# LOWER SEMICONTINUITY OF RESTRICTED HOLONOMY GROUPS

LINUS GÖTZFRIED

Abstract. A result on the monotony of holonomy groups of metric connections in limits is proven. The main outcome is that given a sequence of metric connections that are Lipschitz continuous and converging in $C^0$, if the connections in the sequence all have holonomy contained in a closed group $H$, then also the limit connection has holonomy contained in $H$. In particular, this finds an application for Riemannian holonomy groups, where it is proven that the map assigning to a Riemannian metric the conjugacy class of its restricted holonomy group is lower semicontinuous with respect to the $C^1$-topology on the space of metrics.

## 1. Introduction

An important notion in differential geometry is the one of *holonomy*. Given any connection on a principal or vector bundle and a basepoint, one can define the holonomy group of the connection as the group of endomorphisms of the fibres over this point given by parallel translation along piecewise-differentiable loops. Of particular interest are the holonomy groups of the Levi-Civita connection on a Riemannian manifold, which are subgroups of the special orthogonal group. In many cases, a "special" holonomy of a Riemannian manifold implies interesting consequences for the resulting metric. For example, a smooth $4m$-dimensional Riemannian manifold with holonomy contained in the group $\mathrm{Sp}(m)\mathrm{Sp}(1)$ is automatically an Einstein manifold (see e.g. [3], Theorem 14.39).

In this work, we study the changes of the holonomy group under changes of the connection. We focus on the case of metric connections on a vector bundle and prove a result of monotony of the holonomy group in limits, Theorem 2.4 below. Roughly speaking, it is shown that in the case of $C^0$-convergence of metric connections $\nabla^k$ to a metric connection $\nabla$, the connection $\nabla$ can only have a more special holonomy than the connections $\nabla^k$, but not a more generic one.

In particular, this finds an application for Riemannian holonomy groups. In this case, we also obtain a prettier (but slightly weaker) reformulation of the statement in terms of the lower semicontinuity of the map assigning to a $C^1$-Riemannian metric the conjugacy class of its restricted holonomy group.

The motivation and some of the ideas of this paper are taken from the recent preprint [6] by Müller. However, the proof of the main theorem is a different one than in the cited preprint, and the resulting theorem is a stronger one (note that in [6], $C^2$-convergence is required instead of only $C^1$-convergence).

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## 2. Monotony of the holonomy group in limits

We recall the definition of the holonomy group.

**Definition 2.1.** Let $M$ be a connected manifold of class $C^1$, let $x \in M$ and let $E \to M$ be an orientable vector bundle over $M$ with standard fibre $\mathbb{R}^l$, $l \in \mathbb{N}$. Let $\nabla$ be a connection on $E$.

(1) We denote by $\Omega_{\mathrm{p}-C^1}(x)$ the set of all piecewise-differentiable loops $c : [0,1] \to M$ based at $x$.
(2) Let $c \in \Omega_{\mathrm{p}-C^1}(x)$. We define the *parallel transport* $P_c^{\nabla}$ along $c$ (with respect to the connection $\nabla$ on $E$) as follows: Given any $e \in E_x$, there exists a unique section $s$ of $E|_{c([0,1])}$ with $\nabla_{\dot{c}(t)} s(t) = 0$ for all $t \in [0,1]$, and we let $P_c^{\nabla}(e) := s(1)$. Then
$$\mathrm{Hol}_x(\nabla) := \left\{ P_c^{\nabla} \,\middle|\, c \in \Omega_{\mathrm{p}-C^1}(x) \right\}$$
forms a subgroup of $\mathrm{GL}(E_x)$, the *holonomy group* of $\nabla$ (at $x$).
(3) We define the *restricted holonomy group* of $\nabla$ at $x$,
$$\begin{aligned}\mathrm{Hol}_x^0(\nabla) := &\left\{ P_c^{\nabla} \,\middle|\, c \in \Omega_{\mathrm{p}-C^1}(x) \text{ such that } c \text{ is contractible} \right\} \\ &\subseteq \mathrm{Hol}_x(\nabla).\end{aligned}$$
Both $\mathrm{Hol}_x(\nabla)$ and $\mathrm{Hol}_x^0(\nabla)$ are Lie subgroups of $\mathrm{GL}(E_x)$. It can be shown (see e.g. [1], Section 2, Lemma 4) that $\mathrm{Hol}_x^0(\nabla)$ is the connected component of the identity in $\mathrm{Hol}_x(\nabla)$.
(4) Choosing any basis of $E_x$, we may identify $E_x$ with $\mathbb{R}^l$ and consequently view $\mathrm{Hol}_x(\nabla)$ and $\mathrm{Hol}_x^0(\nabla)$ as subgroups of $\mathrm{GL}(l)$. However, this is only up to conjugation; the choice of different bases yields subgroups conjugate in $\mathrm{GL}(l)$.

