A conjecture on the lower bound of the length-scale critical exponent ν at continuous phase transitions

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A fundamental issue in the renormalization-group (RG) theory of critical phenomena concerns the allowed values of critical exponents that are consistent with the continuous nature of a phase transition. Here we conjecture a lower bound for the length-scale exponent ν that should hold at continuous transitions associated with d-dimensional Landau-Ginzburg-Wilson (LGW) Φ^4 theories with a multicomponent scalar field $\boldsymbol{\varphi}$ (including some extensions with fermionic and gauge fields). If $\Delta_\varphi = (d-2+\eta)/2$ is the dimension of the order parameter—$\boldsymbol{\varphi}$ in LGW models—and $\Delta_\varepsilon = d-1/\nu$ is the RG dimension of the energy operator ε, which can be identified with $[\boldsymbol{\varphi}\cdot\boldsymbol{\varphi}]$ (the squared field with a proper subtraction of the mixing with the identity), we conjecture the inequality $\Delta_\varepsilon - 2\Delta_\varphi \geq 0$, which implies $1/\nu \leq 2-\eta$ and $\gamma = (2-\eta)\nu \geq 1$. These inequalities are supported by general arguments for lattice models, exact relations for two-dimensional minimal conformal field theories, and are consistent with all known (numerical, perturbative, and exact) results for LGW Φ^4 theories. In particular, since unitarity requires $\eta \geq 0$, the above inequality implies $\nu \geq 1/2$ for unitary theories. This lower bound is more restrictive than the bound $\nu > 1/d$, which is derived by noting that $\nu = 1/d$ is the expected behavior at first-order transitions.

A fundamental issue in the renormalization-group (RG) theory of critical phenomena [1–7] concerns the allowed range of values of the critical exponents defined at continuous phase transitions. In this paper we conjecture an exponent inequality that should hold at standard continuous transitions, which admit effective descriptions in terms of d-dimensional Landau-Ginzburg-Wilson (LGW) Φ^4 theories [6–11], with a multicomponent order-parameter scalar field $\boldsymbol{\varphi}$.

We consider critical behaviors characterized by the presence of a single relevant symmetry-preserving perturbation of the critical point, which may be identified with the reduced temperature $t = (T-T_c)/T_c$ in many cases. In LGW theories this corresponds to Hamiltonians with a single quadratic term $r\,\boldsymbol{\varphi}\cdot\boldsymbol{\varphi}$, where r controls the approach to criticality in the absence of external symmetry-breaking sources. The length scale ξ of the critical modes diverges as $\xi \sim |\delta r|^{-\nu}$, where $\delta r = r - r_c$ is the deviation from the critical point and ν is a critical exponent, which only depends on the universality class of the transition. If the transition is continuous, ν should be larger than $1/d$. Indeed, a RG analysis shows that $\nu = 1/d$ characterizes the singular behavior at first-order transitions [12–24]. The lower bound $\nu > 1/d$ has often been used as a criterion to distinguish continuous from first-order transitions in numerical analyses. However, a systematic examination (as far as we know, see below) of the results for the continuous transitions shows that there are no analytic, (sufficiently robust) numerical or experimental estimates of ν with $\nu < 1/2$, suggesting that ν always satisfies $\nu \gtrsim 1/2$, which is stronger than $\nu > 1/d$ for $d > 2$. Here, we address this issue, and conjecture a lower bound for ν, see Eq. (3) below, which explains the apparent absence of continuous transitions with $\nu \lesssim 1/2$.

We consider the most general LGW theory [6–11] for an N-component real field φ_i which has a global symmetry group that forbids the presence of quadratic terms proportional to $\varphi_i\varphi_j$ ($i \neq j$) and of cubic terms in the Hamiltonian; e.g., this is the case if the model is invariant under the parity $\mathbb{Z}_2$ transformations $\varphi_i \to -\varphi_i$. Thus, the Hamiltonian density takes the form [6, 8]

$$\begin{aligned} \mathcal{H} &= \sum_i (\partial_\mu \varphi_i)^2 + P(\boldsymbol{\varphi}), \qquad (1)\\ P(\boldsymbol{\varphi}) &= \sum_i r_i \varphi_i^2 + \sum_{ijkl} u_{ijkl}\, \varphi_i\varphi_j\varphi_k\varphi_l, \end{aligned}$$

where u_{ijkl} is symmetric under any exchange of the indices. The number of independent parameters r_i and u_{ijkl} depends on the symmetry group of the model. Standard continuous transitions are generally described by LGW theories whose global symmetry permits only one invariant quadratic term, $\boldsymbol{\varphi}\cdot\boldsymbol{\varphi} = \sum_i \varphi_i^2$ (LGW theories with multiple quadratic terms describe multicritical behaviors [7, 25–27] which are not considered here). This implies $r_i = r$ for all i, and the trace condition $\sum_i u_{iikl} = U\delta_{kl}$ for the quartic couplings u_{ijkl} [6, 8]. Criticality is approached by tuning the single parameter r. All field components become critical simultaneously and the two-point function in the disordered phase is diagonal: $\langle \varphi_i(\boldsymbol{x})\varphi_j(\boldsymbol{y})\rangle = \delta_{ij} G(\boldsymbol{x}-\boldsymbol{y})$. We will also consider extended LGW Φ^4 theories, including fermionic and gauge fields, such as the Gross-Neveu-Yukawa (GNY) [6] and Abelian Higgs (AH) [6, 46] field theories.

The universality classes of the critical behaviors described by LGW theories are characterized by universal critical exponents [1–7], which can be derived from the RG dimensions Δ_φ and Δ_ε of the field operator $\boldsymbol{\varphi}$ and the *energy* operator $\varepsilon \equiv [\boldsymbol{\varphi}^2]$ corresponding to the square of the field operator after subtracting the mixing with the identity. We conjecture that they satisfy the inequality

$$\Sigma \equiv \Delta_\varepsilon - 2\Delta_\varphi \geq 0. \qquad (2)$$

The equality $\Sigma = 0$ is satisfied by the Gaussian model, where Δ_ε and Δ_φ are given by their naive dimensions. By using the relations $\Delta_\varphi = (d-2+\eta)/2$ (η quantifies the deviation from the Gaussian dimension) and $\Delta_\varepsilon = d-y_r$, where $y_r = 1/\nu$ is the RG dimension of δr [1–7], we can rewrite Eq. (2) as

$$\Sigma = 2 - \eta - y_r \geq 0, \quad \text{thus} \quad \nu \geq (2-\eta)^{-1}, \tag{3}$$

which provides a lower bound for ν. If the theory is unitary (corresponding to reflection positivity in Euclidean space), then $\eta \geq 0$ [5, 28, 29]. Therefore the inequality (3) implies $\nu \geq 1/2$ for unitary LGW theories, which is stronger than $\nu > 1/d$. Moreover, since the critical exponent γ, associated with the divergence $\chi \sim |\delta r|^{-\gamma}$ of the susceptibility $\chi = \int d^d x\, G(\boldsymbol{x})$, is related to ν and η by the scaling relation $\gamma = d - 2\Delta_\varphi = (2-\eta)\nu$ [6, 7], we may equivalently write the inequality (3) as

$$\gamma = (2-\eta)\,\nu \geq 1. \tag{4}$$

We also recall the upper bound $\gamma < d\nu$.

The inequality (2) has a notable implication on the operator-product expansion (OPE) of the operator product $\boldsymbol{\varphi}(\boldsymbol{x}_1)\cdot\boldsymbol{\varphi}(\boldsymbol{x}_2)$ when $|\boldsymbol{x}_{12}| \equiv |\boldsymbol{x}_1 - \boldsymbol{x}_2| \to 0$. Indeed, assuming translation invariance, its OPE reads [5, 6]

$$\boldsymbol{\varphi}(\boldsymbol{x}_1)\cdot\boldsymbol{\varphi}(\boldsymbol{x}_2) \approx F_I(|\boldsymbol{x}_{12}|)\; I + F_\varepsilon(|\boldsymbol{x}_{12}|)\; [\boldsymbol{\varphi}^2](\boldsymbol{X}), \tag{5}$$

where I is the identity, $\boldsymbol{x}_{ij} \equiv \boldsymbol{x}_i - \boldsymbol{x}_j$, $F_I(|\boldsymbol{x}_{12}|) = \langle \boldsymbol{\varphi}(\boldsymbol{x}_1)\cdot \boldsymbol{\varphi}(\boldsymbol{x}_2)\rangle$, $\boldsymbol{X} \equiv (\boldsymbol{x}_1+\boldsymbol{x}_2)/2$, and $F_\varepsilon(|\boldsymbol{x}_{12}|) \propto |\boldsymbol{x}_{12}|^\Sigma$, which is nondiverging if the inequality (2) holds. Moreover, conformal symmetry implies [5, 28, 30, 31]

$$G_{\varphi\varphi\varphi^2} = \langle \boldsymbol{\varphi}(\boldsymbol{x}_1)\cdot\boldsymbol{\varphi}(\boldsymbol{x}_2)\; \varepsilon(\boldsymbol{x}_3)\rangle \propto \frac{|\boldsymbol{x}_{12}|^\Sigma}{|\boldsymbol{x}_{23}|^{\Delta_\varepsilon} |\boldsymbol{x}_{13}|^{\Delta_\varepsilon}} \tag{6}$$

at criticality. Thus, $\Sigma \geq 0$ implies that $G_{\varphi\varphi\varphi^2}$ does not diverge when $\boldsymbol{x}_{12} \to 0$ keeping $\boldsymbol{x}_{13}$ and $\boldsymbol{x}_{23}$ finite.

