GENERALIZING LEE'S CONJECTURE ON THE SUM OF ABSOLUTE VALUES OF MATRICES

QUANYU TANG AND SHU ZHANG

ABSTRACT. Let $\|\cdot\|_p$ denote the Schatten p-norm of matrices and $\|\cdot\|_F$ the Frobenius norm. For a square matrix X, let $|X|$ denote its absolute value. In 2010, Eun-Young Lee posed the problem of determining the smallest constant c_p such that $\|A+B\|_p \le c_p \| \, |A|+|B| \, \|_p$ for all complex matrices A, B. The Frobenius case ($p=2$) conjectured by Lee was proved by Lin and Zhang (2022) [6] and re-proved by Zhang (2025) [7]. In this paper, we extend Lee's conjecture from two matrices to an arbitrary number $m \ge 2$ of complex matrices $A_1, \dots, A_m$, and determine the sharp inequality

$$\left\| \sum_{k=1}^{m} A_k \right\|_F \le \sqrt{\frac{1+\sqrt{m}}{2}} \left\| \sum_{k=1}^{m} |A_k| \right\|_F ,$$

with equality attained by an equiangular rank-one family. We further generalize Lee's problem by seeking the smallest constant $c_p(m)$ such that $\| \sum_{k=1}^m A_k\|_p \le c_p(m) \, \| \sum_{k=1}^m |A_k| \|_p$. It is shown that $c_p(m) \le (\sqrt{m})^{1-1/p}$, and we conjecture a closed-form expression for the optimal value of $c_p(m)$ that recovers all known cases $p = 1, 2, \infty$.

1. Introduction

Let $M_n(\mathbb{C}) = \mathbb{C}^{n\times n}$ denote the space of all complex $n \times n$ matrices. For $A \in M_n(\mathbb{C})$, its *absolute value* is defined by $|A| = (A^*A)^{1/2}$, where A^* denotes the conjugate transpose of A. The singular values of A, arranged in nonincreasing order, are denoted by $s_1(A) \ge s_2(A) \ge \cdots \ge s_n(A) \ge 0$. For $p \ge 1$, the *Schatten p-norm* of A is

$$\|A\|_p = \left(\operatorname{Tr} |A|^p\right)^{1/p} = \left(\sum_{j=1}^{n} s_j(A)^p \right)^{1/p} .$$

The case $p = 2$ gives the *Frobenius norm*,

$$\|A\|_F = \sqrt{\operatorname{Tr} |A|^2} = \left(\sum_{j=1}^{n} s_j(A)^2 \right)^{1/2} .$$

For $A, B \in M_n(\mathbb{C})$, Lee [5] asked for the smallest constant c_p such that

$$\|A+B\|_p \le c_p \| \, |A|+|B| \, \|_p, \tag{1.1}$$

and conjectured that $c_2 = \frac{1+\sqrt{2}}{2}$. The Frobenius case $p = 2$ of (1.1) was later proved in Lin and Zhang [6], with an alternative proof via the matrix Cauchy–Schwarz inequality given in Zhang [7].

In this paper we first establish a sharp generalization of Lee's conjecture for the Frobenius norm involving m matrices.

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Theorem 1.1. *For any $m \geq 2$ and $A_1, \ldots, A_m \in M_n(\mathbb{C})$,*

$$\left\| \sum_{k=1}^{m} A_k \right\|_F \leq \sqrt{\frac{1+\sqrt{m}}{2}} \left\| \sum_{k=1}^{m} |A_k| \right\|_F . \tag{1.2}$$

Moreover, the constant $\sqrt{\frac{1+\sqrt{m}}{2}}$ in inequality (1.2) is optimal.

Remark 1.2. Although we state Theorem 1.1 for matrices in $M_n(\mathbb{C})$, the same inequality (with the same proof) still holds for Hilbert–Schmidt operators on a complex Hilbert space and, more generally, for elements of a von Neumann algebra equipped with a regular trace. For simplicity we work in the finite-dimensional matrix setting.

As a noteworthy special case, taking $m = 9$ in Theorem 1.1 yields the sharp inequality

$$\|A_1 + \cdots + A_9\|_F \leq \sqrt{2}\, \||A_1| + \cdots + |A_9|\|_F ,$$

whose constant $\sqrt{2}$ coincides with the sharp constant in the two-matrix operator norm inequality

$$\|A + B\|_\infty \leq \sqrt{2}\, \||A| + |B|\|_\infty .$$

We then extend Lee's problem to the case of m summands in the Schatten p-norm setting.