Note that since any $C^1$-map is in particular Lipschitz, a manifold of class $C^1$ is a *Lipschitz manifold*, that is, the transition maps of its defining atlas are bi-Lipschitz ([8], Definition 1). It is well-defined to speak of locally Lipschitz continuous functions on such a manifold.

In the present work, we will be concerned with the following setting.

**Situation 2.2.**

- Let $M$ be a connected manifold of class $C^1$, let $x \in M$ and $E \to M$ be an orientable vector bundle with standard fibre $\mathbb{R}^l$. We denote by $E^*$ the dual bundle of $E$ and by $\mathrm{S}^2E^*$ the symmetrized tensor product of the dual bundle.
- Let $g \in C^1(M, \mathrm{S}^2E^*)$ be a bundle metric on $E$ (that remains fixed throughout this section unless otherwise stated). Let $\nabla^g$ be a connection on $E$ whose coefficients are locally Lipschitz continuous and that is compatible with $g$, i. e. $X(g(v,w)) = g(\nabla^g_X v, w) + g(v, \nabla^g_X w)$ for all $X \in TM$, $v, w \in C^1(M,E)$.
- Given any metric $h$ on $E$, let
$$\begin{aligned}\mathrm{SO}_l(E_x)_h := \{A \in \mathrm{End}(E_x) \mid{}& \det(A) = 1, \\ &\forall v, w \in E_x : h|_x(Av, Aw) = h|_x(v,w)\}\end{aligned}$$
be the special orthogonal group of $E_x$ with respect to the metric $h|_x$. Then, since $\nabla^g$ is metric, we have $P_c^{\nabla^g} \in \mathrm{SO}_l(E_x)_g$ for all $c \in \Omega_{\mathrm{p}-C^1}(x)$.

- An orthonormal basis of $E_x$ with respect to some metric $h$ are vectors $e_1, \ldots, e_l$ with
$$\forall i, j = 1, \ldots, n : h(e_i, e_j) = \delta_{ij},$$
where $\delta_{ij}$ is the Kronecker delta. We choose a positively oriented orthonormal basis $(b_1, \ldots, b_l)$ of $E_x$ with respect to the metric $g$ and use it to identify $E_x$ with $\mathbb{R}^l$. This then identifies $\mathrm{SO}_l(E_x)_g$ with $\mathrm{SO}(l) \subset \mathrm{GL}(l)$ and consequently we may view $\mathrm{Hol}_x(\nabla^g)$ as an explicit subgroup of $\mathrm{SO}(l)$.

Now, given a (possibly different) metric $h \in C^1(M, \mathrm{S}^2E^*)$, we may still view $\mathrm{SO}_l(E_x)_h$ as a distinguished subgroup of $\mathrm{GL}(l)$, as follows:

**Definition 2.3.**

(1) Using the basis $(b_1, \ldots, b_l)$ of $E_x$, we may view all bilinear forms $Q \in \mathrm{S}^2E_x^*$ and all endomorphisms $Z \in \mathrm{End}(E_x)$ as $l \times l$-matrices over $\mathbb{R}$. In particular, we may associate to a bilinear form $Q$ its representing matrix $M_Q$ and via this an endomorphism $Z_Q$ of $E_x \cong \mathbb{R}^l$.[1] This defines an embedding $\mathrm{S}^2E_x^* \to \mathrm{End}(E_x)$, $Q \mapsto Z_Q$. Now $\mathrm{End}(E_x)$ can be turned into a normed vector space by using the operator norm induced by the metric $g$. By pullback along the embedding described above, also $\mathrm{S}^2E_x^*$ becomes a normed vector space. Unless otherwise stated, in the following these norms will be used. (As all vector spaces involved are finite-dimensional, the resulting topologies do not depend on $g$ or the basis.)