As we shall see, the conjecture (2), and its implications (3) and (4) on ν and γ, which we call *ν conjecture*, are supported by general arguments for lattice systems, and are consistent with all known results for the critical exponents obtained numerically, by using the $d = 2, 3, 4-\epsilon$ expansions, the large-N expansion and conformal-field theory (CFT) in two dimensions. The ν conjecture should also apply to d-dimensional quantum continuous transitions that can be related to $(d{+}1)$-dimensional LGW field theories, by the quantum-to-classical mapping [32–34].

In the following we summarize the arguments and evidences supporting the ν conjecture. We anticipate that no violations are found. More details are reported in the appendix.

The bound $\gamma \geq 1$ in d-dimensional lattice systems. A proof of the inequality (4) has been reported for Ising models [35–37] in any dimensions, and later extended [38] to the Ashkin-Teller model [39]. The proof was based on the rigorous inequality $\partial_\beta \chi \leq q\chi^2$ for the susceptibility χ (q is the lattice coordination number). We conjecture that an analogous inequality holds in generic ferromagnetic systems whose critical behavior is described by a generic LGW Φ^4 theory. We consider a partition function for an N-component real field $\boldsymbol{\phi}_{\boldsymbol{x}}$ in a d-dimensional lattice of volume V, which we assume cubic for simplicity,

$$Z = \int \prod_{\boldsymbol{x}} d\mu(\boldsymbol{\phi}_{\boldsymbol{x}})\; e^{-\beta H}, \quad H = -\sum_{\langle \boldsymbol{x}\boldsymbol{y}\rangle} \boldsymbol{\phi}_{\boldsymbol{x}}\cdot\boldsymbol{\phi}_{\boldsymbol{y}}, \tag{7}$$

where the sum is over nearest-neighbor sites, $d\mu(\boldsymbol{\phi}) = e^{-P(\boldsymbol{\phi})}d\boldsymbol{\phi}$, and $P(\boldsymbol{\phi})$ is a generic function of the field. We assume the symmetry to be large enough so that $\langle \phi_{i,\boldsymbol{x}}\phi_{j,\boldsymbol{y}}\rangle \propto \delta_{ij}$. Defining $\chi = V^{-1}\sum_{\boldsymbol{x},\boldsymbol{y}}\langle \boldsymbol{\phi}_{\boldsymbol{x}}\cdot\boldsymbol{\phi}_{\boldsymbol{y}}\rangle$, if there is a positive constant c such that [35]

$$W(\beta, c) \equiv \lim_{V\to\infty} \left(\partial_\beta \chi - c\,\chi^2\right) \leq 0, \tag{8}$$

then $\chi \sim |\beta - \beta_c|^{-\gamma}$ with $\gamma \geq 1$ at the critical point. We argue that the inequality (8) is also satisfied in generic lattice systems for some constant c that be determined below. In the low-temperature ($\beta > \beta_c$) magnetized phase, $W = -\infty$ for any $c > 0$ due to the fact that $\chi \approx V m^2$ where m is the spontaneous magnetization. In the high-temperature phase, a straightforward computation (see appendix) gives

$$W = K^2 N\left[(2d - \bar{c}) + 4dK(2d - \bar{c} - 1)\beta + O(\beta^2)\right], \tag{9}$$

where $K\delta_{ij} = \int d\mu(\boldsymbol{\phi})\phi_i\phi_j / \int d\mu(\boldsymbol{\phi})$, thus $K > 0$, and $\bar{c} = cN$. If $\bar{c} \geq 2d$, $W(\beta, c)$ is negative and decreasing, proving the inequality (8) for small β. We have no proof for generic values of $\beta < \beta_c$. However, since for $\bar{c} \geq 2d$ the inequality holds for small β and $W = -\infty$ for $\beta > \beta_c$, it is plausible that the inequality holds for any β. Therefore, although a conclusive proof is still missing, we believe that the above analysis strongly supports the general bound $\gamma \geq 1$ for all transitions that occur in generic LGW lattice models.

LGW Φ^4 theories close to four dimensions. The RG flow of LGW theories can be investigated using the $\epsilon = 4 - d$ expansion [1, 2, 40]. Results for generic LGW Φ^4 theories are reported in Refs. [6, 8, 41, 42]. One can use the one-loop calculations of Refs. [6, 8] to demonstrate the ν conjecture for any LGW Φ^4 theory close to four dimensions. More details are reported in the appendix. The one-loop β functions of the renormalized couplings g_{ijkl} in the minimal-subtraction scheme are [6, 8]

$$\beta_{ijkl}(g) = \frac{\partial g_{ijkl}}{\partial \mu} = -\epsilon g_{ijkl} + \frac{1}{2}\left(g_{ijmn}\, g_{mnkl} + g_{ikmn}\, g_{mnjl} + g_{ilmn}\, g_{mnkj}\right), \tag{10}$$

where a sum is understood over repeated indices. The β functions and coupling are symmetric under exchanges of the indices. Moreover, the trace condition on u_{ijkl} also implies $g_{iikl} = T\delta_{kl}$ for the renormalized coupling. The zeroes of the β functions determine the fixed points (FPs) of the RG flow, which are directly related to the possible critical behaviors of the model. The location of the zeroes g^*_{ijkl} can be determined perturbatively. If we write $g^*_{ijkl} = \epsilon f_{ijkl} + O(\epsilon^2)$ and $T^* = c\,\epsilon + O(\epsilon^2)$, then the FP equation $\beta_{iikk}(g^*) = 0$ implies

$$N c^2 - 2Nc + 2 f_{ijkl} f_{ijkl} = 0, \qquad (11)$$

which can only by satisfied if $0 \le c \le 2$ ($c = 0$ only occurs if $f_{ijkl} = 0$, i.e., for the Gaussian FP). The RG dimensions Δ_φ and Δ_ε can be obtained by evaluating the corresponding RG functions at the FP [8]. In particular, $y_r = 2 - \frac{1}{2}T^* + O(\epsilon^2)$ and $\eta = O(\epsilon^2)$, so $\Sigma = c\,\epsilon/2$. The positivity of c at all FPs implies Eq. (2) close to four dimensions for any continuous transition with an effective LGW description.

The large-N limit. The ν conjecture in confirmed in the large-N limit of some classes of LGW theories. Large-N expansions are known in the O(N) vector models [43], which allow us to obtain

$$\Sigma = 4 - d - Y(d)/N + O(N^{-2}), \qquad (12)$$
$$Y(d) = \frac{4(2d^2 - 7d + 8)\Gamma(d-2)}{d\,\Gamma(d/2)\Gamma(2-d/2)\Gamma(d/2-1)^2}.$$

Therefore $\Sigma \ge 0$ for $d \in [2, 4]$. An analogous computation can be performed in the d-dimensional O(M)⊗O(N)-symmetric LGW theory with quartic potential [44]

$$P = r \sum_{ai} \Phi^2_{ai} + u\Big(\sum_{ai} \Phi^2_{ai}\Big)^2 + v \sum_{aibj} \Phi_{ai}\Phi_{bi}\Phi_{aj}\Phi_{bj}, \quad (13)$$

where Φ_{ai} is an $M \times N$ matrix field. For $N \to \infty$ keeping M fixed, one obtains [45]

$$\Sigma = 4 - d - Y(d)\frac{(M+1)}{2N} + O(N^{-2}) \qquad (14)$$

at the stable chiral FP of the theory. Moreover, one finds $\Sigma = Y(d)(M+2)(M-1)/(2MN)$ at the unstable antichiral FP. For both FPs we have $\Sigma \ge 0$.

An analysis of the 3D continuous transitions. We show that the ν conjecture is consistent with all known results for 3D continuous transitions. We begin with the 3D O(N) vector models, for which many accurate estimates of the critical exponents have been obtained [6, 7, 29, 46–64]. Using the best estimates, we obtain $\Sigma = 0.37565(1)$ for the Ising universality class ($N = 1$), $\Sigma = 0.47312(5)$ for the XY universality class ($N = 2$), $\Sigma = 0.5570(2)$ for Heisenberg universality class ($N = 3$), $\Sigma \approx 1 - 40/(3\pi^2 N)$ for large N, and $\Sigma = 0.26711(4)$ for the self-avoiding walks ($N \to 0$). Another interesting class of continuous transitions are described by LGW theories with cubic anisotropy (relevant for anisotropic magnets and randomly dilute spin models), for which the scalar potential reads [10]

$$P(\boldsymbol{\varphi}) = r\,\boldsymbol{\varphi}^2 + u\,(\boldsymbol{\varphi}^2)^2 + v \sum_{i=1}^{N} \varphi_i^4. \qquad (15)$$

For $N = 2$ the stable FP has an enlarged O(2) symmetry, while for $N \ge 3$ the stable FP is anisotropic, see appendix. The available results for $N \ge 3$ [7, 10, 59, 61, 65–73] show that $\Sigma > 0$: we find $\Sigma = 0.5556(5)$ for $N = 3$, $\Sigma = 0.574(1)$ for $N = 4$, $\Sigma = 0.54981(1)$ in the large-N limit, and $\Sigma = 0.500(4)$ in the $N \to 0$ limit (corresponding to the randomly-diluted Ising universality class [7, 71]). Other interesting critical behaviors are expected to be described by the 3D O(2)⊗O(N) LGW theories [7, 11, 74], cf. Eq. (13). The existence of the chiral O(2)⊗O(2), chiral O(2)⊗O(3) and ^{3}He superfluid universality classes was pointed out in Refs. [7, 22, 74–76] (see also Refs. [77–79] for some contrasting results). Using their results, we find $\Sigma > 0$ in all cases. We also mention the more complex LGW theories discussed in Refs. [80–83]. The corresponding estimates of the critical exponents satisfy $\Sigma > 0$ as well. Finally, the inequality $\Sigma \ge 0$ holds (see appendix) for the percolation problem related to a particular limit of Φ^3 field theories, in any $d \ge 3$, up to its upper critical dimension $d = 6$ [84–90].

