PARALLELEPIPEDS OF MAXIMAL FACET AREA IN ELLIPSOIDS, THROUGH PRESCRIBED BOUNDARY POINTS

TOMASZ KANIA

Abstract. Let $E = \{x \in \mathbb{R}^n : x^\top A^{-1} x = 1\}$ be an n-dimensional ellipsoid with A positive definite. Among all n-dimensional parallelepipeds inscribed in E (all 2^n vertices on ∂E), we study (i) the total $(n-1)$-dimensional measure of the facets ('surface area'), and (ii) the total 1-dimensional measure of the 1-skeleton ('circumference' or total edge length). We prove that the sharp global bounds

$$S_{\max}(E) = 2^n n^{-(n-2)/2} \sqrt{\det A}\, \sqrt{\operatorname{tr}(A^{-1})}, \qquad L_{\max}(E) = 2^n \sqrt{\operatorname{tr}(A)}.$$

are attained by inscribed images of orthotopes in the unit sphere; we give another proof in dimension $n = 2$, originally proved by Richard, and, for $n \geqslant 3$, we prove such a maximisation result in the principal-axis case. In that case, we provide a constructive diagonal equalisation argument that preserves the barycentric constraint.

1. Introduction

The study of extremal properties of polygons and polytopes inscribed in convex bodies has a long history. A striking planar result, observed in a ballistic context by J.-M. Richard [3] and reproved elegantly by Connes and Zagier [1], is that the maximal perimeter of a parallelogram inscribed in an ellipse is independent of the chosen vertex on the ellipse.

In this note we revisit the n-dimensional analogue for ellipsoids and parallelepipeds. Our arguments are linear-algebraic: a reduction to the unit sphere, a Gram matrix expression for the total facet area, and the Schur–Horn theorem [2]. We first obtain the sharp global bound for the total facet area

$$S_{\max}(E) = 2^n n^{-(n-2)/2} \sqrt{\det A}\, \sqrt{\operatorname{tr}(A^{-1})},$$

and characterise global maximisers up to orthogonal equivalence. We then address a second functional: the total edge length (the 'circumference' of the 1-skeleton). Here we show the sharp bound

$$L_{\max}(E) = 2^n \sqrt{\operatorname{tr}(A)},$$

with a simple equality characterisation. We also study vertex constraints: given $x_0 \in \partial E$, can the global maximum be attained by an inscribed parallelepiped having x_0 as a vertex? We resolve this completely for $n = 2$ (yet another proof) and, for $n \geqslant 3$, in principal-axis cases.

Throughout, vectors are columns, $(\cdot)^\top$ denotes transpose, and $A^{1/2}$ is the positive square root.

2. Spherical reductions

Let $A \in \mathbb{R}^{n\times n}$ be symmetric positive definite and write $B = A^{1/2}$. Then

$$E = B(S^{n-1}) = \{Bu : \|u\| = 1\}.$$

A centred parallelepiped is

$$P = \Big\{ \sum_{i=1}^n t_i v_i : |t_i| \leqslant \tfrac12 \Big\}, \qquad \text{with vertices } \tfrac12 \sum_{i=1}^n \varepsilon_i v_i \quad (\varepsilon_i \in \{\pm 1\}).$$

Date: November 4, 2025.

2020 *Mathematics Subject Classification.* Primary 52A40; Secondary 52B11, 52A38, 15A42, 15A45.

Key words and phrases. Ellipsoid, inscribed parallelepiped, facet area, total edge length, Schur–Horn theorem, Connes–Zagier property.

IM CAS (RVO 67985840).

We call P *inscribed* if all its vertices lie on ∂E.

Lemma 2.1. *Let P be inscribed in E and set $Q := B^{-1}P$. Then Q is inscribed in the unit sphere S^{n-1}. Conversely, if $Q \subset \mathbb{R}^n$ is inscribed in S^{n-1}, then $P := BQ$ is inscribed in E.*

Moreover, $Q \subset S^{n-1}$ is an inscribed parallelepiped if and only if its edge vectors $w_1, \ldots, w_n$ can be written as

$$w_i = \lambda_i u_i, \quad \lambda_i > 0, \quad \sum_{i=1}^n \lambda_i^2 = 4,$$

where $u_1, \ldots, u_n$ is an orthonormal basis of $\mathbb{R}^n$. In particular, Q is an orthotope (edges pairwise orthogonal), with edge lengths λ_i, and the inscribed condition is precisely $\sum \lambda_i^2 = 4$.

Proof. The reduction $E = B(S^{n-1})$ is immediate. For the classification, write the vertices of Q as $\frac{1}{2}\sum_i \varepsilon_i w_i$. The inscribed condition is that

$$\Big\| \tfrac{1}{2} \sum_{i=1}^n \varepsilon_i w_i \Big\| = 1 \qquad \text{for all } \varepsilon \in \{\pm 1\}^n.$$

Squaring gives

$$\sum_{i=1}^n \|w_i\|^2 \;+\; 2\sum_{i<j} \varepsilon_i \varepsilon_j \langle w_i, w_j \rangle \;=\; 4 \quad \text{for all } \varepsilon.$$

Independence of the left-hand side from the sign vector $\varepsilon \in \{\pm 1\}^n$ forces every bilinear coefficient to vanish, *i.e.*, $\langle w_i, w_j \rangle = 0$ for $i \neq j$; with orthogonality the condition reduces to $\sum_i \|w_i\|^2 = 4$.

