DEMOTIONS OF IDEALS IN COMMUTATIVE RINGS WITH APPLICATIONS TO NORMALLY TORSION-FREENESS

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Abstract. Let $J \subseteq I$ be ideals in a commutative Noetherian ring R, and $r, s \geq 0$. We say that J is a demotion of I if $I^r J^s = I^{r+s} \cap J^s$ for all $r, s \geq 0$. In this paper, we mainly aim to explore this notion in the polynomial rings. In particular, we investigate the relation between the demotion property and normally torsion-freeness. Furthermore, we compare the reductions of ideals and demotions of ideals.

1. Introduction and Overview

The study of powers and symbolic powers of ideals has long been a central theme in commutative algebra and its interactions with combinatorics and geometry. A classical result due to Brodmann [3] asserts that for any ideal I in a Noetherian ring R, the sequence of associated primes $\{\mathrm{Ass}_R(R/I^s)\}_{s\geq 1}$ stabilizes for sufficiently large s. The limiting set, called the *stable set* of associated primes, plays a fundamental role in the asymptotic behavior of ideals. Closely related to this is the notion of *normally torsion-free* ideals. An ideal I is normally torsion-free if $\mathrm{Ass}_R(R/I^s) \subseteq \mathrm{Ass}_R(R/I)$ for all $s \geq 1$. Although this property is uncommon in full generality, it has attracted considerable attention in the monomial setting. For example, the edge ideal (resp. the cover ideal) of a finite graph G is normally torsion-free precisely when G is bipartite [4, 20]. Further results extend this connection to Alexander duals of path ideals in rooted trees, strongly chordal graphs, t-spread

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principal Borel ideals, t-spread monomial ideals, and other families of monomial ideals, as can be found in [6, 8, 11, 13, 14, 16]. More broadly, it is well-known that normally torsion-free square-free monomial ideals correspond to *Mengerian* clutters, i.e., clutters satisfying the max-flow min-cut property; equivalently, for the edge ideal $I := I(\mathcal{C})$ one has $I^k = I^{(k)}$ for all $k \geq 1$ if and only if $\mathcal{C}$ is Mengerian [21]. This perspective links commutative algebra with combinatorial optimization and polyhedral theory and places the subject in the context of the Conforti–Cornuéjols conjecture.

Motivated by these interactions, this paper pursues three closely related directions. First, we introduce and study the notion of a demotion between ideals. Let $J \subseteq I$ be ideals in a commutative Noetherian ring R, and $r, s \geq 0$. Since $I^rJ^s \subseteq I^r \cap J^s \subseteq J^s$ and $I^rJ^s \subseteq I^{r+s}$, we always have $I^rJ^s \subseteq I^{r+s} \cap J^s$. We are interested in when $I^rJ^s = I^{r+s} \cap J^s$ holds. We say that J is a *demotion* of I if $I^rJ^s = I^{r+s} \cap J^s$ for all $r, s \geq 0$. Furthermore, J is said to be a *proper demotion* of I if J is a demotion of I and $I \neq J$. This equality refines the trivial containment $I^rJ^s \subseteq I^{r+s} \cap J^s$ and captures a precise compatibility between the two ideals. In this paper, we show that in the monomial world this notion admits explicit characterizations and constructions: for instance, prime monomial ideals and several natural classes of monomial ideals give rise to demotions, and principal monomial ideals admit a simple description of all their demotions.

The second aim is to deepen the study of normally torsion-free monomial ideals and to use the demotion framework to produce new normally torsion-free examples. Understanding the structure of normally torsion-free ideals contributes to the broader program of investigating minimal counterexamples to the Conforti–Cornuéjols conjecture and sheds light on how combinatorial properties of hypergraphs reflect algebraic stability phenomena, more information can be found in [2].

The paper is organized as follows. In Section 2 we collect fundamental definitions, results, and facts that will play a key role in the subsequent sections of this paper. Section 3 is concerned with investigating some general properties of demotions of ideals in commutative rings, refer to Propositions 3.1 and 3.2. The main aim of Section 4 is to identify some classes of monomial ideals that have the demotion property, consult Propositions 4.2, 4.3, and 4.4. In Section 5, we probe the behavior of the demotions of monomial ideals under a variety of monomial operations. In particular, we explore how these ideals are affected by operations such as expansion (Proposition 5.12), summation (Proposition 5.2), weighting (Proposition 5.15), monomial multiple (Proposition 5.10), permutation (Proposition 5.9), monomial localization (Proposition 5.4), contraction (Proposition 5.5), and deletion (Proposition 5.7). Building upon these operations, we propose several systematic methods for constructing new monomial ideals whose demotions are determined by those of previously studied monomial ideals. Section 6 is associated with exploring the relationship between demotions of monomial ideals and normally torsion-freeness. particularly, we demonstrate that the demotion property can be used to construct new normally torsion-free monomial ideals, and conversely, that normally torsion-free ideals can inform the generation of ideals with the demotion property, see Theorems 6.2 and 6.3. Finally, Section 7 is devoted to comparing reductions and demotions of ideals such that we establish that a number of its characteristic properties are not preserved in the case of demotions of ideals. For this purpose, we present several counterexamples in this section.

Throughout this text, we let $\mathcal{G}(I)$ denote the unique minimal set of monomial generators of a monomial ideal $I \subset R = K[x_1, \ldots, x_n]$, where R is the polynomial ring over a field K. Moreover, for a monomial $u \in R$, its *support*, denoted by $\mathrm{supp}(u)$, is defined as the set of variables that divide u; we also set $\mathrm{supp}(1) = \emptyset$. For a monomial ideal I, we further define $\mathrm{supp}(I) = \bigcup_{u \in \mathcal{G}(I)} \mathrm{supp}(u)$.

2. Preliminaries

This section aims to consolidate the fundamental definitions, key results, and essential facts and theoretical tools that will serve as the foundation for the developments discussed in the subsequent sections of this study. To establish a firm groundwork, we begin by revisiting the following result.

Fact 2.1. *(*[9, Exercise 6.4]*) Let I and J be ideals of a Noetherian ring A. Prove that if $JA_{\mathfrak{p}} \subseteq IA_{\mathfrak{p}}$ for every $\mathfrak{p} \in \mathrm{Ass}_A(A/I)$, then $J \subseteq I$.*

Fact 2.2. *(*[21, Proposition 4.3.29]*) Let I be an ideal of a ring R. If I has no embedded primes, then I is normally torsion-free if and only if $I^n = I^{(n)}$ for all $n \geq 1$.*

Fact 2.3. *(*[21, Exercise 6.1.25]*) If $\mathfrak{q}_1, \ldots, \mathfrak{q}_r$ are primary monomial ideals of R with non-comparable radicals and I is an ideal such that $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$, then $I^{(n)} = \mathfrak{q}_1^n \cap \cdots \cap \mathfrak{q}_r^n$.*

Fact 2.4. *(*[21, Exercise 6.1.23]*) If I, J, L are monomial ideals, then the following equalities hold:*

(i) $I \cap (J + L) = (I \cap J) + (I \cap L)$.
(ii) $I + (J \cap L) = (I + J) \cap (I + L)$.

Theorem 2.5. *(*[21, Theorem 14.3.6]*) Let $\mathcal{C}$ be a clutter and let A be its incidence matrix. The following are equivalent:*

(i) $\mathrm{gr}_I(R)$ *is reduced, where* $I = I(\mathcal{C})$ *is the edge ideal of* $\mathcal{C}$.
(ii) $R[It]$ *is normal and* $\mathcal{Q}(A)$ *is an integral polyhedron.*
(iii) $x \geq 0$; $xA \geq \mathbf{1}$ *is a* TDI *system.*
(iv) $\mathcal{C}$ *has the max-flow min-cut (MFMC) property.*
(v) $I^i = I^{(i)}$ *for* $i \geq 1$.
(vi) I *is normally torsion-free, i.e.,* $\mathrm{Ass}_R(R/I^i) \subseteq \mathrm{Ass}_R(R/I)$ *for* $i \geq 1$.
(vii) $\mathcal{C}$ *is Mengerian, i.e.,* $\beta_1(\mathcal{C}^a) = \alpha_0(\mathcal{C}^a)$ *for all* $a \in \mathbb{N}^n$.

Proposition 2.6. *Let I and J be two square-free monomial ideals in a polynomial ring $R = K[x_1, \dots, x_n]$ over a field K. Then, for all $k \geq 1$, we have $(I \cap J)^{(k)} = I^{(k)} \cap J^{(k)}$.*

Theorem 2.7. *(*[19, Theorem 2.5]*) Let I be a monomial ideal in $R = K[x_1, \dots, x_n]$ such that $I = I_1R + I_2R$, where $\mathcal{G}(I_1) \subset R_1 = K[x_1, \dots, x_m]$ and $\mathcal{G}(I_2) \subset R_2 = K[x_{m+1}, \dots, x_n]$ for some positive integer m. If I_1 and I_2 are normally torsion-free, then I is so.*

Lemma 2.8. *(*[12, Lemma 2.1]*) Let $I \subset R = K[x_1, \dots, x_r]$ be a square-free monomial ideal and $\mathfrak{q}$ be a prime monomial ideal in R such that $\bigcap_{\mathfrak{p} \in \mathrm{Ass}(I)} \mathfrak{p} \cap \mathfrak{q}$ is a minimal primary decomposition of $I \cap \mathfrak{q}$. Let I and $I \cap \mathfrak{q}$ be normally torsion-free. Then, for all $m, n \geq 1$, we have $I^n(I \cap \mathfrak{q})^m = I^{n+m} \cap \mathfrak{q}^m$.*

Fact 2.9. *(*[10, Exercise 7.9.1]*) Let $J_1, \dots, J_n, I$ be monomial ideals of R, and let $f \in [[R]]$.*

(a) *Prove that $f(\bigcap_{i=1}^n J_i) = \bigcap_{i=1}^n (fJ_i)$.*
(b) *Prove or disprove: $I(\bigcap_{i=1}^n J_i) = \bigcap_{i=1}^n (IJ_i)$. Justify your answer.*

Lemma 2.10. *(*[15, Lemma 2.17]*) Let I be a normally torsion-free square-free monomial ideal in a polynomial ring $R = K[x_1, \dots, x_n]$ with $\mathcal{G}(I) \subset R$. Then the ideal*

$$L := IS \cap (x_n, x_{n+1}, x_{n+2}, \dots, x_m) \subset S = R[x_{n+1}, x_{n+2}, \dots, x_m],$$

is normally torsion-free.

3. Demotions of ideals in commutative rings

In this section, we investigate some general properties of demotions of ideals in commutative rings. To accomplish this, we commence by presenting the following proposition.

Proposition 3.1. *Let $J \subseteq I$ be ideals in a commutative Noetherian ring R. Then the following statements are equivalent:*

(i) *J is a demotion of I;*
(ii) *$W^{-1}J$ is a demotion of $W^{-1}I$ for every multiplicatively closed subset W of R;*
(iii) *$J_\mathfrak{p}$ is a demotion of $I_\mathfrak{p}$ for every prime ideal $\mathfrak{p}$ of R.*

Proof. (i) $\Rightarrow$ (ii). We show that $(W^{-1}I)^r(W^{-1}J)^s = (W^{-1}I)^{r+s} \cap (W^{-1}J)^s$, where $r, s \geq 0$ and W is a multiplicatively closed subset of R. Clearly, $W^{-1}J \subseteq W^{-1}I$. As $I^rJ^s = I^{r+s} \cap J^s$, we get the following equalities

$$\begin{aligned}(W^{-1}I)^r(W^{-1}J)^s &= W^{-1}(I^r)W^{-1}(J^s) \\ &= W^{-1}(I^rJ^s) \\ &= W^{-1}(I^{r+s} \cap J^s) \\ &= (W^{-1}I)^{r+s} \cap (W^{-1}J)^s.\end{aligned}$$

This gives that $W^{-1}J$ is a demotion of $W^{-1}I$, as desired.

