Stability threshold of close-to-Couette shear flows with no-slip boundary conditions in 2D

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Abstract

In this paper, we develop a stability threshold theorem for the 2D incompressible Navier-Stokes equations on the channel, supplemented with the no-slip boundary condition. The initial datum is close to the Couette flow in the following sense: the shear component of the perturbation is small, but independent of the viscosity ν. On the other hand, the x-dependent fluctuation is assumed small in a viscosity-dependent sense, namely, $O(\nu^{\frac{1}{2}}|\log \nu|^{-2})$. Under this setup, we prove nonlinear enhanced dissipation of the vorticity and a time-integrated inviscid damping for the velocity. These stabilizing phenomena guarantee that the Navier-Stokes solution stays close to an evolving shear flow for all time. The analytical challenge stems from a time-dependent nonlocal term that appears in the associated linearized Navier-Stokes equations.

Contents

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1 Introduction

In this paper, we study the quantitative stability threshold for perturbations of two-dimensional nearly-Couette shear flows with no-slip boundary conditions on the channel $\mathbb{T}\times[-1,1]$. Perturbations of the nonzero tangential modes that are assumed small (measured by a power of viscosity, ν) in an appropriate Sobolev space, are shown to undergo enhanced dissipation and inviscid damping. More precisely, we consider the incompressible 2D Navier-Stokes equations

$$\begin{aligned}
&\partial_t\tilde{u}+\tilde{u}\cdot\nabla\tilde{u}-\nu\Delta\tilde{u}+\nabla\tilde{p}=0,\\
&\nabla\cdot\tilde{u}=0,\quad \tilde{u}(t,x,y=\pm1)=(\pm1,0),\\
&\tilde{u}(t=0)=\tilde{u}_{\text{in}}(x,y),
\end{aligned}\tag{1.1}$$

on the domain $\Omega\times[0,\infty)=(\mathbb{T}\times[-1,1])\times[0,\infty)$, where $\nu>0$ denotes the inverse Reynolds number, $\tilde{u}(x,y)$ is the velocity field, and $\tilde{p}(x,y)$ is the pressure. We denote $U_{\text{in}}(y)$ to be the shear-flow component of the initial data and $u_{\text{in}}(x,y)$ to be the x-dependent component:

$$\tilde{u}_{\text{in}}(x,y)=(\tilde{u}^{(1)}_{\text{in}}(x,y),\tilde{u}^{(2)}_{\text{in}}(x,y))=(U_{\text{in}}(y),0)+u_{\text{in}}(x,y),\tag{1.2}$$

where

$$U_{\text{in}}(y):=\frac{1}{2\pi}\int_{-\pi}^{\pi}\tilde{u}^{(1)}_{\text{in}}(x,y)dx,\qquad u_{\text{in}}(x,y)=\tilde{u}_{\text{in}}(x,y)-\frac{1}{2\pi}\int_{-\pi}^{\pi}\tilde{u}_{\text{in}}(x,y)dx.$$

We assume that the shear component, $U_{\text{in}}(y)$, is an $O(\epsilon)$ Sobolev perturbation of the Couette flow (which satisfies the boundary conditions $U_{\text{in}}(y=\pm1)=\pm1$), whereas the x-dependent component, $u_{\text{in}}(x,y)$ is $O(\epsilon\nu^{\frac12}|\log\nu|^{-2})$. Importantly, $\epsilon>0$ is a small universal number and the viscosity is much smaller than this universal number, namely, $0<\nu\ll\epsilon$.

In general, the study of stability thresholds for fluid equations is by now vast, and we discuss related results in more detail below. For now, the reader should keep in mind two "nearest neighbor" results. First, Q. Chen, T. Li, D. Wei and Z. Zhang studied the nonlinear stability threshold of the purely-Couette case, $U_{\text{in}}(y)=y$ in [18]. Our result can be seen as a generalization of [18] to the close-to-Couette shear flow case, i.e., $U_{\text{in}}(y) = y + O(\epsilon)$. Such a generalization is important to obtain from the point of view of potential treatment of much larger, smoother data, such as those in [11,16]. Indeed, in the uniform-in-ν argument of [11], a similar threshold theorem was applied after the enhanced viscous time scale $t \gg O(\nu^{-1/3})$. While nonzero modes have been heavily dissipated by this time, the zero mode has not, and is instead an $O(\epsilon)$ perturbation of the Couette flow. In this case, one needs to understand precisely the stability threshold of nearly-Couette shear flows as opposed to purely-Couette flows, which is the setting of the present paper. The second "nearest-neighbor" result is that of [13], which treats nearly-Couette shear flows with the Navier boundary condition. Therefore, one can also think of the current paper as the no-slip version of [13], which presents significant new difficulties.

For the case when $u_{\text{in}}(x, y) \equiv 0$, the solution of the Navier-Stokes equations (1.1) is given by the heat extension of the initial data, that is

$$(U(t, y), 0) := (y + e^{\nu t\partial_y^2}(U_{\text{in}}(y) - y), 0), \tag{1.3}$$

where $e^{\nu t\partial_y^2}$ is the semi-group generated by the 1D heat equation subject to homogeneous Dirichlet boundary conditions. Naturally, if $U_{\text{in}}(y)$ is regular, then the resulting shear flow $U(t, y)$ is similarly uniform-in-time regular and converges to the Couette flow like $O(e^{-\nu t})$. This is because $U(t, y)-y$ solves the 1D heat equation with the Dirichlet boundary condition and the initial data $U_{\text{in}}(y)-y$, thus all the higher order derivatives of $U(t, y)-y$ are uniformly bounded in time.

For the case when $u_{\text{in}}(x, y)$ is small and nonzero, we write the solution $\tilde{u}(t, x, y)$ as a perturbation of the time-dependent shear profile $(U(t, y), 0)$ and hence define

$$u(t, x, y) := \tilde{u}(t, x, y) - (U(t, y), 0). \tag{1.4}$$

From (1.1) and (1.4), we infer that the exact perturbation $u(t, x, y) = (u^{(1)}, u^{(2)})$ solves

$$\begin{aligned}
&\partial_t u + U\partial_x u + u \cdot \nabla u - \nu\Delta u + (U'u^{(2)}, 0) + \nabla p = 0,\\
&\nabla \cdot u = 0, \quad u(t, x, y = \pm 1) = 0,\\
&u(t = 0, x, y) = u_{\text{in}}(x, y),
\end{aligned} \tag{1.5}$$

where we used the notation $U' = \partial_y U(t, y)$. Taking the curl of (1.5) and denoting

$$\omega = \operatorname{curl} u = -\partial_y u^{(1)} + \partial_x u^{(2)},$$

we obtain the vorticity formulation of the perturbed Navier-Stokes equations

$$\begin{aligned}
&\partial_t\omega + U\partial_x\omega + u \cdot \nabla\omega = U''\partial_x\psi + \nu\Delta\omega,\\
&u = \nabla^\perp\psi, \quad \Delta\psi = \omega, \quad (\partial_x, \partial_y)\psi\big|_{y=\pm 1} = 0,\\
&\omega(t = 0, x, y) = \omega_{\text{in}}(x, y),
\end{aligned} \tag{1.6}$$

where $U'' = \partial_y^2 U(t, y)$ and $\nabla^\perp = (-\partial_y, \partial_x)$.

The *quantitative stability threshold* is, given a norm on the initial data $\|\cdot\|_X$, the smallest $\gamma = \gamma(X) \geq 0$ such that $\|u_{\text{in}}\|_X \ll \nu^\gamma$ implies 'stability' and $\|u_{\text{in}}\|_X \gg \nu^\gamma$ in general permits 'instability', or at least fully nonlinear dynamics. The exact quantification of 'stability' may vary, but generally involves at least the enhanced dissipation characteristic of the linearized problem which shall be discussed further below. See [7] for a review and e.g. [15,18,22,25,33,34] and the references therein for 2D, and for 3D see [6,8,9,37] and the references therein. When studying high Reynolds number stability problems, the mixing induced by the shearing enhances the stability of the equilibrium, leading to (A) *inviscid damping* wherein the perturbation velocity decays in the linearized (or nonlinear) Euler equations and uniformly in Reynolds number and (B) *enhanced dissipation*, wherein the mixing accelerates the viscous dissipation, leading to rapid decay of the x-dependence on time-scales such as e.g. $\approx \nu^{-1/3}$ (rather than $\approx \nu^{-1}$). Many works have studied these in the linearized Navier-Stokes and Euler equations; see [5,26–28,32,35,39–42] and the references therein for inviscid damping results and e.g. [10,19,22,25,29,41] and the references therein for results on enhanced dissipation (some results study both, such as [19,29]). See for example [1–3,23] and the references therein on mixing and enhanced dissipation by laminar flows in passive scalars. The purpose of this paper is to obtain the stability threshold for all shear flows close to Couette, i.e. the shear profile in (1.2) satisfies $\|U_{\text{in}} - y\|_{H^4} \ll 1$.

To state our result, we introduce the x-Fourier transform

$$f_k(y) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x,y)\exp(-ikx)dx, \quad \text{for } k \in \mathbb{Z}.$$

In particular,

$$f_0(y) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x,y)dx.$$

For $T > 0$, we define the energy

$$\begin{aligned} E(T,\omega) = \|\omega_0\|_{L_t^\infty(0,T;L_y^2)} + \sum_{k\neq 0} \big(&\|e^{\zeta\nu^{1/3}t}(1-y^2)^{1/2}\omega_k\|_{L_t^\infty(0,T;L_y^2)} \\ &+ \nu^{1/4}|k|^{1/2}\|e^{\zeta\nu^{1/3}t}\omega_k\|_{L_t^2(0,T;L_y^2)} + |k|\|e^{\zeta\nu^{1/3}t}u_k\|_{L_t^2(0,T;L_y^2)}\big), \end{aligned}$$

where $0 < \zeta \ll 1$ is a small universal constant.

The following theorem is the main result of the paper.

Theorem 1.1. *Consider the solution ω to (1.6) with regular initial data $\omega_{\text{in}} \in H^2(\Omega)$. There exist constants $\nu_0, \kappa, \epsilon > 0$ such that, for any $\nu \in (0,\nu_0)$, if the following conditions are satisfied*

1. *Smallness:*

$$\|U_{\text{in}} - y\|_{H^4([-1,1])} \le \kappa, \tag{1.7}$$

$$\|\omega_{\text{in}}\|_{H^2(\Omega)} \le \epsilon\nu^{1/2}|\log\nu|^{-2}; \tag{1.8}$$

2. *Compatibility:*

$$\int_{-1}^{1} e^{\pm ky}\omega_{\text{in};k}(y)dy = 0, \quad \forall k \in \mathbb{Z}, \tag{1.9}$$

$$\partial_y^j U_{\text{in}}|_{y=\pm 1} = 0, \quad j = 2,4, \tag{1.10}$$

then the solution of (1.6) *satisfies the following global-in-time estimate*

$$\sup_{T\geq 0} E(T,\omega) \leq C\epsilon\nu^{1/2}|\log\nu|^{-1}.$$

Throughout this paper, we always assume that the smallness assumptions (1.7)–(1.8) and compatibility conditions (1.9)–(1.10) hold without mention.

Remark 1.2. The compatibility condition (1.9) is equivalent to the no-slip boundary condition on the initial velocity.

In [15], the analogue of this theorem was proved without boundaries (i.e. $y \in \mathbb{R}$) with initial data in Sobolev spaces $X = H^s$ with $s > 1$ (also without the logarithmic correction in ν) and for Navier boundary conditions in the channel in [13]. For the case of no-slip boundary conditions and $\|U_{\text{in}} - y\| \lesssim \nu^{1/2}$, this theorem was first proved in [18]. It was essentially proved in [17] that this result is sharp up to the logarithmic correction in ν even for $\|U_{\text{in}} - y\| \lesssim \nu^{1/2}$ in any finite regularity due to instabilities in the boundary layer.

In the case of $\mathbb{T}\times\mathbb{R}$, [33] establishes stability of the Couette flow under the assumption $\|U_{\text{in}} - y\| \lesssim \nu^{1/2}$ in the almost-critical space, $H^{\log}_x L^2_y$, whereas [30] proves a complimentary instability result. However, at higher Sobolev regularities, this was later improved to $\gamma \leq 1/3$ for sufficiently high Sobolev regularity [34, 38] and $\gamma \in [0, 1/3]$ for suitable Gevrey classes [31]. These higher regularity results are also expected to be sharp, due to the results of [24]. When one has at least Gevrey-2 regularity on the other hand, the result was proved with $\gamma = 0$ in [14] (i.e. the results are uniform-in-ν). The corresponding uniform-in-ν results with Navier boundary conditions were proved in [11, 12].

2 Outline

2.1 Setup

The main difficulty of this paper is to obtain a sufficiently precise understanding of the *linearized* Navier-Stokes equations associated with (1.6). The relevant linearized Navier-Stokes system for our analysis is the following system

$$\begin{aligned}
&\partial_t\omega + U\partial_x\omega - U''\partial_x\psi - \nu\Delta\omega = \nabla\cdot(\mathbb{f}_1, \mathbb{f}_2),\\
&u = \nabla^\perp\psi, \quad \Delta\psi = \omega, \quad (\partial_x, \partial_y)\psi|_{y=\pm1} = 0,\\
&\omega(t=0,x,y) = \omega_{\text{in}}(x,y).
\end{aligned} \tag{2.1}$$

We emphasize that $U = U(t, y)$ is time-dependent, as defined in (1.3). Taking the Fourier transform in x, the equations (2.1) reduce to the following k-by-k system:

$$\begin{aligned}
&\partial_t\omega_k + ikU\omega_k - U''ik\psi_k - \nu\Delta_k\omega_k = \nabla_k\cdot(\mathbb{f}_{1;k}, \mathbb{f}_{2;k}),\\
&u_k = \nabla_k^\perp\psi_k, \quad \Delta_k\psi_k = \omega_k,\\
&ik\psi_k|_{y=\pm1} = \partial_y\psi_k|_{y=\pm1} = 0,\\
&\omega_k(t=0,y) = \omega_{\text{in};k}(y),
\end{aligned} \tag{2.2a}$$

where $\nabla_k^\perp := (-\partial_y, ik)$, $\nabla_k := (ik, \partial_y)$ and $\Delta_k := \partial_y^2 - |k|^2$. We note that the boundary condition for ψ_k is over-determined. To make the system solvable, we always assume the consistency condition on ω_k:

$$\int_{-1}^{1} e^{\pm ky}\omega_k(t,y)dy = 0, \quad \forall t \geq 0, \quad k \in \mathbb{Z}. \tag{2.2b}$$

One can view this as an integral boundary condition for ω_k.

Next we specify the functional framework. Since the vorticity undergoes a transient growth near the boundary, the desirable norm should balance out this growth. We recall the standard setup here, see, e.g., [18]. The main focus of the estimates on the linearized Navier-Stokes equations (2.2) is to develop the uniform-in-time estimate for the weighted L^2-norm of the vorticity:

$$\|(1-|y|)^{1/2}\omega_k\|_{L_y^2},$$

and the time-integral estimate of the following quantity:

$$\|\omega_k\|_{\mathcal{Z}}^2 := \nu^{1/3}|k|^{2/3}\|\rho_k^{1/2}\omega_k\|_{L_y^2}^2 + |k|^2\|\nabla_k\Delta_k^{-1}\omega_k\|_{L_y^2}^2 + \nu^{1/2}|k|\|\omega_k\|_{L_y^2}^2. \tag{2.3}$$

Here, Δ_k^{-1} is the inverse of the Dirichlet Laplacian and the boundary weight ρ_k is defined as

$$\rho_k := \min\{L(1-|y|), 1\}, \quad L := \nu^{-1/3}|k|^{1/3}.$$

We observe that in the high-frequency regime, i.e., $\nu|k|^2 \gtrsim 1$, the $\mathcal{Z}$-norm is equivalent to the following simplified expression:

$$\|\omega_k\|_{\mathcal{Z}}^2 \approx |k|^2\|\nabla_k\Delta_k^{-1}\omega_k\|_{L_y^2}^2 + \nu^{1/2}|k|\|\omega_k\|_{L_y^2}^2.$$

The main linear estimate we establish for solutions to (2.2) is stated in the following proposition, which will be used to prove Theorem 1.1.

Proposition 2.1. *Let ω_k be the solution of $(2.2)_{k\neq 0}$ and $U(t,y)$ be the heat extension defined in (1.3). There exist constants $\zeta > 0$ and $0 < \nu_0, \kappa \ll 1$ such that if $\|U_{\rm in} - y\|_{H^4} \leq \kappa$ and $\nu \in (0, \nu_0]$, then*

$$\begin{aligned} &\|e^{\zeta\nu^{1/3}t}(1-|y|)^{1/2}\omega_k\|_{L_t^\infty L_y^2}^2 + \|e^{\zeta\nu^{1/3}t}\omega_k\|_{L_t^2\mathcal{Z}}^2 \\ &\lesssim |\log\nu|^2\left(|k|^{-2}\|\omega_{{\rm in};k}\|_{H_k^2}^2 + \min\{\nu^{-1/3}|k|^{4/3}, \nu^{-1}\}\|e^{\zeta\nu^{1/3}t}\mathbb{f}_{1;k}\|_{L_t^2L_y^2}^2 + \nu^{-1}\|e^{\zeta\nu^{1/3}t}\mathbb{f}_{2;k}\|_{L_t^2L_y^2}^2\right). \end{aligned} \tag{2.4}$$

Here, the $\mathcal{Z}$-norm is defined in (2.3) and $\|f_k\|_{H_k^2}^2 := \sum_{\alpha+\beta=2}\||k|^\alpha\partial_y^\beta f_k\|_{L_y^2}^2$.

Throughout this paper, we always assume that $k \neq 0$, unless stated otherwise.

Remark 2.2. The same proof also yields the space-time estimate (2.4) for the system (2.2) with general time-dependent shear flow $\mathbb{U}(t,y)$, provided the flow satisfies the following conditions:

$$\begin{aligned} &\|\mathbb{U} - y\|_{L_t^\infty H_y^4} \leq \kappa, \quad (\mathbb{U}-y)\big|_{y=\pm 1} = 0, \\ &\|\partial_y^2(\mathbb{U}(s,\cdot) - \mathbb{U}(t,\cdot))\|_{L_y^\infty} + \sup_{y\in[-1,1]}\left|\frac{\mathbb{U}(s,y)-\mathbb{U}(t,y)}{1-|y|}\right| \leq C\nu|s-t|, \quad \forall s,t \geq 0. \end{aligned}$$

In proving Proposition 2.1, we treat the high and low frequency regimes separately.
a) High Frequency Regime: We establish the following proposition for the high frequency regime.

