STRUCTURE OF CLOSED SUBIDEALS OF $\mathcal{L}(X)$

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ABSTRACT. The closed subalgebra $\mathcal{J}$ of the Banach algebra $\mathcal{L}(X)$ of bounded linear operators on the Banach space X is a non-trivial closed $\mathcal{I}$-subideal of $\mathcal{L}(X)$ if $\mathcal{I}$ is a closed ideal of $\mathcal{L}(X)$ and $\mathcal{J}$ is an ideal of $\mathcal{I}$, but $\mathcal{J}$ is not an ideal of $\mathcal{L}(X)$. We obtain a variety of examples of non-trivial closed subideals of $\mathcal{L}(X)$ for different spaces X, which highlight further significant differences compared to the class of closed ideals. We study the concept of a closed n-subideal of $\mathcal{L}(X)$ for $n \geq 3$, which is a natural generalization of that of a closed subideal. In particular, we find explicit spaces X for which $\mathcal{L}(X)$ contains a decreasing sequence $(\mathcal{M}_n)_{n\in\mathbb{N}}$ of closed subalgebras, where for all $n \in \mathbb{N}$ the subalgebra $\mathcal{M}_n$ is an $(n+1)$-subideal of $\mathcal{L}(X)$ but not an n-subideal. Moreover, we construct closed n-subideals contained in the compact operators $\mathcal{K}(X)$ for certain Banach spaces X which fail the approximation property.

1. INTRODUCTION

Let $\mathcal{A}$ be a Banach algebra, and suppose that

$$\mathcal{J} \subset \mathcal{I} \subset \mathcal{A}$$

are closed subalgebras. The subalgebra $\mathcal{J}$ is said to be a closed $\mathcal{I}$-*subideal* of $\mathcal{A}$ if $\mathcal{I}$ is an ideal of $\mathcal{A}$ and $\mathcal{J}$ is an ideal of $\mathcal{I}$. The closed $\mathcal{I}$-subideal $\mathcal{J}$ is *non-trivial* if $\mathcal{J}$ is not an ideal of $\mathcal{A}$. For brevity we say that $\mathcal{J}$ is a non-trivial closed subideal of $\mathcal{A}$ if $\mathcal{J}$ is a non-trivial closed $\mathcal{I}$-subideal of $\mathcal{A}$ for some closed ideal $\mathcal{I} \subsetneq \mathcal{A}$.

Let $\theta : \mathcal{A} \to \mathcal{B}$ be a continuous algebra isomorphism, where $\mathcal{A}$ and $\mathcal{B}$ are Banach algebras. Then θ maps non-trivial closed subideals of $\mathcal{A}$ to non-trivial closed subideals of $\mathcal{B}$ (cf. Lemma 2.5 below), so the collection of closed subideals contains finer information about the structure of the algebras compared with the closed ideals.

Our main concern is the class of Banach algebras $\mathcal{L}(X)$ consisting of the bounded operators on the (real or complex) Banach space X. The first examples of non-trivial (and non-closed) subideals of $\mathcal{L}(\ell^2)$ were obtained by Fong and Radjavi [13], though the term subideal was coined later by Patnaik and Weiss [33]. On the other hand, $\mathcal{L}(X)$ fails to have non-trivial closed subideals if X is a Hilbert space, or if X is one of the spaces in

$$\{\ell^p : 1 \leq p < \infty\} \cup \{c_0\}. \tag{1.1}$$

This is seen from the fact that the compact operators $\mathcal{K}(X)$ is the only proper closed ideal of $\mathcal{L}(X)$ for X in (1.1), see e.g. [36, 5.1-5.2], and from [49, Remark 3.2.(i)] for arbitrary Hilbert spaces X. At the other extreme, it

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was shown in [49, Section 3] that $\mathcal{L}(X)$ admits large families of non-trivial closed subideals for many classical Banach spaces X, including $L^p(0,1)$ with $p \neq 2$.

In Section 2 we obtain examples of various intermediary phenomena for non-trivial closed subideals of $\mathcal{L}(X)$ by considering classes of spaces X, where either $\mathcal{L}(X)$ contains relatively few closed ideals or closed ideals of a specific type. In particular, we discuss closed subideals for the class of hereditarily indecomposable spaces X_k constructed by Tarbard [46], [45], as well as the spaces X obtained by Motakis in [28], where the Calkin algebra $\mathcal{L}(X)/\mathcal{K}(X)$ is isomorphic as a Banach algebra to $C(K)$ for compact metric spaces K.

In [48, Theorem 4.5] we found a Banach space Z_p together with a family

$$\mathfrak{F} = \{\mathcal{I}_B : \emptyset \neq B \subsetneq \mathbb{N}\}$$

the size of the continuum, which consists of non-trivial closed $\mathcal{K}(Z_p)$-subideals of $\mathcal{L}(Z_p)$. The family $\mathfrak{F}$ has the rather unexpected property [49, Theorem 2.1] that $\mathcal{I}_{B_1}$ and $\mathcal{I}_{B_2}$ are isomorphic Banach algebras for any non-empty subsets $\emptyset \neq B_1, B_2 \subsetneq \mathbb{N}$. In Section 3 we revisit this setting and construct another large family $\mathfrak{F}_+$ of non-trivial closed $\mathcal{K}(Z_p)$-subideals of $\mathcal{L}(Z_p)$ such that $\mathfrak{F} \cap \mathfrak{F}_+ = \emptyset$. The new subideals from $\mathfrak{F}_+$ are again isomorphic as Banach algebras to the subideals $\mathcal{I}_B$ from $\mathfrak{F}$.

Shulman and Turovskii [42] introduced the natural graded concept of an algebraic n-subideal for Banach algebras. In Section 4, which is the most significant part of this paper, we turn our attention to the closed n-subideals of $\mathcal{L}(X)$ for $n \geq 2$. Here the closed 2-subideals correspond to the subideals studied in Sections 2 and 3. We construct a variety of non-trivial closed n-subideals. Theorem 4.8 demonstrates that for certain direct sums such as $X = \left(\bigoplus_{j=1}^{\infty} \ell^{p_j}\right)_{c_0}$ the algebra $\mathcal{L}(X)$ contains an infinite decreasing sequence $(\mathcal{M}_n)$ of closed subalgebras, where for all $n \in \mathbb{N}$ the subalgebra $\mathcal{M}_n$ is an $(n+1)$-subideal of $\mathcal{L}(X)$ but not an n-subideal. Moreover, Theorem 4.12 produces for each $n \in \mathbb{N}$ a closed subspace $Z_n \subset c_0$ that fails to have the approximation property, for which $\mathcal{L}(Z_n)$ has a closed $(n+1)$-subideal $\mathcal{M}_n \subset \mathcal{K}(Z_n)$ that is not an n-subideal of $\mathcal{L}(Z_n)$.

The identification or even classification of the closed ideals of $\mathcal{L}(X)$ for various spaces X is a topic of longstanding interest, see e.g. [36, Chapter 5]. Recently there has been significant advances, see e.g. [39], [18], [19], [14] and their references. Closed n-subideals of $\mathcal{L}(X)$ for $n \geq 2$ concern a weaker ideal structure, which exhibits many surprising properties.

We refer to [24] and [9] for undefined concepts related to Banach spaces and operator theory, and similarly to [5] and [32] for Banach algebra theory. Throughout the paper $X \approx Y$ denotes linearly isomorphic Banach spaces X and Y, and $\mathcal{A} \cong \mathcal{B}$ denotes (multiplicatively) isomorphic Banach algebras $\mathcal{A}$ and $\mathcal{B}$.

2. Closed subideals associated to H.I. type spaces

In this section we discuss examples which highlight various differences between closed ideals and non-trivial closed subideals of the algebras $\mathcal{L}(X)$. For instance, there are Banach spaces X such that $\mathcal{L}(X)$ has only finitely many closed ideals, but contains a continuum of non-trivial closed subideals

(Theorem 2.3). On the other hand, there are spaces X for which $\mathcal{L}(X)$ has a large class of closed ideals, but fails to contain any non-trivial closed subideals (Example 2.7).

These examples involve a precise understanding of the closed ideals of $\mathcal{L}(X)$, and some of the relevant spaces will be drawn from the class of hereditarily indecomposable spaces. Recall that the Banach space X is *hereditarily indecomposable* (H.I.) if no closed infinite-dimensional subspace $M \subset X$ can be decomposed as a direct sum $M = M_1 \oplus M_2$, where $M_j \subset M$ are closed infinite-dimensional subspaces for $j = 1, 2$. The first examples of H.I. spaces were constructed by Gowers and Maurey [17], and this class appears in many connections, see e.g. the survey [27].

We state the following relevant fact for explicit reference.

Remark 2.1. Let X be an arbitrary Banach space and denote by $\mathcal{F}(X)$ the class of bounded finite-rank operators $X \to X$. If $\mathcal{J}$ is any non-zero closed subideal of $\mathcal{L}(X)$, then $\mathcal{A}(X) \subset \mathcal{J}$, where $\mathcal{A}(X) = \overline{\mathcal{F}(X)}$ is the closed ideal of the approximable operators $X \to X$, see e.g. [5, Theorem 2.5.8.(ii)] or [49, (2.1)]. In particular, if X has the approximation property (A.P.), then $\mathcal{K}(X) = \mathcal{A}(X) \subset \mathcal{J}$ for any such $\mathcal{J} \neq \{0\}$.

Argyros and Haydon [2] constructed a real H.I. Banach space X_{AH} having a Schauder basis, such that

$$\mathcal{L}(X_{AH}) = \{\lambda I_{X_{AH}} + T : \lambda \in \mathbb{R},\ T \in \mathcal{K}(X_{AH})\}.$$

The algebra $\mathcal{L}(X_{AH})$ does not have any non-trivial closed subideals, because X_{AH} has the A.P. so that $\mathcal{K}(X_{AH})$ is the only non-trivial closed ideal of $\mathcal{L}(X_{AH})$. Subsequently Tarbard [46], [45] obtained H.I. spaces X_k for $k \geq 2$, such that $\mathcal{L}(X_k)$ has relatively few operators and the closed ideals of $\mathcal{L}(X_k)$ are explicitly classified (see (2.2) below).

Let X and Y be Banach spaces. Recall that $T \in \mathcal{L}(X, Y)$ is a *strictly singular* operator, denoted $T \in \mathcal{S}(X, Y)$, if the restriction of T does not define a linear isomorphism $M \to TM$ for any closed infinite-dimensional subspace $M \subset X$. The class $\mathcal{S}$ is a closed Banach operator ideal in the sense of Pietsch [36], so that $\mathcal{S}(X)$ is a closed ideal of $\mathcal{L}(X)$ for any X, see e.g. [24, Proposition 2.c.5] or [36, Section 1.9]. Moreover, $\mathcal{K}(X, Y) \subset \mathcal{S}(X, Y)$ for any X and Y.

It is a natural question whether the algebras $\mathcal{L}(X_k)$ admit any non-trivial closed subideals. The answer turns out to depend on k. Recall from [45] that the Tarbard space X_k is a separable real $\mathcal{L}^\infty$-space which has a Schauder basis, the dual space $X_k^* \approx \ell^1$, and that the following properties are satisfied for $k \geq 2$:

(i) There is a strictly singular operator $S \in \mathcal{S}(X_k) \setminus \mathcal{K}(X_k)$ such that $S^{k-1} \notin \mathcal{K}(X_k)$ and $S^k = 0$, and every $U \in \mathcal{L}(X_k)$ has the unique representation

$$U = \alpha I_{X_k} + \sum_{j=1}^{k-1} \alpha_j S^j + T, \tag{2.1}$$

where $\alpha, \alpha_1, \ldots, \alpha_{k-1} \in \mathbb{R}$ and $T \in \mathcal{K}(X_k)$. In particular,

$$\big(I_{X_K} + \mathcal{K}(X_k), S + \mathcal{K}(X_k), \ldots, S^{k-1} + \mathcal{K}(X_k)\big)$$

is a linear basis of the k-dimensional Calkin algebra $\mathcal{L}(X_k)/\mathcal{K}(X_k)$.

(ii) The non-zero closed ideals of $\mathcal{L}(X_k)$ form the chain

$$\mathcal{K}(X_k) \subsetneq [S^{k-1}] \subsetneq [S^{k-2}] \subsetneq \ldots \subsetneq [S^2] \subsetneq [S] = \mathcal{S}(X_k) \subsetneq \mathcal{L}(X_k). \tag{2.2}$$

Here $[S^j]$ denotes the closed ideal of $\mathcal{L}(X_k)$ generated by the operator S^j for $j = 1, \ldots, k-1$. Property (2.1) implies that

$$[S^j] = span(S^j, \ldots, S^{k-1}) + \mathcal{K}(X_k). \tag{2.3}$$

Hence the quotients $\mathcal{L}(X_k)/[S]$, $[S^{k-1}]/\mathcal{K}(X_k)$, as well as $[S^j]/[S^{j+1}]$ for $j = 1, \ldots, k-2$, are 1-dimensional.

We first point out that in the above setting any closed $\mathcal{S}(X_k)$-subideal $\mathcal{J}$ is actually an ideal of $\mathcal{L}(X_k)$.

Lemma 2.2. *Suppose that X is a Banach space, and assume that the closed ideal $\mathcal{I} \subset \mathcal{L}(X)$ has codimension 1 in $\mathcal{L}(X)$. Then every closed $\mathcal{I}$-subideal $\mathcal{J}$ is an ideal of $\mathcal{L}(X)$.*

Proof. Suppose that $V \in \mathcal{J}$ and $U = \alpha I_X + T \in \mathcal{L}(X)$ are arbitrary, where $T \in \mathcal{I}$. (Recall that $I_X \notin \mathcal{I}$.) It follows that $VU = \alpha V + VT \in \mathcal{J}$ and $UV = \alpha V + TV \in \mathcal{J}$, that is, $\mathcal{J}$ is an ideal of $\mathcal{L}(X)$. □

For $n \geq 2$ there are spaces X and closed ideals $\mathcal{I} \subset \mathcal{L}(X)$ having codimension n, such that $\mathcal{L}(X)$ contains non-trivial closed $\mathcal{I}$-subideals $\mathcal{J}$, see Remark 2.4 below.

Theorem 2.3. *(i) The algebras $\mathcal{L}(X_2)$ and $\mathcal{L}(X_3)$ do not have any non-trivial closed subideals.*

(ii) Let $k \geq 4$ and fix $j \in \mathbb{N}$ such that $k/2 \leq j \leq k-2$. Let $S \in \mathcal{S}(X_k)$ be the operator that satisfies (2.1) – (2.3). *Then the family $\mathfrak{F}_j$ consisting of the non-trivial closed $[S^j]$-subideals of $\mathcal{L}(X_k)$ equals the class of closed subalgebras*

$$\mathcal{I}_N := N + \mathcal{K}(X_k), \tag{2.4}$$

where $\{0\} \neq N \subset span(S^j, \ldots, S^{k-1})$ is any linear subspace that satisfies

$$N \neq span(S^r, \ldots, S^{k-1}), \text{ for } r = j, \ldots, k-1.$$

Each family $\mathfrak{F}_j$ has the size of the continuum.

Proof. (i) $\mathcal{K}(X_2)$ and $\mathcal{S}(X_2)$ are the only proper closed ideals $\mathcal{I}$ of $\mathcal{L}(X_2)$, and any non-zero closed subideal $\mathcal{J}$ must satisfy $\mathcal{K}(X_2) \subset \mathcal{J} \subset \mathcal{S}(X_2)$ in view of Remark 2.1. Since $\mathcal{S}(X_2)/\mathcal{K}(X_2)$ is 1-dimensional, there are not even any closed linear subspaces strictly between $\mathcal{K}(X_2)$ and $\mathcal{S}(X_2)$.

Analogously, if $\mathcal{J}$ is a non-trivial closed $\mathcal{I}$-subideal of $\mathcal{L}(X_3)$, then by (2.2) and Lemma 2.2 the intermediate closed ideal $\mathcal{I} = [S^2]$. However, once again there are no closed linear subspaces $\mathcal{K}(X_3) \subsetneq \mathcal{J} \subsetneq [S^2]$, since the quotient $[S^2]/\mathcal{K}(X_3)$ is 1-dimensional.

(ii) Note first that for any $k \geq 4$ it is possible to find $j \in \{2, \ldots, k-2\}$ such that $2j \geq k$. (In fact, choose $j = 2$ for $k = 4$, $j = 3$ for $k = 5$, and for $k \geq 6$ there are progressively more choices for j.)

We will require the cancellation property

$$UV \in \mathcal{K}(X_k) \text{ for } U \in [S^j] \text{ and } V \in [S^m], \tag{2.5}$$

whenever j and m satisfy $j+m \geq k$. We will actually verify a more comprehensive statement, where the case $j+m<k$ will be relevant in Theorem 5.4.

Claim. If $U \in [S^l]$ and $V \in [S^m]$, where $l, m \in \{1, \dots, k-1\}$, then

$$UV \in [S^{l+m}] \ \text{ if } l+m<k, \quad UV \in \mathcal{K}(X_k) \ \text{ if } l+m \geq k. \tag{2.6}$$

In fact, write $U = \sum_{r=l}^{k-1} a_r S^r + R_1$ and $V = \sum_{s=m}^{k-1} b_s S^s + R_2$, where $R_1, R_2 \in \mathcal{K}(X_k)$. If $l+m<k$ then we get from $S^k = 0$ that

$$UV = \sum_{u=0}^{k-1-(l+m)} \Big(\sum_{t=0}^{u} a_{l+t} b_{m+u-t} \Big) S^{l+m+u} + R \in [S^{l+m}],$$

as $R := \Big(\sum_{r=l}^{k-1} a_r S^r \Big) R_2 + R_1 \Big(\sum_{s=m}^{k-1} b_s S^s \Big) + R_1 R_2 \in \mathcal{K}(X_k)$. Moreover, if $l+m \geq k$, then $UV \in \mathcal{K}(X_k)$ since $(\sum_{r=l}^{k-1} a_r S^r)(\sum_{s=m}^{k-1} b_s S^s) = 0$ in view of the property that $S^k = 0$.

Next we make the following elementary observation: if M and N are distinct subspaces of $span(S^j, \dots, S^{k-1})$, then

$$M + \mathcal{K}(X_k) \neq N + \mathcal{K}(X_k). \tag{2.7}$$

This follows from linear algebra and the fact that

$$span(S^j, \dots, S^{k-1}) \cap \mathcal{K}(X_k) = \{0\}.$$

We proceed by showing that $\mathcal{I}_N := N + \mathcal{K}(X_k)$ is a non-trivial closed $[S^j]$-subideal of $\mathcal{L}(X_k)$ for any non-zero linear subspace $N \subset span(S^j, \dots, S^{k-1})$ such that $N \neq span(S^r, \dots, S^{k-1})$ for $r = j, \dots, k-1$. By definition $\mathcal{I}_N$ is a closed linear subspace of $[S^j]$, and (2.5) together with the assumption $2j \geq k$ gives that

$$UA, \ AU \in \mathcal{K}(X_k) \subset \mathcal{I}_N$$

for any $U \in \mathcal{I}_N$ and $A \in [S^j]$. Thus $\mathcal{I}_N$ is a closed ideal of $[S^j]$. On the other hand, $\mathcal{I}_N$ is not a closed ideal of $\mathcal{L}(X_k)$ in view of (2.2), (2.3) and (2.7).

Next, suppose that $\mathcal{M}$ is a non-trivial closed $[S^j]$-subideal of $\mathcal{L}(X_k)$. By Remark 2.1 and (2.3) we have

$$\mathcal{K}(X_k) \subsetneq \mathcal{M} \subsetneq span(S^j, \dots, S^{k-1}) + \mathcal{K}(X_k)$$

and thus $\mathcal{M} = N + \mathcal{K}(X_k)$ for some linear subspace N of $span(S^j, \dots, S^{k-1})$. Moreover, $N \neq span(S^r, \dots, S^{k-1})$ for any $r = j, \dots, k-1$ by (2.2) and (2.3) since $\mathcal{M}$ is not a closed ideal of $\mathcal{L}(X_k)$.

Finally, since $\#(\{j, \dots, k-1\}) \geq 2$ by assumption, there is a continuum of distinct subspaces $N \subset span(S^j, \dots, S^{k-1})$ for which $N \neq span(S^r, \dots, S^{k-1})$ for $r = j, \dots, k-1$. It follows from (2.7) that there is a continuum of non-trivial closed $[S^j]$-subideals of $\mathcal{L}(X_k)$.

Conclude that the family $\mathfrak{F}_j$ in (2.4) has the size of the continuum for each $j \in \mathbb{N}$ that satisfies $k/2 \leq j \leq k-2$. $\square$

Remark 2.4. Let $r \geq 2$. By Theorem 2.3 there are (continuum many) non-trivial closed $[S^r]$-subideals of $\mathcal{L}(X_{2r})$, where $dim(\mathcal{L}(X_{2r})/[S^r]) = r$ in view of (2.1) and (2.3). Hence Lemma 2.2 does not have a general counterpart for the closed $\mathcal{I}$-subideals of $\mathcal{L}(X)$, where $\mathcal{I} \subset \mathcal{L}(X)$ has codimension $r \geq 2$.

The following simple fact provides a systematic tool for producing non-trivial closed (sub)ideals of certain algebras $\mathcal{L}(X)$. Special instances have been used e.g. in [48, Proposition 2.1] and [52, Proposition 5.2].

Lemma 2.5. *Let X be a Banach space and suppose that $q : \mathcal{L}(X) \to \mathcal{A}$ is a continuous surjective algebra homomorphism onto the Banach algebra $\mathcal{A}$.*

(i) If $\mathcal{J}$ is a non-trivial closed $\mathcal{I}$-subideal of $\mathcal{L}(X)$ such that $ker(q) \subset \mathcal{J}$, then $q(\mathcal{J})$ is a non-trivial closed $q(\mathcal{I})$-subideal of $\mathcal{A}$.

(ii) If $\mathcal{J}$ is a non-trivial closed $\mathcal{I}$-subideal of $\mathcal{A}$, then $q^{-1}(\mathcal{J})$ is a non-trivial closed $q^{-1}(\mathcal{I})$-subideal of $\mathcal{L}(X)$ for which $ker(q) \subset q^{-1}(\mathcal{J})$.