By choosing $W = R\backslash\mathfrak{p}$, (ii) implies (iii). To complete the proof, we need to show that (iii) implies (i). To do this, we assume that $J_\mathfrak{p}$ is a demotion of $I_\mathfrak{p}$ for every prime ideal $\mathfrak{p}$ of R. Fix $r, s \geq 0$. Since $I^r J^s \subseteq I^{r+s} \cap J^s$, it remains to prove that $I^{r+s} \cap J^s \subseteq I^r J^s$. Pick an arbitrary element $\mathfrak{q} \in \mathrm{Ass}(R/I^r J^s)$. Due to $J_\mathfrak{p}$ is a demotion of $I_\mathfrak{p}$ for every prime ideal $\mathfrak{p}$ of R, this yields that $I_\mathfrak{q}^{r+s} \cap J_\mathfrak{q}^s \subseteq I_\mathfrak{q}^r J_\mathfrak{q}^s$. This leads to $(I^{r+s} \cap J^s)_\mathfrak{q} \subseteq (I^r J^s)_\mathfrak{q}$. It follows at once from Fact 2.1 that $I^{r+s} \cap J^s \subseteq I^r J^s$, as required. This finishes the proof. $\square$

Recall that if $f : A \to B$ is a ring homomorphism and I is an ideal of A, we will write IB for the ideal $f(I)B$ of B. The ideal IB is called the *extension* of I to B, and is often denoted by I^e.

Proposition 3.2. *Let $J \subseteq I$ be ideals in a commutative Noetherian ring R, and $f : R \to S$ be a ring homomorphism. Denote $I^e := IS$ and $J^e := JS$. Then the following statements hold.*

(i) *If J is a demotion of I, then J^k is a demotion of I^k for all $k \geq 1$.*
(ii) *If S is flat over R and J is a demotion of I in R, then J^e is a demotion of I^e in S.*
(iii) *If S is faithfully flat over R and J^e is a demotion of I^e in S, then J is a demotion of I in R.*

Proof. (i) Let J be a demotion of I. Fix $r, s \geq 0$ and $k \geq 1$. It is clear that $J^k \subseteq I^k$. Because J is a demotion of I, we have $I^a J^b = I^{a+b} \cap J^b$ for all $a, b \geq 0$. This implies that

$$(I^k)^r (J^k)^s = I^{kr} J^{ks} = I^{kr+ks} \cap J^{ks} = (I^k)^{r+s} \cap (J^k)^s.$$

This means that J^k is a demotion of I^k, as claimed.

(ii) Assume S is flat and $I^r J^s = I^{r+s} \cap J^s$ for all $r, s \geq 0$. It is obvious that $J^e \subseteq I^e$. Recall that the extension of ideals is given by tensoring with S and then taking the image in S, that is, for any ideal $K \subseteq R$ we have $K^e = KS = \mathrm{im}(K \otimes_R S \to S)$. Since S is flat, the functor $- \otimes_R S$ is exact. Exactness gives the following standard consequences:

- For all ideals $X, Y \subseteq R$, $(X \cap Y) \otimes_R S \xrightarrow{\cong} (X \otimes_R S) \cap (Y \otimes_R S)$, and passing to images in S yields $(X \cap Y)^e = X^e \cap Y^e$.
- For all ideals $X, Y \subseteq R$, $(XY) \otimes_R S \xrightarrow{\cong} (X \otimes_R S)(Y \otimes_R S)$, and so $(XY)^e = X^e Y^e$, and more generally $(X^n)^e = (X^e)^n$ for $n \geq 0$.

Apply extension to $I^r J^s = I^{r+s} \cap J^s$, we get $(I^r J^s)^e = (I^{r+s} \cap J^s)^e$. By multiplicativity and power-compatibility of extension, we have $(I^e)^r (J^e)^s = (I^r J^s)^e$. Moreover, we obtain $(I^{r+s} \cap J^s)^e = (I^{r+s})^e \cap (J^s)^e = (I^e)^{r+s} \cap (J^e)^s$. Combining the equalities yields $(I^e)^r (J^e)^s = (I^e)^{r+s} \cap (J^e)^s$ for all $r, s \geq 0$, as required.

(iii) Assume S is faithfully flat and that $(I^e)^r (J^e)^s = (I^e)^{r+s} \cap (J^e)^s$ for all $r, s \geq 0$. We want to show that $I^r J^s = I^{r+s} \cap J^s$ for all $r, s \geq 0$. Recall this key fact for faithfully flat property that for every ideal $K \subseteq R$, one has $(K^e)^c = K$, where $(-)^c$ denotes contraction to R. Also contraction commutes with finite intersections, that is, for any two ideals $X, Y \subseteq S$, $(X \cap Y)^c = X^c \cap Y^c$. Apply contraction to $(I^e)^r(J^e)^s = (I^e)^{r+s} \cap (J^e)^s$, we obtain $\big((I^e)^r(J^e)^s\big)^c = \big((I^e)^{r+s} \cap (J^e)^s\big)^c$. Using the fact $(K^e)^c = K$ for any ideal $K \subseteq R$, we have $\big((I^e)^r(J^e)^s\big)^c = \big((I^rJ^s)^e\big)^c = I^rJ^s$. Furthermore,

$$\big((I^e)^{r+s} \cap (J^e)^s\big)^c = \big((I^{r+s})^e\big)^c \cap \big((J^s)^e\big)^c = I^{r+s} \cap J^s.$$

This leads to $I^rJ^s = I^{r+s} \cap J^s$ for all $r, s \geq 0$, which is exactly the demotion property for J with respect to I in R. □

4. Some classes of monomial ideals with the demotion property

The main aim of this section is to identify some classes of monomial ideals that have the demotion property. To achieve this, we present the first such class in Proposition 4.2. The following lemma is essential for completing the proof of Proposition 4.2.

Lemma 4.1. *Let $f_1, \ldots, f_m$ be nonnegative integers and let t be a nonnegative integer such that $\sum_{i=1}^m f_i \geq t$. Then there exist integers $c_1, \ldots, c_m$ satisfying $0 \leq c_i \leq f_i$ for all $1 \leq i \leq m$ and $\sum_{i=1}^m c_i = t$.*

Proof. Set $r_0 := t$ and, for each $i = 1, \ldots, m$, define $c_i := \min\{f_i, r_{i-1}\}$ and $r_i := r_{i-1} - c_i$. By construction, we have $0 \leq c_i \leq f_i$ and $r_i \geq 0$ for all i. Summing the telescoping differences gives $\sum_{i=1}^m c_i = r_0 - r_m = t - r_m$, so it remains to prove $r_m = 0$.

Assume for contradiction that $r_m > 0$. Then $r_{i-1} > 0$ for every i, so if for some index i we had $f_i > r_{i-1}$ we would get $c_i = r_{i-1}$ and hence $r_i = 0$, contradicting $r_m > 0$. Thus $f_i \leq r_{i-1}$ for all i, and consequently $c_i = f_i$ for every i. This implies that $\sum_{i=1}^m c_i = \sum_{i=1}^m f_i \geq t$, but also $\sum_{i=1}^m c_i = t - r_m < t$, a contradiction. Hence, we must have $r_m = 0$ and $\sum_{i=1}^m c_i = t$. This finishes the proof. □

Proposition 4.2. *Let $\mathfrak{p}$ and $\mathfrak{q}$ be monomial prime ideals in $R = K[x_1, \ldots, x_n]$ with $\mathfrak{q} \subseteq \mathfrak{p}$. Then $\mathfrak{q}$ is a demotion of $\mathfrak{p}$.*

Proof. Fix $r, s \geq 0$. Without loss of generality, we may write

$$\mathfrak{p} = (x_1, \ldots, x_c) \quad \text{and} \quad \mathfrak{q} = (x_1, \ldots, x_d), \quad \text{with } 1 \leq d \leq c \leq n.$$

Since $\mathfrak{p}^r\mathfrak{q}^s \subseteq \mathfrak{p}^{r+s} \cap \mathfrak{q}^s$, it remains to show the reverse inclusion. Take a monomial $m = x_1^{\beta_1} \cdots x_n^{\beta_n} \in \mathfrak{p}^{r+s} \cap \mathfrak{q}^s$. Due to $m \in \mathfrak{q}^s$, we have $\sum_{i=1}^d \beta_i \geq s$, and since $m \in \mathfrak{p}^{r+s}$, we get $\sum_{i=1}^c \beta_i \geq r + s$. By virtue of Lemma 4.1, we can choose nonnegative integers $\beta_i'' \leq \beta_i$ for all $1 \leq i \leq d$ with $\sum_{i=1}^d \beta_i'' = s$. For $n \geq i > d$ set $\beta_i'' = 0$, and put $\beta_i' := \beta_i - \beta_i''$ for all $i = 1, \ldots, n$. Now, define monomials

$$m_1 = x_1^{\beta_1'} \cdots x_n^{\beta_n'} \ \text{ and } \ m_2 = x_1^{\beta_1''} \cdots x_d^{\beta_d''}.$$

By construction $m = m_1m_2$, and $m_2 \in \mathfrak{q}^s$ due to $\sum_{i=1}^{d} \beta_i'' = s$. Moreover,

$$\sum_{i=1}^{c} \beta_i' = \sum_{i=1}^{c} \beta_i - \sum_{i=1}^{c} \beta_i'' = \sum_{i=1}^{c} \beta_i - \sum_{i=1}^{d} \beta_i'' \geq (r+s) - s = r,$$

since $\beta_i'' = 0$ for $i > d$. Hence, we obtain $m_1 \in \mathfrak{p}^r$, and so $m = m_1m_2 \in \mathfrak{p}^r\mathfrak{q}^s$. This proves that $\mathfrak{p}^{r+s} \cap \mathfrak{q}^s \subseteq \mathfrak{p}^r\mathfrak{q}^s$. Thus, $\mathfrak{q}$ is a demotion of $\mathfrak{p}$. $\square$

Proposition 4.3. *Let $\mathfrak{p} = (x_1, \ldots, x_m) \subset K[x_1, \ldots, x_n]$, $1 \leq m \leq n$, be a monomial prime ideal and $J = (x_1^m, \ldots, x_m^m)$. Then J is a demotion of $\mathfrak{p}^m$. In particular, if $m > 1$, then J is a proper demotion of $\mathfrak{p}^m$.*

Proof. We should prove $(\mathfrak{p}^m)^r J^s = (\mathfrak{p}^m)^{r+s} \cap J^s$ for all $r, s \geq 0$. Fix $r, s \geq 0$. Since $(\mathfrak{p}^m)^r J^s \subseteq (\mathfrak{p}^m)^{r+s} \cap J^s$, it suffices to check the reverse inclusion. Pick a monomial $u \in (\mathfrak{p}^m)^{r+s} \cap J^s$. Write $u = (\prod_{i=1}^m x_i^{\alpha_i})(\prod_{i=m+1}^n x_i^{\alpha_i})$, and set

$$\alpha = (\alpha_1, \ldots, \alpha_m), \qquad |\alpha| = \sum_{i=1}^{m} \alpha_i, \qquad U_i = \left\lfloor \frac{\alpha_i}{m} \right\rfloor \ (1 \leq i \leq m).$$

In the sequel, we will use the following facts. The first fact is obvious. We justify only the second fact:

- $u \in (\mathfrak{p}^m)^t \iff |\alpha| \geq mt.$
- $u \in J^t \iff \sum_{i=1}^m U_i \geq t.$

Since the monomial ideal $J = (x_1^m, \ldots, x_m^m)$ is generated by the monomials x_i^m $(1 \leq i \leq m)$, a typical minimal monomial generator of J^t has the form $g = \prod_{i=1}^m x_i^{mc_i}$, where $c_i \geq 0$ are integers with $\sum_{i=1}^m c_i = t$. Indeed, such g arises as a product of t minimal monomial generators of J, and conversely every product of t minimal monomial generators is of this form.

