annihilating (22).

By construction, v^{+} is a *highest-weight vector* in the sense of [43, § 4.1], and so in turn M^{+} is a *highest-weight module* (cf. § 4.2 of op. cit.). Moreover, being a quotient of a regular/non-parabolic Verma module of the same highest weight, M^{+} lies in the category $\mathcal{O}^{(\chi_s^{+},\dots,\chi_1^{+})}$ of [43, § 4.3]. (Cf. [42] when $s = 2$.) Analogously, M^{-} is a *lowest-weight module*, generated by the *lowest-weight vector* v^{-}, etc. ♢

4.2.5. — The inclusions are reversed for the Lie-centres of the Levi factors, i.e., one has a flag $\mathfrak{Z}(\mathfrak{g}) \subseteq \mathfrak{Z}(\mathfrak{l}_1) \subseteq \cdots \subseteq \mathfrak{Z}(\mathfrak{l}_s)$; and $M^{\pm}$ are weight modules for the largest one, which acts in semisimple fashion. Moreover, there is a canonical antipode-contragredient $\mathbb{C}$-bilinear form

$$\mathcal{S} = \mathcal{S}_{\mathfrak{p}^{\pm},\chi^{\pm}} : M^{-} \otimes M^{+} \longrightarrow \mathbb{C}, \tag{23}$$

such that $\mathcal{S}(v^{-} \otimes v^{+}) = 1$, which we call after Shapovalov [139]. The simultaneous decomposition into $\mathfrak{Z}(\mathfrak{l}_s)$-weight spaces is $\mathcal{S}$-orthogonal, and the left/right radicals of (23) coincide with the maximal proper submodules of $M^{\pm}$.

What matters here is that $\mathcal{S}$ can be regarded as an extension of the KKS structure on the orbit, and so the deformation quantization of $\mathcal{O}'$ involves (a formal version of) the inverse tensor of (23), generalizing a particular case of [2]. Concretely, denoting as customary by $\hbar$ a formal deformation parameter, there is a natural $*$-product

$$\mathcal{R}_0 \otimes \mathcal{R}_0 \longrightarrow \mathcal{R}_0[\![\hbar]\!], \qquad \mathcal{R}_0 := \mathbb{C}[\mathcal{O}'], \tag{24}$$

on the commutative Poisson ring of regular functions.

Now the construction of (24) requires that $M^{\pm}$ is simple/irreducible, i.e., that the Shapovalov form is nondegenerate, for generic values of $\mathcal{A}'$. To this extent, the most direct link with meromorphic gauge theory seems to be the following:

***4.2.6. Proposition* (cf. [39], Conj. 15.2.9).** — *If $\mathcal{A}' \in \mathfrak{g}_s''$, then the parabolic Verma modules* (21) *are* simple.

Proof. — For the sake of notation (in this argument), let us write the modules of truncation index $s \geqslant 1$ as

$$M_s^{\pm} = M^{(s)}(\mathfrak{p}_s^{\pm},\dots,\mathfrak{p}_1^{\pm};\chi_s^{\pm},\dots,\chi_1^{\pm}).$$

The 'parabolic restriction' functor of [43, Thm. 4.1] takes $M_s^{\pm}$ to the parabolic Verma modules $\widetilde{M}_s^{\pm} = M^{(s)}\big(\widetilde{\mathfrak{p}}_s^{\pm},\dots,\widetilde{\mathfrak{p}}_1^{\pm};\widetilde{\chi}_s^{\pm},\dots,\widetilde{\chi}_1^{\pm}\big)$, for the depth-$s$ TCLA on the reductive Lie subalgebra $\mathfrak{l}_1 \subseteq \mathfrak{g}$, where

$$\widetilde{\mathfrak{p}}_i^{\pm} := \mathfrak{p}_i \cap \mathfrak{l}_1, \qquad \widetilde{\chi}_i^{\pm} := \chi_i^{\pm}\big|_{\widetilde{\mathfrak{p}}_i^{\pm}}, \qquad i \in \{1,\dots,s\}.$$

(This uses the fact that $\widetilde{\mathfrak{p}}_i^{\pm}$ are parabolic subalgebras of $\mathfrak{l}_1$.) Applying the functor of Lem. 3.7 of op. cit. then yields the modules

$$\underline{\widetilde{M}}_s^{\pm} := M^{(s)}\big(\widetilde{\mathfrak{p}}_s^{\pm},\dots,\widetilde{\mathfrak{p}}_2^{\pm},\widetilde{\mathfrak{p}}_1^{\pm};\widetilde{\chi}_s^{\pm},\dots,\widetilde{\chi}_2^{\pm},0\big).$$

Since $\widetilde{\mathfrak{p}}_1 = \mathfrak{l}_1$, the latter are precisely the modules

$$\widetilde{M}_{s-1}^{\pm} = M^{(s-1)}\big(\widetilde{\mathfrak{p}}_s^{\pm},\dots,\widetilde{\mathfrak{p}}_2^{\pm};\widetilde{\chi}_s^{\pm},\dots,\widetilde{\chi}_2^{\pm}\big),$$

for the $(s-1)$-truncated TCLA on $\mathfrak{l}_1$, with an additional zero-action of $\mathfrak{l}_1 \cdot \varpi^{s-1} \subseteq \mathfrak{g}_s$. In particular, since the above functors are (exact) equivalences—between appropriate categories—, the modules $M_s^{\pm}$ are simple if and only if $\widetilde{M}_{s-1}^{\pm}$ are simple. By induction on $s \geqslant 1$, we conclude that $M_s^{\pm}$ are simple if and only the same holds for

$$M_1^{\pm} = M^{(1)}_{\mathfrak{p}_s^{\pm} \cap \mathfrak{l}_{s-1}, \chi_s^{\pm}},$$

with tacit restriction of the characters. Moreover, it follows that $\mathfrak{l}_s \subseteq \mathfrak{p}_s^{\pm} \cap \mathfrak{l}_{s-1}$ is a (common) Levi factor, so that $\chi_s^{\pm}$ are (still) determined by a linear functional $\lambda' \in \mathfrak{Z}(\mathfrak{l}_s)^{\vee}$, extended by zero to the corresponding nilradicals.

Now we conclude the proof for the highest-weight Verma modules M_1^{+}, and omit that of the analogous 'negative' version. Choose a Cartan subalgebra $\mathfrak{t} \subseteq \mathfrak{g}$ such that $\mathfrak{t} \subseteq \mathfrak{l}_s$, so that $\mathfrak{Z}(\mathfrak{l}_i) \subseteq \mathfrak{t}$ for $i \in \{1,\dots,s\}$; and view λ' as a linear function $\mathfrak{t} \to \mathbb{C}$ vanishing on the subspace $\mathfrak{t} \cap [\mathfrak{l}_s,\mathfrak{l}_s] \subseteq \mathfrak{t}$. Denote then by $\phi_s \subseteq \Phi$ the Levi (root) subsystem determined by $\mathfrak{l}_s$, i.e., the subset of roots annihilating $\mathcal{A}'$. Finally, for simplicity, let $\mathfrak{b} \subseteq \mathfrak{g}$ be a Borel subalgebra such that $\mathfrak{t} \subseteq \mathfrak{b} \subseteq \mathfrak{p}_s^{+}$. (Whence $\mathfrak{p}_s^{+} \subseteq \cdots \subseteq \mathfrak{p}_1^{+}$ is a sequence of *standard* parabolic subalgebras of $(\mathfrak{g},\mathfrak{b})$.) Then [82, Thm. 9.12 (a)] implies in particular that the parabolic Verma module $M_{\lambda'} = M_1^{+}$ is simple if

$$\big\langle \lambda' + \rho, \alpha^{\vee} \big\rangle \notin \mathbb{Z} \setminus \{0\}, \qquad \alpha \in \Phi \setminus \phi_s, \tag{25}$$

where $\alpha^{\vee} \in \Phi^{\vee} \subseteq \mathfrak{t}$ are the coroots and $\rho \in \mathfrak{t}^{\vee}$ is the *Weyl vector*.[18] (Noting that $\langle \lambda', \alpha^{\vee} \rangle = 0$ for $\alpha \in \phi_s$, by construction.) The point now is that $\lambda' \in \mathfrak{t}^{\vee}$ matches up with the normal residue $\Lambda' \in \mathfrak{t}$ under the G-invariant duality used above, and that the nonresonance condition of (17) can be rewritten as

$$\big\langle \lambda', \alpha^{\vee} \big\rangle = \big\langle \alpha, \Lambda' \big\rangle \notin \mathbb{Z} \setminus \{0\}, \qquad \alpha \in \Phi \setminus \phi_s, \tag{26}$$

since the root-valuations of Λ' exhaust the set of nonzero eigenvalues of $\mathrm{ad}_{\Lambda'}$. Finally, recalling that $\langle \rho, \Phi^{\vee} \rangle \subseteq \mathbb{Z}$, one sees that (26) implies (25). □

4.2.7. Remark. — Beware that the usual sufficient stability condition for $M_{\lambda'}$ rather reads

$$\big\langle \lambda' + \rho, \alpha^{\vee} \big\rangle \notin \mathbb{Z}_{>0}, \qquad \alpha \in \Phi^{+} \setminus \phi_s, \tag{27}$$

and it is thus *weaker* than (25): there might be positive coroots with negative integer valuation. The fact is that the nonresonance condition (17) does *not* depend on the

(18) I.e., $\rho := \frac{1}{2} \sum_{\Phi^+} \alpha$ is the half-sum of positive roots $\Phi^{+} \subseteq \Phi$ determined by $\mathfrak{b}$.

choice of a system of positive roots, and indeed (by Lem. 3.2.2) the former is a Weyl-invariant notion—upon intersecting $\mathfrak{g}''_s$ with $\mathfrak{t}$. Turning this around, the nonresonance of the residue implies that *all* the induced Verma modules will be simple, regardless of the choice of a parabolic sequence whose terms contain $\mathfrak{t}$.

Nonetheless, recall that If λ' (or, equivalently, Λ') is regular then the wall-avoiding constraints (27) are also *necessary* for $M_{\lambda'}$ to be simple, cf. [82, Thm 9.12 (b)]. In the proper parabolic case, instead, a sharp criterion is given in [84, Satz 4]. ♢

4.3. Quantum comoments. — For later use, we prove that the $*$-product (24) admits a *strong* quantum comoment map, adapting a result of [40].

4.3.1. — Concretely, consider the topologically-free $\mathbb{C}[\![\hbar]\!]$-module $\mathcal{R}_0[\![\hbar]\!]$. Equip it with an $\hbar$-adically-continuous $\mathbb{C}[\![\hbar]\!]$-bilinear product, defined as follows on pure tensors $f \otimes f' \in \mathcal{R}_0 \otimes \mathcal{R}_0$, with $f, f' \in \mathcal{R}_0$: (i) denote by $\widetilde{f}, \widetilde{f}' \in \mathbb{C}[G_s]$ the pullbacks of f, f'—respectively—on the TCLG, along the fibre-bundle projection $G_s \twoheadrightarrow \mathcal{O}'$; (ii) apply to $\widetilde{f} \otimes \widetilde{f}'$ the formal bidifferential operator $F_{\hbar^{-1}}$, obtained as the formal Taylor expansion of the dilated inverse-Shapovalov-form

$$F_c = F_{c,\mathfrak{p}^{\pm},\chi^{\pm}} := \mathcal{S}^{-1}_{\mathfrak{p}^{\pm},c\chi^{\pm}}, \tag{28}$$ [19]

as the dilation parameter $c \in \mathbb{C}^*$ tends to infinity; (iii) compose with the multiplication of $\mathbb{C}[G_s]$—in $\hbar$-graded fashion—to obtain a formal function on G_s; and (iv) note that the latter descends to an element $f * f' \in \mathcal{R}_0[\![\hbar]\!]$, in view of the G_s-invariance of $F_{\hbar^{-1}}$. (The crux of the matter is proving the associativity of this operation.)

Then one also shows that

$$f * f' - ff' \in \hbar\,\mathcal{R}_0[\![\hbar]\!], \qquad f * f' - f' * f - \hbar\{f, f'\} \in \hbar^2\,\mathcal{R}_0[\![\hbar]\!],$$

invoking the commutative product and the Poisson bracket of $\mathcal{R}_0$, respectively.

***4.3.2. Remark.* —** The fact that F_1 is well-defined, i.e., that the starting 'nondilated' Shapovalov form $\mathcal{S}$ is nondegenerate, is immaterial. Rather, it is sufficient that (28) is well-defined for all but countably many values of c, cf. again [39, § 16]. A quicker proof of the latter fact follows from the relation with nonresonance in Prop. 4.2.6: just observe that $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}_{\mathbb{C}}(\mathfrak{g})$ is $\mathbb{C}$-linear, and that the spectra of dilated endomorphisms are also dilated (by the same scalar).

On the same token, choose any orbit $\mathcal{O}' \subseteq \mathfrak{g}^{\vee}_s$ through an UTS principal part $\mathcal{A}'$, and a nonvanishing number $c \in \mathbb{C}$. Then the natural dilation map

$$\mathcal{A} \longmapsto c\mathcal{A} : \mathcal{O}' \longrightarrow \mathcal{O}'_c := \mathcal{O}'_{c\mathcal{A}'} = c\mathcal{O}',$$

[19] This element exists when the Verma modules of parameters $(\mathfrak{p}^{\pm}, c\chi^{\pm})$ are simple, thereby defining a meromorphic function on the complex c-plane: with poles at the degeneracy locus of the Shapovalov form, but always nonsingular at infinity. This involves the completed tensor product of the *completed* modules $M^{\pm} \hookrightarrow \widetilde{M}^{\pm}$, with respect to the grading provided by the semisimple $\mathfrak{z}(\mathfrak{l}_s)$-action.

is an algebraic isomorphism, and the pullback of the KKS symplectic structure of $\mathcal{O}'_c$ equals the c-dilation of the KKS form of $\mathcal{O}'$. Thus, one finds a $\mathbb{C}^*$-family of Poisson brackets on the same underlying affine variety. In turn, their deformation quantization are all canonically isomorphic, by rescaling the formal deformation parameter $\hbar$, and so one can always assume to start from a *NUTS* principal part.[20]

Albeit the quantum theory detects the nonresonance condition only 'asymptotically', the latter is necessary on the semiclassical side: to identify the moduli of principal parts with the (local) moduli of meromorphic connections, cf. § 6 (particularly (40)). ♢

4.3.3. *Remark.* — Consider the nilradicals $\mathfrak{u}_i^{\pm} \subseteq \mathfrak{p}_i^{\pm}$ of the parabolic subalgebras of § 4.2. Then we also assume that the vector subspaces $\boldsymbol{\mathfrak{u}}^{\pm} := \bigoplus_{i=0}^{s-1} (\mathfrak{u}_{s-i}^{\pm} \cdot \varpi^i) \subseteq \mathfrak{g}_s$ are Lie subalgebras, a condition which was referred to as having 'balanced polarizations' in [39]. This is used to expand the formal bidifferential operator $F_{\hbar^{-1}}$, as follows: the choice of Poincaré–Birkhoff–Witt (= PBW) $\mathbb{C}$-bases of the Verma modules yields $U(\boldsymbol{\mathfrak{u}}^{\mp})$-linear isomorphisms $M^{\pm} \simeq U(\boldsymbol{\mathfrak{u}}^{\mp})$, and one can write

(29) $$F_{\hbar^{-1}} = \sum_{i \geqslant 0} F_{\hbar^{-1}}^{(i)} \cdot \hbar^i, \qquad F_{\hbar^{-1}}^{(i)} \in U(\boldsymbol{\mathfrak{u}}^-) \otimes U(\boldsymbol{\mathfrak{u}}^+),$$

regarding as usual $U(\mathfrak{g}_s)$ as the ring of left-invariant differential operators on $\mathbb{C}[G_s]$.

(One can perhaps get rid of this requirement by a more extensive usage of PBW bases, considering $\mathrm{Sym}(\boldsymbol{\mathfrak{u}}^{\mp})$ in general.) ♢

4.3.4. — Now recall that the embedding $\mu = \mu_{\mathcal{O}'} \colon \mathcal{O}' \hookrightarrow \mathfrak{g}_s^{\vee}$ is a moment map for the $\mathrm{Ad}_{G_s}^{\vee}$-action, which contravariantly yields a (Poisson) *comoment map*

(30) $$\mu^* \colon \mathrm{Sym}(\mathfrak{g}_s) \longrightarrow \mathcal{R}_0, \qquad X \longmapsto \mu^*(X) = f_X := X\big|_{\mathcal{O}'},$$

for $X \in \mathrm{Sym}(\mathfrak{g}_s) \simeq \mathbb{C}[\mathfrak{g}_s^{\vee}]$. Then (30) generates the infinitesimal action of $\mathfrak{g}_s$ on $\mathcal{O}'$, via the following commutative triangle of Lie-algebra morphisms:

(31) $$\begin{array}{ccc} & & \mathcal{R}_0 \\ & \overset{\mu^*}{\nearrow} & \big\downarrow \mathrm{ad}_{\mathcal{R}_0} \\ \mathfrak{g}_s & \longrightarrow & \mathfrak{Der}(\mathcal{R}_0) \end{array},$$

with tacit restriction of the comoment map. Here $\mathfrak{Der}(\mathcal{R}_0) \subseteq \mathfrak{gl}_{\mathbb{C}}(\mathcal{R}_0)$ is the Lie subalgebra of Lie-algebra derivations, and we use the Poisson-adjoint action of $(\mathcal{R}_0, \{\cdot,\cdot\})$—on itself.

Analogously, a *quantum comoment map* is a continuous $\mathbb{C}[\![\hbar]\!]$-algebra morphism

(32) $$\widehat{\mu}^* = \widehat{\mu}_{\hbar}^* \colon U_{\hbar}(\mathfrak{g}_s) \longrightarrow \widehat{\mathcal{R}}_{\hbar} := (\mathcal{R}_0[\![\hbar]\!], *),$$

[20] Taking $\hbar$ up to homothety, as one sometimes does, yields a *single* quantization for the whole cone of affine Poisson $\mathbb{C}$-varieties; not just up to isomorphism.

where $\mathrm{U}_\hbar(\mathfrak{g}_s) \subseteq \big(\mathrm{U}(\mathfrak{g}_s)\big)[\![\hbar]\!]$ is the completed Rees algebra of $\mathrm{U}(\mathfrak{g}_s)$ (a.k.a. the *homogenized* universal enveloping algebra [**121**, Exmp. 2.6.2], cf. also [**64**]).[(21)] As in (31), this generates a $\mathfrak{g}_s$-action on $\widehat{\mathcal{R}}_\hbar$ (by $\mathbb{C}[\![\hbar]\!]$-linear derivations); and the latter is by definition a *quantization* of the 'semiclassical' action, provided that there is a commutative square of $\mathbb{C}$-algebra morphisms:

(33)
$$\begin{array}{ccc} \mathrm{U}_\hbar(\mathfrak{g}_s) & \xrightarrow{\widehat{\mu}^*} & \widehat{\mathcal{R}}_\hbar \\ \downarrow & & \downarrow \\ \mathrm{Sym}(\mathfrak{g}_s) & \xrightarrow[\mu^*]{} & \mathcal{R}_0 \end{array},$$

where the vertical arrows are the semiclassical limits $\hbar \to 0$.

But of course in our setting there is a natural 'quantum' action, viz., the infinitesimal version of the G_s-action on the coefficients of $\mathcal{R}_0$-valued formal power series. Then the following statement implies in particular that (45) is G_s-invariant:

4.3.5. Theorem-Definition. — *Consider the (unique) continuous $\mathbb{C}[\![\hbar]\!]$-algebra morphism $\widehat{\mu}^* \colon \mathrm{U}_\hbar(\mathfrak{g}_s) \to \widehat{\mathcal{R}}_\hbar$ extending*

$$X \longmapsto f_X \cdot \hbar^{-1} \in \widehat{\mathcal{R}}_\hbar[\hbar^{-1}], \qquad X \in \mathfrak{g}_s,$$

in the notation of (30)*. Then:*

1. *$\widehat{\mu}^*$ is a* strong *quantum comoment map generating the natural 'quantum' $\mathfrak{g}_s$-action;*
2. *and the corresponding square* (33) *commutes.*

Proof. — Using the fact that μ^* is a Poisson morphism, and a comoment map for the standard 'semiclassical' action, one is left to prove the identity

$$f_X * f' - f' * f_X =: \big[f_X, f'\big]_* = \hbar\{f_X, f'\}, \qquad X \in \mathfrak{g}_s, \quad g \in \mathcal{R}_0.$$

(This is the condition for the quantum comoment map to be 'strong'.)

We will now generalize the proof of [**40**, Prop. 3.2], as follows. The $\mathbb{C}$-linear evaluation function

(34)
$$\mathrm{ev}_X \colon \mathfrak{g}_s^\vee \longrightarrow \mathbb{C}[G_s], \qquad \mathrm{ev}_X(\xi) \colon g \longmapsto \langle \mathrm{Ad}_g^\vee \, \xi, X \rangle,$$

maps $\mathcal{A}'$ to the pullback $\widetilde{f}_X \in \mathbb{C}[G_s]$ of $f_X \in \mathcal{R}_0$. Moreover, the arrow (34) intertwines the $\mathrm{ad}^\vee_{\mathfrak{g}_s}$-action on $\mathcal{A}'$ with the standard infinitesimal action of $\mathfrak{g}_s$ on $\mathbb{C}[G_s]$—by left-invariant vector fields. Hence, for all $c \in \mathbb{C}^*$ such that (28) is defined, and for all $\widetilde{f}' \in \mathbb{C}[G_s]$, one has

$$\begin{aligned} F_c\big(\widetilde{f}' \otimes \widetilde{f}_X\big) &= (1 \otimes \mathrm{ev}_X) F_c\big(\widetilde{f}' \otimes \mathcal{A}'\big) \\ &= \widetilde{f}' \otimes \widetilde{f}_X - \frac{1}{c} \sum_j \big(Y_j.\widetilde{f}'\big) \otimes X_j.\widetilde{f}_X \in \mathbb{C}[G_s] \otimes \mathbb{C}[G_s], \end{aligned}$$

(21) Thus, the arrow (32) contains as much information as a Lie-algebra morphism $\mathfrak{g}_s \to \mathcal{R}_0[\![\hbar]\!][\hbar^{-1}]$, equipping the right-hand side with the commutator bracket, cf. Thm.-Def. 4.3.5.

in view of Lem. 4.3.6. (Identifying $M_c^+ \simeq U(\mathfrak{u}^-)$, and invoking the $U(\mathfrak{u}^-)$-submodule of $\mathbb{C}[G_s]$ generated by $\widetilde{f}'$.) Now take formal Taylor expansions in $\hbar = c^{-1}$, and compose with (the $\mathbb{C}[\![\hbar]\!]$-linear extension of) the multiplication of $\mathbb{C}[G_s]$; this yields the $\mathbf{L}$-invariant formal function

$$m \circ F_{\hbar^{-1}}\big(\widetilde{f}' \otimes \widetilde{f}_X\big) = m\Big(\widetilde{f}' \otimes \widetilde{f}_X - \hbar \sum_j (Y_j.\widetilde{f}') \otimes X_j.\widetilde{f}_X\Big) \in \big(\mathbb{C}[G_s]\big)[\![\hbar]\!].$$

Then, repeating the same computations (verbatim) with the factors swapped, and projecting down to the orbit, one finds

$$[f', f_X]_* = -\hbar \sum_j Y_j \wedge X_j \big(f' \otimes f_X\big) = \hbar \{ f', f_X \} \in \widehat{\mathcal{R}}_\hbar,$$

as (by construction) $\Pi := \sum_j X_j \wedge Y_j \in \mathfrak{u}^+ \otimes \mathfrak{u}^- \subseteq \bigwedge^2(\mathfrak{g}_s/\mathfrak{l})$ is the Poisson bivector field—in the vector-space identification $\mathfrak{g}_s/\mathfrak{l} \simeq \mathfrak{u}^+ \oplus \mathfrak{u}^-$. □

4.3.6. Lemma. — *For any $c \in \mathbb{C}^*$ such that (28) is defined, let $v_c^+ := v^+_{\mathfrak{p}^+, c\chi^+}$ be the cyclic vector of the (simple) Verma module $M_c^+ := M^+_{\mathfrak{p}^+, c\chi^+}$—of highest weight $c\mathcal{A}' \in \mathfrak{g}_s'$. Denote also by $(Y_j)_j$ and $(X_j)_j$ two mutually-dual $\mathbb{C}$-bases of $\mathfrak{u}^-$ and $\mathfrak{u}^+$, respectively, for the nondegenerate pairing $\mathfrak{u}^- \otimes \mathfrak{u}^+ \to \mathbb{C}$ defined by the (nonsingular) character χ^+. Then*

$$F_c(v_c^+ \otimes \mathcal{A}') = v_c^+ \otimes \mathcal{A}' - \frac{1}{c} \sum_j (Y_j v_c^+) \otimes \mathrm{ad}^\vee_{X_j}(\mathcal{A}') \in M_c^+ \otimes \mathfrak{g}_s^\vee, \tag{35}$$

viewing $\mathfrak{g}_s^\vee$ as a $U(\mathfrak{g}_s)$-module for (the universal enveloping version of) the $\mathrm{ad}^\vee_{\mathfrak{g}_s}$-action.

Proof. — Postponed to § C.2. □

4.3.7. Remark. — Again, outside of the proofs we have phrased everything with *no* choice of a Cartan subalgebra $\mathfrak{t} \subseteq \mathfrak{g}$; but one can choose one, as in the proof of Prop. 4.2.6. Then (the coefficients of) the principal part $\mathcal{A}'$ can be regarded as a tuple of linear functions on $\mathfrak{t}$: the latter lies in a certain—root-valuation [**73**]—stratum of $(\mathfrak{t}^\vee)^s \simeq \mathfrak{t}^s$, determined by the nested Levi subsystems of roots which annihilate the irregular tail at each step, cf. [**148**, **56**, **39**], and the proof of Prop. 8.1.1. ♢

4.4. About canonicity. — Besides an $\mathrm{Ad}^\vee_{G_s}$-orbit $\mathcal{O}' \subseteq \mathfrak{g}_s^\vee$ intersecting $\mathfrak{g}_s'$, and a uniformizer, in this sections we have made two further related choices: (i) a marked point $\mathcal{A}' \in \mathcal{O}' \cap \mathfrak{g}_s'$, a.k.a. a *marking of* $\mathcal{O}'$; and (ii) a sequence of parabolic subalgebras of $\mathfrak{g}$.

(In § 5 we will prove that the choice of uniformizer is immaterial.)

4.4.1. — In the viewpoint of meromorphic connections, marking a UTS coadjoint orbit is akin to selecting a normal principal part within a normal principal G-orbit. This is *not* canonical when working on unframed principal bundles, but:

4.4.2. Lemma. — *For any given parabolic sequence $\mathfrak{p}_s^\pm \subseteq \cdots \subseteq \mathfrak{p}_1^\pm$, the above deformation quantization of $\mathcal{R}_0$ is* independent *of the choice of a marking on the orbit.*

Proof. — We have constructed a G_s-invariant $*$-product which admits a strong quantum comoment map, as well as *separation of variables* with respect to the opposite polarizations of $\mathcal{O}'$ determined by the sequences of nested nilradicals $\mathfrak{u}_i^{\pm} \subseteq \mathfrak{p}_i^{\pm}$: the latter follows immediately from the expansion (29), cf. [40, § 2.2 + Rmk. 3.5]. Now Prop. 3.6 of op. cit., rephrased in our setting and without need for density arguments, states that there exists a *unique* such $*$-product. □

4.4.3. — Thus, after establishing the coordinate independence of § 5, choosing a parabolic sequence makes it possible to *canonically* quantize a coadjoint UTS orbit $\mathcal{O}' \subseteq \mathfrak{g}_s^{\vee}$ with prescribed normal principal part; and the latter arises in particular from the formal germ of an untwisted/unramified meromorphic connection, at a marked point of a wild Riemann surface, which is our underlying viewpoint.

On the contrary, the dependence of quantization upon the choice of polarization is in general a difficult problem: in our situation this immediately hinges on the parabolic sequence, and we shall *not* discuss this further in this text. Note however that this issue is typically stated in *geometric* quantization, and that it was solved in the archetype case of compact Chern–Simons gauge theory in [79, 6]. Once more, these ideas revolve around conformal/topological quantum field theory on surfaces, and the present situation should be viewed as a 'wild' generalization of the 'tame' archetypes, i.e., the Hitchin/KZ(B)/TUY connections (cf. again § 1.2).