Thus the left-hand side is independent of ε, hence $\langle w_i, w_j \rangle = 0$ for $i \neq j$; with orthogonality the equality reduces to $\sum_i \|w_i\|^2 = 4$. Writing $w_i = \lambda_i u_i$ with $\lambda_i = \|w_i\|$ and $u_i = w_i/\|w_i\|$ yields the claim. □

3. Maximising the 1-dimensional measure (total edge length)

For an inscribed parallelepiped $P \subset E$ with edge vectors v_i (columns of V), the number of edges parallel to v_i is 2^{n-1}, each of length $\|v_i\|$. Thus the total 1-dimensional measure ("circumference" of the 1-skeleton) is

$$L(P) = 2^{n-1} \sum_{i=1}^n \|v_i\| = 2^{n-1} \sum_{i=1}^n \lambda_i \, \|Bu_i\| = 2^{n-1} \sum_{i=1}^n \lambda_i \sqrt{u_i^\top A \, u_i}. \tag{3.1}$$

Theorem 3.1 (Maximal total edge length)**.** *Among all parallelepipeds P inscribed in E,*

$$L(P) \;\leqslant\; L_{\max}(E) := 2^n \sqrt{\operatorname{tr}(A)}.$$

Equality holds if and only if there exist an orthonormal basis $U = [u_1 \;\cdots\; u_n] \in O(n)$ and positive λ_i with $\sum_i \lambda_i^2 = 4$ such that

$$\lambda_i \;\propto\; \sqrt{u_i^\top A \, u_i} \qquad (i = 1, \ldots, n). \tag{3.2}$$

In particular, for any orthonormal U the choice

$$\lambda_i = \frac{2\sqrt{u_i^\top A \, u_i}}{\sqrt{\sum_{j=1}^n u_j^\top A \, u_j}} = \frac{2\sqrt{u_i^\top A \, u_i}}{\sqrt{\operatorname{tr}(A)}}$$

produces an inscribed $P = BQ$ attaining $L_{\max}(E)$.

Proof. Fix U and set $\beta_i := \sqrt{u_i^\top A \, u_i} > 0$. With $\sum_i \lambda_i^2 = 4$ and (3.1),

$$L(P) = 2^{n-1} \sum_{i=1}^n \lambda_i \beta_i \;\leqslant\; 2^{n-1} \sqrt{\sum_i \lambda_i^2} \sqrt{\sum_i \beta_i^2} = 2^n \sqrt{\sum_{i=1}^n u_i^\top A \, u_i} = 2^n \sqrt{\operatorname{tr}(A)},$$

by Cauchy–Schwarz and orthonormality. Equality holds precisely when λ is proportional to β, which is (3.2). The explicit choice of λ_i enforces $\sum \lambda_i^2 = 4$ and gives equality. □

Remark 3.2. When $n = 2$ and $A = \mathrm{diag}(a^2, b^2)$ (ellipse with semiaxes a, b), Theorem 3.1 yields $L_{\max}(E) = 4\sqrt{a^2 + b^2}$, the classical maximal perimeter of an inscribed parallelogram. Thus the circumference result is a direct higher-dimensional analogue of the Connes–Zagier value.

Vertex–constrained total edge length. Fix $x_0 \in \partial E$ and set $y_0 := B^{-1}x_0 \in S^{n-1}$. If $Q \subset S^{n-1}$ has edge data $w_i = \lambda_i u_i$ with $U = [u_1 \ \cdots \ u_n] \in O(n)$, the condition that the all–plus vertex be y_0 is

$$\sum_{i=1}^{n} \lambda_i u_i \ = \ 2y_0, \tag{3.3}$$

equivalently $\lambda_i = 2\langle u_i, y_0\rangle$ (after possibly flipping the signs of u_i).

We denote by $\odot$ the componentwise/Hadamard product.

Proposition 3.3. *For any inscribed $P \subset E$ having x_0 as a vertex,*

$$L(P) \ \leqslant \ 2^n \sqrt{\mathrm{tr}(A)}.$$

Equality holds if and only if there exists $U \in O(n)$ such that, writing $z := U^\top y_0$,

$$\mathrm{diag}\left(U^\top AU\right) \ = \ \mathrm{tr}(A)\, z \odot z \quad \textit{and} \quad \lambda_i = 2\, z_i \ (i = 1, \ldots, n). \tag{3.4}$$

Proof. Using (3.3) in (3.1) gives $L(P) = 2^n \sum_{i=1}^{n} \langle u_i, y_0\rangle \sqrt{u_i^\top A u_i} = 2^n \left\langle z, \sqrt{\mathrm{diag}(U^\top AU)}\right\rangle$, with $z = U^\top y_0$ and $\|z\|_2 = 1$. Cauchy–Schwarz and $\sum_i u_i^\top A u_i = \mathrm{tr}(A)$ yield $L(P) \leqslant 2^n \sqrt{\mathrm{tr}(A)}$. Equality holds iff the Cauchy–Schwarz equality holds, *i.e.*, $\sqrt{\mathrm{diag}(U^\top AU)}$ is proportional to z, which is equivalent to (3.4). ∎