2D continuous transitions. We now focus on 2D systems. The inequality $\Sigma \ge 0$ can be proved for an important class of CFTs, the minimal models with central charge $c = 1 - 6/[m(m+1)]$, $m \ge 3$ [28, 31, 91, 92]. We assume that the energy operator ε can be identified with the operator of conformal weight $h_{2,1} = (m+3)/(4m)$ or $h_{1,2} = (m-2)/[4(m+1)]$ (we recall that $h_{1,2} = h_{m-1,m-1}$) [28, 31, 91, 92]. This is the case of the Ising and $q = 3$ Potts model ($m = 3$ and $m = 5$, respectively), for which $\Delta_\varepsilon = 2\,h_{2,1}$, and of the tricritical Ising and $q = 3$ Potts models ($m = 4$ and $m = 6$), for which $\Delta_\varepsilon = 2\,h_{1,2}$. At criticality, under the previous conditions, the correlation function defined in Eq. (6) satisfies a differential equation [28, 91] which allows us (see appendix) to relate Δ_φ and Δ_ε as a consistency condition. This implies

$$\Sigma = \Delta_\varepsilon - 2\Delta_\varphi = \frac{3\Delta_\varepsilon^2}{2(1+\Delta_\varepsilon)} \ge 0, \qquad (16)$$

which proves the inequality (2) for all minimal models. One can also check that the ν conjecture holds for the Berezinskii-Kosterlitz-Thouless transition [93–96] ($\Sigma = 7/4$), the O(N) σ models with $N \ge 3$ ($\Sigma = 2$), the O(N) σ models with $N \in [-2, 2]$ [97–100] (Σ increases from $\Sigma = 0$ for $N = -2$ to $\Sigma = 7/4$ for $N = 2$), and the transition line of the 2D Ashkin-Teller model [101–105] where Σ varies in the range [1/4, 5/4].

GNY models. We can also show that the ν conjecture also holds in GNY theories in which a real scalar field φ is coupled with fermionic fields $\psi_f(\boldsymbol{x})$, i.e., [6, 43, 106]

$$\mathcal{H} = (\partial_\mu \varphi)^2 + r\,\varphi^2 + u\,\varphi^4 - \sum_{f=1}^{N_f} \bar\psi_f(\not\partial + m + \varphi)\psi_f. \quad (17)$$

We again identify $\varepsilon = [\varphi^2]$. In the large N_f limit [43, 107–113], see appendix, we find

$$\Sigma \approx \frac{Z(d)}{N_f}, \quad Z(d) = \frac{2(d-1)\Gamma(d-2)}{\Gamma(d/2)\Gamma(2-d/2)\Gamma(d/2-1)^2}, \quad (18)$$

which is positive since $Z(d) \geq 0$ for any $d \in [2,4]$. Large-N_f results are also available for the chiral XY and Heisenberg GNY models [114, 115], which show that $\Sigma \geq 0$ as well. Moreover, close to four dimensions, using the ϵ-expansion results reported in Refs. [116, 117], one can easily check that $\Sigma = B(N_f)\epsilon$ with $B(N_f) > 0$, for all GNY formulations, see appendix.

AH models. The ν conjecture (with some specifications related to the presence of gauge invariance) also holds at the charged continuous transitions of scalar AH gauge models, where both scalar and gauge modes become critical [46], which admit an effective description in terms of the AH field theory [6, 46, 118]

$$\mathcal{H} = \frac{1}{4g^2} F_{\mu\nu}^2 + |D_\mu \boldsymbol{\phi}|^2 + r\, \bar{\boldsymbol{\phi}} \cdot \boldsymbol{\phi} + u\, (\bar{\boldsymbol{\phi}} \cdot \boldsymbol{\phi})^2, \qquad (19)$$

in which a complex N-component scalar field $\boldsymbol{\phi}$ is coupled to a U(1) gauge field A_μ. As for LGW Φ^4 theories, the energy-density operator can be identifed with $\varepsilon \sim [\bar{\boldsymbol{\phi}} \cdot \boldsymbol{\phi}]$, which is well defined as it is gauge invariant. On the other hand, it is less obvious what can be identified as order parameter. In LGW models without gauge invariance one considers the field $\boldsymbol{\phi}$. However, in the absence of a gauge fixing, the field $\boldsymbol{\phi}$ is not gauge invariant, and therefore its correlation functions are ultralocal, forbidding the definition of a corresponding RG dimension Δ_ϕ. This problem has been extensively discussed in Refs. [46, 119–121]. It was shown thar the correct order parameter is a nonlocal gauge-invariant field, which corresponds to the Lagrangian field in the hard Lorentz gauge $\partial_\mu A_\mu = 0$. In the perturbative setting, this implies that $\Delta_\phi = (d - 2 + \eta_\phi)/2$ is the RG dimension of $\boldsymbol{\phi}$ for $\zeta = 0$, where ζ is the Lorentz-gauge parameter (see appendix for details). In AH models charged continuous transitions occur only for $N > N_*(d)$. Close to four dimensions, N_* is large: $N > N_*(4) = 90 + 24\sqrt{15} \approx 183$. Using the perturbative expansions reported in Refs. [118, 122], we find $\Sigma \approx A(N)\,\epsilon$ with $A(N) > 0$ for $N > N_*(4)$. For 3D AH models the Higgs transition can be continuous for $N > N_*(3)$ with $N_*(3) \approx 7$ [122–124]. Using the numerical results of Ref. [123] and the large-N results of Refs. [46, 118, 125, 126], $\Sigma > 0$ is obtained in all cases. For $N < N_*(3)$ the Higgs transitions are expected to be of first order, with the notable exception of the one-component AH model (which is related to superconductivity, see, e.g. Refs. [118, 127–129]). In this case the transition belongs to the inverted XY universality class [46, 130, 131], related to the standard XY universality class by duality. The exponent ν is the same as in the XY model [7, 54–57], $\nu \approx 0.6717$. On the other hand, since the order parameter in the AH model (a nonlocal gauge-invariant operator) differs from the order parameter in the XY model (duality only relates the energy sectors), $\eta_\phi = -0.74(4)$ in the AH model [46] differs from $\eta \approx 0.03$ for the XY model [7]. Using these estimates we obtain $\Sigma = 1.25(4)$, confirming the ν conjecture.

Conclusions. The above arguments and evidences provide a substantial support to the conjectured inequality (2), and its implications (3) and (4), for standard continuous transitions that can be associated with d-dimensional LGW Φ^4 theories, where ν is the exponent associated with the divergence of the correlation length as a function of the symmetry-conserving perturbation of the transition, while η is the anomalous dimension of the order parameter, which is coupled with the leading symmetry-breaking perturbation. In the LGW setting, the order parameter is the local field operator $\boldsymbol{\varphi}$, while the leading symmetry-conserving operator is the energy $\varepsilon = [\boldsymbol{\varphi}^2]$. The ν conjecture extends to continuous transitions of systems with fermionic and gauge fields, with an appropriate identification of the order parameter field. No violations have been found in earlier studies. Of course, additional studies are needed to assess the generality of the inequality, and, if violations are found, to identify the reasons of its failure. In particular, this issue may be further investigated in three dimensions using CFT methods, such as the conformal bootstrap approach [29].

We finally remark that the lower bound (3) for the exponent ν provides an explanation for the phenomenological fact that no continuous transitions with $\nu < 1/2$ have been observed. Moreover, it may be particular useful for the identification of the nature of the transition in numerical analyses in 3D systems. Indeed, if it holds, finite-size behaviors with effective exponents $\nu < 1/2$ should be interpreted as crossover behaviors toward a first-order transition, without the need of verifying that the data scale with exponent $\nu \approx 1/d$, as it should occur asymptotically.

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Appendix A: Arguments and evidences supporting the ν conjecture

In the following sections we present a more detailed discussion of the various arguments and evidences outlined in the main text to support the ν conjecture

$$\Sigma = \Delta_\epsilon - 2\Delta_\phi = 2 - \eta - 1/\nu \geq 0, \qquad (A1)$$

which implies that $\nu \geq (2-\eta)^{-1}$ and $\gamma = (2-\eta)\nu \geq 1$.