5. Coordinate independence

5.1. Changing uniformizer. — In §§ 3–4 we have used a generator ϖ of the maximal ideal $\mathfrak{m} \subseteq \mathscr{O}$. Here we clarify which of the previous definitions/constructions is intrinsically determined by a triple $(\mathbf{G}, \mathscr{O}, s)$, consisting of: (i) a connected reductive algebraic group defined over $\mathbb{C}$; (ii) a complete DVR with residue/coefficient field $\mathbb{C}$; and (iii) a positive integer.

5.1.1. — The set of uniformizers is acted on by continuous $\mathbb{C}$-algebra automorphisms of $\mathbb{C}[\![\varpi]\!]$, in simply-transitive fashion, and their group can (and will) be identified with the group of units $\mathbb{C}[\![\varpi]\!]^{\times} \subseteq \mathbb{C}[\![\varpi]\!]$. Namely, an automorphism F is determined by the image $F(\varpi) = \varpi f \in \varpi\mathbb{C}[\![\varpi]\!]$, for a unique element $f = f(F)$ such that $f(0) \neq 0 \in \mathbb{C}$, denoting by $f \mapsto f(0)$ the quotient onto the residue field—'evaluating' at the origin.

Then let $\mathbb{C}[\![\varpi]\!]^{\times}$ act on $\mathbb{C}(\!(\varpi)\!)\, d\varpi$, on the *right*, by pullback:

$$(36) \qquad \big(\widetilde{f}\, d\varpi\big).F = F^*\big(\widetilde{f}\, d\varpi\big) := \big(\widetilde{f} \circ F(\varpi)\big) \cdot (f + \varpi f')\, d\varpi, \qquad \widetilde{f} \in \mathbb{C}(\!(\varpi)\!),$$

where $f = F(f) \in \mathbb{C}[\![\varpi]\!]$ is as above, and using the formal derivative operator $f \mapsto f'$. This action extends in $\mathbb{C}$-bilinear fashion to $\mathfrak{g}(\!(\varpi)\!)\, d\varpi$, by preserving pole orders, and so it yields an action on the space of pole-order-bounded principal parts.

In the identifications/dualities of § 2.2.2, we thus find a Poisson action on $\mathfrak{g}_s^\vee$, mapping $\mathrm{Ad}^\vee_{G_s}$-orbits onto each other. Moreover, this action fixes the subspace $\mathfrak{g}^\vee$ (of residues) pointwise; but what matters here is that:

***5.1.2. Lemma.* —**

1. *The subspaces $\mathfrak{g}''_s \subseteq \mathfrak{g}'_s \subseteq \mathfrak{g}^\vee_s$ of (17)–(18) are $\mathbb{C}[\![\varpi]\!]^\times$-invariant.*
2. *And if $\widetilde{\mathcal{A}}'$ lies in the $\mathbb{C}[\![\varpi]\!]^\times$-orbit of $\mathcal{A}' \in \mathfrak{g}'_s$, then the sequence $\mathfrak{l}_s \subseteq \cdots \subseteq \mathfrak{l}_1 \subseteq \mathfrak{g}$ of reductive subalgebras determined by $\widetilde{\mathcal{A}}'$ and $\mathcal{A}'$—as in § 4.1.2—coincide.*

Proof postponed to C.3. — □

5.1.3. — Now the restricted action of $\mathbb{C}[\![\varpi]\!]^\times$ on $\mathfrak{g}''_s \subseteq \mathfrak{g}'_s$ corresponds to a permutation of the $\mathrm{Ad}^\vee_{\mathrm{Bir}_s}$-orbits meeting these subspaces; and the same also holds for the larger $\mathrm{Ad}^\vee_{G_s}$-orbits, since the actions of $\mathbb{C}[\![\varpi]\!]^\times$ and G—on $\mathfrak{g}^\vee_s$—commute.

More importantly, it follows from Lem. 5.1.2 that the spaces of UTS/NUTS connections of Deff. 3.1.2 and 3.2.4 admit an invariant description. Namely, one can define subspaces $\mathfrak{g}''_s \subseteq \mathfrak{g}'_s \subseteq \mathfrak{g}(\Omega^1_{\mathscr{K}}/\Omega^1_{\mathscr{O}})$ of pole-order-bounded principal parts by requiring that they coincide with (17)–(18) in *any* choice of uniformizer, and then ask that the $\mathrm{Ad}^\vee_{G_s}$-orbit of the principal part of a connection $\widehat{\mathcal{A}} \in \mathfrak{g}(\Omega^1_{\mathscr{K}})$ intersects them (cf. Rmk. 6.1.2).

Finally, one can associate two opposite parabolic filtrations $\mathfrak{p}^\pm_s \subseteq \cdots \subseteq \mathfrak{p}^\pm_1 \subseteq \mathfrak{g}$ to any element $\mathcal{A}'$ of these intrinsically-determined subspaces, so that the underlying Levi filtration controls the nested Ad_G-stabilizers of the coefficients of $\mathcal{A}'$ in any choice of uniformizer. The Verma modules (21) still depend on ϖ, precisely to select a character $\chi^\pm$ for the Lie subalgebras $\mathfrak{p}^\pm \subseteq \mathfrak{g}_s$; nonetheless, the following fact implies that the isomorphism class of the deformation quantization of § 4 is intrinsic:

***5.1.4. Lemma.* —**

1. *Any element $F \in \mathbb{C}[\![\varpi]\!]^\times$ yields a $U(\mathfrak{g}_s)$-linear isomorphism*

$$_F\varphi \colon M^{(s)}_{\mathfrak{p}^\pm,\chi^\pm} \xrightarrow{\simeq} M^{(s)}_{\mathfrak{p}^\pm,\widetilde{\chi}^\pm}, \qquad \widetilde{\chi}^\pm := \chi^\pm.F.^{(22)}$$

2. *And $_{F'F}\varphi = {}_{F'}\varphi \circ {}_F\varphi$, for any other element $F' \in \mathbb{C}[\![\varpi]\!]^\times$.*

Proof postponed to C.4. — □

***5.1.5. Remark.* —** The group $\mathbb{C}[\![\varpi]\!]^\times$ also acts on $G[\![\varpi]\!]$, and one can consider the corresponding semidirect product: geometrically, the latter can be identified with the group of principal-bundle automorphisms of $\operatorname{Spec}\mathbb{C}[\![\varpi]\!] \times \mathbf{G} \to \operatorname{Spec}\mathbb{C}[\![\varpi]\!]$, covering *any* automorphism of the base. After truncation, this leads to the semidirect product $\left(\mathbb{C}[\![\varpi]\!]/\varpi^s\mathbb{C}[\![\varpi]\!]\right)^\times \ltimes G_s$, which is intimately related with the group of Lie-algebra automorphisms of $\mathfrak{g}_s$: cf. Thm. B.3.3.

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(22) Here we view $\chi^\pm$ as elements of (18). Equivalently, the characters correspond to s-uples of elements in the duals of the Lie-centres of the Levi factors $\mathfrak{l}_i = \mathfrak{p}^+_i \cap \mathfrak{p}^-_i$, and these centres are also preserved—and acted on—by $\mathbb{C}[\![\varpi]\!]^\times$.

In a different viewpoint, note that (36) is the basic abelian version of the gauge action (16), as $\mathbb{C}[\![\varpi]\!]^\times \simeq \mathrm{Hom}_{\mathrm{Sch}}(\mathrm{Spec}\,\mathbb{C}[\![\varpi]\!], \mathbf{G}_m)$, invoking the multiplicative algebraic group $\mathbf{G}_m$. ♢

5.1.6. Remark. — The optional choice of a maximal torus $T \subseteq G$ is compatible with all the above, as the corresponding truncated-current Cartan subalgebra $\mathfrak{t}_s \subseteq \mathfrak{g}_s$ is $\mathbb{C}[\![\varpi]\!]^\times$-invariant. Then Lem. 5.1.2 also implies that each root-valuation stratum of $\mathfrak{t}_s^\vee \simeq (\mathfrak{t}^\vee)^s$ is $\mathbb{C}[\![\varpi]\!]^\times$-invariant,[23] and recall that the $\mathrm{Ad}^\vee_{G_s}$-orbits meeting a given stratum are all (canonically) isomorphic as homogeneous G_s-varieties (this follows from (19)). In principle, they are not symplectomorphic; but they will be, if they are related by a Poisson automorphism—coming from $\mathbb{C}[\![\varpi]\!]^\times$. ♢

6. Wild de Rham spaces, and their quantization

6.1. (Semiclassical) Hamiltonian reduction. — Let us keep all the notation from the introduction in § 1: we work on $\Sigma = \mathbb{C}P^1$, with finitely many marked points $\boldsymbol{a} \subseteq \Sigma$, and consider the 'open part' $\mathcal{C}^*_{\mathrm{dR}} \subseteq \mathcal{C}_{\mathrm{dR}}$ of the (unframed) naive de Rham groupoid; its objects are certain meromorphic connections $\boldsymbol{\mathcal{A}}$, with polar divisor bounded by $D = \sum_{\boldsymbol{a}} s_a[a]$, defined on the trivial holomorphic principal G-bundle $\boldsymbol{\pi}^*$ over Σ. (Here $G = \mathbf{G}(\mathbb{C})$ is as in § 2.)

6.1.1. — Equivalently, $\mathrm{Ob}(\mathcal{C}^*_{\mathrm{dR}})$ consists of suitably constrained (global) $\mathfrak{g}$-valued meromorphic 1-form on the Riemann sphere—still denoted by $\boldsymbol{\mathcal{A}}$. Since at present we do not care about the moduli of (wild) Riemann spheres, we will break part of the Möbius $\mathrm{PSL}_2(\mathbb{C})$-action by regarding Σ as the 1-point compactification of the standard complex plane, and suppose that all marked points $a \in \boldsymbol{a}$ are at finite distance from the origin. If z is the standard global coordinate on the plane, and if $t_a \coloneqq z(a) \in \mathbb{C}$ is the position of each marked point, then one can (uniquely) write

$$\boldsymbol{\mathcal{A}} = \sum_{\boldsymbol{a}} \mathcal{A}_a, \qquad \mathcal{A}_a = \Lambda_a z_a^{-1}\,\mathrm{d}z + \mathrm{d}Q_a, \quad z_a \coloneqq z - t_a, \tag{37}$$

for suitable residues $\Lambda_a = A_{a,0} \in \mathfrak{g}$ (noting that $\mathrm{d}z = \mathrm{d}z_a$ for all $a \in \boldsymbol{a}$), where as usual

$$Q_a = \sum_{i=1}^{s_a - 1} A_{a,i} \frac{z_a^{-i}}{-i}, \qquad A_{a,1}, \dots, A_{a,s_a-1} \in \mathfrak{g}.$$

Indeed, any term in $\mathfrak{g}[\![z]\!]\,\mathrm{d}z$—in (37)—would give an irregular singularity at infinity. Moreover, the identity $\sum_a \Lambda_a = 0 \in \mathfrak{g}$ must hold, lest there is a regular singularity there.

Now z_a yields a uniformizer ϖ_a for $\mathcal{O}_a$, and the principal part of the formal Laurent expansion $\widehat{\mathcal{A}}_a$ of (37)—as in (1)—coincides with $\mathcal{A}_a$, just written in a 'formal'

[23] In particular, this means that it is actually possible to directly stratify the space (18).

variable. The nonsingular part

$$\mathcal{B}_a \coloneqq \widehat{\mathcal{A}}_a - \mathcal{A}_a \in \mathfrak{g}[\![\varpi_a]\!]\, \mathrm{d}\varpi_a, \tag{38}$$

instead, is computed by gathering the (formal) Taylor expansions of $\mathcal{A}_{a'}$ at a, for all $a' \neq a$. The final hypothesis is that $\widehat{\mathcal{A}}_a$ lies in the $G[\![\varpi_a]\!]$-orbit $\widehat{\mathcal{O}}'_a$ of a NUTS principal part $\mathcal{A}'_a$ as in (2), so that in particular (38) can be formally gauged away. As already explained, this is *equivalent* to asking that $\mathcal{A}_a$ lies in the $\mathrm{Ad}^{\vee}_{G_{s_a}}$-orbit $\mathcal{O}'_a \coloneqq G_{s_a}.\mathcal{A}'_a$, invoking a TCLG G_{s_a}—as in (12).

Denote by $\boldsymbol{\mathcal{O}}' = \{ \mathcal{O}'_a \}_{\boldsymbol{a}}$ the multiset of these finite-dimensional orbits.

6.1.2. Remark. — Again recall that $(\boldsymbol{\pi}^*, \boldsymbol{\mathcal{A}})$ only determines the G-orbit of the normal form, as we are not framing the bundle. Conversely, in view of § 5, the fact that (37) admits such a NUTS normal orbit at each pole is an *intrinsic* notion. (One might say that $\boldsymbol{\mathcal{A}}$ itself is NUTS.)

Namely, introduce also the completed local field $\mathscr{K}_a = \mathrm{Frac}(\mathscr{O}_a)$ at each marked point, and the module of continuous Kähler differentials $\mathscr{K}_a \xrightarrow{\mathrm{d}} \Omega^1_{\mathscr{K}_a}$—as in (10). Then choose an orbit for the $\mathbf{G}(\mathscr{O}_a)$-action on $\mathfrak{g}(\Omega^1_{\mathscr{K}_a})$: define it to be *NUTS* (resp., *UTS*) if it meets an element of (17) (resp., of (18)), in some/any choice of uniformizer ϖ_a, which makes sense by Lem. 5.1.2. These NUTS orbits enter into the actual definition of the de Rham groupoids, and this works on arbitrary principal G-bundles (over arbitrary base surfaces). ♢

6.1.3. — Overall, denoting the orbit product by

$$\mathcal{O}'_{\boldsymbol{a}} \coloneqq \prod_{\boldsymbol{a}} \mathcal{O}'_a \subseteq \prod_{\boldsymbol{a}} \mathfrak{g}^{\vee}_{s_a}, \tag{39}$$

there is a natural injective function

$$\mathrm{Ob}(\mathcal{C}^*_{\mathrm{dR}}) \hookrightarrow M = M(\boldsymbol{\mathcal{O}}'; G) \coloneqq \left\{ \prod_{\boldsymbol{a}} \mathcal{A}_a \in \mathcal{O}'_{\boldsymbol{a}} \;\middle|\; \sum_{\boldsymbol{a}} \Lambda_a = 0 \right\}. \tag{40}$$

Conversely, the fact that it is also *surjective* uses the NUTS condition in an essential way (cf. Rmk. 6.1.4.) Finally, the global (holomorphic) gauge transformations of $\boldsymbol{\pi}^*$ are *constant*, and match up with the diagonal $\mathrm{Ad}^{\vee}_G$-action on the principal parts $\mathcal{A}_a \in \mathfrak{g}^{\vee}_{s_a}$ (cf. (15)). Thus, the arrow (40) induces a bijection

$$\mathrm{Ob}(\mathcal{C}^*_{\mathrm{dR}}) / \mathrm{iso.} \xrightarrow{\simeq} M/G.$$

But we will rather keep working in the complex-algebraic category, considering the affine geometric-invariant-theory (= GIT) quotient

$$\mathcal{M}^*_{\mathrm{dR}} \coloneqq M /\!/ \mathbf{G},$$

viewing $M \subseteq \mathcal{O}'_{\boldsymbol{a}}$ as an affine subvariety, via Prop. 3.2.10; recall that we assume that all points of (40) are *stable*, so that in particular this is a genuine geometric quotient (cf. § 10).

***6.1.4. Remark.* —** In principle, one might consider a groupoid of (possibly resonant) meromorphic connections on π^*, whose formal germ at each pole lies in a prescribed full/untruncated formal gauge orbit, passing through a UTS principal part. Reasoning as in (40), its set of objects can still be identified with a subset of the $\mathbb{C}$-points of a finite-dimensional complex affine variety: it is unclear—to the authors—whether this has nice algebro-geometric/symplectic structures. (But cf. again Fn. 3.) $\diamondsuit$

6.2. Quantum Hamiltonian reduction. — In brief, consider the *affine-GIT* $\mathbf{G}$-*Hamiltonian reduction*

$$\mathcal{M}^*_{\mathrm{dR}} = \mathcal{O}'_{\boldsymbol{a}} /\!/\!/_0 \, \mathbf{G} \coloneqq (\mu')^{-1}(0) /\!/ \mathbf{G}, \tag{41}$$

of a product of coadjoint orbits in dual TCLAs, at level zero, with respect to the (algebraic) moment map

$$\mu' = \mu'_{\mathcal{O}'_{\boldsymbol{a}}} : \mathcal{O}'_{\boldsymbol{a}} \longrightarrow \mathfrak{g}^\vee \simeq \mathfrak{g}, \qquad \prod_{\boldsymbol{a}} \mathcal{A}_a \longmapsto \sum_{\boldsymbol{a}} \mathrm{Res}(\mathcal{A}_a). \tag{42}$$

The main aim is to quantize (41), using the material of § 4.

6.2.1. — Denote by $\mathcal{R}_{a,0} \coloneqq \mathbb{C}[\mathcal{O}'_a]$ (resp., by $\mathcal{R}_{\boldsymbol{a},0} \coloneqq \mathbb{C}[\mathcal{O}'_{\boldsymbol{a}}] \simeq \bigotimes_{\boldsymbol{a}} \mathcal{R}_{a,0}$) the ring of global regular functions on each orbit (resp., on the orbit product). Then there is a comoment map $(\mu')^* : \mathrm{Sym}(\mathfrak{g}) \to \mathcal{R}_{\boldsymbol{a},0}$, corresponding to the pullback along (42): upon taking the 'diagonal' restrictions $\mathrm{Sym}(\mathfrak{g}) \subseteq \mathrm{Sym}(\mathfrak{g}_{s_a})$, for $a \in \boldsymbol{a}$, this is the natural tensor-product extension of (30).

Now the algebraic version of (41) goes as follows (cf. again [64]): (i) the extension/pushforward of the *augmentation ideal* $\mathrm{Sym}(\mathfrak{g}) \cdot \mathfrak{g} \subseteq \mathrm{Sym}(\mathfrak{g})$, i.e., the ideal

$$\mathfrak{I}_{\boldsymbol{a},0} \coloneqq \mathcal{R}_{\boldsymbol{a},0} \cdot (\mu')^* \big(\mathrm{Sym}(\mathfrak{g}) \cdot \mathfrak{g}\big) \subseteq \mathcal{R}_{\boldsymbol{a},0}, \tag{43}$$

intersects the subring $\mathcal{R}^{\mathfrak{g}}_{\boldsymbol{a},0} \subseteq \mathcal{R}_{\boldsymbol{a},0}$—of invariant functions—in a Poisson ideal; and (ii) there is an isomorphism

$$\mathcal{M}^*_{\mathrm{dR}} \simeq \mathrm{Spec}\big(\mathcal{R}_{\boldsymbol{a},0} /\!/\!/_0 \, \mathrm{G}\big), \qquad \mathcal{R}_{\boldsymbol{a},0} /\!/\!/_0 \, \mathrm{G} \coloneqq \mathcal{R}^{\mathfrak{g}}_{\boldsymbol{a},0} \big/ \big(\mathfrak{I}_{\boldsymbol{a},0} \cap \mathcal{R}^{\mathfrak{g}}_{\boldsymbol{a},0}\big), \tag{44}$$

of affine Poisson $\mathbb{C}$-varieties.

***6.2.2. Remark.* —** Since G is connected, it would be the same to take G-invariants. Moreover, as G is reductive, the ring of invariants is finitely-generated [107], and there is a Poisson-ring isomorphism

$$\mathcal{R}_{\boldsymbol{a},0} /\!/\!/_0 \, \mathrm{G} \simeq \big(\mathcal{R}_{\boldsymbol{a},0} / \mathfrak{I}_{\boldsymbol{a},0}\big)^{\mathrm{G}}. \qquad \diamondsuit$$

6.2.3. — Now, for each $a \in \boldsymbol{a}$, let $\widehat{\mathcal{R}}_{a,\hbar} \simeq \mathcal{R}_{a,0}[\![\hbar]\!]$ be a deformation quantization of $\mathcal{R}_{a,0}$, e.g., as in § 4. Then the (completed) $\mathbb{C}[\![\hbar]\!]$-multilinear tensor product of the deformation rings over the marked points, viz.,

$$\widehat{\mathcal{R}}_{\boldsymbol{a},\hbar} \coloneqq \big(\widehat{\bigotimes}_{\mathbb{C}[\![\hbar]\!]}\big)_{\boldsymbol{a}} \big(\widehat{\mathcal{R}}_{a,\hbar}\big) \simeq \mathcal{R}_{\boldsymbol{a},0}[\![\hbar]\!], \tag{45}$$

yields a deformation quantization of the orbit product $\mathcal{O}'_{\boldsymbol{a}}$.

Now suppose in addition that (45) carries a *quantization* of the comoment map $(\mu')^*$. By definition, this is a quantum comoment map $(\widehat{\mu}')^*$ as in (32), together with a commutative square as in (33): e.g., upon restriction to $U_\hbar(\mathfrak{g}) \subseteq U_\hbar(\mathfrak{g}_s)$, and taking completed tensor products over $\mathbb{C}[\![\hbar]\!]$, one such can be readily constructed from Thm.-Def. 4.3.5. Then $(\widehat{\mu}')^*$ generates an action of $\mathfrak{g}$ on $\widehat{\mathcal{R}}_{\mathbf{a},\hbar}$, and this is a quantization of the 'semiclassical' $\mathfrak{g}$-action on the orbit product. In this case, the 'quantum' analogue of (44) is the $\mathbb{C}[\![\hbar]\!]$-algebra quotient

$$\widehat{\mathcal{R}}_{\mathbf{a},\hbar} /\!/\!/_0 G \coloneqq \widehat{\mathcal{R}}^{\mathfrak{g}}_{\mathbf{a},\hbar} \big/ \big(\mathfrak{J}_{\mathbf{a},\hbar} \cap \widehat{\mathcal{R}}^{\mathfrak{g}}_{\mathbf{a},\hbar}\big), \tag{46}$$

where $\mathfrak{J}_{\mathbf{a},\hbar} \subseteq \widehat{\mathcal{R}}_{\mathbf{a},\hbar}$ is the extension/pushforward of the augmentation ideal of $U_\hbar(\mathfrak{g})$, along $(\widehat{\mu}')^*$. (Analogously to (43), one takes the left ideal generated by the image, and proves that the latter intersects the subring of $\mathfrak{g}$-invariants in a *bilateral* ideal therein.)

We conclude by the following fact:

6.2.4. Lemma. — *Suppose that* (42) *is* flat*. Then (the* $\hbar$*-separated quotient of)* (46) *is a deformation quantization of* $\mathcal{M}^*_{\mathrm{dR}}$.

Proof. — Since G is reductive—and $\mathbb{C}$ of characteristic zero—, the quotient of (46) modulo the ideal generated by $\hbar$ surjects onto (44), cf. [101]. Moreover, by construction, the reduced 'quantum' tensor-product commutator matches up with the reduced 'semiclassical' direct-product Poisson bracket. (Recall that we started from a quantization of each orbit.)

Thus, by [90, Prop. VXI.2.4], one must only ensure to have a $\mathbb{C}[\![\hbar]\!]$-module which is: (i) complete; (ii) separated; and (iii) torsion-free. But in our setting completeness is automatic. Furthermore, the $\hbar$-adic topology will be separated up to taking the quotient by the intersections of the bilateral ideals generated by $\{\hbar^i\}_{i\geqslant 0}$. Thus, the crux of the matter is (iii), which is well-known to be equivalent to the fact that $\hbar$ is a nonzerodivisor in (46). Finally, if μ is flat, then the ring morphism μ^* maps weakly-regular sequences in $\mathrm{Sym}(\mathfrak{g})$ to weakly-regular sequences in $A_{\mathbf{a},0}$ [38, Prop. 1.1.2]: then one can conclude, e.g., by [101, Prop. 1].[24] □

7. Interlude: preparation for flatness

The upshot of §§ 4–6 is a deformation quantization of wild de Rham spaces, whenever the G-moment map (42) is a flat morphism. In §§ 8–9 we provide sufficient conditions to ensure that it is *faithfully flat* (= surjective and flat). This leads to a proof of the main Thm. 1.3.4.

To this end, in this section we recall further notions/terminology.

7.1. Preliminary observations. —

[24] One actually only needs that the image of the (strongly) regular sequence—in $\mathrm{Sym}(\mathfrak{g})$—obtained from a $\mathbb{C}$-basis of $\mathfrak{g}$ is H_1-*regular*, cf. [132].

7.1.1. — First, note that (39) is an *irreducible* $\mathbb{C}$-variety, being a product of homogeneous spaces.[25]

Second, recall that a morphism of irreducible nonsingular $\mathbb{C}$-varieties is faithfully flat if and only if: (i) it is surjective; and (ii) it has equidimensional fibres. In general, flatness is stronger, and the nontrivial implication is proven, e.g., in [**112**]—under the weaker assumption that the map is *dominant*, i.e., with dense image.

(This is a.k.a. 'miracle flatness', cf. [**138**, Thm. 26.2.11].)

7.1.2. — Consider a UTS principal part $\mathcal{A}' = \Lambda' \varpi^{-1} \mathrm{d}\varpi + \mathrm{d}Q' \in \mathfrak{g}'_s$, for an integer $s \geqslant 1$, as in (18). Consider also the (marked) $\mathrm{Ad}^{\vee}_{G_s}$-orbit $\mathcal{O}' := G_s.\mathcal{A}' \subseteq \mathfrak{g}^{\vee}_s$, and choose a uniformizer ϖ: the latter is w.l.o.g., since changing ϖ just moves to another such orbit, by a symplectomorphism, in G-equivariant fashion. (The action of G commutes with that of $\mathbb{C}[\![\varpi]\!]^{\times}$, and the same works after taking orbit products, cf. § 5).

***7.1.3. Lemma-Definition* (cf. [148] Rmk. 2.9).** — *Let $\mathfrak{g} = \mathfrak{Z}(\mathfrak{g}) \oplus \widetilde{\mathfrak{g}}$ be the Lie-algebra splitting—of $\mathfrak{g}$—into the centre plus the derived Lie subalgebra $\widetilde{\mathfrak{g}} := [\mathfrak{g}, \mathfrak{g}]$ (= the semisimple part); extend it $\mathbb{C}$-bilinearly to $\mathfrak{g}_s$, and then dualize it to $\mathfrak{g}^{\vee}_s$. Denote also by $\mathcal{A}' = \mathcal{A}'_{\mathfrak{Z}} + \widetilde{\mathcal{A}}'$ the corresponding decomposition of the normal form. Then $\mathcal{O}'$ is canonically isomorphic to the $\mathrm{Ad}^{\vee}_{G_s}$-orbit through $\widetilde{\mathcal{A}}'$, as a Hamiltonian G_s-variety.*

Proof postponed to C.5. — □

7.1.4. — In view of Lem. 7.1.3, up to symplectomorphisms of de Rham spaces, it is enough to consider orbits of normal forms with *no* central component. Hence, as far as the quantization is concerned, one might replace G with the derived (connected [**145**]) commutator subgroup $\widetilde{G} := [G, G]$, or with the projectivization $\mathbf{P}(G) := G/Z(G)$. Hereafter, for the sake of notation, let us just suppose that G itself is *semisimple*. (Some of the criteria established below fail if the centre has positive dimension.)

7.2. Birkhoff/extended orbits. — In addition to the data above, introduce also (as in [**23, 148**]): (i) the *Birkhoff orbit*

$$\breve{\mathcal{O}}' = \breve{\mathcal{O}}'_{Q'} := \mathrm{Bir}_s .(\mathrm{d}Q') \subseteq \mathfrak{bir}^{\vee}_s, \tag{47}$$

through the irregular part of $\mathcal{A}'$, by the action of the unipotent radical; and (ii) the *extended orbit*

$$\widetilde{\mathcal{O}}' = \widetilde{\mathcal{O}}'_{Q'} := \left\{ (g, \mathcal{A}) \in G \times \mathfrak{g}^{\vee}_s \;\middle|\; \pi_{\mathrm{irr}}\big(\mathrm{Ad}^{\vee}_g(\mathcal{A})\big) \in \breve{\mathcal{O}}' \right\}, \tag{48}$$

using the canonical projection $\pi_{\mathrm{irr}} : \mathfrak{g}^{\vee}_s \twoheadrightarrow \mathfrak{bir}^{\vee}_s$—along $\mathfrak{g}^{\vee}$, cf. § 2.2.3. (This is intimately related with the choice of a compatible frame $g \in G$, for the trivial principal **G**-bundle over $\mathcal{D} = \mathrm{Spec}\,\mathbb{C}[\![\varpi]\!]$, at the origin.)

