

Bayesian reliability acceptance sampling plans for competing risks data under interval censoring

Biswabrata Pradhan^{a,*}, Rathin Das^a

^a*Statistical Quality Control & Operations Research Unit, Indian Statistical Institute, Kolkata, India*

Abstract

We obtain a reliability acceptance sampling plan for independent competing risk data under interval censoring schemes using the Bayesian approach. At first, the Bayesian reliability acceptance sampling plan is obtained where the decision criteria of accepting a lot is pre-fixed. For large samples, computing Bayes risk is computationally intensive. Therefore, an approximate Bayes risk is obtained using the asymptotic properties of the maximum likelihood estimators. Lastly, the Bayesian reliability acceptance sampling plan is obtained, where the decision function is arbitrary. The manufacturer can derive an optimal decision function by minimizing the Bayes risk among all decision functions. This optimal decision function is known as Bayes decision function. The optimal sampling plan is obtained by minimizing the Bayes risk. The algorithms are provided for the computation of optimum Bayesian reliability acceptance sampling plan. Numerical results are provided and comparisons between the Bayesian reliability acceptance sampling plans are carried out.

Keywords: Asymptotic property, Bayes risk, Bayes decision, exponential distribution, maximum likelihood estimation, reliability

Abbreviation & Notation

ASP	acceptance sampling plan
RASP	reliability acceptance sampling plan
PDF	probability density function
RF	reliability function
RV	random variable
CDF	cumulative distribution function

*Corresponding author

Email address: bis@isical.ac.in (Biswabrata Pradhan)

FIM	Fisher information matrix
JPDF	joint probability density function
MLE	maximum likelihood estimator
BSP	Bayesian RASP
IID	independent and identically distributed
wrt	with respect to
X_i	RV of lifetime of the i^{th} item
$\boldsymbol{\nu}$	vector of parameters
$f(\cdot \mid \boldsymbol{\nu})$	PDF of X_i
C_i	cause of the failure of the i^{th} item
$G(\cdot \mid \boldsymbol{\nu})$	sub-distribution of X_i and C_i
$F(\cdot \mid \boldsymbol{\nu})$	CDF of X_i
n	sample size
k	inspection number
$\tau_1, \tau_2, \dots, \tau_k$	inspection times points
$\boldsymbol{\tau} = (\tau_1, \tau_2, \dots, \tau_k)$	vector of inspection time points
$\boldsymbol{\zeta} = (n, \boldsymbol{\tau}, k)$	vector of sampling parameters
D_{mj}	RV denoting due to cause j , the number of failures in the $(\tau_{m-1}, \tau_m]$ interval
$\boldsymbol{D} = (D_{11}, \dots, D_{1J}, \dots, D_{m1}, \dots, D_{mJ})$	vector of number of failures in m intervals due to each cause
d_{mj}	observed value of D_{mj}
$\boldsymbol{d} = (d_{11}, \dots, d_{1J}, \dots, d_{m1}, \dots, d_{mJ})$	vector of observed number of failures in m intervals due to each cause
$P_{\boldsymbol{D}}$	joint distribution of $\boldsymbol{D}$
$D_t = \sum_{m=1}^k \sum_{j=1}^J D_{mj}$	RV denoting the total number of failures after life test
d_t	observed value of D_t
$E[D_t]$	expected number of failures
$E[M]$	expected inspection number
$E[\tau]$	expected test duration
$E[D_t^*]$	expected number of failures for optimal BSP
$E[\tau^*]$	expected test duration for optimal BSP

$E[M^*]$	expected inspection number for optimal BSP
$P(A)$	probability of accepting the lot for BSP

1. Introduction

Acceptance sampling plan (ASP) is a popular method for deciding on acceptance or rejection of a lot of products. RASP is a special type of ASP where the decision is taken based on product's reliability. In this method, a sample is drawn for life testing from the lot. After life testing, using the lifetime data, a suitable statistic is obtained to take a decision of acceptance or rejection about the lot. In reliability studies, determining an optimal RASP is a crucial task. The methods for choosing an optimal RASP are defense sampling schemes, producer's and consumer's risk method, decision-theoretic approach and Dodge and Roming's plan. From an economic perspective, the most scientific and reasonable method is the decision-theoretic approach because it is determined by certain economic factors, such as minimizing loss or maximizing return.

Due to time and cost constraints, censored life tests are generally conducted in practice. Determination of RASP based on censored data is an important task in reliability studies. The most common type of censoring schemes used in practice are type I, type II and hybrid censoring schemes. Continuous monitoring of life test is required to obtain lifetime data under these schemes. However, in many situations, it is not possible to monitor the test continuously and the items on the test are inspected at some pre-specified inspection times. Here, the failure times are not observed but only the number of failed items is observed at each inspection time. The data are referred to as interval-censored data, and the test is known as the interval-censored test (see Lu and Tsai [1]). In ICS, all items are put on life test at $\tau_0 = 0$ and the items are inspected at pre-determined inspection times $\tau_1 < \tau_2 < \dots < \tau_k$, where the inspection number, k , is also pre-determined.

There have been several works for determining optimal BSPs for exponential distribution under different censoring schemes. Yeh [2] considered the optimal BSP under type-II censoring. A similar approach is considered determination of BSP under different censoring schemes by many authors. Yeh [3] and Yeh and Choy [4] considered for Type-I and random censoring schemes, respectively. Chen et al. [5] and Lin et al. [6] provided methodology for hybrid censoring scheme. These BSPs for censored data have not considered the prior belief of the parameter in the decision function.

Papazoglou [7] analyzed a Bayesian decision problem based on reliability of the product and pointed out that the decision of acceptance or rejection about the lot can be made based on the existing prior information. However, due to the uncertainty of the parameters, the consumer can not be sure about the decision regarding

the lot. Additional information needs to be obtained through life-testing at a cost. The lifetime data can be combined with the existing prior information to update the information. The decision can be taken using the updated information on the product's reliability. Lin et al. [8] considered a decision function called Bayes decision function under type-I censoring. This Bayes decision function was also studied for different censoring schemes. For example, Liang and Yang [9] and Yang et al. [10] used Bayes decision function for the type-I hybrid and modified type-II hybrid censoring schemes, respectively. The Bayes decision function for the ICS scheme was provided by Chen et al. [11].

The BSPs referenced above primarily focused on systems that have a single failure mode. However, product may fail due to more than one cause called competing risks. In the competing risk setup, the data consists of product lifetime and an indicator variable denoting the specific cause of the failure. If we ignore the information on causes of failure, then it may incorrectly infer a product's reliability. For details, see Crowder [12] and Nelson [13]. To analyze the competing risk data, the latent failure rate model proposed by Cox [14] is the most popular model in reliability analysis. The cause of the failure may be statistically independent or dependent. It is generally assumed, however, that these competing risks are statistically independent even though the assumption of statistical dependence on the data is more realistic. Because in the assumption of dependence, the latent failure rate model may face some identifiability issues. To avoid such problems, it is assumed that latent failure times are statistically independent.

The problem of designing RASP for competing risk data is an important task in reliability study. Wu and Huang [15] studied RASP for competing risk data under progressive ICS schemes by the classical approach. Recently, Prajapati et al. [16] studied BSP for competing risk data under type-II and hybrid censoring schemes. We consider designing optimal BSP for competing risk data under ICS.

The main contribution of the paper is to determine the optimal BSP by minimizing the Bayes risk in the case of exponential distribution under ICS. As discussed before, the consumer can take the decision based on the product's reliability. Therefore, one of the purposes of this paper is to develop a BSP using the decision function based on the MLE of the RF. The second purpose of the paper is to approximate the BSP using the asymptotic properties of the MLE of the RF. The third purpose of the paper is to develop a BSP using the Bayes decision function.

The organization of the paper is as follows. In Section 2, the model for exponential distribution in the presence of competing risks data under ICS is discussed and the Bayes risk for the general decision function is derived. In Section 3, the explicit form of the decision function is derived when the decision is taken based on the MLE of the RF and also an approximate Bayes risk is derived using the asymptotic properties of the

MLE of the RF. In Section 4, the Bayes decision function is derived. In Section 5, the numerical study is performed. In Section 6, the conclusion is made along with direction of future work.

