HIGHER TRACES OF LINEAR MAPS ON FINITE-DIMENSIONAL NORMED SPACES

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Abstract. We prove a unified trace–average formula for the k-th higher trace $\lambda_k(A) = \mathrm{tr}(\Lambda^k A)$ of a linear operator A on a finite-dimensional normed space. The formula averages the matrix coefficient $w \mapsto \langle (\Lambda^k A)w, w^* \rangle$ over the unit sphere of $\Lambda^k X$ against a probability measure η; it holds for *all* A if and only if the operator-valued average $T_\eta = \binom{N}{k} \int w \otimes w^* \, \mathrm{d}\eta$ equals the identity. Two natural choices of η satisfy this isotropy: (i) the hypersurface measure when a finite isometry group acts as an orthogonal 2-design on $\Lambda^k \mathbb{R}^N$; and (ii) the cone probability measure (no symmetry needed). We also identify a first-order obstruction for hypersurface averages at $k = 1$: only degree–2 spherical harmonics of the support function contribute.

1. Introduction and main results

Let X be an N-dimensional real normed space and $A \in \mathrm{End}(X)$. For $1 \leqslant k \leqslant N$, the *higher traces*

$$\lambda_k(A) \;:=\; \mathrm{tr}(\Lambda^k A),$$

are the coefficients of the characteristic polynomial

$$\det(I - tA) = \sum_{k=0}^{N} (-1)^k \lambda_k(A)\, t^k, \qquad \lambda_0 = 1.$$

There are several complementary ways to *understand* λ_k:

- **Exterior-power/k-volume distortion:** $\Lambda^k A$ acts on oriented k-volumes; $\lambda_k(A)$ is the trace of this action and equals the sum of all principal $k \times k$ minors of A.
- **Averaging over Grassmannians:** If $X = \mathbb{R}^N$ is Euclidean, then

  $$\lambda_k(A) = \binom{N}{k} \int_{G_{k,N}} \det(P_E A|_E) \; \mathrm{d}\sigma(E),$$

  the $O(N)$-invariant average of the determinant of the compression $P_E A|_E$ to a k-plane E [2, pp. 231–234]. K. Morrison independently observed this viewpoint (unpublished note).
- **Representation theory:** $\lambda_k(A)$ is the character of the Λ^k-representation evaluated at A. Averaging the matrix coefficient $\left\langle (\Lambda^k A)w, w \right\rangle$ over the $O(N)$-orbit of unit simple k-vectors reproduces $\mathrm{tr}(\Lambda^k A)$ after normalisation.

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The main theme of the paper is the question of whether we can realise these averages *for arbitrary norms* by integrating over the unit sphere of the normed exterior power? For $k=1$, Morrison and the author [6] gave such a formula under the hypothesis of the existence of a 1-symmetric basis, that is, the action of a big enough hypooctahedral group on self-isometries of the space; the present work gives a unified framework for all k and all norms by isolating a single isotropy condition on a measure η on the unit sphere of $\Lambda^k X$.

The main result of the paper is that the trace average

$$\lambda_k(A) = n_k \int_{S_V} \langle (\Lambda^k A)w, w^* \rangle \, \mathrm{d}\eta(w), \qquad V := \Lambda^k X, \quad n_k = \binom{N}{k},$$

holds for *all* A if and only if

$$T_\eta := n_k \int_{S_V} w \otimes w^* \, \mathrm{d}\eta(w) = I_V.$$

This extends the familiar trace-average formula (case $k=1$; see [5]) and higher-trace formulae by Eberlein [2].

This isotropy is automatic for the *cone probability measure*, by the Gauss–Green theorem on B_V, and it holds for the *hypersurface measure* whenever a finite isometry group acts on X and induces an orthogonal 2-design on $\Lambda^k \mathbb{R}^N$. We also quantify the obstruction to isotropy for μ_k at $k=1$: only degree–2 spherical harmonics of the support function enter at first order.

2. Preliminaries

We identify X with $\mathbb{R}^N$ via a fixed auxiliary Euclidean inner product $\langle \cdot, \cdot \rangle_2$, used only to define Euclidean volumes, surface measures, and normals. Denote by B_X the unit ball and by $S_X = \partial B_X$ the unit sphere. Let μ be the *normalised* Euclidean hypersurface measure on S_X. At μ-a.e. $x \in S_X$ there is a unique norming functional x^* with $\|x^*\|_* = 1$ and $\langle x, x^* \rangle = 1$ (see [6, Prop. 2.4]).

Orthogonal 2*-designs and symmetry.* Let W be a Euclidean space and let $G \leqslant O(W)$ be a finite subgroup of the orthogonal group. We say that G is an *orthogonal* 2*-design* on W if

$$\frac{1}{|G|} \sum_{Q \in G} Q^\top B\, Q \;=\; \frac{\operatorname{tr} B}{\dim W}\, I_W \qquad \text{for all } B \in \operatorname{End}(W). \tag{2.1}$$

Equivalently, if w is any unit vector and ν_G is the uniform measure on the orbit $G \cdot w$, then

$$\int (u \otimes u) \, \mathrm{d}\nu_G(u) \;=\; \frac{1}{\dim W}\, I_W.$$

In our applications $W = \Lambda^k \mathbb{R}^N$ with the induced Euclidean structure, and the representation is $Q \mapsto \Lambda^k Q$.

Example 2.1 (Hyperoctahedral group)**.** Let B_N be the group of signed permutation matrices. Then for every k the induced action of B_N on $W = \Lambda^k \mathbb{R}^N$ is an orthogonal 2-design. Indeed, the commutant of $\Lambda^k(B_N)$ is scalar: commuting with all sign-flips forces diagonality in the basis $\{e_{i_1} \wedge \cdots \wedge e_{i_k}\}$, while commuting with permutations forces equality of all diagonal entries; hence $\operatorname{End}(W)^{B_N} = \mathbb{R} I_W$. Averaging as in (2.1) then yields $\frac{1}{|B_N|} \sum Q^\top B Q = (\operatorname{tr} B / \dim W)\, I_W$.