Hence there is a bijective correspondence between the non-trivial closed subideals $ker(q) \subset \mathcal{J} \subset \mathcal{L}(X)$ and the non-trivial closed subideals of $\mathcal{A}$.

Proof. (i) We first verify that $q(\mathcal{J})$ is closed in $\mathcal{A}$. Suppose that $q(S) \in \overline{q(\mathcal{J})}$ is arbitrary, and let $(S_n) \subset \mathcal{J}$ be a sequence for which $q(S_n - S) \to 0$ as $n \to \infty$. Since q induces a linear isomorphism $\mathcal{L}(X)/ker(q) \to \mathcal{A}$, there is a sequence $(R_n) \subset ker(q) \subset \mathcal{J}$ for which $||S - S_n - R_n|| \to 0$ in $\mathcal{L}(X)$ as $n \to \infty$. Thus $S \in \mathcal{J}$ since $\mathcal{J}$ is a closed subalgebra and $S_n + R_n \in \mathcal{J}$ for all $n \in \mathbb{N}$. Similarly $q(\mathcal{I})$ is closed in $\mathcal{A}$.

Since q is a Banach algebra homomorphism we get that $q(\mathcal{J}) \subset q(\mathcal{I})$ and $q(\mathcal{I}) \subset \mathcal{A}$ are respective ideals. Finally, $q(\mathcal{J})$ is not an ideal of $\mathcal{A}$. In fact, suppose that $S \in \mathcal{J}$ and $U \in \mathcal{L}(X)$ satisfy $SU \notin \mathcal{J}$. It follows that $q(S)q(U) = q(SU) \notin q(\mathcal{J})$, since otherwise $SU - R \in ker(q)$ for some $R \in \mathcal{J}$ and hence $SU \in \mathcal{J}$. The case where $US \notin \mathcal{J}$ is similar. Conclude that $q(\mathcal{J})$ is a non-trivial closed $q(\mathcal{I})$-subideal of $\mathcal{A}$.

(ii) The argument is an easier variant of part (i), and the details are left to the readers. □

There are interesting variations of Theorem 2.3 for certain spaces introduced by Kania and Laustsen [20]. Let $m_1, \ldots, m_n \in \mathbb{N}$ be given. They constructed in [20, Note added in proof] a space $Z = Y_1^{m_1} \oplus \cdots \oplus Y_n^{m_n}$, for which the Calkin algebra

$$\mathcal{L}(Z)/\mathcal{K}(Z) \cong M_{m_1}(\mathbb{K}) \oplus \cdots \oplus M_{m_n}(\mathbb{K}).$$

Here $M_r(\mathbb{K})$ is the algebra of scalar $r \times r$-matrices. The spaces Y_j were constructed in [2, Theorem 10.4], and they satisfy $\mathcal{L}(Y_j, Y_k) = \mathcal{K}(Y_j, Y_k)$ for $j \neq k$ with $j, k \in \{1, \ldots, n\}$. In particular,

$$\mathcal{L}(Y_j^{m_j}, Y_k^{m_k}) = \mathcal{K}(Y_j^{m_j}, Y_k^{m_k}) \text{ for } j \neq k, \tag{2.8}$$

see (i) below. It is pointed out in [20, page 1023] that the non-zero closed ideals of $\mathcal{L}(Z)$ are precisely

$$\mathcal{I}_N := \{T = [T_{jk}]_{j,k=1}^n : T_{jj} \in \mathcal{K}(Y_j^{m_j}) \text{ for all } j \notin N\}, \tag{2.9}$$

where $N \subset \{1, \ldots, n\}$ is any subset.

We recall that the operator $T \in \mathcal{L}(X, Y)$ between finite direct sums $X = X_1 \oplus \cdots \oplus X_n$ and $Y = Y_1 \oplus \cdots \oplus Y_m$ can be represented as an $m \times n$-operator matrix $T = [T_{jk}]$, where $T_{jk} = P_{Y_j} T J_{X_k} \in \mathcal{L}(X_k, Y_j)$ and $P_{Y_j} : Y \to Y_j$, respectively $J_{X_k} : X_k \to X$, are the natural projections and inclusions related to the components of the direct sums X and Y. The following facts are well known and easy to check by using the identities $I_X = \sum_{r=1}^n J_{X_r} P_{X_r}$ and $I_Y = \sum_{r=1}^m J_{Y_r} P_{Y_r}$.

(i) If $T \in \mathcal{L}(X,Y)$, then

$$T = \sum_{j=1}^{m} \sum_{k=1}^{n} J_{Y_j} T_{jk} P_{X_k}.$$

In particular, $T \in \mathcal{K}(X,Y)$ if and only if $T_{jk} \in \mathcal{K}(X_k, Y_j)$ for all $1 \le k \le n$ and $1 \le j \le m$.

(ii) If $T, U \in \mathcal{L}(X)$, then

$$[TU]_{jk} = \sum_{r=1}^{n} T_{jr} U_{rk} \tag{2.10}$$

for all $1 \le j, k \le n$.

Example 2.6. Let $Z = Y_1^{m_1} \oplus \cdots \oplus Y_n^{m_n}$, where $m_1, \ldots, m_n \in \mathbb{N}$. Then the Banach algebra $\mathcal{L}(Z)$ does not have any non-trivial closed subideals.

Proof. Fix $N \subset \{1, \ldots, n\}$ and let $\mathcal{I}_N$ be the closed ideal of $\mathcal{L}(Z)$ from (2.9). Let $q : \mathcal{L}(Z) \to \mathcal{L}(Z)/\mathcal{K}(Z)$ be the quotient homomorphism. Since $\ker q = \mathcal{K}(Z) \subset \mathcal{I}_N$ by (2.9), it suffices in view of Lemma 2.5 to verify that there are no non-trivial closed $q(\mathcal{I}_N)$-subideals of $\mathcal{A} := \mathcal{L}(Z)/\mathcal{K}(Z)$.

Define $U = [U_{jk}]_{j,k=1}^n \in \mathcal{L}(Z)$ by

$$U_{jk} = \begin{cases} I_j, & \text{if } j = k \in N \\ 0, & \text{otherwise,} \end{cases}$$

where I_j is the identity operator on the component $Y_j^{m_j} \subset Z$. Note that $U \in \mathcal{I}_N$. We claim that $q(U)$ is the unit element of $q(\mathcal{I}_N)$. For this, let $T \in \mathcal{I}_N$. We observe that

$$[T - TU]_{jj} = T_{jj} - [TU]_{jj} = T_{jj} - \sum_{k=1}^{n} T_{jk} U_{kj} = \begin{cases} 0, & \text{if } j \in N \\ T_{jj}, & \text{if } j \notin N \end{cases}$$

for all $1 \le j \le n$. It follows that $T - TU \in \mathcal{K}(Z)$ in view of (2.8) and (2.9). In a similarly way one verifies that $UT - T \in \mathcal{K}(Z)$. Consequently, $q(U)q(T) = q(UT) = q(T)$ and $q(T)q(U) = q(T)$.

This implies that there are no non-trivial closed $q(\mathcal{I}_N)$-subideals of $\mathcal{A}$. In fact, assume that $\mathcal{J}$ is a closed $q(\mathcal{I}_N)$-subideal of $\mathcal{A}$. If $v \in \mathcal{J}$ and $a \in \mathcal{A}$, then

$$va = (vq(U))a = v(q(U)a) \in \mathcal{J}$$

and similarly $av \in \mathcal{J}$. □

Next we provide examples of Banach spaces X, where $\mathcal{L}(X)$ has an infinite family of closed ideals but fails to have any non-trivial closed subideals. For this purpose we will use the spaces recently constructed by Motakis [28, Theorem A]: *for any compact metric space K there is a separable $\mathcal{L}^\infty$-space $X_{C(K)}$, such that $X_{C(K)}^* \approx \ell^1$ and*

$$\mathcal{L}(X_{C(K)})/\mathcal{K}(X_{C(K)}) \cong C(K) \tag{2.11}$$

are isomorphic as Banach algebras. Here $C(K)$ is the sup-normed Banach algebra of continuous scalar-valued functions defined on K. The construction of $X_{C(K)}$ is highly complicated, but we will only require the existence of a surjective Banach algebra homomorphism $\mathcal{L}(X_{C(K)}) \to C(K)$ as given by (2.11). The results in [28] apply to both real and complex scalars.

Recall that if K is a compact metric space (or even a compact Hausdorff topological space), then the family of closed ideals $\mathcal{I} \subset C(K)$ coincides with

$$\{\mathcal{I}(L) : L \subset K \text{ is a closed subset}\}, \tag{2.12}$$

where $\mathcal{I}(L) = \{f \in C(K) : f_{|L} = 0\}$. We refer e.g. to [21, Theorem 1.4.6] for an argument that also applies to real $C(K)$-algebras.

Example 2.7. Let K be any compact metric space and let $X_{C(K)}$ be the space from [28] for which (2.11) holds. Fix a surjective Banach algebra homomorphism $q_K : \mathcal{L}(X_{C(K)}) \to C(K)$ such that $ker(q_K) = \mathcal{K}(X_{C(K)})$.

Then the closed ideals of $\mathcal{L}(X_{C(K)})$ coincide with the family

$$\{q_K^{-1}(\mathcal{I}(L)) : L \subset K \text{ is a closed subset}\}, \tag{2.13}$$

but $\mathcal{L}(X_{C(K)})$ does not have any non-trivial closed subideals.

Proof. According to the construction in [28] the spaces $X_{C(K)}$ have a Schauder basis, so $ker(q_K) = \mathcal{K}(X_{C(K)})$ is the smallest non-zero closed (sub)ideal of $\mathcal{L}(X_{C(K)})$ by Remark 2.1. Hence (2.12) together with a similar argument as in Lemma 2.5 imply that the closed ideals of $\mathcal{L}(X_{C(K)})$ are given by (2.13).

Lemma 2.5 implies that if $\mathcal{J}$ is a non-trivial closed $\mathcal{I}$-subideal of $\mathcal{L}(X_{C(K)})$, then $q_K(\mathcal{J})$ is a non-trivial closed $q_K(\mathcal{I})$-subideal of $C(K)$. On the other hand, the Banach algebras $C(K)$ do not have any non-trivial closed subideals. In fact, the elementary metric version of Urysohn's lemma implies that any closed ideal $\mathcal{I}(L)$ of $C(K)$ has a bounded approximate identity, so that any closed $\mathcal{I}(L)$-subideal $\mathcal{J}$ is an ideal of $C(K)$ by [49, Lemma 3.1]. (Alternatively, for complex scalars one may appeal to e.g. [5, Theorem 3.2.21].) $\square$

Remarks 2.8. (1) The family of Banach spaces

$$\{X_{C(K)} : K \text{ is an infinite compact metric space}\}$$

is fairly large: if K_1 and K_2 are not homeomorphic metric spaces, then $X_{C(K_1)}$ and $X_{C(K_2)}$ are not linearly isomorphic spaces. Namely, if $U : X_{C(K_2)} \to X_{C(K_1)}$ is a linear isomorphism, then $S \mapsto \psi(S) := U^{-1}SU$ defines a Banach algebra isomorphism $\mathcal{L}(X_{C(K_1)}) \to \mathcal{L}(X_{C(K_2)})$ for which $\psi(\mathcal{K}(X_{C(K_1)}) = \mathcal{K}(X_{C(K_2)})$. Thus ψ induces a Banach algebra isomorphism

$$\mathcal{L}(X_{C(K_1)})/\mathcal{K}(X_{C(K_1)}) \to \mathcal{L}(X_{C(K_2)})/\mathcal{K}(X_{C(K_2)})$$

through $S + \mathcal{K}(X_{C(K_1)}) \mapsto \psi(S) + \mathcal{K}(X_{C(K_2)})$, so that $C(K_1) \cong C(K_2)$. Moreover, if K_1 and K_2 are compact metric spaces, then $C(K_1) \cong C(K_2)$ if and only if K_1 and K_2 are homeomorphic spaces. (See e.g. [11, Theorem IV.6.26 and Corollary IV.6.27], which also applies to real scalars.)

(2) For each countable compact metric space K the earlier result [29, Theorem 5.1] constructs a real Banach space X_K of similar type as $X_{C(K)}$ for which (2.11) holds. The analogue of Example 2.7 also holds here.

More restricted conclusions are available for earlier constructions of strange quotient algebras of particular $\mathcal{L}(X)$-algebras.

Remarks 2.9. (1) Mankiewicz [26, Theorem 1.1] constructed a real reflexive Banach space X_M having a finite-dimensional decomposition, such that there is a surjective Banach algebra homomorphism $q : \mathcal{L}(X_M) \to C(\beta\mathbb{N})$, where $\beta\mathbb{N}$ is the Stone-Čech compactification of $\mathbb{N}$. By arguing as in Example 2.7 it follows that there does not exist any non-trivial closed subideals $\mathcal{J}$ of $\mathcal{L}(X_M)$ that satisfy

$$ker(q) \subsetneq \mathcal{J} \subsetneq \mathcal{L}(X_M). \tag{2.14}$$

The construction of X_M involves random methods, and we are not aware of an explicit description of the ideal $ker(q) \subset \mathcal{L}(X_M)$.

(2) Let J_p be the p-James space defined in terms of the p-variation norm for $p \in (1, \infty)$, see e.g. [25, page 344] or (4.25) from Section 4. Let

$$Y = \Big(\bigoplus_{k\in\mathbb{Z}} J_{p_k}\Big)_{\ell^1},$$

where $(p_k)_{k\in\mathbb{Z}} \subset (1, \infty)$ is a strictly increasing sequence. Dales, Loy and Willis [7, Proposition 3.5] found an explicit surjective Banach algebra homomorphism $q : \mathcal{L}(Y) \to \ell^\infty(\mathbb{Z}) \cong C(\beta\mathbb{Z})$, where $\mathcal{K}(Y) \subsetneq \mathcal{W}(Y) \subset ker(q)$. Here $\mathcal{W}(Y)$ is the closed ideal of $\mathcal{L}(Y)$ consisting of the weakly compact operators on Y. Thus the analogue of (2.14) also holds in $\mathcal{L}(Y)$.

3. New examples of non-trivial closed $\mathcal{K}(Z_p)$-subideals of $\mathcal{L}(Z_p)$

A family $\mathfrak{F} = \{\mathcal{I}_B : \emptyset \neq B \subsetneq \mathbb{N}\}$ of non-trivial closed $\mathcal{K}(Z_p)$-subideals was constructed in [48, Theorem 4.5] for certain direct sums Z_p. It was subsequently shown in [49, Theorem 2.1] that under a natural condition on Z_p the family $\mathfrak{F}$ has the property that $\mathcal{I}_{B_1}$ and $\mathcal{I}_{B_2}$ are isomorphic Banach algebras for any subsets $\emptyset \neq B_1, B_2 \subsetneq \mathbb{N}$. It is known, see e.g. [19, Added in Proof], that such a behaviour is not possible among the closed ideals of $\mathcal{L}(X)$ for any Banach space X. These results motivate a more careful analysis of the structure of closed subideals of $\mathcal{L}(Z_p)$, where many interesting phenomena can be seen explicitly. In this section we find a new concrete family of non-trivial closed $\mathcal{K}(Z_p)$-subideals of $\mathcal{L}(Z_p)$, which is disjoint from $\mathfrak{F}$ and which has the size of the continuum. Recall also that non-trivial closed $\mathcal{K}(X)$-subideals can only exist within the class of Banach spaces that fail the approximation property.

We start by recalling the setting of [48, Theorem 4.5]. Let X be a Banach space such that X has the A.P., but X^* fails the A.P. It follows from known results, see e.g. [24, Theorem 1.e.5], that there is a Banach space Y such that $\mathcal{A}(X, Y) \subsetneq \mathcal{K}(X, Y)$. Define

$$Z_p = \Big(\bigoplus_{j=0}^{\infty} X_j\Big)_{\ell^p}, \tag{3.1}$$

where $X_0 = Y$ and $X_j = X$ for $j \geq 1$. Here $1 < p < \infty$ is fixed. Let $\emptyset \neq B \subsetneq \mathbb{N}$ and put

$$\mathcal{I}_B = \{S = [S_{jk}] \in \mathcal{K}(Z_p) : S_{00} \in \mathcal{A}(Y),\ S_{0k} \in \mathcal{A}(X, Y) \text{ for } k \in B\}. \tag{3.2}$$

Here $S_{jk} = P_j S J_k \in \mathcal{K}(X_k, X_j)$ for $j, k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ by the ideal property of $\mathcal{K}$, where $P_j : Z_p \to X_j$ and $J_k : X_k \to Z_p$ are the natural projection and inclusion operators associated to the component spaces of the direct sum Z_p. We will occasionally use the alternative notation $S_{jk} = [S]_{jk}$ if this is more convenient.

It was shown in [48, Theorem 4.5] that the family

$$\mathfrak{F} = \{\mathcal{I}_B : \emptyset \neq B \subsetneq \mathbb{N}\} \tag{3.3}$$

consists of non-trivial closed $\mathcal{K}(Z_p)$-subideals of $\mathcal{L}(Z_p)$, which has the reversed order structure compared to $\mathcal{P}(\mathbb{N})$. Moreover, if X satisfies the linear isomorphism condition

$$\Big(\bigoplus_{j\in\mathbb{N}} X\Big)_{\ell^p} \approx X, \tag{3.4}$$

then $\mathcal{I}_{B_1} \cong \mathcal{I}_{B_2}$ are isomorphic Banach algebras for any $B_1 \neq B_2$, see [49, Theorem 2.1].

For any non-zero sequence $c = (c_j) \in \ell^1$, let

$$supp(c) = \{j \in \mathbb{N} : c_j \neq 0\}$$

be the support of c. We introduce

$$\begin{aligned} \mathcal{I}(c) :=\big\{&S = [S_{jk}] \in \mathcal{K}(Z_p) : S_{00} \in \mathcal{A}(Y) \text{ and} \\ &x \mapsto \sum_{r=1}^{\infty} c_r S_{0r} x \text{ is an approximable operator } X \to Y\big\}. \end{aligned} \tag{3.5}$$

The series $\sum_{r=1}^{\infty} c_r S_{0r}$ is absolutely convergent in the operator norm of $\mathcal{K}(X,Y)$, since $c = (c_j) \in \ell^1$. The support $supp(c)$ can be any finite or infinite set and we will assume that the cardinality $\#(supp(c)) \geq 2$. Namely, if $supp(c) = \{k\}$, then we see by comparing (3.5) and (3.2) that $\mathcal{I}(c) = \mathcal{I}_{\{k\}} \in \mathfrak{F}$.

The following theorem contains the first main result of Section 3. We will repeatedly use the facts that $\mathcal{K}(Y,X) = \mathcal{A}(Y,X)$ and $\mathcal{K}(X) = \mathcal{A}(X)$ as X has the A.P.

Theorem 3.1. *Let Z_p be the space from (3.1), and assume that the non-zero sequence $c = (c_j) \in \ell^1$ satisfies $\#(supp(c)) \geq 2$. Then the following properties hold:*

(i) *$\mathcal{I}(c)$ is a closed ideal of $\mathcal{K}(Z_p)$, as well as a left ideal of $\mathcal{L}(Z_p)$.*
(ii) *$\mathcal{I}(c)$ is not a right ideal of $\mathcal{L}(Z_p)$, so that $\mathcal{I}(c)$ is a non-trivial closed $\mathcal{K}(Z_p)$-subideal of $\mathcal{L}(Z_p)$.*
(iii) *$\mathcal{I}(c) \neq \mathcal{I}_B$ for any subset $\emptyset \neq B \subsetneq \mathbb{N}$.*
(iv) *If $d \in \ell^1$ is a non-zero sequence such that $supp(d) \neq supp(c)$, then $\mathcal{I}(d) \neq \mathcal{I}(c)$.*
(v) *Let $d \in \ell^1$ be a non-zero sequence. Then $\mathcal{I}(c) = \mathcal{I}(d)$ if and only if $c = \lambda d$ for some non-zero scalar λ.*

Hence the family

$$\mathfrak{F}_+ = \{\mathcal{I}(c) : c \in \ell^1 \text{ and } \#(supp(c)) \geq 2\}$$

consists of non-trivial closed $\mathcal{K}(Z_p)$-subideals of $\mathcal{L}(Z_p)$. The family $\mathfrak{F}_+$ has the size of the continuum, and $\mathfrak{F}_+ \cap \mathfrak{F} = \emptyset$.

Proof. (i) Note first that for any $c = (c_j) \in \ell^1$ the map

$$\psi_c : S \mapsto \sum_{j=1}^{\infty} c_j S_{0j}, \quad \mathcal{L}(Z_p) \to \mathcal{L}(X, Y),$$

defines a bounded linear operator, since for $x \in X$ and $S \in \mathcal{L}(Z_p)$ we have

$$\|\sum_{j=1}^{\infty} c_j S_{0j} x\| \le \sum_{j=1}^{\infty} |c_j| \|S_{0j}\| \|x\| \le \|c\|_1 \|S\| \|x\|,$$

as $\|S_{0j}\| = \|P_0 S J_j\| \le \|S\|$ for $j \in \mathbb{N}$. Deduce that

$$\mathcal{I}(c) = \psi_c^{-1}(\mathcal{A}(X, Y)) \cap \{S \in \mathcal{K}(Z_p) : S_{00} \in \mathcal{A}(Y)\}$$

is a closed linear subspace of $\mathcal{K}(Z_p)$.