Proposition 2.3. *Let ω_k be the solution of (2.2) and $U(t,y)$ be the heat extension defined in (1.3). There exists a constant $\zeta > 0$ such that for the high frequency regime $\nu|k|^2 \geq \frac{1}{2}\|\partial_y U\|_{L^\infty_{t,y}}$, the following estimate holds:*

$$\|e^{\zeta\nu^{1/3}t}(1-|y|)^{1/2}\omega_k\|^2_{L^\infty_t L^2_y} + \|e^{\zeta\nu^{1/3}t}\omega_k\|^2_{L^2_t \mathcal{Z}} \lesssim \nu^{-1}(\|e^{\zeta\nu^{1/3}t}\mathbb{f}_{1;k}\|^2_{L^2_t L^2_y} + \|e^{\zeta\nu^{1/3}t}\mathbb{f}_{2;k}\|^2_{L^2_t L^2_y}) + \|\omega_{\mathrm{in},k}\|^2_{L^2_y}.$$

The proof of this proposition is given in Section 3.6 and follows from standard energy estimates. Therefore, we omit further discussion and focus on the challenging low frequency regime.
b) Low Frequency Regime: To develop the linear estimates in the presence of the no-slip boundary condition $(1.6)_2$, we must use resolvent estimates similar to those studied, for example, in [18, 19, 21] among many other papers. Since the low frequency regime is more difficult, we will restrict our discussion to this case for the remainder of the section. Our goal is to establish the following proposition.

Proposition 2.4. *Let ω_k be the solution of (2.2) and $U(t,y)$ be the heat extension defined in (1.3). There exists a constant $\zeta > 0$ such for the low frequency regime $\nu|k|^2 \leq \inf_{t\in\mathbb{R}_+}\|\partial_y U(t,\cdot)\|_{L^\infty_y}$, the following estimate holds:*

$$\begin{aligned}&\|e^{\zeta\nu^{1/3}t}(1-|y|)^{1/2}\omega_k\|^2_{L^\infty_t L^2_y} + \|e^{\zeta\nu^{1/3}t}\omega_k\|^2_{L^2\mathcal{Z}}\\ &\lesssim |\log\nu|^2\left(|k|^{-2}\|\omega_{\mathrm{in};k}\|^2_{H^2_k} + \nu^{-1/3}|k|^{4/3}\|e^{\zeta\nu^{1/3}t}\mathbb{f}_{1;k}\|^2_{L^2_t L^2_y} + \nu^{-1}\|e^{\zeta\nu^{1/3}t}\mathbb{f}_{2;k}\|^2_{L^2_t L^2_y}\right).\end{aligned} \tag{2.5}$$

We will sketch the proof of this proposition in the forthcoming subsections. The complete proof will be concluded in Section 3.5. Furthermore, we observe that combining Propositions 2.3 and 2.4 yields the full linear estimates in Proposition 2.1. This is because when the shear flow is sufficiently close to Couette flow, the high frequency regime $\nu|k|^2 \geq \frac{1}{2}\|\partial_y U\|_{L^\infty_{t,y}}$ and the low frequency regime $\nu|k|^2 \leq \inf_{t\in\mathbb{R}_+}\|\partial_y U(t,\cdot)\|_{L^\infty_y}$ cover all possible $k \neq 0$.

The main challenges in deriving reasonably sharp quantitative estimates for (2.1) in the limit $\nu \to 0$ are the following:

A. The time-dependency of the background shear flow $U(t,y)$;

B. The nonlocal effect encoded by the term $U''\partial_x\psi$ in $(1.6)_1$;

C. The boundary layer vorticity generation induced by the no-slip boundary condition $(1.6)_2$.

We elaborate on these three aspects here.

Challenge A: The first challenge arises from the fact that representing semigroups in terms of resolvents uses a Laplace transform in time, which is not applicable to time-dependent linear operators. To overcome this difficulty, we employ the frozen-time scheme introduced in [21], wherein the analysis is carried out for a family of Navier-Stokes equations linearized around the stationary shear flows $U(t = j\nu^{-1/3}, y)$, where $j \in \mathbb{N}_0$. After establishing suitable estimates for this family, a layer-cake construction (see Section 2.3 for details) enables us to reconstruct the solution of the original system (2.2) with a time-dependent shear flow. Consequently, the frozen-time reduction allows us to focus on the time-independent linearized Navier-Stokes system.

Challenge B: The second difficulty arises from the linear nonlocal term $U''\partial_x\psi$. Although the closeness of the shear flow $U = U(t, y)$ to the Couette profile $(y, 0)$ ensures $U'' = O(\epsilon)$, this linear nonlocal term cannot be controlled directly by viscosity. A suitable inviscid damping mechanism is therefore necessary to absorb its effect. Under the Navier/Lions boundary condition $\omega|_{y=\pm 1} = 0$, one may use a hypocoercivity energy method (see, e.g., [4, 36]), combined with the singular integral operator $\mathfrak{J}_k$ introduced in [13], to derive the necessary estimates as carried out in [13]; in that setting, Challenge A is also avoided. In contrast, under the no-slip boundary condition, such energy-type estimates break down due to boundary contributions. In this work, we instead use the singular integral operator $\mathfrak{J}_k$ to provide the inviscid damping estimates on the resolvent which is one of the most crucial parts of the resolvent bounds (see Proposition 2.7). The introduction of this operator also simplifies the compactness argument presented in [19] in the close-to-Couette setting.

Challenge C: One of the most delicate aspects of the argument is the analysis of the boundary layers. We decompose the solution of (2.1) into two parts: a component ω^I which has zero initial data and accounts for the forcing on the right hand side, and a component ω^H, which incorporates the initial condition but is not subject to any external forcing. The latter can be treated relatively easily if one has a good enough method for the former component ω^I. As established in [18, 19], it is convenient to further decompose ω^I into ω_{Nav}, which satisfies the Navier/Lions boundary condition $\omega_{\mathrm{Nav}}|_{y=\pm 1} = 0$, and a boundary corrector ω_{B}. The corrector ω_{B} is determined by the velocity $u^{(1)}_{\mathrm{Nav}}$ generated by ω_{Nav} at the boundary points $y = \pm 1$, and it will generally have a large L^2 norm but be very well localized near the boundary. The estimates in [19] are insufficient to obtain a sharp enough bound on ω_{B}. In particular, one cannot directly derive the estimate (3.28), which is essential for controlling ω_{B}, from Proposition 2.7. We therefore establish a new estimate in Lemma 3.1 to address this issue.

The remaining part of the section is organized as follows: In Section 2.2, we present the linear estimates for the time-independent shear flows, which serve as the building blocks for the frozen-time scheme. In Section 2.3, we present the frozen-time scheme and upgrade the linear estimates for the time-independent shear flows to the time-dependent case and conclude the proof of the key Proposition 2.4. Finally, in Section 2.4, we provide the sketch of the proof of Theorem 1.1.

2.2 Linear Estimate: Time-independent Shear

In this section, we consider the following frozen-time linearized Navier-Stokes equations

$$\begin{aligned}
&\partial_t\omega_k-\nu\Delta_k\omega_k+\mathcal{U}ik\omega_k-\mathcal{U}''ik\psi_k=\nabla_k\cdot(\mathbb{f}_1,\mathbb{f}_2),\\
&\Delta_k\psi_k=\omega_k,\quad (ik,\partial_y)\psi_k\big|_{y=\pm1}=0,\quad \int_{-1}^{1}e^{\pm ky}\omega_k dy=0,\\
&\omega_k(t=0,y)=\omega_{\mathrm{in};k}(y),
\end{aligned}\tag{2.6}$$

where $\nabla_k=(ik,\partial_y)$ and $\mathcal{U}=\mathcal{U}(y)$ is a time-independent shear profile. We assume the boundary condition

$$(\mathcal{U}-y)|_{y=\pm1}=0.\tag{2.7}$$

Note that the system (2.6) is a general form of the frozen-time systems (2.25) and (2.27), and its estimates apply to both systems. We focus on the following low frequency regime:

$$\nu|k|^2\le\|\mathcal{U}'\|_{L^\infty_y}.\tag{2.8}$$

Our goal is to establish the following key proposition.

Proposition 2.5. *Let ω_k be the solution of (2.6) in the low frequency regime (2.8). There exist constants $\delta>0$ and $0<\nu_0,\kappa\ll1$ such that if $\|\mathcal{U}-y\|_{H^4}\le\kappa$, the boundary condition (2.7) holds, and $\nu\in(0,\nu_0]$, then*

$$\begin{aligned}
&\|e^{\delta\nu^{1/3}t}(1-|y|)^{1/2}\omega_k\|^2_{L^\infty_tL^2_y}+\|e^{\delta\nu^{1/3}t}\omega_k\|^2_{L^2_t\mathcal{Z}}\\
&\quad\lesssim|\log\nu|^2\left(|k|^{-2}\|\omega_{\mathrm{in};k}\|^2_{H^2_k}+\nu^{-1/3}|k|^{4/3}\|e^{\delta\nu^{1/3}t}\mathbb{f}_1\|^2_{L^2_tL^2_y}+\nu^{-1}\|e^{\delta\nu^{1/3}t}\mathbb{f}_2\|^2_{L^2_tL^2_y}\right).
\end{aligned}\tag{2.9}$$

We present the proof of Proposition 2.5 in Section 3.4. To prove it, we first decompose the solution of (2.6) into its homogeneous component ω_k^H and inhomogeneous component ω_k^I, namely,

$$\omega_k=\omega_k^H+\omega_k^I,\tag{2.10}$$

where ω_k^I solves

$$\begin{cases}
\partial_t\omega_k^I+ik\mathcal{U}(y)\omega_k^I-ik\mathcal{U}''(y)\psi_k^I-\nu\Delta_k\omega_k^I=\mathbb{F}_k,\\
\Delta_k\psi_k^I=\omega_k^I,\quad \psi_k^I\big|_{y=\pm1}=0,\quad \int_{-1}^{1}e^{\pm ky}\omega_k^I(t,y)dy=0,\\
\omega_k^I(t=0,y)=0,
\end{cases}\tag{2.11}$$

and ω_k^H solves

$$\begin{cases}
\partial_t\omega_k^H+ik\mathcal{U}(y)\omega_k^H-ik\mathcal{U}''(y)\psi_k^H-\nu\Delta_k\omega_k^H=0,\\
\Delta_k\psi_k^H=\omega_k^H,\quad \psi_k^H\big|_{y=\pm1}=0,\quad \int_{-1}^{1}e^{\pm ky}\omega_k^H(t,y)dy=0,\\
\omega_k^H(t=0,y)=\omega_{\mathrm{in};k}(y).
\end{cases}\tag{2.12}$$

The inhomogeneous component ω_k^I aims to address the external forcing, while the homogeneous component captures the linear evolution of the initial data. Importantly, obtaining suitable estimates for the inhomogeneous problem also provides control over the homogeneous part. We begin with the analysis of ω_k^I in Section 2.2.1, followed by that of ω_k^H in Section 2.2.2.

2.2.1 Decomposition of the Inhomogeneous Solution ω_k^I

As in [10, 18, 19], we use resolvent estimates. To motivate the key equations under consideration, we apply the Fourier transform in time to a suitably normalized solution. Since we expect the solutions to decay with rate $e^{-\zeta\nu^{1/3}k^{2/3}t}$ for some constant $\zeta > 0$, we consider the normalized vorticity, stream function, and forcing with the enhanced dissipation time weight, i.e.,

$$\omega_k^I(t,y)e^{\zeta\nu^{1/3}|k|^{2/3}t}, \quad \psi_k^I(t,y)e^{\zeta\nu^{1/3}|k|^{2/3}t}, \quad \mathbb{F}_k(t,y)e^{\zeta\nu^{1/3}|k|^{2/3}t}.$$

We define the intermediate Fourier variable which corresponds to t, i.e., $\widetilde{\lambda} \in \mathbb{R}$. To be consistent with the literature, we also recall the actual spectral parameter

$$\lambda_r := -\widetilde{\lambda}/k, \quad \lambda_i := -\zeta(\nu k^{-1})^{1/3}, \quad \lambda := \lambda_r + i\lambda_i.$$

This is equivalent to shifting the contour in the complex plane accordingly. Furthermore, we extend the aforementioned functions to be zero for the time regime $t < 0$. This yields a transform which is essentially equivalent to the Laplace transform (though less flexible). The corresponding Fourier transforms hence read as follows

$$\begin{aligned} &w_k^I(\lambda,y) := \int_0^\infty \omega_k^I(t,y)e^{itk\lambda}dt, \quad \phi_k^I(\lambda,y) := \int_0^\infty \psi_k^I(t,y)e^{itk\lambda}dt, \\ &F_k(\lambda,y) := \int_0^\infty \mathbb{F}_k(t,y)e^{itk\lambda}dt. \end{aligned} \tag{2.13}$$

Note that $w_k^I(t=0,y) = 0$, and thus the following relation holds

$$\int_0^\infty \partial_t\omega_k^I e^{itk\lambda}dt = -\omega_k^I(t=0,y) - ik\lambda\int_0^\infty \omega_k^I e^{itk\lambda}dt = -ik\lambda w_k^I.$$

Consequently, after the Fourier transform in the t-variable, the system (2.11) becomes

$$\begin{cases} -\nu\Delta_k w_k^I + ik\,(\mathcal{U}-\lambda)\,w_k^I - ik\,\mathcal{U}''\phi_k^I = F_k, \\ \Delta_k\phi_k^I = w_k^I, \quad (ik,\partial_y)\phi_k^I\Big|_{y=\pm1} = 0, \quad \displaystyle\int_{-1}^1 e^{\pm ky}w_k^I dy = 0. \end{cases} \tag{2.14}$$

The above boundary condition on w_k^I is more difficult than the Navier boundary condition case. A classical idea used in [18] is to decompose the solution into a contribution associated with the Navier boundary conditions and then additional boundary correctors:

$$w_k^I(\lambda,y) = w_{\mathrm{Na};k}(\lambda,y) + \underbrace{c_+(\lambda;k)w_{+;k}(\lambda,y) + c_-(\lambda;k)w_{-;k}(\lambda,y)}_{w_{\mathrm{BC};k}}, \tag{2.15}$$

where $w_{\mathrm{Na};k}$ is the solution of the system with Navier boundary condition and $w_{\mathrm{BC};k}$ is the boundary corrector that ensure that w_k^I satisfies the no-slip boundary condition. More precisely, the Navier component $w_{\mathrm{Na};k}$ solves

$$\begin{cases} -\nu(\partial_y^2-k^2)w_{\mathrm{Na};k}+ik\,(\mathcal{U}-\lambda)\,w_{\mathrm{Na};k}-ik\mathcal{U}''\phi_{\mathrm{Na};k}=F_k, \\ \Delta_k\phi_{\mathrm{Na};k}=w_{\mathrm{Na};k}, \quad \phi_{\mathrm{Na};k}\big|_{y=\pm1}=0, \quad w_{\mathrm{Na};k}\big|_{y=\pm1}=0. \end{cases} \tag{2.16}$$

Note that the boundary conditions $(2.16)_2$ imply that $\partial_y^2\phi_{\mathrm{Na};k}\big|_{y=\pm1}=0$. The elementary parts $w_{\pm;k}$ in the boundary corrector solve

$$\begin{cases} -\nu(\partial_y^2-k^2)w_{\pm;k}+ik\,(\mathcal{U}-\lambda)\,w_{\pm;k}-ik\mathcal{U}''\phi_{\pm;k}=0, \\ \phi_{\pm;k}=\Delta_k^{-1}w_{\pm;k}, \quad \partial_y\phi_{\pm;k}\big|_{y=\pm1}=1, \quad \partial_y\phi_{\pm;k}\big|_{y=\mp1}=0. \end{cases}$$

Here, Δ_k^{-1} is the inverse of the Dirichlet Laplacian. The coefficients $c_\pm(\lambda;k)$ are chosen to enforce the boundary condition

$$\int_{-1}^{1}e^{\pm ky}w_k^I\,dy=0$$

in $(2.14)_2$. Therefore, we take

$$c_+(\lambda;k)=-\partial_y\phi_{\mathrm{Na};k}(y=1), \quad c_-(\lambda;k)=-\partial_y\phi_{\mathrm{Na};k}(y=-1). \tag{2.17}$$

The boundary conditions of $\phi_{\pm;k}$ are chosen so that the elementary boundary corrector corrects one boundary condition while leaving the other intact. The main difficulty is that $\mathcal{U}(y)$ is not the Couette flow, leading to the presence of the nonlocal term $\mathcal{U}''ik\phi_{\pm;k}$. A related problem was addressed in [19], where numerous estimates for similar boundary correctors were established.

The goal of the decomposition is to obtain the following estimates on the inhomogeneous component.

Proposition 2.6. *Let ω_k^I be the solution of (2.11) in the low frequency regime (2.8) with the forcing*

$$\mathbb{F}_k=ik\mathbb{f}_1+\partial_y\mathbb{f}_2+\mathbb{f}_3, \quad \mathbb{f}_1,\mathbb{f}_2\in L^2,\ \mathbb{f}_3\in H_0^1.$$

There exist constants $\delta>0$ and $0<\kappa,\nu_0\ll1$ such that if $\|\mathcal{U}-y\|_{H^4}<\kappa$ and $\nu\in(0,\nu_0)$ then the following estimate holds:

$$\begin{aligned} &\nu^{1/2}|k|\|e^{\delta(\nu k^2)^{1/3}t}\omega_k^I\|_{L^\infty L^2}^2+\|e^{\delta(\nu k^2)^{1/3}t}\omega_k^I\|_{L^2\mathcal{Z}}^2 \\ &\lesssim \log^2 L\big(\nu^{-1/3}|k|^{4/3}\|e^{\delta(\nu k^2)^{1/3}t}\mathbb{f}_1\|_{L^2L^2}^2+\nu^{-1}\|e^{\delta(\nu k^2)^{1/3}t}\mathbb{f}_2\|_{L^2L^2}^2+\|e^{\delta(\nu k^2)^{1/3}t}\nabla_k\mathbb{f}_3\|_{L^2L^2}^2\big). \end{aligned} \tag{2.18}$$

The proof of the proposition requires several preparatory steps and will be completed in Section 3.4. By Plancherel's theorem and the relation (2.13), the problem reduces to establishing estimates for each component of w_k^I in (2.15), namely, the Navier component $w_{\mathrm{Na};k}$, the unit boundary correctors $w_{\pm;k}$, and the coefficients $c_{\pm;k}$. Compared with [18, Proposition 6.5], our analysis includes the additional treatment of the component $\mathbb{f}_3 \in H_0^1$, which is used to control the nonlocal term $ik\mathcal{U}''\psi_k^I$ in (2.11).

Firstly, we consider the Navier component w_{Na} and present the key estimates. For notational simplicity, we omit the k-dependence and define

$$\mathbb{E}[w,\phi] := \nu^{\frac{1}{6}}|k|^{\frac{4}{3}}\|\nabla_k\phi\|_{L^2} + \nu^{\frac{2}{3}}|k|^{\frac{1}{3}}\|\nabla_k w\|_{L^2} + \nu^{\frac{1}{3}}|k|^{\frac{2}{3}}\|w\|_{L^2}.$$

We summarize the key estimates as follows.

