HITTING ALL LONGEST PATHS IN H-FREE GRAPHS AND H-GRAPHS

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Abstract. The *longest path transversal number* of a connected graph G, denoted by $\mathsf{lpt}(G)$, is the minimum size of a set of vertices of G that intersects all longest paths in G. We present constant upper bounds for the longest path transversal number of *hereditary classes of graphs*, that is, classes of graphs closed under taking induced subgraphs.

Our first main result is a structural theorem that allows us to *refine* a given longest path transversal in a graph using domination properties. This has several consequences:

- First, it implies that for every $t \in \{5, 6\}$, every connected P_t-free graph G satisfies $\mathsf{lpt}(G) \leq t - 2$, allowing to answer a question of Gao and Shan (2019).
- Second, it shows that every (*bull*, *chair*)-free graph G satisfies $\mathsf{lpt}(G) \leq 5$, where a *bull* is the graph obtained from a four-vertex path by adding a vertex adjacent to the two middle vertices of the path, and a *chair* is the graph obtained by subdividing an edge of a claw ($K_{1,3}$) exactly once.
- Third, it implies that for every $t \in \mathbb{N}$, every connected chordal graph G with no induced subgraph isomorphic to $K_t \boxminus \overline{K_t}$ satisfies $\mathsf{lpt}(G) \leq t - 1$, where $K_t \boxminus \overline{K_t}$ is the graph obtained from a t-clique and an independent set of size t by adding a perfect matching between them.

Our second main result provides an upper bound for the longest path transversal number in *H-intersection graphs*. For a given graph H, a graph G is called an *H-graph* if there exists a subdivision H' of H such that G is the intersection graph of a family of vertex subsets of H' that each induce connected subgraphs. The concept of H-graphs, introduced by Biró, Hujter, and Tuza, naturally captures interval graphs, circular-arc graphs, and chordal graphs, among others. Our result shows that for every connected graph H with at least two vertices, there exists an integer $k = k(H)$ such that every connected H-graph G satisfies $\mathsf{lpt}(G) \leq k$.

1. Introduction

It is known that every two longest paths in a connected graph share a common vertex. An old question of Gallai [8] asks a generic form of this, whether all longest paths in a connected graph share a common vertex. According to a counterexample due to Walther [22] and Zamfirescu [23], there exists a graph (depicted in Figure 1) such that every vertex is omitted

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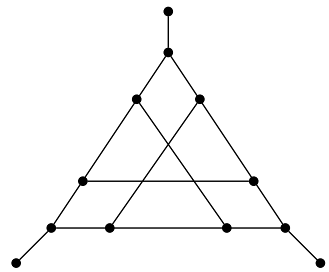

FIGURE 1. The Walther-Zamfirescue graph.

by some longest path of the graph, refuting Gallai's question. However, investigations into Gallai's question led to the formulation of a more general problem, in which one is interested in the size of the smallest set of vertices intersecting every longest path of a graph. Let us be precise. Let G be a connected graph. A *longest path transversal* in G is a set $S \subseteq V(G)$ such that every longest path in G contains at least a vertex in S. The *longest path transversal number* of G, denoted by $\mathsf{lpt}(G)$, is the minimum cardinality of a longest path transversal in G (when we speak of a *longest path*, we mean a path that achieves the maximum possible length among all paths in the graph). In this language, Gallai's question asked whether any graph G is such that $\mathsf{lpt}(G) = 1$. Note that the graph in Figure 1 has a longest path transversal of size two. A graph G with $\mathsf{lpt}(G) = 3$ was constructed in [23]. However, and surprisingly, it is not known whether there exists a connected graph G with $\mathsf{lpt}(G) \geq 4$. Accordingly, an extensive body of work has been devoted to providing non-trivial upper bounds for the longest path transversal number of a general graph [21, 17, 15]. Recently, Norin et al. [20] proved that every connected graph G with n vertices admits $\mathsf{lpt}(G) = \mathcal{O}(\sqrt{n})$, improving the previous best upper bound (see [15]). For the special case where G is chordal, Harvey and Payne [10] showed $\mathsf{lpt}(G) \leq 4\lceil(\omega(G))/5\rceil$, where $\omega(G)$ is the size of the largest clique of G, while Long Jr. et al. [19] showed that if G is an n-vertex connected chordal graph, then $\mathsf{lpt}(G) = \mathcal{O}(\log^2 n)$.

A natural question to ask is:

Question 1 (see also [23])**.** *Is there a constant $c \in \mathbb{N}$ such that every connected graph G satisfies $\mathsf{lpt}(G) \leq c$?*

Surprisingly, Question 1 remains widely open even for special classes of graphs, such as chordal graphs, though it has been resolved positively for some of its subclasses like interval graphs [2], dually chordal graphs [12], well-partitioned chordal graphs [1] and other subclasses of chordal graphs defined by constraints on their tree intersection model [4, 19]. Other graph classes that are known to admit constant size longest path transversals include circular-arc graphs [2, 13] and bipartite permutation graphs [4]. A limited number of results also provide answers for Question 1 when restricted to H-free graphs (for a given graph H, a graph G is said to be H-free if it has no induced subgraph isomorphic to H). Long Jr. et al [18] showed that if H is a linear forest on at most four vertices, then every connected H-free graph G admits $\mathsf{lpt}(G) = 1$. Gao et al. [9] identified multiple subclasses of claw-free graphs that have $\mathsf{lpt} = 1$. Recently, in [16], two of us partially improved upon the results of [18] and [9] by providing a complete classification of $(claw, H)$-free graphs, for H of size at most five, which admit longest path transversals of size one, and by showing that $\mathsf{lpt}(G) = 1$ when G belongs to some subclasses of P_5-free graphs. As far as we know (and curiously enough), Question 1 is still open for the class of P_5-free graphs. In particular, the following question is posed in [9]:

