Highlights

Counterexamples to the conjecture of the upper bound of the derivative of a rational Bézier curve

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- Present counterexamples to the upper bound of the first-order derivative of rational Bézier curves, as proposed by Li et al. in *Applied Mathematics and Computation* 219 (2013) 10425-10433
- Investigate the supremum of derivatives of all orders for rational Bézier curves

Counterexamples to the conjecture of the upper bound of the derivative of a rational Bézier curve

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Abstract

In this paper, we present counterexamples to the upper bound of the first-order derivative of rational Bézier curves, as proposed by Li et al. in *Applied Mathematics and Computation* 219 (2013) 10425-10433, and further investigate the supremum of derivatives of all orders for such curves.

Keywords: Rational Bézier curve, Derivative, Bound, Counterexamples, supremum

1. Introduction

Rational Bézier curves are fundamental in geometric design, with their derivative bounds playing a crucial role in both theory and computation. In computational settings, derivative bounds are vital for maintaining numerical stability in curve evaluation, subdivision, and optimization. They help prevent error accumulation, support adaptive meshing strategies, and enhance collision detection[1, 2, 3, 4].

In general, a rational Bézier curve can be defined as [1]

$$\mathbf{r}(t) = \frac{\mathbf{p}(t)}{\omega(t)} = \frac{\sum\limits_{i=0}^{n} \omega_i \mathbf{p}_i B_i^n(t)}{\sum\limits_{i=0}^{n} \omega_i B_i^n(t)}, \ t \in [0,1], \tag{1}$$

where $B_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$ are Bernstein basis functions of degree n, $\boldsymbol{p}_i \in \mathbb{R}^d$ are control points and ω_i are the corresponding positive weights.

In addition, the first derivative of a rational Bézier curve can be written as [2]

$$\mathbf{r}'(t) = \frac{\sum_{i=0}^{2n-2} B_i^{2n-2}(t) D_i}{\left(\sum_{i=0}^{n} B_i^n(t)\omega_i\right)^2}, \tag{2}$$

where

$$D_i = \frac{1}{\binom{2n-2}{i}} \sum_{j=\max\{0,i-n+1\}}^{\lfloor \frac{i}{2} \rfloor} (i-2j+1)\binom{n}{j}\binom{n}{i-j+1}\omega_j\omega_{i-j+1}(\mathbf{p}_{i-j+1}-\mathbf{p}_i).$$

Apply (2) and let

$$\omega = \max\left\{\max_{0\leq i\leq n-1}\left\{\frac{\omega_i}{\omega_{i+1}}\right\}, \max_{0\leq i\leq n-1}\left\{\frac{\omega_{i+1}}{\omega_i}\right\}\right\}, \tag{3}$$

Zhang and Ma obtain [5]

$$\|\mathbf{r}'(t)\| \leqslant n\omega \max_{0\leqslant i\leqslant n-1}\{\|\mathbf{p}_{i+1}-\mathbf{p}_i\|\}, \qquad (n=2,3).$$

Li et al. [6] conjectured that Zhang and Ma's inequality holds for all rational Bézier curves and that it is stronger than many existing bounds. This conjecture was also endorsed by Zhang [7]. That is

Conjecture 1. *For the rational Bézier curve, it has*

$$\|\mathbf{r}'(t)\| \leqslant n\omega \max_{0\leqslant i\leqslant n-1}\{\|\mathbf{p}_{i+1}-\mathbf{p}_i\|\}, \quad (n\in\mathbb{N}).$$

In this paper, we present counterexamples for the conjecture and discuss the supremum of the derivatives of rational Bézier curves.