We next show that $\mathcal{I}(c)$ is a left ideal of $\mathcal{L}(Z_p)$. Let $S = [S_{jk}] \in \mathcal{I}(c)$ and $U = [U_{jk}] \in \mathcal{L}(Z_p)$ be arbitrary. According to (3.12) from the subsequent Lemma 3.2, for each $j \in \mathbb{N}$

$$[US]_{0j} = \sum_{l=0}^{\infty} U_{0l} S_{lj} = U_{00} S_{0j} + \sum_{l=1}^{\infty} U_{0l} S_{lj} = U_{00} S_{0j} + R_j \tag{3.6}$$

is a norm convergent series in $\mathcal{K}(X, Y)$, where we denote the operator norm limit by

$$R_j := \lim_{r \to \infty} \sum_{l=1}^{r} U_{0l} S_{lj} \in \mathcal{K}(X, Y).$$

Actually $R_j \in \mathcal{A}(X, Y)$ for $j \in \mathbb{N}$, since $S_{lj} \in \mathcal{A}(X)$ for all $l \ge 1$. From (3.6) we get for any fixed $r \in \mathbb{N}$ that

$$x \mapsto \sum_{j=1}^{r} c_j [US]_{0j} x = U_{00}\big(\sum_{j=1}^{r} c_j S_{0j} x\big) + \sum_{j=1}^{r} c_j R_j x, \quad x \in X. \tag{3.7}$$

Since $(c_j) \in \ell^1$ we let $r \to \infty$ in (3.7), and deduce that the pointwise operator

$$x \mapsto \sum_{j=1}^{\infty} c_j [US]_{0j} x = U_{00}\big(\sum_{j=1}^{\infty} c_j S_{0j} x\big) + \sum_{j=1}^{\infty} c_j R_j x$$

is approximable $X \to Y$. Here we use the facts that $x \mapsto \sum_{j=1}^{\infty} c_j S_{0j} x$ defines an approximable operator $X \to Y$ by assumption, and that $R_j \in \mathcal{A}(X, Y)$ for $j \in \mathbb{N}$. Moreover, Lemma 3.2 below implies that

$$[US]_{00} = U_{00} S_{00} + \sum_{l=1}^{\infty} U_{0l} S_{l0} \in \mathcal{A}(Y),$$

as $S_{00} \in \mathcal{A}(Y)$ and $S_{l0} \in \mathcal{A}(Y, X)$ for $l \ge 1$. We conclude that $US \in \mathcal{I}(c)$.

There remains to verify that $\mathcal{I}(c)$ is a right ideal of $\mathcal{K}(Z_p)$. Towards this suppose that $S = [S_{jk}] \in \mathcal{I}(c)$ and $U = [U_{jk}] \in \mathcal{K}(Z_p)$. In order to show that $SU \in \mathcal{I}(c)$ recall first that $U_{lj} \in \mathcal{K}(X) = \mathcal{A}(X)$ for any $j, l \ge 1$.

This implies that for each fixed $j \in \mathbb{N}$ the series

$$[SU]_{0j} = \sum_{l=0}^{\infty} S_{0l} U_{lj} = S_{00} U_{0j} + \sum_{l=1}^{\infty} S_{0l} U_{lj} = S_{00} U_{0j} + T_j \tag{3.8}$$

is norm convergent in $\mathcal{K}(X,Y)$ by (3.12) in Lemma 3.2 below, where as above

$$T_j := \lim_{s\to\infty} \sum_{l=1}^{s} S_{0l}U_{lj} \in \mathcal{A}(X,Y).$$

In particular, $[SU]_{0j} \in \mathcal{A}(X,Y)$ for $j \in \mathbb{N}$ as $S_{00} \in \mathcal{A}(Y)$ by assumption. Thus the operator $x \mapsto \sum_{j=1}^{\infty} c_j[SU]_{0j}x$ is approximable $X \to Y$ because $(c_j) \in \ell^1$. Moreover,

$$[SU]_{00} = S_{00}U_{00} + \sum_{l=1}^{\infty} S_{0l}U_{l0} \in \mathcal{A}(Y),$$

where again $S_{00} \in \mathcal{A}(Y)$ and $U_{l0} \in \mathcal{A}(Y,X)$ for $l \geq 1$. Conclude that $SU \in \mathcal{I}(c)$. In particular, $\mathcal{I}(c)$ is a closed ideal of $\mathcal{K}(Z_p)$.

For the proofs of parts (ii)-(v) we fix an operator

$$S_0 \in \mathcal{K}(X,Y) \setminus \mathcal{A}(X,Y). \tag{3.9}$$

This is possible since $\mathcal{A}(X,Y) \subsetneq \mathcal{K}(X,Y)$ in the construction of Z_p. We also require the following observation: let $r, s \in \mathbb{N}$ where $s \in supp(c)$. Then the operator

$$T_{(r,s)} := J_0S_0P_r - (c_r/c_s)J_0S_0P_s \in \mathcal{I}(c). \tag{3.10}$$

In fact, $T_{(r,s)} \in \mathcal{K}(Z_p)$ by (3.9) and $[T_{(r,s)}]_{00} = 0$. Moreover, for any $x \in X$ we have

$$\begin{aligned}\sum_{j=1}^{\infty} c_j[T_{(r,s)}]_{0j}x &= \sum_{j=1}^{\infty} c_jP_0\big(J_0S_0P_r - (c_r/c_s)J_0S_0P_s\big)J_jx \\ &= c_rS_0x - c_s(c_r/c_s)S_0x = 0.\end{aligned}$$

(ii) Let $r, s \in supp(c)$, $r \neq s$, and consider the operator $T := T_{(r,s)} \in \mathcal{I}(c)$ defined in (3.10). Let $U = J_rP_r \in \mathcal{L}(Z_p)$. It follows that

$$TU = \big(J_0S_0P_r - (c_r/c_s)J_0S_0P_s\big)J_rP_r = J_0S_0P_r \notin \mathcal{I}(c),$$

since

$$x \mapsto \sum_{j=1}^{\infty} c_j[TU]_{0j}x = \sum_{j=1}^{\infty} c_j[J_0S_0P_r]_{0j}x = c_rS_0x$$

is not an approximable operator $X \to Y$ as $c_r \neq 0$. This means that $\mathcal{I}(c)$ is not a right ideal of $\mathcal{L}(Z_p)$, whence $\mathcal{I}(c)$ is a non-trivial closed $\mathcal{K}(Z_p)$-subideal of $\mathcal{L}(Z_p)$.

(iii) Let $\emptyset \neq B \subsetneq \mathbb{N}$ be an arbitrary subset. Pick $r \in B$ and $s \in supp(c) \setminus \{r\}$ and consider again the operator $T := T_{(r,s)} \in \mathcal{I}(c)$ from (3.10). Since

$$T_{0r} = P_0\big(J_0S_0P_r - (c_r/c_s)J_0S_0P_s\big)J_r = S_0 \notin \mathcal{A}(X,Y)$$

we obtain that $T \notin \mathcal{I}_B$. Thus $\mathcal{I}(c) \neq \mathcal{I}_B$.

(iv) Suppose $r \in supp(c) \setminus supp(d)$, and let $R = J_0S_0P_r \in \mathcal{K}(Z_p)$, where S_0 is the operator from (3.9). Clearly $R \in \mathcal{I}(d)$ as $R_{0s} = 0$ for all $s \in supp(d) \cup \{0\}$. On the other hand,

$$x \mapsto \sum_{j=1}^{\infty} c_jR_{0j}x = c_rS_0x$$

is not approximable, so that $R \notin \mathcal{I}(c)$. The case where $s \in supp(d) \setminus supp(c)$ is symmetric.

(v) Note first that $\mathcal{I}(c) = \mathcal{I}(d)$ holds if $c = \lambda d$ for some non-zero scalar λ. In fact, if $S \in \mathcal{I}(d)$, then $x \mapsto \sum_{j=1}^{\infty} d_j S_{0j} x$ defines an approximable operator $X \to Y$, so that

$$x \mapsto \lambda\Big(\sum_{j=1}^{\infty} d_j S_{0j} x\Big) = \sum_{j=1}^{\infty} c_j S_{0j} x$$

is also approximable. Thus $S \in \mathcal{I}(c)$. The reverse inclusion $\mathcal{I}(c) \subset \mathcal{I}(d)$ is similar.

Conversely, assume that $c \neq \lambda d$ for all non-zero scalars λ. If $supp(c) \neq supp(d)$, then $\mathcal{I}(c) \neq \mathcal{I}(d)$ by part (iv). Thus we may assume that $supp(c) = supp(d)$. Let $r \in supp(c)$, that is, $c_r \neq 0$ and $d_r \neq 0$. It follows that there is $s \in supp(c)$ so that

$$\frac{c_r}{d_r} \neq \frac{c_s}{d_s}, \tag{3.11}$$

since otherwise $c = (c_r/d_r)d$, which contradicts the assumptions on c and d. Consider once again the operator

$$T := T_{(r,s)} = J_0 S_0 P_r - (c_r/c_s) J_0 S_0 P_s \in \mathcal{I}(c)$$

from (3.10). In this case the operator

$$x \mapsto \sum_{j=1}^{\infty} d_j T_{0j} x = d_r S_0 x - d_s (c_r/c_s) S_0 x = \frac{d_r c_s - d_s c_r}{c_s} S_0 x$$

is not approximable $X \to Y$, since $S_0 \notin \mathcal{A}(X,Y)$ and $d_r c_s - d_s c_r \neq 0$ in view of (3.11). Hence $T \notin \mathcal{I}(d)$, which demonstrates that $\mathcal{I}(c) \neq \mathcal{I}(d)$.

The unit sphere $\{c \in \ell^1 : \|c\|_1 = 1\}$ is separable, so it follows that the family $\mathfrak{F}_+$ has the size of the continuum. Part (iii) yields that $\mathfrak{F}_+ \cap \mathfrak{F} = \emptyset$.

The argument of Theorem 3.1 will be complete once we have established the subsequent Lemma 3.2, which was used in the proof of part (i). $\square$

The following technical fact is based on [48, Lemma 4.6].

Lemma 3.2. *Let $1 < p < \infty$ and $Z_p = \big(\bigoplus_{j=0}^{\infty} X_j\big)_{\ell^p}$, where X_j is a Banach space for $j \in \mathbb{N}_0$. If $S = [S_{jk}] \in \mathcal{K}(Z_p)$ and $U = [U_{jk}] \in \mathcal{L}(Z_p)$, then*

$$[US]_{0r} = \sum_{l=0}^{\infty} U_{0l} S_{lr} \tag{3.12}$$

for each $r \in \mathbb{N}_0$, where the right hand series converges in the operator norm of $\mathcal{K}(X_r, X_0)$. Moreover, (3.12) *also holds in case $S \in \mathcal{L}(Z_p)$ and $U \in \mathcal{K}(Z_p)$.*

Proof. Assume that $S \in \mathcal{K}(Z_p)$ and $U \in \mathcal{L}(Z_p)$. Let $Q_n = \sum_{s=0}^{n} J_s P_s$ be the natural projection of Z_p onto $\bigoplus_{s=0}^{n} X_s \subset Z_p$ for any $n \in \mathbb{N}_0$. Since S is a compact operator, it follows from the proof of [48, Lemma 4.6] that

$$\|Q_n S - S\| \to 0 \text{ as } n \to \infty.$$

Fix $r \in \mathbb{N}_0$, and observe that $\sum_{l=0}^{n} U_{0l}S_{lr} = P_0UQ_nSJ_r$ for each $n \in \mathbb{N}_0$. Since $[US]_{0r} = P_0USJ_r$ we get that

$$\|\sum_{l=0}^{n} U_{0l}S_{lr} - [US]_{0r}\| = \|P_0U(Q_nS - S)J_r\| \to 0 \tag{3.13}$$

as $n \to \infty$.

Next suppose that $S \in \mathcal{L}(Z_p)$ and $U \in \mathcal{K}(Z_p)$. Since $1 < p < \infty$ and $U \in \mathcal{K}(Z_p)$, it follows again from [48, Lemma 4.6]. that

$$\|UQ_n - U\| \to 0 \text{ as } n \to \infty.$$

An analogous estimate to (3.13) yields that (3.12) holds, with convergence in the operator norm of $\mathcal{K}(X_r, X_0)$. □

The subideals $\mathcal{I}(c)$ have a somewhat stronger property than stated in part (i) of Theorem 3.1. Recall for $n \geq 2$ that the Banach algebra $\mathcal{A}$ is said to be *n-nilpotent* if $a_1 \cdots a_n = 0$ whenever $a_j \in \mathcal{A}$ with $j = 1, \ldots, n$. The closed ideal $\mathcal{I}$ of $\mathcal{A}$ is n-nilpotent if $\mathcal{I}$ is n-nilpotent as a Banach algebra.

Remarks 3.3. (1) The closed ideals $q(\mathcal{I}(c))$ of $\mathcal{K}(Z_p)/\mathcal{A}(Z_p)$ are 2-nilpotent, where $q : \mathcal{K}(Z_p) \to \mathcal{K}(Z_p)/\mathcal{A}(Z_p)$ denotes the quotient homomorphism. In fact, according to [48, Lemma 4.6] the products $SU \in \mathcal{A}(Z_p)$ whenever $S \in \mathcal{I}(c)$ and $U \in \mathcal{K}(Z_p)$, since $[SU]_{jk}$ are approximable operators for each $j, k \in \mathbb{N}_0$ by the argument for part (i) of Theorem 3.1.

(2) The reader may verify that for $c = (c_j) \in \ell^1$ the sets

$$\mathcal{J}(c) = \{S = [S_{jk}] \in \mathcal{K}(Z_p) : S_{00} \in \mathcal{A}(Y) \text{ and } \sum_{r=1}^{\infty} c_r S_{0r} P_r \in \mathcal{A}(Z_p, Y)\}$$

do not give any new closed subideals of $\mathcal{L}(Z_p)$, as $\mathcal{J}(c) = \mathcal{I}_{supp(c)}$. The difference to (3.5) is that

$$(\sum_{r=1}^{\infty} c_j S_{0r} P_r)x = \sum_{r=1}^{\infty} c_j S_{0r} x_r, \quad x = (x_j) \in Z_p,$$

where S_{0r} acts separately on the component $X_r = X$ of Z_p.

The result in [49, Theorem 2.1] suggests the question whether the closed subideals $\mathcal{I}(c)$ from the family $\mathfrak{F}_+$ are isomorphic as Banach algebras to the earlier subideals $\mathcal{I}_B$ from $\mathfrak{F}$. It turns out that, again unexpectedly, this is the case if condition (3.4) holds.

We first recall the strategy behind [49, Theorem 2.1]. Let Z be a Banach space and suppose that $\mathcal{A} \subset \mathcal{L}(Z)$ and $\mathcal{B} \subset \mathcal{L}(Z)$ are closed subalgebras. In order to show that $\mathcal{A}$ and $\mathcal{B}$ are isomorphic Banach algebras, it suffices to find a linear isomorphism $U : Z \to Z$ such that the inner automorphism

$$\theta(S) = U^{-1}SU, \quad S \in \mathcal{L}(Z), \tag{3.14}$$

of $\mathcal{L}(Z)$ satisfies $\theta(\mathcal{A}) \subset \mathcal{B}$ and $\theta^{-1}(\mathcal{B}) \subset \mathcal{A}$.

We note for completeness that (3.14) defines the only possible algebra isomorphism between non-zero closed subideals of $\mathcal{L}(Z)$. Namely, suppose conversely that $\theta : \mathcal{A} \to \mathcal{B}$ is a Banach algebra isomorphism, where the closed subalgebras $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{L}(Z)$ satisfy $\mathcal{F}(Z) \subset \mathcal{A}$ and $\mathcal{F}(Z) \subset \mathcal{B}$. Then a result of Chernoff [4, Corollary 3.2] (see also [32, Section 1.7.15]) implies that there is a linear isomorphism $U : Z \to Z$ such that (3.14) holds.

Theorem 3.4. *The closed subideals $\mathcal{I}(c) \cong \mathcal{I}(d)$ for any non-zero sequences $c, d \in \ell^1$, and $\mathcal{I}(c) \cong \mathcal{I}_{\{j\}}$ for any $j \in supp(c)$.*

In addition, if the component space X in the definition (3.1) *of Z_p satisfies condition* (3.4)*, then*

$$\mathcal{I}(c) \cong \mathcal{I}_B \quad \text{for any } \mathcal{I}(c) \in \mathfrak{F}_+ \text{ and } \mathcal{I}_B \in \mathfrak{F}. \tag{3.15}$$

Proof. The following claim contains the essential step of the argument.

Claim 1. Suppose that $c = (c_j) \in \ell^1$ is a non-zero sequence, and fix $j \in supp(c)$. Then

$$\mathcal{I}(c) \cong \mathcal{I}_{\{j\}}.$$

We define $V : Z_p \to Z_p$ by the conditions

$$[V(x)]_n = \begin{cases} x_0 & \text{if } n = 0, \\ c_j x_j & \text{if } n = j, \\ x_n + c_n x_j & \text{if } n \neq j \text{ and } n \neq 0, \end{cases}$$

for $x = (x_k) \in Z_p$. Here we use the notation $[V(x)]_n = P_n(V(x)) \in X_n$ for the n:th coordinate of $V(x) \in Z_p$.

Observe first that V is bounded on Z_p since $c \in \ell^1$. In fact, it is not difficult to check that

$$V = I_{Z_p} - J_j P_j + \sum_{r=1}^{\infty} c_r J_r I_{r,j} P_j$$

where $I_{r,j} : X_j \to X_r$ denotes the identity map $x \mapsto x$ (recall that $X_j = X_r = X$ for $j, r \in \mathbb{N}$).

It follows that V is a linear isomorphism $Z_p \to Z_p$, whose inverse V^{-1} is defined by

$$[V^{-1}(x)]_n = \begin{cases} x_0 & \text{if } n = 0, \\ c_j^{-1} x_j & \text{if } n = j, \\ x_n - c_n c_j^{-1} x_j & \text{if } n \neq j \text{ and } n \neq 0. \end{cases}$$

In other words,

$$V^{-1} = I_{Z_p} + c_j^{-1} J_j P_j - c_j^{-1} \sum_{r=1}^{\infty} c_r J_r I_{r,j} P_j.$$

We claim that

$$\theta(\mathcal{I}(c)) \subset \mathcal{I}_{\{j\}} \text{ and } \theta^{-1}(\mathcal{I}_{\{j\}}) \subset \mathcal{I}(c),$$

where θ is the inner automorphism of $\mathcal{L}(Z_p)$ defined by $T \mapsto V^{-1}TV$. We will apply the simple observations that

$$P_0 V = P_0 V^{-1} = P_0 \text{ and } V J_r = V^{-1} J_r = J_r \text{ for all } r \in \mathbb{N}_0 \setminus \{j\}. \tag{3.16}$$

Suppose first that $T \in \mathcal{I}(c)$. It is easy to see that

$$[\theta(T)]_{00} = P_0 V^{-1} T V J_0 = P_0 T J_0 = T_{00} \in \mathcal{A}(Y).$$

Next note that

$$VJ_j = \big(I_{Z_p} - J_jP_j + \sum_{r=1}^{\infty} c_r J_r I_{r,j} P_j\big) J_j = \sum_{r=1}^{\infty} c_r J_r I_{r,j}.$$

It follows with the help of (3.16) that

$$\begin{aligned} x \mapsto [\theta(T)]_{0j}x = P_0V^{-1}TVJ_jx &= P_0T\big(\sum_{r=1}^{\infty} c_rJ_rI_{r,j}\big)x \\ &= \sum_{r=1}^{\infty} c_rP_0TJ_rx = \sum_{r=1}^{\infty} c_rT_{0r}x \end{aligned}$$

defines an approximable operator $X \to Y$, since $T \in \mathcal{I}(c)$ by assumption. Thus $\theta(T) \in \mathcal{I}_{\{j\}}$.

Towards the second inclusion $\theta^{-1}(\mathcal{I}_{\{j\}}) \subset \mathcal{I}(c)$, let $T \in \mathcal{I}_{\{j\}}$. Note first that the inverse map θ^{-1} satisfies $\theta^{-1}(T) = VTV^{-1}$. As above we get that $[\theta^{-1}(T)]_{00} \in \mathcal{A}(Y)$. By using (3.16) we see for $j \in \mathbb{N}$ that

$$\begin{aligned} x \mapsto \sum_{r=1}^{\infty} c_r[\theta^{-1}(T)]_{0r}x &= \sum_{r=1}^{\infty} c_rP_0VTV^{-1}J_rx \\ &= c_jP_0TV^{-1}J_jx + \sum_{r\neq j} c_rP_0TV^{-1}J_rx \\ &= P_0T(0, -c_1x, \ldots, -c_{j-1}x, x, -c_{j+1}x, \ldots, -c_{n+1}x, \ldots) + \sum_{r\neq j} c_rP_0TJ_rx \\ &= P_0T\big(J_jx - \sum_{r\neq j} c_rJ_rx\big) + \sum_{r\neq j} c_rP_0TJ_rx \\ &= P_0TJ_jx \end{aligned}$$

defines an approximable operator $X \to Y$, as $T \in \mathcal{I}_{\{j\}}$ by assumption. Consequently, $\theta^{-1}(T) \in \mathcal{I}(c)$. Conclude that $\mathcal{I}(c) \cong \mathcal{I}_{\{j\}}$.

Suppose next that $c \in \ell^1$ and $d \in \ell^1$ are arbitrary non-zero sequences. Fix $j \in supp(c)$ and $k \in supp(d)$. By Claim 1 we know that $\mathcal{I}(c) \cong \mathcal{I}_{\{j\}}$ and $\mathcal{I}(d) \cong \mathcal{I}_{\{k\}}$. Moreover, from [49, Lemma 2.2] we get that $\mathcal{I}_{\{j\}} \cong \mathcal{I}_{\{k\}}$, so that $\mathcal{I}(c) \cong \mathcal{I}(d)$. Recall for completeness that if $\rho : \mathbb{N} \to \mathbb{N}$ is any permutation that interchanges the pair (j, k), then the linear isometry $U : Z_p \to Z_p$ defined by

$$U(y, x_1, x_2, \ldots) = (y, x_{\rho(1)}, x_{\rho(2)}, \ldots), \quad (y, x_1, x_2, \ldots) \in Z_p,$$

induces a Banach algebra isomorphism $T \mapsto \psi(T) = U^{-1}TU$ between $\mathcal{I}_{\{j\}}$ and $\mathcal{I}_{\{k\}}$.

Finally, if the component space X of Z_p satisfies (3.4), then

$$\mathcal{I}_A \cong \mathcal{I}_{\{1\}}$$

for any $\emptyset \neq A \subsetneq \mathbb{N}$ by [49, Theorem 2.1], which implies that (3.15) holds. $\square$

Remarks 3.5. (1) Note that (3.4) guarantees that $\mathcal{I}_A \cong \mathcal{I}_{\{1\}}$ for infinite subsets $A \subsetneq \mathbb{N}$, which is outside Claim 1 of the above proof.