Question 2 (see Problem 6 in [9])**.** *Is there a constant $c \in \mathbb{N}$ such that every connected P_5-free graph G admits $\mathsf{lpt}(G) \leq c$?*

The main results. We prove two main results in this paper, both of which are a step toward understanding the longest path transversal numbers of graphs. The first result, proven in Section 3, allows us to refine a given longest path transversal in a graph using domination properties. Let G be a connected graph and C be a component of G. For $M \subseteq V(G)$ let $\mathsf{bd}(C_i, M)$ denotes the vertices of C_i with at least one neighbor in M. Let t_i denote the length of a longest path of $G[C_i]$ with at least one endpoint in $\mathsf{bd}(C_i, M)$. Let C_{max} be a component of G such that $t_{\mathsf{max}} \geq t_i$, $1 \leq i \leq t$. We call such a component *path-maximal with respect to M*.

Theorem 1.1. *Let G be a connected graph, $M \subseteq V(G)$ be a longest path transversal of G and $D \subseteq M$ be a connected dominating set of $G[M]$. Let $C_1, \ldots, C_t$ be the connected components of $G \setminus M$ and let C_{max} be a path-maximal component with respect to M. Let $S \subseteq M$ be a set that dominates $\mathsf{bd}(C_{\mathsf{max}}, M)$. Then $D \cup S$ is a longest path transversal of G.*

Theorem 1.1 has several consequences. First, combined with the result of [7], it yields the following and thereby answers Question 2.

Theorem 1.2. *Let $t \in \{5, 6\}$. Every connected P_t-free graph G satisfies $\mathsf{lpt}(G) \leq t - 2$.*

Second, paired with the results of [11], it implies the following bound for the class of $(bull, chair)$-free graphs.

Theorem 1.3. *Let G be a connected $(bull, chair)$-free graph. Then $\mathsf{lpt}(G) \leq 5$.*

Theorem 1.1 also represents a step forward toward answering Question 1 on chordal graphs. Kang et al. [14] showed that interval graphs, for which Question 1 has been answered, are $(K_3 \boxminus \overline{K_3})$-free chordal graphs. Theorem 1.1 implies that the longest path transversal number of chordal $(K_t \boxminus \overline{K_t})$-free graphs is bounded by $t - 1$.

Theorem 1.4. *For every $t \in \mathbb{N}$, every connected chordal graph G with no induced subgraph isomorphic to $K_t \boxminus \overline{K_t}$ satisfies $\mathsf{lpt}(G) \leq t - 1$.*

The second main result of this paper, proven in Section 4, contributes to the longest path transversal number of H-intersection graphs, and hence naturally provides an upper bound for the longest path transversal number of interval graphs, circular-arc graphs, and chordal graphs, within the language of intersection graphs (see Observation 1). Note that, for a fixed graph H, our result provides a constant upper bound for the class of H-graphs.

Theorem 1.5. *For every connected graph H with at least two vertices, every connected H-graph G satisfies $\mathsf{lpt}(G) \leq 4(\mathsf{tw}(H) + 1)|E(H)|$.*

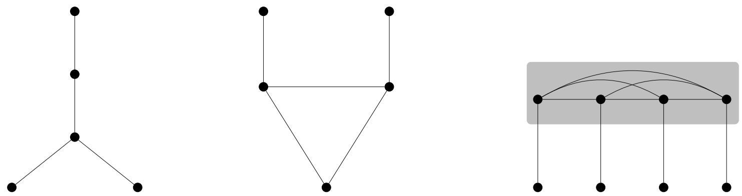

FIGURE 2. From left to right: a chair, a bull, and a $K_4 \boxminus \overline{K_4}$.

2. Preliminaries

Sets and numbers. We use $\mathbb{N}$ to denote the set of positive integers. We let $[n] := \{1, \dots, n\}$ for every $n \in \mathbb{N}$. For a set Ω, we denote by 2^Ω the set of all subsets of Ω. For $S \subseteq \Omega$, we let $\overline{S} = \Omega \setminus S$. Given $\mathcal{S} \subseteq 2^\Omega$, a *hitting set* for $\mathcal{S}$ is a set $X \subseteq \Omega$ such that for all $S \in \mathcal{S}$, $X \cap S \neq \emptyset$. If $X \cap S \neq \emptyset$, we may say that X *hits* S. Let H be a set. A family $\mathcal{F} = \{H_1, \dots, H_k\}$ of subsets of H is said to satisfy the *Helly property* if the following holds: For every subfamily $\mathcal{F}' \subseteq \mathcal{F}$, if every pair $H_i, H_j \in \mathcal{F}'$ satisfies $H_i \cap H_j \neq \emptyset$, then $\bigcap_{H_i \in \mathcal{F}'} \{H_i\} \neq \emptyset$.

Graphs. Graphs in this paper have finite vertex sets and no loops or parallel edges. Let $G = (V, E)$ be a graph. A *clique* in G is a set of pairwise adjacent vertices. A *stable set* or *independent set* in G is a set of pairwise non-adjacent vertices. A subset $D \subseteq V(G)$ is a *dominating set* in G if each vertex of G either belongs to D or is adjacent to some vertex of D. For $X \subseteq V(G)$, we denote the subgraph of G induced by X as $G[X]$, that is $G[X] = (X, \{uv\colon u, v \in X \text{ and } uv \in E\})$. We denote by $N(X)$ vertices of $G \setminus X$ with a neighbor in X, and $N[X] = N(X) \cup X$. For disjoint $X, Y \subseteq G$, we say that X is *complete* to Y if every vertex in X is adjacent to every vertex in Y, and X is *anticomplete* to Y if there are no edges between X and Y. We let $P = p_1 p_2 \dots p_k$ denote a path in G. The length of P is the number of edges in P. We call the vertices p_1 and p_k the endpoints of P and say that P is a path from p_1 to p_k. For a vertex $x \in V(G)$, we say x is *between p_i and p_j*, if x is in the path $p_i \dots p_j$. We also say *x has no neighbor from p_i to p_j*, if there is no vertex y between p_i and p_j such that xy is an edge.