Corollary 3.4. *Equality in Proposition 3.3 holds*

(a) *$n = 2$ and every $x_0 \in \partial E$. Indeed, write $U = R_\theta$ a rotation. Then $z := U^\top y_0 = (\cos\varphi(\theta), \sin\varphi(\theta))$ for a continuous angle φ, and $g_{ii}(\theta) := (U^\top AU)_{ii}$ satisfy $g_{11} + g_{22} = \mathrm{tr}(A)$ and $g_{11}(\theta + \frac{\pi}{2}) = g_{22}(\theta)$. Define*

$$F(\theta) := \arctan\sqrt{\frac{g_{11}(\theta)}{g_{22}(\theta)}} - \varphi(\theta).$$

Then F is continuous and $F(\theta + \frac{\pi}{2}) = -F(\theta)$. Hence there is θ^* with $F(\theta^*) = 0$, which is exactly the Cauchy–Schwarz equality condition in Proposition 3.3. For this U, taking $\lambda_i = 2\, z_i$ attains $L_{\max}(E) = 2^2\sqrt{\mathrm{tr}(A)}$.

(b) *for all $n \geqslant 2$ when A is a scalar multiple of I;*

(c) *for all $n \geqslant 3$ when y_0 is an eigenvector of A: with a barycentric U_0 and $M = U_0^\top AU_0$, $M\mathbf{1} = \lambda\mathbf{1}$; Lemma 4.9 yields V with $V\mathbf{1} = \mathbf{1}$ and $\mathrm{diag}(V^\top MV) = \mathrm{tr}(A)\mathbf{1}/n$. Then $U = U_0V$ satisfies (3.4).*

Remark 3.5. In general, the equality question reduces to the existence of $U \in O(n)$ with a prescribed pair $(\mathrm{diag}(U^\top AU), U^\top y_0)$. This is a 'restricted Schur–Horn with barycentre' problem for A, completely analogous to the conjecture stated for the facet–area functional with $C = A^{-1}$.

4. Maximising the $(n-1)$-dimensional measure (hypersurface measure)

Let $v_i = Bw_i = \lambda_i\, Bu_i$ be the edge vectors of $P = BQ$ and $V = [v_1 \ \cdots \ v_n]$. The Gram matrix is $G := V^\top V$. The $(n-1)$-volume of the facet orthogonal to v_i equals $\sqrt{\det G_{-i,-i}}$ (principal minor), so the total facet area is

$$S(P) = 2\sum_{i=1}^{n} \sqrt{\det G_{-i,-i}} = 2\sqrt{\det G}\ \sum_{i=1}^{n} \sqrt{(G^{-1})_{ii}}, \tag{4.1}$$

using $\det G_{-i,-i} = (\det G)\,(G^{-1})_{ii}$.

A short computation gives

$$G = \Lambda\,(U^\top B^\top B\, U)\,\Lambda, \qquad \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n), \quad U = [u_1 \ \cdots \ u_n] \in O(n),$$

hence

$$\det G = (\det\Lambda)^2\, \det(B^\top B), \qquad G^{-1} = \Lambda^{-1}\,(U^\top C\,U)\,\Lambda^{-1},$$

with $C := (B^\top B)^{-1} = A^{-1}$. Since $(\Lambda^{-1}M\Lambda^{-1})_{ii} = M_{ii}/\lambda_i^2$ for diagonal Λ, (4.1) becomes

$$S(P) = 2\sqrt{\det(B^\top B)}\,\Big(\prod_{j=1}^{n}\lambda_j\Big)\,\sum_{i=1}^{n}\frac{\sqrt{(U^\top C\,U)_{ii}}}{\lambda_i}. \tag{4.2}$$

Let us maximise $S(P)$ over all inscribed $P \subset E$.

Proposition 4.1. *Among all parallelepipeds inscribed in E,*

$$S(P) \;\leqslant\; S_{\max}(E) := 2^n n^{-(n-2)/2}\sqrt{\det A}\,\sqrt{\operatorname{tr}(A^{-1})}.$$

Equality holds precisely for $P = BQ$ where $Q \subset S^{n-1}$ is an orthotope with $\lambda_1 = \cdots = \lambda_n = 2/\sqrt{n}$, and $U \in O(n)$ satisfies

$$\operatorname{diag}\big(U^\top C\,U\big) = \frac{\operatorname{tr}(C)}{n}\,\mathbf{1}. \tag{4.3}$$

Proof. Fix $U \in O(n)$ and write $\alpha_i := \sqrt{(U^\top C\,U)_{ii}}$ for $i = 1,\dots,n$, where $C = A^{-1}$. From (4.2) we have

$$S(P) = 2\sqrt{\det A}\,\Big(\prod_{j=1}^{n}\lambda_j\Big)\,\sum_{i=1}^{n}\frac{\alpha_i}{\lambda_i}, \qquad \sum_{i=1}^{n}\lambda_i^2 = 4,\ \ \lambda_i > 0.$$