(2) It is somewhat surprising that $\mathcal{I}(c) \cong \mathcal{I}(d)$ for any non-zero sequences $c, d \in \ell^1$. For instance, consider $c = (1/j^2)$ and $d = (1/2^j)$, for which

$\lim_{j\to\infty} c_j/d_j = \infty$. Theorem 3.4 yields that the subideals $\mathcal{I}(c) \cong \mathcal{I}(d)$, as well as $\mathcal{I}(\chi_A \cdot c) \cong \mathcal{I}(\chi_B \cdot d)$ for any non-empty sets $\emptyset \neq A, B \subset \mathbb{N}$. Here $\chi_A \cdot c$ denotes the sequence $(\chi_A(j)c_j)_{j\in\mathbb{N}}$ for $c = (c_j) \in \ell^1$ and $\emptyset \neq A \subset \mathbb{N}$.

Write $\mathcal{A} \ncong \mathcal{B}$ for non-isomorphic Banach algebras $\mathcal{A}$ and $\mathcal{B}$. The following question is motivated by the result that there are closed $\mathcal{S}(L^p)$-subideals $\mathcal{J}_1$ and $\mathcal{J}_2$ of $\mathcal{L}(L^p)$ with $\mathcal{J}_1 \ncong \mathcal{J}_2$ for $1 \leq p \leq \infty$ and $p \neq 2$, see [49, Theorem 3.4 and Proposition 3.6].

Problem. Does there exist a Banach space X together with non-trivial closed $\mathcal{K}(X)$-subideals $\mathcal{J}_1$ and $\mathcal{J}_2$ of $\mathcal{L}(X)$ for which $\mathcal{J}_1 \ncong \mathcal{J}_2$? Is this possible for $X = Z_p$? (Note that X has to fail the A.P. in such an example.)

Towards this question we next construct an increasing sequence $(\mathcal{M}_r)$ of closed subideals of $\mathcal{K}(Z_p)$ for which $\mathcal{M}_r \ncong \mathcal{M}_s$ for $r \neq s$. The argument is a variant of [49, Theorem 3.4.(ii)]. In particular, the subalgebras $\mathcal{M}_r \ncong \mathcal{I}$ for $\mathcal{I} \in \mathfrak{F} \cup \mathfrak{F}_+$, where $\mathfrak{F}$ and $\mathfrak{F}_+$ are the families of closed subideals of $\mathcal{L}(Z_p)$ defined by (3.2) and (3.5). The subalgebras $\mathcal{M}_r$ are closed 3-subideals of $\mathcal{L}(Z_p)$ in the subsequent terminology of Section 4, but we do not know if they are actually closed subideals (that is, 2-subideals) of $\mathcal{L}(Z_p)$.

Example 3.6. Let (X, Y) be any pair of Banach spaces that defines the direct sum Z_p in (3.1). Fix $S_0 \in \mathcal{K}(X, Y) \setminus \mathcal{A}(X, Y)$, and put

$$V_s = J_0 S_0 P_s \in \mathcal{I}_{\{1\}}$$

for $s \geq 2$, where $\mathcal{I}_{\{1\}} = \{T \in \mathcal{K}(Z_p) : T_{00} \in \mathcal{A}(Y), T_{01} \in \mathcal{A}(X, Y)\}$ is the closed $\mathcal{K}(Z_p)$-subideal from (3.2). Let

$$\mathcal{M}_r = span(V_2, \ldots, V_r) + \mathcal{A}(Z_p) \quad \text{for } r \geq 2.$$

Then the following properties hold:

(i) $\mathcal{M}_r$ are closed subspaces of $\mathcal{I}_{\{1\}}$, and $\mathcal{A}(Z_p) \subsetneq \mathcal{M}_r \subsetneq \mathcal{M}_s \subsetneq \mathcal{I}_{\{1\}}$ for $2 \leq r < s$.

(ii) $\mathcal{M}_r \subset \mathcal{I}_{\{1\}}$ is a closed ideal for $r \geq 2$, so that $(\mathcal{M}_r)$ is an increasing chain of closed $\mathcal{I}_{\{1\}}$-subideals of $\mathcal{K}(Z_p)$.

(iii) $\mathcal{M}_r \ncong \mathcal{M}_s$ for any $r, s \geq 2$ with $r \neq s$.

Proof. (i) It is straightforward to verify that $\mathcal{M}_r$ are closed subspaces of $\mathcal{I}_{\{1\}}$ for $r \geq 2$. We observe that $(V_2, \ldots, V_r)$ is linearly independent modulo $\mathcal{A}(Z_p)$. Namely, if

$$R := \sum_{j=2}^{r} c_j V_j \in \mathcal{A}(Z_p),$$

then $R_{0k} = P_0\big(\sum_{j=2}^{r} c_j V_j\big) J_k = c_k S_0 \in \mathcal{A}(X, Y)$, which implies that $c_k = 0$ for $k = 2, \ldots, r$ (recall that $S_0 \notin \mathcal{A}(X, Y)$). In particular,

$$\mathcal{A}(Z_p) \subsetneq \mathcal{M}_r \subsetneq \mathcal{M}_s \subsetneq \mathcal{I}_{\{1\}}$$

for $2 \leq r < s$.

(ii) Recall from [48, Theorem 4.5.(i)] that

$$UV \in \mathcal{A}(Z_p)$$

for any $U \in \mathcal{I}_{\{1\}}$ and $V \in \mathcal{K}(Z_p)$. In particular, if $U \in \mathcal{M}_r \subset \mathcal{I}_{\{1\}}$ and $V \in \mathcal{I}_{\{1\}} \subset \mathcal{K}(Z_p)$, then $UV \in \mathcal{A}(Z_p) \subset \mathcal{M}_r$ and $VU \in \mathcal{A}(Z_p) \subset \mathcal{M}_r$. Thus $\mathcal{M}_r$ is a closed $\mathcal{I}_{\{1\}}$-subideal of $\mathcal{K}(Z_p)$ for $r \geq 2$, since the sequence

$$\mathcal{M}_r \subset \mathcal{I}_{\{1\}} \subset \mathcal{K}(Z_p)$$

consists of closed relative ideals.

(iii) Let $2 \leq r < s$, and suppose to the contrary that $\theta : \mathcal{M}_r \to \mathcal{M}_s$ is a Banach algebra isomorphism. Since $\mathcal{M}_r$ and $\mathcal{M}_s$ are closed subalgebras of $\mathcal{L}(Z_p)$ that contain $\mathcal{A}(Z_p)$, Chernoff's theorem [4, Corollary 3.2] implies that there is a linear isomorphism $U : Z_p \to Z_p$ such that

$$S \mapsto \theta(S) = U^{-1}SU, \quad S \in \mathcal{M}_r. \tag{3.17}$$

The map θ from (3.17) defines a Banach algebra isomorphism $\mathcal{L}(Z_p) \to \mathcal{L}(Z_p)$, for which $\theta(\mathcal{M}_r) = \mathcal{M}_s$ by definition and $\theta(\mathcal{A}(Z_p)) = \mathcal{A}(Z_p)$ as $\mathcal{A}(Z_p)$ is a closed ideal of $\mathcal{L}(Z_p)$. Hence θ induces a well-defined Banach algebra isomorphism $\psi : \mathcal{M}_r/\mathcal{A}(Z_p) \to \mathcal{M}_s/\mathcal{A}(Z_p)$ through

$$\psi(S + \mathcal{A}(Z_p)) = \theta(S) + \mathcal{A}(Z_p), \quad S + \mathcal{A}(Z_p) \in \mathcal{M}_r/\mathcal{A}(Z_p).$$

In addition, $\theta(\mathcal{M}_r) = \mathcal{M}_s$ yields that $\psi(\mathcal{M}_r/\mathcal{A}(Z_p)) = \mathcal{M}_s/\mathcal{A}(Z_p)$.

This means that the quotients $\mathcal{M}_r/\mathcal{A}(Z_p)$ and $\mathcal{M}_s/\mathcal{A}(Z_p)$ are linearly isomorphic. However, this is not possible as $dim\big(\mathcal{M}_r/\mathcal{A}(Z_p)\big) = r - 1$ for $r \geq 2$ by the proof of part (i). Conclude that $\mathcal{M}_r \not\approx \mathcal{M}_s$ for $r \neq s$. $\square$

Remark 3.7. Let X be a Banach space for which the algebra $\mathcal{K}(X)/\mathcal{A}(X)$ is not nilpotent. Such spaces X exists by [48, Theorem 2.9]. In fact, there are even closed subspaces $X \subset \ell^p$ with this property for $p \neq 2$. (See also [41, Example 5.24 and Corollary 5.25] as well as [47, Proposition 3.1] for alternative constructions.)

An abstract technique, due to Wojtynski [53, Theorem] and extended by Shulman and Turovskii [41, Theorem 5.22 and Corollary 5.23], produces an infinite chain $(\mathcal{I}_\alpha)$ of closed ideals of $\mathcal{L}(X)$ such that $\mathcal{A}(X) \subsetneq \mathcal{I}_\alpha \subsetneq \mathcal{K}(X)$ for all α. This result applies to Z_p if one replaces the space Y in definition (3.1) by $Y \oplus Y_0$, for which the quotient $\mathcal{K}(Y_0)/\mathcal{A}(Y_0)$ is not nilpotent. In contrast, the families $\mathfrak{F}$ and $\mathfrak{F}_+$ of non-trivial closed $\mathcal{K}(Z_p)$-subideals from this section are very explicit.

4. Closed n-subideals of $\mathcal{L}(X)$

In this section we study closed n-subideals of the Banach algebras $\mathcal{L}(X)$ for Banach spaces X, which provides a graded extension of the concept of a closed subideal. For each $n \geq 2$ we uncover various Banach spaces X for which there are closed $(n+1)$-subideals of $\mathcal{L}(X)$ that fail to be an n-subideal (Theorems 4.5, 4.7 and 4.10). One of our main results (Theorem 4.8) constructs an infinite decreasing chain $(\mathcal{M}_n)$ of such closed $(n+1)$-subideals of $\mathcal{L}(X)$ for certain spaces X. Moreover, Theorem 4.12 and Proposition 4.13 exhibit non-trivial closed n-subideals contained in the compact operators $\mathcal{K}(X)$ (in these cases X fails the A.P).

Let $\mathcal{A}$ be a Banach algebra over the real or complex scalars, $\mathcal{I} \subset \mathcal{A}$ be a subalgebra and $n \in \mathbb{N}$. We say that $\mathcal{I}$ is an (algebraic) *n-subideal* of $\mathcal{A}$ if there is a sequence $\mathcal{J}_0, \dots, \mathcal{J}_n$ of subalgebras of $\mathcal{A}$ such that

$$\mathcal{I} = \mathcal{J}_n \subset \cdots \subset \mathcal{J}_1 \subset \mathcal{J}_0 = \mathcal{A}, \tag{4.1}$$

where $\mathcal{J}_k$ is an ideal of $\mathcal{J}_{k-1}$ for each $k = 1, \dots, n$. The n-subideal $\mathcal{I}$ of $\mathcal{A}$ is *closed*, if in (4.1) the subalgebras $\mathcal{I}$ and $\mathcal{J}_k$, for $k = 1, \dots, n$, are closed in $\mathcal{A}$ (note that by continuity it suffices that $\mathcal{I}$ is closed in $\mathcal{A}$). In this event we will often say that (4.1) is a chain of closed relative ideals associated to $\mathcal{I}$.

We put

$$\mathcal{SI}_n(\mathcal{A}) = \{\mathcal{I} : \mathcal{I} \text{ is a closed } n\text{-subideal of } \mathcal{A}\}.$$

Clearly the closed 1-subideals $\mathcal{I} \subset \mathcal{A}$ are the closed ideals of $\mathcal{A}$, and the closed 2-subideals correspond to the closed subideals discussed in Sections 2 and 3, as well as in [49]. Condition (4.1) allows that $\mathcal{J}_k = \mathcal{J}_{k-1}$ for some indices k, so that $\mathcal{SI}_n(\mathcal{A}) \subset \mathcal{SI}_m(\mathcal{A})$ for $m > n$. However, if $\mathcal{I}$ is a closed m-subideal of $\mathcal{A}$ for some $m \in \mathbb{N}$, then there is a minimal $n \in \mathbb{N}$ such that $\mathcal{I} \in \mathcal{SI}_n(\mathcal{A})$.

Remarks 4.1. (1) Algebraic n-subideals of Banach algebras for $n \geq 2$ appear in [42, 2.2.1] (and earlier implicitly in [40, Theorem 2.24]). For Lie algebras there is a parallel concept of n-step Lie subideals, see e.g. [44]. The latter notion concerns the Lie product $[x, y] = xy - yx$ on $\mathcal{A}$ when applied to a Banach algebra $\mathcal{A}$.

For the Banach algebras $\mathcal{L}(X)$ the corresponding Lie algebraic concepts are very different from the (sub)ideals studied in this paper. Let H be a complex separable infinite-dimensional Hilbert space. It is known that $span(I_H)$, $\mathcal{K}(H)$ and $span(I_H) + \mathcal{K}(H)$ are precisely the proper closed Lie ideals of $\mathcal{L}(H)$, see [12, Corollary 3]. Furthermore, there are non-trivial (non-closed) 2-subideals $\mathcal{J} \subset \mathcal{L}(H)$ that are Lie ideals of $\mathcal{L}(H)$, see [50, Theorem].

(2) Ring theoretic subideals appeared earlier in ring theory (see the discussion before Proposition 5.1), though the term subideal is not in common use.

Any non-zero closed n-subideal of the Banach algebras $\mathcal{L}(X)$ must contain the approximable operators.

Proposition 4.2. *Let X be a Banach space and $n \in \mathbb{N}$. If $\mathcal{I}$ is a non-zero n-subideal of $\mathcal{L}(X)$, then $\mathcal{F}(X) \subset \mathcal{I}$. In particular, if $\mathcal{I} \subset \mathcal{L}(X)$ is a non-zero closed n-subideal, then $\mathcal{A}(X) \subset \mathcal{I}$.*

Proof. The claim is verified by induction. The case $n = 1$ is contained in Remark 2.1, or e.g. [5, Theorem 2.5.8.(i)].

Let $n \in \mathbb{N}$ and assume that $\mathcal{F}(X) \subset \mathcal{J}$ holds for every non-zero n-subideal $\mathcal{J}$ of $\mathcal{L}(X)$. Let $\mathcal{I}$ be a non-zero $(n+1)$-subideal of $\mathcal{L}(X)$. It suffices to check that $x^* \otimes x \in \mathcal{I}$ for any $x^* \in X^*$ and $x \in X$.

Towards this, let $T \in \mathcal{I}$ be a non-zero operator, and pick $y^* \in X^*, y \in X$ such that $y^*(Ty) = 1$. Since $\mathcal{I}$ is an $(n+1)$-subideal of $\mathcal{L}(X)$, there is an n-subideal $\mathcal{J}$ of $\mathcal{L}(X)$ such that $\mathcal{I} \subset \mathcal{J}$ and $\mathcal{I}$ is an ideal of $\mathcal{J}$. By assumption $\mathcal{J}$ contains all rank-one operators, so that

$$x^* \otimes x = y^*(Ty)(x^* \otimes x) = (y^* \otimes x)T(x^* \otimes y) \in \mathcal{I}. \qquad \square$$

It is a natural problem for the class of Banach algebras $\mathcal{A} = \mathcal{L}(X)$ whether

$$\mathcal{SI}_n(\mathcal{L}(X)) \subsetneq \mathcal{SI}_{n+1}(\mathcal{L}(X)) \tag{4.2}$$

holds for any given $n \geq 2$, that is, does $\mathcal{L}(X)$ contain closed $(n+1)$-subideals $\mathcal{I}$ that fail to be n-subideals. Various examples for $n = 1$ can be found in [49], as well as in Sections 2 and 3, but the cases $n \geq 2$ require additional ingredients.

We first observe that if the closed subalgebra $\mathcal{I} \subset \mathcal{L}(X)$ witnesses (4.2) for some $n \in \mathbb{N}$, then $\mathcal{I}$ fails to have an approximate identity. Recall that the net $(e_\alpha) \subset \mathcal{I}$ is an *approximate identity* for $\mathcal{I}$ if for all $x \in \mathcal{I}$ one has

$$\lim_\alpha e_\alpha x = x = \lim_\alpha x e_\alpha.$$

The case $n = 1$ is discussed in [49, Lemma 3.1].

Proposition 4.3. *Suppose that $\mathcal{I}$ is a closed n-subideal of the Banach algebra $\mathcal{A}$, where $n \geq 2$. Let*

$$\mathcal{I} = \mathcal{J}_n \subset \mathcal{J}_{n-1} \subset \cdots \subset \mathcal{J}_1 \subset \mathcal{J}_0 = \mathcal{A} \tag{4.3}$$

be an n-chain of closed relative ideals. If $\mathcal{J}_k$ or $\mathcal{J}_{k-1}$ has an approximate identity for some $k \in \{2, \dots, n\}$, then $\mathcal{I}$ is a closed $(n-k+1)$-subideal of $\mathcal{A}$.

In particular, if $\mathcal{I}$ or $\mathcal{J}_{n-1}$ has an approximate identity, then $\mathcal{I}$ is a closed ideal of $\mathcal{A}$.

Proof. Let $k \in \{2, \dots, n\}$ be fixed. If $\mathcal{J}_k$ has an approximate identity, then [49, Lemma 3.1] implies that $\mathcal{J}_k \subset \mathcal{J}_{k-2}$ is a closed ideal, so that the closed subalgebra $\mathcal{J}_{k-1}$ is redundant in (4.3). By iterating this procedure we get that $\mathcal{J}_k$ is a closed ideal of $\mathcal{A}$.

The same argument shows that, if $\mathcal{J}_{k-1}$ has an approximate identity, then $\mathcal{J}_{k-1}$ is closed ideal of $\mathcal{A}$. Another application of [49, Lemma 3.1] yields that $\mathcal{J}_k$ is a closed ideal of $\mathcal{A}$.

Thus in both cases $\mathcal{I} = \mathcal{J}_n$ is a closed $(n-k+1)$-subideal of $\mathcal{A}$. Note that if $\mathcal{I}$ or $\mathcal{J}_{n-1}$ has an approximative identity, then $\mathcal{I}$ is a closed ideal of $\mathcal{A}$ by choosing $k = n$ above. $\square$

Remarks 4.4. (1) If (4.2) holds for some $n \geq 2$, then according to the definition one has $\mathcal{SI}_k(\mathcal{L}(X)) \subsetneq \mathcal{SI}_{k+1}(\mathcal{L}(X))$ for $k = 1, \dots, n-1$. Moreover, if $\mathcal{SI}_n(\mathcal{L}(X)) = \mathcal{SI}_{n+1}(\mathcal{L}(X))$, then $\mathcal{SI}_m(\mathcal{L}(X)) = \mathcal{SI}_n(\mathcal{L}(X))$ for all $m > n$.

(2) Let $\mathcal{A}$ be a C^*-algebra, and suppose that $\mathcal{I} \subset \mathcal{A}$ is a closed n-subideal for some $n \geq 2$. Then Proposition 4.3 implies that $\mathcal{I}$ is already a closed ideal of $\mathcal{A}$. Namely, any closed ideal $\mathcal{J}$ of a C^*-algebra is itself a C^*-algebra and has an approximate identity, see e.g. [5, Theorem 3.2.21].

(3) We refer to Example 5.6, and the discussion preceding it, for examples of spaces X and closed subalgebras $\mathcal{I} \subset \mathcal{L}(X)$ that fail to be n-subideals of $\mathcal{L}(X)$ for any $n \in \mathbb{N}$.

Let $\mathcal{A}$ be a Banach algebra, and $\mathcal{J} \subset \mathcal{A}$ be a closed ideal. For $k \geq 2$ put

$$\mathcal{J}^k := span\{a_1 \cdots a_k \mid a_1, \dots, a_k \in \mathcal{J}\}.$$

The power ideals $\mathcal{J}^k$ need not be closed in $\mathcal{A}$ for $k \geq 2$, and there are relevant examples among the Banach algebras $\mathcal{A} \subset \mathcal{L}(X)$ for suitable spaces

X. Let $(\mathcal{N}(X), \|\cdot\|_{\mathcal{N}})$ be the Banach operator ideal of the nuclear operators on X. Dales and Jarchow [6, Lemma 3.4] observed that $\mathcal{N}(X)^2$ has infinite codimension in $\mathcal{N}(X)$ for any infinite-dimensional Banach space X, whereas $\mathcal{N}(X)^2$ is $\|\cdot\|_{\mathcal{N}}$-dense in $\mathcal{N}(X)$. Similar properties hold for $\mathcal{A}(P)^2 \subset \mathcal{A}(P)$ by [6, Theorem 4.2], where P is a Pisier space. For general Banach algebras $\mathcal{A}$ there are many further examples, see e.g. [10, p. 198].

The following result contains a general principle underlying many of our examples of closed $(n+1)$-subideals that fail to be n-subideals. Let X be a Banach space, and $\mathcal{I} \subset \mathcal{L}(X)$. We will use the operator matrix notation

$$\begin{bmatrix} \mathcal{I} & \mathcal{I} \\ \mathcal{I} & \mathcal{I} \end{bmatrix} = \left\{ \begin{bmatrix} S_{11} & \mathcal{S}_{12} \\ S_{21} & S_{22} \end{bmatrix} : S_{ij} \in \mathcal{I} \text{ for } i,j = 1,2 \right\},$$

to define classes of operators $S = \begin{bmatrix} S_{11} & \mathcal{S}_{12} \\ S_{21} & S_{22} \end{bmatrix}$ on the direct sum $X \oplus X$.

Theorem 4.5. *Let $n \in \mathbb{N}$. Suppose that X is a Banach space for which there is a closed ideal $\mathcal{I} \subset \mathcal{L}(X)$ and an operator $S \in \mathcal{I}$ such that*

$$S^n \notin \overline{\mathcal{I}^{n+1}}. \tag{4.4}$$

Then $Z = X \oplus X$ admits a non-trivial closed $(n+1)$-subideal

$$\mathcal{M} \in \mathcal{SI}_{n+1}(\mathcal{L}(Z)) \setminus \mathcal{SI}_n(\mathcal{L}(Z)).$$

Proof. Let

$$\mathcal{J} = \begin{bmatrix} \mathcal{I} & \mathcal{I} \\ \mathcal{I} & \mathcal{I} \end{bmatrix} \subset \mathcal{L}(Z).$$

It is straightforward to check that $\mathcal{J}$ is a closed ideal of $\mathcal{L}(Z)$. Put

$$T = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{J},$$

and define

$$\mathcal{M}_k := span(T, \ldots, T^k) + \overline{\mathcal{J}^{k+1}} \subset \mathcal{J} \tag{4.5}$$

for $1 \leq k \leq n$ and $\mathcal{M}_0 := \mathcal{J}$. We first claim that

$$\mathcal{M}_n \subset \mathcal{M}_{n-1} \subset \cdots \subset \mathcal{M}_1 \subset \mathcal{J} \subset \mathcal{L}(Z) \tag{4.6}$$

is a chain of closed relative ideals, so that $\mathcal{M}_n \in \mathcal{SI}_{n+1}(\mathcal{L}(Z))$.