Question 1.3. *Let $p \geq 1$ and $m \geq 2$. Find the smallest constant $c_p(m)$ such that*

$$\left\| \sum_{k=1}^{m} A_k \right\|_p \leq c_p(m) \left\| \sum_{k=1}^{m} |A_k| \right\|_p \quad \textit{for all } A_1, \ldots, A_m \in M_n(\mathbb{C}).$$

Notice that Theorem 1.1 implies $c_2(m) = \sqrt{\frac{1+\sqrt{m}}{2}}$. For $p = 1$ and $p = \infty$, the corresponding optimal constants $c_p(m)$ are also obtained exactly, while for general $p \geq 1$ we establish in Corollary 3.3 a unified upper bound for $c_p(m)$ and further propose in Conjecture 3.1 an explicit formula consistent with all known cases.

This paper is organized as follows. Section 2 contains the proof of Theorem 1.1. In Section 3, we discuss Question 1.3, derive a unified upper bound for $c_p(m)$ in Corollary 3.3, and propose an explicit conjectural formula for $c_p(m)$ in Conjecture 3.1.

2. Main results

We will use the following result from [7, Lemma 2.2].

Lemma 2.1 ([7])**.** *Let X, Y be two $n \times n$ positive semidefinite matrices. Assume Q is a contraction of order n, i.e., $\|Q\|_\infty \leq 1$. Then for any $t > 0$,*

$$4\,|\mathrm{Tr}(QXY)| \leq t\,\mathrm{Tr}(X^2 + Y^2) + \frac{1}{t}\,\mathrm{Tr}(XY + YX).$$

We are now ready to present the following. Here $\Re z$ denotes the real part of $z \in \mathbb{C}$.

Proof of Theorem 1.1. Write the polar decompositions $A_k = U_k|A_k|$. Expanding the Frobenius norm,

$$\left\| \sum_{k=1}^{m} A_k \right\|_F^2 = \sum_{k=1}^{m} \mathrm{Tr}\,|A_k|^2 + 2 \sum_{1 \leq j < k \leq m} \Re\,\mathrm{Tr}\left(U_j^* U_k |A_k||A_j|\right).$$

For each pair (j,k) apply Lemma 2.1 with $X = |A_j|$, $Y = |A_k|$, $Q = U_j^* U_k$; using $|\Re z| \leq |z|$ we get, for every $t > 0$,

$$2\left|\Re\,\mathrm{Tr}(U_j^* U_k |A_k||A_j|)\right| \leq \frac{t}{2}\,\mathrm{Tr}(|A_j|^2 + |A_k|^2) + \frac{1}{2t}\,\mathrm{Tr}(|A_j||A_k| + |A_k||A_j|).$$

Summing over $j < k$ gives

$$\left\|\sum_{k=1}^{m} A_k\right\|_F^2 \leq \left(1 + \frac{m-1}{2}t\right)\sum_{k=1}^{m} \operatorname{Tr}|A_k|^2 + \frac{1}{t}\sum_{1\leq j<k\leq m} \operatorname{Tr}(|A_j||A_k|).$$

Then set $t = \frac{1}{1+\sqrt{m}}$, we have

$$\left\|\sum_{k=1}^{m} A_k\right\|_F^2 \leq \frac{1+\sqrt{m}}{2}\left(\sum_{k=1}^{m} \operatorname{Tr}|A_k|^2 + 2\sum_{1\leq j<k\leq m} \operatorname{Tr}(|A_j||A_k|)\right) = \frac{1+\sqrt{m}}{2}\left\|\sum_{k=1}^{m} |A_k|\right\|_F^2.$$

Taking square roots yields (1.2).

Next we will show that the inequality (1.2) is sharp. Fix an undetermined parameter $s \in (0,1)$. We claim that there exist unit vectors $x_1, \ldots, x_m \in \mathbb{C}^m$ such that

$$\langle x_j, x_k\rangle = \begin{cases} 1, & j = k, \\ s, & j \neq k. \end{cases}$$

To see why such an equiangular system exists, consider the $m \times m$ Gram matrix

$$G = (\langle x_j, x_k\rangle)_{j,k=1}^{m} = (1-s)I_m + sJ_m,$$

where J_m denotes the all–ones matrix. The matrix G is positive semidefinite, so $G = X^*X$ for some X, and with x_k the kth column of X we obtain our unit equiangular system.