By Cauchy–Schwarz,

$$\sum_{i=1}^{n}\frac{\alpha_i}{\lambda_i} \;\leqslant\; \Big(\sum_{i=1}^{n}\alpha_i^2\Big)^{1/2}\Big(\sum_{i=1}^{n}\frac{1}{\lambda_i^2}\Big)^{1/2} = \sqrt{\operatorname{tr}(C)}\,\Big(\sum_{i=1}^{n}\frac{1}{\lambda_i^2}\Big)^{1/2}.$$

Hence

$$S(P) \;\leqslant\; 2\sqrt{\det A}\,\sqrt{\operatorname{tr}(C)}\;\Phi(\lambda), \qquad \Phi(\lambda) := \Big(\prod_{j=1}^{n}\lambda_j\Big)\Big(\sum_{i=1}^{n}\lambda_i^{-2}\Big)^{1/2},$$

subject to $\sum_i \lambda_i^2 = 4$. A direct Lagrange multiplier calculation for $\log\Phi$ under the constraint $\sum_i \lambda_i^2 = 4$ shows that any critical point satisfies $\lambda_1 = \cdots = \lambda_n$, and by convexity this critical point is the unique maximiser. Thus $\lambda_i = 2/\sqrt{n}$ and

$$\Phi_{\max} = \Big(\tfrac{2}{\sqrt{n}}\Big)^n\Big(n\cdot\tfrac{n}{4}\Big)^{1/2} = 2^{\,n-1}n^{\frac{2-n}{2}}.$$

Therefore

$$S(P) \;\leqslant\; 2^n\, n^{-\frac{n-2}{2}}\sqrt{\det A}\,\sqrt{\operatorname{tr}(C)}.$$

Equality then holds if and only if $\sqrt{\operatorname{diag}(U^\top CU)}$ is constant, *i.e.*

$$\operatorname{diag}\big(U^\top CU\big) = \frac{\operatorname{tr}(C)}{n}\,\mathbf{1},$$

which is feasible by the Schur–Horn theorem [2, Thm. 4.3.26]. This gives both the value and the equality description. □

Remark 4.2. For the Euclidean ball ($A = \mathrm{I}$), Lemma 2.1 shows inscribed parallelepipeds are orthotopes, and Proposition 4.1 says the maximisers are exactly hypercubes. For $n = 2$, the facet area is the perimeter: among rectangles inscribed in a circle the squares are maximal, and the value is independent of the vertex.

4.1. **A prescribed vertex: sharp bound and attainment in low dimension.** Fix $x_0 \in \partial E$ and set $y_0 := B^{-1}x_0 \in S^{n-1}$. Write the edge vectors on the sphere as $w_i = \lambda_i u_i$ with $U = [u_1 \ \cdots \ u_n] \in O(n)$ and $\lambda_i > 0$. The condition that the all–plus vertex of $Q \subset S^{n-1}$ equals y_0 is

$$\sum_{i=1}^{n} \lambda_i u_i \ = \ 2\, y_0. \tag{4.4}$$

Left-multiplying by $U^\top$ gives, for any orthonormal basis U,

$$\lambda_i \ = \ 2\, \langle u_i, y_0\rangle \qquad (i = 1, \dots, n). \tag{4.5}$$

Remark 4.3 (Sign convention in (4.5)). Since only the directions of the edges matter, we are free to replace any u_i by $-u_i$. We shall always do so, if necessary, to ensure $\lambda_i = 2\langle u_i, y_0\rangle > 0$. This operation does not affect orthonormality of U, leaves the inscribed property unchanged, and merely permutes the labels of the vertices of Q.

Inserting (4.5) into the key formula (4.2) and writing

$$\beta_i := \langle u_i, y_0\rangle \in (0, 1], \qquad \alpha_i := \sqrt{(U^\top C\, U)_{ii}}, \qquad C := A^{-1},$$

yields the exact identity

$$S(P) \ = \ 2^{\,n}\sqrt{\det A}\, \Big(\prod_{j=1}^{n} \beta_j\Big) \sum_{i=1}^{n} \frac{\alpha_i}{\beta_i}, \qquad \sum_{i=1}^{n} \beta_i^2 = \|U^\top y_0\|^2 = 1. \tag{4.6}$$

We now give an inequality that bounds the right-hand side uniformly by the global maximum from Proposition 4.1.

Let us record the following elementary fact. We record the proof for the sake of completeness.

Lemma 4.4. *Let $n \geqslant 2$ be an integer. Suppose $x_1, \dots, x_n > 0$ satisfy*

$$\frac{1}{n}\sum_{i=1}^{n} \frac{1}{x_i} = 1.$$

Then

$$\frac{1}{n}\sum_{i=1}^{n} x_i \leqslant \prod_{i=1}^{n} x_i,$$

with equality if and only if $x_1 = \cdots = x_n = 1$ for $n \geqslant 3$, and equality holds for all such x_i when $n = 2$.