Example 2.2 (Low-dimensional geometric groups)**.** In $\mathbb{R}^2$, the dihedral group D_m ($m \geqslant 3$) is a 2-design on $\Lambda^1 \mathbb{R}^2 \cong \mathbb{R}^2$. In $\mathbb{R}^3$, the rotation groups of the Platonic solids are 2-designs on $\Lambda^1 \mathbb{R}^3 \cong \mathbb{R}^3$ and, via the Hodge isometry $\star : \Lambda^2 \mathbb{R}^3 \xrightarrow{\cong} \Lambda^1 \mathbb{R}^3$, also on $\Lambda^2 \mathbb{R}^3$.

Exterior powers. For $1 \leqslant k \leqslant N$, write $V = \Lambda^k X$ and $n_k = \dim V = \binom{N}{k}$. We equip V with the exterior projective norm as introduced in [8]:

$$\|w\|_{\wedge,\pi} = \inf\Big\{ \sum_r \|x_1^{(r)}\| \cdots \|x_k^{(r)}\| \ : \ w = \sum_r x_1^{(r)} \wedge \cdots \wedge x_k^{(r)} \Big\}.$$

The canonical Euclidean inner product on V is induced by $\langle x_1 \wedge \cdots \wedge x_k, y_1 \wedge \cdots \wedge y_k \rangle_2 = \det[\langle x_i, y_j \rangle_2]_{i,j}$. Let σ be the Euclidean surface measure on $S_V = \partial B_V$. Denote by μ_k the *normalised* Euclidean hypersurface measure on S_V.

Lemma 2.3. *Let X be a finite-dimensional normed space and equip $\Lambda^k X$ with the exterior projective norm.*

(a) If $Q : X \to X$ is a linear isometry, then $\Lambda^k Q : (\Lambda^k X, \|\cdot\|_{\wedge,\pi}) \to (\Lambda^k X, \|\cdot\|_{\wedge,\pi})$ is an isometry.

(b) More generally, for any $T \in \mathrm{End}(X)$, one has $\|\Lambda^k T\|_{\Lambda^k X \to \Lambda^k X} \leqslant \|T\|^k$.

Proof. (a) Fix $w \in \Lambda^k X$ and $\varepsilon > 0$. Choose a representation $w = \sum_{r=1}^m x_1^{(r)} \wedge \cdots \wedge x_k^{(r)}$ such that

$$\|w\|_{\wedge,\pi} \geqslant \sum_{r=1}^m \|x_1^{(r)}\| \cdots \|x_k^{(r)}\| - \varepsilon.$$

Applying $\Lambda^k Q$ and using that Q is an isometry on X,

$$\Lambda^k Q(w) = \sum_{r=1}^m (Qx_1^{(r)}) \wedge \cdots \wedge (Qx_k^{(r)}), \qquad \prod_{i=1}^k \|Qx_i^{(r)}\| = \prod_{i=1}^k \|x_i^{(r)}\|.$$

Taking the infimum over all such decompositions on the right yields $\|\Lambda^k Q(w)\|_{\wedge,\pi} \leqslant \|w\|_{\wedge,\pi} + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, $\|\Lambda^k Q(w)\|_{\wedge,\pi} \leqslant \|w\|_{\wedge,\pi}$. Applying the same argument to Q^{-1} gives the reverse inequality, hence $\|\Lambda^k Q(w)\|_{\wedge,\pi} = \|w\|_{\wedge,\pi}$ for all w, *i.e.*, $\Lambda^k Q$ is an isometry.

(b) For any representation of w as above,

$$\|\Lambda^k T(w)\|_{\wedge,\pi} \ \leqslant \ \sum_{r=1}^m \prod_{i=1}^k \|Tx_i^{(r)}\| \ \leqslant \ \sum_{r=1}^m \prod_{i=1}^k \|T\| \, \|x_i^{(r)}\| \ = \ \|T\|^k \sum_{r=1}^m \prod_{i=1}^k \|x_i^{(r)}\|.$$

Taking the infimum over all decompositions of w gives $\|\Lambda^k T(w)\|_{\wedge,\pi} \leqslant \|T\|^k \, \|w\|_{\wedge,\pi}$, hence $\|\Lambda^k T\| \leqslant \|T\|^k$. □

Cone measures. Let $K \subset \mathbb{R}^m$ be a convex body with $0 \in \operatorname{int} K$. The *cone probability measure* ν_K on ∂K is

$$\nu_K(A) = \frac{\mathrm{vol}(\{tx : x \in A, \ 0 \leqslant t \leqslant 1\})}{\mathrm{vol}(K)}.$$

If n_e exists a.e., then $\mathrm{d}\nu_K(x) = \frac{\langle n_e(x), x\rangle}{m \ \mathrm{vol}(K)} \ \mathrm{d}\sigma(x)$ (Schneider [7, §§2.2, 4.2, 8.2]; Gardner [4, §§8.4, B.2]).