2. Model & Assumptions

Suppose n items are put on life test. $X_1, \dots, X_n$ be the IID lifetimes of n items. Let us consider that every item may fail due to any one of J causes of failure. Let X_{ij} be the latent lifetime of i^{th} item which fails due to j^{th} cause and therefore $X_i = \min\{X_{i1}, X_{i2}, \dots, X_{iJ}\}$, $i = 1, 2, \dots, n$. It is assumed that X_{ij} follows an exponential distribution with PDF

$$f_j(t \mid \nu_j) = \nu_j \exp(-\nu_j t), \quad \nu_j > 0.$$

In the work, $X_{i1}, \dots, X_{iJ}$ are assumed to be independent. Therefore X_i follows exponential distribution with PDF

$$f(t \mid \boldsymbol{\nu}) = \nu \exp(-\nu t),$$

CDF

$$F(t \mid \boldsymbol{\nu}) = 1 - \exp(-\nu t)$$

and RF

$$\bar{F}(t \mid \boldsymbol{\nu}) = \exp(-\nu t),$$

where $\nu = \nu_1 + \dots + \nu_J$ and $\boldsymbol{\nu} = (\nu_1, \nu_2, \dots, \nu_J)$, for $i = 1, \dots, n$. Now let C_i denote the cause of the failure of the i^{th} item. The sub-distribution of X_i and C_i is defined as

$$G(j, t \mid \boldsymbol{\nu}) = P(C = j, X_i \leq t) = \frac{\nu_j}{\nu} [1 - \exp(-\nu t)].$$

In the above scenario, $t > 0$, $i = 1, \dots, n$ and $j = 1, \dots, J$. Let us consider that the life test is conducted under ICS as described in Introduction. In ICS, we only observe for each cause, the number of failures at pre-specified inspection times $\tau_1, \dots, \tau_k$, where $0 < \tau_1 < \dots < \tau_k$. Suppose D_{mj} denotes number of failures in the interval $(\tau_{m-1}, \tau_m]$ due to the cause j , for $m = 1, \dots, k$ and $j = 1, \dots, J$. Let

$\mathbf{D} = (D_{11}, \dots, D_{1J}, D_{21}, \dots, D_{2J}, \dots, D_{m1}, \dots, D_{mJ})$. The observed data up to inspection time τ_k is $\mathbf{d} = (d_{11}, \dots, d_{1J}, d_{21}, \dots, d_{2J}, \dots, d_{m1}, \dots, d_{mJ})$, where d_{mj} is the observed value of D_{mj} . for $m = 1, \dots, k$ and $j = 1, \dots, J$. Therefore, $d_t = \sum_{m=1}^k \sum_{j=1}^J d_{mj}$ is the total number of failures and $n - d_t$ is the survived items after the life test. Note that $\mathbf{D}$ have multinomial distribution with joint distribution

$$\begin{aligned} P_{\mathbf{D}}(\mathbf{d} \mid \boldsymbol{\nu}) &= \frac{n!}{\prod_{m=1}^k \prod_{j=1}^J d_{mj}! (n - d_t)!} \prod_{m=1}^k \prod_{j=1}^J [G(j, \tau_m \mid \boldsymbol{\nu}) - G(j, \tau_{m-1} \mid \boldsymbol{\nu})]^{d_{mj}} [\bar{F}(\tau_k \mid \boldsymbol{\nu})]^{n-d_t} \\ &= \frac{n!}{\prod_{m=1}^k \prod_{j=1}^J d_{mj}! (n - d_t)!} \prod_{m=1}^k \prod_{j=1}^J \left[\frac{\nu_j}{\nu} \{ \exp(-\nu \tau_{m-1}) - \exp(-\nu \tau_m) \} \right]^{d_{mj}} \\ &\quad \times [\exp(-\nu \tau_k)]^{n-d_t}. \end{aligned} \quad (1)$$

The vector of designing parameters for ICS is denoted by $\boldsymbol{\zeta} = (n, \tau_1, \dots, \tau_k, k)$ and based on the observed data $\mathbf{d}$, the decision function $a(\mathbf{d} \mid \boldsymbol{\zeta})$ is defined as

$$a(\mathbf{d} \mid \boldsymbol{\zeta}) = \begin{cases} 1 & \text{if accepting the lot when data } \mathbf{d} \text{ is observed} \\ 0 & \text{if rejecting the lot when data } \mathbf{d} \text{ is observed,} \end{cases} \quad (2)$$

However, whether the lot is accepted or rejected, there is a loss associated with the decision. If a bad lot is accepted, the manufacturer can bear an acceptance cost $h(\boldsymbol{\nu})$. We take the loss function as $h(\boldsymbol{\nu}) = C_0 + \sum_{j=1}^J C_j \nu_j + \sum_{i=1}^J \sum_{j=1}^J C_{ij} \nu_i \nu_j$, where the coefficient C_i and C_{jl} , for $i = 0, 1, \dots, J$, $j = 1, \dots, l$ and $l = 1, \dots, J$ are taken in such a way that $h(\boldsymbol{\nu}) \geq 0 \forall \boldsymbol{\nu} > \mathbf{0}$. This loss function is known as a quadratic loss function. If the lot is rejected, the lot is returned or discarded. So the cost due to rejection is fixed. Let C_r be the rejection cost. The loss function is considered as

$$\mathcal{L}(a(\mathbf{d} \mid \boldsymbol{\zeta}) \mid \boldsymbol{\nu}) = \begin{cases} h(\boldsymbol{\nu}) + nC_s + \tau C_\tau + MC_I - (n - d_t)r_s & \text{if } a(\mathbf{d} \mid \boldsymbol{\zeta}) \mid \boldsymbol{\nu} = 1 \\ C_r + nC_s + \tau C_\tau + MC_I - (n - d_t)r_s & \text{if } a(\mathbf{d} \mid \boldsymbol{\zeta}) \mid \boldsymbol{\nu} = 0, \end{cases} \quad (3)$$

where C_s is the cost of each item in the sample, C_τ is the cost per unit time, r_s ($< C_s$) is the salvage value of each item that survived after life test and C_I is the cost of each inspection, τ is the duration of life test, M is the inspection number and d_t is the total number of failures in the life testing experiment.

$\mathcal{L}(a(\mathbf{d} \mid \boldsymbol{\zeta}) \mid \boldsymbol{\nu})$ in (3) can be expressed as

$$\begin{aligned}\mathcal{L}(a(\mathbf{d} \mid \boldsymbol{\zeta}) \mid \boldsymbol{\nu}) &= [h(\boldsymbol{\nu}) + nC_s + \tau C_\tau + MC_I - (n - d_t)r_s] a(\mathbf{d} \mid \boldsymbol{\zeta}) \\ &\quad + [C_r + nC_s + \tau C_\tau + MC_I - (n - d_t)r_s] [1 - a(\mathbf{d} \mid \boldsymbol{\zeta})] \\ &= C_r + nC_s + \tau C_\tau + MC_I - (n - d_t)r_s + a(\mathbf{d} \mid \boldsymbol{\zeta}) [h(\boldsymbol{\nu}) - C_r].\end{aligned}$$

Now, the Bayes risk is denoted by $R(\boldsymbol{\zeta}, a)$ and defined by

$$\begin{aligned}R(\boldsymbol{\zeta}, a) &= E_{\boldsymbol{\nu}} E_{\mathbf{d} \mid \boldsymbol{\nu}} [\mathcal{L}(a(\mathbf{d} \mid \boldsymbol{\zeta}) \mid \boldsymbol{\nu})] \\ &= C_r + n(C_s - r_s) + C_\tau E_{\boldsymbol{\nu}}[\tau \mid \boldsymbol{\nu}] + C_I E_{\boldsymbol{\nu}}[M \mid \boldsymbol{\nu}] + r_s E_{\boldsymbol{\nu}}[D_t \mid \boldsymbol{\nu}] + R_1(\boldsymbol{\zeta}, a) \\ &= C_r + n(C_s - r_s) + C_\tau E[\tau] + C_I E[M] + r_s E[D_t] + R_1(\boldsymbol{\zeta}, a),\end{aligned}\tag{4}$$

where

$$\begin{aligned}R_1(\boldsymbol{\zeta}, a) &= E_{\boldsymbol{\nu}} E_{\mathbf{d} \mid \boldsymbol{\nu}} [a(\mathbf{d} \mid \boldsymbol{\zeta}) [h(\boldsymbol{\nu}) - C_r]] \\ &= \sum_{\mathbf{d} \in \mathcal{X}} \int_{\boldsymbol{\nu}} [h(\boldsymbol{\nu}) - C_r] P_{\mathbf{D}}(\mathbf{d} \mid \boldsymbol{\nu}) p(\boldsymbol{\nu}) d\boldsymbol{\nu}\end{aligned}\tag{5}$$

and $\mathcal{X}$ is the set of all possible $\mathbf{d}$ for which the lot is accepted after life testing. Now, we have $E[D_t] = E_{\boldsymbol{\nu}}[E[D_t \mid \boldsymbol{\nu}]] = E_{\boldsymbol{\nu}}[n(1 - \exp(-\nu\tau_k))]$, $E[\tau] = E_{\boldsymbol{\nu}}[\tau \mid \boldsymbol{\nu}] = \sum_{i=1}^k \tau_i P_i$, where $\mathcal{P}_i = P_i - P_{i-1}$ is the probability that the life-test terminates at τ_i with $P_i = [1 - \exp(-\nu\tau_i)]^n$, for $i = 1, \dots, k-1$ and $P_k = 1$ and $E[M \mid \boldsymbol{\nu}] = \sum_{i=1}^k i \mathcal{P}_i$.

Now, the optimal BSP $(\boldsymbol{\zeta}^*, a^*) = (n^*, \tau_1^*, \tau_2^*, \dots, \tau_k^*, k^*, a^*)$ is obtained by maximizing $R(\boldsymbol{\zeta}, a)$, i.e., $(\boldsymbol{\zeta}^*, a^*) = \arg \max_{(\boldsymbol{\zeta}, a)} R(\boldsymbol{\zeta}, a)$. Note that for equal length of interval h , the optimal BSP is $(\boldsymbol{\zeta}^*, a^*) = (n^*, h^*, k^*, a^*)$.