Proposition 2.7. *Let $w_{\mathrm{Na};k}$ be the solution of (2.16) in the low frequency regime (2.8). Assume the parameter constraints $\|\mathcal{U}-y\|_{H^4} \leq \kappa$. There exists a threshold $0 < \kappa_0(\mathcal{U}) = \mathcal{O}(1)$ and $0 < \delta_0 < 1$ such that if $0 < \kappa \leq \kappa_0$ and $k \operatorname{Im}\lambda \geq -\delta_0\nu^{1/3}|k|^{2/3}$, then the following estimates hold:*

$$\mathbb{E}\left[w_{\mathrm{Na};k}, \phi_{\mathrm{Na};k}\right] + \sqrt{|k|(\mathrm{Im}\lambda)_+}\|w_{\mathrm{Na};k}\|_{L^2} + \nu\left\|\Delta_k w_{\mathrm{Na};k}\right\|_{L^2_y} \lesssim \|F_k\|_{L^2_y}; \tag{2.19}$$

$$\mathbb{E}\left[w_{\mathrm{Na};k}, \phi_{\mathrm{Na};k}\right] + \sqrt{|k|(\mathrm{Im}\lambda)_+}\|w_{\mathrm{Na};k}\|_{L^2} + \nu\left\|\Delta_k w_{\mathrm{Na};k}\right\|_{L^2_y} \lesssim \nu^{\frac{1}{6}}|k|^{-\frac{2}{3}}\left\|\nabla_k F_k\right\|_{L^2_y}; \tag{2.20}$$

$$\mathbb{E}\left[w_{\mathrm{Na};k}, \phi_{\mathrm{Na};k}\right] + \sqrt{|k|(\mathrm{Im}\lambda)_+}\|w_{\mathrm{Na};k}\|_{L^2} \lesssim \nu^{-\frac{1}{3}}|k|^{\frac{1}{3}}\|F_k\|_{H^{-1}_y}. \tag{2.21}$$

Here, the H^{-1}-norm is defined as

$$\|F\|_{H^{-1}} := \sup\left\{\langle F, g\rangle : g \in H_0^1([-1,1]),\ \|\nabla_k g\|_{L^2} := (\|\partial_y g\|_{L^2}^2 + |k|^2\|g\|_{L^2}^2)^{1/2} = 1\right\}.$$

Proposition 2.7 is analogous to [19, Proposition 3.13]. In [19], the authors employ a delicate compactness argument to prove these estimates under the assumption that the ambient shear profile $\mathcal{U}(y)$ is spectrally stable. When the shear flow is sufficiently close to the Couette flow, however, the singular integral operator $\mathfrak{J}_k$ introduced in [13] yields a simpler and more quantitative proof. For completeness, we present the proof of Proposition 2.7 in Appendix B.

Next, we discuss the coefficients $c_\pm$ in (2.15). Thanks to the formula (2.17), one might expect that the estimates developed in Proposition 2.7 are sufficient to provide estimates for the coefficients. However, it turns out that these two coefficients are closely related to the L^1_y-bound of w_{Na}, and the $\nu^{1/4}\|w_{\mathrm{Na}}\|_{L^1_y}$ control in [19, Proposition 3.13] is insufficient. To compensate for the loss of information, we keep track of an augmented quantity

$$\|(\mathcal{U}-\lambda)w_{\mathrm{Na}} - \mathcal{U}''\phi_{\mathrm{Na}}\|_{L^2}$$

in (3.2). A suitable control over this quantity implies the desirable bounds on $c_\pm$, see, e.g., Lemma 3.4. This step is basically quantifying some inviscid damping in the linearized Euler equations which will be used to estimate the coefficients. Finally, we follow the argument in [19] to develop bounds on the unit boundary correctors $w_\pm$.

Once the estimates mentioned above are carried out, we combine them to conclude the estimate for w^I.

2.2.2 Decomposition of the Homogeneous Solution ω_k^H

Next, we solve the homogeneous part of the solution ω_k^H. We begin by introducing the decomposition

$$\omega_k^H(t,y) = \omega_k^{(1)}(t,y) + \omega_k^{(2)}(t,y) + \omega_k^{(3)}(t,y),$$

$$\Delta_k \phi_k^{(j)} = \omega_k^{(j)}, \quad \phi^{(j)}\big|_{y=\pm1} = 0, \quad j = 1,2,3.$$

We set $\omega_k^{(1)}$, which can be considered an approximation for the passive scalar evolution, as follows

$$\omega_k^{(1)}(t,y) = \omega_{\mathrm{in};k}(y) \exp\left\{ -it\mathcal{U}(y)k - \frac{1}{3}(\mathcal{U}')^2 \nu k^2 t^3 - \nu k^2 t \right\}. \tag{2.22}$$

The component $\omega_k^{(2)}$ corrects the residual which solves

$$\begin{aligned}
&\partial_t \omega_k^{(2)} + \mathcal{U} ik \omega_k^{(2)} - \mathcal{U}'' ik \psi_k^{(2)} - \nu \Delta_k \omega_k^{(2)} = \nu \partial_y^2 \omega_k^{(1)} + (\mathcal{U}')^2 \nu k^2 t^2 \omega_k^{(1)} + ik\mathcal{U}'' \phi_k^{(1)}, \\
&\phi_k^{(2)} = \Delta_k^{-1} \omega_k^{(2)}, \quad \int_{-1}^{1} \omega_k^{(2)} e^{\pm ky} dy = 0, \\
&\omega_k^{(2)}(t=0,y) = 0.
\end{aligned} \tag{2.23}$$

Note that the first two terms on the right-hand side of $(2.23)_1$ equal

$$\begin{aligned}
(\nu \partial_y^2 + (\mathcal{U}')^2 \nu k^2 t^2) \omega_k^{(1)} &= \nu(\partial_y - i\mathcal{U}' kt)(\partial_y + i\mathcal{U}' kt)\omega_k^{(1)} - \nu i \mathcal{U}'' kt \omega_k^{(1)} \\
&= \nu \Gamma_k^2 \omega_k^{(1)} - 2\nu i \mathcal{U}' kt \Gamma_k \omega_k^{(1)} - \nu i \mathcal{U}'' kt \omega_k^{(1)},
\end{aligned}$$

where we denote the vector field $\Gamma = \partial_y - i\mathcal{U}' kt$, which commutes with the inviscid passive scalar equation. Since $\omega_k^{(1)}$ experiences transport and enhanced dissipation, the term $\Gamma_k^2 \omega_k^{(1)}$ also experiences enhanced dissipation and this quantity remains small. We further remark that if the initial data is smooth enough, then the term $i\mathcal{U}'' \phi_k^{(1)}$ decays like $\langle kt \rangle^{-2}$ (independent of ν) due to inviscid damping. The component $\omega^{(3)}$ corrects the boundary condition which solves

$$\begin{aligned}
&\partial_t \omega_k^{(3)} + \mathcal{U} ik \omega_k^{(3)} - \mathcal{U}'' ik \psi_k^{(3)} - \nu \Delta_k \omega_k^{(3)} = 0, \\
&\int_{-1}^{1} \omega_k^{(3)} e^{\pm ky} dy = -\int_{-1}^{1} \omega_k^{(1)} e^{\pm ky} dy, \qquad \omega_k^{(3)}(t=0,y) = 0.
\end{aligned}$$

Most of the estimates on ω_k^H follow quickly from the corresponding estimates used in the proof of the inhomogeneous problem. This will eventually yield the following proposition.

Proposition 2.8. *Let ω_k^H be the solution of (2.12) in the low frequency regime (2.8). There exist constants $\delta > 0$ and $0 < \kappa, \nu_0 \ll 1$ such that if $\|\mathcal{U} - y\|_{H^4} < \kappa$ and $\nu \in (0, \nu_0)$, then the following estimate holds:*

$$\nu^{1/2} |k| \| e^{\delta (\nu k^2)^{1/3} t} \omega_k^H \|_{L^\infty L^2}^2 + \| e^{\delta (\nu k^2)^{1/3} t} \omega_k^H \|_{L^2 \mathcal{Z}}^2 \lesssim (\log^2 L) |k|^{-2} \| \omega_{\mathrm{in};k} \|_{H_k^2}^2.$$

The proof of the proposition is presented in Section 3.4. Combining the decomposition $\omega_k = \omega_k^I + \omega_k^H$ in (2.10) and the estimates presented in Propositions 2.6 and 2.8, one obtains the key linear estimate (2.9). Hence, the proof for Proposition 2.5 is concluded.

2.3 Linear Estimate: Time-dependent Shear Flows

In this section, we present the frozen-time scheme to upgrade the estimate (2.9) for time-independent shear flows to the bound (2.5) presented in the key Proposition 2.4, following the general framework outlined in [21]. It turns out that to prove the proposition, the key is to control the $L_t^2\mathcal{Z}$-norm of the vorticity. Once this time integral bound is developed, the uniform-in-time estimate of the vorticity in (2.9) is straightforward to achieve. Hence, we will focus on $L_t^2\mathcal{Z}$-norm estimate throughout this section.

We first rewrite the linearized system (2.2) equivalently as

$$\begin{aligned}&\partial_t\omega_k + U(t,y)ik\omega_k - U''(t,y)ik\psi_k - \nu\Delta_k\omega_k = \nabla_k\cdot(\mathbb{f}_1,\mathbb{f}_2),\\&\Delta_k\psi_k=\omega_k,\quad \psi_k|_{y=\pm1}=0,\quad \int_{-1}^{1}e^{\pm ky}\omega_k(t,y)dy=0,\\&\omega_k(t=0,y)=\omega_{\mathrm{in};k}(y).\end{aligned}\tag{2.24}$$

Let $t_j=j\nu^{-1/3}$ and $\mathcal{I}_{[j]}=[t_j,t_{j+1})\cap[0,T]$. There exists some $N\in\mathbb{N}$ such that

$$[0,T]=\bigcup_{j=0}^{N}\mathcal{I}_{[j]}.$$

For $0\le j\le N$, we define the "frozen shear" within each time interval $\mathcal{I}_{[j]}$ as

$$U_{[j]}(y):=U(t_j,y).$$

We introduce the following layer-cake decomposition. Let $\omega_{[0]}$ be the solution of

$$\begin{aligned}&\partial_t\omega_{[0]} + U_{[0]}ik\omega_{[0]} - U''_{[0]}ik\psi_{[0]} - \nu\Delta_k\omega_{[0]} = \nabla_k\cdot(\mathbb{f}_1+\mathbb{f}_{\mathrm{Disc}[0]},\mathbb{f}_2)\mathbb{1}_{t\in\mathcal{I}_{[0]}},\\&\Delta_k\psi_{[0]}=\omega_{[0]},\quad \psi_{[0]}|_{y=\pm1}=0,\quad \int_{-1}^{1}e^{\pm ky}\omega_{[0]}(t,y)dy=0,\\&\omega_{[0]}(t=0,y)=\omega_{\mathrm{in};k}(y),\end{aligned}\tag{2.25}$$

where

$$\mathbb{f}_{\mathrm{Disc}[0]}:=(U_{[0]}(y)-U(t,y))\omega_{[0]}-(U''_{[0]}(y)-U''(t,y))\psi_{[0]}.\tag{2.26}$$

For $1\le j\le N$, let $\omega_{[j]}$ be the solution of

$$\begin{aligned}&\partial_t\omega_{[j]} + U_{[j]}ik\omega_{[j]} - U''_{[j]}ik\psi_{[j]} - \nu\Delta_k\omega_{[j]} = \nabla_k\cdot(\mathbb{f}_1+\mathbb{f}_{\mathrm{Disc}[j]}+\mathbb{f}_{\mathrm{Frozen}[j]},\mathbb{f}_2)\mathbb{1}_{t\in\mathcal{I}_{[j]}},\\&\Delta_k\psi_{[j]}=\omega_{[j]},\quad \psi_{[j]}|_{y=\pm1}=0,\quad \int_{-1}^{1}e^{\pm ky}\omega_{[j]}(t,y)dy=0,\quad \omega_{[j]}(t=0,y)=0,\end{aligned}\tag{2.27}$$

where

$$\mathbb{f}_{\mathrm{Disc}[j]}:=(U_{[j]}(y)-U(t,y))\sum_{j'=0}^{j}\omega_{[j']}-(U''_{[j]}(y)-U''(t,y))\sum_{j'=0}^{j}\psi_{[j']},\tag{2.28}$$

$$\mathbb{f}_{\mathrm{Frozen}[j]} := \sum_{j'=0}^{j-1} \left[(U''_{[j]} - U''_{[j']})\psi_{[j']} - (U_{[j]} - U_{[j']})\omega_{[j']} \right]. \tag{2.29}$$

One can check that since the frozen shear $U_{[j]}(y)$ remains close to $U(t, y)$ on the time interval $\mathcal{I}_{[j]}$, the discretization error $\mathbb{f}_{\mathrm{Disc}[j]}$ and frozen error $\mathbb{f}_{\mathrm{Frozen}[j]}$ are small in the space $L^2_t(\mathcal{I}_{[j]}; L^2_y)$ (see Lemma B.3). On each time interval $\mathcal{I}_{[j]}$ $(0 \le j \le N)$, it is readily checked that the solution of the linearized Navier-Stokes equations (2.24) can be written as

$$\omega_k(t, y) = \sum_{j'=0}^{j} \omega_{[j']}. \tag{2.30}$$

The key observation here is that each component $\omega_{[j]}$ experiences a time-independent shear $U_{[j]}$. Hence, the linear estimate (2.9) developed for *time-independent* shear flows are valid.

Based on the decomposition (2.30), we would like to design two types of functional. The first type of functional ("horizontal functional" $X_{[j]}$) captures the sharp enhanced dissipation and inviscid damping for each $\omega_{[j]}$-component. Bounds on the $X_{[j]}$-functional can be derived from the linear estimate for the *time-independent shear flows* (Proposition 2.5). On the other hand, we need another type of functional ("vertical functional" $Y_{[j]}$) to collect all the contributions from the decomposition (2.30) on each time interval $\mathcal{I}_{[j]}$.

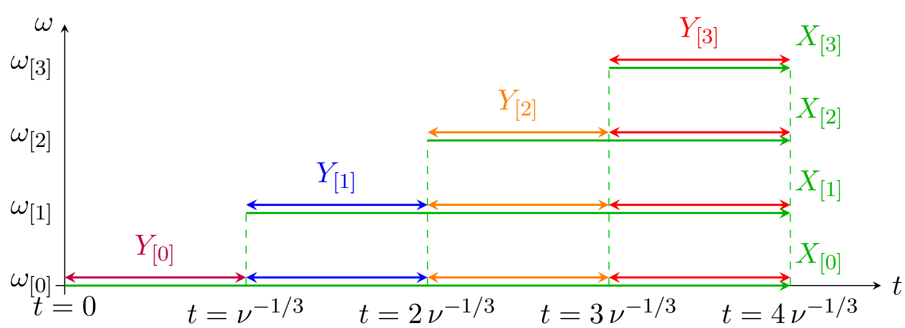

Figure 1: The Frozen Norms

Let us start with the horizontal functional $X_{[j]}$, which measures the size of the solution $\omega_{[j]}$. Since $\omega_{[j]}$ initiates from time t_j, the suitable enhanced dissipation time weight from Proposition 2.5 is $e^{\delta\nu^{1/3}(t-t_j)}$. This leads to the definition:

$$X_{[j]} := \|e^{\delta\nu^{1/3}(t-t_j)}\omega_{[j]}\|_{L^2_t([t_j,T];\mathcal{Z})}, \quad j \ge 0. \tag{2.31}$$

We observe that the $X_{[j]}$-estimate can be naturally derived from Proposition 2.5.

Once we have suitable controls over the horizontal functional $X_{[j]}$, we can bound the following vertical functional (note that the norm is over $\mathcal{I}_{[j]}$)

$$Y_{[j]} := \sum_{j'=0}^{j} (j - j' + 1)\|\omega_{[j']}\|_{L^2(\mathcal{I}_{[j]};\mathcal{Z})}, \quad j \ge 0.$$

Here, the linear-in-j weight is introduced to control the frozen error $\mathbb{f}_{\text{Frozen}}$. We postpone the rigorous estimate of $Y_{[j]}$ to Section 3.5.

Finally, to complete the $L_t^2\mathcal{Z}$ estimate for the vorticity ω_k in (2.30), we collect all the $Y_{[j]}$-functional estimates together to obtain control over the following total energy

$$\mathcal{E} := \sum_{j=0}^{N} e^{2\delta_* j} Y_{[j]}^2. \tag{2.32}$$

Here, the $e^{2\delta_* j}$ is a suitable enhanced dissipation weight and $\delta_* > 0$ is a constant related to the δ in Proposition 2.5. One observes that the total energy $\mathcal{E}$ controls the $L_t^2\mathcal{Z}$-norm of ω_k (Lemma B.1), i.e.,

$$\|e^{\delta_* \nu^{1/3} t}\omega_k\|_{L^2([0,T];\mathcal{Z})}^2 \lesssim \mathcal{E}.$$

This completes the estimate of the $L^2\mathcal{Z}$ norm of ω_k. We postpone the estimate of the quantity $\|(1-|y|)^{1/2}\omega_k\|_{L_t^\infty L_y^2}$ to Section 3.5 because it follows similar line of argument. This concludes the sketch for the proof of Proposition 2.1.

2.4 Nonlinear Stability

Once we establish suitable estimates for the linearized Navier-Stokes equations (2.2), we exploit the quadratic nature of the nonlinearity in the full equation to absorb its effects via the coercive terms in the linear estimates, provided that the perturbation remains sufficiently small. An adaptation of the nonlinear stability arguments in [18, Section 7] yields Theorem 1.1; see Section 4 for details.

3 The Linearized Problem

3.1 Linear Estimates of the Navier Part

In this section, we develop various linear estimates for the Navier part of the inhomogeneous system (2.16).

To estimate the boundary corrector, the following lemma is crucial.