Grassmannian viewpoint and the Λ^k-sphere in the Euclidean case. Let us consider $X = \mathbb{R}^N$ with its Euclidean inner product. Fix $w_0 = e_1 \wedge \cdots \wedge e_k$ and $\mathcal{O}_k := \{(\Lambda^k g) w_0 : g \in O(N)\} \subset S_{\Lambda^k \mathbb{R}^N}$. Then

$$\mathcal{O}_k \ \cong \ O(N)/(O(k) \times O(N-k)),$$

and the map $E \in G_{k,N} \mapsto w_E := v_1 \wedge \cdots \wedge v_k$ (for any orthonormal basis (v_i) of E) identifies the unoriented Grassmannian with $\mathcal{O}_k/\{\pm 1\}$. For such w_E,

$$\langle (\Lambda^k A) w_E, \, w_E \rangle = \det([\langle Av_i, v_j \rangle]_{i,j \leqslant k}) = \det(P_E A|_E),$$

so averaging $\langle(\Lambda^k A)w, w\rangle$ over $\mathcal{O}_k$ (or $G_{k,N}$) yields Eberlein's formula for $\lambda_k(A)$ with factor $\binom{N}{k}$ [2, pp. 231–234].

The Gauss–Green theorem and sets of finite perimeter.

Lemma 2.4 (Gauss–Green Theorem)**.** *Let $\Omega \subset \mathbb{R}^m$ be a bounded set with Lipschitz boundary and let n_e denote its Euclidean outer unit normal (defined $\mathcal{H}^{m-1}$-a.e.). Then for every $F \in C^1(\mathbb{R}^m; \mathbb{R}^m)$,*

$$\int_{\partial\Omega} \langle F, n_e\rangle_2 \; \mathrm{d}\sigma \;=\; \int_{\Omega} \operatorname{div} F \; \mathrm{d}x.$$

This theorem holds more generally for sets of finite perimeter (reduced boundary). Convex bodies are sets of finite perimeter. For more details see [3, Thm. 5.16] and [9, Thm. 5.8.8].

We shall require the following version of the Minkowski identity

Proposition 2.5. *Let $K \subset \mathbb{R}^m$ be a convex body. Then, as operators on $\mathbb{R}^m$,*

$$\int_{\partial K} x \otimes n_e(x) \; \mathrm{d}\sigma(x) \;=\; \mathrm{vol}(K)\, I_m.$$

Equivalently, for all $B \in \mathrm{End}(\mathbb{R}^m)$, $\int_{\partial K} \langle Bx, n_e(x)\rangle_2 \; \mathrm{d}\sigma(x) \;=\; \mathrm{tr}(B) \; \mathrm{vol}(K)$.

Proof. Apply Lemma 2.4 to $F(x) = Bx$, with $\operatorname{div} F = \mathrm{tr}(B)$. Non-degeneracy of the Hilbert–Schmidt pairing yields the operator identity. □

3. Trace average and isotropy

Lemma 3.1 (Hilbert–Schmidt duality)**.** *Let V be a finite-dimensional real inner-product space and $\langle\cdot,\cdot\rangle_{\mathrm{HS}}$ the Hilbert–Schmidt pairing on $\mathrm{End}(V)$. For any probability measure η on S_V for which a measurable choice of norming functionals w^* exists η-a.e., and any $B \in \mathrm{End}(V)$,*

$$n_k \int_{S_V} \langle Bw, w^*\rangle \, \mathrm{d}\eta(w) = \Big\langle B,\ T_\eta \Big\rangle_{\mathrm{HS}}, \qquad T_\eta := n_k \int_{S_V} w \otimes w^* \, \mathrm{d}\eta(w).$$

Hence,

$$\forall\, B \in \mathrm{End}(V): \quad n_k \int_{S_V} \langle Bw, w^*\rangle \, \mathrm{d}\eta(w) = \mathrm{tr}(B) \iff T_\eta = I_V.$$

Proof. Fix an auxiliary Euclidean inner product on V (the one used for $\langle\cdot,\cdot\rangle_{\mathrm{HS}}$). Since all norms on V are equivalent, there exist constants $c_1, c_2 > 0$ with $\|v\|_2 \leqslant c_1\|v\|$ for all $v \in V$, and $\|f\|_2 \leqslant c_2\|f\|_*$ for all $f \in V^*$ (identifying V^* with V via the Euclidean Riesz map). Because $w \in S_V$ and $\|w^*\|_* = 1$, we have $\|w\|_2 \leqslant c_1$ and $\|w^*\|_2 \leqslant c_2$. Hence $\|w \otimes w^*\|_{\mathrm{HS}} = \|w\|_2\, \|w^*\|_2 \leqslant c_1 c_2$, so the map $w \mapsto w \otimes w^*$ is essentially bounded (hence Bochner integrable) on S_V. By assumption there is a measurable choice of norming functionals $w \mapsto w^*$ η-a.e.[1].

For any $u, v \in V$ and $B \in \mathrm{End}(V)$, the Hilbert–Schmidt pairing satisfies $\langle B,\, u \otimes v\rangle_{\mathrm{HS}} = \langle Bu,\, v\rangle$, since both sides equal $\mathrm{tr}\big((u \otimes v)^\top B\big)$.

Using linearity of the Bochner integral and the above,

$$\Big\langle B,\, T_\eta\Big\rangle_{\mathrm{HS}} = \Big\langle B,\ n_k \int_{S_V} w \otimes w^* \, \mathrm{d}\eta(w) \Big\rangle_{\mathrm{HS}} = n_k \int_{S_V} \langle B,\, w \otimes w^*\rangle_{\mathrm{HS}} \, \mathrm{d}\eta(w) = n_k \int_{S_V} \langle Bw,\, w^*\rangle \, \mathrm{d}\eta(w).$$

Since $\mathrm{tr}(B) = \langle B, I_V\rangle_{\mathrm{HS}}$ and the Hilbert–Schmidt pairing is non-degenerate, we have

$$\forall B: \quad n_k \int \langle Bw, w^*\rangle \, \mathrm{d}\eta = \mathrm{tr}(B) \iff \forall B: \quad \langle B, T_\eta\rangle_{\mathrm{HS}} = \langle B, I_V\rangle_{\mathrm{HS}} \iff T_\eta = I_V.$$

This completes the proof. □

[1] For the hypersurface measure, measurability follows from [6, Proposition 2.4].