Fix a unit vector $u \in \mathbb{C}^m$ and set

$$A_k := u\, x_k^* \qquad (k = 1, \ldots, m).$$

Then

$$A_k^* A_k = x_k\, x_k^*, \quad \text{so} \quad |A_k| = (A_k^* A_k)^{1/2} = (x_k x_k^*)^{1/2} = x_k x_k^*.$$

Here $x_k x_k^*$ is the orthogonal projection onto $\operatorname{span}\{x_k\}$ (it is also positive semidefinite), hence equals its own positive square root. Now we compute both sides:

$$\left\|\sum_{k=1}^{m} A_k\right\|_F^2 = \left\|u\sum_{k=1}^{m} x_k^*\right\|_F^2 = \left\|\sum_{k=1}^{m} x_k\right\|_2^2 = m + 2\sum_{1\leq j<k\leq m} \Re\langle x_j, x_k\rangle = m + m(m-1)s,$$

$$\left\|\sum_{k=1}^{m} |A_k|\right\|_F^2 = \left\|\sum_{k=1}^{m} x_k x_k^*\right\|_F^2 = \sum_{j,k=1}^{m} \operatorname{Tr}(x_j x_j^* x_k x_k^*) = m + 2\sum_{1\leq j<k\leq m} |\langle x_j, x_k\rangle|^2 = m + m(m-1)s^2.$$

Therefore

$$\frac{\|\sum_{k=1}^{m} A_k\|_F^2}{\|\sum_{k=1}^{m} |A_k|\|_F^2} = \frac{1+(m-1)s}{1+(m-1)s^2} =: f(s).$$

Optimize $f(s)$ on $(0,1)$:

$$f'(s) = 0 \iff (m-1)s^2 + 2s - 1 = 0 \iff s^* = \frac{\sqrt{m}-1}{m-1} = \frac{1}{1+\sqrt{m}}.$$

Let $s = s^* \in (0,1)$, substituting back,

$$f(s^*) = \frac{1+\sqrt{m}}{2} \quad \Longrightarrow \quad \frac{\|\sum_{k=1}^{m} A_k\|_F}{\|\sum_{k=1}^{m} |A_k|\|_F} = \sqrt{\frac{1+\sqrt{m}}{2}}.$$

This shows that the inequality (1.2) is sharp. The proof is complete. □

Remark 2.2. The extremal rank-one equiangular construction above reduces, when $m = 2$, to the known equality case in Lee's conjecture; see [6].

3. Concluding Remarks

We now turn to the generalized version of Lee's problem, that is, Question 1.3. Supported by numerical experiments, we are led to the following conjectural formula for $c_p(m)$.

Conjecture 3.1. *Let $p > 1$ and $m \geq 2$. Let $c_p(m)$ be the smallest number such that*

$$\left\| \sum_{k=1}^{m} A_k \right\|_p \leq c_p(m) \left\| \sum_{k=1}^{m} |A_k| \right\|_p \quad \textit{for all } A_1, \ldots, A_m \in M_n(\mathbb{C}).$$

Then[1]

$$c_p(m) = \frac{\sqrt{x_{p,m}\,(x_{p,m} + m - 1)}}{(x_{p,m}^p + m - 1)^{1/p}}, \qquad x_{p,m} > 0 \textit{ solves } x^p - 2x - (m-1) = 0. \tag{3.1}$$

A heuristic derivation of Conjecture 3.1. We again employ the construction that realizes equality in the proof of Theorem 1.1. Keep the rank-one equiangular family $A_k = u\,x_k^*$, $\langle x_j, x_k \rangle = s \in (0,1)$. As $\sum_k A_k = u \sum_k x_k^*$ is rank one, its Schatten p-norm equals its unique singular value:

$$\left\| \sum_{k=1}^{m} A_k \right\|_p = \left\| \sum_{k=1}^{m} x_k \right\|_2 = \sqrt{m + m(m-1)s}.$$