Let G be a graph. A *tree decomposition* of G is a pair $(T, \mathcal{X})$ where T is a tree and $\mathcal{X}\colon V(T) \to 2^{V(G)}$ is a map such that:

(i) for every $v \in V(G)$, there exists $t \in V(T)$ such that $v \in \mathcal{X}(t)$;
(ii) for every edge $uv \in E(G)$, there exists $t \in V(T)$ such that $u, v \in \mathcal{X}(t)$;
(iii) for every $v \in V(G)$, the graph $T[\{t \in V(T)\colon v \in \mathcal{X}(t)\}]$ is non-empty and connected.

The sets $\mathcal{X}(t)$ for $t \in V(T)$ are called the *bags* of $(T, \mathcal{X})$. The *treewidth* of G, denoted $\mathsf{tw}(G)$, is the minimum width of a tree decomposition of G, where the *width* of a tree-decomposition $(T, \mathcal{X})$ is defined as $\max_{t \in V(T)} |\mathcal{X}(t)| - 1$. The *pathwidth* of G, is defined analogously with T being a path instead of a tree.

For $X, Y \subseteq G$ we define the *boundary of X with respect to Y*, denoted by $\mathsf{bd}(X, Y)$, as the set of vertices of X that have a neighbor in Y. More precisely:

$$\mathsf{bd}(X, Y) = \{x \in X\colon N_G(x) \cap Y \neq \emptyset\}.$$

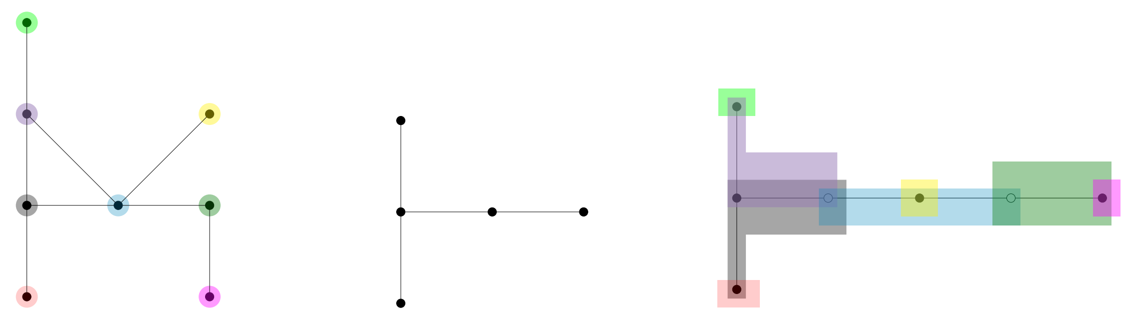

FIGURE 3. From left to right: A graph G, a graph H, and an H-representation of G. Vertices in white color are correspond to the subdivided edges of H.

Special graphs. A graph is *chordal* if it has no induced cycle of length at least four. An *interval intersection representation* for a graph G is a set $\mathcal{I} = \{I_v : v \in V(G)\}$ of intervals, such that $uv \in E(G)$ if and only if $I_u \cap I_v \neq \emptyset$. A graph is an *interval graph* if it admits an interval intersection representation. For given graphs G and H, we say that G is *H-free* if G does not contain H as an induced subgraph. We say G is (H_1, H_2) free if G does not contain H_1 and H_2 as induced subgraphs. We let P_n, C_n, and K_n denote the chordless path, chordless cycle, and the complete graph on n vertices. For integer $t \geq 1$, we denote by tP_n the graph obtained from the disjoint union of t copies of the n-vertex path, and for graphs G_1, G_2, we write G_1+G_2 to denote the disjoint union of G_1 and G_2. A *claw* is a $K_{1,3}$. A *chair* is a graph obtained by subdividing an edge of a claw exactly once. A *bull* is a graph obtained from a four-vertex path by adding a vertex adjacent to the two middle vertices of the path. Let $n \in \mathbb{N}$. We denote by $K_n \boxminus \overline{K_n}$ the graph Λ with $V(\Lambda) = \{v_1^1, \ldots, v_n^1, v_1^2, \ldots, v_n^2\}$ such that for all $i, j \in [n]$ the following holds (see Figure 2):

- $\{v_1^1, \ldots, v_n^1\}$ is a clique, and $\{v_1^2, \ldots, v_n^2\}$ is an independent set; and
- v_i^1 is adjacent v_j^2 if and only if $i = j$.

***H*-graphs.** Let H be a fixed graph. A *subdivision* H' of graph H is a graph obtained from H by replacing each edge $e = xy$ of H by a path P_e of length at least one from x to y such that the interiors of the paths are pairwise disjoint and anticomplete. An *H-representation* of a graph $G = (V, E)$ is a pair (H^Φ, Φ), where H^Φ is a subdivision of H and $\Phi\colon V(G) \to 2^{V(H^\Phi)}$ is a map such that:

(i) for every $v \in V(G)$, the graph $H^\Phi[\Phi(v)]$ is connected; and
(ii) for every distinct $u, v \in V(G)$, $uv \in E(G)$ if and only if $\Phi(u) \cap \Phi(v) \neq \emptyset$.