3. BSP based on reliability criteria decision function

Here we obtain BSP under the assumption that the manufacturer knows the consumer's acceptance criteria. The consumer takes his/her decision based on the reliability criteria. A lot is of acceptable quality when the product's reliability at a specified time point τ_0 , $\bar{F}(\tau_0 \mid \boldsymbol{\nu})$ is greater than R_0 . Let $\hat{\boldsymbol{\nu}}$ is the MLE of $\boldsymbol{\nu}$. Therefore, $\bar{F}(\tau_0 \mid \hat{\boldsymbol{\nu}})$ is the MLE of $R(\tau_0 \mid \boldsymbol{\nu})$. Now, the lot is accepted if $R(\tau_0 \mid \hat{\boldsymbol{\nu}}) > R_0$ and rejected

if $\bar{F}(\tau_0 \mid \hat{\nu}) \leq R_0$. Now, based on the competing risks data under ICS the likelihood function is given by

$$\begin{aligned}
L(\boldsymbol{\nu} \mid \mathbf{d}) &\propto \prod_{m=1}^k \prod_{j=1}^J [G(j, \tau_m \mid \boldsymbol{\nu}) - G(j, \tau_{m-1} \mid \boldsymbol{\nu})]^{d_{mj}} [\bar{F}(\tau_k \mid \boldsymbol{\nu})]^{n-d_t} \\
&\propto \prod_{m=1}^k \prod_{j=1}^J \left[\frac{\nu_j}{\nu} \{ \exp(-\nu \tau_{m-1}) - \exp(-\nu \tau_m) \} \right]^{d_{mj}} [\exp(-\nu \tau_k)]^{n-d_t} \\
&\propto \prod_{m=1}^k \prod_{j=1}^J [\exp(-\nu \tau_{m-1})]^{d_{mj}} \left[\frac{\nu_j}{\nu} \{ 1 - \exp(-\nu(\tau_m - \tau_{m-1})) \} \right]^{d_{mj}} [\exp(-\nu \tau_k)]^{n-d_t}. \tag{6}
\end{aligned}$$

The log-likelihood function is given by

$$\begin{aligned}
l(\boldsymbol{\nu} \mid \mathbf{d}) &= \sum_{m=1}^k \sum_{j=1}^J [-d_{mj} \{ \nu \tau_{m-1} + \log(\nu_j) - \log(\nu) + \log(1 - \exp(-\nu(\tau_m - \tau_{m-1}))) \}] \\
&\quad - (n - d_t) \nu \tau_k.
\end{aligned}$$

The likelihood equations are written as

$$\begin{aligned}
\frac{\partial l(\boldsymbol{\nu} \mid \mathbf{d})}{\partial \nu_j} &= 0 \\
\Rightarrow \sum_{m=1}^k \frac{d_{mj}}{\nu_j} - \sum_{m=1}^k d_{m+} \left[\tau_{m-1} + \frac{1}{\nu} - \frac{(\tau_m - \tau_{m-1}) \exp(-\nu(\tau_m - \tau_{m-1}))}{1 - \exp(-\nu(\tau_m - \tau_{m-1}))} \right] - (n - d_t) \tau_k &= 0, \\
\Rightarrow \frac{d_{+j}}{\nu_j} = \frac{d_t}{\nu} + g(\nu), \tag{7}
\end{aligned}$$

where $g(\nu) = \sum_{m=1}^k d_{m+} \left[\tau_{m-1} - \frac{(\tau_m - \tau_{m-1}) \exp(-\nu(\tau_m - \tau_{m-1}))}{1 - \exp(-\nu(\tau_m - \tau_{m-1}))} \right] + (n - d_t) \tau_k$, $d_{m+} = \sum_{j=1}^J d_{mj}$, $d_{+j} = \sum_{m=1}^k d_{mj}$ and $d_t = \sum_{m=1}^k d_{m+} = \sum_{j=1}^J d_{+j}$, for $j = 1, \dots, J$. Solving (7), for $j = 1, \dots, J$, we get the MLEs $\hat{\boldsymbol{\nu}} = (\hat{\nu}_1, \dots, \hat{\nu}_J)$. Now, from the likelihood equations, we get

$$\begin{aligned}
\frac{\hat{\nu}_1}{d_{+1}} &= \dots = \frac{\hat{\nu}_J}{d_{+J}} = \frac{\hat{\nu}}{d_t} \\
\Rightarrow \hat{\nu}_j &= \frac{d_{+j} \hat{\nu}}{d_t}, \quad j = 1, \dots, J, \tag{8}
\end{aligned}$$

where $\hat{\nu} = \hat{\nu}_1 + \dots + \hat{\nu}_J$. From (7) using (8), we get $g(\hat{\nu}) = 0$. It is easy to show that $\lim_{\nu \rightarrow 0} g(\nu) = -\infty$ and $\lim_{\nu \rightarrow \infty} g(\nu) = \sum_{m=1}^k d_{m+} \tau_{m-1} + (n - d_t) \tau_k > 0$. Also, we can see that $g(\nu)$ is a strictly increasing function

in ν . Hence, we get a unique solution of $\hat{\nu}$. The decision function can be expressed as

$$a(\mathbf{d} \mid \zeta) = \begin{cases} 1 & \exp(-\hat{\nu}\tau_0) > R_0 \\ 0 & \text{Otherwise,} \end{cases} \quad (9)$$

where $\hat{\nu}$ is the root of the equation $g(\nu) = 0$. Note that we only need to estimate ν for our decision function. It is seen that $\hat{\nu}$ exists when $d_t > 0$. When $d_t = 0$, $\hat{\nu}$ is taken as $1/n\tau_k$. If the inspection times are of equal length. i.e, $\tau_m = mh$, $m = 1, \dots, k$, where h is the length of each interval. In that case, the explicit form of the MLE of ν is given by

$$\hat{\nu} = \begin{cases} -\frac{1}{h} \ln \left(1 - \frac{d_t}{\sum_{m=1}^k m d_{m+} + (n - d_t)k} \right) & d_t > 0 \\ \frac{1}{nkh} & d_t = 0. \end{cases} \quad (10)$$

As a result, the decision function is expressed as if $d_t > 0$,

$$a(\mathbf{d} \mid \zeta) = \begin{cases} 1 & \text{if } \left(\frac{d_t}{\sum_{m=1}^k m d_{m+} + (n - d_t)k} \right)^{\tau_0/h} > R_0 \\ 0 & \text{otherwise} \end{cases}$$

and if $d_t = 0$,

$$a(\mathbf{0} \mid \zeta) = \begin{cases} 1 & \text{if } \exp\left(-\frac{\tau_0}{nkh}\right) > R_0 \\ 0 & \text{otherwise.} \end{cases}$$

In this case, the form of the decision function is known. Therefore, to find the optimal decision we need to determine the optimal value of R_0 . Let R_0^* be the optimal value of R_0 . In that case $a^* \equiv R_0^*$. Now, it is assumed that the failure rate of the items $\nu = \nu_1 + \dots + \nu_J$ follows gamma distribution with PDF

$$p(\nu) = \frac{\eta^\alpha}{\Gamma(\alpha)} \nu^{\alpha-1} \exp(-\eta\nu), \quad \nu > 0, \alpha > 0, \eta > 0$$

and it is denoted by $G(\alpha, \eta)$. It is assumed that for given ν , $(\nu_1/\nu, \dots, \nu_J/\nu)$ follows Dirichlet distribution with JPDF

$$p(\nu_1/\nu, \dots, \nu_J/\nu) = \frac{\Gamma(\alpha_0)}{\prod_{j=1}^J \Gamma(\alpha_j)} \prod_{j=1}^J \left(\frac{\nu_j}{\nu} \right)^{\alpha_j-1}, \quad 0 < \frac{\nu_j}{\nu} < 1, \alpha_j > 0, \text{ for } j = 1, \dots, J,$$

where ν_j/ν is the ratio of the failure rate that fails due to j^{th} cause and the failure rate of the item and $\alpha_0 = \sum_{j=1}^J \alpha_j$. Therefore, the prior JPDF of $\boldsymbol{\nu} = (\nu_1, \dots, \nu_J)$ is

$$p(\boldsymbol{\nu}) = \frac{\eta^\alpha}{\Gamma(\alpha)} \nu^{\alpha-\alpha_0} \exp(-\eta\nu) \prod_{j=1}^J \frac{\Gamma(\alpha_j)}{\Gamma(\alpha_j)} \nu_j^{\alpha_j-1}. \quad (11)$$

This is known as the Gamma-Dirichlet distribution and is denoted by $GD(\alpha, \eta, \alpha_1, \dots, \alpha_J)$. We provide the following results needed for computation of Bayes risk.