$(\Rightarrow)$ If $u \in J^t$, then some such g divides u. Divisibility means $\alpha_i \geq mc_i$ for all $1 \leq i \leq m$, hence $\left\lfloor \frac{\alpha_i}{m} \right\rfloor \geq c_i$ for all $1 \leq i \leq m$. Summing over i gives

$$\sum_{i=1}^{m} \left\lfloor \frac{\alpha_i}{m} \right\rfloor \geq \sum_{i=1}^{m} c_i = t.$$

$(\Leftarrow)$ Conversely, suppose $\sum_{i=1}^m \left\lfloor \frac{\alpha_i}{m} \right\rfloor \geq t$. Let $f_i := \lfloor \alpha_i/m \rfloor$ for $i = 1, \ldots, m$. Because $\sum_{i=1}^m f_i \geq t$, Lemma 4.1 permits us to select integers $c_1, \ldots, c_m$ such that $0 \leq c_i \leq f_i$ for all $1 \leq i \leq m$, and $\sum_{i=1}^m c_i = t$. Then the monomial $g = \prod_{i=1}^m x_i^{mc_i}$ is a generator of J^t and divides u, showing $u \in J^t$.

This gives rise to the fact that $u \in J^t \iff \sum_{i=1}^m \left\lfloor \frac{\alpha_i}{m} \right\rfloor \geq t$.

It follows now from $u \in (\mathfrak{p}^m)^{r+s} \cap J^s$ and the above facts that u satisfies both $|\alpha| \geq m(r+s)$ and $\sum_{i=1}^m U_i \geq s$. Since $\sum_{i=1}^m U_i \geq s$, Lemma 4.1 enables us to choose integers $c_1, \ldots, c_m$ such that $0 \leq c_i \leq U_i$ for all $1 \leq i \leq m$, and $\sum_{i=1}^m c_i = s$. In particular, $c_i \leq U_i = \lfloor \alpha_i/m \rfloor$ gives that $\alpha_i \geq mc_i$ for all

$1 \leq i \leq m$. Define, for all $1 \leq i \leq m$, $\gamma_i := mc_i$, $\beta_i := \alpha_i - \gamma_i$, and set

$$b := \prod_{i=1}^{m} x_i^{\gamma_i} \in J^s \text{ and } a := \prod_{i=1}^{m} x_i^{\beta_i} \cdot x_{m+1}^{\alpha_{m+1}} \cdots x_n^{\alpha_n}.$$

Then $u = ab$. Moreover, we have the following

$$\sum_{i=1}^{m} \beta_i = |\alpha| - m \sum_{i=1}^{m} c_i = |\alpha| - ms \geq m(r+s) - ms = mr.$$

Hence, $a \in (\mathfrak{p}^m)^r$, and so $u \in (\mathfrak{p}^m)^r J^s$. This implies $(\mathfrak{p}^m)^{r+s} \cap J^s \subseteq (\mathfrak{p}^m)^r J^s$, and thus J is a demotion of $\mathfrak{p}^m$. Finally, since $J \subsetneq \mathfrak{p}^m$ when $m > 1$, this shows that J is a proper demotion of $\mathfrak{p}^m$. $\square$

This section is concluded with the following proposition, which addresses the demotion of principal monomial ideals.

Proposition 4.4. *Suppose that $I = (m) \subset R = K[x_1, \ldots, x_n]$ is a principal monomial ideal, where $m = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$. Then a monomial ideal $J \subseteq I$ is a* demotion *of I if and only if $J = (mu_1, \ldots, mu_t)$, where $\mathcal{G}(J) = \{mu_1, \ldots, mu_t\}$ with $\operatorname{supp}(u_i) \cap \operatorname{supp}(m) = \emptyset$ for all $i = 1, \ldots, t$. In particular, I admits proper demotions whenever $\operatorname{supp}(m) \subsetneq \{1, \ldots, n\}$.*

Proof. ($\Rightarrow$) Suppose $J \subseteq I$ is a demotion of I. Since I is principal, every generator of J is divisible by m, so we can write $J = (mu_1, \ldots, mu_t)$, where $\mathcal{G}(J) = \{mu_1, \ldots, mu_t\}$. We claim that $\operatorname{supp}(u_i) \cap \operatorname{supp}(m) = \emptyset$ for all $i = 1, \ldots, t$. Suppose, on the contrary, that there exists some $1 \leq i \leq t$ such that $x_j \in \operatorname{supp}(u_i) \cap \operatorname{supp}(m)$ for some $1 \leq j \leq n$. Among all such generators that involve x_j in their supports, assume that $\deg_{x_j} u_i$ is minimum. By virtue of J is a demotion of I, we have $I^{r+s} \cap J^s = I^r J^s$ for all $r, s \geq 0$. Take $r = s = 1$. Then we get $IJ = I^2 \cap J$. Define $M := \operatorname{lcm}(m^2, mu_i) \in I^2 \cap J$. Hence, $M \in IJ$. This implies that there exists some monomial $m^2 u_k \in IJ$, where $1 \leq k \leq t$, such that $m^2 u_k \mid M$. As $x_j \in \operatorname{supp}(u_i) \cap \operatorname{supp}(m)$, we can assume that $\deg_{x_j}(u_i) = b_j > 0$ and $a_j > 0$. Also, let $\deg_{x_j}(u_k) = c_j \geq 0$. Hence, we may consider the following cases:

Case 1. $c_j > 0$. Then the minimality implies that $b_j \leq c_j$, and so we can conclude the following

$$\deg_{x_j}(M) = \max\{2a_j, a_j + b_j\} < 2a_j + b_j \leq 2a_j + c_j = \deg_{x_j}(m^2 u_k).$$

This yields that M is not divisible by $m^2 u_k$, a contradiction.

Case 2. $c_j = 0$. Hence, $x_j \nmid u_k$. Since $mu_i, mu_k \in \mathcal{G}(J)$ and $u_i \neq u_k$, this implies that there exists some x_p such that $x_p \mid u_k$ but $x_p \nmid u_i$. In particular, $x_p \neq x_j$. Then we obtain the following

$$\deg_{x_p}(M) = 2a_p < 2a_p + \deg_{x_p} u_k = \deg_{x_j}(m^2 u_k).$$

This leads to $m^2 u_k \nmid M$, which is a contradiction. Consequently, we can conclude that $\operatorname{supp}(u_i) \cap \operatorname{supp}(m) = \emptyset$ for all $i = 1, \ldots, t$.

($\Leftarrow$) Conversely, let $\mathcal{G}(J) = \{mu_1, \ldots, mu_t\}$ with $\operatorname{supp}(u_i) \cap \operatorname{supp}(m) = \emptyset$ for all $i = 1, \ldots, t$. Then any product in $I^r J^s$ has the form $m^{r+s}(u_{i_1} \cdots u_{i_s})$. Since the variables of m and the u_i are disjoint, for any choice of indices we have $\operatorname{lcm}\big(m^{r+s}, m^s(u_{i_1} \cdots u_{i_s})\big) = m^{r+s}(u_{i_1} \cdots u_{i_s})$. Thus, for all $r, s \geq 0$, $I^r J^s = I^{r+s} \cap J^s$, so J is indeed a demotion of I.

Finally, we observe that if $\operatorname{supp}(m) = \{1, \ldots, n\}$, then no variable is left outside the support of m, so no nontrivial $u_i \neq 1$ exists and the only demotion is $J = I$. On the other hand, if $\operatorname{supp}(m) \subsetneq \{1, \ldots, n\}$, then there exist variables not appearing in m. For any choice of monomials $u_1, \ldots, u_t$ in these complementary variables, the ideal $J = (mu_1, \ldots, mu_t) \subsetneq I$ is a proper demotion of I. $\square$

Corollary 4.5. *any proper monomial ideal in $K[x]$ has no proper demotion ideal.*

5. Demotions of monomial ideals under monomial operations

In this section, we investigate the behavior of the demotions of monomial ideals under a variety of monomial operations. In particular, we explore how these ideals are affected by operations such as expansion, weighting, monomial multiplication, monomial localization, contraction, and deletion. Building upon these operations, we propose several systematic methods for constructing new monomial ideals whose demotions are determined by those of previously studied monomial ideals. To provide a rigorous foundation for our discussion, we begin by examining the demotions of monomial ideals under the sum of ideals.

5.1. **Demotions of monomial ideals under summation.**

Proposition 5.2 asserts that, under certain condition, the sum of two monomial ideals that possess demotions also admits a demotion. Before proceeding, we first mention the following auxiliary lemma.

Lemma 5.1. *Let $J_1 \subseteq I_1$ (respectively, $J_2 \subseteq I_2$) be monomial ideals in $R = K[x_1, \ldots, x_n]$ such that $\mathcal{G}(I_1) \subset R_1 = K[x_1, \ldots, x_m]$ and $\mathcal{G}(I_2) \subset R_2 = K[x_{m+1}, \ldots, x_n]$ for some $m \geq 1$. Let $r, s, c, d \geq 0$ with $c + d = s$. Then*

$$\Big(\sum_{\alpha+\beta=r+s} I_1^{\alpha} I_2^{\beta}\Big) \cap (J_1^c J_2^d) = \sum_{\substack{a+b=r+s\\ a\geq c,\ b\geq d}} \big(I_1^a \cap J_1^c\big)\big(I_2^b \cap J_2^d\big).$$

Proof. Our strategy is to show that each side is contained in the other.

Claim 1: The right-hand side is contained in the left-hand side. Let u be a monomial in the right-hand side. Then for some a, b with $a + b = r + s$, $a \geq c$, $b \geq d$, we have $u \in (I_1^a \cap J_1^c)(I_2^b \cap J_2^d)$. So $u = u_1 u_2$ with $u_1 \in I_1^a \cap J_1^c$ and $u_2 \in I_2^b \cap J_2^d$. Clearly, $u \in I_1^a I_2^b \subseteq \sum_{\alpha+\beta=r+s} I_1^{\alpha} I_2^{\beta}$. Also, $u_1 \in J_1^c$ and $u_2 \in J_2^d$ imply that $u \in J_1^c J_2^d$. Therefore, u is in the left-hand side.

Claim 2: The left-hand side is contained in the right-hand side. Let f be a monomial in the left-hand side. Then $f \in J_1^c J_2^d$ and $f \in \sum_{\alpha+\beta=r+s} I_1^\alpha I_2^\beta$. Since the variables of R_1 and R_2 are disjoint, any monomial f can be uniquely written as $f = f_1 f_2$ with $f_1 \in R_1$ and $f_2 \in R_2$. Because $f \in J_1^c J_2^d$, we have $f_1 \in J_1^c$ and $f_2 \in J_2^d$. Also, since $f \in \sum_{\alpha+\beta=r+s} I_1^\alpha I_2^\beta$, there exist α, β with $\alpha + \beta = r + s$ such that $f \in I_1^\alpha I_2^\beta$. Hence, $f_1 \in I_1^\alpha$ and $f_2 \in I_2^\beta$. Therefore, $f_1 \in I_1^\alpha \cap J_1^c$ and $f_2 \in I_2^\beta \cap J_2^d$. If $\alpha \geq c$ and $\beta \geq d$, then directly $f \in (I_1^\alpha \cap J_1^c)(I_2^\beta \cap J_2^d)$, which is a term in the right-hand sum.