We say that a graph G is an *H-graph* if G admits an H-representation (see Figure 3). We call $\Phi(v)$, $v \in V(G)$, the *segments* of the representation. An H-graph G satisfies the Helly property if the collection $\mathcal{C} = \{\Phi(v) : v \in V(G)\}$ of an H-representation (H^Φ, Φ) of G satisfies the Helly property. Observe that if H is a tree, then every H-representation satisfies the Helly property. The concept of H-graphs introduced by Biró, Hujter and Tuza [3]. We make the following observation, which is not difficult to check.

Observation 1. *The following hold:*

(a) *Every graph H is an H-graph.*
(b) *K_2-graphs are precisely interval graphs.*
(c) *K_3-graphs are precisely circular-arc graphs.*
(d) *$\bigcup_{tree\ T}\{T\text{-graphs}\}$ are precisely chordal graphs.*

3. Refining a transversal

In this section, we first prove Theorem 1.1 which refines a given longest path transversal using domination properties. We will then apply Theorem 1.1 and conclude that: (i) P_6-free graphs admit $\mathsf{lpt} \leq 4$, which answers the open question posed by [9]; (ii) $(bull, chair)$-free graphs admit $\mathsf{lpt} \leq 5$, and (iii) chordal $(K_t \boxminus \overline{K_t})$-free graphs admit $\mathsf{lpt} \leq t-1$ which adds to the extensive list of subclasses of chordal graphs that have been shown to admit constant-size longest path transversals.

We first begin with the following terminology. Given a subset $M \subset V(G)$ and the connected components $C_1, \dots C_t$ of $G \setminus M$, we let t_i be the length of a longest path of $G[C_i]$ with at least one endpoint in $\mathsf{bd}(C_i, M)$. Recall that $\mathsf{bd}(C_i, M)$ denotes the vertices of C_i with at least one neighbor in M. We let C_{max} be a component such that $t_{\mathsf{max}} \geq t_i$ for $1 \leq i \leq t$. We call such a component *path-maximal with respect to M*.

We can now restate and prove Theorem 1.1.

Theorem 3.1. *Let G be a connected graph, $M \subseteq V(G)$ be a longest path transversal of G and $D \subseteq M$ be a connected dominating set of $G[M]$. Let $C_1, \dots, C_t$ be the connected components of $G \setminus M$ and let C_{max} be a path-maximal component with respect to M. Let $S \subseteq M$ be a set that dominates $\mathsf{bd}(C_{\mathsf{max}}, M)$. Then $D \cup S$ is a longest path transversal of G.*

Proof. Let P be a longest path of G. We want to show $V(P) \cap (D \cup S) \neq \emptyset$.

Suppose for a contradiction $V(P) \cap (D \cup S) = \emptyset$. First note that, since M is a longest path transversal of G, $V(P) \cap M \neq \emptyset$. In particular, this implies $V(P)$ cannot be contained in C_i, for $1 \leq i \leq t$. We will show that $V(P) \cap C_{\mathsf{max}} = \emptyset$. We already argued $V(P)$ cannot be contained in C_{max}. Suppose there exists an edge $cc' \in E(P)$ such that $c \in C_{\mathsf{max}}$ and $c' \in \mathsf{bd}(M, C_{\mathsf{max}})$. Recall that $S \subseteq M$ is a set that dominates $\mathsf{bd}(C_{\mathsf{max}}, M)$. Let $s \in S$ be a vertex that dominates c. Since $s \in M$, let $d \in D$ be a vertex that dominates s. Note that it can be that $s = d$, as S and D do not need to be disjoint. Let $d' \in D$ be a vertex that dominates c'. It can also be $c' = d'$. We are now able to replace the edge cc' in P by the path $csd \dots d'c'$, where $d \dots d'$ is a path in $G[D]$ connecting d and d'. Such a path is certain to exist as D is connected. With this replacement, we obtained a path that is longer than P, as at least one vertex was added to it, a contradiction. Hence, $V(P) \cap C_{\mathsf{max}} = \emptyset$.

Note that P cannot have an endpoint in M. Indeed, since D dominates $G[M]$, we would be able to extend P to D, a contradiction. Let p_z be a vertex of P such that $p_z \in M$, but $p_{z+j} \notin M$, for $j \geq 1$. Such a vertex exists since $p_k \notin M$. Moreover, since the C_i's are connected components of $G \setminus M$, the vertices p_{z+j}, for $j \geq 1$, all belong to the same component. Let C_i be this component. Since $p_{z+1} \dots p_k$ is a path in $G[C_i]$, its length is at most t_i. To reach a contradiction, we will modify the path P in order to obtain an

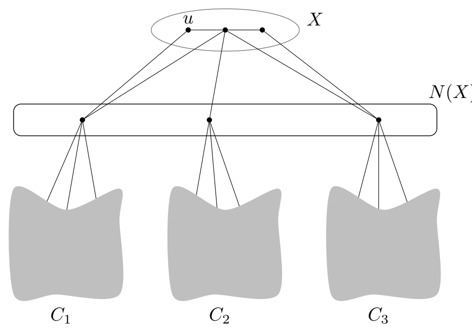

FIGURE 4. Outcome of Theorem 3.2 for $t = 6$. For $i \in [3]$, C_i's are connected components in $G \setminus N_G[X]$.

even longer path. Let $q_1 \dots q_{t_1}$ be a longest path of $G[C_{\mathsf{max}}]$ with at least one endpoint in $\mathsf{bd}(C_{\mathsf{max}}, M)$. Assume $q_1 \in \mathsf{bd}(C_{\mathsf{max}}, M)$. Let $s \in S$ be a vertex that dominates q_1, and let $d \in D$ be a vertex that dominates s. Let $d'' \in D$ be a vertex that dominates p_z. Consider the path $P' = p_1 p_2 \dots p_z d'' \dots d s q_1 \dots q_{t_1}$, where $d'' \dots d$ is a path in $G[D]$ connecting d'' and x, which exists since D is connected. Since $t_{\mathsf{max}} \geq t_i$ and the length of the path $d'' \dots ds$ is at least one, we conclude that P' is longer than P, a contradiction. We can conclude $V(P) \cap (D \cup S) \neq \emptyset$, and hence $D \cup S$ is a longest path transversal of G. This completes the proof of Theorem 3.1 ■

Before we apply Theorem 3.1, we state the following Observation of [5], that will be useful in the remainder of this section.