(2) Let $h \in C^1(M, \mathrm{S}^2E^*)$ with $h|_x \in B_1(g|_x)$. Let $M_h$ be the representing matrix of $h|_x$ and $Z_h$ the corresponding endomorphism as in the previous point. We define
$$\mathrm{SO}(l)_h := \{A \in \mathrm{GL}(l) \mid A^T M_h A - M_h = 0\}$$
(the $x$-dependency is understood). Let $W_h$ be a symmetric positive definite square root of $Z_h$ as defined by the power series of $\sqrt{1+c}$ in $B_1(0)$. Then, the map $\mathrm{SO}(l) \to \mathrm{SO}(l)_h, A \mapsto W_h^{-1} A W_h$ is an isomorphism.[2] Indeed this follows from the following matrix computation: If $A \in \mathrm{SO}(l)$, then
$$\begin{aligned}(W_h^{-1}AW_h)^T h|_x (W_h^{-1}AW_h) - M_h \\ &= W_h^T A^T W_h^{-T} W_h^T W_h W_h^{-1} A W_h - M_h \\ &= W_h^T A^T A W_h - M_h \\ &= W_h^T W_h - M_h = M_h - M_h = 0,\end{aligned}$$
thus $W_h^{-1}AW_h \in \mathrm{SO}(l)_h$; the inverse isomorphism is simply given by $A \mapsto W_h A W_h^{-1}$. We call this isomorphism the *nonstandard embedding* $\phi_h : \mathrm{SO}(l) \to \mathrm{SO}(l)_h \subset \mathrm{GL}(l)$. By the description of $W_h$ in terms of a power series in $h$, we obtain that if we have a sequence $(h_k)_{k\in\mathbb{N}} \subset C^1(M, \mathrm{S}^2E^*)$ with $h_k|_x \in B_1(g|_x) \cap \mathrm{S}^2E_x^*$ for all $k$ and $h_k|_x \to g|_x$ in $\mathrm{S}^2E_x^*$, then $W_{h_k} \to I_l$ (the $l \times l$-identity matrix).

(3) Given any two connections $\nabla, \tilde{\nabla}$ on $E$ with continuous coefficients, its difference $\nabla - \tilde{\nabla}$ is a continuous section of $T^*M \otimes E \otimes E^*$. We equip the

[1] To be very explicit, therefore $Q(X,Y) = g(X, Z_Q Y)$ for all $X, Y \in E_x$, and if we view $X, Y$ also as their respective coordinate vectors with respect to the basis $(b_1, \ldots, b_l)$, then this is nothing but $X^T M_Q Y$.

[2] Here we again use the basis $(b_1, \ldots, b_l)$ of $E_x$ to identify $E_x$ with $\mathbb{R}^l$, $\mathrm{SO}(l)$ with a matrix group, $Z_h$ with $M_h$ and $W_h$ also with an explicit $l \times l$-matrix over $\mathbb{R}$.

space $\Gamma(T^*M\otimes E\otimes E^*)$ of such sections with the *compact-open $C^r$-topology*, $r=0$, which is defined as follows ([4], Chapter 2.1): Given any two charts $\mathbf{x}: U\to V, \mathbf{v}: U'\to V'$ of $M$ respectively $T^*M\otimes E\otimes E^*$, a compact $K\subseteq U$, $\epsilon>0$ and $f\in\Gamma(T^*M\otimes E\otimes E^*)$, we let

$$B(f,\mathbf{x},\mathbf{v},K,\epsilon):=\Big\{g\in\Gamma(T^*M\otimes E\otimes E^*)\,|\,g(K)\subseteq V,\forall 0\le k\le r: \\ \big\|D^k(\mathbf{v}\circ f\circ\mathbf{x}^{-1})-D^k(\mathbf{v}\circ g\circ\mathbf{x}^{-1})\big\|_{C^0}<\epsilon\Big\}.$$

The set of all such $B(f,\mathbf{x},\mathbf{v},K,\epsilon)$ has the basis property and thus defines a topology on $\Gamma(T^*M\otimes E\otimes E^*)$. Analogously, we define compact-open $C^r$-topologies, $r\ge 0$, on the spaces of sufficiently often differentiable sections of other bundles (if $M$ is of class at least $C^r$).

Using this, we can speak of $C^0$-convergence in $\Gamma(T^*M\otimes E\otimes E^*)$, and we say that a sequence of connections $\nabla^k$ converges in $C^0$ to a connection $\nabla$ on $E$ if their difference converges to zero. If this is the case, then the restrictions of $\nabla^k-\nabla$ to any compact $K\subseteq M$ converge to zero uniformly on $K$ (with respect to any set of charts covering $K$).

Now, the announced result can be stated.

**Theorem 2.4.** *Assume Situation 2.2 and let $H\subseteq \mathrm{SO}(l)$ be closed. Moreover, for $k\in\mathbb{N}$, let $g_k\in C^1(M,\mathrm{S}^2E^*)$ be a bundle metric on $E$ and $\nabla^k$ be a connection with locally Lipschitz continuous coefficients compatible with $g_k$, such that $g_k|_x\to g|_x$ in $\mathrm{S}^2E^*_x$ and $\nabla^k-\nabla^g\to 0$ in the compact-open $C^0$-topology on $\Gamma(M,T^*M\otimes E^*\otimes E)$.*