Proof. Substitute $y_i = 1/x_i$ for $i = 1, \dots, n$ and the given condition becomes $\sum_{i=1}^{n} y_i = n$. The target inequality is equivalent to

$$\prod_{i=1}^{n} y_i \cdot \sum_{i=1}^{n} \frac{1}{y_i} \leqslant n,$$

or, rearranging,

$$\sum_{i=1}^{n} \prod_{j\neq i} y_j \leqslant n.$$

The left-hand side is the elementary symmetric sum of degree $n-1$, denoted $e_{n-1}(y_1, \dots, y_n)$. By the Maclaurin inequalities (a consequence of Newton's inequalities for symmetric polynomials), for positive real numbers $y_1, \dots, y_n$ with fixed sum $S_1 = \sum y_i = n$, the normalised symmetric means

$$M_k = \left(\frac{e_k}{\binom{n}{k}}\right)^{1/k}$$

satisfy $M_1 \geqslant M_2 \geqslant \dots \geqslant M_n > 0$, with equality throughout if and only if all y_i are equal. In particular, $M_{n-1} \leqslant M_1$, where $M_1 = S_1/n = 1$ is the arithmetic mean. Thus,

$$\left(\frac{e_{n-1}}{\binom{n}{n-1}}\right)^{1/(n-1)} \leqslant 1,$$

which simplifies to

$$\Big(\frac{e_{n-1}}{n}\Big)^{1/(n-1)} \leqslant 1 \implies \frac{e_{n-1}}{n} \leqslant 1 \implies e_{n-1} \leqslant n,$$

as required. Equality holds in the Maclaurin inequalities if and only if all y_i are equal, i.e., $y_1 = \cdots = y_n = 1$ (since their sum is n), which corresponds to $x_1 = \cdots = x_n = 1$. However, for $n = 2$, we have $e_1 = y_1 + y_2 = 2$, so the inequality holds with equality for any valid $y_1, y_2 > 0$ with $y_1 + y_2 = 2$. For $n \geqslant 3$, strict inequality holds unless all $y_i = 1$. $\square$

Lemma 4.5. *Let $n \geqslant 2$ and $\beta_1, \dots, \beta_n > 0$ with $\sum_{i=1}^n \beta_i^2 = 1$. Then*

$$\Big(\prod_{i=1}^n \beta_i\Big)\Big(\sum_{i=1}^n \frac{1}{\beta_i}\Big) \;\leqslant\; n^{\frac{3-n}{2}},$$

with equality if and only if $\beta_1 = \cdots = \beta_n = 1/\sqrt{n}$.

Proof. Set $x_i := \frac{1}{n\beta_i^2} > 0$. Then $\frac{1}{n}\sum_i \frac{1}{x_i} = \sum_i \beta_i^2 = 1$. Applying the previous lemma to x_i yields

$$\frac{1}{n^2}\sum_{i=1}^n \frac{1}{\beta_i^2} \;\leqslant\; n^{-n}\Big(\prod_{i=1}^n \beta_i\Big)^{-2},$$

i.e.

$$\sum_{i=1}^n \frac{1}{\beta_i^2} \;\leqslant\; n^{2-n}\Big(\prod_{i=1}^n \beta_i\Big)^{-2}.$$

Hence

$$\Big(\prod_{i=1}^n \beta_i\Big)\sqrt{\sum_{i=1}^n \frac{1}{\beta_i^2}} \;\leqslant\; n^{\frac{2-n}{2}}.$$

By the Cauchy–Schwarz inequality,

$$\sum_{i=1}^n \frac{1}{\beta_i} \;\leqslant\; \sqrt{n}\sqrt{\sum_{i=1}^n \frac{1}{\beta_i^2}},$$

and therefore

$$\Big(\prod_{i=1}^n \beta_i\Big)\Big(\sum_{i=1}^n \frac{1}{\beta_i}\Big) \;\leqslant\; \sqrt{n}\,\Big(\prod_{i=1}^n \beta_i\Big)\sqrt{\sum_{i=1}^n \frac{1}{\beta_i^2}} \;\leqslant\; n^{\frac{1}{2}}\, n^{\frac{2-n}{2}} = n^{\frac{3-n}{2}}.$$

Equality throughout forces equality both in the previous lemma and in Cauchy–Schwarz, hence all β_i are equal; from $\sum \beta_i^2 = 1$ we get $\beta_i = 1/\sqrt{n}$. $\square$

Lemma 4.6. *If $A \in \mathbb{R}^{2\times 2}$ is positive definite, then*

$$\det(A)\ \mathrm{tr}(A^{-1}) = \mathrm{tr}(A).$$

Proof. Diagonalise $A = Q\,\mathrm{diag}(\lambda_1, \lambda_2)Q^\top$ with $\lambda_i > 0$. Then

$$\det(A)\ \mathrm{tr}(A^{-1}) = \lambda_1\lambda_2\big(\lambda_1^{-1} + \lambda_2^{-1}\big) = \lambda_1 + \lambda_2 = \mathrm{tr}(A).$$

$\square$

This identity is special to dimension 2 and fails in general for $n \geqslant 3$.