Observation 2 (Cerioli and Lima [5])**.** *Let D be a connected dominating set of G. Then D is a longest path transversal of G.*

We first consider P_5-free graphs and P_6-free graphs. Following [7], we say that $M \subset V(G)$ is a *monitor* in G if for every connected component C of $G \setminus M$ there exists a vertex $w \in M$ that is complete to C. Note that a monitor is a dominating set of G.

The following result of [7] shows that every connected P_t-free graph, $t \in \{4, 5, 6\}$, admits a monitor formed by the closed neighborhood of a path. In particular, this implies that this monitor itself induces a connected subgraph of G and admits a small connected dominating set. More precisely:

Lemma 3.2 (Chudnovsky, King, Pilipczuk, Rzążewski, and Spirkl; see Lemma 5 in [7])**.** *Let $t \in \{4, 5, 6\}$, G be a connected P_6-free graph, and $u \in V(G)$ be a vertex such that in G there is no induced P_t with u being one of the endpoints. Then there exists a subset X of vertices such that $u \in X$, $|X| \leq t - 3$, $G[X]$ is a path whose one endpoint is u, and $N_G[X]$ is a monitor in G.*

We are now able to prove the following.

Theorem 3.3. *Let $t \in \{5, 6\}$. Every connected P_t-free graph G satisfies* $\mathsf{lpt}(G) \leq t - 2$.

Proof. Since G is P_t-free, for $t \in \{5, 6\}$, it follows from Theorem 3.2 that G admits a monitor formed by the closed neighborhood of an induced path X of length at most $t - 3$. Since

$N[X]$ is a monitor and X is connected, $N[X]$ is also a connected dominating set of G, and by Observation 2, it is a longest path transversal of G. We can now apply Theorem 3.1 to refine this transversal. Let $w \in N[X]$ be the vertex that dominates a path-maximal component C_{max} of G with respect to $N[X]$. Since X is a connected dominating set of $N[X]$ and w is a vertex that dominates $\mathsf{bd}(C_{\mathsf{max}}, N[X])$, by Theorem 3.1, $X \cup \{w\}$ is a longest path transversal of G of size at most $t-2$. This completes the proof of Theorem 3.3. ■

We now consider the class of $(bull, chair)$-free graphs, where monitors also play a relevant role. Indeed, the lemma below states that, in this graph class, the closed neighborhood of a maximal induced path is a monitor.

Lemma 3.4 (Hodur, Pilśniak, Prorok, and Rzążewski; see 3.2 in [11])**.** *Let G be a connected (bull, chair)-free graph on at least three vertices. Let $X \subseteq V(G)$ be such that $G[X]$ is a maximal induced path on at least three vertices. Then $N[X]$ is a monitor in G.*

Similarly as in the proof of Theorem 3.3, we want to apply Theorem 3.4 to the monitor $N[X]$. The problem, however, is that Theorem 3.4 does not provide any bound on the length of the maximal induced path, which is crucial to obtain a constant upper bound on lpt. To deal with that, we will use the following characterization of $(bull, chair)$-free graphs that have a maximal induced path with at least seven vertices.

Before we proceed, we need the following definition. A *fat path* (or a *fat cycle*) is a graph whose vertex set can be partitioned into nonempty sets $V_1, V_2, \dots, V_r$, such that the sets V_i, V_j are complete to each other if $j = i+1$ (or $j = i+1 \mod r$), and anticomplete otherwise.

Lemma 3.5 (Hodur, Pilśniak, Prorok, and Rzążewski; see 3.1 in [11])**.** *Let G be a connected $(bull, chair)$-free graph, containing a maximal induced path P with at least 7 vertices. If there exists a vertex $x \in V(G)$ such that x is adjacent to (only) both endpoints of P, then we let Q to be the induced cycle formed by vertices of P and x. Otherwise, we let $Q = P$. In polynomial time one can compute a partition of $V(G)$ into sets R, D, T with the following properties:*

(i) *$G[R]$ is a fat path on at least 7 vertices (or a fat cycle on at least 8 vertices);*
(ii) *D is complete to R, and separates R and T;*
(iii) *Every component of $G-(R \cup D)$ is contained in one component of $G - N_G[Q]$; and*
(iv) *$N[R]$ is a monitor in G.*

We can now consider three separate cases: when G has no induced P_6 (in which case we simply apply Theorem 3.3), when the graph has a maximal induced path on at least seven vertices (Theorem 3.6), and when the graph has only has a maximal induced path on six vertices (Theorem 3.7).

Lemma 3.6. *Let G be a connected $(bull, chair)$-free graph. If G has a maximal induced path on $\ell \geq 7$ vertices, then $\mathsf{lpt}(G) \leq 2$.*

Proof. By Theorem 3.5, there is a partition of $V(G)$ into sets R, D, T satisfying properties (i)-(iv). By property (iv), $N[R]$ is a monitor, and hence it is a dominating set of G. Since R and D are complete to each other, $N[R]$ is connected. By Observation 2, $N[R]$ is a longest path transversal of G. Let $w \in D$ be a vertex that dominates a path-maximal component

C_{max} of G with respect to $N[R]$. Note that w dominates all vertices of R. Let x be a vertex of R. Note that $\{w, x\}$ is a connected dominating set of $G[N[R]]$. Moreover, w dominates $C_{\mathsf{max}} = \mathsf{bd}(C_{\mathsf{max}}, N[R])$. By Theorem 3.1, $\{w, x\}$ is a longest path transversal of G. This completes the proof of Theorem 3.6. ■

Lemma 3.7. *Let G be a connected (bull, chair)-free graph. If any maximal induced path of G has six vertices, then $\mathsf{lpt}(G) \leq 5$.*

Proof. Let $Q = q_1 \dots q_6$, be a maximal induced path of G. By Theorem 3.4, $N[Q]$ is a monitor in G, and hence also a connected dominating set of G. By Observation 2, $N[Q]$ is a longest path transversal of G. Since $N[Q]$ is a monitor, let $w \in N[Q]$ be the vertex that dominates a path-maximal component C_{max} of G with respect to $N[Q]$.