[25] Incidentally, the affine-GIT quotient (41) is also *irreducible*: the subvariety $M = (\mu')^{-1}(0) \subseteq \mathcal{O}'_{\mathfrak{a}}$ is cut out by a single linear equation in (40), so that $\mathbb{C}[M]$ (and its subring $\mathbb{C}[M]^{\mathfrak{g}}$) is an integral domain.

7.2.1. — We will use some general results about the Hamiltonian geometry of (47)–(48). To phrase them, retain the notation of § 4 for the fission sequence $L_s \subseteq \cdots \subseteq L_1 \subseteq G$ of connected reductive subgroups determined by $\mathcal{A}'$, and for the corresponding sequence $\mathfrak{l}_s \subseteq \cdots \mathfrak{l}_1 \subseteq \mathfrak{g}$ of reductive Lie subalgebras.

In addition, let $\pi_i : \mathfrak{g}^\vee \twoheadrightarrow \mathfrak{l}_i^\vee$ be the transposition of the Lie-algebra inclusion $\mathfrak{l}_i \hookrightarrow \mathfrak{g}$, for $i \in \{1, \ldots, s\}$. Composing with the dualities provided by the given nondegenerate pairing on $\mathfrak{g}$, one can view them as linear maps $\mathfrak{g} \twoheadrightarrow \mathfrak{l}_i$. Write in particular

$$\pi' := \pi_{s-1} : \mathfrak{g}^\vee \longrightarrow\!\!\!\!\!\rightarrow (\mathfrak{l}')^\vee, \qquad \mathfrak{l}' = \mathfrak{l}_{Q'} := \mathfrak{l}_{s-1}.$$

(This involves the $\mathrm{ad}_\mathfrak{g}$-stabilizer of the irregular type Q'; if $s = 1$ then $Q' = 0$ and $\pi' = \pi_0$ is the identity of $\mathfrak{g}^\vee = (\mathfrak{l}')^\vee = \mathfrak{l}_0^\vee$.)

Finally, trivialize $T^*G \simeq G \times \mathfrak{g}^\vee$ using left translations: elements will be written (g, Λ), with $g \in G$ and $\Lambda \in \mathfrak{g} \simeq \mathfrak{g}^\vee$, cf. Rmk. 2.3.4.

Then the following statements hold true (cf. [23, 148]):

1. there is a 'decoupling' symplectomorphism $T^*G \times \check{\mathcal{O}}' \xrightarrow{\simeq} \widetilde{\mathcal{O}}'$, i.e.,

$$\big((g, \Lambda), dQ\big) \longmapsto (g, \mathcal{A}), \qquad \mathcal{A} := \Lambda \varpi^{-1} d\varpi + \mathrm{Ad}_g^\vee(dQ); \tag{49}$$

   (The identification (49) is implicit in what follows.)
2. the $\mathrm{Ad}_{L'}^\vee$-action on $\check{\mathcal{O}}'$ is Hamiltonian, with a moment map

$$\check{\mu}' = \check{\mu}'_{\check{\mathcal{O}}'} : \check{\mathcal{O}}' \longrightarrow (\mathfrak{l}')^\vee \simeq \mathfrak{l}', \tag{50}$$

   satisfying $\check{\mu}'(dQ') = 0$;
3. the direct product $G \times L'$ acts on $\widetilde{\mathcal{O}}'$ (on the left), via

$$\big((g, \Lambda), dQ\big) \longmapsto \big(\big(hg\widetilde{g}^{-1}, \mathrm{Ad}_{\widetilde{g}}(\Lambda)\big), \mathrm{Ad}_h(dQ)\big), \qquad \widetilde{g} \in G, \quad h \in L'; \tag{51}$$

4. the action (51) is Hamiltonian, and generated by the moment map

$$\widetilde{\mu}' = (\widetilde{\mu}'_1, \widetilde{\mu}'_2) : \widetilde{\mathcal{O}}' \longrightarrow \mathfrak{g}^\vee \times (\mathfrak{l}')^\vee \simeq \mathfrak{g} \times \mathfrak{l}',$$

   where

$$\widetilde{\mu}'_1 : \big((g, \Lambda), dQ\big) \longmapsto \Lambda,$$

   and

$$\widetilde{\mu}'_2 : \big((g, \Lambda), dQ\big) \longmapsto \check{\mu}'(dQ) - \pi'\big(\mathrm{Ad}_g(\Lambda)\big); \tag{52}$$

5. and there is a G-equivariant symplectomorphism

$$\widetilde{\mathcal{O}}' /\!\!/\!\!/_{(-\Lambda')} L' := (\widetilde{\mu}'_2)^{-1}(-\Lambda') /\!\!/ L_s \xrightarrow{\simeq} \mathcal{O}', \tag{53}$$

   recalling that $L_s \subseteq L'$ is the $\mathrm{Ad}_{L'}$-stabilizer of the normal residue.

8. Criteria for flatness: generic case

It is convenient to first spell out the proof of Cor. 1.3.6, which is anyway of separate interest: the general case is treated in § 9. Recall that an $\mathrm{Ad}^{\vee}_{G_s}$-orbit $\mathcal{O}' \subseteq \mathfrak{g}_s^{\vee}$ is *generic* if the leading coefficient of any of its points $\mathcal{A}' \in \mathcal{O}'$ is regular semisimple; in particular $\mathcal{A}' \in \mathfrak{g}_s''$, and $\mathfrak{l}_s = \cdots = \mathfrak{l}_1 \subseteq \mathfrak{g}$ is the *unique* Cartan subalgebra containing the leading coefficient.

Thus, we must establish flatness in the following cases (in increasing difficulty):

1. there is one generic pole of order at least three (cf. § 8.1);
2. there is one generic pole of order two, and some other generic pole (cf. § 8.2);
3. or there are two simple generic poles, and some other generic pole (cf. § 8.3).

8.1. Flatness with high pole order. — In the first case we can immediately establish the following stronger result, in the nongeneric setting:

***8.1.1. Proposition.* —** *Suppose that* $s \geqslant 3$, *and that* $L_{s-2} \subseteq G$ *is a maximal torus. (Whence* $L_s = L_{s-1} = L_{s-2}$.*) Then the* G*-moment map*

$$\mu' = \mu'_{\mathcal{O}'} \colon \mathcal{O}' \longrightarrow \mathfrak{g}^{\vee} \simeq \mathfrak{g}, \qquad \mathcal{A} \longmapsto \mathrm{Res}(\mathcal{A}), \tag{54}$$

is faithfully flat.

Proof. — We will prove that the moment map (50) is surjective, and omit the proof that it has equidimensional fibres: the conclusion follows from Lem. 8.1.2.

Choose two opposite Borel subalgebras $\mathfrak{b}^{\pm} \subseteq \mathfrak{g}$ containing the Cartan subalgebra $\mathfrak{t} \coloneqq \mathfrak{l}'$, i.e., such that $\mathfrak{b}^+ \cap \mathfrak{b}^- = \mathfrak{t}$, and let $\Phi^{\pm} \subseteq \Phi$ be the subsystems of positive/negative roots of $(\mathfrak{g}, \mathfrak{t})$ determined by $\mathfrak{b}^{\pm}$. For $i \in \{1, \dots, s\}$, denote also by $\phi_i \subseteq \Phi$ the Levi (root) subsystem corresponding to $\mathfrak{l}_i$, whence a decreasing filtration

$$\phi_s = \phi_{s-1} = \phi_{s-2} \subseteq \cdots \subseteq \phi_1 \subseteq \phi_0 \coloneqq \Phi,$$

cf. [39]. Finally, let $\phi_i^{\pm} \coloneqq \Phi^{\pm} \cap \phi_i$ be the corresponding systems of positive/negative roots in ϕ_i. Then there are also: (i) disjoint unions

$$\phi_{i-1} = \nu_i^- \cup \phi_i \cup \nu_i^+, \qquad \nu_i^{\pm} \coloneqq \phi_{i-1}^{\pm} \setminus \phi_i;$$

and (ii) nested triangular decompositions

$$\mathfrak{l}_{i-1} = \mathfrak{n}_i^- \oplus \mathfrak{l}_i \oplus \mathfrak{n}_i^+, \qquad \mathfrak{n}_i^{\pm} \coloneqq \bigoplus_{\nu_i^{\pm}} \mathfrak{g}_{\alpha},$$ [26]

[26] The nilpotent subalgebras $\mathfrak{n}_i \subseteq \mathfrak{l}_{i-1}$ are *not* the nilradicals $\mathfrak{u}_i^{\pm} \subseteq \mathfrak{p}_i^{\pm}$ of the parabolic subalgebras $\mathfrak{p}_i^{\pm} \subseteq \mathfrak{g}$ of § 4: the latter rather satisfy $\mathfrak{g} = \mathfrak{u}_i^- \oplus \mathfrak{l}_i \oplus \mathfrak{u}_i^+$ (and do *not* use the choice of a Borel subalgebra): cf. Cor.-Def. 9.4.2. Also note that $\nu_s^{\pm} = \nu_{s-1}^{\pm} = \varnothing$ and $\mathfrak{n}_s^{\pm} = \mathfrak{n}_{s-1}^{\pm} = (0)$—in the current situation.

invoking the root lines $\mathfrak{g}_\alpha \subseteq \mathfrak{g}$—and setting $\mathfrak{l}_0 := \mathfrak{g}$, cf. [29, Def. 7.2]. Now define in addition

$$\check{\mathfrak{n}}_i^{\pm} := \left\{ X = \sum_{j=1}^{s-i-1} X_j \varpi^j \;\middle|\; X_j \in \mathfrak{n}_i^{\pm} \right\} \subseteq \mathfrak{bir}_{s-i}, \qquad i \in \{1, \dots, s-2\}, \tag{55}$$

and let the maximal torus $T := L'$ act as usual on (the coefficients of) such truncated power series—noting that $[\mathfrak{l}_i, \mathfrak{n}_i^{\pm}] \subseteq \mathfrak{n}_i^{\pm}$. By [148, Cor. 3.2], there is a T-equivariant symplectomorphism

$$\check{\mathcal{O}}' \xrightarrow{\simeq} T^* \check{\mathfrak{n}}^+, \qquad \check{\mathfrak{n}}^+ := \check{\mathfrak{n}}_1^+ \times \cdots \times \check{\mathfrak{n}}_{s-2}^+, \tag{56}$$

and we shall now compute a T-moment map for each factor of $T^*\check{\mathfrak{n}}^+ \simeq \prod_{i=1}^{s-2} (T^* \check{\mathfrak{n}}_i^+)$.

As in § 2.2.2, identify the dual of $\check{\mathfrak{n}}_i^{\pm}$ with a space of (irregular parts of) $\mathfrak{n}_i^{\mp}$-valued meromorphic 1-forms on $\operatorname{Spec} \mathbb{C}[\![\varpi]\!]$, i.e.,

$$(\check{\mathfrak{n}}_i^{\pm})^{\vee} \simeq \left\{ dQ \;\middle|\; Q = \sum_{j=1}^{s-i-1} A_j \frac{\varpi^{-j}}{-j}, \quad A_j \in \mathfrak{n}_i^{\mp} \right\}. \tag{57}$$

Then, in the notation of (55) and (57), the moment map on the i-th factor, which vanishes at the origin, reads

$$\check{\mu}_i' : \check{\mathfrak{n}}_i^+ \times (\check{\mathfrak{n}}_i^+)^{\vee} \simeq T^* \check{\mathfrak{n}}_i^+ \longrightarrow \mathfrak{t}^{\vee} \simeq \mathfrak{t}, \qquad (X, dQ) \longmapsto \sum_{j=1}^{s-i-1} \sum_{\Phi_i^+} [X_j^{(\alpha)}, A_j^{(-\alpha)}],$$

invoking the root-line components of

$$X_j = \sum_{\Phi_i^+} X_j^{(\alpha)} \in \mathfrak{n}_i^+, \qquad A_j = \sum_{\Phi_i^-} A_j^{(\alpha)} \in \mathfrak{n}_i^-. \tag{58}$$

(And noting that $\Phi_i^- = -\Phi_i^+$.) The surjectivity of the moment map for the diagonal action on all factors now follows from the identity

$$\bigoplus_{i=1}^{s-2} \mathfrak{n}_i^{\pm} = [\mathfrak{b}^{\pm}, \mathfrak{b}^{\pm}] = \bigoplus_{\Phi^{\pm}} \mathfrak{g}_\alpha;$$

because, in turn, one has $\mathfrak{t} = \bigoplus_{\Phi^{\pm}} [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$. □

8.1.2. Lemma. — *If (50) is surjective, with equidimensional fibres, then the same holds for (54).*

Proof. — Postponed to C.6 □

8.1.3. — In turn, in order to leverage Prop. 8.1.1 to the case of several marked points, we use the fact that the moment map for a diagonal G-action—on a product of Hamiltonian G-varieties—is the sum of the moment maps on each factor, together with the following observation:

8.1.4. Lemma. — *Let M'_1 and M'_2 be two $\mathbb{C}$-varieties. Consider a finite-dimensional complex vector space V, and two maps $\varphi_i \colon M'_i \to V$ (for $i \in \{1,2\}$), such that φ_1 is faithfully flat. Then the sum*

$$\varphi_1 + \varphi_2 \colon M'_1 \times M'_2 \longrightarrow V, \qquad (x_1, x_2) \longmapsto \varphi_1(x_1) + \varphi_2(x_2), \tag{59}$$

is faithfully flat.

Proof postponed to C.7. — □

8.2. Flatness with a pole of order 2. — If $s = 2$, then $\mathfrak{bir}_s = \mathfrak{bir}_2$ is an *abelian* Lie algebra, and $\check{\mathcal{O}}' = \{ dQ' \}$ is a singleton: the statement of Prop. 8.1.1 does *not* extend—as (50) is never surjective/flat. (Note that here $\widetilde{\mathcal{O}}' \simeq T^*G \times \{ dQ' \}$ is a 'twisted' cotangent bundle.)

Rather, one can use additional orbits (and marked points) to ensure surjectivity and equidimensionality of the fibres, even in the nongeneric case, as follows:

8.2.1. Proposition. — *Suppose that $s = 2$, and let M' be a nonsingular irreducible complex G-variety, equipped with a G-equivariant map $\varphi \colon M' \to \mathfrak{g}^\vee \simeq \mathfrak{g}$, such that the composition $\pi' \circ \varphi \colon M' \to (\mathfrak{l}')^\vee \simeq \mathfrak{l}'$ is surjective, with equidimensional fibres. Then the map*

$$\psi_2 \colon \mathcal{O}' \times M' \longrightarrow \mathfrak{g}^\vee \simeq \mathfrak{g}, \qquad (\mathcal{A}, x) \longmapsto \mathrm{Res}(\mathcal{A}) + \varphi(x), \tag{60}$$

is faithfully flat.

Proof. — We will prove that (60) is surjective, with equidimensional fibres—cf. the proof C.6. Namely, given $X \in \mathfrak{g} \simeq \mathfrak{g}^\vee$, using (the complex-analytic version of) (53) and the G-equivariance of φ, and setting $\varphi' := \pi' \circ \varphi$, the fibre over X can be described as follows:

$$\begin{aligned}
\psi_2^{-1}(X) &\simeq \Big\{ \big((g,\Lambda),x\big) \in T^*G \times M' \;\Big|\; \Lambda + \varphi(x) = X, \quad -\pi'\big(\mathrm{Ad}_g^\vee(\Lambda)\big) = -\Lambda' \Big\} \Big/ L_s \\
&\simeq \Big\{ (g,x) \in G \times M' \;\Big|\; \pi'\big(\mathrm{Ad}_g^\vee\big(X - \varphi(x)\big)\big) = \Lambda' \Big\} \Big/ L_s \\
&= \Big\{ (g,x) \in G \times M' \;\Big|\; \varphi'(g.x) = \pi'\big(\mathrm{Ad}_g^\vee(\Lambda)\big) - \Lambda' \Big\} \Big/ L_s \\
&\simeq \big(G \times (\varphi')^{-1}(X')\big) \big/ L_s, \qquad X' := \pi'\big(\mathrm{Ad}_g^\vee(X)\big) - \Lambda'.
\end{aligned}$$

By hypothesis, this is nonempty, and its dimension does *not* depend on X—as the action of L_s on $G \times M'$ is free. □

8.2.2. — In turn, in order to leverage Prop. 8.2.1, we use the following fact:

8.2.3. Proposition. — *Let again $s \geqslant 1$ be arbitrary, but suppose that $\mathcal{O}'$ is* generic. *Then the composition*

$$\pi_1 \circ \mu' \colon \mathcal{O}' \longrightarrow \mathfrak{l}_1^\vee \simeq \mathfrak{l}_1, \qquad \mathcal{A} \longmapsto \pi_1\big(\mathrm{Res}(\mathcal{A})\big),$$

is surjective, with equidimensional fibres—using the G*-moment map* (54).
Proof postponed to C.8; this is generalized by Cor. 9.3.8. — □

8.2.4. Remark. — In particular, taking $s = 1$ in Prop. 8.2.3 (and keeping the notation of its proof), it follows: (i) that a standard regular semisimple $\mathrm{Ad}^{\vee}_{G}$-orbit $\mathcal{O}' \subseteq \mathfrak{g}^{\vee}$ is covered by open patches, isomorphic to $U^{+} \times U^{-}$ as (symplectic) Hamiltonian T-spaces; and (ii) that $\mathcal{O}'$ projects onto $\mathfrak{t}^{\vee}$ in faithfully flat fashion. Thus, the 'complexified' analogue of Kostant's convexity theorem [97] (cf. [124, 81, 135]) has an easier statement.[27] ♢

8.3. Flatness with simple poles. — Finally, if $s = 1$, then $\mathfrak{bir}_s = \mathfrak{bir}_1$ is *trivial*, and again the above cannot be adapted. (Note that here $\widetilde{\mathcal{O}}' \simeq T^{*}G$ is a genuine cotangent bundle, and so in particular we change the definition of [23].)

Rather, one can 'unfold' a pole of order two to recover the wild situation of § 8.2, by relying on one additional orbit, as follows:

8.3.1. Proposition. — *Suppose that* $s = 1$*, with* Λ' regular *semisimple, and let* $\underline{\mathcal{O}}'$ *be any other* regular *semisimple* $\mathrm{Ad}^{\vee}_{G}$*-orbit. Moreover, let* M' *be a nonsingular irreducible complex* G*-variety, equipped with a* G*-equivariant map* $\varphi \colon M' \to \mathfrak{g}^{\vee} \simeq \mathfrak{g}$*, such that the composition* $\pi_1 \circ \varphi \colon M' \to \mathfrak{l}_1^{\vee} \simeq \mathfrak{l}_1$ *is surjective, with equidimensional fibres. Then the map*

$$\psi_1 \colon \mathcal{O}' \times \underline{\mathcal{O}}' \times M' \longrightarrow \mathfrak{g}^{\vee} \simeq \mathfrak{g}, \qquad (\Lambda, \underline{\Lambda}, x) \longmapsto \Lambda + \underline{\Lambda} + \varphi(x), \tag{61}$$

is faithfully flat.

Proof. — Write for simplicity $T \coloneqq L_1$ and $\mathfrak{t} \coloneqq \mathrm{Lie}(T) = \mathfrak{l}_1$. Recall that $\mathcal{O}'$ and $\underline{\mathcal{O}}'$ intersect $\mathfrak{t}^{\vee}$ in an orbit for the Weyl group W—of cardinality $|W|$ (cf. [45]). Now, by construction, $\Lambda' \in \mathcal{O}' \cap \mathfrak{t}^{\vee}$; choose also a marking $\underline{\Lambda}' \in \underline{\mathcal{O}}' \cap \mathfrak{t}$. Finally, for any element $w \in W$, consider the following NUTS pole-order-2 principal part:

$$\mathcal{A}'_w = \Lambda'_w \varpi^{-1} d\varpi + dQ', \qquad \Lambda'_w \coloneqq \Lambda' + w^{-1}(\underline{\Lambda}') \in \mathfrak{t}, \quad Q' \coloneqq -\Lambda' \varpi^{-1}. \tag{62}$$

Then the result follows from Prop. 8.2.1 + Lem. 8.3.2, because the restriction of (61) to suitable open subspaces is as in (60). (Recall that open immersions are flat.) □

8.3.2. Lemma. — *Let* $\mathcal{O}'_w \coloneqq G_2.\mathcal{A}'_w \subseteq \mathfrak{g}_2^{\vee}$ *be the* $\mathrm{Ad}^{\vee}_{G_2}$*-coadjoint orbit of* (62)*. Then:*

1. *there is a* G*-equivariant symplectic (Zariski-)open immersion* $\iota_w \colon \mathcal{O}'_w \hookrightarrow \mathcal{O}' \times \underline{\mathcal{O}}'$*;*
2. *and these immersions yield an open cover* $\mathcal{O}' \times \underline{\mathcal{O}}' = \bigcup_W \iota_w(\mathcal{O}'_w)$*.*

Proof postponed to C.9; this is generalized by Cor. 9.4.9. — □

9. Criteria for flatness: general case

Putting together the statements of § 8 yields a proof of Cor. 1.3.6.

(27) If G had positive-dimensional centre, then the projection of $\mathcal{O}'$ would be a proper affine subspace of $\mathfrak{t}^{\vee}$, modelled over the complex span of the roots, translated by the central part of—any element of—$\mathcal{O}'$ (cf. Lem.-Def. 7.1.3).

9.1. — Here instead we conclude the proof of the main Thm. 1.3.4. Namely, recall from § 1.3.3 that we associate a number ν_a to the UTS orbits $\mathcal{O}'_a \subseteq \mathfrak{g}^{\vee}_{s_a}$, via

$$\nu_a = |\mathcal{T}_a| \leqslant s, \qquad \mathcal{T}_a = \mathcal{T}(\mathcal{O}'_a) := \left\{ i \in \{1, \dots, s\} \;\middle|\; L_{a,i} \subseteq G \text{ is a max. torus } \right\}, \tag{63}$$

by looking at the Levi/fission sequence $L_{a,s_a} \subseteq \cdots \subseteq L_{a,1} \subseteq L_{a,0} := G$ determined by any UTS principal part $\mathcal{A}'_a = (\Lambda'_a \varpi^{-1} d\varpi + dQ'_a) \in \mathcal{O}'_a$. Thus, $\nu_a \geqslant 2$ means that the centralizer of the irregular type is a maximal torus, which only happens in the irregular-singular case (where $Q'_a \neq 0$), and which was referred to as having 'complete fission' in [39] (cf. [26, 29]); while $\nu_a \geqslant 1$ means that the centralizer of the whole of the principal part is a maximal torus. E.g., if $s_a = 1$ (whence $Q'_a = 0$) this means considering a standard $\mathrm{Ad}^{\vee}_G$-orbit through a regular semisimple element.

Then we must establish flatness in the following cases (in increasing difficulty):

1. there is one pole $a \in \mathbf{a}$ such that $\nu_a \geqslant 3$ (cf. § 9.2);
2. there are two poles $a, a' \in \mathbf{a}$ such that $\nu_a \geqslant 2$ and $\nu_{a'} \geqslant 1$ (cf. § 9.3);
3. or there are three poles $a, a', a'' \in \mathbf{a}$ such that $\nu_a, \nu_{a'}, \nu_{a''} \geqslant 1$ (cf. § 9.4).

9.2. Three maximal tori at a marked point. — The first case, where $\nu_a \geqslant 3$, follows from Prop. 8.1.1 + Lem. 8.1.4.

9.3. Three maximal tori at two marked points. — To treat the second case we establish two results (of separate interest): the construction of Darboux charts on coadjoint orbits for TCLAs (cf. Thm. 9.3.2), and a generalization of Prop. 8.2.3 (cf. Cor. 9.3.8).

9.3.1. — Throughout this section, it will be convenient to fix a maximal torus; we thus refresh the whole Lie-theoretic setup (which is unchanged from § 7).

Let G be a connected *semisimple* complex Lie group, with Lie algebra $\mathfrak{g}$. Choose a maximal torus $T \subseteq G$, with Lie algebra $\mathfrak{t} \subseteq \mathfrak{g}$, and two (opposite) Borel subgroups $B^{\pm} \subseteq G$ intersecting at T, with Lie algebras $\mathfrak{b}^{\pm} \subseteq \mathfrak{g}$. Then consider a $\mathfrak{t}$-*valued* UTS principal part, i.e.,

$$\mathcal{A}' = \sum_{i=0}^{s-1} A'_i \varpi^{-i-1} d\varpi, \qquad \Lambda = A'_0, A'_1, \dots, A'_{s-1} \in \mathfrak{t}, \tag{64}$$

with *leading term*

$$\mathcal{A}'_{\mathrm{top}} := A'_{s-1} \varpi^{-s} d\varpi. \tag{65}$$

(The latter will enter a recursion.) Let again $\mathcal{O}' := G_s.\mathcal{A}'$ be the $\mathrm{Ad}^{\vee}_{G_s}$-orbit of $\mathcal{A}'$.

Analogously to the proof of Prop. 8.1.1, denote by $\mathfrak{n}^{\pm}_i \subseteq \mathfrak{l}_{i-1}$ the (opposite) nilradicals of the parabolic subalgebras of $\mathfrak{l}_{i-1}$ containing $\mathfrak{b}^{\pm} \cap \mathfrak{l}_{i-1}$, whence $\mathfrak{l}_i \subseteq \mathfrak{l}_{i-1}$ is their (common) Levi factor containing $\mathfrak{t}$. Furthermore, if $N^{\pm}_i \subseteq L_{i-1}$ are the unipotent radicals of the parabolic subgroup of L_{i-1} containing $B^{\pm} \cap L_{i-1}$, so that $\mathrm{Lie}(N^{\pm}_i) = \mathfrak{n}^{\pm}_i$, define also the subgroups

$$\widetilde{L}_i := L_i(\mathscr{O}_{s-i}) \subseteq G_{s-i}, \qquad i \in \{0, \dots, s\}, \tag{66}$$

and

$$\widetilde{\mathrm{N}}_i^{\pm} := \mathrm{N}_i^{\pm}(\mathscr{O}_{s-i+1}) \subseteq \widetilde{\mathrm{L}}_{i-1}, \qquad i \in \{1,\dots,s\},$$

with Lie algebras $\widetilde{\mathfrak{l}}_i := \mathrm{Lie}(\widetilde{\mathrm{L}}_i)$ and $\widetilde{\mathfrak{n}}_i^{\pm} := \mathrm{Lie}(\widetilde{\mathrm{N}}_i^{\pm})$. (As usual, L_i and $\mathrm{N}_i^{\pm}$ are tacitly viewed as *algebraic* groups, as needed.)

Then the cotangent splitting (56), of the Birkhoff orbit through dQ', is upgraded as follows:

9.3.2. *Theorem* (cf. [148], Thm. 3.1). — *There are:*

1. *a finite affine (Zariski-)open cover $\mathcal{O}' = \bigcup_i V_i'$, by L_s-invariant symplectic subvarieties $\mathcal{O}_i' \subseteq \mathcal{O}'$;*
2. *and L_s-equivariant (algebraic) symplectomorphisms*

$$V_i' \xrightarrow{\simeq} T^*\widetilde{\mathrm{N}}^+, \qquad \widetilde{\mathrm{N}}^+ := \widetilde{\mathrm{N}}_1^+ \times \cdots \times \widetilde{\mathrm{N}}_s^+. \tag{67}$$

Proof. — Decompose (64) as $\mathcal{A}' = \mathcal{A}'_{s-1} + \mathcal{A}'_{\mathrm{top}}$, with

$$\mathcal{A}'_{s-1} := \sum_{j=0}^{s-2} A_j' \varpi^{-j-1} d\varpi \in \widetilde{\mathfrak{l}}_1^{\vee}, \tag{68}$$

in the notation of (65), and under the usual dualities for TCLAs. Moreover, let $\mathcal{O}'_{s-1} := \widetilde{\mathrm{L}}_1.\mathcal{A}'_{s-1} \subseteq \widetilde{\mathfrak{l}}_1^{\vee}$ be the $\mathrm{Ad}^{\vee}_{\widetilde{\mathrm{L}}_1}$-orbit of the 'subleading' term of $\mathcal{A}'$. Then the statement is proven by induction on the pole order using Prop. 9.3.4, and noting that the images (70) cover $\mathcal{O}'$, upon replacing the normal form $\mathcal{A}'$ by all its Weyl-translated

$$w(\mathcal{A}') \in \mathcal{O}', \qquad w \in W. \qquad \square$$

9.3.3. — Hereafter let $\mathbf{g}(0) \in G \subseteq G \ltimes \mathrm{Bir}_s$ be the 'constant' part of a group element $\mathbf{g} \in G_s$.