Result 1.

$$\int_{\nu_1=0}^{\infty} \dots \int_{\nu_J=0}^{\infty} \nu^{\alpha-\alpha_0} \exp(-\eta\nu) \prod_{j=1}^J \nu_j^{\alpha_j-1} d\nu_1 \dots d\nu_J = \frac{\Gamma(\alpha)}{\eta^\alpha \Gamma(\alpha_0)} \prod_{j=1}^J \Gamma(\alpha_j).$$

Result 2. If a sample of size n is put on the test and inspected at an equal length of the interval h up to the maximum number of inspections k , then the Bayes risk can be expressed as

$$R(\zeta, R_0) = C_r + nC_s + r_s n \left(\frac{\eta}{\eta + kh} \right)^\alpha + (C_I + C_\tau h) \left[k - \sum_{i=1}^{k-1} \sum_{j=1}^n \binom{n}{j} (-1)^j \left(\frac{\eta}{\eta + ijh} \right)^\alpha \right] + R_1(\zeta, R_0),$$

where

$$\begin{aligned} R_1(\zeta, R_0) = & \sum_{\mathbf{d} \in \mathcal{X}} \frac{n!}{d_{11}! d_{12}! \dots d_{m1}! d_{k2}! (n - d_t)!} \sum_{i=0}^{d_t} (-1)^i \binom{d_t}{i} \left[(C_0 - C_r) \frac{\eta^\alpha}{(\eta + ih + sh)^\alpha} \right. \\ & + \Gamma(\alpha_0) \left\{ \frac{\alpha \eta^\alpha}{(\eta + ih + sh)^{\alpha+1}} \sum_{p=1}^J C_p \prod_{j=1}^J \frac{\Gamma(\alpha_j + d_{+j} + \delta_{pj})}{\Gamma(\alpha_j) \Gamma(\alpha_0 + d_t + 1)} \right. \\ & \left. \left. + \frac{\alpha(\alpha+1)\eta^\alpha}{(\eta + ih + sh)^{\alpha+2}} \sum_{p=1}^J \sum_{q=1}^J C_{pq} \prod_{j=1}^J \frac{\Gamma(\alpha_j + d_{+j} + \delta_{pj} + \delta_{qj})}{\Gamma(\alpha_j) \Gamma(\alpha_0 + d_t + 2)} \right\} \right] \end{aligned}$$

$$\text{and } \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The expressions of $E[D_t]$, $E[M]$ and $E[\tau]$ are given in Chen et al. [11]. We only need to prove the term $R_1(\zeta, R_0)$. Using (1), when the inspection intervals are equal, the joint distribution of $\mathbf{D}$ can be expressed

$$\begin{aligned} P_{\mathbf{D}}(\mathbf{d} | \boldsymbol{\nu}) = & \frac{n!}{\prod_{m=1}^k \prod_{j=1}^J d_{mj}! (n - d_t)!} \prod_{m=1}^k \prod_{j=1}^J \left\{ \exp[-\nu d_{mj}(m-1)h] [1 - \exp(-\nu h)]^{d_{mj}} \left(\frac{\nu_j}{\nu} \right)^{d_{mj}} \right\} \\ & \times \exp[-\nu(n - d_t)kh] \end{aligned}$$

$$= \frac{n!}{\prod_{m=1}^k \prod_{j=1}^J d_{mj}!(n-d_t)!} \sum_{i=0}^{d_t} (-1)^i \binom{d_t}{i} \prod_{j=1}^J \left(\frac{\nu_j}{\nu}\right)^{d_{tj}} \exp[-(i+s)\nu_j h], \quad (12)$$

where $s = nk - \sum_{m=1}^k \sum_{j=1}^J (k - (m-1))d_{mj}$.

We know that $R_1(\zeta, a) = \sum_{\mathbf{d} \in \mathcal{X}} \int_{\nu} [h(\nu) - C_r] P_{\mathbf{D}}(\mathbf{d} | \nu) p(\nu) d\nu$. Now using Result 1, we get the required expression of $R_1(\zeta, a)$. (Proved)

It is noted that when length of inspection intervals are equal, the vector of decision variable can be expressed as $\zeta = (n, h, k)$. For different inspection intervals, the term $R_1(\zeta, a)$ cannot be obtained analytically, a simple Monte Carlo integration can be used to evaluate the integration. For this, a large number N_1 of observations are generated from the prior distributions $p(\nu)$. If $p(\nu)$ follows $GD(\alpha, \eta, \alpha_1, \dots, \alpha_J)$, then the data can be generated using Algorithm 1.

Algorithm 1:

- I. Generate $\nu^{(i)}$ from gamma distribution with parameters (α, η) .
- II. For given $\nu^{(i)}$, generate $\frac{\nu^{(i)}}{\nu^{(i)}}$ from Dirichlet distribution with parameters $(\alpha_1, \dots, \alpha_J)$ using the built-in R packages "LaplaceDemon".
- III. Multiplying $\nu^{(i)}$ and $\frac{\nu^{(i)}}{\nu^{(i)}}$, we get the required generated sample $\nu^{(i)}$.

Let $\nu^{(1)}, \dots, \nu^{(N_1)}$ be the generated observations. Then

$$R_1(\zeta, a) \approx \frac{1}{N_1} \sum_{i=1}^{N_1} \left[C_r - h(\nu^{(i)}) \right] \sum_{\mathbf{d} \in \mathcal{X}} P_{\mathbf{D}}(\mathbf{d} | \nu^{(i)}).$$

3.1. Approximate form of Bayes risk using asymptotic properties of the MLE

For large samples and different prior distributions, computing Bayes risk will be intensive and complex. Due to complexity, we approximate the Bayes risk in a simplified form, where we can calculate Bayes risk for any choice of prior. Under certain regularity conditions, for large n , $\hat{\nu}$ follows normal distribution with mean ν and variance-covariance matrix $I^{-1}(\nu)$, where $I^{-1}(\nu)$ is inverse of the FIM of ν . The regularity conditions are stated below:

Regularity Conditions (Das et al. [17]):

- I. $X_1, X_2, \dots, X_n$ are IID unobserved lifetimes with common PDF $f(\cdot | \nu)$ and CDF $F(\cdot | \nu)$.
- II. The number of causes of failure is finite.

III. $(X_1, C_1), (X_2, C_2), \dots, (X_n, C_n)$ are IID with common sub-distribution function $G(\cdot, \cdot \mid \boldsymbol{\nu})$.

IV. The supports of $f(\cdot \mid \boldsymbol{\nu})$ and $G(\cdot \mid \boldsymbol{\nu})$ is independent of $\boldsymbol{\nu}$.

V. In the parameter space $\boldsymbol{\nu}$, the true parameter $\boldsymbol{\nu}^0$ is an interior point of the open set $\boldsymbol{\nu}_0$.

VI. For almost all t and $\forall j = 1, \dots, J$, $F(t \mid \boldsymbol{\nu})$ and $G(j, t \mid \boldsymbol{\nu})$ admit all third order derivatives $\frac{\partial^3 F(t \mid \boldsymbol{\nu})}{\partial \nu_u \partial \nu_v \partial \nu_w}$ and $\frac{\partial^3 G(j, t \mid \boldsymbol{\nu})}{\partial \nu_u \partial \nu_v \partial \nu_w}$, respectively $\forall \boldsymbol{\nu} \in \boldsymbol{\nu}_0$ and $w, u, v = 1, \dots, J$. Also, upto third-order derivatives of $G(j, t \mid \boldsymbol{\nu})$ and $F(t \mid \boldsymbol{\nu})$ wrt $\boldsymbol{\nu}$ are bounded $\forall \boldsymbol{\nu} \in \boldsymbol{\nu}_0$.

VII. In such a way, the τ_m 's are chosen that

(a) $0 < q_{mj} < 1$ and $0 < q_m < 1$, where

$$q_{mj} = \frac{P(X_i \leq \tau_m, I = j) - P(X_i \leq \tau_{m-1}, I = j)}{P(X_i \geq \tau_{m-1})} = \frac{G(j, \tau_m \mid \boldsymbol{\nu}) - G(j, \tau_{m-1} \mid \boldsymbol{\nu})}{\bar{F}(\tau_{m-1} \mid \boldsymbol{\nu})} \quad (13)$$

is the probability of an item failed by the time τ_m due to cause j given that the item is at risk at time τ_{m-1} and

$$q_m = \frac{P(X_i \leq \tau_m) - P(X_i \leq \tau_{m-1})}{P(X_i \geq \tau_{m-1})} = \frac{F(\tau_m \mid \boldsymbol{\nu}) - F(\tau_{m-1} \mid \boldsymbol{\nu})}{\bar{F}(\tau_{m-1} \mid \boldsymbol{\nu})}. \quad (14)$$

is the probability of an item failed by the time τ_m given that the item is at risk at time τ_{m-1} , for $i = 1, \dots, n$, $j = 1, \dots, J$ and $m = 1, \dots, k$.

(b) $\nabla_{\boldsymbol{\nu}} \mathbf{q}$ is matrix of rank l , where $\nabla_{\boldsymbol{\nu}} \mathbf{q} = \left(\frac{\partial q_i}{\partial \nu_u} \right)_{s \times m}$ for $u = 1, \dots, J$ and $i = 1, \dots, m$.