Lemma 3.1. *Let w_{Na} be the solution of (2.16). Let $\lambda = \lambda_r + i\lambda_i$. Assume the parameter constraints $\|\mathcal{U}-y\|_{H^4} \ll 1$, $\nu|k|^2 \le \|\mathcal{U}'\|_{L^\infty}$ and spectral constraint $0 \ge k\mathrm{Im}\lambda \ge -\delta_0\nu^{1/3}|k|^{2/3}$, we have*

$$\|(\mathcal{U}-\lambda_r)w_{\text{Na}} - \mathcal{U}''\phi_{\text{Na}}\|_{L^2} \lesssim \|F_k\|_{L^2}, \tag{3.1}$$

$$\|(\mathcal{U}-\lambda)w_{\text{Na}} - \mathcal{U}''\phi_{\text{Na}}\|_{L^2} \lesssim \|F_k\|_{L^2}. \tag{3.2}$$

Proof. Without loss of generality, we assume that $k > 0$. Let $\lambda_i = -\delta\nu^{1/3}|k|^{-1/3}$, where $\delta \in [0, \delta_0]$. We rewrite (2.16) as

$$\begin{aligned} &-\nu\Delta_k w_{\text{Na}} + ik\left(\mathcal{U}-\lambda_r\right)w_{\text{Na}} - i\mathcal{U}''k\phi_{\text{Na}} - \delta\nu^{1/3}|k|^{2/3}w_{\text{Na}} = F_k,\\ &\phi_{\text{Na}} = \Delta_D^{-1}w_{\text{Na}}, \quad w_{\text{Na}}\big|_{y=\pm1} = 0. \end{aligned} \tag{3.3}$$

We test $(3.3)_1$ by $\overline{(\mathcal{U}-\lambda_r)w_{\mathrm{Na}}-\mathcal{U}''\phi_{\mathrm{Na}}}$ and take the imaginary part to obtain

$$\begin{aligned}
&|k|\int|(\mathcal{U}-\lambda_r)w_{\mathrm{Na}}-\mathcal{U}''\phi_{\mathrm{Na}}|^2dy\\
&\lesssim\left|\int F_k\overline{((\mathcal{U}-\lambda_r)w_{\mathrm{Na}}-\mathcal{U}''\phi_{\mathrm{Na}})}dy\right|+\left|\nu\mathrm{Im}\int\Delta_k w_{\mathrm{Na}}\overline{((\mathcal{U}-\lambda_r)w_{\mathrm{Na}}-\mathcal{U}''\phi_{\mathrm{Na}})}dy\right|\\
&\quad+\delta\nu^{1/3}|k|^{2/3}\left|\int w_{\mathrm{Na}}\overline{((\mathcal{U}-\lambda_r)w_{\mathrm{Na}}-\mathcal{U}''\phi_{\mathrm{Na}})}dy\right|\\
&\lesssim\left(|k|^{-1/2}\|F_k\|_{L^2}+\delta\nu^{1/3}|k|^{1/6}\|w_{\mathrm{Na}}\|_{L^2}\right)|k|^{1/2}\|(\mathcal{U}-\lambda_r)w_{\mathrm{Na}}-\mathcal{U}''\phi_{\mathrm{Na}}\|_{L^2}\\
&\quad+\nu\left|\int\partial_y w_{\mathrm{Na}}\mathcal{U}'\overline{w_{\mathrm{Na}}}\right|+\nu\left|\int\nabla_k w_{\mathrm{Na}}\cdot\overline{\nabla_k(\mathcal{U}''\phi_{\mathrm{Na}})}dy\right|.
\end{aligned}$$

From Young's inequality it follows that

$$\begin{aligned}
&|k|\|(\mathcal{U}-\lambda_r)w_{\mathrm{Na}}-\mathcal{U}''\phi_{\mathrm{Na}}\|_{L^2}^2\\
&\lesssim C_\epsilon|k|^{-1}\|F_k\|_{L^2}^2+\epsilon|k|\|(\mathcal{U}-\lambda_r)w_{\mathrm{Na}}-\mathcal{U}''\phi_{\mathrm{Na}}\|_{L^2}^2+\nu^{4/3}|k|^{-1/3}\|\nabla_k w_{\mathrm{Na}}\|_{L^2}^2\\
&\qquad+C_\epsilon\nu^{2/3}|k|^{1/3}\|w_{\mathrm{Na}}\|_{L^2}^2\\
&\lesssim C_\epsilon|k|^{-1}\|F_k\|_{L^2}^2+\epsilon|k|\|(\mathcal{U}-\lambda_r)w_{\mathrm{Na}}-\mathcal{U}''\phi_{\mathrm{Na}}\|_{L^2}^2,
\end{aligned}\tag{3.4}$$

for any $\epsilon\in(0,1)$, where we used Proposition 2.7 in the last step. We take $\epsilon>0$ in (3.4) sufficiently small to get

$$|k|^2\|(\mathcal{U}-\lambda_r)w_{\mathrm{Na}}-\mathcal{U}''\phi_{\mathrm{Na}}\|_{L^2}^2\lesssim\|F_k\|_{L^2}^2,$$

which completes the proof of (3.1). Finally, we use Proposition 2.7 to obtain

$$\begin{aligned}
&|k|^2\|(\mathcal{U}-\lambda)w_{\mathrm{Na}}-\mathcal{U}''\phi_{\mathrm{Na}}\|_{L^2}^2\\
&\lesssim|k|^2\|(\mathcal{U}-\lambda_r)w_{\mathrm{Na}}-\mathcal{U}''\phi_{\mathrm{Na}}\|_{L^2}^2+\nu^{2/3}|k|^{4/3}\|w_{\mathrm{Na}}\|_{L^2}^2\lesssim\|F_k\|_{L^2}^2.
\end{aligned}$$

This completes the proof of the lemma. □

Lemma 3.2. *Let w_{Na} be the solution of (2.16). Assume the parameter constraints $\|\mathcal{U}-y\|_{H^4}\ll1$, $\nu|k|^2\geq\|\mathcal{U}'\|_{L^\infty}$ and the spectral constraint $0\geq k\mathrm{Im}\lambda\geq-\delta_0\nu^{1/3}|k|^{2/3}$, we have*

$$\nu k^2\left\|w_{\mathrm{Na}}\right\|_{L^2}\lesssim\left\|F_k\right\|_{L^2},\tag{3.5}$$

$$\nu k^2\left\|u_{\mathrm{Na}}\right\|_{L^2}\lesssim\left\|F_k\right\|_{H^{-1}},\tag{3.6}$$

$$\nu|k|\left\|w_{\mathrm{Na}}\right\|_{L^2}\lesssim\left\|F_k\right\|_{H^{-1}}.\tag{3.7}$$

Proof. Without loss of generality, we assume that $k>0$. We test $(2.16)_1$ with $\overline{\phi_{\mathrm{Na}}}$ and integrate by parts to obtain

$$-\nu\left\|w_{\mathrm{Na}}\right\|_{L^2}^2+ik\int(\mathcal{U}-\lambda)w_{\mathrm{Na}}\overline{\phi_{\mathrm{Na}}}\,dy-ik\int\mathcal{U}''\phi_{\mathrm{Na}}\overline{\phi_{\mathrm{Na}}}\,dy=\int F_k\overline{\phi_{\mathrm{Na}}}\,dy.$$

Taking the real part of the above equation, we get

$$\nu\left\|w_{\mathrm{Na}}\right\|_{L^2}^2+\mathrm{Im}k\int(\mathcal{U}-\lambda)w_{\mathrm{Na}}\overline{\phi_{\mathrm{Na}}}\,dy$$

$$
\begin{aligned}
&= \nu \|w_{\mathrm{Na}}\|_{L^2}^2 - \mathrm{Im}k \int \mathcal{U}' \phi_{\mathrm{Na}}' \overline{\phi_{\mathrm{Na}}}\, dy + \mathrm{Im}\lambda k \int |\phi_{\mathrm{Na}}'|^2\, dy + \mathrm{Im}\lambda k^3 \int |\phi_{\mathrm{Na}}|^2\, dy \\
&= -\mathrm{Re} \int F_k \overline{\phi_{\mathrm{Na}}}\, dy,
\end{aligned}
$$

from where

$$
\begin{aligned}
\nu \|w_{\mathrm{Na}}\|_{L^2}^2 \leq \left| \int F_k \overline{\phi_{\mathrm{Na}}}\, dy \right| &+ k \|\mathcal{U}'\|_{L^\infty_y} \|\phi_{\mathrm{Na}}'\|_{L^2} \|\phi_{\mathrm{Na}}\|_{L^2} + \delta_0 \nu^{1/3} |k|^{2/3} \|\phi_{\mathrm{Na}}'\|_{L^2}^2 \\
&+ \delta_0 \nu^{1/3} |k|^{8/3} \|\phi_{\mathrm{Na}}\|_{L^2}^2 .
\end{aligned}
$$

Using $\|\mathcal{U}'\|_{L^\infty} \leq \nu |k|^2$ and the identity

$$
\|w_{\mathrm{Na}}\|_{L^2}^2 = \|\phi_{\mathrm{Na}}''\|_{L^2}^2 + k^4 \|\phi_{\mathrm{Na}}\|_{L^2}^2 + 2k^2 \|\phi_{\mathrm{Na}}'\|_{L^2}^2 , \tag{3.8}
$$

we obtain

$$
\nu \|w_{\mathrm{Na}}\|_{L^2}^2 \lesssim \left| \int F_k \overline{\phi_{\mathrm{Na}}}\, dy \right| \lesssim \|F_k\|_{L^2} \|\phi_{\mathrm{Na}}\|_{L^2} . \tag{3.9}
$$

Note that (3.8) implies

$$
\|\phi_{\mathrm{Na}}\|_{L^2} \leq \frac{1}{k^2} \|w_{\mathrm{Na}}\|_{L^2} ,
$$

and thus (3.5) follows. To prove (3.6), we use (3.8) and (3.9) to obtain

$$
\nu k^2 \|u_{\mathrm{Na}}\|_{L^2}^2 \lesssim \nu \|w_{\mathrm{Na}}\|_{L^2}^2 \lesssim \|F_k\|_{H^{-1}} \|\phi_{\mathrm{Na}}\|_{H^1} \lesssim \|F_k\|_{H^{-1}} \|u_{\mathrm{Na}}\|_{L^2} .
$$

Finally, we note that

$$
\nu \|w_{\mathrm{Na}}\|_{L^2}^2 \lesssim \|F_k\|_{H^{-1}} \|u_{\mathrm{Na}}\|_{L^2} \lesssim \frac{1}{|k|} \|F_k\|_{H^{-1}} \|w_{\mathrm{Na}}\|_{L^2} ,
$$

and (3.7) follows. $\square$

3.2 Estimates of the Coefficients c_+ and c_-

In this section, we establish several estimates on the coefficients $c_\pm$ that were introduced in the decomposition (2.15). We aim to derive some analogous estimates for the coefficients c_1 and c_2 in [18, Lemmas 3.6, 4.1, and 4.2]. In addition, we present the necessary modifications to treat the extra nonlocal term.

Since $\mathcal{U} \in C^3$ and $\|\mathcal{U} - y\|_{H^4} \ll 1$, there exists a strictly increasing function $\widetilde{\mathcal{U}} \in C^3(\mathbb{R})$ such that $\widetilde{\mathcal{U}}(y) = \mathcal{U}(y)$ on $[-1,1]$, $\lim_{y\to-\infty} \widetilde{\mathcal{U}}(y) = -\infty$, and $\lim_{y\to\infty} \widetilde{\mathcal{U}}(y) = \infty$. Hence, for any $\lambda_r \in \mathbb{R}$, there exists a unique $y_\lambda \in \mathbb{R}$ such that $\widetilde{\mathcal{U}}(y_\lambda) = \lambda_r$. For $\lambda_r \in \mathbb{R}$, we define a cutoff function χ on $\mathbb{R}$ such that

$$
\chi(y;\lambda_r) = \frac{1}{\widetilde{\mathcal{U}}(y) - \lambda_r}, \quad \text{if } |y - y_\lambda| \geq L^{-1}. \tag{3.10}
$$

Lemma 3.3. *Let $w_{\rm Na}$ be the solution of (2.16) and $g_\pm \in H^1(-1,1)$ be test functions with the boundary condition $g_+(-1)=0$ and $g_-(+1)=0$. For $\nu|k|^2 \le \|\mathcal{U}'\|_{L^\infty}$ and $0 \ge k{\rm Im}\lambda \ge -\delta_0\nu^{1/3}|k|^{2/3}$, we have*

$$
\begin{aligned}
|\langle w_{\rm Na}, g_\pm\rangle| \lesssim |k|^{-1}\|F_k\|_{H^{-1}}\big(&L^{3/2}\|g_\pm\|_{L^\infty((-1,1)\cap(y_\lambda-L^{-1},y_\lambda+L^{-1}))}\\
&+|g_\pm(\pm1)|(|\mathcal{U}(\pm1)-\lambda_r|+L^{-1})^{-3/4}L^{3/4}+\|g_\pm\chi\|_{H^1}\\
&+L\|g_\pm\chi\|_{L^2}+\nu^{-1/2}\|g_\pm\chi\|_{L^1}\big).
\end{aligned}
\tag{3.11}
$$

The above lemma is analogous to [18, Lemma 3.6]. The main modification relative to [18, Lemma 3.6] lies in the appearance of the L^1-norm of $g_\pm\chi$ on the right-hand side.

Proof. Let $\eta_1 \in H_0^1(-1,1)$ be a test function. We test $(2.16)_1$ with $\overline{\eta_1}$ to obtain

$$
\begin{aligned}
&|k|\,|\langle(\mathcal{U}-\lambda_r)w_{\rm Na}-\mathcal{U}''\phi_{\rm Na},\eta_1\rangle|\\
&\lesssim \|F_k\|_{H^{-1}}\|\nabla_k\eta_1\|_{L^2}+\nu\|\partial_y w_{\rm Na}\|_{L^2}\|\partial_y\eta_1\|_{L^2}+\nu^{1/3}|k|^{2/3}\|w_{\rm Na}\|_{L^2}\|\eta_1\|_{L^2}\\
&\lesssim \|F_k\|_{H^{-1}}(\|\partial_y\eta_1\|_{L^2}+L\|\eta_1\|_{L^2}).
\end{aligned}
\tag{3.12}
$$

where we used (2.21) in the last step. Let G_k be the Green's function for $\Delta_k=-|k|^2+\partial_{yy}$ with homogeneous Dirichlet boundary condition, which has the explicit form

$$
G_k(y,y') = -\frac{1}{k\sinh(2k)}\begin{cases}\sinh(k(1-y'))\sinh(k(1+y)), & y\le y';\\ \sinh(k(1-y))\sinh(k(1+y')), & y\ge y'.\end{cases}
\tag{3.13}
$$

From the symmetric property $G_k(y,z)=G_k(z,y)$ it follows that

$$
\begin{aligned}
\langle\Delta_k^{-1}g,h\rangle &= \int\int G_k(y,z)g(z)dz\overline{h(y)}dy=\int g(z)\overline{\int G_k(z,y)h(y)dy}dz\\
&=\langle g,\Delta_k^{-1}h\rangle.
\end{aligned}
\tag{3.14}
$$

Using (3.12) and (3.14), we conclude

$$
\begin{aligned}
\big|\big\langle w_{\rm Na},(\mathcal{U}-\lambda_r)\eta_1-\Delta_k^{-1}(\mathcal{U}''\eta_1)\big\rangle\big| &= |\langle(\mathcal{U}-\lambda_r)w_{\rm Na}-\mathcal{U}''\phi_{\rm Na},\eta_1\rangle|\\
&\lesssim |k|^{-1}\|F_k\|_{H^{-1}}(\|\partial_y\eta_1\|_{L^2}+L\|\eta_1\|_{L^2}),
\end{aligned}
\tag{3.15}
$$

for all $\eta_1\in H_0^1(-1,1)$.

Let $\eta_2\in H^1(-1,1)$ be a test function with $\eta_2(-1)=0$. Let $L_*\in[L,\infty)$ be a constant to be determined later. Let $\chi_+^*(y):=\max\{0,L_*(y-1+L_*^{-1})\}$ be defined on $[-1,1]$. It follows that

$$
\|(\mathcal{U}-\lambda_r)\chi_+^*\|_{L^\infty}\le\|\mathcal{U}-\lambda_r\|_{L^\infty([1-L_*^{-1},1])}\|\chi_+^*\|_{L^\infty}\le|\mathcal{U}(1)-\lambda_r|+L_*^{-1}\|\mathcal{U}'\|_{L^\infty}. \tag{3.16}
$$

Since $w_{\rm Na}|_{y=\pm1}=0$, the fundamental theorem of calculus implies

$$
|w_{\rm Na}(y)|\le L_*^{-1/2}\|\partial_y w_{\rm Na}\|_{L^2},\quad \forall y\in[1-L_*^{-1},1],
$$

from where

$$
\|w_{\rm Na}\|_{L^1([1-L_*^{-1},1])}\lesssim L_*^{-3/2}\|\partial_y w_{\rm Na}\|_{L^2}. \tag{3.17}
$$

Denote $\eta_2^* = \eta_2 - \eta_2(1)\chi_+^*$. Direct computation gives

$$\begin{aligned}
&\left|\left\langle w_{\mathrm{Na}}, (\mathcal{U}-\lambda_r)\eta_2 - \Delta_k^{-1}(\mathcal{U}''\eta_2)\right\rangle\right| \\
&\le \left|\left\langle w_{\mathrm{Na}}, (\mathcal{U}-\lambda_r)\eta_2^* - \Delta_k^{-1}(\mathcal{U}''\eta_2^*)\right\rangle\right| + |\eta_2(1)|\left\langle w_{\mathrm{Na}}, (\mathcal{U}-\lambda_r)\chi_+^*\right\rangle| \\
&\quad + |\eta_2(1)|\left|\left\langle w_{\mathrm{Na}}, \Delta_k^{-1}(\mathcal{U}''\chi_+^*)\right\rangle\right| \\
&=: T_1 + T_2 + T_3.
\end{aligned} \tag{3.18}$$

Note that $\eta_2^* \in H_0^1$. From (3.15) it follows that

$$\begin{aligned}
|T_1| &\lesssim |k|^{-1}\|F_k\|_{H^{-1}}(\|\partial_y \eta_2^*\|_{L^2} + L\|\eta_2^*\|_{L^2}) \\
&\lesssim |k|^{-1}\|F_k\|_{H^{-1}}(\|\partial_y \eta_2\|_{L^2} + L\|\eta_2\|_{L^2}) \\
&\quad + |\eta_2(1)||k|^{-1}\|F_k\|_{H^{-1}}(\|\partial_y \chi_+^*\|_{L^2} + L\|\chi_+^*\|_{L^2}) \\
&\lesssim |k|^{-1}\|F_k\|_{H^{-1}}(\|\partial_y \eta_2\|_{L^2} + L\|\eta_2\|_{L^2}) \\
&\quad + |\eta_2(1)||k|^{-1}\|F_k\|_{H^{-1}}(L_*^{1/2} + LL_*^{-1/2}).
\end{aligned} \tag{3.19}$$

For the term T_2, we appeal to (3.16), (3.17) and (2.21) to get

$$\begin{aligned}
|T_2| &\lesssim |\eta_2(1)|\|w_{\mathrm{Na}}\|_{L^1([1-L_*^{-1},1])}\|(\mathcal{U}-\lambda_r)\chi_+^*\|_{L^\infty} \\
&\lesssim |\eta_2(1)|L_*^{-3/2}\|\partial_y w_{\mathrm{Na}}\|_{L^2}(|\mathcal{U}(1)-\lambda_r| + L_*^{-1}) \\
&\lesssim |\eta_2(1)|L_*^{-3/2}\nu^{-1}\|F_k\|_{H^{-1}}(|\mathcal{U}(1)-\lambda_r| + L_*^{-1}).
\end{aligned} \tag{3.20}$$

For the term T_3, we use the explicit expression of the Green's function (3.13), yielding

$$\begin{aligned}
&\|\Delta_k^{-1}(\mathcal{U}''\chi_+^*)\|_{L^2} \\
&\lesssim \max_{y\in[-1,1]}\left|\int_{-1}^1 \frac{G_k(y,y')}{1-\max\{y,y'\}}(1-\max\{y,y'\})\mathcal{U}''(y')\chi_+^*(y')dy'\right| \\
&\lesssim \left\|\frac{G_k(y,y')}{1-\max\{y,y'\}}\right\|_{L^\infty_{y,y'}} \left\|(1-\max\{y,y'\})\mathcal{U}''(y')\chi_+^*(y')\right\|_{L^\infty_y L^1_{y'}} \\
&\lesssim \|\mathcal{U}''\|_{L^\infty} L_*^{-1}\|\chi_+^*\|_{L^1} \lesssim L_*^{-2}.
\end{aligned} \tag{3.21}$$