In fact, each $\mathcal{M}_k$ is a closed subspace of $\mathcal{J}$ as a finite-dimensional extension of the closed subspace $\overline{\mathcal{J}^{k+1}}$ of $\mathcal{L}(Z)$. Moreover, $\mathcal{M}_k \subset \mathcal{M}_{k-1}$ for $k = 1, \ldots, n$, since $T^k \in \mathcal{J}^k$ and $\overline{\mathcal{J}^{k+1}} \subset \overline{\mathcal{J}^k}$. We next verify that $\mathcal{M}_k$ is an ideal of $\mathcal{M}_{k-1}$. Towards this fix $k \in \{1, \ldots, n\}$ and let

$$U = \sum_{r=1}^{k} a_r T^r + U_0 \in \mathcal{M}_k, \quad V = \sum_{s=1}^{k-1} b_s T^s + V_0 \in \mathcal{M}_{k-1}$$

be arbitrary, where $U_0 \in \overline{\mathcal{J}^{k+1}}$ and $V_0 \in \overline{\mathcal{J}^k}$. We claim that

$$UV = \sum_{r=1}^{k} \sum_{s=1}^{k-1} a_r b_s T^{r+s} + U_0\big(\sum_{s=1}^{k-1} b_s T^s + V_0\big) + \big(\sum_{r=1}^{k} a_r T^r\big) V_0 \in \mathcal{M}_k.$$

To see this, note that the first term splits as

$$\sum_{r=1}^{k}\sum_{s=1}^{k-1} a_r b_s T^{r+s} := U_1 + V_1 \in span(T,\dots,T^k) + \mathcal{J}^{k+1} \subset \mathcal{M}_k,$$

since $T^j \in \mathcal{J}^{k+1}$ for any $j = k+1,\dots,2k-1$. Moreover,

$$U_0\big(\sum_{s=1}^{k-1} b_s T^s + V_0\big) \in \overline{\mathcal{J}^{k+1}} \subset \mathcal{M}_k$$

as $\overline{\mathcal{J}^{k+1}}$ is an ideal of $\mathcal{L}(Z)$. Finally, $T^j V_0 \in \overline{\mathcal{J}^{k+1}}$ for $j = 1,\dots,k$ by approximation, so that $\big(\sum_{r=1}^{k} a_r T^r\big)V_0 \in \overline{\mathcal{J}^{k+1}} \subset \mathcal{M}_k$.

The fact that $VU \in \mathcal{M}_k$ is verified in an analogous manner. Altogether (4.6) implies that $\mathcal{M}_n \in \mathcal{SI}_{n+1}(\mathcal{L}(Z))$ is a closed $(n+1)$-subideal.

We next claim that $\mathcal{M}_n \notin \mathcal{SI}_n(\mathcal{L}(Z))$. In view of Lemma 4.6 below it suffices to verify that

$$\begin{bmatrix} 0 & 0 \\ S^n & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ I_X & 0 \end{bmatrix} \circ T^n \notin \mathcal{M}_n. \tag{4.7}$$

Suppose to the contrary that $\begin{bmatrix} 0 & 0 \\ S^n & 0 \end{bmatrix} \in \mathcal{M}_n$. Hence there are scalars $a_1,\dots,a_n$ and $U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \in \overline{\mathcal{J}^{n+1}}$ with

$$\begin{bmatrix} 0 & 0 \\ S^n & 0 \end{bmatrix} = \sum_{k=1}^{n} a_k T^k + U = \begin{bmatrix} \sum_{k=1}^{n} a_k S^k + U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}.$$

On the other hand, one verifies by induction that

$$\overline{\mathcal{J}^{n+1}} = \begin{bmatrix} \overline{\mathcal{I}^{n+1}} & \overline{\mathcal{I}^{n+1}} \\ \overline{\mathcal{I}^{n+1}} & \overline{\mathcal{I}^{n+1}} \end{bmatrix}.$$

By combining these facts we get that the component $S^n = U_{21} \in \overline{\mathcal{I}^{n+1}}$, which contradicts the assumption (4.4) of the theorem. □

Above we made use of the following basic lemma, which we state separately for explicit reference.

Lemma 4.6. *Let $n \in \mathbb{N}$ and suppose that $\mathcal{M}_n$ is a closed subalgebra of the Banach algebra $\mathcal{A}$. If $\mathcal{M}_n \in \mathcal{SI}_n(\mathcal{A})$, then $s \cdot t^n \in \mathcal{M}_n$ for all $s \in \mathcal{A}$ and $t \in \mathcal{M}_n$.*

Proof. Let

$$\mathcal{M}_n = \mathcal{J}_n \subset \cdots \subset \mathcal{J}_1 \subset \mathcal{A}$$

be a chain of closed relative ideals associated to $\mathcal{M}_n$. Since $s \cdot t \in \mathcal{J}_1$ and $t \in \mathcal{J}_k$ for all $k = 1,\dots,n$, by using the relative ideal property at each step of the chain, we obtain that $s \cdot t^n \in \mathcal{M}_n$. □

Conceptually the most elementary examples of non-trivial closed $(n+1)$-subideals of $\mathcal{L}(X)$ that fail to be n-subideals can be defined on finite direct sums of different ℓ^p-spaces. For $n \in \mathbb{N}$, let $1 \leq p_1 < p_2 < \cdots < p_{n+1} \leq \infty$ and define

$$X_n = \bigoplus_{j=1}^{n+1} \ell^{p_j}. \tag{4.8}$$

For unicity of notation we use the convention that $\ell^\infty = c_0$ if $p_{n+1} = \infty$. We equip X_n with the max-norm, though our results are also valid for equivalent norms. The strictly singular operators $\mathcal{S}(X_n)$ will be the closed reference ideal $\mathcal{I}$ of $\mathcal{L}(X_n)$ in this application of Theorem 4.5. We will repeatedly use the following facts.

For any pair of different spaces Y and Z from $\{\ell^p : 1 \leq p < \infty\} \cup \{c_0\}$ we have

$$\mathcal{L}(Y, Z) = \mathcal{S}(Y, Z) \tag{4.9}$$

by the total incomparability of these spaces, see e.g. [24, p. 75]. If $X = \ell^q$ with $q > p$ or $X = c_0$, then by Pitt's theorem

$$\mathcal{L}(X, \ell^p) = \mathcal{K}(X, \ell^p), \tag{4.10}$$

see e.g. [24, Proposition 2.c.3]. Moreover,

$$\mathcal{S}(Y) = \mathcal{K}(Y) \tag{4.11}$$

for any Y from $\{\ell^p : 1 \leq p < \infty\} \cup \{c_0\}$, see e.g. [24, p. 76] or [36, Sections 5.1-5.2].

Properties (4.9) - (4.11) imply that $S = [S_{jk}] \in \mathcal{S}(X_n)$ if and only if S_{jk} is compact for $1 \leq j \leq k \leq n+1$ and S_{jk} is bounded for $1 \leq k < j \leq n+1$. In other words, the operator matrix of $S \in \mathcal{S}(X_n)$ modulo $\mathcal{K}(X_n)$ is lower subtriangular and has a 0-diagonal.

Theorem 4.7. *Let X_n be the space from (4.8), where we fix $n \in \mathbb{N}$ and $1 \leq p_1 < \cdots < p_{n+1} \leq \infty$. Then the following properties hold:*

(i) *the quotient algebra $\mathcal{S}(X_n)/\mathcal{K}(X_n)$ is $(n+1)$-nilpotent, and*
$$\overline{\mathcal{S}(X_n)^{n+1}} = \mathcal{K}(X_n),$$
(ii) *there is $S \in \mathcal{S}(X_n)$ such that $S^n \notin \mathcal{K}(X_n)$,*

(iii) *there is a closed subalgebra $\mathcal{M} \subset \mathcal{S}(X_n)$ for which*
$$\mathcal{M} \in \mathcal{SI}_{n+1}(\mathcal{L}(X_n)) \setminus \mathcal{SI}_n(\mathcal{L}(X_n)).$$

Proof. (i) We claim that the products

$$S_{n+1} \cdots S_1 \in \mathcal{K}(X_n) \tag{4.12}$$

whenever $S_j \in \mathcal{S}(X_n)$ for $j = 1, \ldots, n+1$. In particular, this will imply that $\mathcal{S}(X_n)^{n+1} \subset \mathcal{K}(X_n)$, so the quotient algebra $\mathcal{S}(X_n)/\mathcal{K}(X_n)$ is $(n+1)$-nilpotent. Moreover, $\overline{\mathcal{S}(X_n)^{n+1}} = \mathcal{K}(X_n)$ by Remark 2.1, since $\overline{\mathcal{S}(X_n)^{n+1}}$ is a non-zero closed ideal of $\mathcal{L}(X_n)$ and X_n has the A.P.

By iterating (2.10) we obtain that for any $j, k \in \{1, \ldots, n+1\}$ the component $[S_{n+1} \cdots S_1]_{jk}$ of the product in (4.12) is a finite sum of terms of the form

$$U_{n+1} \cdots U_1,$$

where $U_r \in \mathcal{S}(\ell^{k_r}, \ell^{k_{r+1}})$ and $k_r, k_{r+1} \in \{p_1, \ldots, p_{n+1}\}$ for $r = 1, \ldots, n+1$. Thus we pick $k_1, \ldots, k_{n+2}$ from the set $\{p_1, \ldots, p_{n+1}\}$ containing $n+1$

elements, so there are $r < s$ such that $k_r = k_s$. Since the strictly singular operators $\mathcal{S}$ form an operator ideal we obtain that the partial product

$$U_{s-1} \cdots U_r \in \mathcal{S}(\ell^{k_r}) = \mathcal{K}(\ell^{k_r})$$

in view of (4.11). Thus $U_{n+1} \cdots U_1 \in \mathcal{K}(\ell^{k_1}, \ell^{k_{n+2}})$, so that (4.12) holds.

(ii) Recall that $p_k < p_{k+1}$ for $k = 1, \ldots, n$, and let i_k be the corresponding inclusion map $\ell^{p_k} \to \ell^{p_{k+1}}$. (If $p_{n+1} = \infty$, then i_n is the inclusion map $\ell^{p_n} \to c_0$.) We define $S \in \mathcal{L}(X_n)$ by

$$S = \sum_{k=1}^{n} J_{k+1} i_k P_k,$$

where J_k and P_k are the canonical inclusions, respectively projections, related to the component space ℓ^{p_k} of the finite direct sum X_n for $k = 1, \ldots, n+1$. Property (4.9) implies that $S \in \mathcal{S}(X_n)$. The representation

$$Sx = (0, i_1 x_1, \ldots, i_n x_n), \quad x = (x_1, \ldots, x_{n+1}) \in X_n, \tag{4.13}$$

of S acting as a shift operator on the components of X_n implies that

$$S^n x = (0, \ldots, 0, i_n \cdots i_1 x_1), \quad x \in X_n.$$

Thus $P_{n+1} S^n J_1$ is the non-compact inclusion $\ell^{p_1} \to \ell^{p_{n+1}}$, so that

$$S^n \notin \mathcal{K}(X_n) = \overline{\mathcal{S}(X_n)^{n+1}}$$

in view of part (i).

(iii) Let $Z_n = X_n \oplus X_n$. By parts (i) - (ii) and Theorem 4.5 there is a closed subalgebra $\mathcal{M}_0 \subset \mathcal{S}(Z_n)$ such that

$$\mathcal{M}_0 \in \mathcal{SI}_{n+1}(\mathcal{L}(Z_n)) \setminus \mathcal{SI}_n(\mathcal{L}(Z_n)).$$

Recall that $X \oplus X \approx X$ for the spaces X from $\{\ell^p : 1 \le p < \infty\} \cup \{c_0\}$. Thus there is a linear isomorphism $U : Z_n \to X_n$, which induces a Banach algebra isomorphism

$$T \mapsto \theta(T) = UTU^{-1}, \quad \mathcal{L}(Z_n) \to \mathcal{L}(X_n).$$

Note that $\theta(\mathcal{S}(Z_n)) = \mathcal{S}(X_n)$, and it is easy to check that algebra isomorphisms preserve the n-subideal structure (cf. Lemma 2.5 for $n = 2$). It follows that the closed subalgebra $\mathcal{M} = \theta(\mathcal{M}_0) \subset \mathcal{S}(X_n)$ is the desired example associated to the space X_n. (Note that $\mathcal{M}$ will depend on the choice of U.) □

A central aim of this section is to construct an infinite decreasing sequence of non-trivial closed n-subideals in $\mathcal{L}(X)$, where X is an infinite direct sum of ℓ^{p_j}-spaces. This answers positively a query of Niels Laustsen. We remark that it is significantly easier to find a space Z for which there are closed subideals $\mathcal{K}_n \in \mathcal{SI}_{n+1}(\mathcal{L}(Z)) \setminus \mathcal{SI}_n(\mathcal{L}(Z))$ for each $n \in \mathbb{N}$, see the proof of part (ii) of Theorem 4.12. In that construction $(\mathcal{K}_n)_{n \in \mathbb{N}}$ will not be a chain.

Let $(p_j)_{j \in \mathbb{N}}$ be any strictly increasing sequence in $[1, \infty)$. We put $E = \ell^r$ for r satisfying $\sup_j p_j \le r \le \infty$, where the notational convention is again that $\ell^\infty = c_0$. The desired Banach space will be the E-direct sum

$$X = \Big(\bigoplus_{j=1}^{\infty} \ell^{p_j}\Big)_E . \tag{4.14}$$

We will find a decreasing chain $(\mathcal{M}_n)$ of closed $(n+1)$-subideals of $\mathcal{L}(X)$ such that

$$\mathcal{K}(X) \subset \mathcal{M}_n \subset \mathcal{S}(X)$$

for all $n \in \mathbb{N}$. Such an infinite chain lies outside the scope of Theorem 4.5 as well as the subsequent Proposition 5.3. In particular, the quotient algebra $\mathcal{S}(X)/\mathcal{K}(X)$ is non-nilpotent in view of the finite shift operators from (4.13). This obstruction creates serious technical difficulties, which are resolved in the proofs of (4.18) and (4.19).

Theorem 4.8. *Let X be the space in* (4.14). *Then there is a strictly decreasing chain $(\mathcal{M}_n)_{n\in\mathbb{N}_0}$ of closed subalgebras of $\mathcal{L}(X)$ such that $\mathcal{M}_0 = \mathcal{S}(X)$, and $\mathcal{M}_n \subset \mathcal{M}_{n-1}$ is a closed ideal for all $n \in \mathbb{N}$, so that*

$$\mathcal{M}_n \in \mathcal{SI}_{n+1}(\mathcal{L}(X)) \setminus \mathcal{SI}_n(\mathcal{L}(X))$$

for all $n \in \mathbb{N}$.

Proof. For each $n \in \mathbb{N}$ we consider the finite E-direct sum

$$X_n = \ell^{p_1} \oplus \cdots \oplus \ell^{p_{n+1}}.$$

We use $J_n : X_n \to X$ and $P_n : X \to X_n$ for the natural inclusions and the projections related to X_n.

According to the proof of part (ii) of Theorem 4.7 there is $S_n \in \mathcal{S}(X_n)$ such that $S_n^n \notin \mathcal{K}(X_n)$. Next, for each $n \in \mathbb{N}$ let $\theta_n : X_n \to X_n \oplus X_n$ be the linear isomorphism defined by

$$\theta_n(x_1, \ldots, x_{n+1}) := (\phi_1(x_1)_1, \ldots, \phi_{n+1}(x_{n+1})_1) + (\phi_1(x_1)_2, \ldots, \phi_{n+1}(x_{n+1})_2),$$

where $\phi_k : \ell^{p_k} \to \ell^{p_k} \oplus \ell^{p_k}$, $x \mapsto (\phi_k(x)_1, \phi_k(x)_2)$ for $x \in \ell^{p_k}$ is a fixed linear isomorphism for each $k \in \mathbb{N}$.

We define

$$V_k = J_k \theta_k^{-1} \begin{bmatrix} S_k & 0 \\ 0 & 0 \end{bmatrix} \theta_k P_k \in \mathcal{S}(X)$$

for $k \in \mathbb{N}$, see the diagram.

$$\begin{tikzcd}
X \arrow[r, "V_k"] \arrow[d, "P_k"'] & X \\
X_k \arrow[d, "\theta_k"'] & X_k \arrow[u, "J_k"'] \\
X_k \oplus X_k \arrow[r, "\begin{bmatrix} S_k & 0 \\ 0 & 0 \end{bmatrix}"] & X_k \oplus X_k \arrow[u, "\theta_k^{-1}"']
\end{tikzcd}$$

Let

$$\mathcal{A} = \mathcal{A}(\{V_k : k \in \mathbb{N}\}) \tag{4.15}$$

be the subalgebra of $\mathcal{S}(X)$ generated by the set $\{V_k : k \in \mathbb{N}\}$. Finally, for each $n \in \mathbb{N}_0$ we put

$$\mathcal{M}_n = \overline{\mathcal{A} + \mathcal{S}(X)^{n+1}}. \tag{4.16}$$

We first verify that

$$\mathcal{K}(X) \subset \cdots \subset \mathcal{M}_{n+1} \subset \mathcal{M}_n \subset \cdots \subset \mathcal{M}_0 = \mathcal{S}(X),$$

where $\mathcal{M}_n$ is a closed ideal of $\mathcal{M}_{n-1}$ for $n \in \mathbb{N}$. These facts will imply that $\mathcal{M}_n \in \mathcal{SI}_{n+1}(\mathcal{L}(X))$ for $n \in \mathbb{N}$.

Clearly $\mathcal{M}_0 = \overline{\mathcal{A} + \mathcal{S}(X)} = \mathcal{S}(X)$, and $\mathcal{K}(X) \subset \mathcal{M}_n$ for $n \in \mathbb{N}$, since $\mathcal{S}(X)^{n+1}$ is a non-zero ideal of $\mathcal{L}(X)$ and X has the A.P. The inclusion $\mathcal{S}(X)^{n+1} \subset \mathcal{S}(X)^n$ yields that $\mathcal{M}_n \subset \mathcal{M}_{n-1}$ for $n \in \mathbb{N}$. Suppose next that

$$U = A + T_1 \in \mathcal{A} + \mathcal{S}(X)^{n+1}, \quad V = B + T_2 \in \mathcal{A} + \mathcal{S}(X)^n$$

are arbitrary, with $A, B \in \mathcal{A}$, $T_1 \in \mathcal{S}(X)^{n+1}$ and $T_2 \in \mathcal{S}(X)^n$. Hence

$$UV = AB + AT_2 + T_1B + T_1T_2 \in \mathcal{A} + \mathcal{S}(X)^{n+1} \subset \mathcal{M}_n,$$

since $AB \in \mathcal{A}$ and $AT_2 + T_1B + T_1T_2 \in \mathcal{S}(X)^{n+1}$. Similarly the product $VU \in \mathcal{A} + \mathcal{S}(X)^{n+1}$. Thus we obtain by approximation and the continuity of the product that $\mathcal{M}_n$ is a closed ideal of $\mathcal{M}_{n-1}$.

The main step of the argument consists of establishing the following

Claim. $\mathcal{M}_n$ is not a closed n-subideal of $\mathcal{L}(X)$.

Assume towards a contradiction that $\mathcal{M}_n \in \mathcal{SI}_n(\mathcal{L}(X))$. By Lemma 4.6 we have

$$J_n\theta_n^{-1} \begin{bmatrix} 0 & 0 \\ S_n^n & 0 \end{bmatrix} \theta_n P_n = \big(J_n\theta_n^{-1} \begin{bmatrix} 0 & 0 \\ I_{X_n} & 0 \end{bmatrix} \theta_n P_n\big) V_n^n \in \mathcal{M}_n$$

as $V_n \in \mathcal{M}_n$. Above we applied the identity $V_n^n = J_n\theta_n^{-1} \begin{bmatrix} S_n^n & 0 \\ 0 & 0 \end{bmatrix} \theta_n P_n$.

Hence there are sequences $(T_r) \subset \mathcal{A}$ and $(W_r) \subset \mathcal{S}(X)^{n+1}$ such that

$$T_r + W_r \to J_n\theta_n^{-1} \begin{bmatrix} 0 & 0 \\ S_n^n & 0 \end{bmatrix} \theta_n P_n$$

in $\mathcal{S}(X)$ as $r \to \infty$. It follows that

$$\theta_n P_n T_r J_n \theta_n^{-1} + \theta_n P_n W_r J_n \theta_n^{-1} \to \begin{bmatrix} 0 & 0 \\ S_n^n & 0 \end{bmatrix} \tag{4.17}$$

in $\mathcal{S}(X_n \oplus X_n)$ as $r \to \infty$.

We will establish that for each $n \in \mathbb{N}$ and $T \in \mathcal{A}$ there is an operator $U = U(T, n) \in \mathcal{S}(X_n)$ such that

$$\theta_n P_n T J_n \theta_n^{-1} = \begin{bmatrix} U & 0 \\ 0 & 0 \end{bmatrix} \tag{4.18}$$

and that

$$P_n A J_n \in \mathcal{K}(X_n) \tag{4.19}$$

holds for all $A \in \mathcal{S}(X)^{n+1}$. Conditions (4.18) and (4.19) will then imply that (4.17) takes the form

$$\begin{bmatrix} U_r & 0 \\ 0 & 0 \end{bmatrix} + K_r \to \begin{bmatrix} 0 & 0 \\ S_n^n & 0 \end{bmatrix} \text{ as } r \to \infty,$$

where $U_r \in \mathcal{S}(X_n)$ and $K_r = \theta_n P_n W_r J_n \theta_n^{-1} \in \mathcal{K}(X_n \oplus X_n)$ for $r \in \mathbb{N}$. In particular, above the operator norm convergence as $r \to \infty$ of the respective $(2,1)$-components yields that $S_n^n \in \mathcal{K}(X_n)$, which contradicts our original choice of S_n. In the rest of the argument we complete the proof by showing that (4.18) and (4.19) hold.