The positive operator $\sum_k |A_k| = \sum_k x_k x_k^*$ has eigenvalues $1 + (m-1)s$ (multiplicity 1) and $1 - s$ (multiplicity $m-1$), so

$$\left\| \sum_{k=1}^{m} |A_k| \right\|_p = \Big((1 + (m-1)s)^p + (m-1)(1-s)^p \Big)^{1/p}.$$

Maximizing the ratio over $s \in (0,1)$ is equivalent to introducing $y = \frac{1+(m-1)s}{1-s} \in (1, \infty)$ and optimizing

$$R_p(y) = \frac{\sqrt{y(y+m-1)}}{(y^p + m - 1)^{1/p}}.$$

A logarithmic derivative shows the maximizer y satisfies $y^p = 2y + (m-1)$, and substituting $x_{p,m} := y$ gives (3.1). □

In [5], Lee mentioned that "one may easily state a version of [5, Corollaries 2.2 and 2.2a] for finite families of operators", but did not explicitly write down the corresponding p-norm formulation. For completeness, we state and prove such a version here. The argument follows the same structure as Lee's original proof, with minor modifications to handle m summands.

We recall that a function $f : [0, \infty) \to [0, \infty)$ is said to be *geometrically convex* if $f\left(\sqrt{ab}\right) \leq \sqrt{f(a)f(b)}$ for all $a, b > 0$. We also recall that *symmetric norms* refer to *unitarily invariant norms*, namely those matrix norms satisfying $\|UAV\| = \|A\|$ for all unitary matrices U, V. In particular, the Schatten p-norms ($1 \leq p \leq \infty$) are unitarily invariant.

Proposition 3.2. *Let $f : [0, \infty) \to [0, \infty)$ be concave and geometrically convex. Then for any matrices $A_1, \ldots, A_m \in M_n(\mathbb{C})$ and any Schatten norm $\|\cdot\|_q$ ($1 \leq q \leq \infty$),*

$$\|f(|A_1 + \cdots + A_m|)\|_q \leq \left(\sqrt{m}\right)^{1-1/q} \|f(|A_1|) + \cdots + f(|A_m|)\|_q.$$

[1]The limiting case $p = 1$ is discussed separately in Remark 3.4.

Proof. Let

$$S := \sum_{i=1}^{m} A_i, \qquad P := \sum_{i=1}^{m} |A_i^*|, \qquad Q := \sum_{i=1}^{m} |A_i|.$$

By [1, Corollary 1.3.7], for each i, the block matrix $\begin{pmatrix} |A_i^*| & A_i \\ A_i^* & |A_i| \end{pmatrix}$ is positive semidefinite. Summing over i yields $\begin{pmatrix} P & S \\ S^* & Q \end{pmatrix} \geq 0$. By [1, Proposition 1.3.2], there exists a contraction K (i.e. $\|K\|_\infty \leq 1$) such that $S = P^{1/2}\, K\, Q^{1/2}$. Let $s_j(\cdot)$ denote the singular values and $\lambda_j(\cdot)$ the eigenvalues of an operator (both in nonincreasing order). In the same way as in [4, p. 208], for every $k \geq 1$, we have

$$\prod_{j=1}^{k} \lambda_j(|S|) = \prod_{j=1}^{k} s_j(S) \leq \prod_{j=1}^{k} s_j(P^{1/2}) s_j(Q^{1/2}) = \prod_{j=1}^{k} \lambda_j^{1/2}(P)\lambda_j^{1/2}(Q).$$

Since f is geometrically convex and increasing, we observe that $\phi(t) = f(e^t)$ is convex and increasing. The above weak log-majorisation then yields the following weak majorisation: for all $k = 1, 2, \ldots,$

$$\sum_{j=1}^{k} \lambda_j \left(f(|S|)\right) \leq \sum_{j=1}^{k} f\left[\lambda_j^{1/2}(P)\lambda_j^{1/2}(Q)\right] \leq \sum_{j=1}^{k} \lambda_j^{1/2}\left(f(P)\right) \lambda_j^{1/2}\left(f(Q)\right).$$

Let X be the diagonal matrix whose diagonal entries are

$$\lambda_1^{1/2}\left(f(P)\right), \, \lambda_2^{1/2}\left(f(P)\right), \, \ldots,$$

and let Y be the diagonal matrix with diagonal entries

$$\lambda_1^{1/2}\left(f(Q)\right), \, \lambda_2^{1/2}\left(f(Q)\right), \, \ldots.$$