Combining (2.21), (3.21), and the constraint $L_* \ge L$, we obtain

$$\begin{aligned}
|T_3| &\lesssim |\eta_2(1)|\|w_{\mathrm{Na}}\|_{L^2}L_*^{-2} \lesssim |\eta_2(1)|\nu^{-2/3}|k|^{-1/3}\|F_k\|_{H^{-1}}L_*^{-2} \\
&\lesssim |k|^{-1}|\eta_2(1)|\|F_k\|_{H^{-1}}.
\end{aligned} \tag{3.22}$$

Let $L_* := (|\mathcal{U}(1)-\lambda_r| + L^{-1})^{1/2}L^{3/2}$. From (3.18), (3.19), (3.20), and (3.22), we conclude that

$$\begin{aligned}
&\left|\left\langle w_{\mathrm{Na}}, (\mathcal{U}-\lambda_r)\eta_2 - \Delta_k^{-1}(\mathcal{U}''\eta_2)\right\rangle\right| \\
&\lesssim |k|^{-1}\|F_k\|_{H^{-1}}(\|\partial_y \eta_2\|_{L^2} + L\|\eta_2\|_{L^2}) \\
&\quad + |k|^{-1}|\eta_2(1)|\|F_k\|_{H^{-1}}(|\mathcal{U}(1)-\lambda_r| + L^{-1})^{1/4}L^{3/4}.
\end{aligned} \tag{3.23}$$

Let $g_+ \in H^1(-1,1)$ with $g_+(-1) = 0$. Denote $\eta_3 := g_+\chi$, where χ is defined in (3.10). It is clear that $\eta_3(-1) = 0$ and $\eta_3 \in H^1(-1,1)$. We invoke the estimate (3.23) for η_3 to get

$$\begin{aligned}
&|\langle w_{\mathrm{Na}}, g_+\rangle| \\
&\lesssim |\langle w_{\mathrm{Na}}, g_+ - [(\mathcal{U}-\lambda_r)\eta_3 - \Delta_k^{-1}(\mathcal{U}''\eta_3)]\rangle| + |\langle w_{\mathrm{Na}}, (\mathcal{U}-\lambda_r)\eta_3 - \Delta_k^{-1}(\mathcal{U}''\eta_3)\rangle| \\
&\lesssim \|w_{\mathrm{Na}}\|_{L^2}\|g_+ - (\mathcal{U}-\lambda_r)\eta_3\|_{L^2} + |\langle w_{\mathrm{Na}}, \Delta_k^{-1}(\mathcal{U}''\eta_3)\rangle| \\
&\quad + |k|^{-1}\|F_k\|_{H^{-1}}(\|\partial_y\eta_3\|_{L^2} + L\|\eta_3\|_{L^2}) \\
&\quad + |k|^{-1}|\eta_3(1)|\|F_k\|_{H^{-1}}(|\mathcal{U}(1)-\lambda_r| + L^{-1})^{1/4}L^{3/4} \\
&=: T_4 + T_5 + T_6 + T_7.
\end{aligned} \tag{3.24}$$

We bound the term T_4 using (2.21) as

$$\begin{aligned}
T_4 &\lesssim \nu^{-2/3}|k|^{-1/3}\|F_k\|_{H^{-1}}\|g_+\|_{L^\infty([-1,1]\cap[y_\lambda - L^{-1}, y_\lambda + L^{-1}])}L^{-1/2} \\
&\lesssim |k|^{-1}L^{3/2}\|F_k\|_{H^{-1}}\|g_+\|_{L^\infty([-1,1]\cap[y_\lambda - L^{-1}, y_\lambda + L^{-1}])}.
\end{aligned} \tag{3.25}$$

For the term T_5, we appeal to (2.21) and (3.14), yielding

$$\begin{aligned}
T_5 = |\langle \Delta_k^{-1}w_{\mathrm{Na}}, \mathcal{U}''\eta_3\rangle| &\lesssim \|\phi_{\mathrm{Na}}\|_{L^\infty}\|\mathcal{U}''\|_{L^\infty}\|g_+\chi\|_{L^1} \\
&\lesssim \|\nabla_k\phi_{\mathrm{Na}}\|_{L^2}\|g_+\chi\|_{L^1} \\
&\lesssim \nu^{-1/2}|k|^{-1}\|F_k\|_{H^{-1}}\|g_+\chi\|_{L^1}.
\end{aligned} \tag{3.26}$$

Note that the term T_6 appears on the right-hand side of (3.11). Finally, for the term T_7, we note that

$$\begin{aligned}
|\chi(1)| &\lesssim \min\{|\mathcal{U}(1)-\lambda_r|^{-1}, L\} \lesssim \max\{|\mathcal{U}(1)-\lambda_r|, L^{-1}\}^{-1} \\
&\lesssim (|\mathcal{U}(1)-\lambda_r| + L^{-1})^{-1}.
\end{aligned}$$

Thus, we get

$$T_7 \lesssim |g_+(1)|\|F_k\|_{H^{-1}}|k|^{-1}(|\mathcal{U}(1)-\lambda_r| + L^{-1})^{-3/4}L^{3/4}. \tag{3.27}$$

From (3.24), (3.25), (3.26), and (3.27), we conclude the estimate (3.11) for g_+. The proof for g_- is similar and thus we omit the details. ☐

Lemma 3.4. *Let w_{Na} be the solution of (2.16) with $0 \ge k\mathrm{Im}\lambda \ge -\delta_0\nu^{1/3}|k|^{2/3}$. For $\nu k^2 \le \|\mathcal{U}'\|_{L^\infty}$, we have*

$$(1+|k||\lambda_r - \mathcal{U}(1)|)|c_+| + (1+|k||\lambda_r - \mathcal{U}(-1)|)|c_-| \lesssim \nu^{-1/6}|k|^{-5/6}\|F_k\|_{L^2}\log L, \tag{3.28}$$

$$(1+|k||\lambda_r - \mathcal{U}(1)|)|c_+| + (1+|k||\lambda_r - \mathcal{U}(-1)|)|c_-| \lesssim k^{-1/2}\|\nabla_k F_k\|_{L^2}\log L, \tag{3.29}$$

where $c_\pm$ are the coefficients introduced in (2.15).

Proof. Without loss of generality, we assume that $k > 0$. Let $\lambda = \lambda_r - i\delta L^{-1}$, where $\delta \in [0,\delta_0]$. We rewrite $(2.16)_1$ as

$$(\mathcal{U} - \lambda_r + iL^{-1})(\partial_y^2 - k^2)\phi_{\mathrm{Na}} - \mathcal{U}''\phi_{\mathrm{Na}} = \frac{g_k}{ik} \tag{3.30}$$

where

$$g_k = \nu\Delta_k w_{\mathrm{Na}} + F_k - k(1-\delta)L^{-1}w_{\mathrm{Na}}.$$

From (2.19) it follows that $\|g_k\|_{L^2} \lesssim \|F_k\|_{L^2}$. From (2.17) and (3.30) it follows that

$$\begin{aligned}|c_+| &= |\partial_y \phi_{\mathrm{Na}}(y=1)| \\ &= \left|\int_{-1}^{1} \frac{1}{\mathcal{U}-\lambda_r+iL^{-1}}\left(\mathcal{U}''\phi_k + \frac{g_k}{ik}\right)\frac{\sinh(k(y+1))}{\sinh(2k)}dy\right|.\end{aligned} \tag{3.31}$$

Recall that $\mathcal{U}(y_\lambda) = \lambda_r$. Direct computation gives

$$\begin{aligned}\left\|\frac{1}{\mathcal{U}-\lambda_r+iL^{-1}}\right\|_{L^1_y} &= \int_{-1}^{1}\frac{1}{|\mathcal{U}-\lambda_r+iL^{-1}|}dy \\ &\lesssim \int_{[-1,1]\cap[y_\lambda-L^{-1},y_\lambda+L^{-1}]}\frac{1}{L^{-1}}dy + \int_{[-1,1]\setminus[y_\lambda-L^{-1},y_\lambda+L^{-1}]}\frac{1}{|\mathcal{U}-\lambda_r|}dy \\ &\lesssim \log L\end{aligned} \tag{3.32}$$

and

$$\begin{aligned}\left\|\frac{1}{\mathcal{U}-\lambda_r+iL^{-1}}\right\|^2_{L^2_y} &= \int_{-1}^{1}\frac{1}{|\mathcal{U}-\lambda_r|^2+L^{-2}}dy \\ &\lesssim \int_{[-1,1]\cap[y_\lambda-L^{-1},y_\lambda+L^{-1}]}\frac{1}{L^{-2}}dy + \int_{[-1,1]\setminus[y_\lambda-L^{-1},y_\lambda+L^{-1}]}\frac{1}{|\mathcal{U}-\lambda_r|^2}dy \\ &\lesssim L.\end{aligned} \tag{3.33}$$

- **The case $|\lambda_r - \mathcal{U}(1)| \le |k|^{-1}$.** From (3.31) it follows that

$$|c_+| \lesssim \|\phi_k\|_{L^\infty}\left\|\frac{1}{\mathcal{U}-\lambda_r+iL^{-1}}\right\|_{L^1_y} + |k|^{-1}\|g_k\|_{L^2}\left\|\frac{1}{\mathcal{U}-\lambda_r+iL^{-1}}\right\|_{L^2_y}. \tag{3.34}$$

Combining (3.32), (3.33), (3.34), and (2.19), we deduce that

$$\begin{aligned}|c_+| &\lesssim \|\nabla_k\phi_k\|_{L^2}\log L + |k|^{-1}\|F_k\|_{L^2}\nu^{-1/6}|k|^{1/6} \\ &\lesssim \nu^{-1/6}|k|^{-5/6}\log L\|F_k\|_{L^2}.\end{aligned}$$

Suppose $F_k \in H^1_0$. We integrate by parts of (3.31) to obtain

$$\begin{aligned}|c_+| &= \Big|\int_{-1}^{1}\frac{1}{\mathcal{U}-\lambda_r+iL^{-1}}\left(\mathcal{U}''\phi_k + \frac{\nu\Delta_k w_{\mathrm{Na}} + F_k - k(1-\delta)L^{-1}w_{\mathrm{Na}}}{ik}\right) \\ &\quad\times\frac{\sinh(k(y+1))}{\sinh(2k)}dy\Big| \\ &\lesssim \|\phi_k\|_{L^\infty}\left\|\frac{1}{\mathcal{U}-\lambda_r+iL^{-1}}\right\|_{L^1_y} + |k|^{-1}\nu\|\Delta_k w_{\mathrm{Na}}\|_{L^2}\left\|\frac{1}{\mathcal{U}-\lambda_r+iL^{-1}}\right\|_{L^2_y} \\ &\quad + \left|\int_{-1}^{1}\log\left(\mathcal{U}-\lambda_r+iL^{-1}\right)\partial_y\left(\frac{F_k}{\mathcal{U}'ik}\frac{\sinh(k(y+1))}{\sinh(2k)}\right)dy\right| \\ &\quad + L^{-1}\|w_{\mathrm{Na}}\|_{L^2}\left\|\frac{1}{\mathcal{U}-\lambda_r+iL^{-1}}\right\|_{L^2_y}.\end{aligned} \tag{3.35}$$

By the complex natural logarithmic rule we have

$$\left|\log(\mathcal{U}-\lambda_r+iL^{-1})\right| \lesssim \log L, \tag{3.36}$$

for all $y \in [-1,1]$ and $\lambda_r \in [\mathcal{U}(1)-|k|^{-1}, \mathcal{U}(1)+|k|^{-1}]$. Using (2.20), (3.32), (3.33), (3.35), (3.36), we obtain

$$|c_+| \lesssim |k|^{-1/2}\|\nabla_k F_k\|_{L^2}\log L.$$

- **The case $\lambda_r \geq \mathcal{U}(1)+|k|^{-1}$.** Using (2.19) and

$$\frac{1+|k||\mathcal{U}(1)-\lambda_r|}{|\mathcal{U}(y)-\lambda_r|+L^{-1}} \lesssim |k|, \quad y \in [-1,1],$$

we obtain

$$\begin{aligned}
&(1+|k||\mathcal{U}(1)-\lambda_r|)|c_+| \\
&\quad\lesssim \left|\int_{-1}^{1}\frac{1+|k||\mathcal{U}(1)-\lambda_r|}{\mathcal{U}-\lambda_r+iL^{-1}}\left(\mathcal{U}''\phi_k+\frac{g_k}{ik}\right)\frac{\sinh(k(y+1))}{\sinh(2k)}dy\right| \\
&\quad\lesssim \|\nabla_k\phi_k\|_{L^2}\left\|\frac{\sinh(k(y+1))}{\sinh(2k)}\right\|_{L^1} \\
&\qquad\quad + \|g_k\|_{L^2}(|k|^{-1}+|\mathcal{U}(1)-\lambda_r|)\left|\int_{-1}^{1}\left(\frac{\sinh(k(y+1))}{(\mathcal{U}-\lambda_r)\sinh(2k)}\right)^2 dy\right|^{1/2} \\
&\quad\lesssim \nu^{-1/6}|k|^{-5/6}\|F_k\|_{L^2}.
\end{aligned} \tag{3.37}$$

Suppose $F_k \in H_0^1$. We proceed as in (3.37) and use (2.20) instead of (2.19), yielding

$$(1+|k||\mathcal{U}(1)-\lambda_r|)|c_+| \lesssim |k|^{-3/2}\|\nabla_k F_k\|_{L^2}.$$

- **The case $\lambda_r < \mathcal{U}(1)-|k|^{-1}$.** Let $\widetilde{y_\lambda} \in \mathbb{R}$ be such that $\widetilde{\mathcal{U}}(\widetilde{y_\lambda}) = (1+\lambda)/2$. Let $E = [-1,1]\cap[-1,\widetilde{y_\lambda})$ and $E^c = [-1,1]\cap[\widetilde{y_\lambda},1]$. Using (3.31), we have

$$|c_+| =: \int_{-1}^{1}\mathcal{I}\,dy = \int_{E^c}\mathcal{I}\,dy + \int_E \mathcal{I}\,dy. \tag{3.38}$$

For the first term on the right-hand side of (3.38), we use (2.19) and the bound

$$\left(\int_{E^c}\left|\frac{\sinh(k(y+1))}{(\mathcal{U}-\lambda_r)\sinh(2k)}\right|^2 dy\right)^{1/2} \lesssim |k|^{-1/2}|\lambda_r-\mathcal{U}(1)|^{-1}$$

to get

$$\begin{aligned}
\int_{E^c}\mathcal{I}\,dy &\lesssim \nu^{-1/6}|k|^{-11/6}\|F_k\|_{L^2}(|\mathcal{U}(1)-\lambda_r|+|k|^{-1})^{-1} \\
&\qquad + |k|^{-3/2}\|F_k\|_{L^2}(|\mathcal{U}(1)-\lambda_r|+|k|^{-1})^{-1} \\
&\lesssim \nu^{-1/6}|k|^{-5/6}\|F_k\|_{L^2}(|k||\mathcal{U}(1)-\lambda_r|+1)^{-1}.
\end{aligned} \tag{3.39}$$

For the second term on the right-hand side of (3.38), we split into two cases. If $\lambda_r \le -1 - |k|^{-1}$, we have

$$\begin{aligned}
\int_E \mathcal{I}\, dy &\lesssim \|\mathcal{U}''\phi_k + (ik)^{-1} g_k\|_{L^2(E)} \left\| \frac{\sinh(k(y+1))}{\sinh(2k)} \right\|_{L^\infty(E)} (|\mathcal{U}(-1) - \lambda_r| + L^{-1})^{-1} \\
&\lesssim \nu^{-1/6} |k|^{-4/3} e^{-|k|/4} (|\mathcal{U}(-1) - \lambda_r| + 2|k|^{-1})^{-1} \|F_k\|_{L^2} \\
&\lesssim \nu^{-1/6} |k|^{-4/3} e^{-|k|/4} (|\mathcal{U}(-1) - \lambda_r| + |k|^{-1}|\mathcal{U}(1) - \mathcal{U}(-1)| + |k|^{-1})^{-1} \|F_k\|_{L^2} \\
&\lesssim \nu^{-1/6} |k|^{-1/3} e^{-|k|/4} (|\mathcal{U}(-1) - \lambda_r| + |\mathcal{U}(1) - \mathcal{U}(-1)| + 1)^{-1} \|F_k\|_{L^2} \\
&\lesssim (|k||\mathcal{U}(1) - \lambda_r| + 1)^{-1} \nu^{-1/6} |k|^{-5/6} \|F_k\|_{L^2}.
\end{aligned} \tag{3.40}$$

If $-1 - |k|^{-1} < \lambda_r < \mathcal{U}(1) - |k|^{-1}$, we appeal to (3.32) and (3.33) to obtain

$$\begin{aligned}
\int_E \mathcal{I}\, dy \lesssim\, & \|\phi_k\|_{L^\infty} \left\| \frac{1}{|\mathcal{U} - \lambda_r| + L^{-1}} \right\|_{L^1(E)} \left\| \frac{\sinh(k(y+1))}{\sinh(2k)} \right\|_{L^\infty(E)} \\
&+ k^{-1} \|g_k\|_{L^2(E)} \left\| \frac{\sinh(k(y+1))}{\sinh(2k)} \right\|_{L^\infty(E)} \left\| \frac{1}{|\mathcal{U} - \lambda_r| + L^{-1}} \right\|_{L^2(E)} \\
\lesssim\, & \nu^{-1/6} |k|^{-5/6} \|F_k\|_{L^2} (|k||\mathcal{U}(1) - \lambda_r| + 1)^{-1} \log L.
\end{aligned} \tag{3.41}$$

Combining (3.38), (3.39), (3.40), and (3.41), we conclude that

$$|c_+| \lesssim \nu^{-1/6} |k|^{-5/6} \|F_k\|_{L^2} (|k||\lambda_r - \mathcal{U}(1)| + L^{-1})^{-1} \log L. \tag{3.42}$$

For the case when $F_k \in H_0^1$, we have

$$\begin{aligned}
|c_+| \le\, & \left| \int_{-1}^1 \frac{1}{\mathcal{U} - \lambda_r + iL^{-1}} \left(\mathcal{U}''\phi_k + \frac{\nu\Delta_k w_{\mathrm{Na}} - k(1-\delta)L^{-1} w_{\mathrm{Na}}}{ik} \right) \right. \\
& \quad \left. \times \frac{\sinh(k(y+1))}{\sinh(2k)} dy \right| \\
& + \left| \int_{-1}^1 \frac{1}{\mathcal{U} - \lambda_r + iL^{-1}} \frac{F_k}{ik} \frac{\sinh(k(y+1))}{\sinh(2k)} dy \right| \\
=:\, & \mathcal{I}_1 + \mathcal{I}_2.
\end{aligned} \tag{3.43}$$