9.3.4. *Proposition.* — *Denote by $\mathrm{Ad}^{\flat}$ the coadjoint $\widetilde{\mathrm{N}}_1^+$-action,*[28] *and use the left translations of $\widetilde{\mathrm{N}}_1^+$ to trivialize its cotangent bundle. Then:*

1. *the map*

$$\iota : \left(\widetilde{\mathrm{N}}_1^+ \times (\widetilde{\mathfrak{n}}_1^+)^{\vee}\right) \times \mathcal{O}'_{s-1} \simeq T^*\widetilde{\mathrm{N}}_1^+ \times \mathcal{O}'_{s-1} \longrightarrow \mathfrak{g}_s^{\vee}, \tag{69}$$

defined by

$$\big((\mathbf{u}^+, \mathbf{Y}), \mathcal{A}_{s-1}\big) \longmapsto \mathrm{Ad}^{\vee}_{(\mathbf{u}^+)^{-1}}\big(\mathrm{Ad}^{\flat}_{\mathbf{u}^+}(\mathbf{Y}) + \mathcal{A}_{s-1} + \mathcal{A}'_{\mathrm{top}}\big),$$

yields an L_1-equivariant symplectic (Zariski-)open embedding—of the domain—into the orbit $\mathcal{O}'$;

2. *and the image of (69) is*

$$\mathcal{O}'(T) := \left\{ \mathrm{Ad}^{\vee}_{\mathbf{g}^{-1}}(\mathcal{A}') \;\middle|\; \mathbf{g} \in G_s \text{ and } \mathbf{g}(0) \in \mathrm{L}_1 \cdot \mathrm{N}_1^- \cdot \mathrm{N}_1^+ \right\}. \tag{70}$$

(Cf. (105) and (112).)

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(28) The composite of the $\mathrm{Ad}^{\vee}_{G_s}$-action $\mathrm{Ad}^{\vee}_{\mathbf{u}^+} : \mathfrak{g}_s^{\vee} \to \mathfrak{g}_s^{\vee}$ and the projection $\mathfrak{g}_s^{\vee} \twoheadrightarrow (\widetilde{\mathfrak{n}}_1^+)^{\vee}$, for $\mathbf{u}^+ \in \widetilde{\mathrm{N}}_1^+$.

Proof. — Let us first show that (69) takes values into $\mathcal{O}'$. Choose a triple

$$\big((\mathbf{u}^+,\mathbf{Y}),\mathcal{A}_{s-1}\big) \in \big(\widetilde{\mathrm{N}}_1^+ \times (\widetilde{\mathfrak{n}}_1^+)^\vee\big) \times \mathcal{O}'_{s-1},$$

as well as an element $\mathbf{h} \in \widetilde{\mathrm{L}}_1$, such that $\mathcal{A}_{s-1} = \mathrm{Ad}^\vee_{\mathbf{h}^{-1}}(\mathcal{A}'_{s-1})$. Denote also—abusively—by the same symbol a lift of $\mathbf{h}$ in $\underline{\widetilde{\mathrm{L}}}_1 \twoheadrightarrow \widetilde{\mathrm{L}}_1$, in the notation of (73). Then

$$\mathrm{Ad}^\vee_{\mathbf{h}^{-1}}(\mathcal{A}') = \mathrm{Ad}^\vee_{\mathbf{h}^{-1}}(\mathcal{A}'_{s-1}) + \mathcal{A}'_{\mathrm{top}} = \mathcal{A}_{s-1} + \mathcal{A}'_{\mathrm{top}}, \tag{71}$$

as $\mathrm{L}_1 = \mathrm{G}^{\mathrm{A}_{s-1}}$. Moreover, by Lem. 9.3.6, there exists $\mathbf{u}^- \in \widetilde{\mathrm{N}}_1^-$ such that

$$\mathrm{Ad}^\flat_{\mathbf{u}^+}(\mathbf{Y}) + \mathcal{A}_{s-1} + \mathcal{A}'_{\mathrm{top}} = \mathrm{Ad}^\vee_{\mathbf{u}^-}(\mathcal{A}_{s-1} + \mathcal{A}'_{\mathrm{top}}). \tag{72}$$

Then (71)–(72) yield

$$\begin{aligned}\mathrm{Ad}^\vee_{(\mathbf{u}^+)^{-1}}\big(\mathrm{Ad}^\flat_{\mathbf{u}^+}(\mathbf{Y}) + \mathcal{A}_{s-1} + \mathcal{A}'_{\mathrm{top}}\big) &= \mathrm{Ad}^\vee_{(\mathbf{u}^+)^{-1}\mathbf{u}^-}(\mathcal{A}_{s-1} + \mathcal{A}'_{\mathrm{top}}) \\ &= \mathrm{Ad}^\vee_{\mathbf{g}^{-1}}(\mathcal{A}'),\end{aligned}$$

setting $\mathbf{g} \coloneqq \mathbf{h}(\mathbf{u}^-)^{-1}\mathbf{u}^+ \in \widetilde{\mathrm{L}}_1 \cdot \widetilde{\mathrm{N}}_1^- \cdot \widetilde{\mathrm{N}}_1^+ \subseteq \mathrm{G}_s$.

The resulting algebraic morphism $\mathrm{T}^*\widetilde{\mathrm{N}}_1^+ \times \mathcal{O}'_{s-1} \to \mathcal{O}'$ is L_1-equivariant, and we conclude by showing that it maps isomorphically onto (70). (The fact that (69) is a symplectic map follows from a computation quite similar to the proof of [**148**, Thm. 3.1].) In view of Lem. 9.3.6, it is enough to show that the map

$$\widetilde{\mathrm{N}}_1^+ \times \widetilde{\mathrm{N}}_1^- \times \mathcal{O}'_{s-1} \longrightarrow \mathcal{O}'(\mathrm{T}), \qquad (\mathbf{u}^+,\mathbf{u}^-,\mathcal{A}_{s-1}) \longmapsto \mathrm{Ad}^\vee_{(\mathbf{u}^+)^{-1}\mathbf{u}^-}(\mathcal{A}_{s-1} + \mathcal{A}'_{\mathrm{top}}),$$

is an isomorphism. To this end, given a group element $\mathbf{g} \in \mathrm{G}_s$ such that

$$g \coloneqq \mathbf{g}(0) = hu^-u^+ \in \mathrm{L}_1 \cdot \mathrm{N}_1^- \cdot \mathrm{N}_1^+,$$

consider the (unique) factorization $\mathbf{g} = \mathbf{h}(\mathbf{u}^-)^{-1}\mathbf{u}^+$ provided by Lem. 9.3.5. Then define

$$\mathcal{A}_{s-1} = \mathcal{A}_{s-1}(\mathbf{g}) \coloneqq \mathrm{Ad}^\vee_{\mathbf{h}^{-1}}(\mathcal{A}') - \mathcal{A}'_{\mathrm{top}} \in \widetilde{\mathfrak{l}}_1^\vee.$$

This yields an algebraic map

$$\big\{\, \mathbf{g} \in \mathrm{G}_s \;\big|\; g \in \mathrm{L}_1 \cdot \mathrm{N}_1^- \cdot \mathrm{N}_1^+ \,\big\} \longrightarrow \widetilde{\mathrm{N}}_1^+ \times \widetilde{\mathrm{N}}_1^- \times \mathcal{O}'(\mathrm{T}), \qquad \mathbf{g} \longmapsto (\mathbf{u}^+,\mathbf{u}^-,\mathcal{A}_{s-1}),$$

which is invariant under the (left) action of the $\mathrm{Ad}^\vee_{\mathrm{G}_s}$-stabilizer of $\mathcal{A}'$ (as the latter is contained in $\underline{\widetilde{\mathrm{L}}}_1$): thus, it induces the desired inverse. □

9.3.5. Lemma. — *Choose an element $\mathbf{g} \in \mathrm{G}_s$ such that $g \coloneqq \mathbf{g}(0) \in \mathrm{L}_1 \cdot \mathrm{N}_1^- \cdot \mathrm{N}_1^+ \subseteq \mathrm{G}$, and set also*

$$\underline{\widetilde{\mathrm{L}}}_1 \coloneqq \mathrm{L}_1(\mathscr{O}_s) \subseteq \mathrm{G}_s. \tag{73}$$

(Whence (66) yields a quotient $\underline{\widetilde{\mathrm{L}}}_1 \twoheadrightarrow \widetilde{\mathrm{L}}_1$, truncating modulo ϖ^{s-1}.) Then:

1. *there is a* unique *factorization*

$$\mathbf{g} = \mathbf{h}\mathbf{u}^-\mathbf{u}^+, \qquad \mathbf{u}^\pm \in \widetilde{\mathrm{N}}_1^\pm, \quad \mathbf{h} \in \underline{\widetilde{\mathrm{L}}}_1;$$

2. *and the component maps $\mathbf{g} \mapsto \mathbf{h}$ and $\mathbf{g} \mapsto \mathbf{u}^\pm$ are L_1-equivariant algebraic morphisms.*

Proof. — Proof postponed to C.10. □

9.3.6. *Lemma.* — *Choose elements* $\mathbf{u}^- \in \widetilde{N}_1^-$ *and* $\mathcal{A}_{s-1} \in \mathcal{O}'_{s-1} \subseteq \widetilde{\mathfrak{l}}_1^\vee$. *Then:*

1. *the following vector—of* $\mathfrak{g}_s^\vee$*—lies in* $(\widetilde{\mathfrak{n}}_1^+)^\vee$*:*

$$\mathbf{Y}' := \mathrm{Ad}^\vee_{\mathbf{u}^-}\big(\mathcal{A}_{s-1} + \mathcal{A}'_{\mathrm{top}}\big) - \big(\mathcal{A}_{s-1} + \mathcal{A}'_{\mathrm{top}}\big); \tag{74}$$

2. *and the map*

$$\widetilde{N}_1^- \times \mathcal{O}'_{s-1} \longrightarrow (\widetilde{\mathfrak{n}}_1^+)^\vee \times \mathcal{O}'_{s-1}, \qquad (\mathbf{u}^-, \mathcal{A}_{s-1}) \longmapsto (\mathbf{Y}', \mathcal{A}_{s-1}), \tag{75}$$

is an L_1*-equivariant algebraic* isomorphism.

Proof. — Proof postponed to C.11. □

9.3.7. — We now deduce two corollaries from Thm. 9.3.2:

9.3.8. *Corollary* (Cf. Prop. 8.2.3). — *Suppose that* $L_s = T$ *(whence* $\mathfrak{l}_s = \mathfrak{t}$*). Then the composition*

$$\pi_s \circ \mu' : \mathcal{O}' \longrightarrow \mathfrak{t}^\vee \simeq \mathfrak{t}, \qquad \mathcal{A} \longmapsto \pi_s\big(\mathrm{Res}(\mathcal{A})\big),$$

is faithfully flat—using the G*-moment map* (54).

Proof. — In view of Thm. 9.3.2, it is enough to show that the T-moment map

$$T^*\widetilde{N}_1^+ \times \cdots \times T^*\widetilde{N}_s \longrightarrow \mathfrak{t}^\vee,$$

vanishing at the origin, is surjective, with equidimensional fibres: we will establish the former, and omit the proof of the latter.

The exponential map $\widetilde{\mathfrak{n}}_i^+ \to \widetilde{N}_i^+$ is a T-equivariant isomorphism of complex manifolds/varieties, so replace the i-th factor with $T^*\widetilde{\mathfrak{n}}_i^+$. Then, in the usual identifications/dualities, the T-moment map reads

$$\widetilde{\mathfrak{n}}_i^+ \times (\widetilde{\mathfrak{n}}_i^+)^\vee \simeq T^*\widetilde{\mathfrak{n}}_i^+ \longrightarrow \mathfrak{t}^\vee \simeq \mathfrak{t}, \qquad (X, \mathcal{A}) \longmapsto -\sum_{j=0}^{s-i} \sum_{\phi_i^+} \big[X_j^{(\alpha)}, A_j^{(-\alpha)}\big].$$

Here the notation for the Levi (root) subsystems $\phi_i \subseteq \Phi$, for their positive parts $\phi_i^+ = \phi_i \cap \Phi^+$, and for the root-line components of the coefficients of

$$X = \sum_{j=0}^{s-i} X_j \varpi^j, \qquad \mathcal{A} = \sum_{j=0}^{s-i} A_k \varpi^{-j-1} \, d\varpi,$$

is as in the proof of Prop. 8.1.1; and the conclusion follows in the same way. □

9.3.9. *Corollary.* — *Choose two integers* $s, \underline{s} \geqslant 1$. *Let* $\mathcal{O}' = G_s.\mathcal{A}'$ *and* $\underline{\mathcal{O}}' = G_{\underline{s}}.\underline{\mathcal{A}}$ *be two coadjoint orbits through UTS principal parts* $\mathcal{A}'$ *and* $\underline{\mathcal{A}}'$ *(as in* (64)*), of pole orders* s *and* $\underline{s}$*—respectively. Moreover, suppose that* $\nu(\mathcal{O}') \geqslant 2$ *and that* $\nu(\underline{\mathcal{O}}') \geqslant 1$*, in the notation of* (63)*—upon inserting the orbits at two distinct marked points* $a, \underline{a}$ *on the Riemann sphere. Then the* G*-moment map* (42)

$$\mu' : \mathcal{O}' \times \underline{\mathcal{O}}' \longrightarrow \mathfrak{g}^\vee \simeq \mathfrak{g}, \qquad (\mathcal{A}, \underline{\mathcal{A}}) \longmapsto \mathrm{Res}(\mathcal{A}) + \mathrm{Res}(\underline{\mathcal{A}}),$$

is faithfully flat.

Proof. — As usual, we will prove that the moment map is 'miraculously' flat. To this end, let $\check{\mathcal{O}}' \subseteq \mathfrak{bir}_s^\vee$ be the Birkhoff orbit associated with the irregular part of $\mathcal{A}'$, as in (47), with T-moment map (50). (Here $L' = T$ and $\mathfrak{l}' = \mathfrak{t}$.) Denote also by $A_0' \in \mathfrak{t}$ the residue of $\mathcal{A}'$. Then, in view of the symplectic decoupling (49), and of the T-Hamiltonian quotient (53), splitting/dualizing $T^*G \simeq G \times \mathfrak{g}^\vee \simeq G \times \mathfrak{g}$ yields an identification $(\mu')^{-1}(X) \simeq Z/T$ for all $X \in \mathfrak{g} \simeq \mathfrak{g}^\vee$, invoking the subvariety

$$Z = Z_X := \Big\{ \big((g, A_0), dQ, \underline{\widetilde{\mathcal{A}}} \big) \in T^*G \times \check{\mathcal{O}}' \times \underline{\mathcal{O}}' \\ \Big| \, A_0 + \mathrm{Res}\big(\underline{\widetilde{\mathcal{A}}}\big) = X, \quad \pi_s\big(\mathrm{Ad}_g^\vee(A_0)\big) - \check{\mu}'(dQ) = A_0' \Big\} \\ \simeq \Big\{ \big(g, dQ, \underline{\widetilde{\mathcal{A}}} \big) \in G \times \check{\mathcal{O}}' \times \underline{\mathcal{O}}' \, \Big| \, \pi_s\big(X - \mathrm{Res}\big(\underline{\widetilde{\mathcal{A}}}\big)\big) = \check{\mu}'(dQ) + A_0' \Big\}.$$

(The latter isomorphism is obtained by restricting, to Z, the canonical projection $G \times \mathfrak{g} \times \check{\mathcal{O}}' \times \underline{\mathcal{O}}' \twoheadrightarrow G \times \check{\mathcal{O}}' \times \underline{\mathcal{O}}'$.) Thus, each fibre of the projection $Z \to G \times \check{\mathcal{O}}'$ is isomorphic to a fibre of the map

$$\underline{\mathcal{O}}' \longrightarrow \mathfrak{t}^\vee \simeq \mathfrak{t}, \qquad \underline{\widetilde{\mathcal{A}}} \longmapsto \pi_s\big(\mathrm{Res}\big(\underline{\widetilde{\mathcal{A}}}\big)\big).$$

By Cor. 9.3.8, the latter is surjective, with equidimensional fibres; hence the same holds for $Z \to G \times \check{\mathcal{O}}'$, so that $Z \neq \varnothing$ has constant dimension as X varies. As usual, the conclusion follows from the fact that the T-action on $T^*G \times \check{\mathcal{O}}' \times \underline{\mathcal{O}}'$ is free. □

9.4. Three maximal tori at three marked points. —

To treat the last case we establish two results (of separate interest): the construction of an *unfolding* map (cf. Thm. 9.4.6), and a generalization of Lem. 8.3.2 (cf. Cor. 9.4.9). As mentioned in § 1, unfolding relates a single coadjoint G_s-orbit to several coadjoint G-orbits.

9.4.1. — Keep all the notation from §§ 7.2 + 9.3.1: consider a UTS principal part $\mathcal{A}'$ as in (64), its $\mathrm{Ad}_{G_s}^\vee$-orbit $\mathcal{O}' \subseteq \mathfrak{g}_s^\vee$, the extended orbit $\widetilde{\mathcal{O}}' \subseteq G \times \mathfrak{g}_s^\vee$, and the Birkhoff orbit $\check{\mathcal{O}}' \subseteq \mathfrak{bir}_s^\vee$. Recall from (56) that there is an L'-equivariant symplectomorphism of $\check{\mathcal{O}}'$ with the cotangent bundle to the product $\check{\mathfrak{n}}^+ = \prod_{i=1}^{s-2} \check{\mathfrak{n}}_i^+$ (cf. (55)). We will need a different version of it:

9.4.2. Corollary-Definition. — *For* $i \in \{1, \ldots, s\}$ *let*

$$\mathfrak{u}_i^\pm := \bigoplus_{j=1}^{i} \mathfrak{n}_j^\pm \subseteq \mathfrak{g}.^{(29)}$$

(29) These are the nilradical of two (opposite) parabolic subalgebras $\mathfrak{p}_i^\pm \subseteq \mathfrak{g}$, such that $\mathfrak{l}_i = \mathfrak{p}_i^+ \cap \mathfrak{p}_i^-$ is the (common) Levi factor containing $\mathfrak{t}$: they were first used in § 4.

Then there is (also) an L'*-equivariant symplectomorphism*

$$\check{\mathcal{O}}' \xrightarrow{\simeq} T^*\big(\mathfrak{u}_1^+ \times \cdots \times \mathfrak{u}_{s-2}^+\big) \simeq \prod_{i=1}^{s-2} \big(T^*\mathfrak{u}_i^+\big).$$

Proof postponed to C.12. — □

9.4.3. — Following [76], we pose the following:

9.4.4. Definition. — Choose a tuple $\boldsymbol{\varepsilon} = (\varepsilon_0, \dots, \varepsilon_{s-1})$ of distinct numbers $\varepsilon_i \in \mathbb{C}$. Then the $\boldsymbol{\varepsilon}$*-unfolding of* $\mathcal{A}'$ is the following $\mathfrak{t}$-valued logarithmic 1-form on the complex z-plane:

$$\operatorname{Unf}(\mathcal{A}') = \operatorname{Unf}_{\boldsymbol{\varepsilon}}(\mathcal{A}') \coloneqq \sum_{i=0}^{s-1} \frac{A_i' \, dz}{(z-\varepsilon_0)\cdots(z-\varepsilon_i)}. \tag{76}$$

9.4.5. — Note that (76) has poles at $z = \varepsilon_i$, for $i \in \{0, \dots, s-1\}$. Now let also

$$\widehat{\Lambda}_i \coloneqq \operatorname*{Res}_{z=\varepsilon_i}\big(\operatorname{Unf}(\mathcal{A}')\big) = \sum_{j=i}^{s-1} A_j' \cdot \prod_{\substack{l \in \{0,\dots,j\} \\ l \neq i}} (\varepsilon_i - \varepsilon_l)^{-1} \in \mathfrak{t}, \tag{77}$$

which is a $\mathbb{C}$-linear combination of $A_i', \dots, A_{s-1}'$. It follows that

$$\sum_{i=0}^{s-1} \widehat{\Lambda}_i = -\operatorname*{Res}_{z=\infty}\big(\operatorname{Unf}(\mathcal{A}')\big) = \Lambda'. \tag{78}$$

Moreover, one can recursively modify the ordered configuration $\boldsymbol{\varepsilon} \in \mathbb{C}^s$ (starting from ε_{s-1}) so that $G^{\widehat{\Lambda}_i} = L_{s-i} \subseteq G$ for $i \in \{0, \dots, s-1\}$: we always tacitly assume that the latter holds.

Then, using the (bijective) exponential map $\mathfrak{u}_i^{\pm} \to U_i^{\pm} \coloneqq e^{\mathfrak{u}_i^{\pm}} \subseteq G$ (in the notation of Cor.-Def. 9.4.2), the tame/logarithmic version of Prop. 9.3.4 yields L'-equivariant symplectic open immersions

$$T^*\mathfrak{u}_{s-i}^+ \simeq T^*U_{s-i}^+ = T^*U_{s-i}^+ \times \mathcal{O}_{L'}'(\widehat{\Lambda}_i) \hookrightarrow \mathcal{O}_G'(\widehat{\Lambda}_i), \qquad i \in \{2, \dots, s-1\},$$

noting that the $\operatorname{Ad}_{L'}^{\vee}$-orbits of $\widehat{\Lambda}_2, \dots, \widehat{\Lambda}_{s-1}$ are points—by the choice of $\boldsymbol{\varepsilon}$. Taking direct products, by Cor.-Def. 9.4.2 there is then another such (symplectic, L'-equivariant) open immersion

$$\check{\imath} \colon \check{\mathcal{O}}' \hookrightarrow \prod_{i=2}^{s-1} \mathcal{O}_G'(\widehat{\Lambda}_i). \tag{79}$$

The next aim is to upgrade (79) to a G-equivariant symplectic immersion of $\mathcal{O}'$ into the product of *all* the $\operatorname{Ad}_G^{\vee}$-orbits through the residues (77). To this end, one can check that the following is an L'-moment map:

$$\check{\nu} \colon \prod_{i=2}^{s-1} \mathcal{O}_G'(\widehat{\Lambda}_i) \longrightarrow (\mathfrak{l}')^{\vee} \simeq \mathfrak{l}', \qquad (A_2, \dots, A_{s-1}) \longmapsto \sum_{i=2}^{s-1} \pi'(A_i). \tag{80}$$

(We omit the proof of this fact.) It then follows from (78) that

$$\check{\nu} \circ \check{\iota} - \check{\mu}' = \sum_{i=2}^{s-1} \widehat{\Lambda}_i = \Lambda' - \widehat{\Lambda}_0 - \widehat{\Lambda}_1, \tag{81}$$

in the notation of (50) and (79)–(80).

Now we construct an L_s-invariant map

$$T^* G \times \check{\mathcal{O}}' \simeq \widetilde{\mathcal{O}}' \supseteq (\widetilde{\mu}_2')^{-1}(-\Lambda') \xrightarrow{\widetilde{\Psi}} \mathcal{O}'_G(\widehat{\boldsymbol{\Lambda}}) \coloneqq \prod_{i=0}^{s-1} \mathcal{O}'_G(\widehat{\Lambda}_i),$$

in the notation of (49) and (52): by (53), this will induce the desired G-equivariant map

$$\Psi \colon \mathcal{O}' \simeq \widetilde{\mathcal{O}}' /\!\!/\!\!/_{(-\Lambda')} L' \longrightarrow \mathcal{O}'_G(\widehat{\boldsymbol{\Lambda}}). \tag{82}$$

Choose thus a triple $\big((g, \Lambda), dQ\big) \in (G \times \mathfrak{g}) \times \check{\mathcal{O}}' \simeq T^* G \times \check{\mathcal{O}}'$ (in the usual dualities/trivializations) such that

$$\widetilde{\mu}_2'\big((g, \Lambda), dQ\big) = \check{\mu}'(dQ) - \pi'\big(\mathrm{Ad}_g^\vee(\Lambda)\big) = -\Lambda'. \tag{83}$$

Then

$$(A_2, \dots, A_{s-1}) \eqqcolon \check{\iota}(dQ) \in \prod_{i=2}^{s-1} \mathcal{O}'_G(\widehat{\Lambda}_i),$$

and we put a modification of $\widehat{\Lambda}_0$ and $\widehat{\Lambda}_1$ in the 'lowest' factors $\mathcal{O}'_G(\widehat{\Lambda}_0)$ and $\mathcal{O}'_G(\widehat{\Lambda}_1)$ of $\mathcal{O}'_G(\widehat{\boldsymbol{\Lambda}})$—respectively. Namely, consider the element

$$S \coloneqq \mathrm{Ad}_g^\vee(\Lambda) - \widehat{\Lambda}_0 - \widehat{\Lambda}_1 - \sum_{i=2}^{s-1} A_i \in \mathfrak{g}^\vee \simeq \mathfrak{g}. \tag{84}$$

Then, using (81) and (83), compute

$$\begin{aligned}
\pi'(S) &= \pi'\big(\mathrm{Ad}_g^\vee(\Lambda)\big) - \widehat{\Lambda}_0 - \widehat{\Lambda}_1 - \sum_{i=2}^{s-1} \pi'(A_i) \\
&= \pi'\big(\mathrm{Ad}_g^\vee(\Lambda)\big) - \widehat{\Lambda}_0 - \widehat{\Lambda}_1 - \check{\nu}\big(\check{\iota}(dQ)\big) \\
&= \pi'\big(\mathrm{Ad}_g^\vee(\Lambda)\big) - \check{\mu}'(dQ) - \Lambda' = 0.
\end{aligned}$$

Therefore, one can (uniquely) decompose (84) as

$$S = S^+ + S^-, \qquad S^\pm \in \mathfrak{u}_{s-1}^\pm.$$

If we now let

$$A_0 \coloneqq \widehat{\Lambda}_0 + S^+, \qquad A_1 \coloneqq \widehat{\Lambda}_1 + S^-, \tag{85}$$

then we finally get an L_s-invariant map on the level-set of $\widetilde{\mu}_2'$ via

$$\widetilde{\Psi} \colon \big((g, \Lambda), dQ\big) \longmapsto \big(\mathrm{Ad}_{g^{-1}}^\vee(A_0), \mathrm{Ad}_{g^{-1}}^\vee(A_1), \dots, \mathrm{Ad}_{g^{-1}}^\vee(A_{s-1})\big) \in \mathcal{O}'_G(\widehat{\boldsymbol{\Lambda}}). \tag{86}$$

9.4.6. Theorem. — *The (well-)induced* G*-equivariant map* (82) *is a symplectic open immersion.*

Proof. — Denote by $\boldsymbol{\omega}_G = \boldsymbol{\omega}_G(\widehat{\boldsymbol{\Lambda}}) \in \Gamma\big(\mathcal{O}'_G(\widehat{\boldsymbol{\Lambda}}), \bigwedge^2 T^* \mathcal{O}'_G(\widehat{\boldsymbol{\Lambda}})\big)$ the complex symplectic form of the orbit product $\mathcal{O}'_G(\widehat{\boldsymbol{\Lambda}})$. Given $dQ \in \breve{\mathcal{O}}'$, choose elements $u^{\pm} \in U^{\pm}_{s-1}$ such that

$$\mathrm{Ad}^{\vee}_{(u^+)}(A_0) = \widehat{\Lambda}_0, \qquad \mathrm{Ad}^{\vee}_{(u^-)}(A_1) = \widehat{\Lambda}_1,$$

in the notation of (85); moreover, choose elements $b_i \in U^-_{s-i} \cdot U^+_{s-i}$ such that

$$\mathrm{Ad}^{\vee}_{b_i}(A_i) = \widehat{\Lambda}_i, \qquad i \in \{2, \dots, s-1\}.$$

Then one has $\Psi^* \boldsymbol{\omega}_G = d\theta \in \Gamma\big(\mathcal{O}', \bigwedge^2 T^* \mathcal{O}'\big)$, where (in 'matrix' notation for the pullback of the Maurer–Cartan form)

$$\begin{aligned} \theta &:= \big(d(u^+ g) \cdot (u^+ g)^{-1} \mid \widehat{\Lambda}_0\big) + \big(d(u^- g) \cdot (u^- g)^{-1} \mid \widehat{\Lambda}_1\big) \\ &\quad + \sum_{i=2}^{s-1} \big(d(b_i g) \cdot (b_i g)^{-1} \mid \widehat{\Lambda}_i\big) \\ &= \sum_{i=0}^{s-1} \big(dg \cdot g^{-1} \mid A_i\big) + \sum_{i=2}^{s-1} \big(db_i \cdot b_i^{-1} \mid \widehat{\Lambda}_i\big), \end{aligned}$$

using the Ad_G-invariant pairing $(\cdot \mid \cdot) \colon \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$, and noting that

$$\big(du^+ \cdot (u^+)^{-1} \mid \widehat{\Lambda}_0\big) = 0 = \big(du^- \cdot (u^-)^{-1} \mid \widehat{\Lambda}_1\big).$$

(By the choice of $u^{\pm}$.)