*Assume that for all $k\in\mathbb{N}$, there exists a positively oriented $g_k$-orthonormal basis of $E_x$ such that if we use this basis to identify $E_x$ with $\mathbb{R}^l$, we have $\mathrm{Hol}_x(\nabla^k)\subseteq H$. Then, there exists a positively oriented $g$-orthonormal basis $(c_1,\dots,c_l)$ of $E_x$ such that if we use this basis to identify $E_x$ with $\mathbb{R}^l$, we have $\mathrm{Hol}_x(\nabla^g)\subseteq H$.*

*Proof of Theorem 2.4.* Note first that the conclusion of the theorem may be reformulated as follows: If we use our pre-chosen basis $(b_1,\dots,b_l)$ to identify $E_x$ with $\mathbb{R}^l$, then there exists $U\in\mathrm{SO}(l)$ such that $U\mathrm{Hol}_x(\nabla^g)U^{-1}\subseteq H$.[3] Therefore, we fix once and for all the basis $(b_1,\dots,b_l)$ and use it to identify $E_x$ with $\mathbb{R}^l$.

We may assume that $g_k|_x\in B_1(g|_x)\cap \mathrm{S}^2E^*_x$ for all $k\in\mathbb{N}$ by dropping a finite number of elements from the sequence, hence we may define the matrices $W_{g_k}$, the subgroups $\mathrm{SO}(l)_{g_k}$ and the nonstandard embedding $\phi_{g_k}$ for $k\in\mathbb{N}$ as in Definition 2.3. Furthermore, we denote by $H_{g_k}$ the image of the composition of the inclusion $H\hookrightarrow\mathrm{SO}(l)$ and the nonstandard embedding $\mathrm{SO}(l)\to\mathrm{SO}(l)_{g_k}$.

The assumption then means that for all $k\in\mathbb{N}$ there exists an $U_k\in\mathrm{SO}(l)_{g_k}$ such that $U_k\mathrm{Hol}_x(\nabla^k)U_k^{-1}\subseteq H_{g_k}$.[4] Now, the sequence $\phi_{g_k}^{-1}(U_k)$ is contained in the compact group $\mathrm{SO}(l)$ and hence possesses a convergent subsequence. Moreover, using the explicit description of $\phi_{g_k}$ in terms of conjugation by matrices converging to the identity matrix, we see that if $(\phi^{-1}_{g_{k_m}}(U_{k_m}))_{m\in\mathbb{N}}$ converges to some $U\in\mathrm{SO}(l)$, then also $(U_{k_m})_{m\in\mathbb{N}}$ converges to $U$.

Therefore, passing to the subsequence without changing the notation for brevity, we may assume that $(U_k)_{k\in\mathbb{N}}$ converges to some $U\in\mathrm{SO}(l)$. At this point, it suffices

[3] Indeed: It suffices to take $U$ as the matrix such that $\sum_{k=1}^l U_{ki}c_k=b_i$ for $i=1,\dots,l$.

[4] Note that since the connection $\nabla^k$ is metric with respect to $g_k$, we have $\mathrm{Hol}_x(\nabla^k)\subseteq\mathrm{SO}(l)_{g_k}$ for all $k\in\mathbb{N}$.

to prove the following claim:

*Claim.* We have $U\mathrm{Hol}_x(\nabla^g)U^{-1} \subseteq H$.

*Proof of the claim.* Let $c \in \Omega_{\mathrm{p}-C^1}(x)$. By definition of the compact-open topology, the restrictions of the connections to the (compact) image of $c$ converge uniformly in $\Gamma((T^*M \otimes E^* \otimes E)|_{c([0,1])})$. Therefore, the coefficients of the parallel transport equation defining $P_c^{\nabla^k}$ converge uniformly to the coefficients of the parallel transport equation defining $P_c^{\nabla^g}$. As the solutions to ordinary differential equations depend continuously on their coefficients, we have $\lim\limits_{k\to\infty} P_c^{\nabla^k} = P_c^{\nabla^g}$.[5] Thus, also

$$UP_c^{\nabla^g}U^{-1} = \lim_{k\to\infty} U_k P_c^{\nabla^k} U_k^{-1} = \lim_{k\to\infty} W_{g_k} U_k P_c^{\nabla^k} U_k^{-1} W_{g_k}^{-1}.$$

Since $U_k P_c^{\nabla^k} U_k^{-1} \in H_{g_k}$ respectively $W_{g_k} U_k P_c^{\nabla^k} U_k^{-1} W_{g_k}^{-1} \in H$ for all $k \in \mathbb{N}$, we obtain $UP_c^{\nabla^g}U^{-1} \in H$ by continuity of matrix multiplication. This proves the claim and hence the theorem. ☐

*Remark* 2.5. The theorem also holds with the holonomy groups replaced by the restricted holonomy groups, with the same proof.

## 3. Application to Riemannian holonomy groups

The most useful use case of Theorem 2.4 is probably the case of oriented Riemannian manifolds and their Levi-Civita connection.