Now suppose $\alpha < c$ or $\beta < d$. Without loss of generality, assume $\alpha < c$. Clearly, $f \in (I_1^\alpha \cap J_1^c)(I_2^\beta \cap J_2^d) \subseteq J_1^c J_2^d$. Since $\alpha + \beta = r + s$ and $\alpha < c$, we have $\beta = r + s - \alpha > r + s - c$. As $c + d \leq r + s$, it follows that $r + s - c \geq d$, so $\beta \geq d + 1 \geq d$. Now choose $a := c$ and $b := r + s - c$. Note that $b \geq d$. Then, $f_1 \in I_1^\alpha \cap J_1^c \subseteq J_1^c = I_1^c \cap J_1^c$ and $f_2 \in I_2^\beta \cap J_2^d \subseteq I_2^\beta \subseteq I_2^{r+s-c} = I_2^b$ by $\beta > b$. Also, $f_2 \in J_2^d$, so $f_2 \in I_2^b \cap J_2^d$. Therefore, we obtain

$$f = f_1 f_2 \in (I_1^c \cap J_1^c)(I_2^b \cap J_2^d),$$

which is a term in the right-hand sum for $a = c$ and $b = r + s - c$.

The case $\beta < d$ is symmetric. Thus, every element of the left-hand side is contained in the right-hand side. This finishes the proof. □

Recall that if I and J are two monomial ideals in $R = K[x_1, \ldots, x_n]$ such that $\operatorname{supp}(I) \cap \operatorname{supp}(J) = \emptyset$, then $I \cap J = IJ$.

Proposition 5.2. *Let $J_1 \subseteq I_1$ (respectively, $J_2 \subseteq I_2$) be monomial ideals in $R = K[x_1, \ldots, x_n]$ such that $\mathcal{G}(I_1) \subset R_1 = K[x_1, \ldots, x_m]$ and $\mathcal{G}(I_2) \subset R_2 = K[x_{m+1}, \ldots, x_n]$ for some $m \geq 1$. If J_1 is a demotion of I_1 (respectively, J_2 is a demotion of I_2), then $J_1 + J_2$ is a demotion of $I_1 + I_2$.*

Proof. Assume that J_1 is a demotion of I_1 (respectively, J_2 is a demotion of I_2). For simplicity of notation, let us define $J := J_1 + J_2$ and $I := I_1 + I_2$. Fix $r, s \geq 0$. Since the variables in R_1 and R_2 are disjoint, the polynomial ring decomposes as a tensor product $R = R_1 \otimes_K R_2$. It is obvious that $J \subseteq I$. Moreover, based on the binomial expansion theorem, it follows that

$$I^r = \sum_{a+b=r} I_1^a I_2^b \quad \text{and} \quad J^s = \sum_{c+d=s} J_1^c J_2^d,$$

where multiplication is taken inside R. By distributivity, we have

$$I^r J^s = \left(\sum_{a+b=r} I_1^a I_2^b \right) \left(\sum_{c+d=s} J_1^c J_2^d \right) = \sum_{a+b=r,\, c+d=s} I_1^a I_2^b J_1^c J_2^d.$$

Because the variables of R_1 and R_2 are disjoint, these factors commute and can be grouped as $I_1^a J_1^c \cdot I_2^b J_2^d \subseteq R_1 \otimes_K R_2$. Due to J_1 is a demotion of I_1, we get $I_1^a J_1^c = I_1^{a+c} \cap J_1^c$ for all $a, c \geq 0$. Similarly, since J_2 is a demotion of I_2, we have $I_2^b J_2^d = I_2^{b+d} \cap J_2^d$ for all $b, d \geq 0$. Substituting these back,

$$I^r J^s = \sum_{a+b=r,\ c+d=s} (I_1^{a+c} \cap J_1^c)(I_2^{b+d} \cap J_2^d).$$

By virtue of the variables are disjoint, we can readily deduce that

$$\begin{aligned}(I_1^{a+c} I_2^{b+d}) \cap (J_1^c J_2^d) &= (I_1^{a+c} \cap I_2^{b+d}) \cap (J_1^c \cap J_2^d)\\ &= (I_1^{a+c} \cap J_1^c) \cap (I_2^{b+d} \cap J_2^d)\\ &= (I_1^{a+c} \cap J_1^c)(I_2^{b+d} \cap J_2^d).\end{aligned}$$

Thus, we can conclude the following equality

$$I^r J^s = \sum_{a+b=r,\ c+d=s} (I_1^{a+c} I_2^{b+d}) \cap (J_1^c J_2^d).$$

It follows now from Lemma 5.1 that

$$\begin{aligned}I^{r+s} \cap J^s &= \Big(\sum_{\alpha+\beta=r+s} I_1^\alpha I_2^\beta\Big) \cap \Big(\sum_{c+d=s} J_1^c J_2^d\Big)\\ &= \sum_{c+d=s} \Big[\Big(\sum_{\alpha+\beta=r+s} I_1^\alpha I_2^\beta\Big) \cap (J_1^c J_2^d)\Big]\\ &= \sum_{\substack{c+d=s,\ a+b=r+s\\ a\geq c,\ b\geq d}} \big(I_1^a \cap J_1^c\big)\big(I_2^b \cap J_2^d\big).\end{aligned}$$

On account of J_1 is a demotion of I_1 and J_2 is a demotion of I_2, for $a \geq c$ and $b \geq d$, we have $I_1^a \cap J_1^c = I_1^{a-c} J_1^c$ and $I_2^b \cap J_2^d = I_2^{b-d} J_2^d$. Thus, we obtain

$$I^{r+s} \cap J^s = \sum_{\substack{c+d=s,\ a+b=r+s\\ a\geq c,\ b\geq d}} I_1^{a-c} J_1^c\, I_2^{b-d} J_2^d.$$

Let $a' := a-c$ and $b' := b-d$. Then $a'+b' = (a+b)-(c+d) = (r+s)-s = r$, and so we get the following equalities

$$I^{r+s} \cap J^s = \sum_{a'+b'=r,\ c+d=s} I_1^{a'} I_2^{b'} J_1^c J_2^d = \Big(\sum_{a'+b'=r} I_1^{a'} I_2^{b'}\Big)\Big(\sum_{c+d=s} J_1^c J_2^d\Big) = I^r J^s.$$

We can at once conclude that $J = J_1 + J_2$ is a demotion of $I = I_1 + I_2$, as claimed. $\square$

5.2. **Demotions of monomial ideals under monomial localization, contraction, and deletion.**

We now focus on the notion of monomial localization. Assume that $I \subset R = K[x_1, \ldots, x_n]$ is a monomial ideal, where K is a field. We denote by $V^*(I)$ the collection of monomial prime ideals that contain I. Suppose that $\mathfrak{p} = (x_{i_1}, \ldots, x_{i_r})$ is a monomial prime ideal. The *monomial localization* of I with respect to $\mathfrak{p}$, denoted by $I(\mathfrak{p})$, is defined as the ideal of the polynomial ring $R(\mathfrak{p}) = K[x_{i_1}, \ldots, x_{i_r}]$, obtained from I via the K-algebra homomorphism

$$R \longrightarrow R(\mathfrak{p}), \qquad x_j \mapsto 1 \quad \text{for all } x_j \notin \{x_{i_1}, \ldots, x_{i_r}\}.$$

The following lemma will be needed in order to prove Proposition 5.4. For the sake of clarity and ease of reference, we state it here.

Lemma 5.3. *([19, Lemma 3.13]) Let I and J be two monomial ideals in $R = K[x_1, \ldots, x_n]$, and $\mathfrak{p}$ be a monomial prime ideal of R. Then the following statements hold.*

(i) $(I + J)(\mathfrak{p}) = I(\mathfrak{p}) + J(\mathfrak{p})$;
(ii) $(IJ)(\mathfrak{p}) = I(\mathfrak{p})J(\mathfrak{p})$;
(iii) $(I \cap J)(\mathfrak{p}) = I(\mathfrak{p}) \cap J(\mathfrak{p})$;
(iv) $(I :_R J)(\mathfrak{p}) = (I(\mathfrak{p}) :_{R(\mathfrak{p})} J(\mathfrak{p}))$;
(v) *If Q is a $\mathfrak{q}$-primary monomial ideal in R with $\mathfrak{q} \subseteq \mathfrak{p}$, then $Q(\mathfrak{p})$ is a $\mathfrak{q}$-primary monomial ideal in $R(\mathfrak{p})$.*

The following proposition asserts that if a monomial ideal has the demotion property, then any of its monomial localizations with respect to a monomial prime ideal also possesses this property.

Proposition 5.4. *Let $J \subseteq I$ be monomial ideals in $R = K[x_1, \ldots, x_n]$, and $\mathfrak{p} \in V^*(I)$. If J is a demotion of I, then $J(\mathfrak{p})$ is a demotion of $I(\mathfrak{p})$.*

Proof. Suppose that J is a demotion of I, and fix $r, s \geq 0$. Then we have $I^r J^s = I^{r+s} \cap J^s$, and so $(I^r J^s)(\mathfrak{p}) = (I^{r+s} \cap J^s)(\mathfrak{p})$. It follows from Lemma 5.3(i) that $J(\mathfrak{p}) \subseteq I(\mathfrak{p})$. We can now derive from parts (ii) and (iii) of Lemma 5.3 that $I(\mathfrak{p})^r J(\mathfrak{p})^s = I(\mathfrak{p})^{r+s} \cap J(\mathfrak{p})^s$. We therefore obtain $J(\mathfrak{p})$ is a demotion of $I(\mathfrak{p})$, as desired. □

To state the forthcoming result, we first recall the definition of the contraction operation. Let I be a monomial ideal in $R = K[x_1, \ldots, x_n]$ with minimal generating set $\mathcal{G}(I) = \{u_1, \ldots, u_m\}$. For some index $1 \leq j \leq n$, the *contraction* of x_j from I, denoted by I/x_j, is obtained by substituting $x_j = 1$ in each generator u_i, for $i = 1, \ldots, m$. Since the contraction I/x_j coincides with the monomial localization of I with respect to $\mathfrak{p} = \mathfrak{m} \setminus \{x_j\}$, where $\mathfrak{m} = (x_1, \ldots, x_n)$ denotes the graded maximal ideal of R, Proposition 5.4 allows us to immediately deduce the following result.

Corollary 5.5. *Let $J \subseteq I$ be monomial ideals in $R = K[x_1, \ldots, x_n]$, and $1 \leq j \leq n$. If J is a demotion of I, then J/x_j is a demotion of I/x_j.*

We continue by recalling the definition of the deletion operation. Let $I \subset R = K[x_1, \ldots, x_n]$ be a monomial ideal with minimal generating set $\mathcal{G}(I) = \{u_1, \ldots, u_m\}$. For some index $1 \leq j \leq n$, the *deletion* of x_j from I, denoted by $I \setminus \{x_j\}$ (sometimes written as $I \setminus x_j$ for simplicity), is obtained by setting $x_j = 0$ in each generator u_i, for all $i = 1, \ldots, m$.

The lemma below plays a crucial role in the proof of Proposition 5.7. For the reader's convenience, we include it here.

Lemma 5.6. *([17, Lemma 4.12]) Let I and J be two monomial ideals in $R = K[x_1, \ldots, x_n]$ over a field K, and $1 \le j \le n$. Then the following statements hold.*

(i) $(I + J) \setminus x_j = I \setminus x_j + J \setminus x_j$;
(ii) $(IJ) \setminus x_j = (I \setminus x_j)(J \setminus x_j)$;
(iii) $(I \cap J) \setminus x_j = (I \setminus x_j) \cap (J \setminus x_j)$;
(iv) $\sqrt{I \setminus x_j} = \sqrt{I} \setminus x_j$.
(v) *If Q is a $\mathfrak{q}$-primary monomial ideal in R, then $Q \setminus x_j$ is a $(\mathfrak{q} \setminus x_j)$-primary monomial ideal in $R \setminus x_j$;*
(vi) *If $I = Q_1 \cap \cdots \cap Q_s$ is a minimal primary decomposition of I in R, then $I \setminus x_j = (Q_1 \setminus x_j) \cap \cdots \cap (Q_s \setminus x_j)$ is a primary decomposition of $I \setminus x_j$ in $R \setminus x_j$;*
(vii) $\mathrm{Ass}_{R\setminus x_j}((R \setminus x_j)/(I \setminus x_j)) \subseteq \{\mathfrak{q} \setminus x_j \ : \ \mathfrak{q} \in \mathrm{Ass}_R(R/I)\}$.