Now observe that the 2-form $\theta' := \sum_{i=2}^{s-1} \big(db_i \cdot b_i^{-1} \mid \widehat{\Lambda}_i\big)$ is the pullback of the complex symplectic form of the 'partial' orbit product $\prod_{i=2}^{s-1} \mathcal{O}'_G(\widehat{\Lambda}_i) \subseteq \mathcal{O}'_g(\widehat{\boldsymbol{\Lambda}})$, along the symplectic embedding (79); hence, it coincides with the complex symplectic form $\breve{\omega} \in \Gamma\big(\breve{\mathcal{O}}', \bigwedge^2 T^* \breve{\mathcal{O}}'\big)$ on the Birkhoff orbit. Overall

$$\Psi^* \boldsymbol{\omega}_G = \theta' + \sum_{i=0}^{s-1} d\big(dg \cdot g^{-1} \mid A_i\big) = \breve{\omega} + \sum_{i=0}^{s-1} d\big(dg \cdot g^{-1} \mid A_i\big). \tag{87}$$

On the other hand, by construction

$$A_0 + A_1 = \widehat{\Lambda}_0 + \widehat{\Lambda}_1 + S = \mathrm{Ad}^{\vee}_g(\Lambda) - \sum_{i=2}^{s-1} A_i,$$

whence $\sum_{i=0}^{s-1} A_i = \mathrm{Ad}^{\vee}_g(\Lambda)$, and in turn

$$\sum_{i=0}^{s-1} d\big(dg \cdot g^{-1} \mid A_i\big) = d\big(dg \cdot g^{-1} \mid \Lambda\big),$$

which is precisely the complex symplectic form of T^*G. In view of the symplectic decoupling (49), the equality (87) proves that (82) intertwines the symplectic structures.

We conclude by showing that the fibres of the map (86) are precisely the L_s-orbits through $(\widetilde{\mu}'_2)^{-1}(-\Lambda') \subseteq \widetilde{\mathcal{O}}'$, so that Ψ is injective: the fact that it is an open immersion now follows from [114, Thm. 4.2, p. 187] (cf. Cor. 9.4.7 for a description of the image in the generic case). Suppose thus that

$$\widetilde{\Psi}\big((g_1,\Lambda_1),dQ_1\big) = \widetilde{\Psi}\big((g_2,\Lambda_2),dQ_2\big) \in \mathcal{O}'_G\big(\widehat{\mathbf{\Lambda}}\big),$$

for suitable elements $g_, g_2 \in G$, $\Lambda_1,\Lambda_2 \in \mathfrak{g}$, and $dQ_1, dQ_2 \in \check{\mathcal{O}}'$, then comparing the components in $\mathcal{O}'_G\big(\widehat{\Lambda}_0\big) \times \mathcal{O}'_G\big(\widehat{\Lambda}_1\big) \subseteq \mathcal{O}'_G\big(\widehat{\mathbf{\Lambda}}\big)$ shows that the group element $h \coloneqq g_2 \cdot g_1^{-1}$ lies in $L' \cdot U^+_{s-1} \cap L' \cdot U^-_{s-1} = L'$, and that $\mathrm{Ad}_h\big(\widehat{\Lambda}_0\big) = \widehat{\Lambda}_0$. Thus, $h \in (L')^{\Lambda'} = L_s$. □

9.4.7. Corollary. — *Suppose that* $A'_{s-1} \in \mathfrak{t}_{\mathrm{reg}}$, *i.e., that* $L_1 = T$ *(the* generic case*). If* $U^\pm \subseteq B^\pm$ *are the unipotent radicals of the two Borel subgroups intersecting at* T*, then the open image of the unfolding map is given by*

$$\Psi(\mathcal{O}') = \Big\{ \big(\mathrm{Ad}^\vee_{g_0^{-1}}\big(\widehat{\Lambda}_0\big),\ \ldots, \mathrm{Ad}^\vee_{g_{s-1}^{-1}}\big(\widehat{\Lambda}_{s-1}\big)\big) \in \mathcal{O}'_G\big(\widehat{\mathbf{\Lambda}}\big) \ \Big|\ g_i g_0^{-1} \in T \cdot U^- \cdot U^+ \textit{ for } i \in \{1,\ldots,s-1\} \Big\}.$$

Proof omitted. — □

9.4.8. — We now deduce the following corollary, which makes it possible to establish the last sufficient condition for the (faithful) flatness of the G-moment map (42).

***9.4.9. Corollary* (Cf. Lem. 8.3.2).** — *Choose an integer* $s \geqslant 1$*, and let* $\mathcal{O}'_{G,0},\ldots,\mathcal{O}'_{G,s-1} \subseteq \mathfrak{g}^\vee$ *be* regular *semisimple* $\mathrm{Ad}^\vee_G$*-orbits. Then there exist:*

1. *a finite cover*

$$\prod_{j=0}^{s-1} \mathcal{O}'_{G,j} = \bigcup_{i\in I} V'_{G,i},$$

   by (Zariski-)open subspaces $V'_{G,i} \subseteq \prod_{j=0}^{s-1} \mathcal{O}'_{G,j}$;

2. *and* G*-equivariant symplectomorphisms*

$$V'_{G,i} \xrightarrow{\simeq} \mathcal{O}'_i \coloneqq G_s.\mathcal{A}'_i \subseteq \mathfrak{g}_s^\vee, \qquad i \in I,$$

   for suitable UTS principal parts $\mathcal{A}'_i \in \mathfrak{g}'_s$.

Proof. — Choose markings $\Lambda'_j \in \mathcal{O}'_{G,j} \cap \mathfrak{t}^\vee$, for $j \in \{0,\ldots,s-1\}$, as well as distinct numbers $\varepsilon_0,\ldots,\varepsilon_{s-1} \in \mathbb{C}$. Then, given an s-tuple $\boldsymbol{w} = (w_0,\ldots,w_{s-1})$ of elements of the Weyl group W, define a UTS principal part

$$\mathcal{A}'_{\boldsymbol{w}} = \sum_{j=0}^{s-1} A_{\boldsymbol{w},j}\varpi^{-j-1}\,d\varpi, \qquad A_{\boldsymbol{w},j} \in \mathfrak{t},$$

by the condition that

$$\widehat{\Lambda}_j = w_j(\Lambda'_j) \in \mathfrak{t}, \qquad j \in \{0,\ldots,s-1\},$$

where $\widehat{\Lambda}_j \in \mathfrak{t}$ is the residue associated to the ε-unfolding $\mathrm{Unf}(\mathcal{A}'_w)$ of $\mathcal{A}'_w$, as prescribed by (77)—with $\varepsilon \coloneqq (\varepsilon_0, \dots, \varepsilon_{s-1})$. (The coefficients $A_{w,j}$ are then uniquely determined as $\mathbb{C}$-linear combinations of $\widehat{\Lambda}_0, \dots, \widehat{\Lambda}_{s-1}$.)

It follows that the leading coefficient $A'_{w,s-1}$ is a nonzero multiple of a Weyl-translated of Λ'_{s-1}, and so it is *regular* semisimple. By Cor. 9.4.7, the open image $V_w \coloneqq \Psi(\mathcal{O}'_w)$ of the corresponding unfolding map

$$\Psi : \mathcal{O}'_w \hookrightarrow \prod_{j=0}^{s-1} \mathcal{O}'_{G,j}, \qquad \mathcal{O}'_w \coloneqq G_s.\mathcal{A}'_w,$$

consists of tuples of the form

$$\left(\mathrm{Ad}^\vee_{g_0^{-1}}\left(w_0(\Lambda'_0)\right), \dots, \mathrm{Ad}^\vee_{g_{s-1}^{-1}}\left(w_{s-1}(\Lambda'_{s-1})\right)\right), \qquad g_1 g_0^{-1}, \dots, g_{s-1} g_0^{-1} \in T \cdot U^- \cdot U^+.$$

In turn, using (again) that $G = \bigcup_W \dot{w}(T \cdot U^- \cdot U^+)$ (cf. (104)), the open subspaces $V_w \subseteq \prod_{j=0}^{s-1} \mathcal{O}'_{G,j}$ cover the orbit product. □

9.4.10. — The following statement concludes the proof of the main Thm. 1.3.4.

9.4.11. Corollary. — *Choose three integers $s_1, s_2, s_3 \geqslant 1$. For $i \in \{1,2,3\}$, let $\mathcal{O}'_i \coloneqq G_{s_i}.\mathcal{A}'_i \subseteq \mathfrak{g}^\vee_{s_i}$ be a coadjoint orbit through a UTS principal part $\mathcal{A}'_i$ (as in (64)), of pole order s_i. Moreover, suppose that $\nu(\mathcal{O}'_i) \geqslant 1$, in the notation of (63)—upon inserting the orbits at three distinct marked points a_1, a_2, a_3 on the Riemann sphere.*[30] *Then the G-moment map (42)*

$$\mu' : \mathcal{O}'_1 \times \mathcal{O}'_2 \times \mathcal{O}'_3 \longrightarrow \mathfrak{g}^\vee \simeq \mathfrak{g}, \qquad (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3) \longmapsto \mathrm{Res}(\mathcal{A}_1) + \mathrm{Res}(\mathcal{A}_2) + \mathrm{Res}(\mathcal{A}_3),$$

is faithfully flat.

Proof. — The case where $s_1 = s_2 = s_3 = 1$ follows from Prop. 8.3.1 (taking $\mathcal{O}' \coloneqq \mathcal{O}'_1$, $\underline{\mathcal{O}}' \coloneqq \mathcal{O}'_2$, and $M' \coloneqq \mathcal{O}'_3$) + Cor. 9.3.8.

For the general case, given $i \in \{1,2,3\}$ consider the corresponding unfolding map (82), viz.,

$$\Psi_i : \mathcal{O}'_i \hookrightarrow \prod_{j=0}^{s_i-1} \mathcal{O}'_G\left(\widehat{\Lambda}^{(i)}_j\right),$$

in the notation of (77). (So that $\widehat{\Lambda}^{(i)}_0, \dots, \widehat{\Lambda}^{(i)}_{s_i-1} \in \mathfrak{t}$ are the residues of a suitable ε-unfolding of $\mathcal{A}'_i$.) Then the corresponding product map, i.e.,

$$\boldsymbol{\Psi} \coloneqq (\Psi_1, \Psi_2, \Psi_3) : \mathcal{O}'_1 \times \mathcal{O}'_2 \times \mathcal{O}'_3 \hookrightarrow \prod_{i=1}^{3} \prod_{j=0}^{s_i-1} \mathcal{O}'_G\left(\widehat{\Lambda}^{(i)}_j\right),$$

is (still) a G-equivariant symplectic open immersion. Thus, up to a constant, the corresponding G-moment maps are intertwined by $\boldsymbol{\Psi}$; and this constant vanishes by (78). Now flatness follows from the tame/logarithmic case, together with Lem. 8.1.4: by

[30] The notation does *not* imply that we fix/use an ordering of the marked points.

construction $\widehat{\Lambda}_0^{(i)} \in \mathfrak{t}$ is (only) centralized by the maximal torus T, for $i \in \{1,2,3\}$—and so it is regular semisimple.

To establish surjectivity, choose an element $\Lambda \in \mathfrak{g} \simeq \mathfrak{g}^\vee$. Consider the canonical projection $\pi_{\mathfrak{t}} \colon \mathfrak{g} \twoheadrightarrow \mathfrak{t}$, along the direct sum $\mathfrak{u}^- \oplus \mathfrak{u}^+ \subseteq \mathfrak{g}$ of the (opposite) nilradicals $\mathfrak{u}^\pm \subseteq \mathfrak{b}^\pm$. In view of Cor. 9.3.8, there exists a principal part $\mathcal{A}_3 \in \mathcal{O}_3'$ such that

$$\pi_{\mathfrak{t}}(\mathcal{A}_3) = \pi_{\mathfrak{t}}(\Lambda) - A_0^{(1)} - A_0^{(2)} \in \mathfrak{t},$$

writing

$$\mathcal{A}_i' = \sum_{j=0}^{s_i-1} A_j^{(i)} \varpi^{-j-1} d\varpi, \qquad A_j^{(i)} \in \mathfrak{t}, \quad i \in \{1,2,3\}.$$

We shall now construct two principal parts $\mathcal{A}_1 \in \mathcal{O}_1'$ and $\mathcal{A}_2 \in \mathcal{O}_2'$ such that $\mu'(\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3) = \Lambda$, by adding on the 'off-diagonal' components of $\Lambda - \mathrm{Res}(\mathcal{A}_3) \in \mathfrak{g}$.

Let us start from $i = 2$. Denote by $T = L_{s_i}^{(2)} \subseteq \cdots \subseteq \cdots L_1^{(2)} \subseteq G$ the fission sequence determined by $\mathcal{A}_2'$, and introduce the usual nilpotent Lie subalgebras $(\mathfrak{n}_j^\pm)^{(2)} \subseteq \mathfrak{l}_{j-1}^{(2)} := \mathrm{Lie}\big(L_{j-1}^{(2)}\big)$. It follows that the direct sums $(\mathfrak{u}_{s_2}^\pm)^{(2)} := \bigoplus_{j=1}^{s_2} (\mathfrak{n}_j^\pm)^{(2)}$—in the notation of Cor.-Def. 9.4.2—coincide with the above nilradicals $\mathfrak{u}^\pm \subseteq \mathfrak{b}^\pm$. Let then $\pi_{\mathfrak{u}^\pm} \colon \mathfrak{g} \twoheadrightarrow \mathfrak{u}^\pm$ be the canonical projections along $\mathfrak{t} \oplus \mathfrak{u}^\mp \subseteq \mathfrak{g}$, and set in addition $Y^{(2)} := \pi_{\mathfrak{u}^+}\big(\Lambda - \mathrm{Res}(\mathcal{A}_3)\big)$. Decompose the latter (uniquely) as

$$Y^{(2)} = Y_1^{(2)} + \cdots + Y_{s_2}^{(2)}, \qquad Y_j^{(2)} \in (\mathfrak{n}_j^+)^{(2)},$$

and choose $X_j \in (\mathfrak{n}_j^+)^{(2)}$ such that $\big[A_{s-j}^{(2)}, X_j\big] = Y_j^{(2)}$. (Recall that the adjoint map $\mathrm{ad}_{A_{s-j}^{(2)}} \colon \mathfrak{g} \to \mathfrak{g}$ restricts to a linear automorphism of $(\mathfrak{n}_j^\pm)^{(2)}$.) Now, since $A_0^{(2)}$ is a regular semisimple element of the reductive Lie algebra $\mathfrak{l}_{s_2-1}^{(2)}$, there exists an element $u^+ \in (N_{s_2}^+)^{(2)} := \exp\big((\mathfrak{n}_{s_2}^+)^{(2)}\big)$ such that $\mathrm{Ad}_{u^+}\big(A_0^{(2)} + Y_{s_2}^{(2)}\big) = A_0^{(2)}$. Then consider the group element

$$g := u^+ \cdot e^{X_{s_2-1} \varpi_{a_2}} \cdots e^{X_1 \varpi_{a_2}^{s_2-1}} \in G_{s_2},$$

and finally let $\mathcal{A}_2 := \mathrm{Ad}_{g^{-1}}^\vee\big(\mathcal{A}_2'\big) \in \mathcal{O}_2'$. By construction, $X_j \in \mathfrak{l}_{j-1}^{(2)}$ commutes with $A_{s_2-j+1}^{(2)}, \ldots, A_{s_2-1}^{(2)}$, and Ad_{u^+} stabilizes $A_1^{(2)}, \ldots, A_{s_2-1}^{(2)}$; thus:

$$\mathcal{A}_2 = \mathcal{A}_2' + Y_{s_2}^{(2)} - \sum_{j=1}^{s_2-1} \big[X_j, A_{s_2-j}^{(2)}\big] \frac{d\varpi}{\varpi} = \mathcal{A}_2' + Y^{(2)} \frac{d\varpi}{\varpi}.$$

Analogously, one can construct an element $\mathcal{A}_1 \in \mathcal{O}_1'$ of the form

$$\mathcal{A}_1 = \mathcal{A}_1' + Y^{(1)} \frac{d\varpi}{\varpi}, \qquad Y^{(1)} := \pi_{\mathfrak{u}^-}\big(\Lambda - \mathrm{Res}(\mathcal{A}_3)\big).$$

Finally, compute

$$\mathrm{Res}(\mathcal{A}_1) + \mathrm{Res}(\mathcal{A}_2) + \mathrm{Res}(\mathcal{A}_3) = A_0^{(1)} + Y^{(1)} + A_0^{(2)} + Y^{(2)} + \mathrm{Res}(\mathcal{A}_3) = \Lambda. \qquad \square$$

10. About stability

Let again G be *reductive*. In this section we provide a sufficient condition on the choice of truncated-current orbits $\boldsymbol{\mathcal{O}}' = \{ \mathcal{O}'_a \}_{\boldsymbol{a}}$ so that the level-zero set of the moment map (42) only contains *stable* points—after modding out the centre.

(Recall that $\boldsymbol{a} \subseteq \Sigma = \mathbb{C}P^1$ is a finite set of marked points.)

10.1. — Fix again a maximal torus $T \subseteq G$ and a Borel subgroup $B \subseteq G$ containing T. Choose also UTS principal parts $\mathcal{A}'_a$, for $a \in \boldsymbol{a}$, as in (2), with pole orders $s_a \geqslant 1$ and coefficients in $\mathfrak{t} := \mathrm{Lie}(T)$. Let $\mathcal{O}'_{\boldsymbol{a}} \subseteq \prod_{\boldsymbol{a}} \mathfrak{g}^{\vee}_{s_a}$ denote the corresponding orbit product (as in (39)), and cut out the affine subvariety $M = (\boldsymbol{\mu}')^{-1}(0) \subseteq \mathcal{O}'_{\boldsymbol{a}}$ (as in (40)).

By construction, the diagonal action of G factors through the group quotient $\mathbf{P}(G) = G/Z(G)$, as the centre acts trivially. (And analogously for the algebraic group $\mathbf{G}$.) Now recall that a ($\mathbb{C}$-)point of M is defined to be $\mathbf{P}(G)$-*stable* if: (i) its stabilizer in $\mathbf{P}(G)$ is finite; and (ii) its $\mathbf{P}(G)$-orbit—or, equivalently, its G-orbit—is Zariski-closed.

Then in this section we will prove the following stability criterion, which we view as a G-bundle generalization of [98, 99]:

10.1.1. Proposition. — *Choose: (i) a proper parabolic subgroup $P \subsetneq G$ containing B, with Lie algebra $\mathfrak{p} := \mathrm{Lie}(P)$; and (ii) an element w_a of the Weyl group W, for $a \in \boldsymbol{a}$. Moreover, suppose that for all such choices there exists a character $\chi \colon \mathfrak{p} \to \mathbb{C}$ such that*

$$\sum_{\boldsymbol{a}} \chi\big(\widetilde{\Lambda}'_a\big) \neq 0, \qquad \widetilde{\Lambda}'_a := w_a(\Lambda'_a) \in \mathfrak{t}. \tag{88}$$

Then all the points of M are $\mathbf{P}(G)$-stable.

Proof. — Suppose that (88) holds, and—by contradiction—that there exists a point $\boldsymbol{\mathcal{A}} = (\mathcal{A}_a)_{\boldsymbol{a}} \in M$ which is *not* stable. By the Hilbert–Mumford criterion [75], there exists a 1-parameter subgroup $\lambda \colon \mathbb{C}^{\times} \to G$, with $\lambda(\mathbb{C}^{\times}) \nsubseteq Z(G)$, such that $\lim_{t \to 0}\big(\lambda(t).\boldsymbol{\mathcal{A}}\big)$ exists in $\prod_{\boldsymbol{a}} \mathfrak{g}^{\vee}_{s_a}$. This implies that the coefficients of $\mathcal{A}_a$ lie in the Lie algebra $\mathfrak{p} := \mathrm{Lie}(P)$ of the (proper, algebraic) parabolic subgroup $P = P_{\lambda} \subsetneq G$ determined by λ, for $a \in \boldsymbol{a}$. Up to acting by G on $\boldsymbol{\mathcal{A}}$, one can also assume that $B \subseteq P$.

Now, for $a \in \boldsymbol{a}$, choose a group element $g_a \in G$ such that $\mathrm{Ad}^{\vee}_{g_a}(\mathcal{A}_a) \in \mathfrak{g}^{\vee}_{s_a}$ lies in the $\mathrm{Ad}^{\vee}_{\mathrm{Bir}_{s_a}}$-orbit of $\mathcal{A}'_a$. Lem. 10.1.2 (for the subgroup $H := g_a P g_a^{-1} \subseteq G$) implies that the coefficients of the latter lie in the Lie subalgebra $\mathrm{Ad}_{g_a}(\mathfrak{p}) \subseteq \mathfrak{g}$, and that there exists $\mathbf{h}_a \in g_a P_{s_a} g_a^{-1} \subseteq G_{s_a}$ such that: (i) $\mathbf{h}_a \in \mathrm{Bir}_{s_a}$; and (ii) $\mathrm{Ad}^{\vee}_{\mathbf{h}_a g_a}\big(\mathcal{A}_a\big) = \mathcal{A}'_a$. Write now

$$g_a^{-1} \mathbf{h}_a g_a = e^{-\mathbf{X}}, \qquad \mathbf{X} = \sum_{j=1}^{s_a - 1} X_j \varpi_a^j, \quad X_j \in \mathfrak{p}.$$

Then the equality $\mathrm{Ad}^{\vee}_{g_a^{-1}h_a g_a}(\mathcal{A}_a) = \mathrm{Ad}^{\vee}_{g_a^{-1}}(\mathcal{A}'_a) \in \mathfrak{g}^{\vee}_{s_a}$ yields in particular

$$\mathrm{Res}(\mathcal{A}_a) = \mathrm{Ad}_{g_a^{-1}}(\Lambda'_a) + \sum_{j=1}^{s_a-1} \sum_{n>0} \frac{1}{n!} X_{j,n},$$

where

$$X_{j,n} := \sum_{\substack{i_1,\dots,i_n>0 \\ i_1+\dots+i_n=s_a-j}} \mathrm{ad}_{X_{i_1}} \cdots \mathrm{ad}_{X_{i_n}} \left(\mathrm{Ad}_{g_a^{-1}}(A'_{a,s_a-j})\right) \in [\mathfrak{p},\mathfrak{p}].$$

Choose now a character $\chi \in \mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{p},\mathbb{C}) \simeq \big(\mathfrak{p}/[\mathfrak{p},\mathfrak{p}]\big)^{\vee}$. It follows that

$$0 = \chi(0) = \chi\left(\sum_a \mathrm{Res}(\mathcal{A}_a)\right) = \sum_a \chi(\underline{\Lambda}'_a), \qquad \underline{\Lambda}'_a := \mathrm{Ad}_{g_a^{-1}}(\Lambda'_a).$$

But $\underline{\Lambda}'_a$ is a semisimple element of $\mathfrak{p}$, and so it lies in the Ad_P-orbit of an element $\widetilde{\Lambda}'_a \in \mathfrak{t}$. On the other hand, by construction, $\underline{\Lambda}'_a$ lies in the Ad_G-orbit of $\Lambda'_a \in \mathfrak{t}$, so that $\widetilde{\Lambda}'_a = w_a(\Lambda'_a)$ for a suitable group element $w_a \in W$: by the Ad_P-invariance of χ (P being connected), one now has $\sum_a \chi(\widetilde{\Lambda}'_a) = 0$. □

10.1.2. Lemma. — *Choose a UTS principal part $\mathcal{A}' \in \mathfrak{g}'_s$ as in (64). Consider also a connected (Zariski-)closed subgroup $H \subseteq G$, with Lie algebra $\mathfrak{h} := \mathrm{Lie}(H)$, and let $\mathcal{A} = \sum_{j=0}^{s-1} A_j \varpi^{-j-1}\, d\varpi \in \mathfrak{g}^{\vee}_s$ be an element of the $\mathrm{Ad}^{\vee}_{\mathrm{Bir}_s}$-orbit of $\mathcal{A}'$. If $A_0,\dots,A_{s-1} \in \mathfrak{h}$, then:*

1. *the same holds for the coefficients of $\mathcal{A}'$;*
2. *and there exists a group element $h \in H_s := H(\mathscr{O}_s) \subseteq G_s$ such that:*
    (a) $h \in \mathrm{Bir}_s$;
    (b) *and* $\mathrm{Ad}^{\vee}_h(\mathcal{A}) = \mathcal{A}' \in \mathfrak{g}^{\vee}_s$.

Proof postponed to C.13. — □

10.1.3. Remark. — There are finitely many parabolic subgroups containing B, because the *standard* parabolic subalgebras $\mathfrak{p}$ of $(\mathfrak{g},\mathfrak{b})$ correspond to parabolic subsets of roots $\psi \subseteq \Phi$ which contain the subsystem of positive roots determined by $\mathfrak{b}$. In turn, if $\phi := \psi \cap (-\psi) \subseteq \Phi$ is the corresponding Levi subsystem of roots, then the characters of $\mathfrak{p}$ correspond to linear maps $\mathfrak{t} \to \mathbb{C}$ vanishing on the coroots $\alpha^{\vee} \in \mathfrak{t}$, for $\alpha \in \phi$. (E.g., if $\mathfrak{p} = \mathfrak{b}$ then $\phi = \varnothing$, and there are no conditions.) Equivalently, they correspond to linear functionals on the centre of the unique Levi factor $\mathfrak{l} \subseteq \mathfrak{p}$ which contains $\mathfrak{t}$.

Explicitly, let $\Delta \subseteq \Phi$ be the base of simple roots determined by $\mathfrak{b}$. Then the relevant (proper) Levi subsystems are of the form

$$\phi(\Delta') := \mathrm{span}_{\mathbb{C}}(\Delta') \cap \Phi \subsetneq \Phi,$$

for any (proper) subset $\Delta' \subsetneq \Delta$. Now let

$$V' = \bigcup_{\Delta' \subsetneq \Delta} V(\Delta') \subsetneq \mathfrak{t}, \qquad V(\Delta') \coloneqq \bigoplus_{\phi(\Delta')} (\mathbb{C} \cdot \alpha^{\vee}). \tag{89}$$

(It is enough to take direct sums over $\Delta' \subseteq \phi(\Delta')$.) Thus, V' is (also) the union over the *maximal* proper subsets of the base, obtained by leaving out precisely one simple root.[31] Finally, the condition (88) is that none of the (finite) sums of vectors of the set $\prod_a W.(\Lambda'_a) \subseteq \mathfrak{t}^a$ lies in the union (89), invoking the Weyl-orbits through the residues of the chosen principal parts. (The latter can be achieved for sufficiently generic choices.) ♢

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Appendix A. More about connections on discs

A.1. Computation of the normal form. — Here we keep the notation from § 3.1.