In order to refine $N[Q]$ using Theorem 3.1, we will find a small connected set dominating $N[Q]$. Let $\widetilde{Q} = Q \setminus \{q_1, q_6\}$. More precisely, we now show $D = \widetilde{Q} \cup \{w\}$ is a set that is connected and that dominates $N[Q]$. We first show w is complete to Q. Suppose not. If w is adjacent to two consecutive vertices of Q, but not complete to Q, then there is always a set of three consecutive vertices $q_i, q_{i+1}, q_{i+2} \in Q$ such that wq_{i+1} is an edge and exactly one of q_i, q_{i+2} is a neighbor of w. Then, the set $\{c, w, q_i, q_{i+1}, q_{i+2}\}$ induces a bull in G for some vertex $c \in C_{\mathsf{max}}$ (recall that w dominates C_{max}). If w has no consecutive neighbors in Q but is adjacent to a vertex $q_i \in Q$ with $2 \leq i \leq 5$, then the set $\{w, c, q_{i-1}, q_i, q_{i+1}\}$ induces a chair in G. Since Q is maximal, w cannot be adjacent only to q_1 nor only to q_6. If w is adjacent to both q_1 and q_6 (and only these two), then $\{q_5, q_6, w, c, q_1\}$ induces a chair in G, a contradiction. Hence, w is complete to Q, and $G[D]$ is connected. Let v be a vertex of $N[Q] \setminus D$. If $v = q_1$ or $v = q_6$, then v is clearly dominated by D. If v has a neighbor in $\widetilde{Q}$ or is adjacent to w, then it is also dominated by D. Otherwise, note that since Q is maximal, v cannot be dominated only by q_1 nor only by q_6. So v must be adjacent to both q_1 and q_6 (and only these). If v has a neighbor $u \notin N[Q]$, then $q_2q_3q_4q_5q_6vu$ is an induced path of length at least seven, a contradiction to the assumption that every maximal induced path of G has length six. Let $c \in C_{\mathsf{max}}$. Note that $cv \notin E(G)$, as v has no neighbor outside $N[Q]$. Then $\{q_1, q_2, w, c, v\}$ induces a bull in G, a contradiction. Hence D is indeed a connected dominating set of $N[Q]$ and, by Theorem 3.1, D is a longest path transversal of G. This completes the proof of Theorem 3.7. ■

We are now able to conclude the following.

Theorem 3.8. *Let G be a connected (bull, chair)-free graph. Then $\mathsf{lpt}(G) \leq 5$.*

Proof. If G is P_6-free, then from Theorem 1.2 we have $\mathsf{lpt}(G) \leq 4$ and the claim follows. Therefore, we may assume that there exists a set $X \subseteq V(G)$ such that $G[X]$ is an induced path on six vertices. Consider a maximal induced path Q of G that contains $G[X]$. If Q has length at least 7, by Theorem 3.6, $\mathsf{lpt}(G) \leq 2$. Otherwise, by Theorem 3.7, $\mathsf{lpt}(G) \leq 5$. This completes the proof of Theorem 3.8. ■

As a last application of Theorem 3.1, we now consider chordal $(K_t \boxminus \overline{K_t})$-free graphs. We will make use of the observation below.

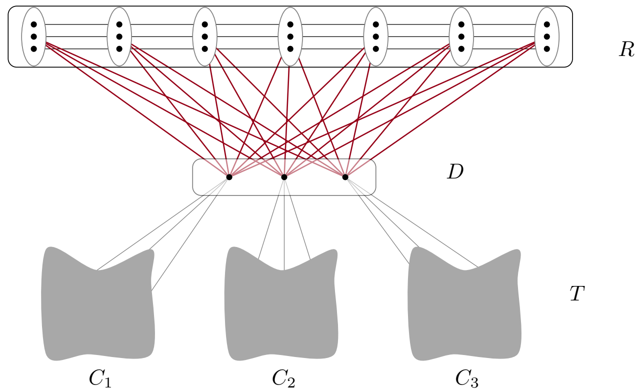

FIGURE 5. An (R, D, T) decomposition. C_i's are components in $G - (R \cup D)$. Red lines represent complete adjacency between R and D.

Observation 3 (Balister, Györi, Lehel, and Schelp [2])**.** *If G is a connected chordal graph, then there exists a clique K in G such that K is a longest path transversal of G.*

Theorem 3.9. *For every $t \in \mathbb{N}$, if G is a connected chordal graph with no induced subgraph isomorphic to $K_t \boxminus \overline{K_t}$, then $\mathsf{lpt}(G) \leq t - 1$.*

Proof. Let G be a chordal $(K_t \boxminus \overline{K_t})$-free graph. By Observation 3, there exists a clique K that is a longest path transversal of G. Let C_{max} be a path-maximal component of $G \setminus K$. Let $D \subseteq K$ be a minimal set with the property that it dominates $\mathsf{bd}(C_{\mathsf{max}}, K)$. Note that D also dominates K, as K is a clique. By Theorem 3.1, D is a longest path transversal of G. We now show that since G is $(K_t \boxminus \overline{K_t})$-free, then $|D| \leq t - 1$.