1. The bound $\gamma \geq 1$ in lattice systems

We argue that $\gamma \geq 1$ for generic ferromagnetic systems at their continuous transitions. Using the relation $\gamma = (2-\eta)\nu$, this result implies the conjectured bound on ν. The relation $\gamma \geq 1$ has been proved for Ising systems in any dimension using a simple inequality for the magnetic susceptibility [35]. Here, we conjecture that an analogous inequality also holds in generic ferromagnetic systems, implying $\gamma \geq 1$ in all cases.

We consider a generic system in which the fundamental fields are multicomponent variables $\boldsymbol{\phi}_{\boldsymbol{x}}$ defined on the sites of a d-dimensional lattice, which we assume to be cubic-like for simplicity. We consider real fields, but the extension to complex variables is straightforward. We define a local integration measure at each site $\boldsymbol{x}$,

$$d\mu(\boldsymbol{\phi}_{\boldsymbol{x}}) = \mathcal{N}\, e^{-P(\boldsymbol{\phi}_{\boldsymbol{x}})} d\boldsymbol{\phi}_{\boldsymbol{x}}, \tag{A2}$$

where $P(\boldsymbol{\phi}_{\boldsymbol{x}})$ is a local potential, and $\mathcal{N}$ a normalizing factor so that $\int d\mu(\boldsymbol{\phi}) = 1$. We do not make any assumption on $P(\boldsymbol{\phi})$, which may depend on $\boldsymbol{\phi}$ in an arbitrary manner, which is only constrained by the global symmetry of the model. For example, it may be obtained from a lattice discretization of a corresponding LGW theory. The partition function is given by

$$Z = \int \prod_{\boldsymbol{x}} d\mu(\boldsymbol{\phi}_{\boldsymbol{x}})\, e^{-\beta H}, \quad H = -\sum_{\langle \boldsymbol{x}\boldsymbol{y}\rangle} \boldsymbol{\phi}_{\boldsymbol{x}} \cdot \boldsymbol{\phi}_{\boldsymbol{y}}, \tag{A3}$$

where the sum runs over all lattice bonds $\langle \boldsymbol{x}\boldsymbol{y}\rangle$. We assume that the model undergoes a continuous transition at $\beta = \beta_c$, where all components of the field become critical. The latter condition is verified if the model is invariant under a transformation group G such that the field transforms irreducibly under G. This implies

$$\int d\mu(\boldsymbol{\phi})\, \boldsymbol{\phi}^a \boldsymbol{\phi}^b = K\delta^{ab}, \qquad K > 0. \tag{A4}$$

For the Ising model, whose lattice variables are $\sigma_i = \pm 1$, the proof of $\gamma \geq 1$ starts from the following inequality proved by Lebowitz [37]:

$$\langle \sigma_i\sigma_j\sigma_h\sigma_k\rangle \leq \langle \sigma_i\sigma_j\rangle\langle\sigma_h\sigma_k\rangle + \langle\sigma_i\sigma_h\rangle\langle\sigma_j\sigma_k\rangle + \langle\sigma_i\sigma_k\rangle\langle\sigma_j\sigma_h\rangle. \tag{A5}$$

The proof of this relation relies heavily on the Ising nature $\sigma_i = \pm 1$ of the spin variables, and it has been generalized to Ashkin-Teller systems in Ref. [38]. If $\chi = \frac{1}{V}\sum_{ij}\langle\sigma_i\sigma_j\rangle$ (V is the volume and the two sums extend over all lattice sites), we have

$$\begin{aligned}\frac{\partial\chi}{\partial\beta} &= \frac{1}{V}\sum_{ij}\sum_{\langle hk\rangle} \left[\langle\sigma_i\sigma_j\sigma_h\sigma_k\rangle - \langle\sigma_i\sigma_j\rangle\langle\sigma_h\sigma_k\rangle\right] \\ &\leq \frac{1}{V}\sum_{ij}\sum_{\langle hk\rangle} \left[\langle\sigma_i\sigma_h\rangle\langle\sigma_j\sigma_k\rangle + \langle\sigma_i\sigma_k\rangle\langle\sigma_j\sigma_h\rangle\right] = q\chi^2,\end{aligned} \tag{A6}$$

where we used Eq. (A5) in the second step. For cubic lattices $q = 2d$. The inequality holds for generic regular lattices, with q being the lattice coordination number.

We conjecture that a similar inequality holds in the general model introduced above. If we define

$$\chi = \frac{1}{V}\sum_{\boldsymbol{x},\boldsymbol{y}} \langle \boldsymbol{\phi}_{\boldsymbol{x}} \cdot \boldsymbol{\phi}_{\boldsymbol{y}}\rangle, \tag{A7}$$

(V is the volume) we conjecture the existence of positive constants c such that

$$\frac{\partial\chi}{\partial\beta} \leq c\chi^2. \tag{A8}$$

Eq. (A8) allows one to prove $\gamma \geq 1$. Indeed, if the transition is continuous, for $\beta < \beta_c$ close to the transition, we have $\chi \approx a_0(\beta_c - \beta)^{-\gamma}$ with $a_0 > 0$. Eq. (A8) implies

$$a_0\gamma(\beta_c - \beta)^{-\gamma-1} \leq c a_0^2 (\beta_c - \beta)^{-2\gamma},$$

and therefore

$$0 \leq (\beta_c - \beta)^{\gamma - 1} \leq \frac{c a_0}{\gamma}, \tag{A9}$$

which is only possible if $\gamma \geq 1$. Note that also the reverse statement holds: if $\gamma \geq 1$, close to the critical point the inequality (A8) holds.

Let us now provide some arguments supporting the conjectured inequality. In the low-temperature ferromagnetic phase, the inequality is verified for any positive c. As the system magnetizes, $\chi \approx m(\beta)^2 V$ in the large-volume limit. Thus, the left-hand side scales as V, while the right-hand side scales as V^2, so Eq. (A8) is certainly true for large values of V. In the high-temperature phase, for β small, we have

$$\frac{1}{KN}\chi = 1 + 2dK\beta + 2d(2d-1)K^2\beta^2 + O(\beta^3), \tag{A10}$$

which implies

$$\frac{1}{K^2 N}\left[\frac{\partial\chi}{\partial\beta} - c\chi^2\right] = (2d - \bar{c}) + 4dK(2d - \bar{c} - 1)\beta + O(\beta^2) \tag{A11}$$

with $\bar{c} = cN$. Thus, if $\bar{c} \geq 2d$, the considered quantity is negative and decreasing, supporting the inequality also for small values of β.

Inequality (A8) can also be used as a tool to try to understand the nature of the transition in those cases in which fits of numerical data give estimates of ν that are not compatible with the bound $\nu \geq (2-\eta)^{-1}$. If one finds that the inequality (A8) is satisfied close to the critical point in a region where infinite-volume results can be reliably obtained, the inconsistent result for ν should be taken as an indication that the transition is not continuous.

2. LGW theories close to four dimensions

The RG flow of LGW Φ^4 theories,

$$\mathcal{H} = \sum_i (\partial_\mu \varphi_i)^2 + r\sum_i \varphi_i^2 + \sum_{ijkl} u_{ijkl}\, \varphi_i\varphi_j\varphi_k\varphi_l, \tag{A12}$$

close to four dimensions can be investigated in the framework of the $\epsilon = 4 - d$ expansion [1, 2, 40]. In this approach, one considers dimensional regularization and the minimal-subtraction (MS) renormalization scheme. The corresponding one-loop β functions in terms of the renormalized couplings g_{ijkl} are given by [8]

$$\begin{aligned}\beta_{ijkl}(g) = \frac{\partial g_{ijkl}}{\partial\mu} &= -\epsilon g_{ijkl} \\ &+ \frac{1}{2}(g_{ijmn}\, g_{mnkl} + g_{ikmn}\, g_{mnjl} + g_{ilmn}\, g_{mnkj}),\end{aligned} \tag{A13}$$

where we use the convention that a sum is understood over repeated indices. Moreover, we choose an appropriate normalization of the coupling to simplify the expressions, see, e.g., Ref. [6]. Note that, like the quartic couplings u_{ijkl} of the Φ^4 Hamiltonian density, the MS renormalized couplings g_{ijkl} are completely symmetric with respect to interchanges of the indices, and satisfy the trace condition

$$g_{iikl} = T\delta_{kl}. \tag{A14}$$

The fixed points (FPs) are obtained from the zeroes of the β function. The stable FP generally determines the critical behavior of systems with the same symmetry as the LGW model, as long as the FP is related by RG transformations to bare LGW models that have the same symmetry-breaking pattern observed at the transition. The unstable FPs are instead associated with critical behaviors that can only be observed by performing appropriate tunings of the model parameters.

At any FP $g^*_{ijkl} = \epsilon f_{ijkl} + O(\epsilon^2)$ and

$$T^* = c\,\epsilon + O(\epsilon^2). \tag{A15}$$

The FP equation $\beta_{iikk}(g^*) = 0$ gives

$$Nc^2 - 2Nc + 2f_{ijkl}f_{ijkl} = 0, \tag{A16}$$

where N is the number of components of the field. This equation can only be satisfied if

$$0 \le c \le 2, \qquad f_{ijkl}f_{ijkl} \le \frac{N}{2}. \tag{A17}$$

Note that $c = 0$ is only possible if $f_{ijkl} = 0$. Relation (A17) implies that T^* is nonnegative close to four dimensions [8].