A.1.1. — One can construct the group element $h \in G_1[\![\varpi]\!]$, which puts a NUTS connection $\widehat{\mathcal{A}}$ in normal form $\mathcal{A}' = \widehat{\mathcal{A}}.h$, as a product $h = h' \cdot h''$, where: (i) the factor h' turns the starting principal part $\mathcal{A}$ into $\mathcal{A}'$, and moreover the new nonsingular part $\mathcal{B}' \coloneqq \widehat{\mathcal{A}}.h - \mathcal{A}'$ takes coefficients in $\mathfrak{l}_{Q'}$; and (ii) the factor h'' removes $\mathcal{B}'$, and can be chosen inside the subgroup $\exp(\mathfrak{l}_{Q'} \otimes \mathfrak{m}) \subseteq G_1[\![\varpi]\!]$. (Cf., e.g., [7]; one might envision different algorithms to compute τ''_s, but the fact that the action is free implies that h is uniquely determined.)

Both h' and h'' are typically infinite products, constructed recursively. For the latter, at each stage one needs to choose a suitable vector in the image of the adjoint action of the residue, shifted by a positive integer κ and restricted to $\mathfrak{l}_{Q'}$, whence the surjectivity condition in (17). (Cf. Rmk. 3.1.8.)

The continuity of the map $\widehat{\mathcal{A}} \mapsto (\mathcal{A}', h)$ (for the ϖ-adic topology) is essentially due to the fact that the lowest factors $X_i \in \mathfrak{g}$ of $h = \prod_{i>0} e^{X_i \varpi^i}$ only depend on the lowest coefficients of the input $\widehat{\mathcal{A}}$.

[31] E.g., if $\mathfrak{g}$ is semisimple then $\mathfrak{t} \setminus V' \subseteq \mathfrak{t}$ is a hyperplane complement.

A.2. Resonant normal forms. — More generally, choose a (possibly resonant) UTS connection $\widehat{\mathcal{A}}$. Then the group element $\mathbf{h}'$ and the connection $\widehat{\mathcal{A}}.\mathbf{h}' = \mathcal{A}' + \mathcal{B}'$ are still uniquely determined—as in § A.1.1.

A.2.1. — Moreover, if we let $\boldsymbol{\kappa} = \{ \kappa_1, \dots, \kappa_l \}$ be the set of positive integer eigenvalues of $\operatorname{ad}_{\Lambda'}(Q', 0)$, for some integer $l \geqslant 0$, then one can formally gauge away all the terms of the nonsingular part whose ϖ-degree does *not* lie in $\boldsymbol{\kappa}$. The result is a (polynomial) normal form, i.e.,

$$\widehat{\mathcal{A}}' = \mathcal{A}' + \widetilde{\mathcal{B}}', \qquad \widetilde{\mathcal{B}}' = \sum_{i=1}^{l} \widetilde{A}_i \varpi^{\kappa_i} \in \mathfrak{l}_{Q'}[\varpi] \, d\varpi. \tag{90}$$

The leftover coefficients $\widetilde{A}_1, \dots, \widetilde{A}_l$ are also uniquely determined, provided that $\widetilde{A}_i$ is chosen within the vector subspace $\ker\big(\operatorname{ad}_{\Lambda'}(Q', \kappa_i)\big) \subseteq \mathfrak{l}_{Q'}$, since (by hypothesis) $\operatorname{ad}_{\Lambda'}(Q', \kappa)$ acts in semisimple fashion on $\mathfrak{l}_{Q'}$, for any $\kappa \in \mathbb{Z}$—and so its kernel/image are in direct sum. Finally, the $G_1[\![\varpi]\!]$-stabilizer of the starting connection $\widehat{\mathcal{A}}$ has finite complex dimension, equal to

$$\sum_{\kappa > 0} \operatorname{coker}\big(\operatorname{ad}_{\Lambda'}(Q', \kappa)\big) = \sum_{i=1}^{l} \dim_{\mathbb{C}} \operatorname{coker}\big(\operatorname{ad}_{\Lambda'}(Q', \kappa_i)\big).$$

Hence, one can also consider the larger topological subspace $X'_s \subseteq \varpi^{-s}\mathfrak{g}[\![\varpi]\!] \, d\varpi$ of UTS connections, as well as the—continuous—map $\tau'_s \colon X'_s \twoheadrightarrow \mathfrak{g}'_s$ which takes the 'normal' principal part $\mathcal{A}'$ (in the notation of (18)). The latter is still determined by the based gauge orbits, and so τ'_s is also $G_1[\![\varpi]\!]$-invariant: but it does *not* lead to a complete invariant for the (resonant) connections lying over the complement $\mathfrak{g}'_s \setminus \mathfrak{g}''_s$, as the fibres thereon split into uncountably many orbits, parameterized by the normal forms (90); i.e., equivalently, parameterized by the finite-dimensional vector space

$$\bigoplus_{\kappa > 0} \ker\big(\operatorname{ad}_{\Lambda'}(Q', \kappa)\big) = \bigoplus_{i=1}^{l} \ker\big(\operatorname{ad}_{\Lambda'}(Q', \kappa_i)\big). \tag{91}$$

In the end, only the orbit corresponding to the origin of (91) consists of connections whose nonsingular part can be gauged away—in nonunique fashion. We will *not* use any of this in this text, but note that one still has a homeomorphism $X'_s / G_1[\![\varpi]\!] \simeq \mathfrak{g}'_s$—in the ϖ-adic topologies; as usual, this topological quotient forgets the stabilizers.

Appendix B. Automorphisms of TCLAs

Here we describe the automorphism group of a truncated-current Lie algebra

$$\mathfrak{g}_s = \mathfrak{g}(\mathscr{O}_s) \simeq \mathfrak{g}[\![\varpi]\!] / \varpi^s \mathfrak{g}[\![\varpi]\!],$$

for any integer truncation order $s \geqslant 1$.

B.1. Reduction to the simple case. — Assume that $\mathfrak{g}$ is *simple* (but cf. Rmk. B.3.4); if $\mathfrak{g}$ splits into (mutually-commuting) simple ideals $\mathfrak{I}_i \subseteq \mathfrak{g}$, then $\mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{g}_s)$ consists of the direct product of the automorphism groups of the TCLAs $\mathfrak{I}_i(\mathscr{O}_s)$, extended by the permutations of isomorphic simple summands.

B.2. Filtrations. — Hereafter, for an integer $i \in \{0, \dots, s-1\}$ write $\mathfrak{g}_s^{(i)} := \mathfrak{g} \otimes \varpi^i \subseteq \mathfrak{g}$, so that there is a decreasing Lie-algebra filtration of $\mathfrak{g}_s$, with filtered pieces

$$\mathfrak{g}_s^{(\geqslant i)} := \bigoplus_{j=i}^{s-1} \mathfrak{g}_s^{(j)}, \qquad i \in \{0, \dots, s-1\}. \tag{92}$$

(In particular $\mathfrak{g}_s^{(\geqslant 1)} = \mathfrak{bir}_s \subseteq \mathfrak{g}_s$, in the notation of § 2.)

Then note that:

B.2.1. Lemma. — *The Lie-algebra automorphisms of* $\mathfrak{g}_s$ *preserve the filtration* (92).

Proof. — By induction on $i \in \{1, \dots, s-1\}$. The base follows from the fact that $\mathfrak{g}_s^{(\geqslant 1)}$ is the nilradical,[32] and the inductive step from the equality $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. □

B.3. Semidirect factors. — We now define three subgroups of $\mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{g}_s)$, and later prove that they generate it. They consist, respectively, of: (i) automorphisms of $\mathfrak{g}$, acting in ϖ-graded fashion on $\mathfrak{g}_s$; (ii) inner automorphisms of $\mathfrak{g}_s$, coming from the unipotent radical $\mathrm{Bir}_s = \mathrm{Rad}_u(G_s)$ of the corresponding truncated-current Lie group G_s; and (iii) (ring) automorphisms of $\mathscr{O}_s$, acting 'polynomially' on $\mathfrak{g}_s$.

B.3.1. — Precisely, let:

$$\mathrm{Aut}_{\mathrm{diag}}(\mathfrak{g}_s) := \{ \varphi \otimes \mathrm{Id}_{\mathscr{O}_s} \mid \varphi \in \mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{g}) \}, \tag{93}$$

and

$$\mathrm{Aut}_{\mathrm{conj}}(\mathfrak{g}_s) := \{ \mathrm{Ad}_g \mid g \in \mathrm{Bir}_s \}, \tag{94}$$

and

$$\mathrm{Aut}_{\mathrm{poly}}(\mathfrak{g}_s) := \{ \mathrm{Id}_{\mathfrak{g}} \otimes F \mid F \in \mathscr{O}_s^{\times} \}.$$

(Recall from § 5 that $\mathscr{O}_s^{\times} \simeq \mathrm{Aut}_{\mathbb{C}-\mathrm{alg}}(\mathscr{O}_s)$ consists of the morphisms mapping $\varpi \mapsto f\varpi$, where $f \in \mathscr{O}_s$ has nonvanishing 'evaluation' $f(0) \in \mathbb{C} \simeq \mathscr{O}_s/\mathfrak{m}$.)

B.3.2. Remark. — Identify $\mathrm{Bir}_s = \exp(\mathfrak{bir}_s) \subseteq G_s$ with its (nilpotent) Lie algebra. Then (93) acts on (94), and there is a group embedding

$$G_s \hookrightarrow \mathrm{Aut}_{\mathrm{diag}}(\mathfrak{g}_s) \ltimes \mathrm{Aut}_{\mathrm{conj}}(\mathfrak{g}_s).$$ ◊

B.3.3. Theorem. — *There is a nested semidirect group factorization*

$$\mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{g}_s) = \big(\mathrm{Aut}_{\mathrm{diag}}(\mathfrak{g}_s) \ltimes \mathrm{Aut}_{\mathrm{conj}}(\mathfrak{g}_s)\big) \rtimes \mathrm{Aut}_{\mathrm{poly}}(\mathfrak{g}_s). \tag{95}$$

[32] Incidentally, it is also the intersection of the maximal ideals of $\mathfrak{g}_s$.

Proof. — For $i \in \{0,\dots,s-1\}$ let $p_i : \mathfrak{g}_s \twoheadrightarrow \mathfrak{g}_s^{(i)} \simeq \mathfrak{g}$ be the canonical projection. Then, given an automorphism $\Phi \in \mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{g}_s)$, define

$$\Phi_{i,j} \coloneqq p_i \circ \Phi\big|_{\mathfrak{g}_s^{(j)}}, \qquad i,j \in \{0,\dots,s-1\},$$

and regard it as a $\mathbb{C}$-linear map $\mathfrak{g} \to \mathfrak{g}$. (Lem. B.2.1 implies that $\Phi_{i,j} = 0$ if $i < j$.)

Now, by hypothesis:

$$\Phi\big([X,Y]\varpi^{i+j}\big) = \big[\Phi(X\varpi^i), \Phi(Y\varpi^j)\big], \qquad X,Y \in \mathfrak{g},$$

which is only relevant if $i+j < s$. Applying p_k to both sides, with $k \in \{0,\dots,s-1\}$, yields

$$\Phi_{k,i+j}\big([X,Y]\big) = \sum_{k_1+k_2=k} \big[\Phi_{k_1,i}(X), \Phi_{k_2,j}(Y)\big]. \tag{96}$$

Taking $i = j = k \coloneqq 0$ in (96) shows that $\Phi_{0,0}$ is actually a Lie-algebra endomorphism of $\mathfrak{g}$; and it must be an automorphism (by Lem. B.2.1, else Φ would *not* be surjective). To conclude, since—by construction—$\Phi_{0,0} \otimes \mathrm{Id}_{\mathscr{O}_s} \in \mathrm{Aut}_{\mathrm{diag}}(\mathfrak{g}_s)$, it is enough to prove that if $\Phi_{0,0} = \mathrm{Id}_{\mathfrak{g}}$ then Φ lies in the subgroup generated by $\mathrm{Aut}_{\mathrm{conj}}(\mathfrak{g}_s)$ and $\mathrm{Aut}_{\mathrm{poly}}(\mathfrak{g}_s)$.

To this end, fix $k \in \{1,\dots,s-1\}$, and suppose in addition that $\Phi_{0,0} = \mathrm{Id}_{\mathfrak{g}}$ and $\Phi_{l,0} = 0$ for all $l \in \{1,\dots,k-1\}$. Then, taking $i = j \coloneqq 0$ in (96) yields

$$\Phi_{k,0}\big([X,Y]\big) = \big[\Phi_{k,0}(X), Y\big] + \big[X, \Phi_{k,0}(Y)\big],$$

i.e., $\Phi_{k,0}$ is actually a derivation of $\mathfrak{g}$. But by hypothesis all derivations are *inner*, and so there exists $Z \in \mathfrak{g}$ such that $\Phi_{k,0} = \mathrm{ad}_Z \in \mathfrak{Der}(\mathfrak{g})$. Letting $\mathbf{g} \coloneqq e^{Z\varpi^k} \in \mathrm{Bir}_s$, the automorphism $\Phi' \coloneqq \mathrm{Ad}_{\mathbf{g}^{-1}} \circ \Phi$ of $\mathfrak{g}_s$ (still) satisfies $\Phi'_{0,0} = \mathrm{Id}_{\mathfrak{g}}$, as well as the identities $\Phi'_{1,0} = \cdots = \Phi'_{k,0} = 0$.

Since—by construction—$\mathrm{Ad}_{\mathbf{g}} \in \mathrm{Aut}_{\mathrm{conj}}(\mathfrak{g}_s)$, it now suffices to show the following: if $\Phi_{0,0} = \mathrm{Id}_{\mathfrak{g}}$, and $\Phi_{k,0} = 0$ for $k > 0$, then $\Phi \in \mathrm{Aut}_{\mathrm{poly}}(\mathfrak{g}_s) \subseteq \mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{g}_s)$. With these hypotheses, taking $(i,j) \coloneqq (0,1)$ in (96) yields

$$\Phi_{k,1}\big([X,Y]\big) = \big[X, \Phi_{k,1}(Y)\big],$$

i.e., $\Phi_{k,1}$ is actually a $\mathfrak{g}$-module endomorphism of $\mathfrak{g}$. By Schur's Lemma, there exist numbers $\lambda_k \in \mathbb{C}$ such that $\Phi_{k,1} = \lambda_k \cdot \mathrm{Id}_{\mathfrak{g}} \in \mathrm{End}_{\mathbb{C}}(\mathfrak{g})^{\mathfrak{g}} \simeq \mathbb{C}$. Moreover, one necessarily has $\lambda_1 \neq 0$, else Φ would *not* be surjective (since $\Phi_{1,j} = 0$ for $j \neq 1$). Finally, consider the ring automorphism $F \colon \mathscr{O}_s \to \mathscr{O}_s$ given by $\varpi \mapsto \sum_{k=1}^{s-1} \lambda_k \varpi^k$ (whence $f \coloneqq \sum_{k=1}^{s-1} \lambda_k \varpi^{k-1} \in \mathscr{O}_s^{\times}$): then $\Phi = \mathrm{Id}_{\mathfrak{g}} \otimes F$, because Φ is determined by $\Phi_{k,0}$ and $\Phi_{k,1}$, since—in turn—$\mathfrak{g}_s$ is generated as a Lie algebra by the vector subspace $\mathfrak{g}_s^{(0)} \oplus \mathfrak{g}_s^{(1)} \subseteq \mathfrak{g}_s$. □

B.3.4. Remark. — The fact that $\mathfrak{g}$ is simple is only used at the very last step of the proof of Thm. B.3.3—invoking Schur's lemma. Everything else in this sections applies verbatim to a semisimple Lie algebra. ♢

B.3.5. Remark. — Equivalently, the splitting (95) amounts to the direct-product decomposition $\mathrm{Out}(\mathfrak{g}_s) \simeq \mathrm{Out}(\mathfrak{g}) \times \mathscr{O}_s^{\times}$, for the groups of outer automorphisms, noting that the two subgroups $\mathrm{Aut}_{\mathrm{diag}}(\mathfrak{g}_s), \mathrm{Aut}_{\mathrm{poly}}(\mathfrak{g}_s) \subseteq \mathrm{Aut}_{\mathrm{Lie}}(\mathfrak{g}_s)$ commute. ♢

Appendix C. Deferred proofs

C.1. Proof of Lem. 3.2.2. — The first statement follows from: (i) the fact that the Ad_G-action is by Lie-algebra automorphisms of $\mathfrak{g}$ (which preserve semisimple elements, and their mutual commutation relations); (ii) the identification of the $\mathrm{Ad}_G^{\vee}$-action on $\mathfrak{g}_s^{\vee}$ with the diagonal Ad_G-action on the coefficients of principal parts; and (iii) the identities

$$\mathrm{ad}_{X'} = \mathrm{Ad}_g(\mathrm{ad}_X)\,\mathrm{Ad}_g^{-1} \in \mathrm{GL}_{\mathbb{C}}(\mathfrak{g}), \qquad g \in G, \quad X \in \mathfrak{g},$$

where $X' := \mathrm{Ad}_g(X)$.

For the second statement, an intersection of the form $\mathcal{O}' \cap \mathfrak{g}_s'$ or $\mathcal{O}' \cap \mathfrak{g}_s''$ must now be a union of G-orbits: the conclusion follows from the splitting $G_s = G \ltimes \mathrm{Bir}_s$, together with Prop.-Def. 3.1.7.

C.2. Proof of Lem. 4.3.6. — Recall that $F_c \in \widetilde{M}_c^+ \widehat{\otimes} \widetilde{M}_c^-$ is $\mathfrak{g}_s$-invariant. (Where we also introduce the 'opposite' Verma modules and take completions.) Then the evaluation map

$$\mathrm{ev}_{\mathcal{A}'} \colon U(\mathfrak{u}^+) \longrightarrow \mathfrak{g}_s^{\vee}, \qquad X \longmapsto \mathrm{ad}_X^{\vee}(\mathcal{A}'),$$

yields a $U(\mathfrak{u}^+)$-linear composition

$$\widetilde{M}_c^+ \widehat{\otimes} \widetilde{M}_c^- \xrightarrow{\simeq} \widetilde{M}_c^+ \widehat{\otimes} N(\mathfrak{u}^+) \xrightarrow{1 \otimes \mathrm{ev}_{\mathcal{A}'}} M_c^+ \otimes \mathfrak{g}_s^{\vee}.$$

The left-hand side of (35) is just the image of F_c under this arrow: in particular it is a $\mathfrak{u}^+$-invariant element, which is thus uniquely determined by its zeroth-component—inside $M_c^+[0] \otimes \mathfrak{g}_s^{\vee}$, cf. [39]. Hence, it is enough to prove that the right-hand side of (35) is also $\mathfrak{u}^+$-invariant.

Let thus $X_j \in \mathfrak{u}^+$ be an element of the chosen basis, and (using the coproduct) compute

$$\Delta(X_j)(v_c^+ \otimes \mathcal{A}') = v_c^+ \otimes \mathrm{ad}_{X_j}^{\vee}(\mathcal{A}'), \tag{97}$$

and

$$\Delta(X_j)\big((Y_i v_c^+) \otimes \mathrm{ad}_{X_i}^{\vee}(\mathcal{A}')\big) = \big([X_j, Y_i] v_c^+\big) \otimes \mathrm{ad}_{X_i}^{\vee}(\mathcal{A}') + (Y_i v_c^+) \otimes \mathrm{ad}_{X_j}^{\vee}\,\mathrm{ad}_{X_i}^{\vee}(\mathcal{A}'). \tag{98}$$

Now denote by $\pi^- \colon \mathfrak{g}_s \twoheadrightarrow \mathfrak{u}^-$ and $\pi_0 \colon \mathfrak{g}_s \twoheadrightarrow \mathfrak{l}$ the canonical projections, in the vector-space splitting $\mathfrak{g}_s = \mathfrak{u}^+ \oplus \mathfrak{l} \oplus \mathfrak{u}^-$. Then

$$[X_j, Y_i] v_c^+ = \pi_0\big([X_j, Y_i]\big) v_c^+ + \pi^-\big([X_j, Y_i]\big) v_c^+ = c\,\delta_{ji} v_c^+ + \sum_k c_{ji}^k Y_k v_c^+,$$

expanding the second summand as a linear combination with coefficients $c^k_{ji} \in \mathbb{C}$, in the given basis of $\mathfrak{u}^-$, and because by construction

$$\Big\langle \mathcal{A}', \pi_0\big([X_j, Y_i]\big) \Big\rangle = \big\langle \mathcal{A}', [X_j, Y_i] \big\rangle = \delta_{ji}.$$

Hence, combining (97)–(98) (and reindexing), it is enough to prove that

$$\sum_k (Y_k v_c^+) \otimes \Big(\mathrm{ad}^\vee_{X_j} \mathrm{ad}^\vee_{X_k} + \sum_i c^k_{ji} \mathrm{ad}^\vee_{X_i} \Big) \mathcal{A}' = 0 \in M_c^+ \otimes \mathfrak{g}_s^\vee;$$

but we will actually show the following stronger identity:

$$\Big(\mathrm{ad}^\vee_{X_j} \mathrm{ad}^\vee_{X_k} + \sum_i c^k_{ji} \mathrm{ad}^\vee_{X_i} \Big) \mathcal{A}' = 0 \in \mathfrak{g}_s^\vee. \tag{99}$$

To this end, note that the left-hand side of (99) vanishes on $\mathfrak{p}^+ \subseteq \mathfrak{g}_r$, since $\mathcal{A}'$ is the restriction of the character χ^+: so we will regard it as an element of the subspace $(\mathfrak{u}^-)^\vee \simeq (\mathfrak{p}^+)^\perp \subseteq \mathfrak{g}_r^\vee$. Then one computes

$$\Big\langle \mathrm{ad}^\vee_{X_i}(\mathcal{A}'), Y_l \Big\rangle = -\big\langle \mathcal{A}', [X_i, Y_l] \big\rangle = -\delta_{il}, \tag{100}$$

and analogously

$$\begin{aligned} \Big\langle \mathrm{ad}^\vee_{X_j} \mathrm{ad}^\vee_{X_k}(\mathcal{A}'), Y_l \Big\rangle &= \Big\langle \mathcal{A}', \big[X_k, [X_j, Y_l]\big] \Big\rangle = \sum_m c^m_{jl} \big\langle \mathcal{A}', [X_k, Y_m] \big\rangle \\ &= \sum_m c^m_{jl} \delta_{km} = c^k_{jl}. \end{aligned} \tag{101}$$

as (by definition) $\mathrm{ad}^\vee_{\mathfrak{l}}(\mathcal{A}') = (0)$. Finally, the identity (99) follows from linear combinations of (100)–(101).

C.3. Proof of Lem. 5.1.2. — Consider first an irregular type Q', with semisimple commuting coefficients $A'_1, \dots, A'_{s-1} \in \mathfrak{g}$. Compute

$$F^*(dQ') = d\big(Q'\big(F(\varpi)\big)\big) = \sum_{i=1}^{s-1} A'_i \varpi^{-i-1} \cdot \widetilde{f}_i \, d\varpi, \qquad \widetilde{f}_i := f^{-i}(f + \varpi f'), \tag{102}$$

using (36), and commuting F^* past the structural derivation $d\colon \mathfrak{g}(\!(\varpi)\!) \to \mathfrak{g}(\!(\varpi)\!)\, d\varpi$. Now $\widetilde{f}_i(0) \neq 0$ for $i \in \{1, \dots, s-1\}$, and (102) has no residue.[33] Thus, there exists a new tuple $\big(\widetilde{A}'_1, \dots, \widetilde{A}'_{s-1}\big) \in \mathfrak{g}^s$ such that: (i) $\widetilde{A}'_i$ is a $\mathbb{C}$-linear combination of $A'_i, \dots, A'_{s-1}$—only; (ii) the coefficient of A'_i in that linear combination is nonzero; and (iii) one has

$$F^*(dQ') - d\widetilde{Q}' \in \mathfrak{g}[\![\varpi]\!]\, d\varpi, \qquad \widetilde{Q}' := \sum_{i=1}^{s-1} \widetilde{A}'_i \frac{\varpi^{-i}}{-i}.$$

Thus, the elements $\widetilde{A}'_1, \dots, \widetilde{A}'_{s-1}$ (still) lie in a Cartan subalgebra $\mathfrak{g}$.

[33] Beware of a possible confusion: $f^{-i} := (f^{-1})^i$ has *no* pole, as we take the inverse in the ring $\mathbb{C}[\![\varpi]\!]$.

Now choose a semisimple residue $\Lambda' \in \mathfrak{g}$ which commutes with (the coefficients of) Q', and let $\mathcal{A}' := \Lambda' \varpi^{-1} \, d\varpi + dQ' \in \mathfrak{g}'_s$, so that

$$\widetilde{\mathcal{A}}' := F^*(\mathcal{A}') = \Lambda' \varpi^{-1} \, d\varpi + d\widetilde{Q}'.$$

The above (also) implies that Λ' commutes with $\widetilde{Q}'$, whence $\widetilde{\mathcal{A}}' \in \mathfrak{g}'_s$; moreover, it follows that the nested $\mathrm{ad}_{\mathfrak{g}}$-stabilizers are invariant (at each step), and so: (i) the second statement is proven; and (ii) $\mathfrak{l}_{Q'} = \mathfrak{l}_{\widetilde{Q}'}$, preserving nonresonance.

C.4. Proof of Lem. 5.1.4. — The $\mathbb{C}[\![\varpi]\!]^\times$-action on $\mathfrak{g}_s^\vee$ (also) comes by dualizing an action on $\mathfrak{g}_s$, by Lie-algebra automorphisms. (Cf. § B; again, this fixes the Lie subalgebra $\mathfrak{g}$ pointwise and the nilradical $\mathfrak{bir}_s$ setwise.) Then the latter extends—in universal-enveloping fashion—to an action on $U(\mathfrak{g}_s)$, by $\mathbb{C}$-algebra automorphisms, mapping the left-ideals annihilating the cyclic vectors (22) onto each other. (The compatibility with compositions is automatic.)

C.5. Proof of Lem./Def. 7.1.3. — The $\mathrm{Ad}^\vee_{G_s}$-action is trivial on $\mathcal{A}'_3$, and so subtracting the latter yields a Poisson automorphism of $\mathfrak{g}_s^\vee$, which restricts to a G_s-equivariant symplectomorphism mapping $\mathcal{O}'$ onto $G_s.\widetilde{\mathcal{A}}'$.

C.6. Proof of Lem. 8.1.2. — Choose a vector $X \in \mathfrak{g} \simeq \mathfrak{g}^\vee$. Using (the complex-analytic version of) (53), the fibre over X can be described as follows:

(103)

$$\begin{aligned}(\mu')^{-1}(X) &= \Big\{ \big((g,\Lambda), dQ\big) \in T^*G \times \breve{\mathcal{O}}' \;\Big|\; \Lambda = X, \quad \breve{\mu}'(dQ) - \pi'\big(\mathrm{Ad}^\vee_g(\Lambda)\big) = -\Lambda' \Big\} \big/ L_s \\ &\simeq \Big\{ (g, dQ) \in G \times \breve{\mathcal{O}}' \;\Big|\; \breve{\mu}'(dQ) = \pi'\big(\mathrm{Ad}^\vee_g(X)\big) - \Lambda' \Big\} \big/ L_s \\ &= \big(G \times (\breve{\mu}')^{-1}(X')\big) \big/ L_s, \qquad X' := \pi'\big(\mathrm{Ad}^\vee_g(X)\big) - \Lambda'.\end{aligned}$$

Hence, by hypothesis, $(\mu')^{-1}(X) \neq \varnothing$. Moreover, the L_s-action on $G \times \breve{\mathcal{O}}'$ is free, and so the dimension of (103) does *not* depend on X.

C.7. Proof of Lem. 8.1.4. — The map (59) is the composite of the following arrows, which are faithfully flat: (i) the base-change $(\varphi_1, \mathrm{Id}_{M'_2}) \colon M'_1 \times M'_2 \to V \times M'_2$—of φ_1; (ii) the isomorphism $V \times M'_2 \xrightarrow{\simeq} V \times M'_2$, mapping $(X, x_2) \mapsto \big(X + \varphi_2(x_2), x_2\big)$; and (iii) the canonical projection $V \times M'_2 \twoheadrightarrow V$—onto the first factor.