The FIM is given by $I(\boldsymbol{\nu}) = (I_{u,v}(\boldsymbol{\nu}))$, where

$$I_{u,v}(\boldsymbol{\nu}) = E \left[- \frac{\partial^2 l(\boldsymbol{\nu})}{\partial \nu_u \partial \nu_v} \right] \quad (15)$$

For interval-censored data, the FIM for $\boldsymbol{\nu}$ can be expressed as

$$I(\boldsymbol{\nu}) = \sum_{m=1}^k \left[\sum_{j=1}^J \frac{n[R(\tau_{m-1} \mid \boldsymbol{\nu})]}{q_{mj}} \nabla_{\boldsymbol{\nu}}(q_{mj}) \nabla_{\boldsymbol{\nu}}(q_{mj})^T + \frac{n[R(\tau_{m-1} \mid \boldsymbol{\nu})]}{(1 - q_m)} \nabla_{\boldsymbol{\nu}}(q_m) \nabla_{\boldsymbol{\nu}}(q_m)^T \right]. \quad (16)$$

The derivation of the form of the FIM and proof of the normal approximation of the MLEs are given in Das et al. [17].

Now we find the approximate expression of $P(\exp(-\hat{\nu}\tau_0) > R_0)$. By using the delta method, we have $c(\hat{\nu}) \sim \mathcal{N}(c(\boldsymbol{\nu}), S(\boldsymbol{\nu})^2)$, where $c(\boldsymbol{\nu}) = \exp(-\nu\tau_0)$, $S(\boldsymbol{\nu})^2 = \nabla_{\boldsymbol{\nu}}c(\boldsymbol{\nu})^T I^{-1}(\boldsymbol{\nu}) \nabla_{\boldsymbol{\nu}}c(\boldsymbol{\nu})$ and $\nabla_{\boldsymbol{\nu}}c(\boldsymbol{\nu}) = \left(\frac{\partial c(\boldsymbol{\nu})}{\partial \nu_1}, \dots, \frac{\partial c(\boldsymbol{\nu})}{\partial \nu_J}\right)$. Therefore,

$$P(\exp(-\hat{\nu}\tau_0) > R_0) = P\left(\frac{c(\hat{\nu}) - c(\boldsymbol{\nu})}{S(\boldsymbol{\nu})} > \frac{R_0 - c(\boldsymbol{\nu})}{S(\boldsymbol{\nu})}\right) \approx 1 - \Phi\left(\frac{R_0 - c(\boldsymbol{\nu})}{S(\boldsymbol{\nu})}\right) = \Phi\left(\frac{c(\boldsymbol{\nu}) - R_0}{S(\boldsymbol{\nu})}\right),$$

where $\Phi(\cdot)$ is the CDF of standard normal distribution.

For computation of $S(\boldsymbol{\nu})$, we need to compute $\nabla_{\boldsymbol{\nu}}(q_{mj})$ and $\nabla_{\boldsymbol{\nu}}(q_m)$ to obtain $I(\boldsymbol{\nu})$. For exponential distribution, q_{mj} given in (13) can be written as $q_{mj} = \frac{\nu_j}{\nu} [1 - \exp\{-\nu(\tau_m - \tau_{m-1})\}]$ and q_m given in (14) can be written as $q_i = 1 - \exp[-\nu(\tau_m - \tau_{m-1})]$, for $j = 1, \dots, J$ and $m = 1, \dots, k$. Therefore,

$$\frac{\partial q_{mj}}{\partial \nu_l} = \begin{cases} \frac{\nu_j(\tau_m - \tau_{m-1})(1 - q_m)}{\nu} - \frac{q_m \nu_j}{\nu^2}, & l \neq j \\ \frac{\nu_j(\tau_m - \tau_{m-1})(1 - q_m)}{\nu} + \frac{q_m(\nu - \nu_j)}{\nu^2}, & l = j, \end{cases}$$

$$\frac{\partial q_m}{\partial \nu_j} = (\tau_m - \tau_{m-1}) \exp[-\nu(\tau_m - \tau_{m-1})]$$

and

$$\frac{\partial c(\boldsymbol{\nu})}{\partial \nu_j} = \tau_0 \exp[-\nu\tau_0],$$

for $m = 1, \dots, k$ and $l, j = 1, \dots, J$.

3.2. Finding optimal BSP

Now, we provide upper bounds of n , k and last inspection time point τ_k . This shows that we get finite value of ζ^* . Since all cost-coefficients are non-negative and $C_s \geq r_s \geq 0$, this implies that $R(\zeta^*, a)$ is non-negative. Therefore, the following equalities hold:

$$\begin{aligned} R(\zeta^*, a) &\geq n^*(C_s - r_s) + C_\tau E[\tau] + C_I E[I] \\ &\geq n^*(C_s - r_s) + C_\tau \tau_k + C_I E[I] \end{aligned} \tag{17}$$

$$\geq n^*(C_s - r_s) + C_I k \tag{18}$$

$$\geq n^*(C_s - r_s) \tag{19}$$

When the decision of acceptance or rejection about the lot is taken without life testing, that scenario is called no sampling. For no sampling case, when the lot is accepted, $R(\mathbf{0}, 1) = E_\nu[h(\nu)]$ and when the lot is rejected, $R(\mathbf{0}, 0) = C_r$. Therefore, $R(\mathbf{0}, a) = \min\{E_\nu[h(\nu)], C_r\}$. Now,

$$R(\zeta^*, a) \leq \min\{E_\nu[h(\nu)], C_r\}. \quad (20)$$

From (19) and (20), we get

$$n^* \leq \frac{\min\{E_\nu[h(\nu)], C_r\}}{(C_s - r_s)}.$$

Therefore, n_0 can be taken as

$$n_0 = \left\lfloor \frac{\min\{E_\nu[h(\nu)], C_r\}}{C_s - r_s} \right\rfloor,$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x .

From (18) and (20), we get

$$k^*(n) = \frac{\min\{E_\nu[h(\nu)], C_r\} - (C_s - r_s)n}{C_I}.$$

Therefore, $k_0(n)$ can be taken as

$$k_0(n) = \left\lfloor \frac{\min\{E_\nu[h(\nu)], C_r\} - (C_s - r_s)n}{C_I} \right\rfloor.$$

From (17) and (20), we get $\tau_k^*(n, k(n)) = \lfloor (\min\{E_\nu[h(\nu)], C_r\} - (C_s - r_s)n - C_I k) / C_\tau \rfloor$. Therefore, $\tau_k(n, k(n))$ can be taken as

$$\tau_k(n, k(n)) = \left\lfloor \frac{\min\{E_\nu[h(\nu)], C_r\} - (C_s - r_s)n - C_I k}{C_\tau} \right\rfloor$$

We consider the following algorithm to determine the optimal BSP.

Algorithm 2:

1. For each pair of values of $(n, k(n))$, (4) is minimized wrt $(\boldsymbol{\tau}, R_0)$, where $\boldsymbol{\tau} = (\tau_1, \dots, \tau_k)$. Since $\tau_1 < \tau_2 < \dots < \tau_k < \tau_0(n, k(n))$, we consider a vector $\mathbf{h} = (h_1, \dots, h_k)$ such that $\tau_1 = h_1$, $\tau_i = \tau_{i-1} + h_i$, for $i = 2, \dots, k$ and $\sum_{i=1}^k h_i < \tau_0(n, k(n))$. For equal length of intervals, we take $h_i = h$, for $i = 1, \dots, k$ and $h < \tau_0(n, k(n))/k(n)$. Now, minimize (4) wrt $\mathbf{h}$, where $h_i > 0$, for $i = 1, \dots, k$, $R_0 > 0$ and $\sum_{i=1}^k h_i < \tau_0(n, k(n))$. The optimal $\boldsymbol{\tau}$ is given by $\boldsymbol{\tau}^*(n, k(n)) = (\tau_1^*(n, k(n)), \tau_2^*(n, k(n)), \dots, \tau_k^*(n, k(n)))$, where for $i = 1, 2, \dots, k$, $\tau_i^*(n, k) = \tau_{i-1}^*(n, k) + h_i^*(n, k)$. Let $h_i^*(n, k(n))$ denotes optimal value of $h_i(n, k(n))$,

$i = 1, \dots, k$ and $\tau_0^*(n, k) = 0$. Then we have

$$R(n, \tau^*(n, k(n)), k, R_0^*(n, k(n))) = \min_{(\tau, R_0), \tau_1 < \dots, \tau_k < \tau_0(n, k(n))} R(n, \tau, k, R_0).$$

2. By minimizing (4) wrt k , we find an integer $k^*(n)$, $0 \leq k^*(n) \leq k_0(n)$, for each value of n . i.e.,

$$R(n, \tau(n, k^*(n)), k^*(n), R_0^*(n, k^*(n))) = \min_{0 \leq k \leq k(n)} R(n, \tau^*(n, k(n)), k(n), R_0^*(n, k(n))).$$

3. Finally, by minimizing (4), we find n^* , $0 \leq n^* \leq n_0$. i.e.,

$$R(n^*, \tau^*(n^*, k^*(n^*)), k^*(n^*), R_0^*(n^*, k^*(n^*))) = \min_{0 \leq n \leq n_0} R(n, \tau^*(n, k^*(n)), k^*(n), R_0^*(n, k^*(n))).$$

Now, we write $R_0^*(n^*, k^*(n^*)) = R_0^*$, $\tau^*(n^*, k^*(n^*)) = \tau^*$ and $k^*(n^*) = k^*$. Therefore, (ζ^*, R_0^*) is the optimal BSP when the form of the decision function is known.