2. Counterexample

For the rational Bézier curves (1), if $n = 11$,

$$\boldsymbol{\omega} = \{1, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{16}, \tfrac{1}{32}, \tfrac{1}{64}, \tfrac{1}{128}, \tfrac{1}{256}, \tfrac{1}{512}, \tfrac{1}{1024}, \tfrac{1}{512}\},$$

$$\mathbf{p} = \{(0,0),(1,0),(2,0),(3,0),(4,0),(5,0),(6,0),(7,0),(8,0),(9,0),(10,0),(11,0)\}.$$

The rational Bézier function can be expressed as (See Fig. 1)

$$r(t) = \frac{22t\left(t^{10}-5t^9+45t^8-240t^7+840t^6-2016t^5+3360t^4-3840t^3+2880t^2-1280t+256\right)}{t^{11}+11t^{10}-110t^9+660t^8-2640t^7+7392t^6-14784t^5+21120t^4-21120t^3+14080t^2-5632t+1024}$$

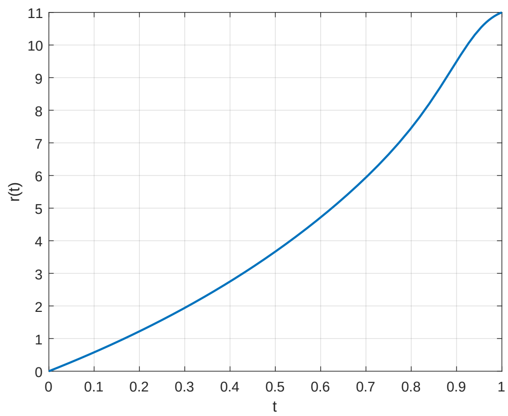

Figure 1: The rational Bézier curve $r(t)$ ($n = 11$).

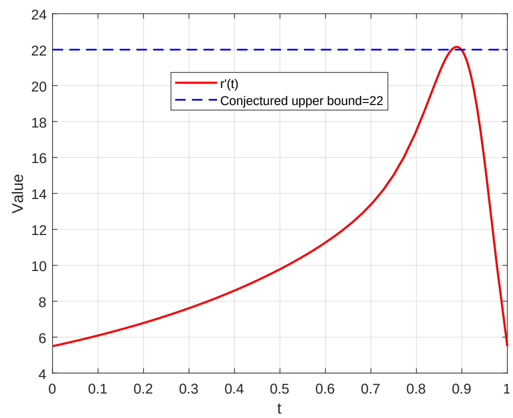

Figure 2: The derivative curve $r'(t)$ and the reference line of the conjectured upper bound.

The corresponding first derivative is

$$r'(t) = \frac{352\left(t^{11} - \frac{11}{8}t^{10} + \frac{55}{16}t^9 - \frac{165}{8}t^8 + \frac{165}{2}t^7 - 231t^6 + 462t^5 - 660t^4 + 660t^3 - 440t^2 + 176t - 32\right)(t-2)^9}{(t^{11} + 11t^{10} - 110t^9 + 660t^8 - 2640t^7 + 7392t^6 - 14784t^5 + 21120t^4 - 21120t^3 + 14080t^2 - 5632t + 1024)^2}$$

With the help of mathematical software, we can easily get $max\ c'(t) = 22.152423$ at $t = 0.888645$.

By (3), it yields $\omega = 2$. So the right side of the conjecture is 22. This contradicts conjecture (See Fig. 2). More numerical conclusions can be found in Example 1.

3. Supremum

Shi [8] presented the explicit formulas for the high-order derivatives of rational Bézier curves and their bounds. We employ the approach to derive explicit expressions for the first-order derivative of rational Bézier curves and establish its supremum. The method can be conveniently extended to the calculation of the supremum of any-order derivatives.

Theorem 1. *The explicit expression for the first derivative of a rational Bézier curve is*

$$\mathbf{r}'(t) = \frac{n\sum_{i=0}^{2n}\hat{\mathbf{P}}_{i,2n}^{[1]}B_i^{2n}(t)}{\sum_{i=0}^{2n}\omega_{i,2n}^{[1]}B_i^{2n}(t)} = \frac{\sum_{i=0}^{2n}\mathbf{Q}_{i,2n}^{[1]}\omega_{i,2n}^{[1]}B_i^{2n}(t)}{\sum_{i=0}^{2n}\omega_{i,2n}^{[1]}B_i^{2n}(t)},$$

where the parameters are calculated as follows
1. Weights $\omega_{i,2n}^{[1]}$ *$(i = 0, \ldots, 2n)$*

$$\omega_{i,2n}^{[1]} = \sum_{j=\max(0,i-n)}^{\min(i,n)} \frac{\binom{n}{j}\binom{n}{i-j}}{\binom{2n}{i}}\omega_j\omega_{i-j}.$$