Indeed, since D is minimal, for every $d \in D$ there exists $d' \in \mathsf{bd}(C_{\mathsf{max}}, K)$ such that d' is a *private neighbor* of d, that is, d' is dominated *only* by d in D. Let $\{d'_1, \ldots d'_r\}$ be the set of private neighbors of the vertices $d_1, \ldots, d_r$ in D. Since $G[D]$ is a clique, there are no edges between vertices in $\{d'_1, \ldots d'_r\}$, otherwise, there would be an induced cycle of length four in G, which violates the assumption that G is chordal. Thus, $\{d_1, \ldots, d_r\} \cup \{d'_1, \ldots, d'_r\}$ induces a $K_r \boxminus \overline{K_r}$ in G. Since G has no induced subgraph isomorphic to $K_t \boxminus \overline{K_t}$, we conclude that $r \leq t - 1$. This completes the proof of Theorem 3.9. ■

4. H-GRAPHS

Before proving Theorem 1.5 in this section, we introduce some notation that will be used later.

H-representation of graphs. Let (H^Φ, Φ) be an H-representation of a graph G (see Section 2). We say that (H^Φ, Φ) is a *nice* H-representation of G if the following hold:

(a) for every $x \in V(H^\Phi)$, there is a vertex $v \in V(G)$ such that $x \in \Phi(v)$; and
(b) for every edge $xy \in E(H^\Phi)$, there is $v \in V(G)$ such that $x, y \in \Phi(v)$.

It is known that an H-representation of G can be turned into a nice one [6]. Henceforth, whenever we talk about an H-representation of G, we may assume that it is a nice one.

For a vertex x of H^Φ, let V_x denote the set $\{v \in V(G)\colon x \in \Phi(v)\}$. Note that for each $x \in V(H^\Phi)$, the set V_x is a clique in G.

Consider an edge $ab \in E(H)$ and fix its orientation from a to b. Note that the edge ab of H corresponds to a path in H^Φ, denote the consecutive vertices of this path by $s_1, \ldots, s_\ell$, where $a = s_1$ and $b = s_\ell$. For a vertex $v \in V_a$, let $\mathsf{reach}_{a\to b}(v)$ be the maximum index i such that $s_1, \ldots, s_i \in \Phi(v)$. We emphasize that the intersection of $\Phi(v)$ with $\{s_1, \ldots, s_\ell\}$ does not have to be connected, so $\Phi(v)$ may contain some s_j for $j > \mathsf{reach}_{a\to b}(v)$.

Tree decompositions of H. For an H-graph G, by an *H-profile of G* we denote a tuple $((H^\Phi, \Phi); (T, \mathcal{X}))$, where (H^Φ, Φ) is a nice H-representation of G and $(T, \mathcal{X})$ is a tree decomposition of H^Φ. Note that since H^Φ is a subdivision of H, we have $\mathsf{tw}(H^\Phi) = \mathsf{tw}(H)$.

By the properties of a tree decomposition, for each $u \in V(H^\Phi)$, the set $\{t \in V(T)\colon u \in \mathcal{X}(t)\}$ induces a subtree of T. As, for every $v \in V(G)$, the set $\Phi(v)$ induces a connected subgraph of H^Φ, we conclude that the set $\{t \in V(T)\colon \Phi(v) \cap \mathcal{X}(t) \neq \emptyset\}$ corresponds to a subtree of T. Denote this subtree by $T_{\Phi(v)}$. For a path P in G, we write $T_{\Phi(P)}$ as a shorthand for $\bigcup_{v\in V(P)} T_{\Phi(v)}$.

The following lemma guarantees that, given an H-profile $((H^\Phi, \Phi); (T, \mathcal{X}))$ of a connected graph G, one can always find a *bag* in $(T, \mathcal{X})$ such that those vertices of G represented by the vertices of the bag form a longest path transversal set in G. More precisely:

Lemma 4.1. *Let G be a connected graph and $((H^\Phi, \Phi); (T, \mathcal{X}))$ be an H-profile of G. There exists $t \in V(T)$ such that the set $\bigcup_{x\in\mathcal{X}(t)} V_x$ is a longest path transversal of G.*

Proof. Let $\mathcal{F}$ be a family of all longest paths in G. Since G is connected, for every pair $P_1, P_2 \in \mathcal{F}$, there exists a vertex $x \in V(P_1) \cap V(P_2)$. Since the set $\{T_{\Phi(P)}\colon P \in \mathcal{F}\}$ has the Helly property (as the subtrees of a tree admit the Helly property), it follows that there exists a vertex $t \in V(T)$ such that for every $P \in \mathcal{F}$, we have $t \in T_{\Phi(P)}$. This implies that the set $\{v \in V(G)\colon t \in V(T_{\Phi(v)})\} = \bigcup_{x\in\mathcal{X}(t)} V_x$ forms a longest path transversal in G. ∎