4. BSP based on Bayes decision function

Here, we consider that the decision criteria is unknown to the manufacturer. Therefore, the decision function is arbitrary. It is also assumed that the manufacturer and consumer use a single prior assessment. An optimal decision function is derived that minimizes the Bayes risk among all the decision functions. This optimal function is known as a Bayes decision function. In (4), it is seen that the decision function $a(\cdot | \zeta)$ is involved only in the term $R_1(\zeta, a)$ of the Bayes risk $R(\zeta, a)$. Therefore, to find the Bayes decision function, $R_1(\zeta, a)$ is minimized wrt a for fixed ζ . Note that $R_1(\zeta, a)$ in (5) can be expressed as

$$\begin{aligned} R_1(\zeta, a) &= E_{\nu} E_{d | \nu} [a(d | \zeta)(h(\nu) - C_r)] \\ &= E_d E_{\nu | d} [a(d | \zeta)(h(\nu) - C_r)] \\ &= \sum_{d \in \mathcal{X}} [a(d | \zeta)] E_{\nu | d} [h(\nu) - C_r] P_D(d) \\ &= \sum_{d \in \mathcal{X}} [a(d | \zeta)] \left\{ \int_{\nu} h(\nu) p_{\nu | D}(\nu | d) d\nu \right\} P_D(d) \\ &= \sum_{d \in \mathcal{X}} [a(d | \zeta)] [\varphi(d) - C_r] P_D(d), \end{aligned}$$

where $p_{\nu | D}(\nu | \mathbf{d})$ is the posterior JPDP of ν and it is given by

$$p_{\nu | D}(\nu | \mathbf{d}) = \frac{L(\nu | \mathbf{d}) p(\nu)}{\int_{\nu} L(\nu | \mathbf{d}) p(\nu) d\nu},$$

$$\varphi(\mathbf{d}) = \int_{\nu} h(\nu) p_{\nu | D}(\nu | \mathbf{d}) d\nu \text{ and } P_D(\mathbf{d}) = \int_{\nu} P_D(\mathbf{d} | \nu) p(\nu) d\nu.$$

Now we consider two cases:

Case 1: $\varphi(\mathbf{d}) \leq C_r$:

If $a(\mathbf{d} | \zeta) = 1$, then $E_{\mathbf{d}} E_{\nu} |_{\mathbf{d}} [a(\mathbf{d} | \zeta)(h(\nu) - C_r)] \leq 0$ and if $a(\mathbf{d} | \zeta) = 0$, then $E_{\mathbf{d}} E_{\nu} |_{\mathbf{d}} [a(\mathbf{d} | \zeta)(h(\nu) - C_r)] = 0$

Case 2: $\varphi(\mathbf{d}) > C_r$:

If $a(\mathbf{d} | \zeta) = 1$, then $E_{\mathbf{d}} E_{\nu} |_{\mathbf{d}} [a(\mathbf{d} | \zeta)(h(\nu) - C_r)] \geq 0$ and if $a(\mathbf{d} | \zeta) = 0$, then $E_{\mathbf{d}} E_{\nu} |_{\mathbf{d}} [a(\mathbf{d} | \zeta)(h(\nu) - C_r)] = 0$

Therefore, for each fixed ζ , if we take $\varphi(\mathbf{d}) - C_r \leq 0$ when $a(\mathbf{d} | \zeta) = 1$ and $\varphi(\mathbf{d}) - C_r \geq 0$ when $a(\mathbf{d} | \zeta) = 0$, then $E_{\nu} E_{\mathbf{d}} |_{\nu} [a(\mathbf{d} | \zeta)(h(\nu) - C_r)]$ is minimized wrt $a(\cdot | \zeta)$. Therefore, for fixed ζ , the Bayes decision function $a^*(\mathbf{d} | \zeta)$ is given by,

$$a^*(\mathbf{d} | \zeta) = \begin{cases} 1 & \text{if } C_r - \varphi(\mathbf{d}) \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

For no sampling case, the designing parameter ζ and observed data $\mathbf{d}$ are taken as $\mathbf{0}$. The prior expectation of $h(\nu)$ is

$$\begin{aligned} E[h(\nu)] &= C_0 + \sum_{p=1}^J C_p \frac{\alpha(\alpha_p + 1)}{\eta \alpha_0} + \frac{\alpha(\alpha + 1)}{\eta^2 \alpha_0(\alpha_0 + 1)} \left[\sum_{\substack{p=1 \\ p < q}}^J \sum_{q=1}^J C_{pq} (\alpha_p + 1)(\alpha_q + 1) + \sum_{p=1}^J C_{pp} (\alpha_p + 2) \right] \\ &= \varphi(\mathbf{0}). \end{aligned}$$

Therefore, the Bayes decision for the no sampling case can be expressed as

$$a^*(\mathbf{0} | \mathbf{0}) = \begin{cases} 1 & \text{if } C_r - \varphi(\mathbf{0}) \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Result 3. If a sample of size n is put on the life test and the items are inspected at an equal length of the

interval h , then the posterior expectation of the loss function $h(\boldsymbol{\nu})$ can be expressed as

$$\begin{aligned} \phi(\mathbf{d}) = & C_0 + N_C \sum_{p=1}^J C_p \left[\sum_{i=0}^{d_t} (-1)^i \binom{d_t}{i} \frac{\Gamma(\alpha+1)}{[(i+s)h+\eta]^{\alpha+1}} \prod_{j=1}^J \frac{\Gamma(\alpha_j + d_{+j} + \delta_{pj})}{\Gamma(\alpha_0 + d_t + 1)} \right] \\ & + N_C \sum_{p=1}^J \sum_{\substack{q=1 \\ p \leq q}}^J C_{pq} \left[\sum_{i=0}^{d_t} (-1)^i \binom{d_t}{i} \frac{\Gamma(\alpha+2)}{[(i+s)h+\eta]^{\alpha+2}} \prod_{j=1}^J \frac{\Gamma(\alpha_j + d_{+j} + \delta_{pj} + \delta_{qj})}{\Gamma(\alpha_0 + d_t + 2)} \right], \end{aligned}$$

where

$$N_C = \left[\sum_{i=0}^{d_t} (-1)^i \binom{d_t}{i} \frac{\Gamma(\alpha)}{[(i+s)h+\eta]^\alpha} \prod_{j=1}^J \frac{\Gamma(\alpha_j + d_{+j})}{\Gamma(\alpha_0 + d_t)} \right]^{-1}.$$

Proof: Using (6), when the inspection intervals are equal, the likelihood function can be expressed as

$$\begin{aligned} L(\boldsymbol{\nu} \mid \mathbf{d}) & \propto \prod_{m=1}^k \prod_{j=1}^J \left\{ \exp[-\nu d_{mj}(m-1)h] [1 - \exp(-\nu h)]^{d_{mj}} \left(\frac{\nu_j}{\nu}\right)^{d_{mj}} \right\} \exp[-\nu(n-d_t)kh] \\ & \propto \sum_{i=0}^{d_t} (-1)^i \binom{d_t}{i} \prod_{j=1}^J \left(\frac{\nu_j}{\nu}\right)^{d_{+j}} \exp[-(i+s)\nu_j h], \end{aligned} \quad (21)$$

where $s = nk - \sum_{m=1}^k \sum_{j=1}^J (k - (m-1))d_{mj}$. The prior of $\boldsymbol{\nu}$ follows Gamma-Dirichlet distribution with JPDPF

$$p(\boldsymbol{\nu}) = \frac{\eta^\alpha}{\Gamma(\alpha)} \nu^{\alpha-\alpha_0} \exp(-\eta\nu) \prod_{j=1}^J \frac{\Gamma(\alpha_0)}{\Gamma(\alpha_j)} \nu_j^{\alpha_j-1}.$$

The posterior JPDPF of $\boldsymbol{\nu}$ is

$$p(\boldsymbol{\nu} \mid \mathbf{d}) = N_C \sum_{i=0}^{d_t} (-1)^i \binom{d_t}{i} \nu^{\alpha-(\alpha_0+d_t)} \prod_{j=1}^J \nu_j^{\alpha_j+d_{+j}-1} \exp[-(\eta + (i+s)h)\nu_j],$$

where N_C is the normalizing constant of the posterior distribution, which is given by

$$N_C = \left[\sum_{i=0}^{d_t} (-1)^i \binom{d_t}{i} \frac{\Gamma(\alpha)}{[(i+s)h+\eta]^\alpha} \prod_{j=1}^J \frac{\Gamma(\alpha_j + d_{+j})}{\Gamma(\alpha_0 + d_t)} \right]^{-1}.$$

Using Result 1, we get the desired result. (Proved)

For the sampling case, when the length of inspection intervals are unequal, the posterior expectation of $h(\boldsymbol{\nu})$ cannot be obtained analytically. A simple Monte Carlo integration can be used to evaluate it. For this, a large number N_2 of observations are generated from the prior distributions $p(\boldsymbol{\nu})$. Let $\boldsymbol{\nu}^{(1)}, \dots, \boldsymbol{\nu}^{(N_2)}$ be the

generated observations. Then

$$\varphi(\mathbf{d}) \approx \frac{\sum_{j=1}^{N_2} h(\boldsymbol{\nu}^{(j)}) L(\boldsymbol{\nu}^{(j)} | \mathbf{d})}{\sum_{j=1}^{N_2} L(\boldsymbol{\nu}^{(j)} | \mathbf{d})} \quad (22)$$

After finding the Bayes decision function, we obtain Bayes risk and the procedure of finding Bayes risk is similar to Section 4. Next, we provide an algorithm for finding optimal BSP under this scenario.