Theorem 3.2. *Let $V := \Lambda^k X$, $n_k = \binom{N}{k}$, and η a probability measure on S_V with measurable norming functionals w^* a.e. Then*

$$\forall\, B \in \operatorname{End}(V): \quad \operatorname{tr}(B) \;=\; n_k \int_{S_V} \langle Bw, w^* \rangle \; \mathrm{d}\eta(w) \iff T_\eta = I_V. \tag{3.1}$$

Consequently, if $T_\eta = I_V$, then for every $A \in \operatorname{End}(X)$,

$$\boxed{\lambda_k(A) = \operatorname{tr}(\Lambda^k A) \;=\; n_k \int_{S_V} \left\langle (\Lambda^k A) w, w^* \right\rangle \; \mathrm{d}\eta(w).}$$

This isotropy holds in the following cases:

(i) If a finite $G \leqslant O(N)$ acts by norm isometries on X and its induced action on $\Lambda^k \mathbb{R}^N$ is an orthogonal 2-design, then $T_{\mu_k} = I_V$.

(ii) Cone measure: *For $\mathrm{d}\nu_k(w) = \frac{\langle n_e(w), w\rangle_2}{n_k \ \operatorname{vol}(B_V)} \, \mathrm{d}\sigma(w)$, one has $T_{\nu_k} = I_V$.*

Proof. Lemma 3.1 is the equivalence. For (i): T_{μ_k} commutes with the Λ^k-action, hence $T_{\mu_k} = cI$; since $\operatorname{tr}(T_{\mu_k}) = n_k$, we get $c = 1$. For (ii): using $w^* = n_e / \langle n_e, w\rangle_2$,

$$T_{\nu_k} = \frac{1}{\operatorname{vol}(B_V)} \int_{\partial B_V} w \otimes n_e(w) \; \mathrm{d}\sigma(w).$$

Then, for any B,

$$\langle B, T_{\nu_k} \rangle_{\mathrm{HS}} = \frac{1}{\operatorname{vol}(B_V)} \int_{\partial B_V} \langle Bw, n_e \rangle_2 \; \mathrm{d}\sigma = \frac{1}{\operatorname{vol}(B_V)} \int_{B_V} \operatorname{div}(Bw) \; \mathrm{d}w = \operatorname{tr}(B),$$

by Lemma 2.4. $\square$

Corollary 3.3. *If $G \leqslant O(N)$ induces an orthogonal 2-design on $\Lambda^k \mathbb{R}^N$ for every k, then the hypersurface–measure trace formula holds simultaneously for all k. This includes the hyperoctahedral group B_N.*

Corollary 3.4 (Discrete trace formula for polyhedral norms)**.** *Let $B_X \subset \mathbb{R}^N$ be a polytope with facets $F_1, \dots, F_m$, Euclidean unit outer normals n_j, and areas $\mathcal{H}^{N-1}(F_j)$. Then for every $A \in \operatorname{End}(\mathbb{R}^N)$,*

$$\operatorname{tr}(A) = \frac{1}{\operatorname{vol}(B_X)} \sum_{j=1}^{m} \int_{F_j} \langle Ax, n_j \rangle \, \mathrm{d}\mathcal{H}^{N-1}(x) = \frac{1}{\operatorname{vol}(B_X)} \sum_{j=1}^{m} \mathcal{H}^{N-1}(F_j) \, \langle Ac_j, n_j \rangle, \tag{3.2}$$

where $c_j := \mathcal{H}^{N-1}(F_j)^{-1} \int_{F_j} x \, \mathrm{d}\mathcal{H}^{N-1}(x)$ is the (Euclidean) centroid of F_j.

Proof. By the cone–measure formula (or, equivalently, the matrix-valued Minkowski identity applied with $K = B_X$ and $B = A$), we have

$$\operatorname{tr}(A) \;=\; \frac{1}{\operatorname{vol}(B_X)} \int_{\partial B_X} \langle Ax, n_e(x) \rangle \, \mathrm{d}\sigma(x). \tag{3.3}$$

Since B_X is a polytope, its boundary is a union of flat facets F_j whose Euclidean outer unit normal is constant and equals n_j on F_j; the union of all ridges and vertices has $\mathcal{H}^{N-1}$-measure zero and does not contribute to the integral. Hence the right-hand side of (3.3) splits as

$$\frac{1}{\operatorname{vol}(B_X)} \sum_{j=1}^{m} \int_{F_j} \langle Ax, n_j \rangle \, \mathrm{d}\mathcal{H}^{N-1}(x),$$

which proves the first equality in (3.2). For the second, the integrand is linear in x, so

$$\int_{F_j}\langle Ax, n_j\rangle \,\mathrm{d}\mathcal{H}^{N-1}(x) = \Big\langle A\Big(\int_{F_j} x\,\mathrm{d}\mathcal{H}^{N-1}(x)\Big),\, n_j\Big\rangle = \mathcal{H}^{N-1}(F_j)\,\langle Ac_j, n_j\rangle.$$

Substituting this back into (3.2) yields the result. □

Remark 3.5. Identity (3.3) is the $k = 1$ instance of the unified cone–measure trace formula and may also be viewed as the scalar pairing of the matrix-valued Minkowski identity $\int_{\partial B_X} x\otimes n_e(x)\,\mathrm{d}\sigma = \mathrm{vol}(B_X)\, I_N$ with A (by the Gauss–Green theorem argument). For context on Grassmannian trace averages in the Euclidean case, see Eberlein [2, pp. 231–236].