The following proposition states that whenever a monomial ideal satisfies the demotion property, each of its deletions also retains this property.

Proposition 5.7. *Let $J \subseteq I$ be monomial ideals in $R = K[x_1, \ldots, x_n]$, and $1 \le j \le n$. If J is a demotion of I, then $J \setminus x_j$ is a demotion of $I \setminus x_j$.*

Proof. Assume that J is a demotion of I, and fix $r, s \geq 0$. This implies that $I^r J^s = I^{r+s} \cap J^s$, and hence $(I^r J^s) \setminus x_j = (I^{r+s} \cap J^s) \setminus x_j$. According to Lemma 5.6(i), we obtain $J \setminus x_j \subseteq I \setminus x_j$. It can be concluded from parts (ii) and (iii) of Lemma 5.6 that $(I \setminus x_j)^r (J \setminus x_j)^s = (I \setminus x_j)^{r+s} \cap (J \setminus x_j)^s$. This gives rise to $J \setminus x_j$ is a demotion of $I \setminus x_j$, as claimed. □

5.3. **Demotions of monomial ideals under permutation and monomial multiple.**

In this subsection, we investigate how the demotion property behaves under permutations and the monomial multiple operation in Propositions 5.9 and 5.10, respectively. Prior to presenting Proposition 5.9, we first need to recall the following definition.

Definition 5.8. *([18, Definition 11])* Let $I \subset R = K[x_1, \ldots, x_n]$ be a square-free monomial ideal and σ a permutation on $\mathrm{supp}(I)$. Then we define the $\sigma(I)$ as the square-free monomial ideal in R such that $x_{i_1} \cdots x_{i_s} \in \mathcal{G}(I)$ if and only if $x_{\sigma(i_1)} \cdots x_{\sigma(i_s)} \in \mathcal{G}(\sigma(I))$.

Proposition 5.9. *With the notation of Definition 5.8, a square-free monomial ideal J is a demotion of a square-free monomial ideal I if and only if $\sigma(J)$ is a demotion of $\sigma(I)$.*

Proof. Let $\mathrm{supp}(I) = \{x_1, \ldots, x_n\}$, and fix $r, s \geq 0$. Consider the K-algebra homomorphism $\varphi : R = K[x_1, \ldots, x_n] \to R$ given by $\varphi(x_i) = x_{\sigma(i)}$ for all $i = 1, \ldots, n$. It is straightforward to see that φ is an automorphism of R with $\varphi(I) = \sigma(I)$. Particularly, it is clear that $J \subseteq I$ if and only if $\sigma(J) \subseteq \sigma(I)$.

($\Rightarrow$) Assume $I^{r+s} \cap J^s = I^r J^s$ holds. We want to show that

$$(\sigma(I))^{r+s} \cap (\sigma(J))^s = (\sigma(I))^r(\sigma(J))^s.$$

Apply φ to both sides of $I^{r+s} \cap J^s = I^r J^s$, we get $\varphi(I^{r+s} \cap J^s) = \varphi(I^r J^s)$. Due to φ is an automorphism of R, we have the following equalities

$$\varphi(I^{r+s} \cap J^s) = \varphi(I^{r+s}) \cap \varphi(J^s) = (\sigma(I))^{r+s} \cap (\sigma(J))^s, \text{and}$$

$$\varphi(I^r J^s) = \varphi(I^r)\varphi(J^s) = (\sigma(I))^r(\sigma(J))^s.$$

Thus, we can conclude that $(\sigma(I))^{r+s} \cap (\sigma(J))^s = (\sigma(I))^r(\sigma(J))^s$.

($\Leftarrow$) Assume $(\sigma(I))^{r+s} \cap (\sigma(J))^s = (\sigma(I))^r(\sigma(J))^s$. Our aim is to establish $I^{r+s} \cap J^s = I^r J^s$. As φ is an isomorphism, it has an inverse φ^{-1}. Apply φ^{-1} to both sides, we obtain $\varphi^{-1}\left((\sigma(I))^{r+s} \cap (\sigma(J))^s\right) = \varphi^{-1}\left((\sigma(I))^r(\sigma(J))^s\right)$. This implies that $\varphi^{-1}((\sigma(I))^{r+s}) \cap \varphi^{-1}((\sigma(J))^s) = I^{r+s} \cap J^s$ and moreover $\varphi^{-1}((\sigma(I))^r)\varphi^{-1}((\sigma(J))^s) = I^r J^s$. Hence, $I^{r+s} \cap J^s = I^r J^s$. □

Proposition 5.10. *Let $J \subseteq I$ be monomial ideals in $R = K[x_1, \ldots, x_n]$, and h be a monomial in R such that* $\operatorname{supp}(h) \cap (\operatorname{supp}(I) \cup \operatorname{supp}(J)) = \emptyset$. *Then J is a demotion of I if and only if hJ is a demotion of hI.*

Proof. Since K is a field, we obtain $R = K[x_1, \ldots, x_n]$ is an integral domain, and so it is easy to see that $J \subseteq I$ if and only if $hJ \subseteq hI$. Fix $r, s \geq 0$. Due to $\operatorname{supp}(h) \cap (\operatorname{supp}(I) \cup \operatorname{supp}(J)) = \emptyset$, we get the following equalities

$$\begin{aligned} h^{r+s}(I^{r+s} \cap J^s) &= (h^{r+s}) \cap (I^{r+s} \cap J^s) \\ &= ((h)^{r+s} \cap I^{r+s}) \cap ((h)^s \cap J^s) \\ &= (hI)^{r+s} \cap (hJ)^s. \end{aligned}$$

On the other hand, we always have $(hI)^r(hJ)^s = h^{r+s}I^r J^s$. On account of $R = K[x_1, \ldots, x_n]$ is an integral domain, we can rapidly deduce that $I^{r+s} \cap J^s = I^r J^s$ if and only if $(hI)^{r+s} \cap (hJ)^s = (hI)^r(hJ)^s$. This means that J is a demotion of I if and only if hJ is a demotion of hI. □

5.4. **Demotions of monomial ideals under expansion.**

We begin this subsection by recalling the definition of the expansion operation on monomial ideals, as introduced in [1].

Let K be a field and consider the polynomial ring $R = K[x_1, \ldots, x_n]$ over K in the variables $x_1, \ldots, x_n$. Fix an ordered n-tuple $(i_1, \ldots, i_n)$ of positive integers, and define the polynomial ring $R^{(i_1,\ldots,i_n)}$ over K with variables

$$x_{11}, \ldots, x_{1i_1},\ x_{21}, \ldots, x_{2i_2},\ \ldots,\ x_{n1}, \ldots, x_{ni_n}.$$

For each $j = 1, \ldots, n$, let $\mathfrak{p}_j$ denote the monomial prime ideal

$$\mathfrak{p}_j = (x_{j1}, x_{j2}, \ldots, x_{ji_j}) \subseteq R^{(i_1,\ldots,i_n)}.$$

Now, let $I \subset R$ be a monomial ideal with a minimal generating set $\mathcal{G}(I) = \{\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_m}\}$, where $\mathbf{x}^{\mathbf{a}_i} = x_1^{a_i(1)} \cdots x_n^{a_i(n)}$, and $\mathbf{a}_i = (a_i(1), \ldots, a_i(n))$ with $a_i(j)$ denoting the jth component of $\mathbf{a}_i$ for $i = 1, \ldots, m$.

The *expansion of I with respect to the n-tuple* $(i_1, \dots, i_n)$, denoted by $I^{(i_1,\dots,i_n)}$, is defined as the monomial ideal

$$I^{(i_1,\dots,i_n)} = \sum_{i=1}^{m} \mathfrak{p}_1^{a_i(1)} \cdots \mathfrak{p}_n^{a_i(n)} \subseteq R^{(i_1,\dots,i_n)}.$$

For simplicity, we will often write R^* and I^* instead of $R^{(i_1,\dots,i_n)}$ and $I^{(i_1,\dots,i_n)}$, respectively.

For example, assume that $R = K[x_1, x_2, x_3, x_4]$ and consider the ordered 4-tuple $(2, 3, 1, 2)$. We then obtain

$$\mathfrak{p}_1 = (x_{11}, x_{12}), \quad \mathfrak{p}_2 = (x_{21}, x_{22}, x_{23}), \quad \mathfrak{p}_3 = (x_{31}), \quad \mathfrak{p}_4 = (x_{41}, x_{42}).$$

Now, for the monomial ideal $I = (x_1^2x_2, x_1x_3, x_2x_4^2) \subset R$, the expanded ideal $I^* \subseteq K[x_{11}, x_{12}, x_{21}, x_{22}, x_{23}, x_{31}, x_{41}, x_{42}]$ is given by $I^* = \mathfrak{p}_1^2\mathfrak{p}_2 + \mathfrak{p}_1\mathfrak{p}_3 + \mathfrak{p}_2\mathfrak{p}_4^2$, namely,

$$\begin{aligned} I^* = (&x_{11}^2x_{21},\, x_{11}^2x_{22},\, x_{11}^2x_{23},\, x_{12}^2x_{21},\, x_{12}^2x_{22},\, x_{12}^2x_{23},\\ &x_{11}x_{31},\, x_{12}x_{31},\, x_{21}x_{41}^2,\, x_{21}x_{41}x_{42},\, x_{21}x_{42}^2,\, x_{22}x_{41}^2,\\ &x_{22}x_{41}x_{42},\, x_{22}x_{42}^2,\, x_{23}x_{41}^2,\, x_{23}x_{41}x_{42},\, x_{23}x_{42}^2).\end{aligned}$$

To establish Proposition 5.12, we first make use of the following auxiliary result.

Lemma 5.11. *([1, Lemma 1.1]) Let I and J be monomial ideals in a polynomial ring S. Then*

(i) *$f \in I^*$ if and only if $\pi(f) \in I$, for all $f \in S^*$;*
(ii) *$(I + J)^* = I^* + J^*$;*
(iii) *$(IJ)^* = I^*J^*$;*
(iv) *$(I \cap J)^* = I^* \cap J^*$;*
(v) *$(I : J)^* = (I^* : J^*)$;*
(vi) *$\sqrt{I^*} = (\sqrt{I})^*$;*
(vii) *If the monomial ideal Q is $\mathfrak{p}$-primary, then Q^* is $\mathfrak{p}^*$-primary.*

The next proposition states that a monomial ideal has the demotion property if and only if its expansion has the demotion property.

Proposition 5.12. *Let $J \subseteq I$ be monomial ideals in $R = K[x_1, \dots, x_n]$. Then J is a demotion of I if and only if J^* is a demotion of I^*, where I^* denotes the expansion of I.*

Proof. Suppose that J is a demotion of I, and fix $r, s \geq 0$. Then we have $I^rJ^s = I^{r+s} \cap J^s$, and so $(I^rJ^s)^* = (I^{r+s} \cap J^s)^*$. We also deduce from Lemma 5.11(ii) that $J^* \subseteq I^*$. In light of parts (iii) and (iv) of Lemma 5.11, one can derive that $(I^*)^r(J^*)^s = (I^*)^{r+s} \cap (J^*)^s$. Accordingly, J^* is a demotion of I^*, as required.