The RG dimensions Δ_φ and Δ_ε of the order-parameter field φ_i and of the composite energy-density operator $\varepsilon = [\boldsymbol{\varphi}^2]$ can be obtained by evaluating the corresponding RG functions at the FP. The RG functions associated with η and y_r are given by[1] [6, 8]

$$\eta(g) = \frac{1}{24N} g_{iklm}\, g_{iklm} + O(g^3), \tag{A24}$$

$$y_r(g) = 2 - \frac{1}{2N} g_{iikk} + O(g^2) = 2 - \frac{1}{2}T + O(g^2),$$

where we used the trace condition (A14). At the FPs, we obtain

$$y_r = \frac{1}{\nu} = 2 - \frac{c}{2}\,\epsilon + O(\epsilon^2). \tag{A25}$$

Since $\eta = O(\epsilon^2)$, we have

$$\Sigma = 2 - \eta - y_r = \frac{c}{2}\,\epsilon + O(\epsilon^2) \ge 0. \tag{A26}$$

confirming the conjecture (A1).

We note that the perturbative expansions of the β-function can be written as the gradient of a function $Q(g)$ of the coupling: [41]

$$\beta_{ijkl}(g) = \frac{\partial Q(g)}{\partial g_{ijkl}}, \tag{A27}$$

$$Q(g) = -\frac{\epsilon}{2} g_{ijkl}\, g_{ijkl} + \frac{1}{2} g_{ijkl}\, g_{klmn}\, g_{mnij} + O(g^4).$$

A peculiar property of the RG trajectories is that the potential $Q(g)$ decreases along RG trajectories, thus the FPs are extrema of the potential [6, 41]. Therefore, if two FPs are connected by a RG trajectory, the most stable FP corresponds to the lowest value of the potential. At the FP $g^* \equiv g^*_{ijkl}$, where

$$\beta_{ijkl}(g^*) = \frac{\partial Q(g^*)}{\partial g_{ijkl}} = 0, \tag{A28}$$

the potential can be written as [6] [2]

$$Q(g^*) = -\frac{\epsilon}{6}\, g^*_{ijkl}\, g^*_{ijkl} + O(\epsilon^4). \tag{A30}$$

---

[1] By replacing

$$g_{ijkl} = \frac{g}{3}(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{A18}$$

one obtains the RG functions for the O(N)-symmetric Φ^4 theory:

$$\beta(g) = -\epsilon g + \frac{N+8}{6} g^2 + O(g^3), \tag{A19}$$

$$\eta(g) = \frac{N+2}{72} g^2 + O(g^3), \tag{A20}$$

$$y_r(g) = 2 - \frac{N+2}{6} g + O(g^2). \tag{A21}$$

Computing $\eta(g)$ and $y_r(g)$ at the FP

$$g^* = \frac{6}{N+8}\epsilon + O(\epsilon^2), \tag{A22}$$

we obtain

$$\Sigma = \frac{N+2}{N+8}\epsilon + O(\epsilon^2) \ge 0. \tag{A23}$$

[2] This relation can be derived by noting that at the FP

$$\epsilon\, g^*_{ijkl}\, g^*_{ijkl} = \frac{3}{2}\, g^*_{ijkl}\, g^*_{klmn}\, g^*_{mnij}, \tag{A29}$$

This leads to a notable leading-order relation between the exponent η and the potential:

$$\eta(g^*) = \frac{1}{24N}\, g^*_{ijkl}\, g^*_{ijkl} = -\frac{1}{N\epsilon} Q(g^*). \tag{A31}$$

Since the stable FP corresponds to the lowest value of $Q(g)$, the above relation implies that it must have the largest value of the exponent η [42], or equivalently the larger value of the RG dimension $\Delta_\varphi = (d-2+\eta)/2$ of the order-parameter field, also implying that the two-point correlation function corresponding to the stable FP have the fastest large distance decay, $G(\boldsymbol{x}) \sim |\boldsymbol{x}|^{-2\Delta_\varphi}$, at the critical point.

3. LGW theories in the large-N limit

In this section, we present some analytic results obtained for large values of N, which confirm again the validity of the conjectured inequality (A1).

a. O(N)-vector models

We first consider LGW theories with N-component fields $\boldsymbol{\varphi}(\boldsymbol{x})$ and O(N) global symmetry, which are obtained by taking

$$u_{ijkl} = \frac{u}{3}\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \tag{A32}$$

in Eq. (A12).

The large-N expansions of the critical exponents for any $2 < d < 4$ are reported in Refs. [6, 43] and references therein. Setting

$$X(d) = \frac{4\,\Gamma(d-2)}{\Gamma(d/2)\Gamma(2-d/2)\Gamma(d/2-1)^2}, \tag{A33}$$

to order $1/N$ we have

$$\eta = X(d)\frac{4-d}{dN} + O(N^{-2}), \tag{A34}$$

$$y_r = d - 2 + X(d)\frac{2(d-1)(d-2)}{dN} + O(N^{-2}), \tag{A35}$$

which implies

$$\Sigma = 4 - d - X(d)\frac{2d^2 - 7d + 8}{dN} + O(N^{-2}). \tag{A36}$$

The quantity Σ is always positive for $2 \le d < 4$, varying from $\Sigma = 0$ for $d = 4$ to $\Sigma = 2$ for $d = 2$. In three dimensions, using $X(d=3) = 8/\pi^2$, we obtain

$$\Sigma = 1 - \frac{40}{3\pi^2 N} + O(N^{-2}). \tag{A37}$$

which is obtained exploiting the symmetry of g_{ijkl} under interchanges of the indices.

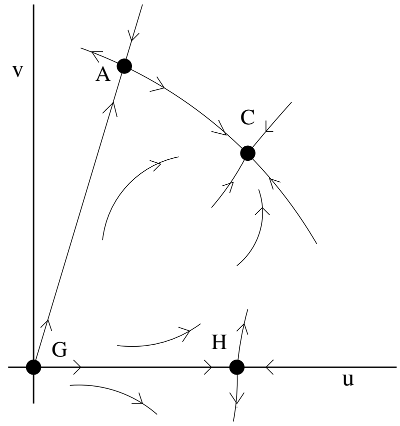

FIG. 1: RG flow and FPs of the O(M)⊗O(N) model in the large-N limit. Tere are four FPs: the unstable Gaussian (G) and O(N) symmetric (H) FPs, the stable chiral (C) FP and the unstable antichiral (A) FP.

b. O(M)⊗O(N) LGW theories

We now focus on the d-dimensional O(M)⊗O(N) LGW Φ^4 theory defined by the Hamiltonian density

$$\begin{aligned}\mathcal{H} &= \sum_{a,i}(\partial_\mu \Phi_{ai})^2 + r\sum_{a,i}\Phi_{ai}^2 + u\Big(\sum_{a,i}\Phi_{ai}^2\Big)^2 \\ &+ v\sum_{a,i,b,j}\Big(\Phi_{ai}\Phi_{bi}\Phi_{aj}\Phi_{bj} - \Phi_{ai}^2\Phi_{bj}^2\Big)\end{aligned} \tag{A38}$$

where Φ_{ai} is an $M \times N$ real matrix ($a = 1,...,M$ and $i = 1,...,N$). The symmetry group of the model is O(M)⊗ O(N).

The O(M)⊗O(N) Φ^4 model can be solved in the large-N limit for any fixed M [44, 45]. For any $M \geq 2$, there are four FPs: the Gaussian FP, the Heisenberg O($M \times N$) FP, and two new FPs which we call chiral (C) and antichiral (A). Fig. 1 shows a sketch of the RG flow in the quartic-coupling space. In the large-N limit, for any $M \geq 2$ and for any $2 < d < 4$, the stable FP is the chiral one, and all other FPs are unstable.

It is useful to define the d-dimensional constants

$$A(d) = -\frac{4\Gamma(d-2)}{\Gamma(2-d/2)\Gamma(d/2-1)\Gamma(d/2-2)\Gamma(d/2+1)},$$

$$B(d) = \frac{(d-1)(d-2)}{(4-d)}A(d). \tag{A39}$$

At the stable chiral FP we have

$$\eta_c = A(d)\frac{M+1}{2N} + O(N^{-2}), \tag{A40}$$

$$y_{t,c} = d - 2 + B(d)\frac{M+1}{N} + O(N^{-2}), \tag{A41}$$

$$\Sigma_c = 4 - d - Y(d)\frac{M+1}{2N} + O(N^{-2}). \tag{A42}$$

TABLE I: We report estimates of the critical exponents ν and η, and of the difference $\Sigma = \Delta_\varepsilon - 2\Delta_\varphi$, for some 3D continuous transitions.