For the term $\mathcal{I}_1$, we proceed in a similar fashion as in (3.38)–(3.42) and use (2.20) to obtain

$$\mathcal{I}_1 \lesssim k^{-1/2} \|\nabla_k F_k\|_{L^2} (k|\lambda_r - \mathcal{U}(1)| + L^{-1})^{-1} \log L. \tag{3.44}$$

For the term $\mathcal{I}_2$, we split into two cases. If $\lambda_r \le \mathcal{U}(-1) - k^{-1}$, we use the bound

$$\max_{y \in [-1,1]} \left| \frac{1 + k|\mathcal{U}(1) - \lambda_r|}{\mathcal{U}(y) - \lambda_r + iL^{-1}} \right| \lesssim k^2$$

to get

$$\begin{aligned}
\mathcal{I}_2 &\lesssim \frac{k^2}{(1 + k|\mathcal{U}(1) - \lambda_r|)} \left| \int_{-1}^1 \frac{F_k}{ik} \frac{\sinh(k(y+1))}{\sinh(2k)} dy \right| \\
&\lesssim k^{-1/2} (1 + k|\mathcal{U}(1) - \lambda_r|)^{-1} \|\nabla_k F_k\|_{L^2}.
\end{aligned} \tag{3.45}$$

If $\mathcal{U}(-1)-k^{-1}\le\lambda_r\le\mathcal{U}(1)-k^{-1}$, we use the bound $(1+|k||\mathcal{U}(1)-\lambda_r|)\lesssim|k|$ and (3.36) to obtain

$$\begin{aligned}\mathcal{I}_2&\lesssim\frac{k}{(1+k|\mathcal{U}(1)-\lambda_r|)}\left|\int_{-1}^1\log\left(\mathcal{U}-\lambda_r+i\delta_0L^{-1}\right)\partial_y\left(\frac{F_k}{ik\mathcal{U}'}\frac{\sinh(k(y+1))}{\sinh(2k)}\right)dy\right|\\&\lesssim k^{-1/2}(1+k|\mathcal{U}(1)-\lambda_r|)^{-1}\|\nabla_kF_k\|_{L^2}\log L.\end{aligned}\tag{3.46}$$

Combining (3.43), (3.44), (3.45), and (3.46), we conclude

$$|c_+|\lesssim|k|^{-1/2}\|\nabla_kF_k\|_{L^2}(|k||\lambda_r-\mathcal{U}(1)|+L^{-1})^{-1}\log L.$$

The estimates for $|c_-|$ in (3.28) and (3.29) are similar and thus we omit the further details. $\square$

Lemma 3.5. *Let w_{Na} be the solution of (2.16) with $0\ge k\mathrm{Im}\lambda\ge-\delta_0\nu^{1/3}|k|^{2/3}$. For $F_k\in H^{-1}$, we have*

$$\begin{aligned}&(1+|k||\lambda_r-\mathcal{U}(1)|)^{3/4}|c_+|+(1+|k||\lambda_r-\mathcal{U}(-1)|)^{3/4}|c_-|\\&\quad\lesssim\nu^{-1/2}|k|^{-1/2}\|F_k\|_{H^{-1}}\log L.\end{aligned}$$

Proof. Let

$$g_+(y)=\frac{\sinh(k(y+1))}{\sinh(2k)},\quad g_-(y)=\frac{\sinh(k(1-y))}{\sinh(2k)}.$$

It is clear that $g_+\in H^1(-1,1)$ and $g_+(-1)=0$. Using (2.17) and Lemma 3.3, we get

$$\begin{aligned}|c_+|=|\langle w_{\mathrm{Na}},g_+\rangle|&\lesssim|k|^{-1}\|F_k\|_{H^{-1}}\Big(L^{3/2}\|g_+\|_{L^\infty((-1,1)\cap(y_r-L^{-1},y_r+L^{-1}))}\\&\quad+(|\mathcal{U}(1)-\lambda_r|+L^{-1})^{-3/4}L^{3/4}+\|g_+\chi\|_{H^1}+L\|g_+\chi\|_{L^2}+\nu^{-1/2}\|g_+\chi\|_{L^1}\Big),\end{aligned}\tag{3.47}$$

where χ is a smooth cutoff function defined in (3.10). To estimate the last term on the right side of (3.47), we aim to prove

$$(1+|k||\lambda_r-\mathcal{U}(1)|)^{3/4}\|g_+\chi\|_{L^1}\lesssim|k|^{1/2}\log L,\tag{3.48}$$

which concludes that the L^1-contribution on the right-side of (3.47).

• **The case $\lambda_r-\mathcal{U}(1)\ge2k^{-1}$.** Using the mean value theorem, we obtain $y_\lambda-1\ge k^{-1}\ge L^{-1}$. Hence, we deduce that

$$\|g_+\chi\|_{L^1}\lesssim\|\chi\|_{L^\infty}\|g_+\|_{L^1}\lesssim\frac{1}{|k|^{-1}+|\mathcal{U}(1)-\lambda_r|}\frac{1}{|k|}=\frac{1}{1+|k||\mathcal{U}(1)-\lambda_r|}.$$

• **The case $|\lambda_r-\mathcal{U}(1)|\le2k^{-1}$.** We use (3.32) to get

$$\|g_+\chi\|_{L^1}\lesssim\|g_+\|_{L^\infty}\|\chi\|_{L^1}\lesssim\int_{-1}^1\frac{1}{|\mathcal{U}-\lambda_r|+L^{-1}}dy\lesssim\log L.$$

- **The case $\lambda_r \in [-1-2|k|^{-1}, \mathcal{U}(1)-2|k|^{-1}]$.** Let $\widetilde{y_\lambda} = \widetilde{\mathcal{U}}^{-1}((1+\lambda_r)/2)$. Let $E = [-1,1]\cap[-1,\widetilde{y_\lambda})$ and $E^c = [-1,1]\cap[\widetilde{y_\lambda},1]$. It is readily checked that

$$\|\chi\|_{L^\infty(E^c)} \lesssim \left(|k|^{-1} + |\mathcal{U}(1)-\lambda_r|\right)^{-1}.$$

Using (3.32), we obtain

$$\begin{aligned}\|g_+\chi\|_{L^1} &\le \|g_+\|_{L^\infty(E)}\|\chi\|_{L^1(E)} + \|g_+\|_{L^1(E^c)}\|\chi\|_{L^\infty(E^c)} \\ &\lesssim e^{-|k|(1-\widetilde{y_\lambda})}\log L + |k|^{-1}\left(|k|^{-1}+|\mathcal{U}(1)-\lambda_r|\right)^{-1} \\ &\lesssim (1+|k||\mathcal{U}(1)-\lambda_r|)^{-1}\log L.\end{aligned}$$

- **The case $\lambda_r \le -1-2|k|^{-1}$.** It is clear that

$$|k||\mathcal{U}(1)-\lambda| = |k||\mathcal{U}(1)-\mathcal{U}(-1)| + |k||\mathcal{U}(-1)-\lambda_r| \lesssim k^2|\mathcal{U}(-1)-\lambda_r|.$$

Therefore,

$$\begin{aligned}\|g_+\chi\|_{L^1} &\le \|g_+\|_{L^1(E)}\|\chi\|_{L^\infty(E)} + \|g_+\|_{L^1(E^c)}\|\chi\|_{L^\infty(E^c)} \\ &\lesssim \frac{e^{-|k|/2}}{|\mathcal{U}(-1)-\lambda_r| + |k|^{-1}} + \frac{1}{|k|}\frac{1}{|\mathcal{U}(-1)-\lambda_r|} \\ &\lesssim \frac{e^{-|k|/4}}{1+|k||\mathcal{U}(1)-\lambda_r|} + \frac{1}{1+|k||\mathcal{U}(1)-\lambda_r|} \\ &\lesssim \frac{1}{1+|k||\mathcal{U}(1)-\lambda_r|}.\end{aligned}$$

This concludes the proof of (3.48). The rest of the terms on the right side of (3.47) are estimated in a similar fashion as in [18, Lemma 4.2]. Since the estimates for c_- is similar, we conclude the proof of the lemma. □

3.3 Boundary Corrector

In this section, we provide the estimates for the boundary corrector. Most of the estimates are carried out in the works [18, 19]. For the sake of completeness, we decide to review their main results and key estimates.

Recall that the elementary boundary correctors introduced in (2.15) satisfy

$$\begin{cases} -\nu(\partial_y^2 - k^2)w_{\pm;k} + ik\left(\mathcal{U}-\lambda\right)w_{\pm;k} - \mathcal{U}'' ik\phi_{\pm;k} = 0, \\ \phi_{\pm;k} = \Delta_k^{-1} w_{\pm;k}, \quad \partial_y\phi_{\pm;k}(y=\pm 1) = 1, \quad \partial_y\phi_{\pm;k}(y=\mp 1) = 0. \end{cases} \tag{3.49}$$

Let $Ai(y)$ be the Airy function, which is a nontrivial solution of $f'' - yf = 0$. For convenience, we denote

$$L_\pm := \left(\frac{\nu}{\mathcal{U}'(\pm 1)k}\right)^{-1/3}, \quad d_\pm := \frac{\mathcal{U}(\pm 1) \mp \mathcal{U}'(\pm 1) - \lambda - i\nu k}{\mathcal{U}'(\pm 1)}.$$

and define the approximate boundary correctors

$$W_{\pm;\mathrm{ap}}(y) = Ai\left(L_\pm(y+d_\pm)\exp\left(i\left(\frac{\pi}{2}\pm\frac{\pi}{3}\right)\right)\right).$$

It is easy to check that

$$L_{\pm}^{-3}\partial_y^2 W_{\pm;\mathrm{ap}} - i\left(y+d_{\pm}\right)W_{\pm;\mathrm{ap}} = 0,$$

which is equivalent to

$$-\nu(\partial_y^2 - k^2)W_{\pm;\mathrm{ap}} + ik\left(\mathcal{U}(\pm 1) + \mathcal{U}'(\pm 1)(y\mp 1) - \lambda\right)W_{\pm;\mathrm{ap}} = 0.$$

Let $W_{\pm;\mathrm{re}}$ be the solution of

$$\begin{cases} -\nu(\partial_y^2 - k^2)W_{\pm;\mathrm{re}} + ik\left(\mathcal{U}-\lambda\right)W_{\pm;\mathrm{re}} - \mathcal{U}'' ik\Phi_{\pm;\mathrm{re}} \\ \qquad = \mathcal{U}'' ik\Phi_{\pm;\mathrm{ap}} - ik[\mathcal{U}(y) - (\mathcal{U}(\pm 1) + \mathcal{U}'(\pm 1)(y\mp 1)]W_{\pm;\mathrm{ap}}, \\ \Phi_{\pm;\mathrm{ap}} = \Delta_k^{-1}W_{\pm;\mathrm{ap}}, \quad \Phi_{\pm;\mathrm{re}} = \Delta_k^{-1}W_{\pm;\mathrm{re}}, \qquad W_{\pm;\mathrm{re}}\big|_{y=\pm 1} = 0. \end{cases}$$

Here, the expression $\mathcal{U}(\pm 1)+\mathcal{U}'(\pm 1)(y\mp 1)$ is the linear approximation of the strictly monotone shear flow on the upper and lower boundaries. Define

$$W_{\pm} = W_{\pm;\mathrm{ap}} + W_{\pm;\mathrm{re}}, \quad \Phi_{\pm} = \Delta_k^{-1}W_{\pm}.$$

It is clear that

$$-\nu(\partial_y^2 - k^2)W_{\pm} + ik(\mathcal{U}(y)-\lambda)W_{\pm} - \mathcal{U}''(y)ik\Phi_{\pm} = 0.$$

We assume that the boundary correctors solution (w_+, w_-) to (3.49) can be written as

$$w_+ = \mathfrak{C}_+^- W_-(y) + \mathfrak{C}_+^+ W_+(y), \quad w_- = \mathfrak{C}_-^- W_-(y) + \mathfrak{C}_-^+ W_+(y). \tag{3.50}$$

It is easy to see that

$$\int_{-1}^{1} \frac{\sinh(k(y\pm 1))}{\sinh(2k)} w_+ dy = \int_{-1}^{1} \frac{\sinh(k(y\pm 1))}{\sinh(2k)} (\partial_y^2 - k^2)\phi_+ dy = \partial_y\phi_+(y=\pm 1)$$

and

$$\int_{-1}^{1} \frac{\sinh(k(y\pm 1))}{\sinh(2k)} w_- dy = \int_{-1}^{1} \frac{\sinh(k(y\pm 1))}{\sinh(2k)} (\partial_y^2 - k^2)\phi_- dy = \partial_y\phi_-(y=\pm 1).$$

Therefore, the boundary conditions on $\partial_y\phi_{\pm}$ in (3.49) are satisfied by setting

$$\int_{-1}^{1} w_+(y)\frac{\sinh(k(y+1))}{\sinh(2k)}\,dy = 1 \quad \text{and} \quad \int_{-1}^{1} w_+(y)\frac{\sinh(k(y-1))}{\sinh(2k)}\,dy = 0$$

and

$$\text{and} \quad \int_{-1}^{1} w_-(y)\frac{\sinh(k(y+1))}{\sinh(2k)}\,dy = 0, \quad \int_{-1}^{1} w_-(y)\frac{\sinh(k(y-1))}{\sinh(2k)}\,dy = 1.$$

Using (3.50), we obtain

$$\mathfrak{C}_+^- \underbrace{\int_{-1}^1 W_-(y)\frac{\sinh(k(y-1))}{\sinh(2k)}\,dy}_{=:A_{--}} + \mathfrak{C}_+^+ \underbrace{\int_{-1}^1 W_+(y)\frac{\sinh(k(y-1))}{\sinh(2k)}\,dy}_{=:A_{+-}} = 0,$$

$$\mathfrak{C}_+^- \underbrace{\int_{-1}^1 W_-(y)\frac{\sinh(k(y+1))}{\sinh(2k)}\,dy}_{=:A_{-+}} + \mathfrak{C}_+^+ \underbrace{\int_{-1}^1 W_+(y)\frac{\sinh(k(y+1))}{\sinh(2k)}\,dy}_{=:A_{++}} = 1.$$

Similarly, we have

$$\mathfrak{C}_-^- A_{--} + \mathfrak{C}_-^+ A_{+-} = 1, \quad \mathfrak{C}_-^- A_{-+} + \mathfrak{C}_-^+ A_{++} = 0$$

If $\mathfrak{D} := A_{--}A_{++} - A_{+-}A_{-+} \neq 0$, then w_+ and w_- are linearly independent and

$$(\mathfrak{C}_+^-, \mathfrak{C}_+^+) = \frac{(-A_{+-}, A_{--})}{\mathfrak{D}}, \quad (\mathfrak{C}_-^-, \mathfrak{C}_-^+) = \frac{(A_{++}, -A_{-+})}{\mathfrak{D}}. \tag{3.51}$$

We shall prove that $\mathfrak{D} \neq 0$ after Lemma 3.7 in (3.61). To ease the notation in forthcoming arguments, we further denote

$$A_{s\pm;\mathrm{ap}} := \int_{-1}^1 W_{s;\mathrm{ap}}(y)\frac{\sinh(k(y\pm 1))}{\sinh(2k)}dy, \quad A_{s\pm;\mathrm{re}} := \int_{-1}^1 W_{s;\mathrm{re}}(y)\frac{\sinh(k(y\pm 1))}{\sinh(2k)}dy,$$

where $s \in \{+,-\}$ and

$$A_0(z) = \int_{e^{i\pi/6}z}^\infty Ai(t)dt = e^{i\pi/6}\int_z^\infty Ai(e^{i\pi/6}t)dt.$$

It is easy to see that

$$A_0'(L_-(y+d_-)) = -e^{i\pi/6}Ai(L_-(y+d_-)e^{i\pi/6}) = -e^{i\pi/6}W_{-;\mathrm{ap}}(y). \tag{3.52}$$

Lemma 3.6. *Let $\rho_k = \min\{1, L(1-|y|)\}$. For $\nu k^2 \lesssim 1$, we have the following bounds*

$$\|W_-\|_{L_y^\infty} \lesssim \nu^{-1/2}L^{-1}(1+|k(\lambda - \mathcal{U}(-1))|)^{1/2}|A_0(L_-(d_- - 1))|, \tag{3.53}$$

$$\|W_+\|_{L_y^\infty} \lesssim \nu^{-1/2}L^{-1}(1+|k(\lambda - \mathcal{U}(1))|)^{1/2}|A_0(-L_+(\overline{d_+}+1))|, \tag{3.54}$$

$$\frac{L\|W_-\|_{L_y^1}}{|A_0(L_-(d_- - 1))|} + \frac{L\|W_+\|_{L_y^1}}{|A_0(-L_+(\overline{d_+}+1)|} \lesssim 1, \tag{3.55}$$

$$\frac{L^{1/2}\|\rho_k^{1/2}W_-\|_{L_y^2}}{|A_0(L_-(d_- - 1))|} + \frac{L^{1/2}\|\rho_k^{1/2}W_+\|_{L_y^2}}{|A_0(-L_+(\overline{d_+}+1))|} \lesssim 1. \tag{3.56}$$

Proof. The estimates (3.53) and (3.54) are similar to [19, Lemma 4.3] and thus we omit the details. To prove (3.55), we use (3.52) and [19, Lemma B.3] to obtain

$$\begin{aligned}\left|\frac{W_{-;\mathrm{ap}}(y)}{A_0(L_-(d_- - 1))}\right| &= \left|\frac{A_0'(L_-(y+d_-))}{A_0(L_-(y+d_-))}\right|\left|\frac{A_0(L_-(y+d_-))}{A_0(L_-(d_- - 1))}\right| \\ &\lesssim (1+|L_-(d_- - 1)|)^{1/2}\exp\left\{-cL_-(y+1)(1+|L_-(d_- - 1)|)^{1/2}\right\},\end{aligned} \tag{3.57}$$

which implies

$$\begin{aligned}\|W_{-;\mathrm{ap}}\|_{L^1_y} &\lesssim |A_0(L_-(d_- - 1))|(1+|L_-(d_- - 1)|)^{1/2}\\ &\qquad\times\int_{-1}^{1}\exp\left\{-cL_-(y+1)(1+|L_-(d_- - 1)|)^{1/2}\right\}dy\\ &\lesssim \frac{1}{L_-(1+|L_-(d_- - 1)|)^{1/2}}|A_0(L_-(d_- - 1))|(1+|L_-(d_- - 1)|)^{1/2}\\ &\lesssim \frac{1}{L}|A_0(L_-(d_- - 1))|.\end{aligned}$$

From [19, Lemma 4.2] it follows that

$$\|W_{-;\mathrm{re}}\|_{L^1_y} \lesssim \frac{1}{L}|A_0(L_-(d_- - 1))|$$

Hence, we conclude the proof of (3.55) for the term $\|W_-\|_{L^1}$. The estimate for $\|W_+\|_{L^1_y}$ is similar and thus we omit the details.