Proof of (4.18).

By definition (4.15) and linearity it is enough verify (4.18) for operators $T \in \mathcal{A}$ which have the form $T = V_{n_1} \cdots V_{n_k}$ where $n_1, \ldots, n_k \in \mathbb{N}$.

Note that the inverses of θ_n satisfy

$$\theta_n^{-1} : (x_1, \ldots, x_{n+1}) + (y_1, \ldots, y_{n+1}) \mapsto (\phi_1^{-1}(x_1, y_1), \ldots, \phi_{n+1}^{-1}(x_{n+1}, y_{n+1}))$$

for $(x_1, \ldots, x_{n+1}), (y_1, \ldots, y_{n+1}) \in X_n$. For $t \geq s$ we let $J_{t,s} : X_s \to X_t$ and $P_{s,t} : X_t \to X_s$ be the natural inclusion and projection maps. The following identities are straightforward to check:

$$J_s\theta_s^{-1} = J_t\theta_t^{-1}(J_{t,s} \oplus J_{t,s}),$$
$$\theta_s P_s = (P_{s,t} \oplus P_{s,t})\theta_t P_t$$

and

$$(J_{t,s} \oplus J_{t,s}) \begin{bmatrix} S_s & 0 \\ 0 & 0 \end{bmatrix} (P_{s,t} \oplus P_{s,t}) = \begin{bmatrix} J_{t,s}S_sP_{s,t} & 0 \\ 0 & 0 \end{bmatrix}.$$

Here $(P_{s,t} \oplus P_{s,t})(x, y) = (P_{s,t}x, P_{s,t}y)$ and $J_{t,s} \oplus J_{t,s}$ is defined in a similar manner. By using these identities we obtain that

$$\begin{aligned} V_s &= J_s\theta_s^{-1} \begin{bmatrix} S_s & 0 \\ 0 & 0 \end{bmatrix} \theta_s P_s \\ &= J_t\theta_t^{-1}(J_{t,s} \oplus J_{t,s}) \begin{bmatrix} S_s & 0 \\ 0 & 0 \end{bmatrix} (P_{s,t} \oplus P_{s,t})\theta_t P_t \\ &= J_t\theta_t^{-1} \begin{bmatrix} J_{t,s}S_sP_{s,t} & 0 \\ 0 & 0 \end{bmatrix} \theta_t P_t, \end{aligned}$$

where $J_{t,s}S_sP_{s,t} \in \mathcal{S}(X_t)$.

It follows that if $t \geq max\{r, s\}$, then both the products V_rV_s and V_sV_r have the form

$$J_t\theta_t^{-1} \begin{bmatrix} U & 0 \\ 0 & 0 \end{bmatrix} \theta_t P_t \tag{4.20}$$

for some $U \in \mathcal{S}(X_t)$. By iteration, any finite product $V_{n_1} \cdots V_{n_k}$ has the form (4.20) if $t \geq max\{n_1, \ldots, n_k\}$.

Suppose next that $T = V_{n_1} \cdots V_{n_k}$, where $n_1, \ldots, n_k \in \mathbb{N}$ are arbitrary. If $max\{n_1, \ldots, n_k\} \leq n$, then according to (4.20) we have

$$T = J_n\theta_n^{-1} \begin{bmatrix} U & 0 \\ 0 & 0 \end{bmatrix} \theta_n P_n$$

for some $U \in \mathcal{S}(X_n)$, so $\theta_n P_n T J_n \theta_n^{-1} = \begin{bmatrix} U & 0 \\ 0 & 0 \end{bmatrix}$ has the desired form (4.18).

On the other hand, if $s := max\{n_1, \ldots, n_k\} > n$, then

$$T = J_s\theta_s^{-1} \begin{bmatrix} U & 0 \\ 0 & 0 \end{bmatrix} \theta_s P_s$$

for some $U \in \mathcal{S}(X_s)$ by (4.20). By using the simple identities

$$\theta_s P_s J_n \theta_n^{-1} = J_{s,n} \oplus J_{s,n}, \quad \theta_n P_n J_s \theta_s^{-1} = P_{n,s} \oplus P_{n,s},$$

we get that

$$\begin{aligned}\theta_n P_n T J_n \theta_n^{-1} &= \theta_n P_n (J_s \theta_s^{-1} \begin{bmatrix} U & 0 \\ 0 & 0 \end{bmatrix} \theta_s P_s) J_n \theta_n^{-1} \\ &= (P_{n,s} \oplus P_{n,s}) \begin{bmatrix} U & 0 \\ 0 & 0 \end{bmatrix} (J_{s,n} \oplus J_{s,n}) \\ &= \begin{bmatrix} P_{n,s} U J_{s,n} & 0 \\ 0 & 0 \end{bmatrix}.\end{aligned}$$

Thus we also obtain the desired identity (4.18) in the second case since $P_{n,s} U J_{s,n} \in \mathcal{S}(X_n)$.

Proof of (4.19). We claim that $P_n A J_n \in \mathcal{K}(X_n)$ for all $A \in \mathcal{S}(X)^{n+1}$.

By linearity we may assume that $A = A_1 \cdots A_{n+1}$, where $A_i \in \mathcal{S}(X)$ for $i \in \{1, \ldots, n+1\}$. Let $Q_k : X \to \ell^{p_k}$ and $q_k : X_n \to \ell^{p_k}$ be the natural projections, and $R_k : \ell^{p_k} \to X$ and $r_k : \ell^{p_k} \to X_n$ be the natural inclusion maps for $1 \le k \le n+1$. Thus $P_n = r_1 Q_1 + \ldots + r_{n+1} Q_{n+1}$ and $J_n = R_1 q_1 + \ldots + R_{n+1} q_{n+1}$, so that

$$P_n A J_n = \sum_{s,t=1}^{n+1} r_t Q_t A R_s q_s.$$

In order to verify (4.19) it will suffice to show that

$$Q_t A R_s = Q_t A_1 \cdots A_{n+1} R_s \in \mathcal{K}(\ell^{p_s}, \ell^{p_t}) \tag{4.21}$$

holds for all $1 \le s, t \le n+1$. Note that (4.21) holds for $s \ge t$ by (4.9) and (4.10), and thus we may assume that $1 \le s < t \le n+1$.

We will apply the following result from [22, Corollary 5.2] in order to verify (4.21): If $U, V \in \mathcal{L}(X)$ and $s, t \in \mathbb{N}$ then

$$Q_t U V R_s = \sum_{k=1}^{\infty} Q_t U R_k Q_k V R_s \ , \tag{4.22}$$

where the series converges in the operator norm of $\mathcal{L}(\ell^{p_s}, \ell^{p_t})$. Note that [22, Corollary 5.2] (see also condition (ii) on [22, page 166]) applies here since $\mathcal{L}(E, \ell^{p_j}) = \mathcal{K}(E, \ell^{p_j})$ for all $j \in \mathbb{N}$ by Pitt's theorem (4.10) for $E = c_0$ or $E = \ell^r$ with $r \ge \sup_j p_j$.

Let $U_m = \sum_{k=1}^m R_k Q_k$ for all $m \in \mathbb{N}$, and define for all $r = 1, \ldots, n$ and $m_1, \ldots, m_r \in \mathbb{N}$ the auxiliary operators

$$B_{(m_1,\ldots,m_r)} := Q_t A_1 U_{m_1} \cdots A_r U_{m_r} A_{r+1} A_{r+2} \cdots A_{n+1} R_s.$$

We first claim that for all $\varepsilon > 0$ there are $m_1, \ldots, m_n \in \mathbb{N}$ such that

$$||B_{(m_1,\ldots,m_n)} - T|| < \varepsilon \tag{4.23}$$

where $T = Q_t A_1 \cdots A_{n+1} R_s$ is the operator in (4.21).

To see this let $\varepsilon > 0$. By (4.22) (with $U = A_1$ and $V = A_2 \cdots A_{n+1}$) there is $m_1 \in \mathbb{N}$ such that $||B_{(m_1)} - T|| < \varepsilon/n$. If $n \ge 2$, we continue by picking $m_2 \in \mathbb{N}$ such that

$$||B_{(m_1)} - B_{(m_1,m_2)}|| < \varepsilon/n$$

(here we again use (4.22) with $U = A_1 U_{m_1} A_2$ and $V = A_3 \cdots A_{n+1}$). Assume that $2 \le r \le n-1$ and we have picked $m_1, \ldots, m_r \in \mathbb{N}$ such that

$$||B_{(m_1,\ldots,m_{k-1})} - B_{(m_1,\ldots,m_k)}|| < \varepsilon/n$$

for $k = 2, \ldots, r$. Next we use (4.22) (with $U = A_1 U_{m_1} A_2 U_{m_2} \cdots A_r U_{m_r} A_{r+1}$ and $V = A_{r+2} \cdots A_{n+1}$) to pick $m_{r+1} \in \mathbb{N}$ such that

$$||B_{(m_1,\ldots,m_r)} - B_{(m_1,\ldots,m_{r+1})}|| < \varepsilon/n.$$

Hence we get (4.23) from the triangle inequality.

Next we note that

$$\begin{aligned} B_{(m_1,\ldots,m_n)} &= Q_t A_1 U_{m_1} A_2 U_{m_2} \cdots A_n U_{m_n} A_{n+1} R_s \\ &= \sum_{k_1=1}^{m_1} \cdots \sum_{k_n=1}^{m_n} Q_t A_1 R_{k_1} Q_{k_1} A_2 R_{k_2} Q_{k_2} \cdots A_n R_{k_n} Q_{k_n} A_{n+1} R_s. \end{aligned}$$

In order to show (4.21) it will be enough in view of (4.23) to verify that

$$Q_t A_1 R_{k_1} Q_{k_1} A_2 R_{k_2} Q_{k_2} \cdots A_n R_{k_n} Q_{k_n} A_{n+1} R_s \in \mathcal{K}(\ell^{p_s}, \ell^{p_t}) \tag{4.24}$$

holds for all $k_1, \ldots, k_n \in \mathbb{N}$.

If $k_r \ge t$ or $k_r \le s$ for some $r = 1, \ldots, n$, then (4.10) - (4.11) imply that (4.24) holds. In fact, if $k_r \ge t$, then

$$Q_t A_1 R_{k_1} Q_{k_1} \cdots A_r R_{k_r} \in \mathcal{K}(\ell^{p_{k_r}}, \ell^{p_t})$$

and thus (4.24) holds. The case $k_r \le s$ follows similarly by observing that

$$Q_{k_r} A_r R_{k_r} Q_{k_r} \cdots A_{n+1} R_s \in \mathcal{K}(\ell^{p_s}, \ell^{p_{k_r}}).$$

On the other hand, if $s < k_r < t$ for all $r = 1, \ldots, n$, we must have $k_l = k_m$ for some $1 \le l < m \le n$ since $\#(\{s+1, \ldots, t-1\}) < n$ (recall that $1 \le s < t \le n+1$). By (4.11) we have

$$Q_{k_l} A_{l+1} R_{k_{l+1}} Q_{k_{l+1}} \cdots A_m R_{k_m} \in \mathcal{K}(\ell^{p_{k_l}})$$

which again implies (4.24).

As pointed out above this establishes the second claim (4.19). Altogether we have completed the argument that $\mathcal{M}_n \notin \mathcal{SI}_n(\mathcal{L}(X))$, and consequently that of Theorem 4.8. □

Remark 4.9. The result of Theorem 4.8 also holds for the direct ℓ^r-sum

$$Y = \Big(\bigoplus_{j=1}^{\infty} \ell^{p_j}\Big)_{\ell^r},$$

where (p_j) is any strictly increasing sequence in $(1,\infty)$ and $1 \le r < p_1$. In this case condition (i) from [22, page 166] is satisfied by (4.10), so that the convergence formula (4.22) from [22, Corollary 5.2] is also valid in $\mathcal{L}(Y)$.

Next we return to the setting of Theorem 4.5 and include a variant of Theorem 4.7 for finite sums of different p-James spaces. The underlying spaces are quasi-reflexive and the reference ideals $\mathcal{I}$ in the application of Theorem 4.5 can be chosen differently from Theorem 4.7. (See Remarks 4.11 for further comments.)

Let $1 < p < \infty$. The p-James space J_p consists of the scalar-valued sequences $x = (x_j)$ for which $\lim_{j\to\infty} x_j = 0$ and the p-variation norm

$$\|x\|_{J_p} = \sup\{\big(\sum_{k=1}^{n} |x_{j_{k+1}} - x_{j_k}|^p\big)^{1/p} : 1 \le j_1 < \cdots < j_{n+1}, n \in \mathbb{N}\} \tag{4.25}$$

is finite. Here J_p^{**} is identified with the space of scalar sequences $x = (x_j)$ for which $\|x\|_{J_p} < \infty$. In fact, $J_p^{**} = J_p \oplus \mathbb{K}e$, where $e = (1, 1, 1, \dots) \in J_p^{**}$.

Let $n \in \mathbb{N}$ and $1 < p_1 < p_2 < \cdots < p_{n+1} < \infty$. We will consider the max-normed direct sums

$$Y_n = \bigoplus_{j=1}^{n+1} J_{p_j}. \tag{4.26}$$

Let $\mathcal{W}(X, Y)$ denote the class of weakly compact operators $X \to Y$ for Banach spaces X and Y. The class $\mathcal{W}$ is a closed Banach operator ideal, so that $\mathcal{W}(X)$ is a closed ideal of $\mathcal{L}(X)$ for any X.

We will require the following properties of the spaces J_p and their bounded operators:

(i) For $1 < p < \infty$ we have $\mathcal{L}(J_p) = span(I_{J_p}) + \mathcal{W}(J_p)$ and

$$\mathcal{S}(J_p) = \mathcal{K}(J_p) \subsetneq \mathcal{W}(J_p), \tag{4.27}$$

see [25, p. 344] and [23, Proposition 4.9]. Moreover, there is a complemented subspace $R_p \subset J_p$ such that

$$R_p \approx \ell^p, \tag{4.28}$$

see e.g. [23, Corollary 4.7].

(ii) For $1 < p < q < \infty$ the inclusion map $i_{p,q} : J_p \to J_q$ is not weakly compact and

$$\mathcal{L}(J_p, J_q) = span(i_{p,q}) + \mathcal{W}(J_p, J_q), \tag{4.29}$$

see [25, p. 344]. In addition,

$$\mathcal{L}(J_p, J_q) = \mathcal{S}(J_p, J_q) \tag{4.30}$$

since the spaces J_p and J_q are totally incomparable, which follows from [23, Corollary 4.7].

(iii) For $1 < q < p < \infty$ we have

$$\mathcal{L}(J_p, J_q) = \mathcal{K}(J_p, J_q), \tag{4.31}$$

see [25, Theorem 4.5].

Let Z be a Banach space. In the following results we use the standard notation $\mathcal{I}(Z)$ for the closed ideal of $\mathcal{L}(Z)$ consisting of the operators $Z \to Z$ which belong to the closed Banach operator ideal $\mathcal{I} = \mathcal{S} \cap \mathcal{W}$ or $\mathcal{I} = \mathcal{S}$.

Theorem 4.10. *Let $n \in \mathbb{N}$ and Y_n be the direct sum space from* (4.26)*, where $1 < p_1 < \cdots < p_{n+1} < \infty$. If $\mathcal{I}(Y_n) = \mathcal{S}(Y_n) \cap \mathcal{W}(Y_n)$ or $\mathcal{I}(Y_n) = \mathcal{S}(Y_n)$, then*

(i) *the quotient algebra $\mathcal{I}(Y_n)/\mathcal{K}(Y_n)$ is $(n+1)$-nilpotent, and*

$$\overline{\mathcal{I}(Y_n)^{n+1}} = \mathcal{K}(Y_n),$$

(ii) *there is $T \in \mathcal{I}(Y_n)$ for which $T^n \notin \mathcal{K}(Y_n)$,*

(iii) *for* $Z_n = Y_n \oplus Y_n$ *there is a closed subalgebra* $\mathcal{M} \subset \mathcal{I}(Z_n)$ *such that*

$$\mathcal{M} \in \mathcal{SI}_{n+1}(\mathcal{L}(Z_n)) \setminus \mathcal{SI}_n(\mathcal{L}(Z_n)).$$

Proof. The argument follows the outline from Theorem 4.7. Let P_k and I_k denote the canonical projection and inclusion related to the component J_{p_k} of Y_n for $k = 1, \dots, n+1$.

(i) This is a modification of the pigeon-hole argument used in part (i) of Theorem 4.7. One requires the additional facts that

$$\mathcal{S}(J_{p_r}) = \mathcal{S}(J_{p_r}) \cap \mathcal{W}(J_{p_r}) = \mathcal{K}(J_{p_r})$$

for $1 \le r \le n+1$ in view of (4.27), where $\mathcal{S} \cap \mathcal{W}$ is an operator ideal. Recall further that the spaces J_p have a Schauder basis, so Y_n has the A.P.

(ii) For $\mathcal{I}(Y_n) = \mathcal{S}(Y_n) \cap \mathcal{W}(Y_n)$ we use (4.28) to pick a complemented subspace $R_{p_k} \subset J_{p_k}$ such that $R_{p_k} \approx \ell^{p_k}$ for $k = 1, \dots, n+1$. Fix a linear isomorphism $U_k : R_{p_k} \to \ell^{p_k}$, and let $q_k : J_{p_k} \to R_{p_k}$ be a projection and j_k the inclusion map $R_{p_k} \to J_{p_k}$, so that $q_k j_k = I_{R_{p_k}}$ for $k = 1, \dots, n+1$. We define

$$r_k = j_{k+1} U_{k+1}^{-1} i_k U_k q_k \in \mathcal{L}(J_{p_k}, J_{p_{k+1}}),$$

where $i_k : \ell^{p_k} \to \ell^{p_{k+1}}$ is the inclusion map for $k = 1, \dots, n$.

$$\begin{tikzcd}
J_{p_k} \arrow[r, "r_k"] \arrow[d, "q_k"'] & J_{p_{k+1}} \\
R_{p_k} \arrow[d, "U_k"'] & R_{p_{k+1}} \arrow[u, "j_{k+1}"'] \\
\ell^{p_k} \arrow[r, "i_k"] & \ell^{p_{k+1}} \arrow[u, "U_{k+1}^{-1}"']
\end{tikzcd}$$

Recall that $r_k \in \mathcal{S}(J_{p_k}, J_{p_{k+1}}) \cap \mathcal{W}(J_{p_k}, J_{p_{k+1}})$ by the reflexivity of ℓ^{p_k} and the total incomparability of ℓ^p and ℓ^q for $p \neq q$. Hence

$$T = \sum_{k=1}^{n} I_{k+1} r_k P_k \in \mathcal{S}(Y_n) \cap \mathcal{W}(Y_n).$$

By iteration we get that

$$T^n x = (0, \dots, 0, r_n \cdots r_1 x_1), \quad x = (x_1, \dots, x_{n+1}) \in Y_n.$$

It follows that $P_{n+1} T^n I_1 = r_n \cdots r_1 \notin \mathcal{K}(J_{p_1}, J_{p_{n+1}})$, as the restriction

$$(r_n \cdots r_1)_{|R_{p_1}} = j_{n+1} U_{n+1}^{-1} (i_n \cdots i_1) U_1$$

is up to linear isomorphisms the inclusion map $\ell^{p_1} \to \ell^{p_{n+1}}$. Conclude that $T^n \notin \mathcal{K}(Y_n)$.

Clearly $T \in \mathcal{S}(Y_n) \cap \mathcal{W}(Y_n)$ also works for $\mathcal{I}(Y_n) = \mathcal{S}(Y_n)$. (The choice

$$S = \sum_{k=1}^{n} I_{k+1} i_{p_k, p_{k+1}} P_k \in \mathcal{S}(Y_n)$$

is a simpler one, since the inclusion maps $i_{p_k, p_{k+1}} : J_{p_k} \to J_{p_{k+1}}$ are strictly singular for $k = 1, \dots, n$ and $i_{p,q} \notin \mathcal{W}(J_p, J_q)$ for $q > p$ by (4.29).)

Finally, (iii) follows immediately from the proof of Theorem 4.5 and parts (i) - (ii). $\square$

Remarks 4.11. (1) Properties (4.27), (4.30) and (4.31) yield that

$$\mathcal{S}(Y_n) = \{[U_{jk}] \in \mathcal{L}(Y_n) : U_{jk} \text{ is compact for } 1 \le j \le k \le n+1, \\ U_{jk} \text{ is bounded for } 1 \le k < j \le n+1\},$$

$$\mathcal{S}(Y_n) \cap \mathcal{W}(Y_n) = \{[U_{jk}] \in \mathcal{L}(Y_n) : U_{jk} \text{ is compact for } 1 \le j \le k \le n+1, \\ U_{jk} \text{ is weakly compact for } 1 \le k < j \le n+1\}.$$

It is known that $\mathcal{S}(Y_n) \cap \mathcal{W}(Y_n) \subsetneq \mathcal{S}(Y_n)$, so the two choices for $\mathcal{I}$ in Theorem 4.10 are different. In fact, the inclusion maps $i_{p_r,p_s} : J_{p_r} \to J_{p_s}$ extend to strictly singular non-weakly compact operators on Y_n for $1 \le r < s \le n+1$ in view of (4.29) and (4.30).

(2) Theorem 4.5 cannot be applied to the closed ideal $\mathcal{I}(Y_n) = \mathcal{W}(Y_n)$, since $\mathcal{W}(Y_n) = \mathcal{W}(Y_n)^k$ for all $k \ge 2$. The reason is that $\mathcal{W}(Y_n)$ has a bounded left approximate identity, that is, there is a bounded net $(V_\alpha) \subset \mathcal{W}(Y_n)$ such that

$$\lim_\alpha V_\alpha V = V \tag{4.32}$$

for all $V \in \mathcal{W}(Y_n)$. Hence the Cohen-Hewitt factorisation theorem, see e.g. [5, Corollary 2.9.26], implies that $\mathcal{W}(Y_n)^2 = \mathcal{W}(Y_n)$. The fact (4.32) is seen by combining [30, Theorem 2.2 and Proposition 2.5] with the comment before [30, Problem 2.4] and [30, Proposition 5.3]. In particular, the choice of the closed reference ideal $\mathcal{I}$ is essential in Theorem 4.5.