Model	ν	η	Σ
Gaussian	1/2	0	0
Ising [50]	0.629971(4)	0.03698(2)	0.37565(1)
O(2), XY [57]	0.67172(2)	0.03816(2)	0.47312(5)
O(3), H [60]	0.71164(10)	0.03784(5)	0.5570(2)
$O(N\to\infty)$	$1-\frac{32}{3\pi^2 N}$	$\frac{8}{3\pi^2 N}$	$1-\frac{40}{3\pi^2 N}$
SAW [$O(N\to 0)$] [62, 63]	0.587597(7)	0.03104(2)	0.26711(4)
Cubic $N=4$ [66]	0.7202(7)	0.0371(2)	0.574(1)
Cubic $N\to\infty$	0.707902(6)	0.03698(2)	0.54981(1)
Cubic $N\to 0$, RDIs [73]	0.683(2)	0.036(1)	0.500(4)
Percolation [85]	0.8762(12)	−0.0458(2)	0.905(2)

At the unstable antichiral FP we have

$$\eta_{ac} = A(d)\frac{(M+2)(M-1)}{2MN} + O(N^{-2}), \quad \text{(A43)}$$

$$y_{t,ac} = 2 - B(d)\frac{(M+2)(M-1)}{MN} + O(N^{-2}), \quad \text{(A44)}$$

$$\Sigma_{ac} = Y(d)\frac{(M+2)(M-1)}{2MN} + O(N^{-2}). \quad \text{(A45)}$$

where $Y(d) = A(d) + 2B(d)$. One can easily check that Σ_c and Σ_{ac} are both positive for any $2 < d < 4$ and any $M \geq 2$. In particular, for $d = 3$ we have

$$\Sigma_c = 1 - \frac{20(M+1)}{3\pi^2 N} + O(N^{-2}), \quad \text{(A46)}$$

$$\Sigma_{ac} = \frac{40(M+2)(M-1)}{3\pi^2 MN} + O(N^{-2}).$$

4. 3D continuous transitions

We now verify that the conjecture (A1) holds at all known transitions that are described by LGW theories.

a. 3D O(N)-vector models

Accurate estimates of the critical exponents ν and η for the 3D O(N) vector models are reported in Table I. [3] In all cases, $\Sigma = 2 - \eta - y_r$ is positive as a consequence of the smallness of η and of the fact that ν is significantly larger than 1/2.

[3] Results for $N = 1$, 2, and 3 are reported in Refs. [47–53], Refs. [47, 51, 54–57], and Refs. [47, 51, 58–61], respectively. Refs. [62, 63] report estimates for the 3D self-avoiding walk (SAW) universality class ($N = 0$), Ref. [64] for $N \geq 4$.

b. LGW theories with cubic anisotropy

Another interesting class of continuous transitions are described by LGW theories with cubic anisotropy. The Hamiltonian density is [10]

$$\mathcal{H} = \sum_i [(\partial_\mu \varphi_i)^2 + r\varphi_i^2] + u(\sum_i \varphi_i^2)^2 + v\sum_i \varphi_i^4, \quad \text{(A47)}$$

where φ_i is an N-component real field. The analysis of the RG flow of the model shows the appearance of a cubic FP with $u^* \neq 0$ and $v^* \neq 0$ beside the FPs present in O(N) models. The qualititative features of the RG flow as well as the stability of the cubic FP depend on the number of components, see Fig. 2. For $N < N_c$, the O(N)-symmetric FP is stable and the cubic one is unstable, while the opposite occurs for $N > N_c$. In three dimensions, $N_c \approx 2.9$, see, e.g., Refs. [7, 10, 59, 61, 65–67] and references therein.

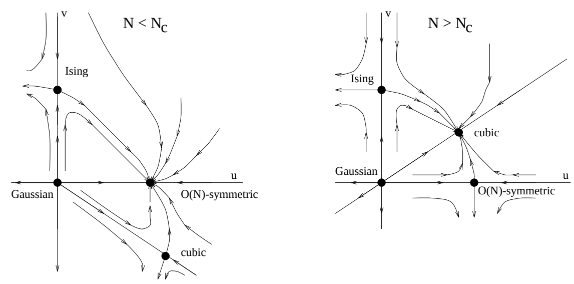

FIG. 2: The three-dimensional RG flow of the cubic model for $N = 2$ (left) and $N \geq 3$ (right), because $N_c \approx 2.9$ in three dimensions [7, 10, 59, 61, 65–67].

The presently known estimates of the critical exponents for the cubic FP all confirm $\Sigma > 0$. Indeed, for $N = 3$ the cubic critical exponents are very close to th O(3) exponents. Indeed, one has [66] $\nu_{\rm cubic} - \nu_{\rm H} = -0.0006(1)$ and $-0.00001 \lesssim \eta_{\rm cubic} - \eta_{\rm H} \lesssim 0.00007$, where $\nu_{\rm H}$ and $\eta_{\rm H}$ are the critical exponents of the Heisenberg universality class. For $N = 4$ results are reported in Table I. For $N \to \infty$ one obtains [10, 69, 70]

$$\eta = \eta_{\rm Is} + O(1/N), \quad \nu = \frac{\nu_{\rm Is}}{1 - \alpha_{\rm Is}} + O(1/N), \quad \text{(A48)}$$

where $\eta_{\rm Is}$, $\nu_{\rm Is}$, and $\alpha_{\rm Is} = 2 - 3\nu_{\rm Is}$ are the critical exponents of the Ising model. It follows $\Sigma \approx 0.55 > 0$.

The model (A47) is also interesting in the formal limit $N \to 0$. In this case it describes the univeral features of randomly-diluted Ising (RDIs) systems [7, 71]. The critical exponents (see Table I and Refs. [7, 72, 73] for a more complete list of results) again satisfy $\Sigma > 0$.

c. O(2)⊗O(N) LGW theories

We finally consider the 3D O(2)⊗O(N) LGW theories with Hamiltonian density (A38) for small values of N. For $N = 2$ and 3 these models may describe transitions in noncollinear frustrated magnets and the normal-to-planar superfluid transition in ^{3}He, see, e.g., Refs. [7, 11, 74] and references therein. The existence of stable FPs in these models, and therefore continuous transitions belonging to corresponding universality classes, is still controversial. While the analyses of high-order perturbative expansions and some numerical simulations favor the existence of stable FPs associated with chiral universality classes for $N = 2, 3$ [7, 22, 74–76], other theoretical studies do not confirm their existence, or at least their stability [77–79]. Note that these FPs for $N = 2$, 3 are not related with the FPs present close to four dimensions for larger values of N, a phenomenon that also occurs in the one-component Abelian Higgs theory, as mentioned below.

We now verify the ν conjecture at the FPs found in Refs. [7, 22, 75, 76] for the chiral O(2)⊗O(2) and O(2)⊗O(3) transitions and of the ^{3}He superfluid universality classes (the last one has the same symmetry but a different symmetry breaking pattern with respect to the chiral transition), and verify that the corresponding estimates of the critical exponents satisfy the conjecture (A1). A field-theoretical perturbative analysis gives $\nu \approx 0.6$ and $\eta \approx 0.09$ for the O(2)⊗O(2) and O(2)⊗O(3) chiral FPs [7, 22], and $\nu = 0.60(5)$ and $\eta = 0.08(1)$ for the ^{3}He superfluid universality class [76]. In all cases $\Sigma \approx 0.3 > 0$.

Other more complex LGW theories with multiparameter quartic terms have also been considered in the literature. We mention here the theories relevant for spin-density waves in cuprates and for the finite-temperature chiral hadronic transition (U(N)⊗U(N) and SU(4) LGW theories). Field-theoretical analyses have identified the relevant FPs and provided estimates of the corresponding critical exponents [80–83], which satisfy $\Sigma > 0$ in all cases.

d. Percolation

The percolation problem can be related to the limit $q \to 1$ of the Potts model, and also to a particular limit of Φ^3 field theories, for which the upper critical dimension is $d = 6$. Therefore, the ν conjecture is expected to hold as well. The is confirmed by results for the 3D percolation [84, 85] reported in Table I. For $d > 3$ the percolation results of Refs. [86–89] give $\Sigma \approx 0.65$ for $d = 4$, $\Sigma \approx 0.32$ for $d = 5$, confirming $\Sigma \geq 0$ in all cases. Moreover, close to the upper critical dimension ($d = 6$)

TABLE II: Critical exponents ν, η and γ for some 2D critical behaviors, and corresponding difference $\Sigma = \Delta_\varepsilon - 2\Delta_\varphi$.

2D Model	ν	η	γ	Σ
Ising ($c = 1/2$)	1	1/4	7/4	3/4
$q = 3$ Potts ($c = 4/5$)	5/6	4/15	13/9	8/15
Tricr. Ising ($c = 7/10$)	5/9	3/20	37/36	1/20
Tricr. $q = 3$ Potts ($c = 6/7$)	7/12	4/21	19/18	2/21
$q = 4$ Potts ($c = 1$)	2/3	1/4	7/6	1/4
XY [BKT, O(2) σ]	∞	1/4	∞	7/4
O($N \geq 3$) σ	∞	0	∞	2
SAW [O($N \to 0$) σ]	3/4	5/24	43/32	11/24
Percolation ($q \to 1$ Potts)	4/3	5/24	43/18	25/24

one can perform a $6-d$ expansion, using an appropriate Φ^3 field theory. At one loop [87, 90] one obtains

$$y_r = 2 - \tfrac{5}{21}\epsilon_6 + O(\epsilon_6^2), \qquad \eta = -\frac{1}{21}\epsilon_6 + O(\epsilon_6^2),$$
$$\gamma = 1 + \tfrac{2}{21}\epsilon_6 + O(\epsilon_6^2), \qquad \Sigma = \frac{2}{7}\epsilon_6 + O(\epsilon_6^2), \quad \text{(A49)}$$

where $\epsilon_6 \equiv 6 - d$.