Intermediate graph. Let H be a connected graph with at least two edges. Take an H-profile $((H^\Phi, \Phi); (T, \mathcal{X}))$ of a connected graph G, where $(T, \mathcal{X})$ is a tree decomposition of H^Φ of width $\mathsf{tw}(H^\Phi) = \mathsf{tw}(H)$. Let t be given by Theorem 4.1. We aim to show that from $\bigcup_{x\in\mathcal{X}(t)} V_x$ we might pick a subset of size bounded in terms of H only, that is still a longest path transversal of G. Here we stumble on the first technical issue. Note that the set $\mathcal{X}(t)$ might contain some vertices that are not in H. On the other hand, the size of H^Φ can be arbitrarily large, not necessarily bounded in terms of H. For this reason, we will work neither with H nor with H^Φ, but rather the *intermediate graph* $\widehat{H}$. We obtain it from H^Φ by contracting all edges with at least one endpoint that is not in $V(H) \cup \mathcal{X}(t)$. Consequently, we have $V(\widehat{H}) = V(H) \cup \mathcal{X}(t)$ and thus $|V(\widehat{H})| \leq |V(H)| + \mathsf{tw}(H) + 1 \leq 2|V(H)|$. On the other hand, every vertex of $\mathcal{X}$ adds at most one new edge to $E(\widehat{H}) \setminus E(H)$, and thus

$$|E(\widehat{H})| \leq |E(H)| + \mathsf{tw}(H) + 1 \leq 2|E(H)| \tag{S1}$$

(here we use that $|E(H)| \geq 2$). Observe that H^Φ is a subdivision of $\widehat{H}$ and thus G is an $\widehat{H}$-graph with $\widehat{H}$-profile $((H^\Phi, \Phi); (T, \mathcal{X}))$. Note that for each $v \in \bigcup_{x\in\mathcal{X}(t)} V_x$, where $t \in V(T)$

is an outcome of applying Theorem 4.1 on G, there is a vertex x of $\widehat{H}$ such that $x \in \Phi(v)$. This property will be useful later.

We can now restate and prove Theorem 1.5.

Theorem 4.2. *For every connected graph H with at least two vertices, every connected H-graph G satisfies $\mathsf{lpt}(G) \leq 4(\mathsf{tw}(H)+1)|E(H)|$.*

Proof. Let $((H^\Phi, \Phi); (T, \mathcal{X}))$ be an H-profile of G. Let $t \in V(T)$ be the outcome of applying Theorem 4.1 on G, and let $\widehat{H}$ be the intermediate graph. For simplicity of notation, we let $X = \mathcal{X}(t)$ and denote $S = \bigcup_{x \in X} V_x$. Note that $|X| \leq \mathsf{tw}(H) + 1$ and each set V_x, for $x \in X$, is a clique in G, thus S is partitioned into X cliques. Furthermore, recall that S is a longest path transversal in G.

For $x \in X$, let $A_x \subseteq V(\widehat{H})$ be the set of all vertices a of $\widehat{H}$ for which $V_x \cap V_a \neq \emptyset$. For any edge $ab \in E(\widehat{H})$, where $a \in A_x$, let $s_{x,a\to b}$ denote a vertex $v \in V_x \cap V_a$ with maximum value of $\mathsf{reach}_{a\to b}(v)$; the ties are resolved arbitrarily. Let Q be the set consisting of all vertices selected this way, that is,

$$Q = \bigcup_{x\in X} \bigcup_{a \in A_x} \bigcup_{ab \in E(\widehat{H})} \{s_{x,a\to b}\}.$$

We observe that since $\widehat{H}$ is connected and has at least two vertices, for each $x \in X$, the set A_x is non-empty. Consequently, for each $x \in X$, the set Q intersects V_x.

Clearly we have $Q \subseteq S$ and, using (S1) we get

$$|Q| \leq |X| \cdot 2 \cdot |E(\widehat{H})| \leq 4(\mathsf{tw}(H)+1)|E(H)|. \tag{S2}$$

We claim the following.

(1) *For every $x \in X$, the set $Q \cap V_x$ dominates $\mathsf{bd}(V(G) \setminus S, S \cap V_x)$.*

For contradiction, suppose there is $x \in V$ and a vertex $u \notin S$ adjacent to some $s \in S \cap V_x$ but non-adjacent to every vertex of $Q \cap V_x$. Since $su \in E(G)$, there is some $z \in V(H^\Phi)$ such that $z \in \Phi(u) \cap \Phi(s)$. Let $ab \in E(\widehat{H})$ be an edge such that z belongs to the path P_{ab} in H^Φ obtained by subdividing the edge ab. Note that ab might not be unique if z is a vertex of $V(\widehat{H})$. Since $\Phi(s)$ is a connected subgraph of H^Φ and $\Phi(s)$ contains a vertex of $V(\widehat{H})$, we conclude that $\Phi(s)$ contains the whole subpath of P_{ab} from a to z or the whole subpath of P_{ab} from z to b. By symmetry, assume the former. As $s \in V_x \cap V_a$, the vertex $s_{x,a\to b}$ is defined and belongs to $Q \cap V_x$. Since $\Phi(s)$ contains the whole subpath of P_{ab} from a to z and $s_{x,a\to b}$ is a vertex from $V_x \cap V_a$ with the maximum value of $\mathsf{reach}_{a\to b}$, we observe that $z \in \Phi(s_{x,a\to b})$. Consequently, $us_{x,a\to b} \in E(G)$, a contradiction. This proves (1).

(2) *Q is a longest path transversal of G.*

For contradiction, suppose there is a longest path $P = p_1 \ldots p_k$ in G that avoids Q, that is, $V(P) \cap Q = \emptyset$. By Theorem 4.1, S is a longest path transversal in G. Let i be the minimum index such that $p_i \in S$ and let $x \in X$ be such that $p_i \in V_x$. First, suppose that $i = 1$. Let q_x be any vertex from $V_x \cap Q$; it exists as $V_x \cap Q$ is non-empty. As V_x is a clique, we obtain that $q_x, p_1 \ldots p_k$ is a path in G, contradicting the maximality of P. So suppose that $i > 1$. Since

$p_{i-1} \in \mathsf{bd}(V(G) \setminus S, S \cap V_x)$, by (1) there is some $q_x \in Q \cap V_x$, such that $p_{i-1}q_x \in E(G)$. Again using that V_x is a clique, we obtain that $p_1, \ldots, p_{i-1}, q_x, p_i \ldots p_k$ is a path in G, contradicting the maximality of P. This proves (2).

From (2) and (S2) we conclude the proof of Theorem 4.2. ■

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