Next we prove the weighted L^2 estimates in (3.56). Using (3.57), we obtain

$$\begin{aligned}&\frac{L}{|A_0(L_-(d_- - 1))|}\|\rho_k^{1/2}W_{-;\mathrm{ap}}\|_{L^2}\\ &=\frac{L}{|A_0(L_-(d_- - 1))|}\left(\int_{-1}^{1}\rho_k(y)|A_0'(L_-(y+d_-))|^2\,dy\right)^{1/2}\\ &=L\left(\int_{-1}^{1}\rho_k(y)\frac{|A_0'(L_-(y+d_-)|^2}{|A_0(L_-(y+d_-)|^2}\frac{|A_0(L_-(y+d_-)|^2}{|A_0(L_-(d_- - 1))|^2}\,dy\right)^{1/2}\\ &\lesssim L\left(\int_{-1}^{1}\rho_k(y)(1+|L_-(d_- - 1)|)\exp\left\{-2cL_-(y+1)(1+|L_-(d_- - 1)|)^{1/2}\right\}dy\right)^{1/2}\end{aligned}\tag{3.58}$$

• **The case $|L_-(d_- - 1)|\le 1$.** From (3.58) it follows that

$$\begin{aligned}&\frac{L}{|A_0(L_-(d_- - 1))|}\|\rho_k^{1/2}W_{-;\mathrm{ap}}\|_{L^2}\\ &\qquad\lesssim L\left(\int_{-1}^{1}(1+|L_-(y+d_-)|)e^{-C(L_-(1+y))}\,dy\right)^{1/2}\lesssim L^{1/2}.\end{aligned}\tag{3.59}$$

• **The case $|L_-(d_- - 1)|>1$.** From (3.58) it follows that

$$\begin{aligned}&\frac{L}{|A_0(L_-(d_- - 1))|}\|\rho_k^{1/2}W_{-;\mathrm{ap}}\|_{L^2}\\ &\quad\lesssim L\left(\int_{-1}^{1}\rho_k(y)(1+|L_-(y+d_-)|)e^{-CL_-(1+y)|L_-(d_- - 1)|^{1/2}}\,dy\right)^{1/2}\\ &\quad\lesssim L\left(\int_{-1}^{1}(1+L_-(1+y)+L(y+1)|L_-(d_- - 1)|)e^{-CL_-(1+y)|L_-(d_- - 1)|^{1/2}}\,dy\right)^{1/2}\\ &\quad\lesssim L^{1/2}.\end{aligned}\tag{3.60}$$

Combining (3.59) and (3.60), we conclude the estimate for $\|\rho_k^{1/2} W_{-;\mathrm{ap}}\|_{L^2}$ in (3.56). For the reminder term $\|\rho_k^{1/2} W_{-;\mathrm{re}}\|_{L^2}$, we have

$$\|\rho_k^{1/2} W_{-;\mathrm{re}}\|_{L^2} \lesssim \|W_{-;\mathrm{re}}\|_{L^2} \lesssim \|W_{-;\mathrm{re}}\|_{L^1}^{1/2} \|W_{-;\mathrm{re}}\|_{L^\infty}^{1/2}.$$

Using [19, Lemma 4.2], we obtain

$$\begin{aligned}\|\rho_k^{1/2} W_{-;\mathrm{re}}\|_{L^2} &\lesssim \left(L_-^{-7/4}|k|^{-1/4}|A_0(L_-(d_- - 1))|\right)^{1/2} |A_0(L_-(d_- - 1))|^{1/2} \\ &\lesssim L^{-7/8}|k|^{-1/8}|A_0(L_-(d_- - 1))| \lesssim L^{-1/2}|A_0(L_-(d_- - 1))|.\end{aligned}$$

Similar arguments prove the $\|\rho_k^{1/2} W_{+;\mathrm{ap}}\|_{L_y^2}$ part in (3.56) and thus we conclude the proof of (3.56). $\square$

We recall the estimates for the coefficients $A_{\pm\pm}$ and $A_{\pm\mp}$ from [19].

Lemma 3.7. [19, Lemma 4.4] *There exists a constant $K > 0$ such that if* $\min\{L_+, L-\} \geq K$, *then*

$$\begin{aligned}&|A_{--}| \geq \frac{1}{CL_-}|A_0(L_-(d_- - 1))|, \quad |A_{++}| \geq \frac{1}{CL_+}|A_0(-L_+(\overline{d_+} + 1))|; \\ &\left|\frac{A_{-+}}{A_{--}}\right| \leq \frac{\sqrt{2}}{2}, \quad \left|\frac{A_{+-}}{A_{++}}\right| \leq \frac{\sqrt{2}}{2}.\end{aligned}$$

Moreover, the following two estimates hold

$$\begin{aligned}|A_{-+}| &\lesssim (L_-^{-7/4}|k|^{-1/4} + L_-^{-2})|A_0(L_-(d_- - 1))|, \\ |A_{+-}| &\lesssim (L_+^{-7/4}|k|^{-1/4} + L_+^{-2})|A_0(-L_+(\overline{d_+} + 1))|.\end{aligned}$$

A direct consequence of the lemma is that the following lower bound for the Evan's function $\mathfrak{D}$

$$\begin{aligned}|\mathfrak{D}| = |A_{--}A_{++} - A_{-+}A_{+-}| &\geq \frac{|A_{--}A_{++}|}{2} \\ &\geq \frac{|A_0(L_-(d_- - 1))||A_0(-L_+(\overline{d_+} + 1))|}{2CL^2}.\end{aligned} \tag{3.61}$$

This lower bound will turn out to be crucial to estimate the coefficients $\mathfrak{C}_\pm^\pm$, and $\mathfrak{C}_\mp^\pm$ introduced in (3.50).

Lemma 3.8. *There exists a constant $K > 0$ such that if* $\min\{L_+, L-\} \geq K$ *and* $\nu k^2 \lesssim 1$, *then*

$$\frac{\nu^{1/2}\|w_-\|_{L^\infty}}{(1 + |k||\lambda - \mathcal{U}(-1)|)^{1/2}} + \|w_-\|_{L^1} \lesssim 1, \tag{3.62}$$

$$\frac{\nu^{1/2}\|w_+\|_{L^\infty}}{(1 + |k||\lambda - \mathcal{U}(1)|)^{1/2}} + \|w_+\|_{L^1} \lesssim 1, \tag{3.63}$$

$$\|\rho_k^{-1/4} w_+\|_{L^2} \lesssim \nu^{-7/24}k^{-1/12}(1 + |k||\lambda - \mathcal{U}(1)|)^{3/8}, \tag{3.64}$$

$$\|\rho_k^{-1/4} w_-\|_{L^2} \lesssim \nu^{-7/24}k^{-1/12}(1 + |k||\lambda - \mathcal{U}(-1)|)^{3/8}, \tag{3.65}$$

$$\|\rho_k^{1/2} w_-\|_{L^2} + \|\rho_k^{1/2} w_+\|_{L^2} \lesssim L^{1/2}. \tag{3.66}$$

Proof. First we prove the estimate (3.62). Thanks to Lemma 3.7, (3.51), and the lower bound on the Evan's function (3.61), we obtain

$$|\mathfrak{C}_-^-| = \left|\frac{A_{++}}{\mathfrak{D}}\right| \lesssim \left|\frac{A_{++}}{A_{++}A_{--}}\right| \lesssim \frac{L}{|A_0(L_-(d_- - 1))|} \tag{3.67}$$

and

$$|\mathfrak{C}_-^+| \lesssim \left|\frac{A_{-+}}{A_{++}A_{--}}\right| \lesssim \frac{L^2(L^{-7/4}|k|^{-1/4} + L^{-2})}{|A_0(-L_+(\overline{d_+} + 1))|} \lesssim \frac{\nu^{-1/12}|k|^{-1/6}}{|A_0(-L_+(\overline{d_+} + 1))|}, \tag{3.68}$$

where we used $\nu k^2 \lesssim 1$ in the last step. Using Lemma 3.6, (3.50), (3.53), (3.54), (3.67), and (3.68), we arrive at

$$\begin{aligned}
\|w_-\|_{L^\infty} &\le |\mathfrak{C}_-^-|\|W_-\|_{L^\infty} + |\mathfrak{C}_-^+|\|W_+\|_{L^\infty} \\
&\lesssim \nu^{-1/2}(1 + |k||\lambda - \mathcal{U}(-1)|)^{1/2} + \nu^{-1/4}k^{-1/2}(1 + |k||\lambda - \mathcal{U}(1)|)^{1/2} \\
&\lesssim \nu^{-1/2}(1 + |k||\lambda - \mathcal{U}(-1)|)^{1/2}.
\end{aligned}$$

For the L^1 norm, we use (3.55), (3.67), and (3.68) to get

$$\|w_-\|_{L^1} \le |\mathfrak{C}_-^-|\|W_-\|_{L^1} + |\mathfrak{C}_-^+|\|W_+\|_{L^1} \lesssim 1.$$

This completes the proof of (3.62). The proof of (3.63) is similar and thus we omit the details.

The proofs of the estimates (3.64), (3.65), and (3.66) are similar to [18, Proposition 4.4] and thus we omit the details. □

3.4 Space-time Estimates

In this section, we prove the space-time estimates for the Navier-Stokes equations linearized around a time-independent shear flow.

Proof of Proposition 2.6. Without loss of generality, we assume that $k > 0$. Let $\lambda = \lambda_r + i\lambda_i$ where $\lambda_i = -\delta_0\nu^{1/3}k^{-1/3}$. We take the Fourier transform in t to get

$$\begin{aligned}
&w_k(\lambda, y) := \int_0^\infty \omega_k^I(t, y)e^{itk\lambda}dt, \quad \phi_k(\lambda, y) := \int_0^\infty \psi_k^I(t, y)e^{itk\lambda}dt, \\
&F_j(\lambda, y) := \int_0^\infty \mathbb{f}_j(t, y)e^{itk\lambda}dt, \quad j = 1, 2, 3.
\end{aligned}$$

Thus, from (2.11) it follows that

$$\begin{aligned}
&-\nu\Delta_k w_k + ik(\mathcal{U} - \lambda)w_k - ik\mathcal{U}''\phi_k = ikF_1 + \partial_y F_2 + F_3, \\
&\phi_k = \Delta_k^{-1}w_k, \quad \int_{-1}^1 w_k e^{\pm ky}dy = 0.
\end{aligned}$$

We further decompose w_k as

$$w_k = w_{\mathrm{Na}}^{(1)} + w_{\mathrm{Na}}^{(2)} + w_{\mathrm{Na}}^{(3)} + (c_+^{(1)} + c_+^{(2)} + c_+^{(3)})w_{+;k} + (c_-^{(1)} + c_-^{(2)} + c_-^{(3)})w_{-;k},$$

where $w_{\rm Na}^{(1)}$, $w_{\rm Na}^{(2)}$, and $w_{\rm Na}^{(3)}$ solves (2.16) with the right-hand side forcing ikF_1, $\partial_y F_2$, and F_3, respectively, and

$$c_+^{(j)} = -\partial_y \phi_{\rm Na}^{(j)}(y=1), \quad c_-^{(j)} = -\partial_y \phi_{\rm Na}^{(j)}(y=-1), \quad j=1,2,3.$$

Using the Plancherel's theorem, we obtain

$$\begin{aligned}
\int_0^\infty \|e^{-tk\lambda_i}\omega_k^I(t)\|_{L^2}^2 dt &\sim \int_{\mathbb{R}} \|w_k(\lambda_r)\|_{L^2}^2 d\lambda_r,\\
\int_0^\infty \|e^{-tk\lambda_i}\psi_k^I(t)\|_{L^2}^2 dt &\sim \int_{\mathbb{R}} \|\phi_k(\lambda_r)\|_{L^2}^2 d\lambda_r,\\
\int_0^\infty \|e^{-tk\lambda_i}u_k^I(t)\|_{L^2}^2 dt &\sim \int_{\mathbb{R}} \|(ik,\partial_y)\Delta_k^{-1} w_k\|_{L^2}^2 d\lambda_r,\\
\int_0^\infty \|e^{-tk\lambda_i}\mathbb{f}_i(t)\|_{L^2}^2 dt &\sim \int_{\mathbb{R}} \|F_j\|_{L^2}^2 d\lambda_r, \quad j=1,2,3.
\end{aligned}$$

The proofs of the components $(w_{\rm Na}^{(1)}, c_+^{(1)} w_{+;k}, c_-^{(1)} w_{-;k})$ and $(w_{\rm Na}^{(2)}, c_+^{(2)} w_{+;k}, c_-^{(2)} w_{-;k})$ are analogous to [18, Proposition 6.5] which rely on Proposition 2.7, Lemmas 3.4, 3.5 and 3.8, thus we omit the details.

Next we estimate the contribution of the component $(w_{\rm Na}^{(3)}, c_+^{(3)} w_{+;k}, c_-^{(3)} w_{-;k})$. From (3.29) and Lemma 3.8, we obtain

$$\begin{aligned}
&|c_+^{(3)}|\|w_+\|_{L^2} + |c_+^{(3)}|\|w_+\|_{L^2} \lesssim |k|^{-1/2}\nu^{-1/4}\log L\|\nabla_k F_3\|_{L^2},\\
&|c_+^{(3)}|\|\rho_k^{1/2} w_+\|_{L^2} + |c_+^{(3)}|\|w_+\|_{L^2} \lesssim |k|^{-1/3}\nu^{-1/6}\log L\|\nabla_k F_3\|_{L^2},\\
&|c_+^{(3)}|\|w_+\|_{L^1} + |c_+^{(3)}|\|w_+\|_{L^1} \lesssim |k|^{-1/2}\log L\|\nabla_k F_3\|_{L^2}.
\end{aligned}$$

Let $\omega_{\rm Na}^3$ be the solution of

$$\begin{aligned}
&\partial_t \omega_{\rm Na}^{(3)} - \nu\Delta_k \omega_{\rm Na}^{(3)} + ik\mathcal{U}\omega_{\rm Na}^{(3)} - ik\mathcal{U}''\psi_{\rm Na}^{(3)} = \mathbb{f}_3,\\
&\omega_{\rm Na}^{(3)}|_{t=0} = 0, \quad \omega_{\rm Na}^{(3)}|_{y=\pm1} = 0.
\end{aligned}$$

By Plancherel's theorem and (2.20), we have

$$\begin{aligned}
&\nu^{1/3}|k|^{2/3}\|e^{-tk\lambda_i}\omega_{\rm Na}^{(3)}(t)\|_{L^2L^2}^2 \lesssim \nu^{1/3}|k|^{2/3}\Big\|\|w_{\rm Na}^{(3)}(\lambda_r)\|_{L_y^2}\Big\|_{\mathbb{R}}^2\\
&\lesssim |k|^{-2}\Big\|\|\nabla_k F_3\|_{L^2}\Big\|_{\mathbb{R}}^2 \lesssim \|e^{-tk\lambda_i}\nabla_k \mathbb{f}_3\|_{L^2L^2}^2.
\end{aligned}$$

Hence, the bounds on $\nu^{1/3}|k|^{2/3}\|e^{-tk\lambda_i}\rho_k^{1/2}\omega^I\|_{L^2L^2}^2$ and $\nu^{1/2}|k|\|e^{-tk\lambda_i}\omega^I\|_{L^2L^2}^2$ are consistent with the right-hand side of (2.18). By (2.20), (3.29), [18, Lemma 9.3], and Lemma 3.8, we have

$$\begin{aligned}
&\|(ik,\partial_y)\Delta_k^{-1}(w_{\rm Na}^{(3)} + c_+^{(3)} w_+ + c_-^{(3)} w_-)\|_{L^2}\\
&\lesssim |k|^{-2}\|\nabla_k F_3\|_{L^2} + |k|^{-1/2}|c_+^{(3)}|\|w_+\|_{L^1} + |k|^{-1/2}|c_-^{(3)}|\|w_-\|_{L^1}\\
&\lesssim |k|^{-1}\|\nabla_k F_3\|_{L^2}.
\end{aligned}$$

This completes the proof of the estimate of $|k|^2\|e^{-tk\lambda_i}u_k^I\|_{L^2L^2}^2$.

The proof of the term $\nu^{1/2}|k|\|e^{-tk\lambda_i}\omega_k^I\|_{L^\infty L^2}^2$ is similar to [18, Proposition 6.5] and thus we omit the details. $\square$

Now we are ready to prove Proposition 2.8.

Proof of Proposition 2.8. We recall the decomposition of the homogeneous component of the solution ω^H in Section 2.2.2:

$$\begin{cases} \omega_k^H(t,y) = \omega_k^{(1)}(t,y) + \omega_k^{(2)}(t,y) + \omega_k^{(3)}(t,y), \\ (\partial_y^2 - k^2)\psi_k^{(i)} = \omega_k^{(i)}, \quad \psi^{(i)}\big|_{y=\pm 1} = 0, \quad i = 1,2,3. \end{cases}$$

First we estimate the term $\omega_k^{(1)}(t,y)$. Since $\nu k^2 \lesssim 1$, we have

$$\begin{aligned} &\nu^{1/3}|k|^{2/3}\|e^{\delta(\nu k^2)^{1/3}t}\rho_k^{1/2}\omega_k^{(1)}\|_{L^2L^2}^2 + \nu^{1/2}|k|\|e^{\delta(\nu k^2)^{1/3}t}\omega_k^{(1)}\|_{L^2L^2}^2 \\ &\lesssim \nu^{1/3}|k|^{2/3}\|e^{\delta(\nu k^2)^{1/3}t}\omega_k^{(1)}\|_{L^2L^2}^2. \end{aligned}$$

Direct computation of the explicit expression (2.22) gives

$$\begin{aligned} \nu^{1/3}|k|^{2/3}\|e^{\delta(\nu k^2)^{1/3}t}\omega_k^{(1)}\|_{L^2L^2}^2 &\lesssim \nu^{1/3}|k|^{2/3}\|\omega_{\mathrm{in};k}\|_{L^2}^2 \int_0^\infty e^{2\delta(\nu k^2)^{1/3}t - C\nu k^2 t^3 - 2\nu k^2 t} dt \\ &\lesssim \nu^{1/3}|k|^{2/3}\|\omega_{\mathrm{in};k}\|_{L^2}^2 \left(\int_0^{(\nu k^2)^{-1/3}} e^{2\delta(\nu k^2)^{1/3}t} dt + \int_{(\nu k^2)^{-1/3}}^\infty e^{-C\nu k^2 t^3} dt \right) \\ &\lesssim \|\omega_{\mathrm{in};k}\|_{L^2}^2 \end{aligned}$$

and

$$\nu^{1/2}|k|\|e^{\delta(\nu k^2)^{1/3}t}\omega_k^{(1)}\|_{L^\infty L^2}^2 \lesssim \nu^{1/2}|k|\|\omega_{\mathrm{in};k}\|_{L^2}^2 \lesssim \|\omega_{\mathrm{in};k}\|_{L^2}^2.$$