**Definition 3.1.** Let $M$ be a manifold of class $C^1$.

(1) We say that a map $f : M \to X$ into a normed vector space $X$ or a section $f : M \to X$ of a vector bundle with norm over $M$ is *locally Lipschitz-$C^1$*, if $f$ is of class $C^1$ and $df : TM \to TX$ is locally Lipschitz continuous.
(2) Let $\mathrm{Met}(M)$ be the set of all locally Lipschitz-$C^1$-Riemannian metrics on $M$, equipped with the compact-open $C^1$-topology.
(3) Given a Riemannian metric $h \in \mathrm{Met}(M)$ with Levi-Civita connection $\nabla^h$ and $x \in M$, we denote $\mathrm{Hol}_x(h) := \mathrm{Hol}_x(\nabla^h)$ and $\mathrm{Hol}_x^0(h) := \mathrm{Hol}_x^0(\nabla^h)$.

Now assume that we have a sequence of locally Lipschitz-$C^1$ Riemannian metrics converging in $C^1$. Then the condition of convergence $\nabla^k \to \nabla^g$ is automatically satisfied, as well as the condition on Lipschitz continuity of the connections, and we get the following corollary.

**Corollary 3.2.** *Let $(M, g)$ be an oriented $n$-dimensional Riemannian manifold of class $C^1$, let $g \in \mathrm{Met}(M)$ and for $k \in \mathbb{N}$, let $g_k \in \mathrm{Met}(M)$. Let $H \subseteq \mathrm{SO}(n)$ be a closed subgroup and let $x \in M$. Assume that $g_k \to g$ in $\mathrm{Met}(M)$ and for all $k \in \mathbb{N}$, we have $\mathrm{Hol}_x(g_k) \subseteq H$ up to conjugation in $\mathrm{SO}(n)$. Then, $\mathrm{Hol}_x(g) \subseteq H$ up to conjugation in $\mathrm{SO}(n)$.*

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[5] See e. g. [7] Satz 4.1.2; to be precise, the "continuous dependence on the coefficients" is the statement that if the coefficients converge in $C^0$ (and are all locally Lipschitz continuous), then the solutions converge in $C^0$. Note that we can assume that the coefficients of the parallel transport equations along $c$ are continuous since we can parametrize $c$ such that it has vanishing speed at the points where its parametrization by arc-length would not be differentiable. Moreover, then the pullback of the locally Lipschitz continuous coefficients of $\nabla^k$ along the differentiable path $c$ is again locally Lipschitz continuous.

As a further corollary, we get:

**Corollary 3.3.** *Let $M$ be an oriented $n$-dimensional Riemannian manifold of class $C^1$, let $H \subseteq \mathrm{SO}(n)$ be a closed subgroup and let $x \in M$. Then, the set*

$$\{h \in \mathrm{Met}(M) \,|\, \mathrm{Hol}_x(h) \subseteq H \text{ up to conjugation}\}$$

*is closed in* $\mathrm{Met}(M)$.

A useful consequence of Theorem 2.4 (respectively, the variant for the restricted holonomy groups) is the observation that the map assigning to a Riemannian metric the conjugacy class of the restricted holonomy group of its Levi-Civita connection is lower semicontinuous with respect to the $C^1$-topology and the inclusion order relation. The proof will follow as a straightforward application of Theorem 2.4, once it has been shown that this statement is actually well-defined.

**Definition 3.4.**

(1) Let $X$ be a topological space and let $(Y, \leq)$ be a partially ordered set. We say that a function $f : X \to Y$ is *lower semicontinuous at* $x \in X$ if there exists a neighbourhood $U$ of $x$ such that $f(x) \leq f(x')$ for all $x' \in U$. We say that $f : X \to Y$ is *lower semicontinuous*, if it is lower semicontinuous at all $x \in X$.

(2) Let $G$ be a compact Lie group and let $\mathrm{CSC}(G)$ be the set of closed subgroups of $G$ modulo the conjugation equivalence relation
$$A \sim B :\Leftrightarrow \exists_{g\in G} : gAg^{-1} = B.$$
Let "$\leq$" be the relation on $\mathrm{CSC}(G)$ defined by inclusion of representatives, that is
$$[A] \leq [B] :\Leftrightarrow \exists_{A\in[A],B\in[B],g\in G} : gAg^{-1} \subseteq B.$$

The relation "$\leq$" on the set $\mathrm{CSC}(G)$ defined as above (for $G$ any compact Lie group) will be shown to define a partial order on $\mathrm{CSC}(G)$. Hence it is then well-defined to speak of lower semi-continuous functions to $\mathrm{CSC}(G)$. To prove antisymmetry, the following Cantor-Bernstein-like theorem is needed, whose proof has its origins in the one from the article [6] by Müller.