The converse can be shown by a similar argument and recalling this fact that for any two monomial ideals L_1, L_2, we always have $L_1 = L_2$ if and only if $L_1^* = L_2^*$. $\square$

5.5. **Demotions of monomial ideals under weighting.**

This subsection is devoted to analyzing the demotion property in the context of the weighting operation. For this purpose, we first revisit the relevant definition.

Definition 5.13. A *weight* over a polynomial ring $R = K[x_1, \ldots, x_n]$ is a function $W : \{x_1, \ldots, x_n\} \to \mathbb{N}$ such that $w_i = W(x_i)$ is called the *weight* of the variable x_i. For a monomial ideal $I \subset R$ and a weight W, we define the *weighted ideal*, denoted by I_W, to be the ideal generated by $\{h(u) : u \in \mathcal{G}(I)\}$, where h is the unique homomorphism $h : R \to R$ given by $h(x_i) = x_i^{w_i}$.

As an example, let $R = K[x_1, x_2, x_3, x_4]$ and consider the monomial ideal $I = (x_1^2x_2,\ x_2^3x_3^2,\ x_1x_3x_4^2) \subset R$. Also, let $W : \{x_1, x_2, x_3, x_4\} \to \mathbb{N}$ be a weight over R defined by $W(x_1) = 2, W(x_2) = 3, W(x_3) = 1$, and $W(x_4) = 4$. Then, the weighted ideal I_W is given by $I_W = (x_1^4x_2^3,\ x_2^9x_3^2,\ x_1^2x_3x_4^8)$.

The proof of Proposition 5.15 relies on the application of the following lemma.

Lemma 5.14. *([19, Lemma 3.5]) Let I and J be two monomial ideals of a polynomial ring $R = K[x_1, \ldots, x_n]$, and W a weight over R. Then the following statements hold.*

(i) $(I + J)_W = I_W + J_W$;
(ii) $(IJ)_W = I_WJ_W$;
(iii) $(I \cap J)_W = I_W \cap J_W$;
(iv) $(I :_R J)_W = (I_W :_R J_W)$.

A monomial ideal has the demotion property if and only if its weighted ideal does, as stated in the following proposition.

Proposition 5.15. *Let $I \subset R = K[x_1, \ldots, x_n]$ be a monomial ideal, and W a weight over R. Then J is a demotion of I if and only if J_W is a demotion of I_W, where I_W denotes the weighted ideal.*

Proof. Assume that J is a demotion of I, and fix $r, s \geq 0$. We thus get $I^rJ^s = I^{r+s} \cap J^s$, and hence $(I^rJ^s)_W = (I^{r+s} \cap J^s)_W$. On account of Lemma 5.14(i), we obtain $J_W \subseteq I_W$. We can immediately conclude from parts (ii) and (iii) of Lemma 5.14 that $(I_W)^r(J_W)^s = (I_W)^{r+s} \cap (J_W)^s$. Therefore, J_W is a demotion of I_W, as claimed.

By remembering this fact that for any two monomial ideals L_1, L_2, we always have $L_1 = L_2$ if and only if $(L_1)_W = (L_2)_W$, a similar discussion can be used to establish the converse. □

As a consequence of Propositions 5.12 and 5.15, the theorem below ensures that there exist infinitely many monomial ideals in $R = K[x_1, \ldots, x_n]$ such that have proper demotion ideals.

Theorem 5.16. *Let $R = K[x_1, \dots, x_n]$ be a polynomial ring in n variables, where $n \geq 2$, with coefficients in a field K. Then there exist infinitely many monomial ideals in R such that have proper demotion ideals.*

Proof. Let $S = K[x, y]$, and consider the monomial ideals $I = (x)$ and $J = (xy)$ is S. Clearly $J \subsetneq I$. We claim J is a demotion of I. Fix $r, s \geq 0$. It is easy to check that $I^r J^s = (x^r)(x^s y^s) = (x^{r+s} y^s)$. On the other hand,

$$I^{r+s} \cap J^s = (x^{r+s}) \cap (x^s y^s) = (\mathrm{lcm}(x^{r+s}, x^s y^s)) = (x^{r+s} y^s).$$

Thus, $I^r J^s = I^{r+s} \cap J^s$, and so J is a proper demotion of I. Now, let $\mathfrak{p}_1 := (x_{i_1}, \dots, x_{i_a})$ and $\mathfrak{p}_2 := (x_{i_{a+1}}, \dots, x_{i_b})$ with $\mathrm{supp}(\mathfrak{p}_1) \cap \mathrm{supp}(\mathfrak{p}_2) = \emptyset$ and $\mathrm{supp}(\mathfrak{p}_1) \cup \mathrm{supp}(\mathfrak{p}_2) = \{x_1, \dots, x_n\}$. One can promptly deduce from Proposition 5.12 that J^* is a demotion of I^* in $R = K[x_1, \dots, x_n]$. Now, assume that W is a weight over $R = K[x_1, \dots, x_n]$ with $W(x_i) = \alpha_i$ such that $\alpha_i \geq 1$ for all $i = 1, \dots, n$. In light of Proposition 5.15, we obtain $(J^*)_W$ is a demotion of $(I^*)_W$. This indicates that there exist infinitely many monomial ideals in $R = K[x_1, \dots, x_n]$ such that have proper demotion ideals, as claimed. □

6. Demotions of monomial ideals and normally torsion-freeness

The primary aim of this section is to investigate the relationship between demotions of monomial ideals and normally torsion-freeness. In particular, we demonstrate that the demotion property can be used to construct new normally torsion-free monomial ideals, and conversely, that normally torsion-free ideals can inform the generation of ideals with the demotion property. To do this, we commence with the following proposition.

Proposition 6.1. *Let $I \subset R = K[x_1, \dots, x_n]$ be a square-free monomial ideal and $\mathfrak{q}$ be a prime monomial ideal in R such that $\bigcap_{\mathfrak{p} \in \mathrm{Ass}(I)} \mathfrak{p} \cap \mathfrak{q}$ is a minimal primary decomposition of $I \cap \mathfrak{q}$. Let I and $I \cap \mathfrak{q}$ be normally torsion-free. Then $I \cap \mathfrak{q}$ is a demotion of I.*

Proof. Set $J := I \cap \mathfrak{q}$. We must show that $I^r J^s = I^{r+s} \cap J^s$ for all $r, s \geq 0$. To do this, fix $r, s \geq 0$. Assume that $\mathrm{Ass}(I) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_t\}$. Then $I = \bigcap_{i=1}^t \mathfrak{p}_i$ and $J = \bigcap_{i=1}^t \mathfrak{p}_i \cap \mathfrak{q}$, and hence $J \subseteq I$. In addition, Facts 2.2, 2.3 and Theorem 2.5 yield that $I^{r+s} = \bigcap_{i=1}^t \mathfrak{p}_i^{r+s}$, $I^r = \bigcap_{i=1}^t \mathfrak{p}_i^r$, and $J^s = \bigcap_{i=1}^t \mathfrak{p}_i^s \cap \mathfrak{q}^s$. Here, by using Lemma 2.8, we obtain the following equalities

$$\begin{aligned} I^{r+s} \cap J^s &= \left(\bigcap_{i=1}^{t} \mathfrak{p}_i^{r+s}\right) \cap \left(\bigcap_{i=1}^{t} \mathfrak{p}_i^{s} \cap \mathfrak{q}^s\right) \\ &= \left(\bigcap_{i=1}^{t} \mathfrak{p}_i^{r+s}\right) \cap \mathfrak{q}^s \\ &= \left(\bigcap_{i=1}^{t} \mathfrak{p}_i^{r}\right) \left(\bigcap_{i=1}^{t} \mathfrak{p}_i^{s} \cap \mathfrak{q}^s\right) = I^r J^s. \end{aligned}$$

This gives that J is a demotion of I, as claimed. $\square$

Theorem 6.2. *Suppose that I and J are normally torsion-free square-free monomial ideals in $R = K[x_1, \ldots, x_n]$ satisfying the following*

(i) *J is a proper demotion of I;*
(ii) *if $\mathfrak{q} \in \mathrm{Ass}(R/J) \setminus \mathrm{Ass}(R/I)$, then $\mathfrak{q} \nsubseteq \mathfrak{p}$ for every $\mathfrak{p} \in \mathrm{Ass}(R/I)$.*

Then, for all $r, s \geq 1$, $I^r J^s$ is normally torsion-free.

Proof. Fix $r, s \geq 1$. As J is a demotion of I, we obtain $I^r J^s = I^{r+s} \cap J^s$. Let $\mathrm{Ass}(R/I) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_t\}$ and put $\Gamma := \mathrm{Ass}(R/J) \setminus \mathrm{Ass}(R/I)$. Let $\mathrm{Ass}(R/J) = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_m, \mathfrak{q}_{m+1}, \ldots, \mathfrak{q}_\ell\}$ and $\Gamma = \{\mathfrak{q}_1, \ldots, \mathfrak{q}_m\}$. In what follows, we claim that $\mathrm{Ass}(I^{r+s} \cap J^s) = \mathrm{Min}(I) \cup \Gamma$. We can conclude from Facts 2.2, 2.3 and Theorem 2.5 that $I^{r+s} = \bigcap_{i=1}^t \mathfrak{p}_i^{r+s}$ and $J^s = \bigcap_{i=1}^\ell \mathfrak{q}_i^s$. Let $\mathfrak{q} \in \mathrm{Ass}(R/J) \setminus \Gamma$. Then $\mathfrak{q} \in \mathrm{Ass}(R/J) \cap \mathrm{Ass}(R/I)$, say $\mathfrak{q} = \mathfrak{p}_1$. This implies that $\mathfrak{p}_1^{r+s} \cap \mathfrak{q}^s = \mathfrak{p}_1^{r+s}$. This allows us to obtain $I^{r+s} \cap J^s = \bigcap_{i=1}^t \mathfrak{p}_i^{r+s} \cap \bigcap_{i=1}^m \mathfrak{q}_i^s$. To show our claim, it is enough to establish $\bigcap_{i=1}^t \mathfrak{p}_i^{r+s} \cap \bigcap_{i=1}^m \mathfrak{q}_i^s$ is a minimal primary decomposition of $I^{r+s} \cap J^s$. To achieve this, we have to prove the following statements:

(i) $\bigcap_{i=1}^t \mathfrak{p}_i^{r+s} \cap \bigcap_{i=1, i\neq j}^m \mathfrak{q}_i^s \nsubseteq \mathfrak{q}_j^s$ for all $j = 1, \ldots, m$.
(ii) $\bigcap_{i=1, i\neq j}^t \mathfrak{p}_i^{r+s} \cap \bigcap_{i=1}^m \mathfrak{q}_i^s \nsubseteq \mathfrak{p}_j^{r+s}$ for all $j = 1, \ldots, t$.