Geometric interpretation of obstructions for the trace–average. In the first–variation analysis for $k = 1$ we linearise the operator–valued map

$$g \;\longmapsto\; T_{\mu_1}(g) \;=\; N\int_{S^{m-1}} x(g;u)\otimes x^*(g;u)\,\mathrm{d}\mu_g(u),$$

at the Euclidean ball in the direction of an even function $g\in C^\infty_{\mathrm{even}}(S^{m-1})$, where $m=\dim X$, $x(g;u)$ is the boundary point with outer normal u, $x^*(g;u)$ the associated norming functional, and μ_g the normalised hypersurface measure. Naturality of the construction implies

$$\mathcal{L}(g\circ Q^\top) \;=\; Q\,\mathcal{L}(g)\,Q^\top, \qquad Q\in O(m),$$

so the first variation $\mathcal{L} : C^\infty_{\mathrm{even}}(S^{m-1}) \to \mathrm{Sym}^2(\mathbb{R}^m)$ is an $O(m)$-*equivariant* linear map. Representation theory therefore constrains which spherical harmonic components of g can affect $\mathcal{L}(g)$. The domain $C^\infty_{\mathrm{even}}(S^{m-1})$ decomposes into even spherical harmonics $\bigoplus_{\ell \text{ even}}\mathcal{H}_\ell$, while $\mathrm{Sym}^2(\mathbb{R}^m) = \mathbb{R}I\oplus \mathrm{Sym}^2_0(\mathbb{R}^m)$ splits as the sum of the trivial representation and an irreducible module isomorphic to $\mathcal{H}_2$. Consequently, only the $\ell = 0$ and $\ell = 2$ parts of g can contribute; all higher degrees are annihilated. Concretely, this selection rule can be read off either from Schur's lemma or from the Funk–Hecke formula, which shows that convolution with a zonal kernel acts by scalars on each $\mathcal{H}_\ell$ (see [1, Ch. 2]).

Lemma 3.6. *Let $m \geqslant 2$. Any $O(m)$-equivariant continuous linear operator*

$$\mathcal{L} : C^\infty_{\mathrm{even}}(S^{m-1}) \to \mathrm{Sym}^2(\mathbb{R}^m)$$

factors through the projections onto $\mathcal{H}_0$ and $\mathcal{H}_2$ (the degree 0 and 2 spherical harmonics). More precisely,

$$\mathcal{L}(g) \;=\; \beta_0\Big(\int_{S^{m-1}} g\,\mathrm{d}\omega\Big)\, I \;+\; \beta_2\,\Psi(P_2 g),$$

where P_2 is the orthogonal projection onto $\mathcal{H}_2$, and $\Psi : \mathcal{H}_2 \to \mathrm{Sym}^2_0(\mathbb{R}^m)$ is the unique $O(m)$-equivariant isomorphism (given on pure quadratics by $u\mapsto uu^\top - \frac{1}{m}I$). In particular, all harmonic components $\mathcal{H}_\ell$ with $\ell\notin\{0,2\}$ are annihilated.

Proof. We view $C^\infty_{\mathrm{even}}(S^{m-1})$ as an $O(m)$-module via $(Q\cdot g)(u) = g(Q^\top u)$, and $\mathrm{Sym}^2(\mathbb{R}^m)$ via $Q\cdot S = Q\,S\,Q^\top$. The Peter–Weyl decomposition gives

$$C^\infty_{\mathrm{even}}(S^{m-1}) \;=\; \widehat{\bigoplus_{\ell \text{ even}}}\;\mathcal{H}_\ell,$$

where each $\mathcal{H}_\ell$ is the irreducible space of degree–ℓ spherical harmonics, pairwise non-isomorphic. On the target side,

$$\mathrm{Sym}^2(\mathbb{R}^m) \;=\; \mathbb{R}I \;\oplus\; \mathrm{Sym}^2_0(\mathbb{R}^m),$$

with $\mathbb{R}I$ the trivial representation and $\mathrm{Sym}^2_0(\mathbb{R}^m)$ irreducible and (canonically) isomorphic to $\mathcal{H}_2$: the map

$$\Theta:\ \mathrm{Sym}^2_0(\mathbb{R}^m)\ \longrightarrow\ \mathcal{H}_2,\qquad \Theta(S)(u)=u^\top S\,u$$

is $O(m)$-equivariant and injective (if $u^\top Su=0$ for all $u\in S^{m-1}$, then $S=0$), hence an isomorphism of irreducible modules. Its inverse may be written explicitly, up to a non-zero constant c_m, as

$$\Psi(q)\ :=\ c_m\int_{S^{m-1}} q(u)\left(u\,u^\top-\frac{1}{m}I\right)\,\mathrm{d}\omega(u),\qquad q\in\mathcal{H}_2,$$

which is $O(m)$-equivariant by construction; c_m is fixed by normalisation (e.g. making $\Psi\circ\Theta=\mathrm{id}$).

Now let $\mathcal{L}:C^\infty_{\mathrm{even}}(S^{m-1})\to\mathrm{Sym}^2(\mathbb{R}^m)$ be $O(m)$-equivariant. By Schur's lemma,

$$\mathrm{Hom}_{O(m)}(\mathcal{H}_\ell,\mathbb{R}I)\cong\begin{cases}\mathbb{R}, & \ell=0,\\ 0, & \ell\neq 0,\end{cases}\qquad \mathrm{Hom}_{O(m)}(\mathcal{H}_\ell,\mathrm{Sym}^2_0)\ \cong\ \mathrm{Hom}_{O(m)}(\mathcal{H}_\ell,\mathcal{H}_2)\cong\begin{cases}\mathbb{R}, & \ell=2,\\ 0, & \ell\neq 2.\end{cases}$$