5. 2D continuous transitions

We now consider continuous transitions in two dimensions. In Table II we report the critical exponents ν, η, γ, and the difference $\Sigma = 2 - \eta - y_r$, for several 2D models. We consider the Ising model, the $q = 3$ and $q = 4$ Potts model, the XY model, which undergoes a Berezinskii-Kosterlitz-Thouless (BKT) transition [93–96], the self-avoiding walk (SAW) model, the tricritical Ising and $q = 3$ Potts models, and percolation, which corresponds to the $q \to 1$ limit of the Potts model. See, e.g., Refs. [6, 7, 28, 39, 91, 92]. They confirm the conjectured inequality $\Sigma = 2 - \eta - y_r \geq 0$. The O($N$) models for $N \geq 3$ do not undergo finite-temperature transitions in two dimensions. A transition occurs for $d = 2 + \epsilon_2$, with exponents [6] $\nu = 1/\epsilon_2$ and $\eta = \epsilon_2/(N-2)$ (they formaly converge to ∞ and 0, as reported in Table II, for $\epsilon_2 \to 0$). It follows

$$\Sigma = 2 - \frac{N-1}{N-2}\epsilon_2 + O(\epsilon_2^2) > 0. \qquad \text{(A50)}$$

a. Proof of the ν conjecture within 2D comformal minimal models

Using conformal field theory the ν conjecture can be proved for the conformal minimal unitary models [28, 31, 91, 92] with central charge $c = 1 - 6/[m(m+1)]$ ($m \geq 3$), when the energy operator ε can be associated with one of the primary operators with dimension[4] $2h_{m-1,m-1} = 2h_{1,2}$ and $2h_{2,1}$. This is the case of the Ising and three-state Potts model, corresponding to $m = 3$ and $m = 5$, for which $\Delta_\varepsilon = 2\,h_{2,1}$, and of the tricritical Ising and three-state Potts model, corresponding to $m = 4$ and $m = 6$, respectively, for which $\Delta_\varepsilon = 2\,h_{1,2}$. For this class of models one can derive an exact relation between Δ_ε and Δ_φ. For this purpose, one considers the three-point correlaton function of the energy operator ϵ and two field operators φ. At the critical point, because of conformal invariance, it takes the form [5, 30]

$$A_{123} \equiv \langle \varepsilon(z_1,\bar z_1)\varphi(z_2,\bar z_2)\varphi(z_3,\bar z_3)\rangle \propto \frac{z_{23}^{\Delta_\varepsilon/2-\Delta_\varphi}}{z_{12}^{\Delta_\varepsilon/2} z_{13}^{\Delta_\varepsilon/2}} \times (\text{c.c.}), \tag{A52}$$

where $z_{ij} = z_i - z_j$. When the energy operator ε is a primary operator with weight $h_{1,2}$ or $h_{2,1}$ [28, 91], the three-point function A_{123} satisfies the differential equation

$$\left[\frac{3}{2(\Delta_\varepsilon+1)}\partial^2_{z_1} - \sum_{k=2}^{3}\left(\frac{\Delta_\varphi}{2(z_1-z_k)^2} + \frac{\partial_{z_k}}{z_1-z_k}\right)\right] A_{123} = 0. \tag{A53}$$

Substituting the expression of A_{123} reported in Eq. (A52), one can straightforwardly derive the consistency condition

$$2\Delta_\varphi = \frac{2-\Delta_\varepsilon}{2(1+\Delta_\varepsilon)}\Delta_\varepsilon. \tag{A54}$$

Then, we have

$$\Sigma = \Delta_\varepsilon - 2\Delta_\varphi = \frac{3\Delta_\varepsilon^2}{2(1+\Delta_\varepsilon)} \geq 0, \tag{A55}$$

which proves the ν conjecture for this class of models.

It is interesting to note that Eq. (A54) can also be used to identify the conformal weight of the field φ. Indeed, if $\Delta_\varepsilon = 2h_{1,2}$, then we can find p, q such that $\Delta_\varphi = 2h_{p,q}$ only for even values of m: in this case $\Delta_\varphi = h_{m/2,m/2}$. On the other hand, if $\Delta_\varepsilon = 2h_{2,1}$, m should be odd and $\Delta_\varphi = h_{(m+1)/2,(m+1)/2}$. Note that relation (A54) also applies to the four-state Potts model (apart from logarithmic corrections). In this case the scaling dimensions are obtained by taking the limit $m \to \infty$.

b. The square-lattice Ashkin-Teller model

The ν conjecture (A1) also holds in the square-lattice Ashkin-Teller (AT). One considers two interacting sets of Ising spin variables, $\sigma_{1,i}$ and $\sigma_{2,i}$, with Hamiltonian

$$\begin{aligned} H_{\rm AT} \;=\; & -J\sum_{\langle ij\rangle}(\sigma_{1,i}\sigma_{1,j} + \sigma_{2,i}\sigma_{2,j}) \\ & -K\sum_{\langle ij\rangle}\sigma_{1,i}\sigma_{1,j}\sigma_{2,i}\sigma_{2,j}, \end{aligned} \tag{A56}$$

where the sum extends over the bonds $\langle ij\rangle$ of a square lattice. Note that the AT model can be formally associated with the 2D two-component LGW theory (A47) with cubic anisotropy. The 2D AT model presents a critical line $[J_c(K), K]$, from $K = -\infty$ to $K = \frac14 \ln 3$ (where the 2D AT model can be mapped into the 2D $q = 4$ Potts model) with $\sinh(J_c) = \exp(-2K)$. Along the line the exponent η is constant, $\eta = 1/4$, while the exponent ν varies continuously, as [101–105]

$$y_r(K) = 1/\nu(K) = \frac{4w-3\pi}{2(w-\pi)}, \quad \cos w = e^{2K}\sinh(2K). \tag{A57}$$

Therefore, along the critical line y_r monotonically increases from $y_r = 1/2$ for $K = -\infty$ to $y_r = 3/2$ for $K = \frac14 \ln 3$. Ccorrespondingly Σ monotonically decreases from $\Sigma = 5/4$ to $\Sigma = 1/4$, remaining always positive. It is interesting to observe that in the AT model the conjecture is rigorously valid, as a consequence of the results of Ref. [38], that proves $\gamma \geq 1$ for the susceptibility.

c. 2D O(N) σ models for $N \in [-2, 2]$

Exact results have also been obtained for the O(N) σ model for $N \in [-2, 2]$ [97–100]. Defining the parameter z through the relation

$$N = -2\cos(2\pi/z), \quad 1 \leq z \leq 2, \quad -2 \leq N \leq 2, \tag{A58}$$

then

$$y_r = 4 - 2z, \qquad \eta = \frac{-z^2+4z-3}{2z}, \tag{A59}$$

implying

$$\Sigma = 2 - \eta - y_r = \frac{5z^2-8z+3}{2z}, \tag{A60}$$

which is always positive for $z \geq 1$: it varies from zero for $N = -2$ ($z = 1$) to 7/4 for $N = 2$ ($z = 2$).

6. Models with fermionic fields

We now consider models with scalar fields coupled with fermionic, showing that the conjecture (A1) extend to these more complex field theories. We focus on the Gross-Neveu-Yukawa (GNY) model defined by the Hamiltonian density [6]

$$\begin{aligned} \mathcal{H}_{\rm GNY} \;=\; & (\partial_\mu\varphi)^2 + r\,\varphi^2 + u\,\varphi^4 \\ & - \sum_{f=1}^{N_f}\bar\psi_f(\partial\!\!\!/ + m + \varphi)\psi_f. \end{aligned} \tag{A61}$$

---

[4] In minimal models, there is a finite number of primary operators with scaling dimensions parametrized by [91, 92]

$$h_{p,q} = h_{m-p,m+1-q} = \frac{[(m+1)^2p - mq]^2 - 1}{4m(m+1).} \tag{A51}$$

Because of the symmetry, one needs only to consider $1 \leq p \leq m-1$ and $1 \leq q \leq p$.

where φ is a real scalar field and $\psi_f(\boldsymbol{x})$, $f = 1, \ldots, N_f$, is a fermionic field. Each *flavor* component ψ_f is a four-dimensional spinor, so that the total number N of fermionic components is given by $N = 4N_f$, and the matrices γ_μ are the usual Euclidean 4×4 matrices used in 4 dimensions [106]. As discussed in Ref. [6, 43], the GNY theory can be related to the Gross-Neveu model

$$\mathcal{H}_{\rm GN} = -\sum_{f=1}^{N_f} \bar{\psi}_f (\not{\partial} + m)\psi_f - \frac{g^2}{2N_f}\Big(\sum_{f=1}^{N_f} \bar{\psi}_f\psi_f\Big)^2, \quad \text{(A62)}$$

with attractive four-fermion interactions. The scalar field φ of the GNY model can be considered as an auxiliary real scalar field associated with the bilinear fermionic operator $\sum_f \bar{\psi}_f\psi_f$. In model (A61) the approach to criticality is controlled by the quadratic term of the scalar field, thus the operator $[\varphi^2]$ can be again identified as the energy-density operator. Therefore we can check the validity of the ν-conjecture also in this model.

We now report the known leading terms of the large-N_f expansion of the exponents $y_r = 1/\nu$ and η_φ for the d-dimensional GNY model [43]:[5]

$$\eta_\varphi = 4 - d - X(d)\frac{(d-1)}{dN_f} + O(N_f^{-2}), \quad \text{(A65)}$$

$$y_r = d - 2 - X(d)\frac{(d-2)(d-1)}{2dN_f} + O(N_f^{-2}). \quad \text{(A66)}$$

Therefore,

$$\Sigma = 2 - \eta_\phi - y_r = X(d)\frac{d-1}{2N_f} + O(N_f^{-2}), \quad \text{(A67)}$$

which is positive for any $2 < d < 4$.