(3) It is not difficult to check that $dim(Y_n^{**}/Y_n) = n+1$, so Y_n are quasi-reflexive spaces. This implies that $Y_n \oplus Y_n \not\approx Y_n$, see e.g. [3].

The non-trivial closed $(n+1)$-subideals in Theorems 4.7, 4.8 and 4.10 are defined on spaces X which have the A.P., where $\mathcal{K}(X)$ is the smallest non-zero subideal. The earlier examples of non-trivial 2-subideals from [49], as well as Sections 2 and 3, raise the question whether there are spaces X (without the A.P.) that allow non-trivial closed subideals $\mathcal{J} \in \mathcal{SI}_{n+1}(\mathcal{L}(X))$ for $n \ge 2$ such that $\mathcal{A}(X) \subsetneq \mathcal{J} \subsetneq \mathcal{K}(X)$. It turns out that this is possible, though more sophisticated tools and concepts are involved.

Let X and Y be Banach spaces and $p \in [1,\infty)$. Recall from [35] that $T \in \mathcal{L}(X,Y)$ is a *quasi p-nuclear* operator, denoted $T \in \mathcal{QN}_p(X,Y)$, if there is a strongly p-summable sequence (x_k^*) in X^* such that

$$\|Tx\| \le \Big(\sum_{k=1}^\infty |x_k^*(x)|^p\Big)^{1/p}, \quad x \in X. \tag{4.33}$$

It is known [35, Section 4] that $(\mathcal{QN}_p, |\cdot|_p)$ defines a Banach operator ideal, where the complete ideal norm is given by

$$|T|_p = \inf\Big\{\Big(\sum_{k=1}^\infty \|x_k^*\|^p\Big)^{1/p} : (x_k^*) \text{ satisfies } (4.33)\Big\}.$$

Let Z be any Banach space. We will require the monotonicity and the Hölder properties of the classes $\mathcal{QN}_p$, that is,

$$\mathcal{QN}_p(Z) \subset \mathcal{QN}_q(Z) \text{ and } \|\cdot\| \le |\cdot|_q \le |\cdot|_p \tag{4.34}$$

for $p < q$ by [35, Satz 21 and 24], and for $1/r = 1/p + 1/q \le 1$ one has

$$\text{if } S \in \mathcal{QN}_p(Z) \text{ and } T \in \mathcal{QN}_q(Z), \text{ then } ST \in \mathcal{QN}_r(Z), \tag{4.35}$$

see [35, Satz 48]. Moreover, $\mathcal{QN}_p(Z) \subset \mathcal{K}(Z)$ for any p by [35, Satz 25], and

$$\mathcal{QN}_2(Z) \subset \mathcal{A}(Z), \tag{4.36}$$

see [52, Remarks 3.2.(ii)] for references to the ingredients of the argument.

The following theorem contains another central result of this section. It makes use of a closed subspace of c_0 and an associated quasi p-nuclear operator from [52, Section 4], which in turn depend on constructions of Davie [8] of Banach spaces without the A.P. and a factorization result of Reinov [38, Lemma 1.1]. Part (i) of the theorem produces for $n = 1$ a closed subspace $Z_1 \subset c_0$ and a non-trivial closed 2-subideal $\mathcal{I}$ of $\mathcal{L}(Z_1)$, which differs from the examples in [49] and Section 2. In part (i) the closed subspaces $Z_n \subset c_0$ and $\mathcal{M}_n \in \mathcal{SI}_{n+1}(\mathcal{L}(Z_n))$ depend on $n \in \mathbb{N}$, but in (ii) we lift them from $\mathcal{L}(Z_n)$ to $\mathcal{L}(Z)$ for the direct sum $Z = \big(\bigoplus_{n=1}^\infty Z_n\big)_{c_0}$.

Theorem 4.12. *(i) Fix $n \in \mathbb{N}$, and assume that $2n < p \le 2(n+1)$. Then there is a closed subspace $Z_n \subset c_0$ without the A.P. and a closed subalgebra $\mathcal{M}_n$ of $\overline{\mathcal{QN}_p(Z_n)}$ such that*

$$\mathcal{M}_n \in \mathcal{SI}_{n+1}(\mathcal{L}(Z_n)) \setminus \mathcal{SI}_n(\mathcal{L}(Z_n)).$$

(ii) Let $Z = \big(\bigoplus_{n=1}^\infty Z_n\big)_{c_0}$. Then

$$\mathcal{SI}_n(\mathcal{L}(Z)) \subsetneq \mathcal{SI}_{n+1}(\mathcal{L}(Z))$$

for all $n \in \mathbb{N}$.

Proof. (i) Fix $n \in \mathbb{N}$ and let $p \in (2n, 2(n+1)]$. According to [52, Proposition 4.3] there is a closed subspace $X_n \subset c_0$ together with $S \in \mathcal{QN}_p(X_n)$ such that $S^n \notin \mathcal{A}(X_n)$. By construction X_n fails the A.P.

In order to apply Theorem 4.5 we require the following observation, which is essentially contained in [52, Proposition 4.1 and Remarks 4.2.(ii)]: if $2n < p \le 2(n+1)$, then

$$\overline{\mathcal{QN}_p(Y)^{n+1}} = \mathcal{A}(Y) \tag{4.37}$$

holds for any Banach space Y. In fact, the Hölder property (4.35) together with (4.34) and (4.36) imply that

$$\mathcal{QN}_p(Y)^{n+1} \subset \mathcal{QN}_{p/(n+1)}(Y) \subset \mathcal{QN}_2(Y) \subset \mathcal{A}(Y),$$

as $p/(n+1) \le 2$ by assumption. Since $\mathcal{F}(Y) \subset \mathcal{QN}_p(Y)^{n+1}$ we obtain (4.37) by passing to the closures.

Hence the preceding operator $S \in \mathcal{QN}_p(X_n)$ satisfies $S^n \notin \overline{\mathcal{QN}_p(X_n)^{n+1}}$ by (4.37). Let $Z_n = X_n \oplus X_n \subset c_0 \oplus c_0 \simeq c_0$ (isometric isomorphism). Thus $Z_n \subset c_0$ is a closed subspace that fails the A.P. We put

$$T = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{QN}_p(Z_n)$$

and define as in Theorem 4.5

$$\mathcal{M}_n = span(T, \dots, T^n) + \overline{\mathcal{QN}_p(Z_n)^{n+1}}. \tag{4.38}$$

We may now apply Theorem 4.5 for the closed ideal $\mathcal{J} = \overline{\mathcal{QN}_p(Z_n)}$, and deduce that the closed subalgebra

$$\mathcal{M}_n \in \mathcal{SI}_{n+1}(\mathcal{L}(Z_n)) \setminus \mathcal{SI}_n(\mathcal{L}(Z_n)).$$

(ii) We again use $J_n : Z_n \to Z$ and $P_n : Z \to Z_n$ for the canonical inclusions, respectively the canonical projections associated to Z.

For all $n \in \mathbb{N}$ pick $p_n \in (2n, 2(n+1)]$ and let $T_n = \begin{bmatrix} S_n & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{QN}_{p_n}(Z_n)$ be the operator from the proof of part (i). (Note that the operators T_n and S_n as well as p_n depend on n in that argument, even though the dependence was not displayed explicitly.) Consider

$$R_n = J_n T_n P_n \in \mathcal{QN}_{p_n}(Z)$$

for all $n \in \mathbb{N}$ and put

$$\mathcal{K}_{n+1} := span(R_n, \dots, R_n^n) + \mathcal{A}(Z).$$

Claim. The closed subalgebra $\mathcal{K}_{n+1}$ of $\mathcal{L}(Z)$ satisfies

$$\mathcal{K}_{n+1} \in \mathcal{SI}_{n+1}(\mathcal{L}(Z)) \setminus \mathcal{SI}_n(\mathcal{L}(Z))$$

for all $n \in \mathbb{N}$.

To see this we introduce the intermediary closed linear subspaces

$$\mathcal{L}_k := span(R_n, \dots, R_n^k) + \overline{\mathcal{QN}_{p_n}(Z)^{k+1}} \subset \overline{\mathcal{QN}_{p_n}(Z)}$$

for $k = 1, \dots, n$. We know that $\overline{\mathcal{QN}_{p_n}(Z)^{n+1}} = \mathcal{A}(Z)$ in view of (4.37) as $2n < p_n \le 2(n+1)$, so that $\mathcal{L}_n = \mathcal{K}_{n+1}$. By arguing as in the proof of Theorem 4.5 we obtain that the chain

$$\mathcal{K}_{n+1} = \mathcal{L}_n \subset \mathcal{L}_{n-1} \subset \cdots \subset \mathcal{L}_1 \subset \overline{\mathcal{QN}_{p_n}(Z)} \subset \mathcal{L}(Z)$$

consists of closed relative ideals. It follows that $\mathcal{K}_{n+1}$ is a closed n-subideal of $\overline{\mathcal{QN}_{p_n}(Z)}$, so that $\mathcal{K}_{n+1} \in \mathcal{SI}_{n+1}(\mathcal{L}(Z))$.

Suppose next to the contrary that $\mathcal{K}_{n+1} \in \mathcal{SI}_n(\mathcal{L}(Z))$. Hence, by Lemma 4.6, we have

$$J_n \begin{bmatrix} 0 & 0 \\ S_n^n & 0 \end{bmatrix} P_n = J_n \left(\begin{bmatrix} 0 & 0 \\ I_{X_n} & 0 \end{bmatrix} T_n^n \right) P_n = \left(J_n \begin{bmatrix} 0 & 0 \\ I_{X_n} & 0 \end{bmatrix} P_n \right) R_n^n \in \mathcal{K}_{n+1}.$$

Above we have applied the identities $P_n J_n = I_{Z_n}$ and $R_n^n = J_n T_n^n P_n$. Observe next that if

$$U = \sum_{k=1}^n a_k R_n^k + V \in \mathcal{K}_{n+1} = span(R_n, \dots, R_n^n) + \mathcal{A}(Z)$$

for some scalars $a_1, \dots, a_n$ and $V \in \mathcal{A}(Z)$, then

$$P_n U J_n = \sum_{k=1}^n a_k P_n R_n^k J_n + P_n V J_n = \sum_{k=1}^n a_k T_n^k + P_n V J_n \in \mathcal{M}_n,$$

where $\mathcal{M}_n \subset \mathcal{L}(Z_n)$ is the closed $(n+1)$-subideal defined in (4.38). In particular, $\begin{bmatrix} 0 & 0 \\ S_n^n & 0 \end{bmatrix} \in \mathcal{M}_n$, which was already shown to be impossible during the proof of Theorem 4.5 (see the verification of (4.7)). $\square$

The Banach operator ideal component $\mathcal{B} := (\mathcal{QN}_p(X), |\cdot|_p)$ is also a Banach algebra in its own right, where $|\cdot|_p$ is the associated complete ideal norm. We include a variant of part (i) of Theorem 4.12 for $\mathcal{B}$, where the subalgebras and n-subideals are closed in the $|\cdot|_p$-norm. This setting is somewhat outside of Theorem 4.5 (but Lemma 4.6 applies).

Proposition 4.13. *For any $n \geq 2$ and $p \in (2n, 2(n+1)]$ there is a closed subspace $Z_n \subset c_0$ and a closed subalgebra $\mathcal{J}_n \subset \mathcal{B} := (\mathcal{QN}_p(Z_n), |\cdot|_p)$ such that*

$$\mathcal{J}_n \in \mathcal{SI}_n(\mathcal{B}) \setminus \mathcal{SI}_{n-1}(\mathcal{B}).$$

Proof. We sketch the main differences to Theorems 4.5 and 4.12.

Fix $n \geq 2$ and $p \in (2n, 2(n+1)]$. As in the proof of Theorem 4.12 we pick a closed linear subspace $X_n \subset c_0$ and an operator $S \in \mathcal{QN}_p(X_n)$ such that $S^n \notin \mathcal{A}(X_n)$. Let $Z_n = X_n \oplus X_n$ and consider $T = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{QN}_p(Z_n)$. For $1 \leq k \leq n$ we define

$$\mathcal{J}_k := span(T, \ldots, T^k) + \overline{\mathcal{B}^{k+1}}^{|\cdot|_p}.$$

By imitating the proof of Theorem 4.5 one verifies that the chain

$$\mathcal{J}_n \subset \mathcal{J}_{n-1} \subset \cdots \subset \mathcal{J}_1 \subset \mathcal{B} = \mathcal{QN}_p(Z_n) \tag{4.39}$$

consists of $|\cdot|_p$-closed relative ideals. Recall here that $|\cdot|_p$ is a Banach algebra norm, so that multiplication is continuous on $\mathcal{B}$. The explicit chain (4.39) implies that $\mathcal{J}_n \in \mathcal{SI}_n(\mathcal{B})$.

Next, suppose to the contrary that $\mathcal{J}_n \in \mathcal{SI}_{n-1}(\mathcal{B})$. By Lemma 4.6 we have

$$\begin{bmatrix} 0 & 0 \\ S^n & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ S & 0 \end{bmatrix} \circ T^{n-1} \in \mathcal{J}_n, \tag{4.40}$$

since $\begin{bmatrix} 0 & 0 \\ S & 0 \end{bmatrix} \in \mathcal{B}$. We may thus write

$$\begin{bmatrix} 0 & 0 \\ S^n & 0 \end{bmatrix} = \sum_{k=1}^n a_k T^k + U = \begin{bmatrix} \sum_{k=1}^n a_k S^k + U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix},$$

where $U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \in \overline{\mathcal{B}^{n+1}}^{|\cdot|_p}$. We observe next that

$$\overline{\mathcal{B}^{n+1}}^{|\cdot|_p} \subset \mathcal{A}(Z_n). \tag{4.41}$$

In fact, from (4.35), (4.34) and (4.36) we obtain that

$$\mathcal{B}^{n+1} = \mathcal{QN}_p(Z_n)^{n+1} \subset \mathcal{QN}_{p/(n+1)}(Z_n) \subset \mathcal{QN}_2(Z_n) \subset \mathcal{A}(Z_n)$$

since $(n+1)/p \leq 1$ and $p/(n+1) \leq 2$. This implies (4.41), since the operator norm satisfies $\|\cdot\| \leq |\cdot|_p$ by (4.34).

By combining these facts we reach the contradiction that $S^n \in \mathcal{A}(X_n)$. Thus $\mathcal{J}_n \notin \mathcal{SI}_{n-1}(\mathcal{B})$. $\square$

Theorem 4.5 also applies to a class of p-compact operators, which is related to $\mathcal{QN}_p$ by duality. Let X and Y be Banach spaces, and $1 < p < \infty$. We say that $T \in \mathcal{L}(X, Y)$ is a *(Sinha-Karn) p-compact* operator, denoted

$T \in \mathcal{SK}_p(X,Y)$, if there is a strongly p-summable sequence (y_k) in Y such that

$$T(B_X) \subset \Big\{ \sum_{k=1}^{\infty} a_k y_k : (a_k) \in B_{\ell^{p'}} \Big\}.$$

Here $1/p + 1/p' = 1$ and B_Z is the closed unit ball of the space Z. There is a complete norm $|\cdot|_{\mathcal{SK}_p}$ such that $(\mathcal{SK}_p, |\cdot|_{\mathcal{SK}_p})$ defines a Banach operator ideal, see e.g. [43, Theorem 4.2]. (The classes $\mathcal{SK}_p$ were introduced in [43] as the p-compact operators, but we use the above terminology to distinguish them from an earlier notion of p-compactness in the literature, see e.g. [37, p. 529].) Properties of $\mathcal{SK}_p$ have been studied quite intensively, see e.g. [31], [15], [52, Section 3] and their references. By [15, Theorem 2.8] one has

- $T \in \mathcal{SK}_p(X,Y)$ if and only if $T^* \in \mathcal{QN}_p(Y^*, X^*)$,
- $T \in \mathcal{QN}_p(X,Y)$ if and only if $T^* \in \mathcal{SK}_p(Y^*, X^*)$,

where the respective ideal norms are preserved. Hence the classes $\mathcal{SK}_p$ satisfy the exact analogues of (4.34), (4.35) and (4.36).

We formulate a variant for $\mathcal{SK}_p$ of part (i) of Theorem 4.12, which does not directly follow by duality from the analogous result for $\mathcal{QN}_p$.

Example 4.14. Let $n \in \mathbb{N}$ and $p \in (2n, 2(n+1)]$. Then there is a closed subspace $Z_n \subset c_0$ and a closed subalgebra $\mathcal{M}_n$ of $\overline{\mathcal{SK}_p(Z_n)}$ such that

$$\mathcal{M}_n \in \mathcal{SI}_{n+1}(\mathcal{L}(Z_n)) \setminus \mathcal{SI}_n(\mathcal{L}(Z_n)).$$

Proof. The argument follows the general outline of Theorem 4.12.(i). According to [52, Proposition 4.3] there is a closed subspace $Y_n \subset c_0$ as well as $V \in \mathcal{SK}_p(Y_n)$ such that $V^n \notin \mathcal{A}(Y_n)$.

We will also require the fact that if $2n < p \leq 2(n+1)$, then

$$\overline{\mathcal{SK}_p(Y)^{n+1}} = \mathcal{A}(Y) \tag{4.42}$$

holds for any Banach space Y. This equality is verified exactly as in the proof of (4.37) by using the $\mathcal{SK}_p$-versions of properties (4.34) - (4.36) (cf. [52, Remarks 4.2.(ii)]).

Let $Z_n = Y_n \oplus Y_n$. The result follows by applying Theorem 4.5 for the closed ideal $\mathcal{J} = \overline{\mathcal{SK}_p(Z_n)}$ of $\mathcal{L}(Z_n)$. The desired $(n+1)$-subideal has the form

$$\mathcal{M}_n = span(T, \dots, T^n) + \overline{\mathcal{SK}_p(Z_n)^{n+1}},$$

where $T = \begin{bmatrix} V & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{SK}_p(Z_n)$. $\square$

5. Concluding remarks

We include here a mixed collection of results that clarify various aspects of Section 4. For instance, in the context of Theorem 4.5 there is additional information on certain powers of the closed subideals constructed therein. We formulate a sufficient condition that yields closed n-subideals of general Banach algebras, and apply it in Theorem 5.4 to find higher closed subideals related to the Tarbard spaces.

Let $\mathcal{A}$ be a Banach algebra. The ideal of $\mathcal{A}$ generated by the subset $M \subset \mathcal{A}$ is denoted by

$$[M]_{\mathcal{A}} := \bigcap \{\mathcal{I} : \ \mathcal{I} \text{ is an ideal of } \mathcal{A} \text{ containing } M\},$$

and the closure

$$\overline{[M]}_{\mathcal{A}} = \bigcap\{\mathcal{I} : \ \mathcal{I} \text{ is a closed ideal of } \mathcal{A} \text{ containing } M\}$$

is the corresponding closed ideal of $\mathcal{A}$ generated by M. It is well known that

$$[M]_{\mathcal{A}} = \mathcal{A}M + M\mathcal{A} + \mathcal{A}M\mathcal{A} + M, \tag{5.1}$$

where $\mathcal{A}M = span\{am : a \in \mathcal{A}, m \in M\}$, and $M\mathcal{A}$ as well as $\mathcal{A}M\mathcal{A}$ are defined similarly. We abbreviate our notation as $[M] := [M]_{\mathcal{L}(X)}$ and $\overline{[M]} := \overline{[M]}_{\mathcal{L}(X)}$ for $\mathcal{A} = \mathcal{L}(X)$, where X is a Banach space and $M \subset \mathcal{L}(X)$.

We first discuss powers of closed n-subideals for Banach algebras $\mathcal{A}$. By way of motivation we recall a lemma of Andrunakievich concerning (ring theoretic) subideals from ring theory, where the setting need not allow for scalar multiplication.

Let $\mathcal{R}$ be a ring, and suppose that $\mathcal{J} \subset \mathcal{I} \subset \mathcal{R}$ are subrings. We say here that $\mathcal{J}$ is a (ring) *subideal* of $\mathcal{R}$ if $\mathcal{J}$ is a ring ideal of $\mathcal{I}$ and $\mathcal{I}$ is a ring ideal of $\mathcal{R}$. Such subideals occur in the following observation of Andrunakievich (restated in this terminology): *if $\mathcal{J}$ is a subideal of the ring $\mathcal{R}$, then*

$$(\mathcal{J})^3_{\mathcal{R}} \subset \mathcal{J}, \tag{5.2}$$

see e.g. [1, Lemma 4] or [16, p. 11]. Above $(\mathcal{J})_{\mathcal{R}}$ is the ring ideal of $\mathcal{R}$ generated by $\mathcal{J}$, and the power $(\mathcal{J})^3_{\mathcal{R}}$ is the ring ideal of $\mathcal{R}$ generated by the set $\{a_1a_2a_3 : a_j \in (\mathcal{J})_{\mathcal{R}} \text{ for } j = 1, 2, 3\}$.

The following result is a version of (5.2) for Banach algebras. It is known that $\mathcal{K}(\ell^2)$ contains ring ideals that are not algebra ideals, see [34, Example 4.1], so Proposition 5.1 concerns a different type of ideals compared to (5.2).

Proposition 5.1. *If $\mathcal{A}$ is a Banach algebra and $\mathcal{J} \in \mathcal{SI}_2(\mathcal{A})$, then*

$$\overline{[\mathcal{J}]^3_{\mathcal{A}}} \subset \mathcal{J}.$$

Proof. By linearity and continuity it is enough to verify that $s_1s_2s_3 \in \mathcal{J}$ for any $s_1, s_2, s_3 \in [\mathcal{J}]_A$, since $\mathcal{J}$ is a closed subalgebra of $\mathcal{A}$. Recall from (5.1) that

$$[\mathcal{J}]_{\mathcal{A}} = \mathcal{A}\mathcal{J} + \mathcal{J}\mathcal{A} + \mathcal{A}\mathcal{J}\mathcal{A} + \mathcal{J}.$$

We may again assume by linearity that $s_2 = u_1t_1 + t_2u_2 + u_3t_3u_4 + t_4$ where $u_i \in \mathcal{A}$ and $t_i \in \mathcal{J}$ for $i = 1, \ldots, 4$. Thus

$$s_1s_2s_3 = s_1u_1t_1s_3 + s_1t_2u_2s_3 + s_1u_3t_3u_4s_3 + s_1t_4s_3. \tag{5.3}$$

Here $s_1u_1 \in [\mathcal{J}]_{\mathcal{A}}$ as $[\mathcal{J}]_{\mathcal{A}}$ is an ideal of $\mathcal{A}$. By assumption $\mathcal{J} \subset \mathcal{I} \subset \mathcal{A}$, where $\mathcal{I}$ is a closed ideal of $\mathcal{A}$ and $\mathcal{J}$ is an ideal of $\mathcal{I}$. Thus $[\mathcal{J}]_{\mathcal{A}} \subset \mathcal{I}$, so that $\mathcal{J}$ is an ideal of $[\mathcal{J}]_{\mathcal{A}}$. This implies that $t_1s_3 \in \mathcal{J}$, and thus $s_1u_1t_1s_3 \in \mathcal{J}$.