2. Intermediate control points $\boldsymbol{P}_{j,2n}^{[1]}$ *$(j = 0, \ldots, 2n-1)$*

$$\boldsymbol{P}_{j,2n}^{[1]} = \sum_{h=\max(0,j-n)}^{\min(n-1,j)} \frac{\binom{n-1}{h}\binom{n}{j-h}}{\binom{2n-1}{j}}\left[\omega_{h+1}\omega_{j-h}(\mathbf{p}_{h+1} - \mathbf{p}_{j-h}) + \omega_h\omega_{j-h}(\mathbf{p}_{j-h} - \mathbf{p}_h)\right],$$

and

$$\hat{\boldsymbol{P}}_{i,2n}^{[1]} = \sum_{j=\max(0,i-1)}^{\min(2n-1,i)} \frac{\binom{2n-1}{j}}{\binom{2n}{i}}\boldsymbol{P}_{j,2n}^{[1]}.$$

3. Control points $\mathbf{Q}_{i,2n}^{[1]}$ *$(i = 0, \ldots, 2n)$*

$$\mathbf{Q}_{i,2n}^{[1]} = \frac{n\hat{\mathbf{P}}_{i,2n}^{[1]}}{\omega_{i,2n}^{[1]}}.$$

The process of calculating the derivative in Theorem 1 is the same as the method used by Pigel, so the calculation method is stable [9][10].

Using the degree elevation algorithm [11], it yields

$$\mathbf{r}'(t) = \frac{n\sum_{i=0}^{2n+e}\hat{\mathbf{P}}_{i,2n+e}^{[1]}B_i^{2n+e}(t)}{\sum_{i=0}^{2n+e}\omega_{i,2n+e}^{[1]}B_i^{2n+e}(t)},$$

where

$$\hat{\mathbf{P}}_{i,2n+e}^{[1]} = \frac{1}{\binom{2n+e}{i}}\sum_{j=\max(0,i-e)}^{\min(2n,i)}\binom{2n}{j}\binom{e}{i-j}\hat{\mathbf{P}}_{j,2n}^{[1]}$$

and

$$\omega_{i,2n+e}^{[1]} = \frac{1}{\binom{2n+e}{i}} \sum_{j=\max(0,i-e)}^{\min(2n,i)} \binom{2n}{j}\binom{e}{i-j}\omega_{j,2n}^{[1]}.$$

Since when $e \to +\infty$, the control polygon formed by the control points

$$\mathbf{Q}_{i,2n+e}^{[1]} = \frac{n\hat{\mathbf{P}}_{i,2n+e}^{[1]}}{\omega_{i,2n+e}^{[1]}}$$

approximates the rational Bézier curve. Then, based on the following inequality [12] [13]

$$\frac{\sum a_k c_k}{\sum b_k c_k} \leqslant \max_k \frac{a_k}{b_k},$$

where $a_k \in \mathbb{R}$, $b_k > 0$, $c_k > 0$ and $1 \le k \le n$, we have

Theorem 2.

$$\|\boldsymbol{r}'(t)\|_{l^p} \le \max_{i=0,\ldots,2n+e} \frac{n\left\|\sum\limits_{j=\max(0,i-e)}^{\min(2n,i)} \binom{2n}{j}\binom{e}{i-j}\hat{\boldsymbol{P}}_{j,2n}^{[1]}\right\|_{l^p}}{\sum\limits_{j=\max(0,i-e)}^{\min(2n,i)} \binom{2n}{j}\binom{e}{i-j}\omega_{j,2n}^{[1]}}, \tag{4}$$

where l^p is the vector norm. When $e \to +\infty$. The boundary of $\|\mathbf{r}'(t)\|_{l^p}$ is its supremum.