Hence $\mathcal{L}$ vanishes on $\mathcal{H}_\ell$ for $\ell\notin\{0,2\}$, and on $\mathcal{H}_0$ and $\mathcal{H}_2$ it is (respectively) a scalar multiple of the canonical maps

$$\mathcal{H}_0\ni g\mapsto\Big(\int g\,\mathrm{d}\omega\Big)I,\qquad \mathcal{H}_2\ni g\mapsto\Psi(g)\in\mathrm{Sym}^2_0(\mathbb{R}^m).$$

Therefore there exist scalars β_0,β_2 with $\mathcal{L}=\beta_0\,\mathrm{avg}\cdot I+\beta_2\,\Psi\circ P_2$, as claimed. □

Remark 3.7. One could view the proof above through the Funk–Hecke formula lens and embed Sym^2_0 into functions via $S\mapsto u\mapsto\langle S,\,uu^\top-\frac{1}{m}I\rangle$. For S fixed, the scalar functional $g\mapsto\langle\mathcal{L}(g),S\rangle_{\mathrm{HS}}$ is $O(m)$-equivariant and thus equals $\int g(u)\,\phi_S(\langle u,\xi\rangle)\,\mathrm{d}\omega(u)$ for some zonal kernel ϕ_S. By the Funk–Hecke formula [1, Ch. 2], such convolutions act as scalars on each $\mathcal{H}_\ell$, whence only $\ell=0$ and $\ell=2$ can contribute when the output lies in $\mathbb{R}I\oplus\mathrm{Sym}^2_0$.

Let $X=\mathbb{R}^N$. For an even $g\in C^\infty(S^{N-1})$ and $|\varepsilon|\ll 1$, consider the perturbation with support function $h_\varepsilon(u)=1+\varepsilon g(u)$. The Gauss map gives $x(u)=h(u)u+\nabla_S h(u)$, with norming functional $x^*(u)=u/h(u)$. After normalising μ to probability,

$$T_{\mu_1}(\varepsilon)=N\int_{S_X}x\otimes x^*\,\mathrm{d}\mu=I+\varepsilon\,\mathcal{L}(g)+O(\varepsilon^2).\tag{3.4}$$

By $O(N)$-equivariance and Funk–Hecke, only $\ell=0,2$ harmonics can contribute; the $\ell=0$ part cancels after normalisation. One finds

$$\mathcal{L}(g)=\alpha_N\int_{S^{N-1}}g(u)\Big(uu^\top-\tfrac{1}{N}I\Big)\,\mathrm{d}\omega(u),$$

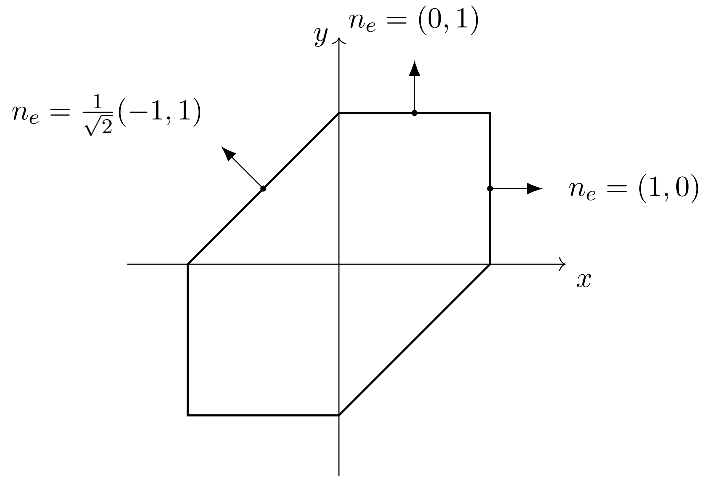
with $\alpha_N=-N$. (A short proof is given in Appendix A.)

3.0.1. *Explicit counterexamples for the hypersurface measure.* The following explicit examples correct the closing remark of [6] where the hypersurface measure was normalised incorrectly.

Example 3.8. Let $\|(x,y)\|=\max\{|x|,|y|,|y-x|\}$. The unit sphere is the hexagon with vertices $(1,1),(0,1),(-1,0),(-1,-1),(0,-1),(1,0)$. A direct calculation yields

$$T_{\mu_1}:=2\int_{S_X}x\otimes x^*\,\mathrm{d}\mu=\begin{pmatrix}1 & 2-\frac{3\sqrt{2}}{2}\\ 2-\frac{3\sqrt{2}}{2} & 1\end{pmatrix}\neq I.$$

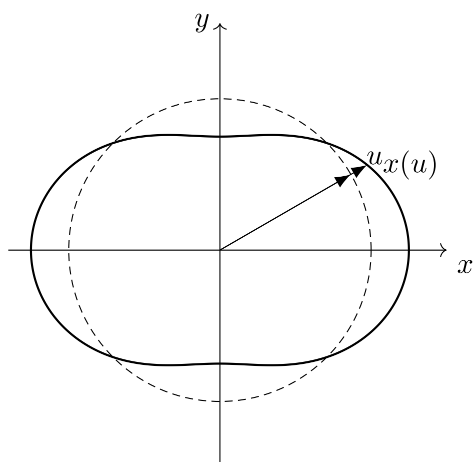
For $A=\left(\begin{smallmatrix}0&1\\0&0\end{smallmatrix}\right)$ with $\mathrm{tr}(A)=0$, $\frac{1}{2}\,\mathrm{tr}(AT_{\mu_1})=1-\frac{3\sqrt{2}}{4}\neq 0$.



Example 3.9. Let $h(\varphi) = 1 + \varepsilon \cos(2\varphi)$ in $\mathbb{R}^2$. By (3.4),

$$T_{\mu_1} = I + \frac{\varepsilon}{2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + O(\varepsilon^2).$$

For $A = \operatorname{diag}(1, -1)$ with $\operatorname{tr}(A) = 0$, $\frac{1}{2} \operatorname{tr}(AT_{\mu_1}) = -\varepsilon + O(\varepsilon^2) \neq 0$.