(i) On the contrary, assume that $\bigcap_{i=1}^t \mathfrak{p}_i^{r+s} \cap \bigcap_{i=1, i\neq j}^m \mathfrak{q}_i^s \subseteq \mathfrak{q}_j^s$ for some $1 \leq j \leq m$. Hence, we obtain $\sqrt{\bigcap_{i=1}^t \mathfrak{p}_i^{r+s} \cap \bigcap_{i=1, i\neq j}^m \mathfrak{q}_i^s} \subseteq \sqrt{\mathfrak{q}_j^s}$. This leads to $\bigcap_{i=1}^t \mathfrak{p}_i \cap \bigcap_{i=1, i\neq j}^m \mathfrak{q}_i \subseteq \mathfrak{q}_j$. We thus get $\mathfrak{p}_\alpha \subseteq \mathfrak{q}_j$ for some $1 \leq \alpha \leq t$, or $\mathfrak{q}_\beta \subseteq \mathfrak{q}_j$ for some $1 \leq \beta \neq j \leq m$. We first assume that $\mathfrak{q}_\beta \subseteq \mathfrak{q}_j$ for some $1 \leq \beta \neq j \leq m$. Due to J is a square-free monomial ideal, this implies that $\mathrm{Ass}(J) = \mathrm{Min}(J)$, and hence $\mathfrak{q}_\beta \nsubseteq \mathfrak{q}_j$ for every $1 \leq \beta \neq j \leq m$, a contradiction. So, let $\mathfrak{p}_\alpha \subseteq \mathfrak{q}_j$ for some $1 \leq \alpha \leq t$. Since $J \subsetneq I \subseteq \mathfrak{p}_\alpha \subseteq \mathfrak{q}_j$ and $\mathfrak{q}_j \in \mathrm{Min}(J)$, we must have $\mathfrak{q}_j = \mathfrak{p}_\alpha$. This gives that $\mathfrak{q}_j \in \mathrm{Ass}(R/I)$, which is a contradiction. Consequently, we have $\bigcap_{i=1}^t \mathfrak{p}_i^{r+s} \cap \bigcap_{i=1, i\neq j}^m \mathfrak{q}_i^s \nsubseteq \mathfrak{q}_j^s$ for all $j = 1, \ldots, m$.

(ii) Suppose, on the contrary, that there exists some $1 \leq j \leq t$ such that $\bigcap_{i=1, i\neq j}^t \mathfrak{p}_i^{r+s} \cap \bigcap_{i=1}^m \mathfrak{q}_i^s \subseteq \mathfrak{p}_j^{r+s}$. Without loss of generality, assume that

$\bigcap_{i=1}^{t-1}\mathfrak{p}_i^{r+s}\cap\bigcap_{i=1}^{m}\mathfrak{q}_i^{s}\subseteq\mathfrak{p}_t^{r+s}$. Since $\bigcap_{i=1}^{t}\mathfrak{p}_i$ is a minimal primary decomposition of I, we can derive that $\bigcap_{i=1}^{t-1}\mathfrak{p}_i\nsubseteq\mathfrak{p}_t$. This yields that there exists some monomial $u\in\bigcap_{i=1}^{t-1}\mathfrak{p}_i\setminus\mathfrak{p}_t$. As $(\bigcap_{i=1}^{t-1}\mathfrak{p}_i)^{r+s}\subseteq\bigcap_{i=1}^{t-1}\mathfrak{p}_i{}^{r+s}$, we have $u^{r+s}\in\bigcap_{i=1}^{t-1}\mathfrak{p}_i^{r+s}\setminus\mathfrak{p}_t^{r+s}$. We here claim that $\bigcap_{i=1}^{m}\mathfrak{q}_i^{s}\nsubseteq\mathfrak{p}_t^{r+s}$. Otherwise, $\bigcap_{i=1}^{m}\mathfrak{q}_i^{s}\subseteq\mathfrak{p}_t^{r+s}$. Hence, $\bigcap_{i=1}^{m}\mathfrak{q}_i\subseteq\mathfrak{p}_t$. This implies that $\mathfrak{q}_z\subseteq\mathfrak{p}_t$ for some $1\leq z\leq m$. Because $\mathfrak{q}_z\in\Gamma$, so we must have $\mathfrak{q}_z\nsubseteq\mathfrak{p}_t$, a contradiction. Accordingly, $\bigcap_{i=1}^{m}\mathfrak{q}_i^{s}\nsubseteq\mathfrak{p}_t^{r+s}$. Let $v\in\bigcap_{i=1}^{m}\mathfrak{q}_i^{s}\setminus\mathfrak{p}_t^{r+s}$. Since $u^{r+s}\in\bigcap_{i=1}^{t-1}\mathfrak{p}_i^{r+s}$, this gives that $u^{r+s}v\in\bigcap_{i=1}^{t-1}\mathfrak{p}_i^{r+s}\cap\bigcap_{i=1}^{m}\mathfrak{q}_i^{s}$, and so $u^{r+s}v\in\mathfrak{p}_t^{r+s}$. Based on [21, Corollary 6.1.8], it is well-known that $\mathfrak{p}_t^{r+s}$ is $\mathfrak{p}_t$-primary. It follows now from $u^{r+s}v\in\mathfrak{p}_t^{r+s}$ that $v\in\mathfrak{p}_t^{r+s}$ or $u^{r+s}\in\sqrt{\mathfrak{p}_t^{r+s}}$. Since $v\notin\mathfrak{p}_t^{r+s}$, we must have $u^{r+s}\in\sqrt{\mathfrak{p}_t^{r+s}}$. On account of $\sqrt{\mathfrak{p}_t^{r+s}}=\mathfrak{p}_t$ and $\mathfrak{p}_t$ is a monomial prime ideal, we get $u\in\mathfrak{p}_t$, which is a contradiction. Therefore, $\bigcap_{i=1}^{t}\mathfrak{p}_i^{r+s}\cap\bigcap_{i=1}^{m}\mathfrak{q}_i^{s}$ is a minimal primary decomposition of $I^{r+s}\cap J^s$, and thus $\mathrm{Ass}(I^rJ^s)=\mathrm{Ass}(I^{r+s}\cap J^s)=\mathrm{Min}(I)\cup\Gamma$ for all $r,s\geq 1$. To complete the proof, put $L:=I^rJ^s$ and let $\ell\geq 1$. Because $L^\ell=I^{r\ell}J^{s\ell}$, the argument above implies that $\mathrm{Ass}(L^\ell)=\mathrm{Min}(I)\cup\Gamma=\mathrm{Ass}(L)$. Consequently, we can deduce that L is normally torsion-free, as required. □

Here, we present the next result from this section, which deals with the relation between the demotion property and normally torsion-freeness.

Theorem 6.3. *Suppose that I and J are square-free monomial ideals in $R=K[x_1,\ldots,x_n]$ such that $I=J+(x_\alpha x_\beta)$, where $1\leq\alpha\neq\beta\leq n$. Then the following statements hold:*

(i) *if J is normally torsion-free, then J is a proper demotion of I.*
(ii) *if both I and J are normally torsion-free, $h:=\prod_{i=1}^{m}x_{n+i}$ and $S=R[x_{n+1},\ldots,x_{n+m}]$, then $L:=JS+hx_\alpha x_\beta S$ is a normally torsion-free square-free monomial ideal in S.*

Proof. (i) Set $J:=(u_1,\ldots,u_t)$. We want to show that $I^rJ^s=I^{r+s}\cap J^s$ for all $r,s\geq 0$. Clearly, the claim is true for $r=0$ or $s=0$ or both of them, so we fix $r,s\geq 1$. Due to $I=J+(x_\alpha x_\beta)$, this implies the following equalities

$$\begin{aligned}I^{r+s}\cap J^s=&(J+(x_\alpha x_\beta))^{r+s}\cap J^s\\=&(\sum_{c=0}^{r+s}J^c(x_\alpha x_\beta)^{r+s-c})\cap J^s\\=&\sum_{c=0}^{r+s}(J^c(x_\alpha x_\beta)^{r+s-c}\cap J^s)\\=&\sum_{c=0}^{s-1}(J^c(x_\alpha x_\beta)^{r+s-c}\cap J^s)+\sum_{c=s}^{r+s}(J^c(x_\alpha x_\beta)^{r+s-c}\cap J^s).\end{aligned}$$

Let $s \leq c \leq r+s$. Hence, $J^c(x_\alpha x_\beta)^{r+s-c} \subseteq J^s$, and we therefore get

$$\sum_{c=s}^{r+s}(J^c(x_\alpha x_\beta)^{r+s-c} \cap J^s) = \sum_{c=s}^{r+s} J^c(x_\alpha x_\beta)^{r+s-c} = J^s \sum_{\rho=0}^{r} J^\rho (x_\alpha x_\beta)^{r-\rho} = J^s I^r.$$

We claim that $J^c(x_\alpha x_\beta)^{r+s-c} \cap J^s \subseteq J^s(x_\alpha x_\beta)^r$ for each $0 \leq c \leq s$. Let $0 \leq c \leq s$. By virtue of J is a normally torsion-free square-free monomial ideal, Facts 2.2, 2.3 and Theorem 2.5 imply that $J^c = J^{(c)} = \bigcap_{\mathfrak{p} \in \mathrm{Min}(J)} \mathfrak{p}^c$ and $J^s = J^{(s)} = \bigcap_{\mathfrak{p} \in \mathrm{Min}(J)} \mathfrak{p}^s$. By part (a) of Fact 2.9, we can deduce $J^c(x_\alpha x_\beta)^{r+s-c} = \bigcap_{\mathfrak{p} \in \mathrm{Min}(J)}((x_\alpha x_\beta)^{r+s-c}\mathfrak{p}^c)$, and so the following equalities

$$\begin{aligned} J^c(x_\alpha x_\beta)^{r+s-c} \cap J^s &= \left((x_\alpha x_\beta)^{r+s-c} \bigcap_{\mathfrak{p} \in \mathrm{Min}(J)} \mathfrak{p}^c\right) \cap \left(\bigcap_{\mathfrak{p} \in \mathrm{Min}(J)} \mathfrak{p}^s\right) \\ &= \left(\bigcap_{\mathfrak{p} \in \mathrm{Min}(J)} ((x_\alpha x_\beta)^{r+s-c}\mathfrak{p}^c)\right) \cap \left(\bigcap_{\mathfrak{p} \in \mathrm{Min}(J)} \mathfrak{p}^s\right) \\ &= \bigcap_{\mathfrak{p} \in \mathrm{Min}(J)} \left(((x_\alpha x_\beta)^{r+s-c}\mathfrak{p}^c) \cap \mathfrak{p}^s\right). \end{aligned}$$

Choose an arbitrary element $\mathfrak{p} \in \mathrm{Min}(J)$. In what follows, our aim is to prove $((x_\alpha x_\beta)^{r+s-c}\mathfrak{p}^c) \cap \mathfrak{p}^s \subseteq (x_\alpha x_\beta)^r \mathfrak{p}^s$. Take $v \in \mathcal{G}(((x_\alpha x_\beta)^{r+s-c}\mathfrak{p}^c) \cap \mathfrak{p}^s)$. Then $v = \mathrm{lcm}((x_\alpha x_\beta)^{r+s-c} f, g)$, where $f \in \mathcal{G}(\mathfrak{p}^c)$ and $g \in \mathcal{G}(\mathfrak{p}^s)$. Because $f \in \mathcal{G}(\mathfrak{p}^c)$ (respectively, $g \in \mathcal{G}(\mathfrak{p}^s)$), one may write $f = \prod_{i=1}^n x_i^{e_i}$ (respectively, $g = \prod_{i=1}^n x_i^{d_i}$) such that $e_i \geq 0$ for all $i = 1, \ldots, n$, $\sum_{i=1}^n e_i = c$, and $x_i \in \mathfrak{p}$ when $e_i > 0$ (respectively, $d_i \geq 0$ for all $i = 1, \ldots, n$, $\sum_{i=1}^n d_i = s$, and $x_i \in \mathfrak{p}$ when $d_i > 0$). It follows readily from $v = \mathrm{lcm}((x_\alpha x_\beta)^{r+s-c} f, g)$ that

$$v = \left(\prod_{i=1, i \notin \{\alpha,\beta\}}^{n} x_i^{\max\{e_i, d_i\}}\right) x_\alpha^{\max\{r+s-c+e_\alpha, d_\alpha\}} x_\beta^{\max\{r+s-c+e_\beta, d_\beta\}}.$$

Since $\deg_{x_\alpha} v = \max\{r+s-c+e_\alpha, d_\alpha\}$ and $\deg_{x_\beta} v = \max\{r+s-c+e_\beta, d_\beta\}$, we get $\deg_{x_\alpha} v \geq r+s-c+e_\alpha \geq r$ and $\deg_{x_\beta} v \geq r+s-c+e_\beta \geq r$. By virtue of $\sum_{i=1}^n e_i = c$ and $\deg_{x_i} v = \max\{e_i, d_i\}$ for all $i \in \{1, \ldots, n\} \setminus \{\alpha, \beta\}$, we can derive

$$\sum_{i=1}^{n} \deg_{x_i} v \geq \sum_{i=1, i \notin \{\alpha,\beta\}}^{n} e_i + r + s - c + e_\alpha + r + s - c + e_\beta \geq s + 2r.$$

From $\deg_{x_\alpha} v \geq r$, $\deg_{x_\beta} v \geq r$, and $\sum_{i=1}^n \deg_{x_i} v \geq s + 2r$, we have $v \in (x_\alpha x_\beta)^r \mathfrak{p}^s$. This gives $((x_\alpha x_\beta)^{r+s-c}\mathfrak{p}^c) \cap \mathfrak{p}^s \subseteq (x_\alpha x_\beta)^r \mathfrak{p}^s$ for all $\mathfrak{p} \in \mathrm{Min}(J)$, and so $J^c(x_\alpha x_\beta)^{r+s-c} \cap J^s \subseteq J^s(x_\alpha x_\beta)^r$ for all $0 \leq c \leq s$. This yields that $I^r J^s = I^{r+s} \cap J^s$ for all $r, s \geq 0$, that is, J is a proper demotion of I.