In a similar manner one verifies that the other three terms on the right hand side of (5.3) belong to $\mathcal{J}$, whence $s_1s_2s_3 \in \mathcal{J}$. □

Concerning the algebras $\mathcal{L}(X)$ the following extension of Theorem 4.5 provides further information for e.g. the closed $(n+1)$-subideals obtained in Theorems 4.7, 4.10 and 4.12 by applying Theorem 4.5.

Theorem 5.2. *Let $n \in \mathbb{N}$. Suppose that X is a Banach space for which there is a closed ideal $\mathcal{I} \subset \mathcal{L}(X)$ and an operator $S \in \mathcal{I}$ such that*

$$S^n \notin \overline{\mathcal{I}^{n+1}}. \tag{5.4}$$

Put $Z = X \oplus X$ *and let*

$$\mathcal{M}_n \in \mathcal{SI}_{n+1}(\mathcal{L}(Z)) \setminus \mathcal{SI}_n(\mathcal{L}(Z))$$

be the closed $(n+1)$*-subideal defined in* (4.5)*. Then*

(i) $\overline{[\mathcal{M}_n]^{n+1}} \subsetneq \mathcal{M}_n$.
(ii) $[\mathcal{M}_n]^n \not\subset \mathcal{M}_n$.

Proof. (i) Recall from (4.5) and the proof of Theorem 4.5 that

$$\mathcal{M}_n = span(T, T^2, \ldots, T^n) + \overline{\mathcal{J}^{n+1}},$$

where

$$\mathcal{J} := \begin{bmatrix} \mathcal{I} & \mathcal{I} \\ \mathcal{I} & \mathcal{I} \end{bmatrix} \subset \mathcal{L}(Z) \text{ and } T := \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{J}.$$

Since $\mathcal{J}$ is a closed ideal of $\mathcal{L}(Z)$ and $\mathcal{M}_n \subset \mathcal{J}$ by definition, we get that $[\mathcal{M}_n] \subset \mathcal{J}$ and $[\mathcal{M}_n]^{n+1} \subset \mathcal{J}^{n+1}$. After passing to the closures it follows that

$$\overline{[\mathcal{M}_n]^{n+1}} \subset \overline{\mathcal{J}^{n+1}} \subset \mathcal{M}_n.$$

Here $\overline{[\mathcal{M}_n]^{n+1}} \subsetneq \mathcal{M}_n$, since $\overline{[\mathcal{M}_n]^{n+1}}$ is a closed ideal of $\mathcal{L}(Z)$, while $\mathcal{M}_n$ fails to be a closed ideal of $\mathcal{L}(Z)$ by construction.

(ii) For $n = 1$ it suffices to note that $\mathcal{M}_1$ is not an ideal of $\mathcal{L}(Z)$, so that $\mathcal{M}_1 \subsetneq [\mathcal{M}_1]$. For $n \geq 2$ recall that

$$\begin{bmatrix} 0 & 0 \\ S^n & 0 \end{bmatrix} = \left(\begin{bmatrix} 0 & 0 \\ I_X & 0 \end{bmatrix} \circ T \right) \circ T^{n-1} \in [\mathcal{M}_n]^n,$$

since $\begin{bmatrix} 0 & 0 \\ I_X & 0 \end{bmatrix} \circ T \in [\mathcal{M}_n]$. It was shown during the proof of Theorem 4.5 (see (4.7)) that $\begin{bmatrix} 0 & 0 \\ S^n & 0 \end{bmatrix} \notin \mathcal{M}_n$. Thus $[\mathcal{M}_n]^n \not\subset \mathcal{M}_n$ for $n \geq 2$. □

In view of Theorem 5.2.(i) there are Banach spaces X and non-trivial closed 4-subideals $\mathcal{M} \in \mathcal{SI}_4(\mathcal{L}(X)) \setminus \mathcal{SI}_3(\mathcal{L}(X))$ for which $\overline{[\mathcal{M}]^4} \subsetneq \mathcal{M}$ and $[\mathcal{M}]^3 \not\subset \mathcal{M}$. A comparison with Proposition 5.1 suggests the following

Problem. Does there exist a space X and $\mathcal{M} \in \mathcal{SI}_3(\mathcal{L}(X))$ for which

$$[\mathcal{M}]^3 \not\subset \mathcal{M} \ ? \tag{5.5}$$

More generally, is there a Banach algebra $\mathcal{A}$ and $\mathcal{M} \in \mathcal{SI}_3(\mathcal{A})$ such that (5.5) holds?

In Section 4 we have studied non-trivial closed higher subideals of $\mathcal{L}(X)$. It is a pertinent query for which other basic classes of Banach algebras

$$\mathcal{SI}_n(\mathcal{A}) \subsetneq \mathcal{SI}_{n+1}(\mathcal{A})$$

for some $n \in \mathbb{N}$. We did not pursue this here, but we include a nilpotency criterion for a subalgebra of a non-unital Banach algebra to be an n-subideal. Condition (5.6) only applies to $\mathcal{M} = \mathcal{A}$ if $\mathcal{A}$ has a unit.

Proposition 5.3. Let $n \geq 2$ and suppose that $\mathcal{A}$ is a non-unital Banach algebra. If $\mathcal{M} \subsetneq \mathcal{A}$ is a subalgebra such that

$$\mathcal{A}^{n+1} \subset \mathcal{M}, \tag{5.6}$$

then $\mathcal{M}$ is an n-subideal of $\mathcal{A}$. If $\mathcal{M}$ is also closed in $\mathcal{A}$, then $\mathcal{M} \in \mathcal{SI}_n(\mathcal{A})$.

Proof. Let $\mathcal{I}_0 := \mathcal{A}$ and define successively

$$\mathcal{I}_k = [\mathcal{M} \cap \mathcal{I}_{k-1}]_{\mathcal{I}_{k-1}} \quad \text{for } k = 1, \dots, n-1.$$

One verifies by induction that $\mathcal{M} \subset \mathcal{I}_k$ for $k = 0, \dots, n-1$, so that actually $\mathcal{I}_k = [\mathcal{M}]_{\mathcal{I}_{k-1}}$ for $k = 1, \dots, n-1$. Hence,

$$\mathcal{M} \subset \mathcal{I}_{n-1} \subset \cdots \subset \mathcal{I}_2 \subset \mathcal{I}_1 \subset \mathcal{I}_0 = \mathcal{A}.$$

Here $\mathcal{I}_k$ is an ideal of $\mathcal{I}_{k-1}$ by definition for $k = 1, \dots, n-1$, so it remains to verify that $\mathcal{M}$ is an ideal of $\mathcal{I}_{n-1}$. Towards this, we first check the following auxiliary result by induction.

Claim.

$$\mathcal{I}_k \subset \mathcal{A}^{k+1} + \mathcal{M} \quad \text{for all } k = 0, \dots, n-1. \tag{5.7}$$

The case $k = 0$ is clear since $\mathcal{I}_0 = \mathcal{A}$. The induction assumption is that (5.7) holds for some $k \in \{0, \dots, n-2\}$. Since

$$\mathcal{I}_{k+1} = [\mathcal{M}]_{\mathcal{I}_k} = \mathcal{I}_k\mathcal{M} + \mathcal{M}\mathcal{I}_k + \mathcal{I}_k\mathcal{M}\mathcal{I}_k + \mathcal{M}$$

according to (5.1), we will verify in (i) - (iii) below that the products

$$a_1m, \ ma_2, \ a_1ma_2 \in \mathcal{A}^{k+2} + \mathcal{M}$$

whenever $a_1, a_2 \in \mathcal{I}_k$ and $m \in \mathcal{M}$.

(i) The induction hypothesis $\mathcal{I}_k \subset \mathcal{A}^{k+1} + \mathcal{M}$ yields that $a_1 = a + b$, where $a \in \mathcal{A}^{k+1}$ and $b \in \mathcal{M}$. Since $m \in \mathcal{M} \subset \mathcal{A}$ we get that $am \in \mathcal{A}^{k+2}$. Moreover, $bm \in \mathcal{M}$ since $\mathcal{M}$ is a subalgebra of $\mathcal{A}$. Consequently

$$a_1m = am + bm \in \mathcal{A}^{k+2} + \mathcal{M}. \tag{5.8}$$

(ii) A symmetrical argument to (i) ensures that $ma_2 \in \mathcal{A}^{k+2} + \mathcal{M}$.

(iii) From (5.8) we get that $a_1m = c + d$, where $c \in \mathcal{A}^{k+2}$ and $d \in \mathcal{M}$. Moreover, $a_2 \in \mathcal{A}^{k+1} + \mathcal{M}$ by the induction hypothesis, so that $a_2 = c_1 + c_2$ with $c_1 \in \mathcal{A}^{k+1}$ and $c_2 \in \mathcal{M}$. It follows that $dc_1 \in \mathcal{A}^{k+2}$ and $dc_2 \in \mathcal{M}$. Thus $da_2 = dc_1 + dc_2 \in \mathcal{A}^{k+2} + \mathcal{M}$, so that

$$a_1ma_2 = (c + d)a_2 = ca_2 + da_2 \in \mathcal{A}^{k+2} + \mathcal{M}$$

as $ca_2 \in \mathcal{A}^{k+3} \subset \mathcal{A}^{k+2}$. Thus $\mathcal{I}_{k+1} \subset \mathcal{A}^{k+2} + \mathcal{M}$ by linearity which completes the proof of the Claim (5.7).

To conclude, suppose that $m \in \mathcal{M}$ and $a \in \mathcal{I}_{n-1}$ are arbitrary. From (5.7) with $k = n-1$ we get that $a = b + c$, where $b \in \mathcal{A}^n$ and $c \in \mathcal{M}$. It follows that $bm \in \mathcal{A}^{n+1} \subset \mathcal{M}$ by the assumption (5.6). Moreover, $cm \in \mathcal{M}$ as $\mathcal{M}$ is a subalgebra of $\mathcal{A}$, so that $am = bm + cm \in \mathcal{M}$. A symmetrical argument yields that $ma \in \mathcal{M}$. Consequently $\mathcal{M}$ is an ideal of $\mathcal{I}_{n-1}$, so that $\mathcal{M}$ is an n-subideal of $\mathcal{A}$.

Finally, suppose that the closed subalgebra $\mathcal{M}$ satisfies (5.6). The first part of the argument gives a chain $\mathcal{M} = \mathcal{J}_n \subset \mathcal{J}_{n-1} \subset \cdots \subset \mathcal{J}_1 \subset \mathcal{J}_0 = \mathcal{A}$ of relative ideals. It follows that

$$\mathcal{M} = \overline{\mathcal{J}_n} \subset \overline{\mathcal{J}_{n-1}} \subset \cdots \subset \overline{\mathcal{J}_1} \subset \overline{\mathcal{J}_0} = \mathcal{A}$$

is a chain of closed relative ideals associated to $\mathcal{M}$, so that $\mathcal{M} \in \mathcal{SI}_n(\mathcal{A})$. $\square$

The nilpotency criterion (5.6) can be used in some arguments in Section 4. However, we prefer to define the relevant higher subideals as explicitly as possible, whereas the chain of closed relative ideals found in the proof of Proposition 5.3 is rather implicit. We next apply Proposition 5.3 to the closed n-subideals in the setting of the Tarbard spaces from Section 2, where the fact (ii) formulated in the proof is quite unexpected.

Theorem 5.4. *Let Y be the Tarbard space X_{2n} or X_{2n+1}, where $n \geq 2$. Then*

$$\mathcal{SI}_1(\mathcal{L}(Y)) \subsetneq \cdots \subsetneq \mathcal{SI}_{n-1}(\mathcal{L}(Y)) \subsetneq \mathcal{SI}_n(\mathcal{L}(Y)) = \mathcal{SI}_{n+1}(\mathcal{L}(Y)) = \cdots$$

Proof. According to part (1) of Remarks 4.4 it is enough to verify that

(i) $\mathcal{SI}_{n-1}(\mathcal{L}(Y)) \subsetneq \mathcal{SI}_n(\mathcal{L}(Y))$ and
(ii) $\mathcal{SI}_n(\mathcal{L}(Y)) = \mathcal{SI}_{n+1}(\mathcal{L}(Y))$.

Let $S \in \mathcal{S}(Y)$ be the operator constructed by Tarbard [45], for which $S^{2n-1} \notin \mathcal{K}(Y)$ and $S^{2n+1} = 0$. (For $Y = X_{2n}$ we have $S^{2n} = 0$, but this fact will play no role here.)

Towards (i) we define

$$\mathcal{I} = span(S^2, S^4, \ldots, S^{2n}) + \mathcal{K}(Y),$$

which is a closed subalgebra of $[S^2]$.

Recall from (2.3) that $[S^j] = span(S^j, \ldots, S^{k-1}) + \mathcal{K}(Y)$ for $j = 1, \ldots, k-1$. Hence it follows by iteration from (2.6) that

$$[S^2]^n \subset span(S^{2n}) + \mathcal{K}(Y) \subset \mathcal{I}. \tag{5.9}$$

Thus $\mathcal{I} \in \mathcal{SI}_{n-1}([S^2])$ by Proposition 5.3, and $\mathcal{I} \in \mathcal{SI}_n(\mathcal{L}(Y))$.

It remains to observe that $\mathcal{I} \notin \mathcal{SI}_{n-1}(\mathcal{L}(Y))$. However, this follows directly from Lemma 4.6 since $S \cdot (S^2)^{n-1} = S^{2n-1} \notin \mathcal{I}$ by (2.1).

(ii) The inclusion $\mathcal{SI}_n(\mathcal{L}(Y)) \subset \mathcal{SI}_{n+1}(\mathcal{L}(Y))$ follows from the definition. Conversely, suppose that $\mathcal{J} \in \mathcal{SI}_{n+1}(\mathcal{L}(Y))$ and let

$$\mathcal{J} = \mathcal{J}_{n+1} \subset \cdots \subset \mathcal{J}_1 \subset \mathcal{L}(Y)$$

be an associated chain of closed relative ideals. We first recall that $\mathcal{K}(Y) \subset \mathcal{J}$ by Proposition 4.2 as Y has the A.P.

If $\mathcal{J}_1 = [S]$, then $\mathcal{J}_2$ is a closed ideal of $\mathcal{L}(Y)$ by Lemma 2.2, so that $\mathcal{J} \in \mathcal{SI}_n(\mathcal{L}(Y))$. Thus we may assume that $\mathcal{J}_1 \neq [S]$, whence $\mathcal{J}_1 \subset [S^2]$ by (2.2).

Observe next that if $\mathcal{J} \subset [S^3]$, then we deduce from (2.6) that

$$[S^3]^n \subset \mathcal{K}(Y) \subset \mathcal{J}$$

as $3n \geq 2n + 1$. Thus $\mathcal{J} \in \mathcal{SI}_{n-1}([S^3])$ by Proposition 5.3, and $\mathcal{J} \in \mathcal{SI}_n(\mathcal{L}(Y))$.

On the other hand, if $\mathcal{J} \not\subset [S^3]$ then there is an operator

$$T = \sum_{j=2}^{2n} a_j S^j + K_0 \in \mathcal{J},$$

where $a_2 \neq 0$ and K_0 is compact, since $\mathcal{J} \subset \mathcal{J}_1 \subset [S^2]$. By a similar calculation as in the proof of (2.6) we have

$$T^n = a_2^n S^{2n} + K \in \mathcal{J}$$

for some $K \in \mathcal{K}(Y)$, and thus $S^{2n} \in \mathcal{J}$ by linearity. It follows that

$$[S^2]^n \subset span(S^{2n}) + \mathcal{K}(Y) \subset \mathcal{J}$$

where the first inclusion was observed in (5.9).

Finally, Proposition 5.3 yields that $\mathcal{J} \in \mathcal{SI}_{n-1}([S^2])$, so $\mathcal{J} \in \mathcal{SI}_n(\mathcal{L}(Y))$. This concludes the argument. □

Remark 5.5. Let X_n be the Tarbard space for $n \geq 2$. We note that Theorem 4.5 yields a closed subalgebra $\mathcal{M} \subset \mathcal{S}(Z_n)$ such that

$$\mathcal{M} \in \mathcal{SI}_{n+1}(\mathcal{L}(Z_n)) \setminus \mathcal{SI}_n(\mathcal{L}(Z_n)),$$

where $Z_n := X_{n+1} \oplus X_{n+1}$ for $n \in \mathbb{N}$. Towards this one requires the above operator $S \in \mathcal{S}(X_{n+1})$, for which $S^n \notin \mathcal{K}(X_{n+1})$ and $S^{n+1} = 0$, as well as

$$\overline{\mathcal{S}(X_{n+1})^{n+1}} = \mathcal{K}(X_{n+1}). \tag{5.10}$$

In fact, $V_1 \cdots V_{n+1} \in \mathcal{K}(X_{n+1})$ for any $V_1, \ldots, V_{n+1} \in \mathcal{S}(X_{n+1})$ by an iteration of (2.6), so condition (5.10) is valid as X_{n+1} has the A.P.

Above Z_n is not an H.I. space, whereas in Theorem 5.4 the spaces X_k are H.I. for all k.

Theorem 4.8 suggests the following problem.

Problem. Let $X_n = \bigoplus_{j=1}^{n+1} \ell^{p_j}$ be the direct sum space from (4.8). Is there a decreasing sequence $(\mathcal{M}_k)$ of closed subalgebras of $\mathcal{L}(X_n)$ such that

$$\mathcal{M}_k \in \mathcal{SI}_{k+1}(\mathcal{L}(X_n)) \setminus \mathcal{SI}_k(\mathcal{L}(X_n)), \quad k \in \mathbb{N}?$$

If such a sequence $(\mathcal{M}_k)$ exists, then Proposition 5.3 implies that each quotient algebra $\mathcal{M}_k/\mathcal{K}(X_n)$ must be non-nilpotent, which lies outside of the setting of the $\mathcal{S}(X_n)$-subideals considered in Theorem 4.7. However, there are closed ideals $\mathcal{I} \subset \mathcal{L}(X_n)$ for which $\mathcal{I}/\mathcal{K}(X_n)$ is not nilpotent. (In fact, $\mathcal{L}(X_n)$ has precisely $n+1$ maximal closed ideals which are non-nilpotent modulo $\mathcal{K}(X_n)$, see [51].)

Finally, for completeness we display a few examples of closed subalgebras of $\mathcal{L}(X)$ that fail to be n-subideals for any $n \in \mathbb{N}$. Recall from Proposition 4.2 that $\mathcal{A}(X) \subset \mathcal{M}$ for any non-zero closed n-subideal $\mathcal{M} \subset \mathcal{L}(X)$. Let X be any infinite-dimensional Banach space and put

$$\mathcal{M} = span(x^* \otimes x) \subset \mathcal{L}(X),$$

where $x \in X$ and $x^* \in X^*$ are normalised vectors. It follows that the closed subalgebra $\mathcal{M} \notin \mathcal{SI}_n(\mathcal{L}(X))$ for any $n \in \mathbb{N}$. For another large class of such examples, let

$$\mathcal{M} = span(I_X) + \mathcal{B},$$

where $\mathcal{B} \subset \mathcal{L}(X)$ is any closed subalgebra such that the quotient space $\mathcal{L}(X)/\mathcal{B}$ is at least 2-dimensional. In this case one applies the fact that if $\mathcal{I}$ is an n-subideal of $\mathcal{L}(X)$ for some $n \in \mathbb{N}$ and $I_X \in \mathcal{I}$, then $\mathcal{I} = \mathcal{L}(X)$ by Lemma 4.6. In particular, if $\mathcal{L}(X)/\mathcal{K}(X)$ is at least 2-dimensional, then $\mathcal{M} = span(I_X) + \mathcal{K}(X)$ is a closed Lie ideal of $\mathcal{L}(X)$ which is not an n-subideal for any $n \in \mathbb{N}$.

There are subtler examples of this phenomenon, including closed subalgebras $\mathcal{M}$ that satisfy $\mathcal{A}(X) \subsetneq \mathcal{M} \subsetneq \mathcal{K}(X)$ for certain spaces X.

Example 5.6. There is a Banach space Z without the A.P. and a closed subalgebra $\mathcal{A}(Z) \subset \mathcal{M} \subset \mathcal{K}(Z)$, such that $\mathcal{M} \notin \mathcal{SI}_n(\mathcal{L}(Z))$ for any $n \in \mathbb{N}$.

Proof. Let X be a Banach space without the A.P. for which there is an operator $S \in \mathcal{K}(X)$ such that $S^n \notin \mathcal{A}(X)$ for any $n \in \mathbb{N}$. Recall from Remark 3.7 that such spaces X exist.

Put $Z = X \oplus X$ and $T = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix} \in \mathcal{K}(Z)$. Let

$$M_0 := span(\{T^n : n \in \mathbb{N}\}) + \mathcal{A}(Z).$$

It is straightforward to verify that $\mathcal{M} := \overline{M_0}$ is a closed subalgebra of $\mathcal{L}(Z)$, for which $\mathcal{A}(Z) \subset \mathcal{M} \subset \mathcal{K}(Z)$. Suppose to the contrary that $\mathcal{M} \in \mathcal{SI}_n(\mathcal{L}(Z))$ for some $n \in \mathbb{N}$. Then

$$\begin{bmatrix} 0 & 0 \\ S^{n+1} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ S & 0 \end{bmatrix} \circ T^n \in \mathcal{M}$$

by Lemma 4.6. On the other hand, if $U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix} \in \mathcal{M}$, then by approximation the component $U_{21} \in \mathcal{A}(Z)$. This entails that $S^{n+1} \in \mathcal{A}(X)$, which contradicts the choice of S. □

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