C.8. Proof of Prop. 8.2.3. — Choose two opposite Borel subgroups $B^\pm \subseteq G$ containing the maximal torus $T := L_1$, i.e., such that $B^+ \cap B^- = T$. W.l.o.g., assume that $\mathcal{A}' \in \mathfrak{t}(\!(\varpi)\!)\, d\varpi$—up to nonsingular terms—, where $\mathfrak{t} := \mathrm{Lie}(T) = \mathfrak{l}_{s-1}$, and let $\Phi^\pm \subseteq \Phi$ be the subsystems of positive/negative roots corresponding to $B^\pm$. Denote also by $U^\pm$ the unipotent radical of $B^\pm$. Moreover, for any element w of the Weyl group W, fix (noncanonically) a lift $\dot{w} \in N_G(T)$, and consider the open cell

$$G(w) := \dot{w}(T \cdot U^- \cdot U^+) \subseteq G. \tag{104}$$

(It does *not* depend on the choice of lift.) Then add on the Birkhoff subgroup to get a subspace of the TCLG, in the notation of (12): i.e., consider the product $G_s(w) \coloneqq G(w) \times \mathrm{Bir}_s \subseteq G_s$. Finally, define the (symplectic) subspace

(105)
$$\mathcal{O}'(w) \coloneqq \left\{ \mathrm{Ad}^\vee_{g^{-1}}(\mathcal{A}') \;\middle|\; g \in G_s(w) \right\} = \left\{ \mathrm{Ad}^\vee_{g^{-1}}\big(w^{-1}(\mathcal{A}')\big) \;\middle|\; g \in G_s(T) \right\} \subseteq \mathcal{O}'.$$

By construction, there are open covers

$$G_s = \bigcup_W G_s(w), \qquad \mathcal{O}' = \bigcup_W \mathcal{O}'(w).$$

Now, although G does *not* act on the subspaces (105), the action of $N_G(T) \subseteq G$ permutes them, and $T \subseteq N_G(T)$ preserves each of them. Moreover, this T-action on $\mathcal{O}'(w)$ is Hamiltonian, and a moment map is given by the restriction

$$\mu'_{\mathfrak{t}}(w) = \mu'_{\mathfrak{t}}\big|_{\mathcal{O}'(w)} : \mathcal{O}'(w) \longrightarrow \mathfrak{t}^\vee, \qquad \mu'_{\mathfrak{t}} \coloneqq \pi_1 \circ \mu', \tag{106}$$

invoking the composition of the statement. We will show that (106) is surjective, and omit the proof that is has equidimensional fibres.

To this end, a particular case of (67) yields a T-equivariant symplectomorphism $\mathcal{O}'(w) \simeq (T^*U^+)^s$ (cf. also (56)). Furthermore, since the unipotent group U^+ is isomorphic to its Lie algebra as a T-manifold/variety (via the exponential map), we can—and will—replace its cotangent bundle with

$$T^*\mathfrak{u}^+ \simeq \mathfrak{u}^+ \times \mathfrak{u}^-, \qquad \mathfrak{u}^\pm \coloneqq \mathrm{Lie}(U^\pm) \subseteq \mathfrak{g},$$

in the usual G-invariant duality $(\mathfrak{u}^+)^\vee \simeq \mathfrak{u}^-$. In these identifications, the T-moment map for (105), vanishing at the origin, reads

$$\mu'_{\mathfrak{t}}(w) : (T^*\mathfrak{u}^+)^s \simeq \mathcal{O}(w) \longrightarrow \mathfrak{t}^\vee \simeq \mathfrak{t}, \qquad (X, \mathcal{A}) \longmapsto -\sum_{j=0}^{s-1} \sum_{\Phi^+} \big[X_j^{(\alpha)}, A_j^{(-\alpha)}\big],$$

where $X = (X_0, \ldots, X_{s-1}) \in (\mathfrak{u}^+)^s$ and $\mathcal{A} = (A_0, \ldots, A_{s-1}) \in (\mathfrak{u}^-)^s$, and taking the root-line components of each term (cf. (58)). The conclusion follows as in (the end of) the proof of Prop. 8.1.1.

C.9. Proof of Lem. 8.3.2. — Keep the notation from the proof of Prop. 8.3.1. Moreover, introduce as usual Borel subgroups/subalgebras $\mathfrak{b}^\pm \coloneqq \mathrm{Lie}(B^\pm)$—opposite with respect to T—, their unipotent/nilpotent radicals $\mathfrak{u}^\pm \coloneqq \mathrm{Lie}(U^\pm)$, and the positive/negative systems of roots $\Phi^\pm \subseteq \Phi$.

For the first statement, choose an element $\mathcal{A} \in \mathcal{O}'_w$. Then there exist a group element $g \in G$, and a vector $X = X^+ + X^- \in \mathfrak{u}^+ \oplus \mathfrak{u}^-$, such that

$$\mathrm{Ad}^\vee_g(\mathcal{A}) = (\Lambda'_w + X)\varpi^{-1}\,\mathrm{d}\varpi + \mathrm{d}Q'.^{(34)} \tag{107}$$

(34)The action of a based gauge transformation $e^{Y\varpi}$ ($Y \in \mathfrak{g}$) on $\mathcal{A}_w$ reduces to $\mathcal{A}_w \mapsto \mathcal{A}_w + \mathrm{ad}_{Y\varpi}(\mathcal{A}_w)$, as we truncate modulo ϖ^2 (cf. below). Also, the adjoint image $\mathrm{ad}_{\Lambda'}(\mathfrak{g}) \subseteq \mathfrak{g}$ is precisely the 'off-diagonal' part $\mathfrak{u}^+ \oplus \mathfrak{u}^-$, and the (semisimple) endomorphisms $\mathrm{ad}_{\Lambda'}$ restricts to a linear automorphism thereon.

Given such a triple (g, X^+, X^-), let

$$\iota_w(\mathcal{A}) = \iota_{w,g,X}(\mathcal{A}) := \big(\mathrm{Ad}^\vee_{g^{-1}}(\Lambda' + X^+), \mathrm{Ad}^\vee_{g^{-1}}\big(w^{-1}(\underline{\Lambda}') + X^-\big)\big) \subseteq \mathfrak{g}^\vee \times \mathfrak{g}^\vee. \tag{108}$$

Now note that (108) does *not* depend on the choice of g and (X^+, X^-)—and it is then tautologically G-equivariant. Indeed, if $(\widetilde{g}, \widetilde{X}^+, \widetilde{X}^-)$ satisfies (107), then upon acting by $\mathrm{Ad}^\vee_{g^{-1}}$ and $\mathrm{Ad}^\vee_{\widetilde{g}^{-1}}$ one finds $h := \widetilde{g}g^{-1} \in G^{\Lambda'} = T$, and so the coadjoint actions of g^{-1} and $\widetilde{g}^{-1}$ on $\mathfrak{t}$ coincide; then equating the coefficients of $\varpi^{-1}\mathrm{d}\varpi$ also yields

$$\mathrm{Ad}^\vee_h(X^+ + X^-) = \widetilde{X}^+ + \widetilde{X}^- \in \mathfrak{u}^+ \oplus \mathfrak{u}^-,$$

and the conclusion follows from the uniqueness of the—T-invariant—decomposition.

Moreover, there are *unique* (ad-)nilpotent elements $Y^\pm \in \mathfrak{u}^\pm$ such that

$$\Lambda' + X^+ = e^{\mathrm{ad}^\vee_{Y^+}}(\Lambda') = \mathrm{Ad}^\vee_{u^+}(\Lambda') \in \mathfrak{b}^+, \qquad u^+ := e^{Y^+} \in U^+, \tag{109}$$

and analogously

$$w^{-1}(\underline{\Lambda}') + X^- = e^{\mathrm{ad}^\vee_{Y^-}}\big(w^{-1}(\underline{\Lambda}')\big) = \mathrm{Ad}^\vee_{u^-}\big(w^{-1}(\underline{\Lambda}')\big) \in \mathfrak{b}^-, \qquad u^- := e^{Y^-} \in U^-, \tag{110}$$

whence $\iota_w(\mathcal{A}) \in \mathcal{O}' \times \underline{\mathcal{O}}'$. (The element $Y^+ = \sum_{\Phi^+} Y^{(\alpha)}$ can be constructed by looking at the *finite* sums $\sum_{i>0} \frac{1}{i!}\mathrm{ad}^i_{Y^{(\alpha)}}(\Lambda')$, and then using BCH, where as usual $Y^{(\alpha)} \in \mathfrak{g}_\alpha$ are the root-line components.)

Finally, the crux of the matter is proving that (108) intertwines the symplectic forms

$$\omega'_w \in \Gamma(\mathcal{O}'_w, \wedge^2 T^*\mathcal{O}'_w), \qquad \Omega' := \omega' + \underline{\omega}' \in \Gamma\big(\mathcal{O}' \times \underline{\mathcal{O}}', \wedge^2 T^*(\mathcal{O}' \times \underline{\mathcal{O}}')\big), \tag{111}$$

invoking the KKS structures on $\mathcal{O}'_w$, $\mathcal{O}'$, and $\underline{\mathcal{O}}'$—and summing the latter two. Indeed, the second statement then follows from: (i) the (Zariski-)open cover $G = \bigcup_W G(w)$, in the notation of (104); and (ii) the explicit description

$$\iota_w(\mathcal{O}'_w) = \Big\{\big(\mathrm{Ad}^\vee_{(g')^{-1}}(\Lambda'),\ \mathrm{Ad}^\vee_{(\underline{g}')^{-1}}\big(w^{-1}(\underline{\Lambda}')\big)\big) \ \Big|\ g', \underline{g}' \in G \text{ and } \underline{g}'(g')^{-1} \in T \cdot U^- \cdot U^+\Big\}, \tag{112}$$

showing that the image of (108) is open and nonsingular. Thus, the corresponding bijection is an isomorphism, by Grothendieck's formulation of Zariski's main theorem [114, Thm. 4.2, p. 187] (cf. [133]; and one can explicitly compute an inverse $\iota_w(\mathcal{O}'_w) \xrightarrow{\simeq} \mathcal{O}'_w$, cf. the end of the proof of Prop. 9.3.4.)

Now, to prove that ι_w is symplectic, consider the fibre bundles

$$\pi'_w : G \times (\mathfrak{u}^+ \oplus \mathfrak{u}^-) \twoheadrightarrow \mathcal{O}'_w, \qquad (g, X) \longmapsto \mathrm{Ad}^\vee_{g^{-1}}\big((\Lambda'_w + X)\varpi^{-1}\mathrm{d}\varpi + \mathrm{d}Q'\big),$$

and

$$\underline{\pi}' : G \times G \twoheadrightarrow \mathcal{O}' \times \underline{\mathcal{O}}', \qquad (g', \underline{g}') \longmapsto \big(\mathrm{Ad}^\vee_{(g')^{-1}}(\Lambda'), \mathrm{Ad}^\vee_{(\underline{g}')^{-1}}\big(w^{-1}(\underline{\Lambda}')\big)\big).$$

Then, mapping

$$\widehat{\iota}_w : G \times (\mathfrak{u}^+ \oplus \mathfrak{u}^-) \longrightarrow G \times G, \qquad (g, X) \longmapsto (u^+ g, u^- g), \tag{113}$$

in the notation of (109)–(110), yields a commutative square:

$$\begin{array}{ccc} G \times (\mathfrak{u}^+ \oplus \mathfrak{u}^-) & \xrightarrow{\widehat{\iota}_w} & G \times G \\ {\scriptstyle \pi'_w} \downarrow & & \downarrow {\scriptstyle \underline{\pi}'} \\ \mathcal{O}'_w & \xrightarrow[\iota_w]{} & \mathcal{O}' \times \underline{\mathcal{O}}' \end{array} .$$

We can (and will) conclude by showing that (113) intertwines the pullbacks $(\pi'_w)^* \omega'_w$ and $(\underline{\pi}')^* \Omega'$ of the 2-forms in (111).

Let us start by computing the latter, which only involves the usual 'tame' structures. Using the given Ad_G-invariant pairing $(\cdot \mid \cdot) \in \mathrm{Sym}^2(\mathfrak{g}^\vee)$ yields

$$(\underline{\pi}')^* \Omega' = d\big(dg' \cdot (g')^{-1} \mid \Lambda'\big) + d\big(d\underline{g}' \cdot (\underline{g}')^{-1} \mid w^{-1}(\underline{\Lambda}')\big).$$

(In 'matrix' notation for the pullback of the Maurer–Cartan forms.) Now, about the first summand, observe that:

$$\widehat{\iota}^*_w\big(dg' \cdot (g')^{-1}\big) = \mathrm{Ad}_{u^+}(dg \cdot g^{-1}) + du^+ (u^+)^{-1},$$

whence

$$\begin{aligned} \widehat{\iota}^*_w\big(dg' \cdot (g')^{-1} \mid \Lambda'\big) &= \big(dg \cdot g^{-1} \mid \mathrm{Ad}^\vee_{(u^+)^{-1}}(\Lambda')\big) + \big(du^+ \cdot (u^+)^{-1} \mid \Lambda'\big) \\ &= (dg \cdot g^{-1} \mid \Lambda' + X^+) + (du^+ \cdot (u^+)^{-1} \mid \Lambda'). \end{aligned}$$

But the 1-form $du^+ \cdot (u^+)^{-1}$ takes values in $\mathfrak{u}^+$—while $\Lambda' \in \mathfrak{t}$—, and so the last term vanishes. Analogously, on the Weyl-twisted factor:

$$\widehat{\iota}^*_w\big(d\underline{g}' \cdot (\underline{g}')^{-1} \mid w^{-1}(\underline{\Lambda}')\big) = \big(dg \cdot g^{-1} \mid w^{-1}(\underline{\Lambda}') + X^-\big).$$

Overall, commuting the de Rham differential past the second pullback yields

$$\widehat{\iota}^*_w (\underline{\pi}')^* \Omega' = d\big(dg \cdot g^{-1} \mid \Lambda' + w^{-1}(\underline{\Lambda}') + X\big) = d(dg \cdot g^{-1} \mid \Lambda'_w + X). \tag{114}$$

Now let us compute the 2-form $(\pi'_w)^* \omega'_w$. Given a point $(g, X) \in G \times (\mathfrak{u}^+ \oplus \mathfrak{u}^-)$, define $Y := \mathrm{ad}^{-1}_{\Lambda'}(X) \in \mathfrak{u}^+ \oplus \mathfrak{u}^-$, and let $b := e^{Y\varpi} \in \mathrm{Bir}_2$, so that

$$\mathrm{Ad}^\vee_{bg}(\mathcal{A}) = \mathcal{A}_w, \qquad \mathcal{A} := \pi'_w(g, X) \in \mathcal{O}'_w.$$

Then the 'wild' pullback reads

$$(\pi'_w)^* \omega'_w = d\,\mathrm{Res}\big(d(bg) \cdot (bg)^{-1} \mid \mathcal{A}_w\big). \tag{115}$$

In turn, one has

$$d(bg) \cdot (bg)^{-1} = \mathrm{Ad}_b(dg \cdot g^{-1}) + db \cdot b^{-1},$$

and so (dualizing the Adjoint action) the primitive of the right-hand side of (115) becomes

$$\begin{aligned}\operatorname{Res}\big(\mathrm{d}(bg)\cdot(bg)^{-1} \mid \mathcal{A}_w\big). &= \operatorname{Res}\big(\mathrm{d}g\cdot g^{-1} \mid \operatorname{Ad}^{\vee}_{b^{-1}}(\mathcal{A}_w)\big) + \operatorname{Res}(\mathrm{d}b\cdot b^{-1} \mid \mathcal{A}_w) \\ &= \operatorname{Res}\big(\mathrm{d}g\cdot g^{-1} \mid (\Lambda'_w + X)\varpi^{-1}\mathrm{d}\varpi\big) + \operatorname{Res}(\mathrm{d}b\cdot b^{-1} \mid \mathcal{A}_w),\end{aligned}$$

using the fact that $\operatorname{Ad}^{\vee}_{b^{-1}}$ fixes $\mathrm{d}Q'$, and that $(\mathrm{d}g\cdot g^{-1} \mid \mathrm{d}Q')$ has no residue. Moreover, since by construction $\mathrm{d}b\cdot b^{-1} = Y\mathrm{d}\varpi$, and Y is 'off-diagonal'—while $\mathcal{A}_w \in \varpi^{-2}\mathfrak{t}[\![\varpi]\!]\mathrm{d}\varpi$—, it follows that $\operatorname{Res}(\mathrm{d}b\cdot b^{-1} \mid \mathcal{A}_w) = 0$. Overall, using (114):

$$(\pi'_w)^*\omega'_w = \mathrm{d}(\mathrm{d}g\cdot g^{-1} \mid \Lambda'_w + X) = \widehat{\iota}^*_w(\underline{\pi}')^*\Omega'.$$

C.10. Proof of Lem. 9.3.5. — For the first statement, consider the (truncated) Taylor expansion

$$\mathbf{g} = \exp\left(\sum_{i=1}^{s-1} X_i\varpi^i\right)\cdot g, \qquad X_1,\dots,X_{s-1}\in\mathfrak{g},$$

and analogously take the Ansätze

$$\mathbf{u}^{\pm} = \exp\left(\sum_{i=1}^{s-1} Y_i^{\pm}\varpi^i\right)\cdot u^{\pm}, \qquad Y_1^{\pm},\dots,Y_{s-1}^{\pm}\in\widetilde{\mathfrak{n}}_1^{\pm}, \quad u^{\pm}\in N_1^{\pm},$$

and

$$\mathbf{h} = \exp\left(\sum_{i=1}^{s-1} X'_i\varpi^i\right)\cdot h, \qquad X'_1,\dots,X'_{s-1}\in\mathfrak{l}_1, \quad h\in L_1.$$

Then we assume that $\mathbf{g} = \mathbf{h}\mathbf{u}^-\mathbf{u}^+$, and must solve the equations

$$X_i + R_i = \operatorname{Ad}_{hu^-}(Y_i^+) + \operatorname{Ad}_h(Y_i^-) + X'_i, \qquad i\in\{1,\dots,s-1\}, \tag{116}$$

with initial condition $R_1 = 0$, where the element $R_i\in\mathfrak{g}$ only depends on

$$\left\{ g, u^{\pm}, X_j, Y_j^{\pm}, X'_j \;\middle|\; j\in\{1,\dots,i-1\}\right\}, \qquad i\in\{2,\dots,s-1\}.$$

But there is a vector-space splitting

$$\mathfrak{g} = \operatorname{Ad}_{u^-}(\widetilde{\mathfrak{n}}_1^+)\oplus\widetilde{\mathfrak{n}}_1^-\oplus\mathfrak{l}_1,$$

and the Lie subalgebras $\widetilde{\mathfrak{n}}_1^{\pm}$ are Ad_{L_1}-invariant: thus, the equations (116) uniquely (inductively) determine $Y_i^{\pm}$ and X'_i, for $i\in\{1,\dots,s-1\}$.

The second statement follows from this explicit construction of the factorization.

C.11. Proof of Lem. 9.3.6. — Choose $u^- \in \widetilde{N}_1^-$ and $\mathcal{A}_{s-1} \in \mathcal{O}'_{s-1}$, and let $\underline{\mathcal{A}}_{s-1} := \mathcal{A}_{s-1} + \mathcal{A}'_{\mathrm{top}} \in \mathfrak{g}_s^\vee$. Moreover, noting that $\widetilde{N}_1^-$ is a unipotent group, write $u^- = e^{X}$ for a (unique) $X \in \widetilde{\mathfrak{n}}_1^-$. Then, in the notation of (74):

$$\mathbf{Y}' = \mathrm{Ad}^\vee_{u^-}\big(\underline{\mathcal{A}}_{s-1}\big) - \underline{\mathcal{A}}_{s-1} = \sum_{k>0} \frac{1}{k!}(\mathrm{ad}^\vee_X)^k\big(\underline{\mathcal{A}}_{s-1}\big), \tag{117}$$

using the $\mathrm{ad}^\vee_{\mathfrak{g}_s}$-action. (This sum is finite.) The first statement follows, since $\big[\mathfrak{l}_1, \mathfrak{n}_1^\pm\big] \subseteq \mathfrak{n}_1^\pm$, identifying as usual the dual of $\widetilde{\mathfrak{n}}_1^+$ with a space of (principal parts of) $\mathfrak{n}_1^-$-valued 1-forms on $\operatorname{Spec} \mathbb{C}[\![\varpi]\!]$.

For the second statement, let us construct an (algebraic) inverse of (75). Choose thus $\mathbf{Y}' \in (\widetilde{\mathfrak{n}}_1^+)^\vee$ and $\mathcal{A}_{s-1} \in \mathcal{O}'_{s-1}$: the point is to construct an element $u^- = e^X \in \widetilde{N}_1$ such that (117) holds. To this end, consider a 1-parameter subgroup $\lambda \colon \mathbb{C}^\times \to L_1$—of L_1—such that

$$L_1 = \big\{ g \in G \;\big|\; \lambda(t) \cdot g = g \cdot \lambda(t) \text{ for } t \in \mathbb{C}^\times \big\},$$

and

$$N_1^+ = \left\{ g \in G \;\middle|\; \lim_{t \to 0}\big(\lambda(t) \cdot g \cdot \lambda(t)^{-1}\big) = 1_G \right\}.\text{[35]}$$

Letting $\theta := -\frac{d\lambda}{dt}\big|_{t=1} \in \mathfrak{l}_1$, we obtain a Lie-algebra $\mathbb{Z}$-grading

$$\mathfrak{g} = \bigoplus_{\mathbb{Z}} \mathfrak{g}_k, \qquad \mathfrak{g}_k := \big\{ X \in \mathfrak{g} \;\big|\; \mathrm{ad}_\theta(X) = kX \big\},$$

and it follows that

$$\mathfrak{l}_1 = \mathfrak{g}_0, \qquad \mathfrak{n}_1^\pm = \mathfrak{g}_{\gtrless 0} := \bigoplus_{\pm \mathbb{Z}_{>0}} \mathfrak{g}_k.$$

Now take an integer $N > 0$ which is greater than all the eigenvalues of ad_θ, and define also a $\mathbb{Z}$-grading on the (algebraic) loop algebra $\mathcal{L}\mathfrak{g} := \mathfrak{g}\big(\mathbb{C}[\varpi^{\pm 1}]\big)$ via

$$\deg(\mathfrak{g}_k \otimes \mathbb{C}\varpi^i) := k + iN, \qquad i, k \in \mathbb{Z}.$$

Denote by $(\mathcal{L}\mathfrak{g})_j \subseteq \mathcal{L}\mathfrak{g}$ the graded pieces, and again by $(\mathcal{L}\mathfrak{g})_{\gtrless} 0$ the positive/negative part of the grading. Then (by construction) $\mathfrak{n}_1^-[\varpi] \subseteq (\mathcal{L}\mathfrak{g})_{>0}$: write therefore

$$\underline{\mathcal{A}}_{s-1} = \sum_{j \geqslant 0} \underline{A}_j \varpi^{-s}\, d\varpi, \quad \mathbf{Y}' = \sum_{j>0} Y'_j \varpi^{-s}\, d\varpi, \qquad \underline{A}_j, Y'_j \in (\mathcal{L}\mathfrak{g})_j, \tag{118}$$

by regarding (again) $\mathfrak{g}_s^\vee$ as a quotient of the subspace

$$\varpi^{-s}\mathfrak{g}[\varpi]\, d\varpi = \mathfrak{g}[\varpi](\varpi^{-s}\, d\varpi) \subseteq \mathcal{L}\mathfrak{g}(\varpi^{-s}\, d\varpi),$$

---

[35] Cf. [119, Prop. 2.3]; it also follows as usual that the parabolic subgroup $L_1 \ltimes N_1 \subseteq G$ can be described as the subset of elements such that the limit exists—in G.

consisting of pole-order-bounded 1-forms. Now the expansion (117), together with the graded components (118), yields the following equalities *modulo* $\varpi^s\mathfrak{g}[\varpi]$:

$$Y'_j = \sum_{k>0} \frac{1}{k!} \sum_{\substack{i_1,\dots,i_k>0,\\ i_1+\cdots+i_k\leqslant j}} \mathrm{ad}_{X_{i_1}}\cdots\mathrm{ad}_{X_{i_k}}\big(\underline{A}_{j-i_1-\cdots-i_k}\big), \qquad j>0, \tag{119}$$

writing also

$$X = \sum_{i>0} X_i, \qquad X_i \in (\mathcal{L}\mathfrak{g})_i.$$

To conclude, observe that the right-hand side of (119) has the form $[X_j,\underline{A}_0]+R_j$, with initial condition $R_1=0$, and where $R_j\in(\mathcal{L}\mathfrak{g})_j$ only depends on $\{\underline{A}_i,X_i \mid i<j\}$, for all $j>0$. Moreover, since by construction $\underline{A}_0 = A'_{s-1}$ is the leading coefficient of $\mathcal{A}'$, it follows that $\mathrm{ad}_{\underline{A}_0}\colon \mathfrak{g}\to\mathfrak{g}$ restricts to a linear automorphism of $\mathfrak{g}_k$ for any integer k, whence it yields a linear automorphism of $(\mathcal{L}\mathfrak{g})_j$ for any integer j. Thus, the equations (119) *uniquely* (inductively) determine the elements X_i, modulo $\varpi^s\mathfrak{g}[\varpi]$. Furthermore, one can prove by induction that

$$X_i \in \mathfrak{n}_1^-[\varpi] + \varpi^s\mathfrak{g}[\varpi], \qquad i>0,$$

so that indeed $u^-\in\widetilde{N}_1^-$.

Finally, the L_1-equivariance follows from the above explicit construction of the map (75)—and its inverse.

C.12. Proof of Cor.-Def. 9.4.2. — There are isomorphisms of L'-modules

$$\check{\mathfrak{n}}_i^\pm \simeq (\mathfrak{n}_i^\pm)^{s-i-1}, \qquad i\in\{1,\dots,s-2\},$$

so that $\check{\mathfrak{n}}_1^+\times\cdots\times\check{\mathfrak{n}}_{s-2}^+ \simeq \mathfrak{u}_1^+\times\cdots\times\mathfrak{u}_{s-2}^+$—by gathering the identical factors.

C.13. Proof of Lem. 10.1.2. — If $s=1$ the statement is trivial, as $\mathfrak{bir}_s=\mathfrak{bir}_1=(0)$; choose thus $s>1$.

By hypothesis $A_{s-1}=A'_{s-1}\in\mathfrak{t}$, because the Birkhoff action cannot modify the leading coefficient. Moreover, by [7, Prop. 9.3.2], there exists $h\in H_s\cap \mathrm{Bir}_s\subseteq G_s$ such that the coefficients of $\underline{\mathcal{A}}\coloneqq \mathrm{Ad}^\vee_h(\mathcal{A})$ commute with $A_{s-1}=A'_{s-1}$. Up to replacing $\mathcal{A}$ by $\underline{\mathcal{A}}$, we may then assume that $A_0,\dots,A_{s-1}$ commute with A'_{s-1}.

Then start again under this assumption, and choose an element

$$h=e^X\in\mathrm{Bir}_s, \qquad X=\sum_{j=1}^{s-1}X_j\varpi^{-j}, \quad X_j\in\mathfrak{g}, \tag{120}$$

such that $\mathrm{Ad}^\vee_h(\mathcal{A})=\mathcal{A}'\in\mathfrak{g}_s^\vee$. Comparing coefficients yields the equalities

$$A_{s-i} = \sum_{j=1}^{i}\sum_{n>0}\frac{1}{n!}\cdot \sum_{\substack{i_1,\dots,i_n>0\\ i_1+\cdots+i_n=i-j}} \mathrm{ad}_{X_{i_1}}\cdots\mathrm{ad}_{X_{i_n}}(\mathcal{A}'_{s-j}), \qquad i\in\{1,\dots,s\}. \tag{121}$$

Introduce as usual the Adjoint/adjoint stabilizers $L_1 \subseteq G$ and $\mathfrak{l}_1 = \mathrm{Lie}(L_1)$ of A'_{s-1}. Then we show that $X_k \in \mathfrak{l}_1$, in the notation of (120), for $k \in \{1, \dots, s-1\}$; by induction on k. For the base, if $i := 2$ then (121) reads

$$A_{s-2} = A'_{s-2} + [X_1, A'_{s-1}] \in \mathfrak{g}.$$

But $\mathfrak{g} = \mathfrak{l}_1 \oplus \mathrm{ad}_{A'_{s-1}}(\mathfrak{g})$ (as A'_{s-1} is semisimple), and $A_{s-2} \in \mathfrak{l}_1$ (by assumption), so that $[X_1, A'_{s-1}] = 0$. For the inductive step, suppose that $X_1, \dots, X_{k-1} \in \mathfrak{l}_1$ for some integer $k \in \{1, \dots, s-1\}$. Then, taking $i := k+1$ in (121), the right-hand side lies in the affine subspace $[X_k, A'_{s-1}] + \mathfrak{l}_1 \subseteq \mathfrak{g}$. Since $A_{s-k-1} \in \mathfrak{l}_1$, we find that $[X_k, A'_{s-1}] = 0$.