Algorithm 3

1. Using the derivation of the Bayes decision function discussed in Section 4, we find the Bayes decision function $a^*(\cdot | \boldsymbol{\zeta})$ that minimized $R(\boldsymbol{\zeta})$ among all class of decision functions $a(\cdot | \boldsymbol{\zeta})$ for each $\boldsymbol{\zeta} = (n, \boldsymbol{\tau}, k)$.
2. For each pair of values of $(n, k(n))$, (4) is minimized wrt $\boldsymbol{\tau}$. The procedure is similar to step 2 of Algorithm 2. that is,

$$R(n, \boldsymbol{\tau}^*(n, k(n)), k, a^*(\cdot | (n, \boldsymbol{\tau}^*(n, k), k))) = \min_{\boldsymbol{\tau}: \tau_1 < \dots, \tau_k} R(n, \boldsymbol{\tau}, k, a^*(\cdot | (n, \boldsymbol{\tau}, k))).$$

3. By minimizing (4) wrt k , we find an integer $k^*(n)$, $0 \leq k^*(n) \leq k_0(n)$, for each value of n , that is,

$$\begin{aligned} & R(n, \boldsymbol{\tau}(n, k^*(n)), k^*(n), a^*(\cdot | (n, \boldsymbol{\tau}(n, k^*(n)), k^*(n)))) \\ &= \min_{0 \leq m \leq n} R(n, \boldsymbol{\tau}^*(n, k(n)), k(n), a^*(\cdot | (n, \boldsymbol{\tau}^*(n, k(n)), k(n)))). \end{aligned}$$

4. Finally, by minimizing (4), we find n^* , $0 \leq n^* \leq n_0$, that is,

$$\begin{aligned} & R(n^*, \boldsymbol{\tau}^*(n^*, k^*(n^*)), k^*(n^*), a^*(\cdot | (n^*, \boldsymbol{\tau}^*(n^*, k^*(n^*)), k^*(n^*)))) \\ &= \min_{0 \leq n \leq n_0} R(n, \boldsymbol{\tau}^*(n, k^*(n)), k^*(n), a^*(\cdot | (n, \boldsymbol{\tau}^*(n, k^*(n)), k^*(n)))). \end{aligned}$$

Now, we write $a^*(\cdot | (n^*, \boldsymbol{\tau}^*(n^*, k^*(n^*)), k^*(n^*))) = a^*$, $\boldsymbol{\tau}^*(n^*, k^*(n^*)) = \boldsymbol{\tau}^*$ and $k^*(n^*) = k^*$. Therefore, $(\boldsymbol{\zeta}^*, a^*) = (n^*, \boldsymbol{\tau}^*, k^*, a^*)$ is the optimal BSP when the decision function is arbitrary.

Theorem 1. *The BSP $(\boldsymbol{\zeta}^*, a^*)$ is the optimal BSP among all BSPs.*

Proof. It is enough to prove that for any BSP $\boldsymbol{\zeta}$, the following inequality holds:

$$R(\boldsymbol{\zeta}, a) \geq R(\boldsymbol{\zeta}^*, a^*).$$

Now,

$$\begin{aligned} R(\zeta, a) - R(\zeta^*, a^*) = & [R(n, \tau, m, a) - R(n, \tau, m, a^*)] + [R(n, \tau, m, a^*) - R(n, \tau^*, m, a^*)] \\ & + [R(n, \tau^*, m, a^*) - R(n, \tau^*, k^*, a^*)] + [R(n, \tau^*, k^*, a^*) - R(n^*, \tau^*, k^*, a^*)]. \end{aligned}$$

From Algorithm 3, we get $[R(n, \tau, m, a) - R(n, \tau, m, a^*)] \geq 0$, $[R(n, \tau, m, a^*) - R(n, \tau^*, m, a^*)] \geq 0$, $[R(n, \tau^*, m, a^*) - R(n, \tau^*, k^*, a^*)] \geq 0$ and $[R(n, \tau^*, k^*, a^*) - R(n^*, \tau^*, k^*, a^*)] \geq 0$. Therefore, $[R(\zeta, a) - R(\zeta^*, a^*)] \geq 0$. Hence, the proof is complete. (Proved)

5. Numerical Example

Here we provide some examples of determining optimum BSPs. We consider two causes of failure, that is, $J = 2$.

Example 1: We consider the hyperparameter values of prior distribution as $\alpha = 2.8$, $\eta = 1$, $\alpha_1 = 1.5$ and $\alpha_2 = 1.8$. The acceptance cost coefficient are taken as $C_0 = 2$, $C_1 = C_2 = 4$ and $C_{11} = C_{12} = C_{22} = 4$. The other cost coefficients are taken as $C_r = 40$, $C_s = 0.5$, $r_s = 0.25$, $C_\tau = 0.3$ and $C_I = 0.1$. $\tau_0 = 0.1$. Here we denote the BSP as type-I when the decision function is arbitrary and it as type-II when the form of the decision function is known. The optimal sampling schemes corresponding to type I and II are provided in Table 2. Also, in Table 2, we provide the probability of accepting the lot for the optimal BSP, which is denoted by $P(A)$ and is given by

$$P(A) = \sum_{d \in \mathcal{X}} \int_{\nu} P_D(d | \nu) p(\nu) d\nu.$$

Bayes risk, $E[\tau^*]$, $E[M^*]$ and $E[D_t^*]$ for the optimal BSP are provided in Table 2.

Table 2: Optimal BSP

Type	$\zeta^* = (n^*, h^*, k^*)$	R_0^*	$P(A)$	$E[D_t^*]$	$E[\tau^*]$	$E[I^*]$	Bayes Risk
I	(4, 0.30, 3)	-	0.489	3.337	0.740	2.467	33.90826
II	(4, 0.30, 3)	0.76	0.489	3.337	0.740	2.467	33.90826

5.1. Effect of the parameters

Here, we study the effect of parameters on designing variables of optimal BSP. The effect of the parameters C_b , C_I , C_τ , η and α are tabulated in Tables 3, 4, 5, 6 and 7, respectively.

In Table 3, it is seen that the optimal solution is $(0, 0, 0)$ when $C_b = 20$ and $C_b = 90$. This is the no sampling

Table 3: Optimal BSPs for different values of C_b

C_b	Type	$\zeta^* = (n^*, h^*, k^*)$	R_0^*	$P(A)$	$E[D_t^*]$	$E[\tau^*]$	$E[I^*]$	Bayes Risk
20	I	(0, 0, 0)	-	0	0	0	0	20
	II	(0, 0, 0)	1	0	0	0	0	20
30	I	(3, 0.41, 3)	-	0.473	2.682	0.855	2.085	28.07
	II	(3, 0.41, 3)	0.80	0.333	2.682	0.855	2.085	28.07
50	I	(6, 0.27, 2)	-	0.528	4.209	0.525	1.944	38.21502
	II	(6, 0.27, 2)	0.72	0.620	4.209	0.525	1.944	38.21517
60	I	(7, 0.21, 2)	-	0.529	4.378	0.416	1.982	41.44479
	II	(7, 0.22, 2)	0.68	0.714	4.478	0.435	1.978	41.44624
70	I	(3, 0.19, 2)	-	0.726	2.98	0.38	1.96	44.69447
	II	(3, 0.19, 2)	0.52	0.726	2.98	0.38	1.96	44.69447
90	I	(0, 0, 0)	-	1				47.6619
	II	(0,0,0)	0	1	0	0	0	47.6619

case. For type-I BSP, when $C_b = 20$, the Bayes risk is equal to C_r , this means that the lot is rejected without life testing. When $C_b = 90$, the Bayes risk is equal to $\varphi(0, 0, 0)$, which means that the lot is accepted without life testing. In Table 3, it is seen that the sample size increases up to a certain value and after that it decreases, when C_b increases. This is due to the fact that the distance between C_b and prior expectation $\varphi(\mathbf{0})$ is decreasing with C_b on $(-\infty, 47.66)$ and increasing with C_b on $(47.66, \infty)$. Also, it is seen that the probability of acceptance and Bayes risk increases with C_b as expected.