**Theorem 3.5.**

(1) *Let $K, H$ be Lie groups with finitely many connected components. Let $f : K \to H$ and $g : H \to K$ be injective Lie group homomorphisms. Then $f$ and $g$ are Lie group isomorphisms.*

(2) *Let $K, H$ be closed Lie subgroups of a compact Lie group $G$. If there exist $g, g' \in G$ such that $gKg^{-1} \subseteq H$ and $g'Hg'^{-1} \subseteq K$, then in fact $gKg^{-1} = H$ and $g'Hg'^{-1} = K$.*

*Proof. Ad 1.* Let $K, H$ be Lie groups with finitely many connected components. Let $f : K \to H$ and $g : H \to K$ be injective Lie group homomorphisms. We prove that $f$ is a Lie group isomorphism, the proof for $g$ is analogous.

At first, we prove that the linear map $d_1 f : T_1K \to T_1H$ is injective. This can be seen as follows: It is well known that for the exponential maps $\exp_K$ and $\exp_H$ of $K$ resp. $H$ we have $\exp_H \circ d_1 f = f \circ \exp_K$. Furthermore, both exponential maps are local diffeomorphisms, in particular injective if restricted to a suitable neighbourhood of zero in the respective tangent spaces.

Now let $v \in T_1K$ with $d_1f(v) = 0$. Let $U \subseteq T_1K$ be a neighbourhood of zero such that $\exp_K|_U$ is a diffeomorphism onto its image. In particular, $(f \circ \exp_K)|_U$ is injective by assumption on $f$. Let furthermore $\lambda \in (0, \infty)$ be so small that $w := \lambda v \in U$. By linearity we have $d_1f(w) = 0$. Thus

$$f(\exp_K(w)) = \exp_H(d_1f(w)) = \exp_H(0) = 1 = f(\exp_K(0))$$

and therefore $w = 0$ by injectivity of $(f \circ \exp_K)|_U$. By linearity of $d_1f$, also $v = 0$.

Thus $d_1f$ is injective, and analogously one shows that $d_1g : T_1H \to T_1K$ is also injective. By the Cantor-Bernstein theorem in the category of finite-dimensional vector spaces, both tangent spaces have the same dimension and both linear maps are isomorphisms.

Now let $H_0, K_0$ be the connected component of $1 \in H_0$ and $1 \in K_0$, respectively. It is well known that $\exp_H$ surjects onto $H_0$; since furthermore $d_1f$ is surjective by the previous paragraph and by the formula $\exp_H \circ d_1f = f \circ \exp_K$ again, one concludes that $H_0 \subseteq f(K_0)$. Thus $f|_{K_0} : K_0 \to H_0$ is an isomorphism. The map $f$ induces a map $\overline{f} : K/K_0 \to H/H_0$ which fits into the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
K_0 & \longrightarrow & K & \longrightarrow & K/K_0 & \longrightarrow & 1 \\
\big\downarrow{\scriptstyle f|_{K_0}} & & \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle \overline{f}} & & \big\downarrow \\
H_0 & \longrightarrow & H & \longrightarrow & H/H_0 & \longrightarrow & 1
\end{array}$$

(the horizontal maps are the obvious ones). In this diagram, the first vertical map is therefore surjective and the second and fourth vertical maps are injective. By one of the four-lemmas, the third vertical map is hence injective. Analogously, one proves that the map $\overline{g} : H/H_0 \to K/K_0$ induced by $g$ is injective. Now $H/H_0$ and $K/K_0$ are in bijection with the sets of connected components of $H$ and $K$, thus they are finite. Therefore, $\overline{f}$ and $\overline{g}$ must be isomorphisms.

Thus for every $h \in H$, there exist $k_1 \in K$ and $h_0 \in H_0$ such that $h = f(k_1)h_0$. On the other hand, due to surjectivity of $f|_{K_0} : K_0 \to H_0$, there exists $k_0 \in K_0$ with $h_0 = f(k_0)$, and then $h = f(k_1)f(k_0) = f(k_1k_0)$ lies in the image of $f$. One concludes that $f$ is surjective, thus bijective.

Now to finish the proof, it suffices to note that $f^{-1}$ is also a Lie group homomorphism, which is well known and follows from the fact that $f$ induces an isomorphism of the Lie algebras as well and an application of the implicit function theorem.