4. Example

For rational Bézier curves with degrees from 2 to 20, weights $\{\omega_i\}_{i=0}^{n-1} = \left\{\frac{1}{2^i}\right\}_{i=0}^{n-1}$ and $\omega_n = \frac{1}{2^{n-2}}$, and control points $\{\mathbf{p}_i\}_{i=0}^{n} = \{(i,0)\}_{i=0}^{n}$. Table 1 presents the maximum values of the first derivatives, their corresponding parameter t, the conjectured upper bounds, the upper bounds derived from our method, and the computation time of our method. A comparison of the three methods is illustrated in Fig. 3, while the running times are depicted in Fig. 4.

Table 1: Summary of calculation results for degrees n from 2 to 20.

n	Max first derivative	Corresponding t	Conjectured upper bounds	Our method (e=1000)	Running times (sec)
2	2.666667	0.500000	4.000000	2.669326	0.42
3	4.466444	0.656248	6.000000	4.473876	0.60
4	6.464102	0.732052	8.000000	6.478938	1.03
5	8.571204	0.778664	10.000000	8.595967	1.03
6	10.748531	0.810729	12.000000	10.785651	1.06
7	12.974278	0.834352	14.000000	13.026093	1.05
8	15.235043	0.852540	16.000000	15.303826	1.02
9	17.522048	0.866991	18.000000	17.609929	1.06
10	19.829270	0.878795	20.000000	19.938505	1.05
11	22.152423	0.888645	22.000000	22.285016	1.09
12	24.488370	0.896975	24.000000	24.646413	1.06
13	26.834755	0.904127	26.000000	27.020171	1.07
14	29.189773	0.910321	28.000000	29.404333	1.04
15	31.552017	0.915748	30.000000	31.798333	1.14
16	33.920374	0.920566	32.000000	34.199561	1.07
17	36.293948	0.924842	34.000000	36.608912	1.14
18	38.672013	0.928682	36.000000	39.023699	1.06
19	41.053972	0.932145	38.000000	41.445157	1.04
20	43.439332	0.935269	40.000000	43.871119	1.05

5. Conclusion

This paper presents several counterexamples to the conjecture on the upper bounds of derivatives of rational Bézier curves. Nevertheless, we remain interested in the following issues related to this conjecture:

- What is the maximum value of n for which the conjecture fails?
- What is the mathematical principle underlying its failure? Could it be related to the non-uniform convergence of rational Bézier curves with respect to weights [14]?

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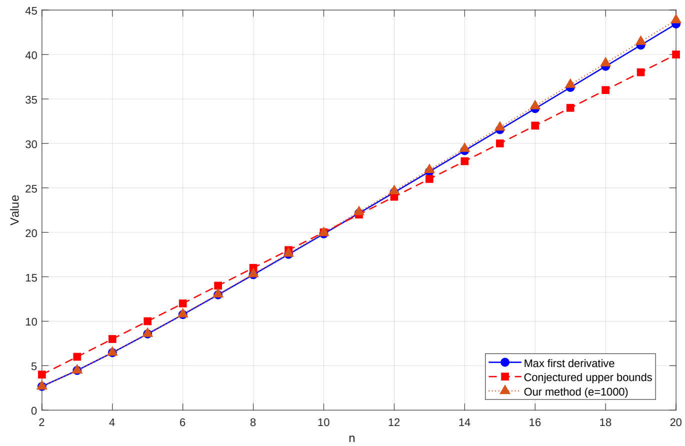

Figure 3: Comparison of the three methods

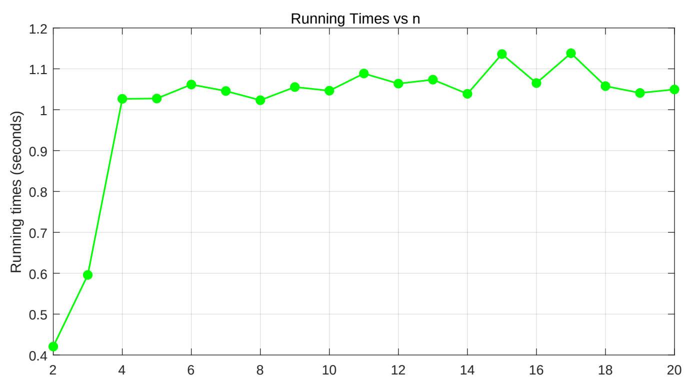

Figure 4: The running time of our method (reference value)

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