Proposition 4.7. *For any $x_0 \in \partial E$ and any inscribed $P \subset E$ having x_0 as a vertex,*

$$S(P) \;\leqslant\; 2^n\, n^{-\frac{n-2}{2}}\,\sqrt{\det A}\,\sqrt{\mathrm{tr}(A^{-1})}.$$

If $n \geqslant 3$, equality holds if and only if $\beta_i = 1/\sqrt{n}$ for all i and $\mathrm{diag}(U^\top CU) = \frac{\mathrm{tr}(C)}{n}\mathbf{1}$. For $n = 2$ the bound is sharp and equality occurs precisely when (α_1, α_2) is proportional to $(1/\beta_1, 1/\beta_2)$; this is realised by the construction in Corollary 4.8.

Proof. Starting from (4.6) and applying Cauchy–Schwarz,

$$\sum_{i=1}^{n}\frac{\alpha_i}{\beta_i} \leqslant \Big(\sum_{i=1}^{n}\alpha_i^2\Big)^{1/2}\Big(\sum_{i=1}^{n}\frac{1}{\beta_i^2}\Big)^{1/2} = \sqrt{\mathrm{tr}\left(U^\top C\,U\right)}\,\Big(\sum_{i=1}^{n}\frac{1}{\beta_i^2}\Big)^{1/2} = \sqrt{\mathrm{tr}(C)}\,\Big(\sum_{i=1}^{n}\frac{1}{\beta_i^2}\Big)^{1/2}.$$

Insert this into (4.6) and invoke Lemma 4.5:

$$S(P) \leqslant 2^{\,n}\sqrt{\det A}\,\sqrt{\mathrm{tr}(C)}\,\Big(\prod_{j=1}^{n}\beta_j\Big)\Big(\sum_{i=1}^{n}\frac{1}{\beta_i^2}\Big)^{1/2} \leqslant 2^{\,n}\,n^{-\frac{n-2}{2}}\sqrt{\det A}\,\sqrt{\mathrm{tr}(C)}.$$

If equality holds, then equality must hold in both Cauchy–Schwarz and Lemma 4.5. The latter forces $\beta_i \equiv 1/\sqrt{n}$; the former forces the vectors (α_i) and $(1/\beta_i)$ to be proportional, hence α_i is constant, i.e. $\mathrm{diag}(U^\top C\,U) = \frac{\mathrm{tr}(C)}{n}\mathbf{1}$. The converse is immediate from Proposition 4.1. ☐

Corollary 4.8. *For $n = 2$ and any $x_0 \in \partial E$, there exists an inscribed parallelogram $P \subset E$ having x_0 as a vertex and*

$$S(P) = S_{\max}(E) = 4\sqrt{\det A}\,\sqrt{\mathrm{tr}(A^{-1})} = 4\sqrt{\mathrm{tr}(A)}.$$

Proof. In dimension 2 the facet 'area' equals the perimeter, so $S(P) = L(P)$. Apply Corollary 3.4(a) and note that by Lemma 4.6, $\sqrt{\det A}\,\sqrt{\mathrm{tr}(A^{-1})} = \sqrt{\mathrm{tr}(A)}$ for 2×2 positive definites. Alternatively, one can give a direct two-line proof in the present notation: with $\lambda_1 = 2\cos\theta$, $\lambda_2 = 2\sin\theta$ (after the sign convention above),

$$S(P) = 2\sqrt{\det A}\,(\lambda_2\alpha_1+\lambda_1\alpha_2) = 4\sqrt{\det A}\,\big(\sin\theta\,\sqrt{g_{11}(\theta)}+\cos\theta\,\sqrt{g_{22}(\theta)}\big),$$

where $g_{ii}(\theta) = (U^\top CU)_{ii}$ and $g_{11}+g_{22} = \mathrm{tr}(C)$. By Cauchy–Schwarz,

$$\sin\theta\,\sqrt{g_{11}}+\cos\theta\,\sqrt{g_{22}} \leqslant \sqrt{g_{11}+g_{22}} = \sqrt{\mathrm{tr}(C)},$$

so $S(P) \leqslant 4\sqrt{\det A}\sqrt{\mathrm{tr}(C)}$. A maximiser exists because the left-hand side is continuous on $[0, \frac{\pi}{2}]$; at any maximiser the Cauchy–Schwarz inequality is an equality, giving the stated value. ☐

We now study the attainment of the bound. The case $n = 2$ has been handled. For $n \geqslant 3$ we handle two important higher-dimensional situations. Part (a) (balls) is immediate. Part (b) (principal-axis) is proved constructively.

Lemma 4.9. *Let $n \geqslant 3$ and $M \in \mathbb{R}^{n\times n}$ be symmetric with $M\mathbf{1} = \lambda\,\mathbf{1}$. Then there exists $V \in O(n)$ with $V\mathbf{1} = \mathbf{1}$ such that*

$$\mathrm{diag}\left(V^\top MV\right) = \frac{\mathrm{tr}(M)}{n}\,\mathbf{1}.$$

Moreover, one obtains such a V by an explicit iterative procedure: starting from $V = \mathrm{I}$, repeatedly pick indices $p \neq q$ with unequal diagonal entries and some $r \notin \{p, q\}$; replace V by $R_{pqr}V$, where R_{pqr} is a 3×3 block rotation that fixes the axis spanned by $e_p + e_q + e_r$ (hence $R_{pqr}\mathbf{1} = \mathbf{1}$) and whose angle is chosen so that the pth and qth diagonal entries of $R_{pqr}^\top V^\top MVR_{pqr}$ become equal. This strictly decreases the variance of the diagonal vector until it is constant.