An analogous analysis can be done for the chiral XY and Heisenberg GNY model, using the large-N_f results of Refs. [114, 115]. For the Heisenberg GNY model we have in three dimensions [115]

$$\eta_\varphi = 1 + \frac{4(3\pi^2+16)}{3\pi^4 N_f^2} + O(N_f^{-3}), \quad \text{(A68)}$$

$$y_r = 1 - \frac{4}{\pi^2 N_f} + \frac{9\pi^2 + 104}{3\pi^4 N_f^2} + O(N_f^{-3}), \quad \text{(A69)}$$

thus

$$\Sigma = 2 - \eta_\varphi - y_r = \frac{4}{\pi^2 N_f} + O(N_f^{-2}) > 0. \quad \text{(A70)}$$

[5] In three dimensions, higher-order terms also also known [43, 107–109] (see also Refs. [110–113] for comparisons with Monte Carlo estimates):

$$\eta_\varphi = 1 - \frac{16}{3\pi^2 N_f} - \frac{4(27\pi^2 - 304)}{27\pi^4 N_f^2} + O(N_f^{-3}), \quad \text{(A63)}$$

$$y_r = 1 - \frac{8}{3\pi^2 N_f} + \frac{4(27\pi^2 + 632)}{27\pi^4 N_f^2} + O(N_f^{-3}). \quad \text{(A64)}$$

Finally, we can use field theory to verify the conjecture close to four dimensions. In this limit the critical exponents of the GNY theory with Lagrangian (17) corresponding to the stable FP are [116, 117]

$$\eta_\varphi = \epsilon\frac{2N_f}{3+2N_f}, \quad \text{(A71)}$$

$$y_r = 2 - \epsilon\frac{10N_f + 3 + \sqrt{4N_f^2 + 132N_f + 9}}{6(3+2N_f)}.$$

Thus,

$$\Sigma = \epsilon\frac{3 - 2N_f + \sqrt{4N_f^2 + 132N_f + 9}}{6(3+2N_f)}, \quad \text{(A72)}$$

which is positive for any N_f. Analogous results can be obtained for the chiral XY and Heisenberg GNY models, using the results reported in Refs. [116, 117].

7. Model with gauge fields

The SU(N)-symmetric Abelian Higgs (AH) gauge field theory is obtained by minimally coupling an N-component complex scalar field $\boldsymbol{\phi}(\boldsymbol{x})$ with an electromagnetic U(1) gauge field $A_\mu(\boldsymbol{x})$. The Lagrangian density reads [6, 118]

$$\mathcal{L}_{\rm AH} = \mathcal{L}_{\rm AH}(\boldsymbol{\phi}, \boldsymbol{A}) + \mathcal{L}_{\rm gf}(\boldsymbol{A}), \quad \text{(A73)}$$

$$\mathcal{L}_{\rm AH} = \frac{1}{4g^2}F_{\mu\nu}^2 + |D_\mu\boldsymbol{\phi}|^2 + r\,\bar{\boldsymbol{\phi}}\cdot\boldsymbol{\phi} + u\,(\bar{\boldsymbol{\phi}}\cdot\boldsymbol{\phi})^2,$$

$$\mathcal{L}_{\rm gf}(\boldsymbol{A}) = \frac{1}{2\zeta}(\partial_\mu A_\mu)^2,$$

where the gauge-fixing term $\mathcal{L}_{\rm gf}$ is necessary to obtain a well-defined theory in perturbation theory. To go beyond perturbation theory, one must consider a nonperturbative regularization and show that the properly renormalized theory admits a finite limit when the regularization is eliminated. If we use the lattice regularization, this is possible only if the lattice model admits a continuous transition with a diverging length scale [46].

As in LGW theories, the exponent ν is associated with the quadratic scalar term, i.e., the energy-density operator $\varepsilon \sim [\bar{\boldsymbol{\phi}}\cdot\boldsymbol{\phi}]$. We compare its RG dimension $\Delta_\varepsilon = d - y_r$ with that of the Lagrangan field Δ_ϕ, i.e. $\Delta_\phi = (d - 2 + \eta_\phi)/2$. Note that the latter is not gauge invariant, but one can argue that the relevant value is that obtained in the Lorenz gauge $\zeta = 0$, where the complex field operator $\boldsymbol{\phi}(\boldsymbol{x})$ corresponds to a nonlocal gauge-invariant vector order-parameter field [46, 119–121]. We now compute the difference $\Sigma = 2 - y_r - \eta_\phi$ and show that it is positive at the stable FPs.

Close to four dimensions, a stable FP exists only for $N > N_*(4) = 90 + 24\sqrt{15} \approx 183$. At order ϵ the critical exponents are [118, 122]

$$\eta_\phi = -\epsilon\frac{6+3\zeta}{N}, \qquad (A74)$$
$$y_r = 2-\epsilon\frac{N^2+N-54+(N+1)\sqrt{N^2-180N-540}}{2N(N+4)}.$$

For $\zeta=0$ we obtain

$$\Sigma = \epsilon\frac{N^2+13N-6+(N+1)\sqrt{N^2-180N-540}}{2N(N+4)}, \qquad (A75)$$

which is positive for $N > N_*(4) = 90+24\sqrt{15}$ (actually, Σ positive for any value of the gauge-fixing parameter ζ).

The difference $\Sigma = 2-y_r-\eta_\phi$ of the 3D AH theory can be also computed in the large-N limit, using the large-N results reported in Refs. [118, 125, 126],

$$\eta_\phi = -\frac{20+8\zeta}{\pi^2 N}+O(N^{-2}), \qquad (A76)$$
$$y_r = 1+\frac{48}{\pi^2 N}+O(N^{-2}), \qquad (A77)$$

see also Refs. [46, 123] for comparisons with numerical Monte Carlo results. For $\zeta=0$ we obtain

$$\Sigma = 1-\frac{28}{\pi^2 N}+O(N^{-2}). \qquad (A78)$$

To obtain finite-N estimates in three dimensions one should consider the related lattice models, which admit several transition lines belonging to different universality classes. We focus here on the Coulomb-Higgs charged transitions that are controlled by the FPs of the AH field theory. These transitions are continuous for $N \geq N_*(3)$. Monte Carlo simulations indicate $N_*(3) = 7(2)$ [123]. Consistent estimates are obtained from field-theoretical analyses, $N_* = 12(4)$ [122], and from an analysis of the Rényi entanglement entropy, $N_* \approx 7$ [124]. The estimates of y_r and η_ϕ are close to the large-N estimates and satisfy the bound $\Sigma > 0$.

For $N < N_*(3)$ the Coulomb-Higgs transitions are first order, with the notable exception of the one-component AH model (relevent for superconductivity, see, e.g. Refs. [118, 127–129]). In this case the model presents a stable FP that is not connected with the stable FP found in the ϵ-expansion analysis. It belongs to the IXY universality class, related to the standard XY universality class by duality [130, 131] (more precisely, duality relates energy observables in the two models). Thus, the exponent ν is the same as in the XY model, $\nu = 0.6717$. On the other hand, the exponent η_ϕ differs from that of the standard XY universality class, as the order parameter in the AH model is a nonlocal gauge-invariant operator which is unrelated with the field in the standard XY model [46, 121]. Numerically, one finds $\eta_\phi = -0.74(4)$, so $\Sigma = 2-\eta_\phi-y_r = 1.25(4) > 0$.

8. Uniaxial systems with strong dipolar forces

Another class of interesting d-dimensional spin systems are those in which d-component spins $\boldsymbol{s_x}$ interact both through short range and dipolar forces, see for example Refs. [6, 10, 132–134]. Assuming that the lattice is strongly anisotropic in such a way that only one component of the spin $\boldsymbol{s_x}$ is critical, the corresponding effective field-theoretical Hamiltonian can be written as

$$\mathcal{H} = \int d^dq\,\phi(-\boldsymbol{q})\left(\boldsymbol{q}_\perp^2 + a_0^2\frac{q_z^2}{\boldsymbol{q}_\perp^2}+r\right)\phi(\boldsymbol{q}) \qquad (A79)$$
$$+u_0\int\prod_{i=1}^4[d^dq_i]\,\delta\left(\sum_{i=1}^4\boldsymbol{q}_i\right)\phi(\boldsymbol{q}_1)\phi(\boldsymbol{q}_2)\phi(\boldsymbol{q}_3)\phi(\boldsymbol{q}_4).$$

The upper critical dimention of this field theory is $d=3$. The RG flow can be investigated in $d<3$ by using an expansion in powers of $\epsilon_3 \equiv 3-d$ [6, 10, 69, 132, 134], obtaining

$$y_r = 2-\tfrac{1}{3}\epsilon_3+O(\epsilon_3^2), \qquad \eta = \frac{4}{243}\epsilon_3^2+O(\epsilon_3^3),$$
$$\gamma = 1+\tfrac{21}{6}\epsilon_3+O(\epsilon_3^2), \qquad \Sigma = \frac{1}{3}\epsilon_3+O(\epsilon_3^2). \qquad (A80)$$

Therefore, we find again that the ν conjecture holds.

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