Table 4: Optimal BSPs for different values of C_I

C_I	Type	$\zeta^* = (n^*, h^*, m^*)$	R_0^*	$P(A)$	$E[D_t^*]$	$E[\tau^*]$	$E[I^*]$	Bayes Risk
0	I	(5, 0.15, 6)	-	0.510	4.171	0.732	4.878	33.57594
	II	(5, 0.16, 5)	0.76	0.515	4.036	0.680	4.252	33.63195
0.2	I	(5, 0.47, 1)	-	0.50	3.30	0.47	1.00	34.09155
	II	(5, 0.47, 1)	0.72	0.50	3.30	0.47	1.00	34.09155
0.3	I	(5, 0.47, 1)	-	0.50	3.30	0.47	1.00	34.19155
	II	(5, 0.47, 1)	0.72	0.50	3.30	0.47	1.00	34.19155
1	I	(5, 0.47, 1)	-	0.50	3.30	0.47	1.00	34.89155
	II	(5, 0.47, 1)	0.72	0.50	3.30	0.47	1.00	34.89155

Table 5: Optimal BSPs for different values of C_τ

C_τ	Type	$\zeta^* = (n^*, h^*, m^*)$	R_0^*	$P(A)$	$E[D_t^*]$	$E[\tau^*]$	$E[I^*]$	Bayes Risk
0	I	(4, 0.36, 3)	-	0.486	3.485	0.838	2.329	33.66994
	II	(4, 0.36, 3)	0.74	0.486	3.485	0.838	2.329	33.66994
0.1	I	(4, 0.36, 3)	-	0.486	3.485	0.838	2.329	33.75377
	II	(4, 0.36, 3)	0.74	0.486	3.485	0.838	2.329	33.75377
0.5	I	(5, 0.29, 2)	-	0.502	3.611	0.553	1.908	34.04424
	II	(5, 0.29, 2)	0.76	0.502	3.611	0.553	1.908	34.04424
1	I	(5, 0.25, 2)	-	0.500	3.393	0.484	1.936	34.31483
	II	(5, 0.25, 2)	0.76	0.500	3.393	0.484	1.936	34.31483

Table 6: Optimal BSPs for different values of η

η	Type	$\zeta^* = (n^*, h^*, m^*)$	R_0^*	$P(A)$	$E[D_t^*]$	$E[\tau^*]$	$E[I^*]$	Bayes Risk
0.5	I	(0, 0, 0)	-	0	0	0	0	40
	II	(0, 0, 0)	1	0	0	0	0	40
0.75	II	(3, 0.37, 3)	-	0.261	2.764	0.716	1.934	37.74172
	II	(3, 0.37, 3)	0.79	0.261	2.764	0.716	1.934	37.74172
1.25	I	(5, 0.38, 1)	-	0.703	2.622	0.380	1.000	29.46731
	II	(5, 0.38, 1)	0.66	0.703	2.622	0.380	1.000	29.46731
1.5	I	(0, 0, 0)	-	1	0	0	0	24.78307
	II	(0, 0, 0)	0	1	0	0	0	24.78307

Table 7: Optimal BSPs for different values of α

α	Type	$\zeta^* = (n^*, h^*, m^*)$	R_0^*	$P(A)$	$E[D_t^*]$	$E[\tau^*]$	$E[I^*]$	Bayes Risk
2.2	I	(5, 0.25, 2)	-	0.690	2.951	0.490	1.961	28.28359
	II	(5, 0.40, 1)	0.670	0.691	2.615	0.400	1.000	28.288779
2.5	I	(5, 0.27, 3)	-	0.599	3.866	0.727	2.691	31.31123
	II	(5, 0.43, 1)	0.690	0.601	2.955	0.430	1.000	32.25544
3	I	(4, 0.31, 3)	-	0.434	3.444	0.741	2.390	35.38329
	II	(5, 0.30, 2)	0.77	0.446	3.779	0.566	1.887	35.45693
3.3	I	(4, 0.40, 3)	-	0.343	3.703	0.836	2.090	37.22465
	II	(5, 0.32, 2)	0.78	0.360	4.023	0.590	1.844	37.36195

In Table 5, it is observed that the expected duration of life test decreases when C_τ increases as expected. In Table 7, we find that for type-II BSP, the sample size n remains unchanged for the different values of α . However, the time interval h and inspection number change with α . In Table 6, a similar fact is seen as in Table 3. Tables 1-6 show that in most cases, optimum BSPs do not change with the types of BSP.

5.2. Illustrative example using approximate Bayes risk

Example 2: We consider the hyperparameter values of prior distribution as $\alpha = 2.8$, $\eta = 1$, $\alpha_1 = 1.5$ and $\alpha_2 = 1.8$. The acceptance cost coefficient are taken as $C_0 = 2$, $C_1 = C_2 = 4$ and $C_{11} = C_{12} = C_{22} = 4$. The other cost coefficients are taken as $C_r = 40$, $C_s = 0.15$, $r_s = 0.1$, $C_\tau = 0.3$, and $C_I = 0$ and $\tau_0 = 0.1$. The optimal BSP is provided in Table 8. For different values of C_I and C_s , the optimal BSPs using approximate

Table 8: Optimal BSP using approximate Bayes Risk function

$\zeta^* = (n^*, h^*, k^*)$	R_0^*	$P(A)$	$E[D_t^*]$	$E[\tau^*]$	$E[I^*]$	Bayes Risk
(13, 0.102, 5)	0.761	0.489	3.337	0.740	2.467	31.4662

Bayes Risk function are tabulated in Table 9. It is seen that when C_I is fixed, the sample size decreases with C_s and when C_s is fixed the sample size increases with C_I . Also, the acceptance limit R_0^* increases with C_s .

Table 9: Optimal BSP for different value of C_s and C_I using approximate Bayes Risk function

C_s	C_I	$\zeta^* = (n^*, h^*, k^*)$	R_0^*	$P(A)$	$E[D_t^*]$	$E[\tau^*]$	$E[I^*]$	Bayes Risk
0.12	0.01	(17, 0.100, 4)	0.759	0.576	10.373	0.399	3.990	31.0691087
0.12	0.05	(19, 0.159, 2)	0.759	0.575	10.230	0.318	2.000	31.1659256
0.12	0.1	(19, 0.159, 2)	0.759	0.575	10.230	0.318	2.000	31.26591
0.15	0.01	(13, 0.103, 5)	0.761	0.573	8.938	0.507	4.924	31.5154582
0.15	0.05	(14, 0.149, 3)	0.761	0.572	9.025	0.445	2.986	31.6515661
0.15	0.1	(15, 0.194, 2)	0.761	0.570	9.010	0.388	1.999	31.7571885
0.18	0.01	(11, 0.117, 5)	0.763	0.569	7.971	0.568	4.850	31.8755930
0.18	0.05	(12, 0.166, 3)	0.763	0.568	8.130	0.493	2.969	32.0293011
0.18	0.1	(12, 0.216, 2)	0.763	0.567	7.609	0.431	1.995	32.1584409

5.3. Illustration of the decision function after life testing

In all cases, after determining the value of ζ , the manufacturer conducts the life testing to take a decision about the lot. In Example 1, the optimal plan is (4, 0.30, 3) for the equal inspection intervals. Therefore, in that case, the manufacturer conducts a life test with a sample size of 4 and after a time length of 0.30, the number of failures due to each cause is observed. The decision about the lot is taken based on that data. In that scenario, the acceptance probability and optimal sampling parameter both are same. Therefore, we can say that the acceptance set for both types of BSPs are same. Now, we generated six observations using the optimal design (4, 0.30, 3) which are tabulated in Table 10. For type-I BSP, the decision is taken for the observed data $\mathbf{d}$ based on the function $\varphi(\mathbf{d})$ and for type-II BSP, the decision is taken based on estimated reliability. In Table 10, both the functions are calculated for the observed data $\mathbf{d}$. Lastly, the decision for the observed data $\mathbf{d}$ are tabulated in Table 10. For type-I BSP, the lot is accepted when $\varphi(\mathbf{d}) < C_r$ and for type-II BSP, the lot is accepted when the estimated reliability is greater than 0.76.

Table 10: The data sets, estimated reliability and corresponding decision about the lot

i	Data			Estimated Reliability	$\varphi(\mathbf{d})$	Decision
	(d_{11}, d_{12})	(d_{21}, d_{22})	(d_{31}, d_{32})			
1	(0, 0)	(0, 0)	(0, 1)	0.971	8.439	Accept the lot
2	(0, 0)	(0, 0)	(0, 0)	0.973	6.063	Accept the lot
3	(0, 0)	(1, 0)	(2, 1)	0.860	22.219	Accept the lot
4	(1, 1)	(1, 0)	(1, 0)	0.753	42.531	Reject the lot
5	(2, 0)	(2, 0)	-	0.693	55.898	Reject the lot
6	(2, 1)	(0, 0)	(0, 1)	0.693	53.155	Reject the lot

In Table 10, it is seen that for 2nd observed data, all 4 items failed after the 2nd inspection. Therefore, no samples are surviving for 3rd inspection. For that reason, $E[M]$ is less than k and $E[\tau]$ is less than τ_k .

6. Conclusion

In this work, we have considered determining the optimal BSP under ICS for competing risk data. The advantage of using ICS is that continuous monitoring is not required. The work can be extended to progressive type-I ICS. We have considered an exponential distribution for illustration. In this work, it is observed that for ICS, the optimal BSP based on reliability acceptance criteria is similar to the optimal BSP based on the Bayes decision function. Therefore, for a large sample, using asymptotic properties of MLEs, the approximate form of Bayes risk can be considered to find an optimal BSP. The proposed methodology of the approximate form of Bayes risk can be extended to other lifetime distributions.

Also, in these works, a single prior assessment is used for the parameters of the lifetime distribution. However, the consumer and the manufacturer have different prior knowledge about the parameters of the lifetime distributions. Different prior assessment for the parameters of the lifetime distribution for single failure mode items was studied by Das and Pradhan [18, 19]. Using this concept, the work can be extended to a model where the consumer and the manufacturer have different prior knowledge about the parameters of the lifetime distributions for multiple failure mode items.

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