4. Quantitative anisotropy and α-cone measures

Proposition 4.1. *Let X be an N-dimensional normed space. Define the* anisotropy tensor

$$\mathcal{A}_X := N \int_{S_X} x \otimes x^* \, \mathrm{d}\mu(x) - I_N \in \operatorname{Sym}^2(\mathbb{R}^N).$$

Then:

(i) $\mathcal{A}_X = 0$ *iff the trace–average formula holds for all* $A \in \operatorname{End}(X)$.
(ii) $\mathcal{A}_X$ *is traceless and symmetric, with* $\|\mathcal{A}_X\|_{\mathrm{HS}} \leqslant N + \sqrt{N}$.
(iii) *For any* $A \in \operatorname{End}(X)$,

$$N \int_{S_X} \langle Ax, x^* \rangle \, \mathrm{d}\mu(x) - \operatorname{tr}(A) = \langle A, \mathcal{A}_X \rangle_{\mathrm{HS}}.$$

(iv) If $h_\varepsilon = 1 + \varepsilon g$ with $\int g \, \mathrm{d}\omega = 0$,

$$\mathcal{A}_X(\varepsilon) = -\frac{2\varepsilon}{N+2} \int_{S^{N-1}} g(u)\Big(uu^\top - \tfrac{1}{N} I\Big)\mathrm{d}\omega(u) + O(\varepsilon^2).$$

Equivalently, for $g(u) = u^\top S u$ with $\operatorname{tr} S = 0$, $\mathcal{A}_X(\varepsilon) = -\frac{2\varepsilon}{N+2} S + O(\varepsilon^2)$.

Theorem 4.2 (α-cone isotropy)**.** *Let $V = \Lambda^k X$ with $m = \dim V$. For $\alpha > 0$ define the α-cone measure on $S_V = \partial B_V$ by*

$$\mathrm{d}\nu_k^\alpha(w) = \frac{\langle n_e(w), w\rangle^\alpha}{C_\alpha} \, \mathrm{d}\sigma(w), \qquad C_\alpha := \int_{\partial B_V} \langle n_e, w\rangle^\alpha \, \mathrm{d}\sigma(w).$$

Then:

(i) *For every convex B_V (no symmetry), $T_{\nu_k^1} = I_V$.* (Global sufficiency of $\alpha = 1$).

(ii) *Let B_V be C^2 and strictly convex. Consider a smooth even perturbation of the Euclidean ball with support function $h_\varepsilon(u) = 1 + \varepsilon g(u)$, $u \in S^{m-1}$, with $\int g \, \mathrm{d}\omega = 0$. Then*

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0} \langle B, T_{\nu_k^\alpha}(h_\varepsilon)\rangle_{\mathrm{HS}} \;=\; (\alpha - 1) \int_{S^{m-1}} g(u) \, \langle Bu, u\rangle \, \mathrm{d}\omega(u) \qquad (B \in \mathrm{End}(V)).$$

In particular, for $\alpha \neq 1$ and any degree-2 harmonic $g \not\equiv 0$, the derivative is non-zero for some B, hence $T_{\nu_k^\alpha} \neq I_V$ for all sufficiently small ε. (Local necessity of $\alpha = 1$ for non-spherical bodies).

Proof. (i) For $\alpha = 1$, using $w^* = n_e / \langle n_e, w\rangle$,

$$T_{\nu_k^1} = n_k \int w \otimes w^* \, \mathrm{d}\nu_k^1 = \frac{n_k}{C_1} \int w \otimes n_e \, \mathrm{d}\sigma = \frac{n_k}{C_1} \left(\mathrm{vol}(B_V) \, I_V \right)$$

by the matrix-valued Minkowski identity (Proposition 2.5). Moreover $C_1 = \int \langle n_e, w\rangle \, \mathrm{d}\sigma = m \, \mathrm{vol}(B_V) = n_k \, \mathrm{vol}(B_V)$, hence $T_{\nu_k^1} = I_V$.

(ii) We write everything in the Gauss parametrisation. For a C^2 strictly convex body with support function h, the boundary point with outer normal $u \in S^{m-1}$ is

$$x(u) = h(u) \, u + \nabla_S h(u), \qquad n_e(x(u)) = u,$$

and the Euclidean surface element pulls back as

$$\mathrm{d}\sigma_h(u) = \det\left(\nabla_S^2 h(u) + h(u) I\right) \, \mathrm{d}\omega(u).$$

Set $m = \dim V$. A direct computation gives, for small ε and $h_\varepsilon = 1 + \varepsilon g$,

$$\begin{aligned} x_\varepsilon(u) &= u + \varepsilon\big(g(u) \, u + \nabla_S g(u)\big) + O(\varepsilon^2), \\ \langle n_e, x_\varepsilon\rangle &= h_\varepsilon(u) = 1 + \varepsilon g(u) + O(\varepsilon^2), \\ \det(\nabla_S^2 h_\varepsilon + h_\varepsilon I) &= 1 + \varepsilon\big(\Delta_S g(u) + (m-1) g(u)\big) + O(\varepsilon^2). \end{aligned}$$

Therefore

$$\langle B, T_{\nu_k^\alpha}(h_\varepsilon)\rangle_{\mathrm{HS}} = \frac{n_k}{C_\alpha(h_\varepsilon)} \int_{S^{m-1}} \underbrace{\langle B \, x_\varepsilon(u), \, u\rangle}_{\mathrm{I}} \underbrace{h_\varepsilon(u)^{\alpha-1}}_{\mathrm{II}} \underbrace{\det(\nabla_S^2 h_\varepsilon + h_\varepsilon I)}_{\mathrm{III}} \, \mathrm{d}\omega(u).$$