*Ad 2.* Let $K, H$ be closed Lie subgroups of a compact Lie group $G$ and $g, g' \in G$ with $gKg^{-1} \subseteq H$ and $g'Hg'^{-1} \subseteq K$.We can view $K, H$ as Lie groups for their own right; since they are closed in the compact set $G$, they are compact as well. In particular, they have only finitely many connected components. Now part 2 follows from part 1, applied to the injective Lie group homomorphisms $K \to H, k \mapsto gkg^{-1}$ and $H \to K, h \mapsto g'hg'^{-1}$. □

**Corollary 3.6.** *Let $G$ be a compact Lie group. The relation "$\leq$" on* $\mathrm{CSC}(G)$ *from Definition 3.4 is a partial order on* $\mathrm{CSC}(G)$.

*Proof.* Reflexivity of the relation "$\leq$" is clear. Transitivity follows from the fact that if there exist closed subgroups $A, B, B', C$ of $G$ and $g_1, g_2, g \in G$ with $g_1Ag_1^{-1} \subseteq B$,

$gBg^{-1} = B'$ and $g_2 B' g_2^{-1} \subseteq C$, then $(g_2 g g_1) A (g_2 g g_1)^{-1} \subseteq C$. Finally, antisymmetry of the relation "$\leq$" follows directly from Theorem 3.5, part 2. $\square$

Now the announced reformulation of Theorem 2.4 for restricted Riemannian holonomy groups can be stated.

**Theorem 3.7.** *Let $M$ be an oriented $n$-dimensional manifold of class $C^1$ and let $x \in M$. The map*

$$\mathrm{Met}(M) \to \mathrm{CSC}(\mathrm{SO}(n)), \qquad g \mapsto [\mathrm{Hol}_x^0(g)],$$

*is lower-semicontinuous.*

*Proof.* It is well known (cf. e.g. [5], Theorem 3.2.8) that for any $h \in \mathrm{Met}(M)$, the group $\mathrm{Hol}_x^0(h)$ is closed in $\mathrm{SO}(n)$. Hence the map from the statement of the theorem is actually well-defined. Now the statement follows from Theorem 2.4. $\square$

Theorem 2.4 and Theorem 3.7 are in a certain sense rigidity results: As already stated in the introduction, the holonomy can only become more special, not more generic. However, full continuity of the map from Theorem 3.7 is in general not true. For example, for Ricci-flat deformations of a Ricci-flat metric the conjugacy class of the holonomy group is known to be constant if the universal covering of $M$ admits a parallel spinor (see [2]). In more general cases, one can easily construct explicit counterexamples.

**Example 3.8.** Consider subsets of scaled Poincaré disk models, $M := B_{1/2}(0) \subset \mathbb{R}^2$ equipped with the family of metrics $g_k(x^1, x^2) := \frac{4}{(1-((x^1)^2+(x^2)^2)/k^2)^2}\, dx^1 \otimes dx^2$ in cartesian coordinates. These converge in $\mathrm{Met}(M)$ to the limit metric $g(x^1, x^2) := 4\, dx^1 \otimes dx^2$.

For any $k \in \mathbb{N}$, $(M, g_k)$ is up to scaling describable as a subset of the hyperbolic space of dimension 2, which is the symmetric space $\mathrm{SO}_0(1,2)/(\{1\} \times \mathrm{SO}(2))$, where $\{1\} \times \mathrm{SO}(2)$ is the isotropy subgroup (cf. [3], Chapter 10.K, Table 3). Hence, as is well known, $\mathrm{Hol}_x(g_k) = \mathrm{Hol}_x^0(g_k) = \{1\} \times \mathrm{SO}(2) \cong \mathrm{SO}(2)$ for all $x \in M$ (cf. [3], Proposition 10.79). However $\mathrm{Hol}_x^0(g) = \{I_2\} \subset \mathrm{SO}(2)$ because $g$ is flat.

It is also easy to perturb a Calabi-Yau metric without leaving the realm of Kähler metrics, that is, obtain a sequence of metrics with holonomy $\mathrm{U}(n)$ converging to a metric with holonomy $\mathrm{SU}(n)$:

**Example 3.9.** Let $M$ be the Fermat quartic, which is a K3 surface in $\mathbb{C}\mathrm{P}^3$. Let $\omega$ be the Kähler form of some Kähler metric on $M$. Let $\alpha \in \ker(\partial) \setminus \ker(\overline{\partial}) \subset C^\infty(M, \Lambda^{1,0} T^* M)$.[6] Let $\rho := \overline{\partial}\alpha = d\alpha \in C^\infty(M, \Lambda^{1,1} T^* M)$; by construction $\rho$ represents zero in cohomology.