Now consider the principal part $\mathcal{A}_{s-1} := \mathcal{A} - A_{s-1}\varpi^{-s}\, d\varpi$, of pole order $s-1$, in the notation of (68). The above implies that

$$\mathcal{A}_{s-1} = \mathrm{Ad}^{\vee}_{\mathbf{h}}(\mathcal{A}') - A'_{s-1}\varpi^{-s}\, d\varpi = \mathrm{Ad}^{\vee}_{\mathbf{h}}(\mathcal{A}'_{s-1}), \qquad \mathcal{A}'_{s-1} := \mathcal{A}' - A'_{s-1}\varpi^{-s}\, d\varpi.$$

Thus: (i) $\mathcal{A}_{s-1}$ lies in the $\mathrm{Ad}^{\vee}_{\mathrm{Bir}_{s-1}}$-orbit of $\mathcal{A}'_{s-1}$; and (ii) $\mathcal{A}_{s-1}$ takes coefficients in $\mathfrak{h} \cap \mathfrak{l}_1 \subseteq \mathfrak{l}_1$. Finally, one can argue recursively on the pole order, working within the (connected, closed) subgroup $H \cap L_1 \subseteq L_1$.

References

[1] *Schémas en groupes. III: Structure des schémas en groupes réductifs*, Lecture Notes in Mathematics, vol. 153, Springer-Verlag, Berlin-New York, 1970, Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). 14

[2] A. Y. Alekseev and A. Lachowska, *Invariant ∗-products on coadjoint orbits and the Shapovalov pairing*, Comment. Math. Helv. **80** (2005), no. 4, 795–810. 9, 25

[3] E. Artin, *Theorie der Zöpfe*, Abh. Math. Sem. Univ. Hamburg **4** (1925), no. 1, 47–72. 6

[4] M. F. Atiyah, *Complex analytic connections in fibre bundles*, Trans. Amer. Math. Soc. **85** (1957), 181–207. 3

[5] M. F. Atiyah and R. Bott, *The Yang–Mills equations over Riemann surfaces*, Philos. Trans. Roy. Soc. London Ser. A **308** (1983), no. 1505, 523–615. 7

[6] S. Axelrod, S. Della Pietra, and E. Witten, *Geometric quantization of Chern–Simons gauge theory*, J. Differential Geom. **33** (1991), no. 3, 787–902. 7, 31

[7] D. G. Babbitt and V. S. Varadarajan, *Formal reduction theory of meromorphic differential equations: a group theoretic view*, Pacific J. Math. **109** (1983), no. 1, 1–80. 13, 18, 56, 68

[8] D. G. Babbitt and V. S. Varadarajan, *Local moduli for meromorphic differential equations*, Astérisque (1989), no. 169-170, 217. 18

[9] W. Balser, W. B. Jurkat, and D. A. Lutz, *A general theory of invariants for meromorphic differential equations. I. Formal invariants*, Funkcial. Ekvac. **22** (1979), no. 2, 197–221. 19

[10] P. Baumann, *The q-Weyl group of a q-Schur algebra*, hal-00143359, 25pp. 6

[11] A. Beauville, *Monodromie des systèmes différentiels linéaires à pôles simples sur la sphère de Riemann (d'après A. B olibruch)*, Astérisque (1993), no. 216, Exp. no. 765, 4, 103–119, Séminaire Bourbaki, vol. 1992/93. 7

[12] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, *Infinite conformal symmetry in two-dimensional quantum field theory*, Nuclear Phys. B **241** (1984), no. 2, 333–380. 7

[13] D. Bernard, *On the Wess–Zumino–Witten models on Riemann surfaces*, Nuclear Phys. B **309** (1988), no. 1, 145–174. 6

[14] ______, *On the Wess–Zumino–Witten models on the torus*, Nuclear Phys. B **303** (1988), no. 1, 77–93. 6

[15] I. N. Bernšteĭn, I. M. Gel'fand, and S. I. Gel'fand, *Structure of representations that are generated by vectors of highest weight*, Funckcional. Anal. i Priložen. (1971), no. 1, 1–9. 6

[16] ______, *Differential operators on the base affine space and a study of* $\mathfrak{g}$*-modules*, Lie groups and their representations (Proc. Summer School , Bolyai János Math. Soc., Budapest), 1975, pp. 21–64. 6

[17] M. Bertola, M. Cafasso, and V. Rubtsov, *Noncommutative Painlevé equations and systems of Calogero type*, Comm. Math. Phys. **363** (2018), no. 2, 503–530. 11

[18] O. Biquard and P. P. Boalch, *Wild non-abelian Hodge theory on curves*, Compos. Math. **140** (2004), no. 1, 179–204. 5, 7, 8

[19] I. Biswas, S. Mukhopadhyay, and R. A. Wentworth, *A Hitchin connection on nonabelian theta functions for parabolic* G*-bundles*, J. Reine Angew. Math. **803** (2023), 137–181. 7

[20] P. P. Boalch, *Hyperkähler manifolds and nonabelian Hodge theory of (irregular) curves*, arXiv:1203.6607, 16pp. 8

[21] ______, *Irregular connections and Kac–Moody root systems*, arXiv:0806.1050, 31pp. 8

[22] ______, *Symplectic geometry and isomonodromic deformations*, Ph.D. thesis, University of Oxford, 1999, tel-02357236, 179pp. 8

[23] ______, *Symplectic manifolds and isomonodromic deformations*, Adv. Math. **163** (2001), no. 2, 137–205. 2, 3, 7, 15, 37, 38, 42

[24] ______, G*-bundles, isomonodromy, and quantum Weyl groups*, Int. Math. Res. Not. (2002), no. 22, 1129–1166. 3, 6

[25] ______, *Quasi-Hamiltonian geometry of meromorphic connections*, Duke Math. J. **139** (2007), no. 2, 369–405. 2, 7

[26] ______, *Through the analytic Halo: fission via irregular singularities*, Ann. Inst. Fourier (Grenoble) **59** (2009), no. 7, 2669–2684. 5, 10, 19, 23, 43

[27] ______, *Riemann–Hilbert for tame complex parahoric connections*, Transform. Groups **16** (2011), no. 1, 27–50. 8

[28] ______, *Simply-laced isomonodromy systems*, Publ. Math. Inst. Hautes études Sci. **116** (2012), 1–68. 6, 8, 11

[29] ______, *Geometry and braiding of Stokes data; fission and wild character varieties*, Ann. of Math. (2) **179** (2014), no. 1, 301–365. 5, 10, 20, 22, 23, 40, 43

[30] ______, *Wild character varieties, meromorphic Hitchin systems and Dynkin diagrams*, Proceedings, Nigel Hitchin's 70th Birthday Conference, Oxford University Press, 2017, pp. 433–454. 3

[31] ______, *Topology of the Stokes phenomenon*, Integrability, quantization, and geometry. I. Integrable systems, Proc. Sympos. Pure Math., vol. 103.1, Amer. Math. Soc., Providence, RI, 2021, pp. 55–100. 8

[32] P. P. Boalch, J. Douçot, and G. Rembado, *Twisted local wild mapping class groups: configuration spaces, fission trees and complex braids*, Publ. Res. Inst. Math. Sci. **61** (2025), no. 3, 391–445. 5, 6, 19

[33] P. P. Boalch and D. Yamakawa, *Twisted wild character varieties*, arXiv:1512.08091, 26pp. 5, 19

[34] ______, *Diagrams for nonabelian Hodge spaces on the affine line*, C. R. Math. Acad. Sci. Paris **358** (2020), no. 1, 59–65. 6

[35] T. Bridgeland and V. Toledano Laredo, *Stability conditions and Stokes factors*, Invent. Math. **187** (2012), no. 1, 61–98. 3

[36] ______, *Stokes factors and multilogarithms*, J. Reine Angew. Math. **682** (2013), 89–128. 3

[37] E. Brieskorn, *Die Fundamentalgruppe des Raumes der regulären Orbits einer endlichen komplexen Spiegelungsgruppe*, Invent. Math. **12** (1971), 57–61. 6

[38] W. Bruns and J. Herzog, *Cohen–Macaulay rings*, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993. 36

[39] D. Calaque, G. Felder, G. Rembado, and R. A. Wentworth, *Wild orbits and generalised singularity modules: stratifications and quantisation*, arXiv:2402.03278, 112pp. 2, 5, 6, 9, 21, 23, 24, 25, 27, 28, 30, 39, 43, 60

[40] D. Calaque and F. Naef, *A trace formula for the quantization of coadjoint orbits*, Int. Math. Res. Not. IMRN (2015), no. 21, 11236–11252. 2, 27, 29, 31

[41] A. Cannas da Silva, *Lectures on symplectic geometry*, Lecture Notes in Mathematics, vol. 1764, Springer-Verlag, Berlin, 2001. 11

[42] M. Chaffe, *Category $\mathcal{O}$ for Takiff Lie algebras*, Math. Z. **304** (2023), no. 1, Paper no. 14, 35. 2, 6, 9, 25

[43] M. Chaffe and L. Topley, *Category $\mathcal{O}$ for truncated current Lie algebras*, Canadian Journal of Mathematics **75** (2023), no. 4, 1032–1074. 2, 6, 9, 24, 25

[44] I. S. Cohen, *On the structure and ideal theory of complete local rings*, Trans. Amer. Math. Soc. **59** (1946), 54–106. 13

[45] D. H. Collingwood and W. M. McGovern, *Nilpotent orbits in semisimple Lie algebras*, Van Nostrand Reinhold Mathematics Series, Van Nostrand Reinhold Co., New York, 1993. 42

[46] B. Conrad, *Reductive group schemes*, Autour des schémas en groupes. Vol. I, Panor. Synthèses, vol. 42/43, Soc. Math. France, Paris, 2014, pp. 93–444. 14

[47] K. Corlette, *Flat G-bundles with canonical metrics*, J. Differential Geom. **28** (1988), no. 3, 361–382. 8

[48] W. Crawley-Boevey, *Geometry of the moment map for representations of quivers*, Compositio Math. **126** (2001), no. 3, 257–293. 8, 11

[49] ______, *On matrices in prescribed conjugacy classes with no common invariant subspace and sum zero*, Duke Math. J. **118** (2003), no. 2, 339–352. 8

[50] C. De Concini, V. G. Kac, and C. Procesi, *Quantum coadjoint action*, J. Amer. Math. Soc. **5** (1992), no. 1, 151–189. 6

[51] P. Deligne, *Les immeubles des groupes de tresses généralisés*, Invent. Math. **17** (1972), 273–302. 6

[52] S. K. Donaldson, *Twisted harmonic maps and the self-duality equations*, Proc. London Math. Soc. (3) **55** (1987), no. 1, 127–131. 8

[53] J. Douçot, *Basic representations of genus zero nonabelian Hodge spaces*, arXiv:2409.12864, 36pp. 6

[54] ______, *Diagrams and irregular connections on the Riemann sphere*, arXiv:2107.02516, 42pp. 6

[55] J. Douçot and G. Rembado, *Topology of irregular isomonodromy times on a fixed pointed curve*, Transform. Groups **30** (2025), no. 2, 591–631. 5, 6, 22

[56] J. Douçot, G. Rembado, and M. Tamiozzo, *Local wild mapping class groups and cabled braids*, Annales de l'Institut Fourier (in press). 5, 6, 20, 30

[57] ______, *Moduli spaces of untwisted wild Riemann surfaces*, arXiv:2403.18505, 15pp. 5, 6, 11, 13, 20

[58] J. Douçot, G. Rembado, and D. Yamakawa, *Twisted G-local wild mapping class groups*, arXiv:2504.01701, 52pp. 5, 6, 19

[59] V. G. Drinfel'd, *Quantum groups*, Proceedings of the International Congress of M athematicians , Vol. 1, 2 (Berkeley, Calif., 1986), Amer. Math. Soc., Providence, RI, 1987, pp. 798–820. 6

[60] ______, *Quasi-Hopf algebras*, Algebra i Analiz **1** (1989), no. 6, 114–148. 6

[61] ______, *Quasi-Hopf algebras and Knizhnik–Zamolodchikov equations*, Problems of modern quantum field theory (Alushta, 1989), Res. Rep. Phys., Springer, Berlin, 1989, pp. 1–13. 6

[62] E. B. Dynkin, *Calculation of the coefficients in the Campbell–Hausdorff formula*, Doklady Akad. Nauk SSSR (N.S.) **57** (1947), 323–326. 16

[63] J. K. Egsgaard, *Hitchin connections for genus 0 quantum representations*, Ph.D. thesis, Centre for Quantum Geometry of Moduli Spaces, University of Aarhus, 2015. 7

[64] P. I. Etingof, *Calogero–Moser systems and representation theory*, Zurich Lectures in Advanced Mathematics, European Mathematical Society (EMS), Zürich, 2007. 9, 29, 35

[65] G. Faltings, *A proof for the Verlinde formula*, J. Algebraic Geom. **3** (1994), no. 2, 347–374. 7

[66] G. Felder, Y. Markov, V. Tarasov, and A. N. Varchenko, *Differential equations compatible with KZ equations*, Math. Phys. Anal. Geom. **3** (2000), no. 2, 139–177. 6

[67] G. Felder and G. Rembado, *Singular modules for affine Lie algebras, and applications to irregular WZNW conformal blocks*, Selecta Math. (N.S.) **29** (2023), no. 1, article no. 15 (55 pp.). 2, 6, 9, 24

[68] E. Frenkel and D. Ben-Zvi, *Vertex algebras and algebraic curves*, Mathematical Surveys and Monographs, vol. 88, American Mathematical Society, Providence, RI, 2001. 3

[69] I. Gaiur, M. Mazzocco, and V. Rubtsov, *Isomonodromic deformations: confluence, reduction and quantisation*, Comm. Math. Phys. **400** (2023), no. 2, 1385–1461. 6

[70] F. Geoffriau, *Sur le centre de l'algèbre enveloppante d'une algèbre de Takiff*, Ann. Math. Blaise Pascal **1** (1994), no. 2, 15–31. 2

[71] ______, *Homomorphisme de Harish–Chandra pour les algèbres de Takiff généralisées*, J. Algebra **171** (1995), no. 2, 444–456. 2

[72] W. M. Goldman, *The symplectic nature of fundamental groups of surfaces*, Adv. in Math. **54** (1984), no. 2, 200–225. 7

[73] M. Goresky, R. Kottwitz, and R. MacPherson, *Codimensions of root valuation strata*, Pure Appl. Math. Q. **5** (2009), no. 4, 1253–1310. 5, 30

[74] J. Harnad, *Dual isomonodromic deformations and moment maps to loop algebras*, Comm. Math. Phys. **166** (1994), no. 2, 337–365. 6, 11

[75] D. Hilbert, *Ueber die vollen Invariantensysteme*, Math. Ann. **42** (1893), no. 3, 313–373. 54

[76] K. Hiroe, *Deformation of moduli spaces of meromorphic* G*-connections on* $\mathbb{P}^1$ *via unfolding of irregular singularities*, arXiv:2407.20486, 88pp. 6, 48

[77] K. Hiroe and D. Yamakawa, *Moduli spaces of meromorphic connections and quiver varieties*, Adv. Math. **266** (2014), 120–151. 6, 22

[78] N. J. Hitchin, *Stable bundles and integrable systems*, Duke Math. J. **54** (1987), no. 1, 91–114. 8

[79] ______, *Flat connections and geometric quantization*, Comm. Math. Phys. **131** (1990), no. 2, 347–380. 7, 31

[80] ______, *Frobenius manifolds*, Gauge theory and symplectic geometry (Montreal, PQ, 1995), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 488, Kluwer Acad. Publ., Dordrecht, 1997, pp. 69–112. 6

[81] A. Horn, *Doubly stochastic matrices and the diagonal of a rotation matrix*, Amer. J. Math. **76** (1954), 620–630. 42

[82] J. E. Humphreys, *Representations of semisimple Lie algebras in the BGG category* $\mathcal{O}$, Graduate Studies in Mathematics, vol. 94, American Mathematical Society, Providence, RI, 2008. 6, 24, 26, 27

[83] Fernandez Herrero A. J., *Reduction theory for connections over the formal punctured disc*, arXiv:2003.00008, 45pp. 13, 16

[84] J. C. Jantzen, *Kontravariante Formen auf induzierten Darstellungen halbeinfacher Lie-Algebren*, Math. Ann. **226** (1977), no. 1, 53–65. 27

[85] M. Jimbo, *A* q*-difference analogue of* $U(\mathfrak{g})$ *and the Yang–Baxter equation*, Lett. Math. Phys. **10** (1985), no. 1, 63–69. 6

[86] M. Jimbo and T. Miwa, *Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. II*, Phys. D **2** (1981), no. 3, 407–448. 10

[87] ———, *Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. III*, Phys. D **4** (1981/82), no. 1, 26–46. 10

[88] M. Jimbo, T. Miwa, Y. Môri, and M. Sato, *Density matrix of an impenetrable Bose gas and the fifth Painlevé transcendent*, Phys. D **1** (1980), no. 1, 80–158. 6, 11

[89] M. Jimbo, T. Miwa, and K. Ueno, *Monodromy preserving deformation of linear ordinary differential equations with rational coefficients. I. General theory and* τ*-function*, Phys. D **2** (1981), no. 2, 306–352. 10, 19

[90] C. Kassel, *Quantum groups*, Graduate Texts in Mathematics, vol. 155, Springer-Verlag, New York, 1995. 36

[91] A. A. Kirillov, *Unitary representations of nilpotent Lie groups*, Uspehi Mat. Nauk **17** (1962), no. 4 (106), 57–110. 4

[92] A. A. Kirillov., *Lectures on the orbit method*, Graduate Studies in Mathematics, vol. 64, American Mathematical Society, Providence, RI, 2004. 13

[93] A. N. Kirillov and N. Y. Reshetikhin, q*-Weyl group and a multiplicative formula for universal* R*-matrices*, Comm. Math. Phys. **134** (1990), no. 2, 421–431. 6

[94] V. G. Knizhnik and A. B. Zamolodchikov, *Current algebra and Wess–Zumino model in two dimensions*, Nuclear Phys. B **247** (1984), no. 1, 83–103. 6

[95] T. Kohno, *Monodromy representations of braid groups and Yang–Baxter equations*, Ann. Inst. Fourier (Grenoble) **37** (1987), no. 4, 139–160. 6

[96] B. Kostant, *Quantization and unitary representations. I. Prequantization*, (1970), 87–208. Lecture Notes in Mathematics, vol. 170. 4

[97] ———, *On convexity, the Weyl group and the Iwasawa decomposition*, Ann. Sci. École Norm. Sup. (4) **6** (1973), 413–455. 42

[98] V. P. Kostov, *On the Deligne–Simpson problem*, C. R. Acad. Sci. Paris Sér. I Math. **329** (1999), no. 8, 657–662. 8, 54

[99] ———, *On the Deligne–Simpson problem and its weak version*, Bull. Sci. Math. **128** (2004), no. 2, 105–125. 8, 54

[100] Y. Laszlo, *Hitchin's and WZW connections are the same*, J. Differential Geom. **49** (1998), no. 3, 547–576. 7

[101] I. Losev, *Quantization commutes with reduction*, 2021, 'Algebra and Geometry lecture series', Lect. 1. 12, 36

[102] G. Lusztig, *Quantum groups at roots of* 1, Geom. Dedicata **35** (1990), no. 1-3, 89–113. 6

[103] B. Malgrange, *Sur les déformations isomonodromiques. II. Singularités irrégulières*, Mathematics and physics (Paris, 1979/1982), Progr. Math., vol. 37, Birkhäuser Boston, Boston, MA, 1983, pp. 427–438. 19

[104] J. Martinet and J.-P. Ramis, *Elementary acceleration and multisummability. I*, Ann. Inst. H. Poincaré Phys. Théor. **54** (1991), no. 4, 331–401. 19

[105] J. J. Millson and V. Toledano Laredo, *Casimir operators and monodromy representations of generalised braid groups*, Transform. Groups **10** (2005), no. 2, 217–254. 6

[106] D. Mumford, J. Fogarty, and F. Kirwan, *Geometric invariant theory*, third ed., Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)], vol. 34, Springer-Verlag, Berlin, 1994. 22

[107] M. Nagata, *Invariants of a group in an affine ring*, J. Math. Kyoto Univ. **3** (1963/64), 369–377. 35

[108] H. Nagoya and J. Sun, *Confluent primary fields in the conformal field theory*, J. Phys. A **43** (2010), no. 46, 465203, 13. 6

[109] ———, *Confluent KZ equations for $\mathfrak{sl}_N$ with Poincaré rank 2 at infinity*, J. Phys. A **44** (2011), no. 28, 285205, 17. 6

[110] M. S. Narasimhan, *Geometry of moduli spaces of vector bundles*, Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2, Gauthier-Villars, Paris, 1971, pp. 199–201. 7

[111] S. P. Novikov, *The Hamiltonian formalism and a multivalued analogue of Morse theory*, Uspekhi Mat. Nauk **37** (1982), no. 5(227), 3–49, 248. 5

[112] K. J. Nowak, *Flat morphisms between regular varieties*, Univ. Iagel. Acta Math. (1997), no. 35, 243–246. 37

[113] K. Okamoto, *The Painlevé equations and the Dynkin diagrams*, Painlevé transcendents (Sainte-Adèle, PQ, 1990), NATO Adv. Sci. Inst. Ser. B Phys., vol. 278, Plenum, New York, 1992, pp. 299–313. 6

[114] A. N. Parshin and I. R. Shafarevich (eds.), *Algebraic geometry. IV*, Encyclopaedia of Mathematical Sciences, vol. 55, Springer-Verlag, Berlin, 1994. 51, 64

[115] C. Pauly, J. Martens, M. Bolognesi, and T. Baier, *The Hitchin connection in arbitrary characteristic*, J. Inst. Math. Jussieu **22** (2023), no. 1, 449–492. 7

[116] V. L. Popov and E. B. Vinberg, *Invariant theory*, pp. 123–278, Springer Berlin Heidelberg, Berlin, Heidelberg, 1994. 22

[117] G. Rembado, *Simply-laced quantum connections generalising KZ*, Comm. Math. Phys. **368** (2019), no. 1, 1–54. 6

[118] ———, *Symmetries of the simply-laced quantum connections and quantisation of quiver varieties*, SIGMA Symmetry Integrability Geom. Methods Appl. **16** (2020), article no. 103 (44 pp.). 6, 9

[119] R. W. Richardson, *Conjugacy classes of $\mathfrak{n}$-tuples in Lie algebras and algebraic groups*, Duke Math. J. **57** (1988), no. 1, 1–35. 67

[120] C. Sabbah, *Harmonic metrics and connections with irregular singularities*, Ann. Inst. Fourier (Grenoble) **49** (1999), no. 4, 1265–1291. 7

[121] T. Schedler, *Deformations of algebras in noncommutative geometry*, arXiv:1212.0914, 112pp. 29

[122] L. Schlesinger, *über die Lösungen gewisser linearer D ifferentialgleichungen als Funktionen der singulären Punkte*, J. Reine Angew. Math. **129** (1905), 287–294. 6, 10

[123] ———, *Über eine Klasse von Differentialsystemen beliebiger Ordnung mit festen kritischen Punkten*, J. Reine Angew. Math. **141** (1912), 96–145. 6, 10

[124] I. Schur, *Über eine Klasse von Mittelbildungen mit Anwendungen auf die Determinantentheorie*, Sitzungsberichte der Berliner Mathematischen Gesellschaft **22** (1923), 9–20. 42

[125] C. T. Simpson, *Harmonic bundles on noncompact curves*, J. Amer. Math. Soc. **3** (1990), no. 3, 713–770. 8
[126] ______, *Products of matrices*, Differential geometry, global analysis, and topology (Halifax, NS, 1990), CMS Conf. Proc., vol. 12, Amer. Math. Soc., Providence, RI, 1991, pp. 157–185. 8
[127] ______, *Higgs bundles and local systems*, Inst. Hautes études Sci. Publ. Math. (1992), no. 75, 5–95. 8
[128] ______, *Moduli of representations of the fundamental group of a smooth projective variety. I*, Inst. Hautes études Sci. Publ. Math. (1994), no. 79, 47–129. 8
[129] ______, *Moduli of representations of the fundamental group of a smooth projective variety. II*, Inst. Hautes études Sci. Publ. Math. (1994), no. 80, 5–79 (1995). 8
[130] J.-M. Souriau, *Structure des systèmes dynamiques*, Maîtrises de mathématiques, Dunod, Paris, 1970. 4
[131] Y. S. Soĭbel'man, *Algebra of functions on a compact quantum group and its representations*, Algebra i Analiz **2** (1990), no. 1, 190–212. 6
[132] The Stacks project authors, *The Stacks project (Koszul regular sequences)*, Def. 15.30.1, 2025. 36
[133] J. Starr, *Answer to mathoverflow question 264204*, 2017. 64
[134] S. J. Takiff, *Rings of invariant polynomials for a class of Lie algebras*, Trans. Amer. Math. Soc. **160** (1971), 249–262. 2
[135] C. J. Thompson, *Inequalities and partial orders on matrix spaces*, Indiana Univ. Math. J. **21** (1971/72), 469–480. 42
[136] V. Toledano Laredo, *A Kohno–Drinfeld theorem for quantum Weyl groups*, Duke Math. J. **112** (2002), no. 3, 421–451. 6
[137] A. Tsuchiya, K. Ueno, and Y. Yamada, *Conformal field theory on universal family of stable curves with gauge symmetries*, Integrable systems in quantum field theory and statistical mechanics, Adv. Stud. Pure Math., vol. 19, Academic Press, Boston, MA, 1989, pp. 459–566. 6
[138] R. Vakil, *The rising sea (Foundations of Algebraic Geometry)*, draft, 2024. 37
[139] N. N. Šapovalov, *A certain bilinear form on the universal enveloping algebra of a complex semisimple Lie algebra*, Funkcional. Anal. i Priložen. **6** (1972), no. 4, 65–70. 25
[140] J. Wess and B. Zumino, *Consequences of anomalous Ward identities*, Phys. Lett. B **37B** (1971), 95–97. 5
[141] B. J. Wilson, *Highest-weight theory for truncated current Lie algebras*, J. Algebra **336** (2011), 1–27. 9, 24
[142] E. Witten, *Global aspects of current algebra*, Nuclear Phys. B **223** (1983), no. 2, 422–432. 5
[143] ______, *Nonabelian bosonization in two dimensions*, Comm. Math. Phys. **92** (1984), no. 4, 455–472. 5
[144] ______, *Gauge theory and wild ramification*, Anal. Appl. (Singap.) **6** (2008), no. 4, 429–501. 3
[145] E. Wofsey, *Answer to Mathematics Stack Exchange question 4508475*, 2022. 37
[146] D. Yamakawa, *Middle convolution and Harnad duality*, Math. Ann. **349** (2011), no. 1, 215–262. 6
[147] ______, *Fourier–Laplace transform and isomonodromic deformations*, Funkcial. Ekvac. **59** (2016), no. 3, 315–349. 6
[148] ______, *Fundamental two-forms for isomonodromic deformations*, J. Integrable Syst. **4** (2019), no. 1, 1–35. 2, 3, 5, 7, 11, 20, 22, 30, 37, 38, 40, 44, 45

[149] ______, *Quantization of simply-laced isomonodromy systems by the quantum spectral curve method*, SUT J. Math. **58** (2022), no. 1, 23–50. 6

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