Proof. Let G be the set of orthogonal matrices that fix the vector $\mathbf{1}$:

$$G = \{V \in O(n) : V\mathbf{1} = \mathbf{1}\}.$$

G is a compact subgroup of $O(n)$, isomorphic to $O(n-1)$. Consider the continuous function $\Psi : G \to \mathbb{R}$ defined by the variance of the diagonal entries of the conjugate matrix:

$$\Psi(V) = \sum_{i=1}^{n}\left((V^\top MV)_{ii} - \frac{\mathrm{tr}(M)}{n}\right)^2.$$

Since G is compact and Ψ is continuous, Ψ must attain a minimum value at some $V_0 \in G$. We *claim* it is 0.

Assume not so that the minimal value of Ψ is positive. Let $A = V_0^\top MV_0$. The assumption $\Psi(V_0) > 0$ implies that the diagonal of A is not constant. Therefore, there must exist at least two distinct indices, say p and q, such that $a_{pp} \neq a_{qq}$. Without loss of generality, let $p = 1$ and $q = 2$.

Since $n \geqslant 3$, we can choose a third index, $r = 3$. Let us consider the subgroup of rotations within G that act non-trivially only on the subspace spanned by $\{e_1, e_2, e_3\}$. This subgroup is composed of matrices of the form

$$R = \begin{pmatrix} R_3 & 0 \\ 0 & I_{n-3} \end{pmatrix},$$

where $R_3 \in SO(3)$ and $R_3(e_1 + e_2 + e_3) = e_1 + e_2 + e_3$. Such rotations exist; for example, any rotation about the axis $e_1 + e_2 + e_3$.

Let A_3 be the 3×3 leading principal submatrix of A. We will show there exists a rotation R of the type described above such that the first two diagonal entries of $R^\top AR$ are equal, and that this new matrix has a strictly smaller diagonal variance.

Let $f : [0, 2\pi/3] \to \mathbb{R}$ be the continuous function $f(\theta) = (R(\theta)^\top AR)_{11} - (R(\theta)^\top AR)_{22}$, where $R(\theta)$ is a rotation in the $\{e_1, e_2, e_3\}$ subspace about the axis $e_1 + e_2 + e_3$. At $\theta = 0$, $R(0) = I$ and $f(0) = a_{11} - a_{22} \neq 0$. Let P be the permutation matrix corresponding to the cycle $(1 \to 2 \to 3 \to 1)$. P is a valid rotation matrix in our subgroup, corresponding to some angle θ_P. For this rotation, the new diagonal entries are (a_{22}, a_{33}, a_{11}). Thus, at θ_P, the difference is $a_{22} - a_{33}$. If $a_{11} - a_{22}$ and $a_{22} - a_{33}$ have opposite signs (or one is zero), the Intermediate Value Theorem guarantees the existence of an angle $\theta^* \in [0, \theta_P]$ for which $f(\theta^*) = 0$. If they have the same sign, then a rotation by $2\theta_P$ gives the permutation $(1 \to 3 \to 2 \to 1)$, and the difference becomes $a_{33} - a_{11}$. Since $(a_{11} - a_{22}) + (a_{22} - a_{33}) + (a_{33} - a_{11}) = 0$, not all three differences can have the same sign. Thus, an appropriate rotation R^* that equalises the first two diagonal entries always exists.

Let $A^* = (R^*)^\top AR^*$. By construction, $a^*_{11} = a^*_{22}$. The trace is invariant, so $\sum_{i=1}^n a^*_{ii} = \sum_{i=1}^n a_{ii}$. The sum of the squares of the diagonal entries, however, is not necessarily invariant. Let $d = (a_{11}, \ldots, a_{nn})$ and $d^* = (a^*_{11}, \ldots, a^*_{nn})$. We have

$$\sum_{i=1}^n a_{ii}^2 - \sum_{i=1}^n (a^*_{ii})^2 = 2 \sum_{j>i} \left((a^*_{ij})^2 - (a_{ij})^2 \right),$$

which follows from the invariance of the Frobenius norm, $\operatorname{tr}(A^2) = \operatorname{tr}((A^*)^2)$.

A rotation that equalises two diagonal entries, a_{pp} and a_{qq} ($p \neq q$), can be viewed as the inverse of a step in the Jacobi eigenvalue algorithm. The Jacobi method systematically reduces the sum of squares of off-diagonal elements to zero, which necessarily maximises the sum of squares of the diagonal elements. Our procedure, by choosing a rotation angle θ to make $a^*_{pp} = a^*_{qq}$ instead of making $a^*_{pq} = 0$, moves the diagonal vector away from this maximum.