(ii) According to Theorem 2.5, it is enough to show that $L^k = L^{(k)}$ for all $k \geq 1$. Fix $k \geq 1$. Since $L = JS + hx_\alpha x_\beta S$ and $\mathrm{supp}(h) \cap \mathrm{supp}(I) = \emptyset$, it follows from Fact 2.4 that $L = (J, h) \cap (J, x_\alpha x_\beta) = J + hI$. On account of Proposition 2.6, we get $L^{(k)} = (J, h)^{(k)} \cap (J, x_\alpha x_\beta)^{(k)}$. In light of J is normally torsion-free and $\mathrm{supp}(h) \cap \mathrm{supp}(J) = \emptyset$, one can deduce from Theorem 2.7 that (J, h) is normally torsion-free, and hence $(J, h)^{(k)} = (J, h)^k$. Since I is normally torsion-free, we get $(J, x_\alpha x_\beta)^{(k)} = (J, x_\alpha x_\beta)^k$. This gives rise to $L^{(k)} = (J, h)^k \cap (J, x_\alpha x_\beta)^k$. It follows now from part (i), Fact 2.4 and the binomial expansion theorem that

$$\begin{aligned}(J,h)^k \cap (J, x_\alpha x_\beta)^k =&(J,h)^k \cap I^k = \left(\sum_{\alpha=0}^{k} h^{k-\alpha} J^\alpha\right) \cap I^k \\ =&\sum_{\alpha=0}^{k} (h^{k-\alpha} J^\alpha \cap I^k) \overset{\text{(i)}}{=} \sum_{\alpha=0}^{k} h^{k-\alpha} J^\alpha I^{k-\alpha} \\ =&(J+hI)^k \\ =&L^k.\end{aligned}$$

Therefore, we get $L^k = L^{(k)}$, and Theorem 2.5 yields that L is normally torsion-free, as required. $\square$

Example 6.4. Let G be a bipartite graph with $V(G) = \{x_1, \ldots, x_n\}$ and $I = (u_1, \ldots, u_t, u_{t+1})$ be its edge ideal. Put $J := (u_1, \ldots, u_t)$. Based on Theorem 6.3, J is a demotion of I.

7. Comparing reductions and demotions of ideals

This section is mainly devoted to a comparison between reductions and demotions of ideals. We start by revisiting the definition of reductions of ideals, and then establish that a number of its characteristic properties are not preserved in the case of demotions of ideals. For this purpose, we present several counterexamples.

Definition 7.1. *([7, Definition 1.2.1])* Let $J \subseteq I$ be ideals. Then J is said to be a *reduction* of I if there exists a nonnegative integer n such that $I^{n+1} = JI^n$.

Example 7.2. (i) Let $R = K[x_1, \ldots, x_7]$, $J := (x_2x_4, x_2x_5, x_1x_4, x_5x_6, x_4x_7)$ and $I := J + (x_1x_3)$. Consider the graph $G = (V(G), E(G))$ with vertex set $V(G) = \{x_1, x_2, x_4, x_5, x_6, x_7\}$ and the following edge set

$$E(G) = \{\{x_2, x_4\}, \{x_2, x_5\}, \{x_1, x_4\}, \{x_5, x_6\}, \{x_4, x_7\}\}.$$

Since J is the edge ideal of the bipartite graph G, this implies that J is normally torsion-free. It follows now from Theorem 6.3 that J is a demotion of I. In what follows, we show $I^{n+1} \neq I^nJ$ for every $n \geq 0$. Consider the monomial $m = x_1^{n+1}x_3^{n+1} \in I^{n+1}$. Suppose, for contradiction, that $m \in I^nJ$. Then there exist monomials $f \in I^n$ and $g \in J$ such that $m = fg$. Note that f is a product of n generators of I, and each generator of I is either x_1x_3 or one of the generators of J. Therefore, f can only contain variables from $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$, but to produce $x_1^{n+1}x_3^{n+1}$ exactly, f must be a power of x_1x_3 alone. On the other hand, every generator of J contains at least one variable outside $\{x_1, x_3\}$, that is, x_2x_4, x_2x_5, x_1x_4, x_5x_6, x_4x_7. Hence, any $g \in J$ necessarily introduces a variable not in $\{x_1, x_3\}$. Multiplying f by such a g would produce a monomial containing an "extra" variable, which contradicts the fact that m involves only x_1 and x_3. Therefore, no such factorization $m = fg$ exists, and we conclude $m \notin I^nJ$. Consequently, $I^{n+1} \neq I^nJ$ for all $n \geq 0$. This shows that J is not a reduction of I.

(ii) Suppose that $I = (x^3, y^3, z^3, x^2y, xy^2, y^2z, yz^2, x^2z, xz^2)$ and $J = (x^3, y^3, z^3)$ in the polynomial ring $R = K[x, y, z]$. Using *Macaulay2* [5], we find that $I^3 = JI^2$, while $IJ^3 \neq I^4 \cap J^3$. Hence, we can deduce that J is a reduction of I, but J is not a demotion of I.

Proposition 7.3. *([7, Proposition 1.2.4]) Let $K \subseteq J \subseteq I$ be ideals in R.*

(i) *If K is a reduction of J and J is a reduction of I, then K is a reduction of I.*
(ii) *If K is a reduction of I, then J is a reduction of I.*
(iii) *If I is finitely generated, $J = K + (r_1, \ldots, r_k)$, and K is a reduction of I, then K is a reduction of J.*

Example 7.4. Let $I = (x_3, x_1x_2, x_4x_5x_6)$ be a square-free monomial ideal in $R = K[x_1, \ldots, x_8]$. According to Theorem 2.7, we can derive that I is normally torsion-free. Moreover, it is easy to check that

$$\begin{aligned} I =&(x_1, x_3, x_4) \cap (x_1, x_3, x_5) \cap (x_1, x_3, x_6) \cap (x_2, x_3, x_4) \cap (x_2, x_3, x_5) \\ &\cap (x_2, x_3, x_6). \end{aligned}$$

Now, set $J := I \cap (x_1, x_7)$ and $L := J \cap (x_4, x_8)$. It follows readily from Lemma 2.10 that both J and L are normally torsion-free as well. Using Proposition 6.1, we can conclude that J is a demotion of I, and L is a demotion of J. Since $(x_1x_2x_4x_5x_6)^2 \in I^4 \cap L^2 \setminus I^2L^2$, this implies that L is not a demotion of I. This demonstrates that statement (i) in Proposition 7.3 does not hold for demotions of ideals in general.

Example 7.5. Let $R = K[x_1, \ldots, x_7]$ and consider the following square-free monomial ideal in R

$$\begin{aligned} I =&(x_3, x_1x_2, x_4x_5x_6) = (x_1, x_3, x_4) \cap (x_1, x_3, x_5) \cap (x_1, x_3, x_6) \\ &\cap (x_2, x_3, x_4) \cap (x_2, x_3, x_5) \cap (x_2, x_3, x_6). \end{aligned}$$

Moreover, assume that $J := I \cap (x_4, x_5, x_7)$ and $L := I \cap (x_4, x_7)$. It is routine to check that $L \subseteq J \subseteq I$. We can deduce from Theorem 2.7 that I is normally torsion-free, and by Lemma 2.10, we obtain L is also normally torsion-free. It follows immediately from Proposition 6.1 that L is a demotion of I. On account of $x_3^3x_4x_5x_6 \in I^4 \cap J^2 \setminus I^2J^2$, we can derive that J is not a demotion of I. This shows that statement (ii) in Proposition 7.3 is, in general, not valid for demotions of ideals.

Lemma 7.6. *([7, Lemma 8.1.10]) Let $J \subseteq I$ be a reduction. Then $\sqrt{I} = \sqrt{J}$, $\mathrm{Min}(R/I) = \mathrm{Min}(R/J)$, and $\mathrm{ht}(I) = \mathrm{ht}(J)$.*

Example 7.7. Let $R = K[x_1, \ldots, x_7]$, $J := (x_2x_4, x_2x_5, x_1x_4, x_5x_6, x_4x_7)$ and $I := J + (x_1x_3)$. According to Example 7.2(i), J is a demotion of I. On the other hand, using *Macaulay2* [5], we detect that

$$\mathrm{Ass}(R/J) = \{(x_5, x_4), (x_6, x_4, x_2), (x_7, x_5, x_2, x_1), (x_7, x_6, x_2, x_1)\} \text{ and}$$

$$\mathrm{Ass}(R/I) = \{(x_5, x_4, x_1), (x_5, x_4, x_3), (x_6, x_4, x_2, x_1), (x_7, x_5, x_2, x_1),\\ (x_7, x_6, x_2, x_1), (x_6, x_4, x_3, x_2)\}.$$

Hence, we get $\mathrm{ht}(J) = 2$, $\mathrm{ht}(I) = 3$, $(x_4, x_5) \in \mathrm{Min}(J) \setminus \mathrm{Min}(I)$, and $(x_1, x_4, x_5) \in \mathrm{Min}(I) \setminus \mathrm{Min}(J)$. It is also obvious that $\sqrt{I} \neq \sqrt{J}$. This indicates that Lemma 7.6 may generally fail for demotions of ideals.

Proposition 7.8. *([7, Proposition 8.1.5]) Let $J = (a_1, \ldots, a_k) \subseteq I$ be ideals in a ring R.*

(1) *If J is a reduction of I, then for any positive integer m, $(a_1^m, \ldots, a_k^m)$ and J^m are reductions of I^m.*
(2) *If for some positive integer m, $(a_1^m, \ldots, a_k^m)$ or J^m is a reduction of I^m, then J is a reduction of I.*

We close this paper with the following example.

Example 7.9. Let $J = (x_2x_4, x_2x_5, x_1x_4, x_5x_6, x_4x_7)$ and $I = J + (x_1x_3)$ in $R = K[x_1, \ldots, x_7]$. It follows from Example 7.2(i) that J is a demotion of I. Now, put $L := ((x_2x_4)^2, (x_2x_5)^2, (x_1x_4)^2, (x_5x_6)^2, (x_4x_7)^2)$. Because $(x_1x_3)(x_2x_4)^2 \in I^3 \cap L \setminus I^2L$, we can conclude that L is not a demotion of I^2. This states that it is possible $J = (a_1, \ldots, a_k)$ is a demotion of I, but $(a_1^m, \ldots, a_k^m)$ is not a demotion of I^m for some $m \geq 1$ (see statement (1) in Proposition 7.8).

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