Let us expand terms I–III to the first order:

$$\begin{aligned} \mathrm{I} &= \langle Bu, u\rangle + \varepsilon\big(\, g \, \langle Bu, u\rangle + \langle B \, \nabla_S g, \, u\rangle \, \big) + O(\varepsilon^2), \\ \mathrm{II} &= 1 + \varepsilon(\alpha - 1) g + O(\varepsilon^2), \\ \mathrm{III} &= 1 + \varepsilon\big(\Delta_S g + (m-1) g\big) + O(\varepsilon^2). \end{aligned}$$

Multiplying and keeping $O(\varepsilon)$-terms we get

$$\langle Bu,u\rangle+\varepsilon\Big(\underbrace{\langle B\,\nabla_S g,\,u\rangle}_{A_1}+\underbrace{(\alpha+m-1)g\,\langle Bu,u\rangle}_{A_2}+\underbrace{(\Delta_S g)\,\langle Bu,u\rangle}_{A_3}\Big)+O(\varepsilon^2).$$

Now, we integrate over S^{m-1} and use two identities:

- *Tangential integration by parts.* Since $\nabla_S g$ is tangential,

$$\int\langle B\,\nabla_S g,\,u\rangle\ \mathrm{d}\omega=\int\nabla_S g\cdot P_T(B^\top u)\ \mathrm{d}\omega=-\int g\ \mathrm{div}_S\big(P_T(B^\top u)\big)\ \mathrm{d}\omega,$$

and (see Appendix A or [1, Ch. 2]) $\mathrm{div}_S\big(P_T(B^\top u)\big)=-\,m\,\langle Bu,u\rangle+\mathrm{tr}(B)$. Hence, for $\int g\ \mathrm{d}\omega=0$, $\int A_1\ \mathrm{d}\omega=m\int g\,\langle Bu,u\rangle\ \mathrm{d}\omega$.

- *Spherical Laplacian on quadratics.* For $f(u)=\langle Bu,u\rangle$, $\Delta_S f=-2m\left(\langle Bu,u\rangle-\frac{1}{m}\,\mathrm{tr}(B)\right)$. Thus, again using $\int g\ \mathrm{d}\omega=0$, $\int A_3\ \mathrm{d}\omega=-2m\int g\,\langle Bu,u\rangle\ \mathrm{d}\omega$.

Combining A_1,A_2,A_3 we obtain for the numerator:

$$\int(\cdots)\ \mathrm{d}\omega=\int\langle Bu,u\rangle\ \mathrm{d}\omega+\varepsilon\left(\alpha-1\right)\int g\,\langle Bu,u\rangle\ \mathrm{d}\omega+O(\varepsilon^2).$$

Next, the normalising constant $C_\alpha(h_\varepsilon)=\int h_\varepsilon^\alpha\det(\nabla_S^2 h_\varepsilon+h_\varepsilon I)\ \mathrm{d}\omega$ satisfies $C_\alpha(h_\varepsilon)=1+O(\varepsilon^2)$ under $\int g=0$. Therefore (using $\int\langle Bu,u\rangle\ \mathrm{d}\omega=\frac{1}{m}\,\mathrm{tr}(B)$),

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\bigg|_{\varepsilon=0}\big\langle B,\,T_{\nu_k^\alpha}(h_\varepsilon)\big\rangle_{\mathrm{HS}}\ =\ (\alpha-1)\int_{S^{m-1}}g(u)\,\langle Bu,u\rangle\ \mathrm{d}\omega(u).$$

If g has a non-zero degree-2 component (equivalently, B_V is a genuinely non-spherical perturbation), choose B so that $u\mapsto\langle Bu,u\rangle$ matches $P_2 g$; then the integral is non-zero unless $\alpha=1$. This proves (ii). ☐

Appendix A. Fourth-moment computation for the degree–2 coefficient

Let ω be the *probability* surface measure on S^{N-1}. The second and fourth moments are

$$\int u\,u^\top\ \mathrm{d}\omega(u)=\frac{1}{N}I,\qquad\int u_iu_ju_ku_\ell\ \mathrm{d}\omega(u)=\frac{\delta_{ij}\delta_{k\ell}+\delta_{ik}\delta_{j\ell}+\delta_{i\ell}\delta_{jk}}{N(N+2)}.$$

Let $g(u)=u^\top Su$ with $\mathrm{tr}\,S=0$. Then

$$\int g(u)\,u\,u^\top\ \mathrm{d}\omega(u)=\sum_{k,\ell}S_{k\ell}\int u_ku_\ell\,u\,u^\top\ \mathrm{d}\omega=\frac{2}{N(N+2)}\,S,$$

while $\int g(u)\ \mathrm{d}\omega=\frac{1}{N}\,\mathrm{tr}\,S=0$. Hence

$$\int g(u)\Big(u\,u^\top-\frac{1}{N}I\Big)\,\mathrm{d}\omega(u)=\frac{2}{N(N+2)}\,S.$$

Comparing with $\mathcal{L}(g)=\alpha_N\int g(u)(uu^\top-\frac{1}{N}I)\ \mathrm{d}\omega$, and using the calibration from $N=2$ (Example 3.9), we obtain $\alpha_N=-N$. Therefore,

$$\mathcal{A}_X(\varepsilon)\ =\ \varepsilon\,\mathcal{L}(g)+O(\varepsilon^2)\ =\ -\frac{2\varepsilon}{N+2}\int_{S^{N-1}}g(u)\Big(uu^\top-\tfrac{1}{N}I\Big)\ \mathrm{d}\omega(u)\ +\ O(\varepsilon^2),\tag{A.1}$$

as used in Proposition 4.1(iv).

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