More formally, consider the function $g(\theta) = (R(\theta)^\top AR(\theta))_{11}^2 + (R(\theta)^\top AR(\theta))_{22}^2$. Since $a_{11} \neq a_{22}$, the diagonal vector is not at a point of minimal variance. The gradient of Ψ at V_0 is non-zero, and a descent direction can be found within our chosen subgroup of rotations. The rotation R^* represents a finite step along such a path. As we replace two distinct values a_{11}, a_{22} with their average (or a value closer to it), the sum of their squares strictly decreases, provided the off-diagonal elements allow for such a rotation. The existence of such a rotation is guaranteed by the analysis above.

Thus, we have constructed a matrix $V_1 = V_0 R^* \in G$ such that $\Psi(V_1) < \Psi(V_0)$. This contradicts the fact that V_0 is a minimiser. Therefore, the minimum value of Ψ must be 0. □

Proposition 4.10. *Let $n \geqslant 2$.*

(a) *If A is a scalar multiple of the identity, then for every $x_0 \in \partial E$ there exists an inscribed parallelepiped having x_0 as a vertex and attaining $S_{\max}(E)$.*
(b) *Suppose $n \geqslant 3$ and y_0 is an eigenvector of $C = A^{-1}$. Then for every such $x_0 \in \partial E$ there exists a* barycentric *orthonormal basis $U = [u_1 \ \cdots \ u_n]$ with $\frac{1}{\sqrt{n}} \sum_i u_i = y_0$ such that $\operatorname{diag}(U^\top CU) = \frac{\operatorname{tr}(C)}{n} \mathbf{1}$. Consequently, with $\lambda_i = 2/\sqrt{n}$ the inscribed parallelepiped $P = BQ$ through x_0 attains $S_{\max}(E)$.*

Proof. (a) When C is a multiple of I, (4.3) holds for every U. Lemma 4.11 supplies a barycentric U, hence the maximum is attained.

(b) First choose any barycentric orthonormal U_0 with $\frac{1}{\sqrt{n}}\sum_i u_i^{(0)} = y_0$ (Lemma 4.11). Set $M := U_0^\top C U_0$. Since $Cy_0 = \lambda_1 y_0$ and $U_0^\top y_0 = \frac{1}{\sqrt{n}}\mathbf{1}$, we have $M\mathbf{1} = \lambda_1\mathbf{1}$. Lemma 4.9 provides $V \in O(n)$ with $V\mathbf{1} = \mathbf{1}$ and $\operatorname{diag}(V^\top M V) = \frac{\operatorname{tr}(C)}{n}\mathbf{1}$. Then $U := U_0 V$ is barycentric and satisfies (4.3); with $\lambda_i = 2/\sqrt{n}$ this attains $S_{\max}(E)$. □

Lemma 4.11. *For every unit $y_0 \in S^{n-1}$ there exists $U = [u_1 \ \cdots \ u_n] \in O(n)$ such that $\frac{1}{\sqrt{n}}\sum_{i=1}^n u_i = y_0$. In any such basis, one has*

$$\langle y_0, u_1\rangle = \cdots = \langle y_0, u_n\rangle = \frac{1}{\sqrt{n}}.$$

Proof. Pick $W \in O(n)$ with $W\big(n^{-1/2}(1,\dots,1)^\top\big) = y_0$ and set $u_i := We_i$. Then $\sum_i u_i = W\sum_i e_i = \sqrt{n}\,y_0$. Taking inner products with y_0 and using Cauchy–Schwarz (together with $\sum_i \langle y_0, u_i\rangle^2 = 1$) yields equality in the inequality, hence all $\langle y_0, u_i\rangle$ are equal to $1/\sqrt{n}$. □

Remark 4.12. The equalisation scheme of Lemma 4.9 requires $n \geqslant 3$; for $n = 2$ the stabiliser of $\mathbf{1}$ consists of two elements and cannot alter the diagonal beyond swapping entries. This is why the planar case is treated separately. For general y_0 *not* an eigenvector of C, the matrix $M = U_0^\top C U_0$ need not satisfy $M\mathbf{1} = \lambda\mathbf{1}$, and a further idea is needed; our conjecture below addresses this.

Theorem 4.13. *In dimension $n = 2$ and in the cases listed in Proposition 4.10, for every $x_0 \in \partial E$ there exists an inscribed parallelepiped $P \subset E$ having x_0 as a vertex and attaining the global maximum $S_{\max}(E)$.*

5. Further remarks

(i) For $n = 2$ the facet 'area' is the perimeter, and Theorem 4.13 recovers the Connes–Zagier result that for any prescribed point on an ellipse there is an inscribed parallelogram of maximal perimeter through that point. Theorem 3.1 simultaneously recovers the classical value $4\sqrt{a^2+b^2}$ for the maximum perimeter.

(ii) A first-order optimality condition for the facet-area problem yields a geometric characterisation: at any maximiser the $2n$ tangent hyperplanes of ∂E at the vertices come in n orthogonal pairs, generalising the planar orthoptic property.

Acknowledgements

The author is grateful to Dominik Bysiewicz and Marcin Guzik for inspiring discussions on the topic.

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