The above weak majorisation implies that $\|f(|S|)\| \leq \|XY\|$ for all symmetric norms. By the Cauchy–Schwarz inequality for symmetric norms,

$$\|XY\| \leq \|X^*X\|^{1/2} \|Y^*Y\|^{1/2},$$

and hence

$$\|f(|S|)\| \leq \|f(P)\|^{1/2} \|f(Q)\|^{1/2}. \tag{3.2}$$

Now, since $f : [0, \infty) \to [0, \infty)$ is concave, we apply the Bourin–Uchiyama subadditivity theorem (see [2, Theorem 1.1] and the remarks in [2, p. 516]). For all symmetric norms, we have

$$\|f(P)\|_q \leq \left\| \sum_{i=1}^{m} f(|A_i^*|) \right\|_q, \qquad \|f(Q)\|_q \leq \left\| \sum_{i=1}^{m} f(|A_i|) \right\|_q. \tag{3.3}$$

Combining (3.2) and (3.3) yields

$$\|f(|S|)\|_q \;\leq\; \left\| \sum_{i=1}^{m} f(|A_i^*|) \right\|_q^{1/2} \left\| \sum_{i=1}^{m} f(|A_i|) \right\|_q^{1/2}. \tag{3.4}$$

By [3, Lemma 2.2], for arbitrary positive semidefinite $X_1, \ldots, X_m$, we have

$$\left\| \sum_{i=1}^{m} X_i \right\|_q \leq \sum_{i=1}^{m} \|X_i\|_q \;\leq\; m^{1-1/q} \left(\sum_{i=1}^{m} \|X_i\|_q^q \right)^{1/q} = m^{1-1/q} \left\| \bigoplus_{i=1}^{m} X_i \right\|_q. \tag{3.5}$$

and

$$\left\| \bigoplus_{i=1}^{m} X_i \right\|_q \le \left\| \sum_{i=1}^{m} X_i \right\|_q . \tag{3.6}$$

Since $|A_i^*|$ and $|A_i|$ have identical singular values, combining inequalities (3.5) and (3.6) with (3.4) yields

$$\begin{aligned} \|f(|S|)\|_q &\le \left(m^{1-1/q} \left\| \bigoplus_{i=1}^{m} f(|A_i|) \right\|_q \right)^{1/2} \left\| \sum_{i=1}^{m} f(|A_i|) \right\|_q^{1/2} \\ &\le m^{(1-1/q)/2} \left\| \sum_{i=1}^{m} f(|A_i|) \right\|_q^{1/2} \left\| \sum_{i=1}^{m} f(|A_i|) \right\|_q^{1/2} \\ &= \left(\sqrt{m}\right)^{1-1/q} \left\| \sum_{i=1}^{m} f(|A_i|) \right\|_q . \end{aligned}$$

This completes the proof. □

As a direct corollary of Proposition 3.2, we obtain the following unified upper bound for $c_p(m)$.

Corollary 3.3. *For any $p \ge 1$ and $m \ge 2$,*

$$c_p(m) \le \left(\sqrt{m}\right)^{1-1/p} . \tag{3.7}$$

Remark 3.4. The conjectural formula (3.1) agrees with the exact values at $p = 1, 2, \infty$:

- $p = 1$: letting $p \to 1^+$ in (3.1) forces $x_{p,m} \to \infty$, so $c_p(m) \to 1$; in fact $c_1(m) = 1$ exactly, since Corollary 3.3 gives $c_1(m) \le 1$.
- $p = 2$: $x_{2,m} = 1 + \sqrt{m}$ and hence $c_2(m) = \sqrt{\frac{1+\sqrt{m}}{2}}$, in agreement with Theorem 1.1.
- $p = \infty$: as $p \to \infty$ we have $x_{p,m} \to 1$ and $(x_{p,m}^p + m - 1)^{1/p} \to 1$, hence (3.1) yields $c_p(m) \to \sqrt{m}$; in fact $c_\infty(m) = \sqrt{m}$ exactly, since Corollary 3.3 implies $c_\infty(m) \le \sqrt{m}$.

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School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, P. R. China
Email address: tang_quanyu@163.com

School of Mathematics and Statistics, Xi'an Jiaotong University, Xi'an 710049, P. R. China
Email address: 2213715573@stu.xjtu.edu.cn