By the Calabi conjecture (solved by Yau in [11], [12]), there exists a unique Kähler metric $g_1$ on $M$ with Ricci form $\rho$ and such that its Kähler form is contained in the cohomology class of $\omega$. Moreover, there exists a unique Ricci-flat Kähler metric $g_\infty$ on $M$ with Kähler form contained in the cohomology class of $\omega$. Let $g_k := \frac{1}{k} g_1 + \frac{k-1}{k} g_\infty$ for $k \in \mathbb{N}_{\geq 2}$. Then, the metrics $g_k$ are Kähler since

[6] For example, let $h \in C^\infty(M, \mathbb{R})$ be such that $\overline{\partial}\partial h$ is not identically zero; such a function can be defined in a chart. Then, define $\alpha := \partial h$.

convex combinations of Kähler metrics are again Kähler, and the sequence $(g_k)_{k\in\mathbb{N}}$ converges in $\mathrm{Met}(M)$ to $g_\infty$. Moreover, for all $k \in \mathbb{N}$ the metric $g_k$ is not Ricci-flat.[7]

Thus, the metrics $g_k$ all have holonomy group contained in $\mathrm{U}(2)$ (by e. g. [5], proposition 4.4.2), but not in $\mathrm{SU}(2)$ (see e. g. [5], proposition 6.1.1). However, $g_\infty$ is Ricci-flat and hence has its holonomy group contained in $\mathrm{SU}(2) \subset \mathrm{U}(2)$.

As a final example, it shall be noted that at least on incomplete manifolds, one can also obtain quaternion-Kähler-metrics (with holonomy in $\mathrm{Sp}(m)\mathrm{Sp}(1)$) converging to a hyperkähler metric (with holonomy in $\mathrm{Sp}(m)$). This is due to Swann ([9]).

**Example 3.10.** Let $N$ be a quaternion-Kähler manifold of dimension $4m$ with positive scalar curvature. Let $M$ be the Swann bundle over $N$, which is an incomplete manifold fibring over $N$ with fibre $(\mathbb{H}\backslash\{0\})/\mathbb{Z}_2$ that has been constructed in [9], Section 2.1. In Theorem 2.1.7 of [9], it is shown that $M$ admits an family of Riemannian quaternion-Kähler metrics $g_k$ converging to a hyperkähler metric $g_\infty$ in the limit where the scalar curvature tends to zero. Hence $[\mathrm{Hol}^0_x(g_\infty)] \leq [\mathrm{Sp}(m+1)]$ is strictly less than all $[\mathrm{Hol}^0_x(g_k)]$ with respect to the order relation on $\mathrm{CSC}(\mathrm{SO}(4m+4))$ from Definition 3.4.[8] Moreover, the metrics $g_k$ and $g_\infty$ can be described explicitly and from this description it follows that the convergence is also with respect to the compact-open $C^1$-topology. Thus the map from Theorem 3.7 is not upper semicontinuous at $g_\infty$.

On the other hand, it would be impossible to construct an example as the previous one on a closed manifold: On a compact manifold $M$ of dimension $n$, all Einstein metrics are stationary points of the Einstein-Hilbert functional $S : \mathrm{Met}(M) \to \mathbb{R}$, $g \mapsto \frac{\int_M \mathrm{scal}^g \, \mathrm{dvol}^g}{\left(\int_M \mathrm{dvol}^g\right)^{(n-2)/n}}$ (where $\mathrm{scal}^g$ is the scalar curvature of the metric $g$ and $\mathrm{dvol}^g$ is its volume form). In particular, one sees from this that it is impossible to deform Ricci-flat Einstein metrics (such as hyperkähler ones) on a compact manifold to non-Ricci-flat Einstein metrics (such as quaternion-Kähler ones), since $S$ needs to remain constant along any deformation which does not leave the space of Einstein metrics. Contrarily, Theorem 3.7 holds without any compactness assumption on the manifold.

To conclude this paper, it shall be remarked that one would assume that the conditions on Theorem 2.4 can actually be weakened. The most apparent restriction is the one to connections compatible with positive definite metrics, which makes the theorem, e.g., inapplicable in the Lorentzian setting. Moreover, it would be interesting to see whether Theorem 3.7 also holds for the full holonomy group (which is, as is known due to an example made by Wilking in [10], in general not closed in the special orthogonal group).

However, the relaxation of these conditions would require entirely different proofs. It would be interesting to study such generalizations.

[7]If it were, it would contradict the uniqueness statement from the Calabi conjecture: The metrics $g_k$ also all have Kähler forms contained in the cohomology class of $\omega$. However, there exists only a unique such one which is Ricci-flat, namely $g_\infty$.

[8]Note that $[\mathrm{Hol}^0_x(g_k)]$ is not less than $[\mathrm{Sp}(m+1)]$ for all $k \in \mathbb{N}$ since the metrics $g_k$ have nonzero scalar curvature and hence can not be hyperkähler ([3], Theorem 14.45).

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L. Götzfried, Universität Regensburg, 93040 Regensburg, Germany
*Email address*: `linus.goetzfried@gmx.de`