Next we estimate the term $|k|^2\|e^{\delta(\nu k^2)^{1/3}t}u^{(1)}\|_{L^2L^2}^2$. Recall that G_k is the Green's function for $\Delta_k = -|k|^2 + \partial_{yy}$ with homogeneous Dirichlet boundary conditions, which has the explicit form (3.13). Using integration by parts, we obtain

$$\begin{aligned} u_k^{(1)} &= (-\partial_y, ik)\Delta_k^{-1}\omega^{(1)} \\ &= \int_{-1}^1 (-\partial_y, ik)G_k(y,\widetilde{y})\omega_{\mathrm{in};k}(\widetilde{y}) \exp\left\{-\frac{1}{3}(\mathcal{U}')^2\nu k^2 t^3 - \nu k^2 t\right\} \\ &\quad \times \frac{\partial_{\widetilde{y}} \exp\{-it\mathcal{U}k\}}{-it\mathcal{U}'k} d\widetilde{y} \\ &= \frac{1}{ikt}\int_{-1}^1 \partial_{\widetilde{y}}\left(\frac{\omega_{\mathrm{in};k}(\widetilde{y})}{\mathcal{U}'}(-\partial_y, ik)G_k(y,\widetilde{y})\exp\left\{-\frac{1}{3}(\mathcal{U}')^2\nu k^2 t^3 - \nu k^2 t\right\}\right) \\ &\quad \times \exp\{-it\mathcal{U}k\}\, d\widetilde{y}, \end{aligned}$$

where we used $G_k(y,\widetilde{y} = \pm 1) = \partial_y G_k(y,\widetilde{y} = \pm 1) = 0$ in the last step. Direct calculation gives

$$\|e^{\delta(\nu k^2)^{1/3}t}u_k^{(1)}(t)\|_{L_y^2} \lesssim \frac{1}{k\langle t\rangle}\|\omega_{\mathrm{in};k}\|_{H_k^1},$$

which leads to

$$|k|^2\|e^{\delta(\nu k^2)^{1/3}t}u_k^{(1)}\|^2_{L^2_tL^2_y} \lesssim \|\omega_{\mathrm{in};k}\|^2_{H^1_k}. \tag{3.69}$$

Next we estimate the term $\omega_k^{(2)}(t,y)$. Note that $\omega_k^{(2)}(t,y)$ is the solution of (2.23), which can be considered as the solution of (2.11) with the right-hand side forcing

$$\mathbb{F}_k = ik\mathbb{f}_1 + \partial_y\mathbb{f}_2 + \mathbb{f}_3,$$

where

$$\mathbb{f}_1 = \frac{1}{ik}(\nu\partial_y^2\omega_k^{(1)} + (\mathcal{U}')^2\nu k^2t^2\omega_k^{(1)}), \quad \mathbb{f}_2 = 0, \quad \mathbb{f}_3 = ik\mathcal{U}''\psi_k^{(1)}.$$

Using the explicit expression of $\omega_k^{(1)}$ in (2.22), we write

$$\begin{aligned}
\mathbb{f}_1 = &\frac{1}{ik}\exp\left\{-it\mathcal{U}k - \frac{1}{3}(\mathcal{U}')^2\nu k^2t^3 - \nu k^2 t\right\}\\
&\times\left[\nu\left(\partial_y^2\omega_{\mathrm{in};k} - \frac{4}{3}\partial_y\omega_{\mathrm{in};k}\mathcal{U}'\mathcal{U}''\nu k^2t^3 - \frac{2}{3}(\mathcal{U}'\mathcal{U}'')'\nu|k|^2t^3\omega_{\mathrm{in};k} + \frac{4}{9}(\mathcal{U}'\mathcal{U}'')^2\nu^2|k|^4t^6\omega_{\mathrm{in};k}\right)\right.\\
&\qquad\left. - 2\nu i\mathcal{U}'kt\left(\partial_y\omega_{\mathrm{in};k} - \frac{2}{3}\omega_{\mathrm{in};k}\mathcal{U}'\mathcal{U}''\nu k^2t^3\right) - \nu i\mathcal{U}''kt\omega_{\mathrm{in};k}\right].
\end{aligned}$$

Direct computation gives

$$\|e^{\delta(\nu k^2)^{1/3}t}\mathbb{f}_1\|^2_{L^2_tL^2_y} \lesssim \nu^{1/3}|k|^{-10/3}\|\omega_{\mathrm{in};k}\|^2_{H^2_k}.$$

Using (3.69), we obtain

$$\|e^{\delta(\nu k^2)^{1/3}t}\nabla_k\mathbb{f}_3\|^2_{L^2L^2} = \|e^{\delta(\nu k^2)^{1/3}t}\nabla_k(i\mathcal{U}''k\psi_k^{(1)})\|^2_{L^2L^2} \lesssim \|\omega_{\mathrm{in};k}\|^2_{H^1_k} \lesssim k^{-2}\|\omega_{\mathrm{in};k}\|^2_{H^2_k}.$$

An application of Proposition 2.6 yields

$$\begin{aligned}
&\nu^{1/3}|k|^{2/3}\|e^{\delta(\nu k^2)^{1/3}t}\rho_k^{1/2}\omega_k^{(2)}\|^2_{L^2L^2} + |k|^2\|e^{\delta(\nu k^2)^{1/3}t}u_k^{(2)}\|^2_{L^2L^2} + \nu^{1/2}|k|\|e^{\delta(\nu k^2)^{1/3}t}\omega_k^{(2)}\|^2_{L^2L^2}\\
&\quad + \nu^{1/2}|k|\|e^{\delta(\nu k^2)^{1/3}t}\omega_k^{(2)}\|^2_{L^\infty L^2} \lesssim (\log^2 L)k^{-2}\|\omega_{\mathrm{in};k}\|^2_{H^2_k}.
\end{aligned}$$

Finally, we estimate the term $\omega_k^{(3)}(t,y)$. We use similar arguments in [18, Proposition 6.6] and replace the estimates [18, (6.12) (6.13)] by [19, (7.21)]. We omit further details for the sake of brevity. Therefore, we conclude the proof of the Proposition. □

Next, we derive the space-time estimates of the Dirichlet problem

$$\begin{cases}
\partial_t\omega_k - \nu\Delta_k\omega_k + \mathcal{U}ik\omega_k - \mathcal{U}''ik\psi_k = ik\mathbb{f}_1 + \partial_y\mathbb{f}_2,\\
\Delta_k\psi_k = \omega_k, \quad \psi_k\big|_{y=\pm1} = 0, \quad \displaystyle\int_{-1}^1 e^{\pm ky}\omega_k dy = 0,\\
\omega_k(t=0,y) = \omega_{\mathrm{in};k}(y).
\end{cases} \tag{3.70}$$

Proposition 3.9. *Let ω_k be the solution of (3.70). For $\nu|k|^2 \le \|\mathcal{U}'\|_{L^\infty}$, we have*

$$\begin{aligned}
&\|e^{\delta(\nu k^2)^{1/3}t}\rho_k^{3/2}\omega_k\|_{L^\infty L^2}^2 + \nu^{1/3}|k|^{2/3}\|e^{\delta(\nu k^2)^{1/3}t}\rho_k^{1/2}\omega_k\|_{L^2L^2}^2 + k^2\|e^{\delta(\nu k^2)^{1/3}t}u_k\|_{L^2L^2}^2 \\
&\quad + \nu^{1/2}|k|\|e^{\delta(\nu k^2)^{1/3}t}\omega_k\|_{L^2L^2}^2 + \nu^{1/2}|k|\|e^{\delta(\nu k^2)^{1/3}t}\omega_k\|_{L^\infty L^2}^2 \\
&\quad \lesssim |\log\nu|^2\big(|k|^{-2}\|\omega_{\mathrm{in};k}\|_{H_k^2}^2 + \nu^{-1/3}|k|^{4/3}\|e^{\delta(\nu k^2)^{1/3}t}\mathbb{f}_1\|_{L^2L^2}^2 + \nu^{-1}\|e^{\delta(\nu k^2)^{1/3}t}\mathbb{f}_2\|_{L^2L^2}^2\big).
\end{aligned} \tag{3.71}$$

Proof. We decompose the solution of (3.70) as $\omega_k = \omega_k^H + \omega_k^I$, where ω_k^I solves (2.11) with $\mathbb{F}_k = ik\mathbb{f}_1 + \partial_y\mathbb{f}_2$ and ω_k^H solves (2.12). Using Propositions 2.6 and 2.8, we obtain

$$\begin{aligned}
&\nu^{1/3}|k|^{2/3}\|e^{\delta(\nu k^2)^{1/3}t}\rho_k^{1/2}\omega_k\|_{L^2L^2}^2 + |k|^2\|e^{\delta(\nu k^2)^{1/3}t}u_k\|_{L^2L^2}^2 \\
&\quad + \nu^{1/2}|k|\|e^{\delta(\nu k^2)^{1/3}t}\omega_k\|_{L^2L^2}^2 + \nu^{1/2}|k|\|e^{\delta(\nu k^2)^{1/3}t}\omega_k\|_{L^\infty L^2}^2 \\
&\quad \lesssim |\log\nu|^2\big(\nu^{1/3}|k|^{-4/3}\|\partial_y\omega_{\mathrm{in};k}\|_{L^2}^2 + |k|^{-2}\|\omega_{\mathrm{in};k}\|_{H_k^2}^2 \\
&\qquad + \nu^{-1/3}|k|^{4/3}\|e^{\delta(\nu k^2)^{1/3}t}\mathbb{f}_1\|_{L^2L^2}^2 + \nu^{-1}\|e^{\delta(\nu k^2)^{1/3}t}\mathbb{f}_2\|_{L^2L^2}^2\big).
\end{aligned} \tag{3.72}$$

It remains to estimate the first term on the left-hand side of (3.71). We perform an L^2 energy estimate as in [18, Proposition 6.7], where a mollified version $\widetilde{\rho}_k$ of the function ρ_k is introduced in [18, (6.17)]. The only modification lies in the additional term in the energy estimate, which can be estimated as

$$\left|\int \widetilde{\rho}_k^3\omega_k i\mathcal{U}''\overline{\psi_k}dy\right| = \left|\int \widetilde{\rho}_k^3\Delta_k\psi_k\mathcal{U}''\overline{\psi_k}dy\right| \lesssim \|\mathcal{U}\|_{W^{3,\infty}}|k|\|\nabla_k\psi_k\|_{L^2}^2.$$

We integrate in time to get

$$\begin{aligned}
&\int_0^\infty \|\mathcal{U}\|_{W^{3,\infty}}|k|\|\nabla_k\psi_k\|_{L^2}^2 dt \\
&\lesssim |\log\nu|^2\big(\nu^{1/3}|k|^{-4/3}\|\partial_y\omega_{\mathrm{in};k}\|_{L^2}^2 + \|\omega_{\mathrm{in};k}\|_{H_k^2}^2 + \nu^{-1/3}|k|^{4/3}\|\mathbb{f}_1\|_{L^2L^2}^2 + \nu^{-1}\|\mathbb{f}_2\|_{L^2L^2}^2\big).
\end{aligned}$$

Hence, we obtain

$$\begin{aligned}
&\|e^{\delta(\nu k^2)^{1/3}t}\rho_k^{3/2}\omega_k\|_{L^\infty L^2}^2 \\
&\lesssim |\log\nu|^2\big(k^{-2}\|\omega_{\mathrm{in};k}\|_{H_k^2}^2 + \nu^{-1/3}|k|^{4/3}\|e^{\delta(\nu k^2)^{1/3}t}\mathbb{f}_1\|_{L^2L^2}^2 + \nu^{-1}\|e^{\delta(\nu k^2)^{1/3}t}\mathbb{f}_2\|_{L^2L^2}^2\big).
\end{aligned} \tag{3.73}$$

Combining (3.72) with (3.73), we conclude the proof of the proposition. □

Finally, we are ready to prove Proposition 2.5.

Proof of Proposition 2.5. It remains to estimate the term $\|e^{\delta\nu^{1/3}t}(1-|y|)^{1/2}\omega_k\|_{L^\infty L^2}$. The proof is similar to [18, Proposition 6.1] by appealing to Proposition 3.9, thus we omit the details. □

3.5 The Frozen-time Scheme

In this section, we prove Proposition 2.4. We will derive the following more refined version:

Proposition 3.10. *Let ω_k be the solution of (2.2) and $U(t,y)$ be the heat extension defined in (1.3). For any $\delta_* \in (0, \delta/2)$ and $\nu|k|^2 \le \inf_{t\ge 0} \|U'(t,\cdot)\|_{L^\infty_y}$, we have*

$$\begin{aligned}
&\|e^{\delta_*\nu^{1/3}t}(1-|y|)^{1/2}\omega_k\|^2_{L^\infty L^2} + \nu^{1/3}|k|^{2/3}\|e^{\delta_*\nu^{1/3}t}\rho_k^{1/2}\omega_k\|^2_{L^2L^2} \\
&\quad + |k|^2\|e^{\delta_*\nu^{1/3}t}u_k\|^2_{L^2L^2} + \nu^{1/2}|k|\|e^{\delta_*\nu^{1/3}t}\omega_k\|^2_{L^2L^2} \\
&\quad \lesssim_{\delta_*} |\log\nu|^2\left(|k|^{-2}\|\omega_{\mathrm{in};k}\|^2_{H^2_k} + \nu^{-1/3}|k|^{4/3}\|e^{\delta_*\nu^{1/3}t}\mathbb{f}_1\|^2_{L^2L^2} + \nu^{-1}\|e^{\delta_*\nu^{1/3}t}\mathbb{f}_2\|^2_{L^2L^2}\right).
\end{aligned} \tag{3.74}$$

Proposition 2.4 is a direct consequence of Proposition 3.10 by taking $\zeta = \delta_*$ in (3.74).

Proof of Proposition 3.10. **The $X_{[j]}$ estimates.** We recall that $X_{[0]}$ is defined in (2.31). Since $\omega_{[0]}$ is a solution of the frozen-time system (2.25), we use Proposition 2.5 to obtain

$$\begin{aligned}
X^2_{[0]} &\lesssim |\log\nu|^2\Big(\|\omega_{\mathrm{in};k}\|^2_{H^2_k} + \nu^{1/3}|k|^{-4/3}\|\partial_y\omega_{\mathrm{in};k}\|^2_{L^2} \\
&\quad + \nu^{-1/3}|k|^{4/3}\|e^{\delta\nu^{1/3}t}(\mathbb{f}_1 + \mathbb{f}_{\mathrm{Disc}[0]})\|^2_{L^2(\mathcal{I}_{[0]};L^2)} + \nu^{-1}\|e^{\delta\nu^{1/3}t}\mathbb{f}_2\|^2_{L^2(\mathcal{I}_{[0]};L^2)}\Big) \\
&\lesssim |\log\nu|^2\left(|k|^{-2}\|\omega_{\mathrm{in};k}\|^2_{H^2_k} + \nu^{-1/3}|k|^{4/3}\|\mathbb{f}_1 + \mathbb{f}_{\mathrm{Disc}[0]}\|^2_{L^2(\mathcal{I}_{[0]};L^2)} + \nu^{-1}\|\mathbb{f}_2\|^2_{L^2(\mathcal{I}_{[0]};L^2)}\right).
\end{aligned} \tag{3.75}$$

Similarly, for $j \ge 1$, since $\omega_{[j]}$ is a solution of (2.27), Proposition 2.5 implies

$$\begin{aligned}
X^2_{[j]} &\lesssim |\log\nu|^2\left(\nu^{-1/3}|k|^{4/3}\|\mathbb{f}_1 + \mathbb{f}_{\mathrm{Disc}[j]} + \mathbb{f}_{\mathrm{Frozen}[j]}\|^2_{L^2(\mathcal{I}_{[j]};L^2)} + \nu^{-1}\|\mathbb{f}_2\|^2_{L^2(\mathcal{I}_{[j]};L^2)}\right) \\
&\lesssim |\log\nu|^2\Big(\nu^{-1/3}|k|^{4/3}\|\mathbb{f}_1 + \mathbb{f}_{\mathrm{Disc}[j]}\|^2_{L^2(\mathcal{I}_{[j]};L^2)} + \nu^{-1}\|\mathbb{f}_2\|^2_{L^2(\mathcal{I}_{[j]};L^2)} + \nu^{1/3}Y^2_{[j]}\Big),
\end{aligned} \tag{3.76}$$

where we used Lemma B.3 and $\nu k^2 \lesssim 1$ in the last step.
The $Y_{[j]}$-estimates. First of all, we observe from the definition (2.31) that

$$\|\omega_{[j']}\|_{L^2_t(\mathcal{I}_{[j]};\mathcal{Z})} \le e^{-\delta|j-j'|}X_{[j']}, \quad \text{for all } j' \le j, \tag{3.77}$$

(and of course, $\|\omega_{[j']}\|_{L^2_t(\mathcal{I}_{[j]};\mathcal{Z})} = 0$ if $j' > j$). For $j \ge 0$, using (3.77) and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
Y^2_{[j]} &= \left(\sum_{j'=0}^{j}(j-j'+1)\|\omega_{[j']}\|_{L^2(\mathcal{I}_{[j]};\mathcal{Z})}\right)^2 \lesssim \left(\sum_{j'=0}^{j}(j-j'+1)e^{-\delta|j-j'|}X_{[j']}\right)^2 \\
&\lesssim \left(\sum_{j'=0}^{j}(j-j'+1)^2 e^{-2\delta(j-j')+7\delta_*(j-j')/3}\right)\left(\sum_{j'=0}^{j} e^{-7\delta_*(j-j')/3}X^2_{[j']},\right) \\
&\lesssim \sum_{j'=0}^{j} e^{-7\delta_*(j-j')/3}X^2_{[j']}.
\end{aligned} \tag{3.78}$$

From the previous estimates (3.75), (3.76), and (3.78) it follows that

$$\begin{aligned} Y_{[j]}^2 \lesssim & |\log\nu|^2 \sum_{j'=0}^{j} e^{-7\delta_*(j-j')/3}\Big[\nu^{-1/3}|k|^{4/3}\|\mathbb{f}_1+\mathbb{f}_{\mathrm{Disc}[j']}\|_{L^2(\mathcal{I}_{[j']};L^2)}^2+\nu^{-1}\|\mathbb{f}_2\|_{L^2(\mathcal{I}_{[j']};L^2)}^2 \\ & +\nu^{1/3}Y_{[j']}^2\mathbb{1}_{j'\geq 1}\Big]+e^{-7\delta_* j/3}|\log\nu|^2|k|^{-2}\|\omega_{\mathrm{in};k}\|_{H_k^2}^2. \end{aligned} \tag{3.79}$$

For $\mathcal{E}$ defined (2.32), we use (3.79) to obtain

$$\begin{aligned} \mathcal{E} &= \sum_{j=0}^{N} e^{2\delta_* j} Y_{[j]}^2 \\ &\lesssim \sum_{j=0}^{N} |\log\nu|^2 \sum_{j'=0}^{j} e^{-\delta_*(j-j')/3+2\delta_* j'}\Big[\nu^{-1/3}|k|^{4/3}\|\mathbb{f}_1+\mathbb{f}_{\mathrm{Disc}[j']}\|_{L^2(\mathcal{I}_{[j']};L^2)}^2 \\ &\quad +\nu^{-1}\|\mathbb{f}_2\|_{L^2(\mathcal{I}_{[j']};L^2)}^2+\nu^{1/3}Y_{[j']}^2\mathbb{1}_{j'\geq 1}\Big]+\sum_{j=0}^{N} e^{-\delta_* j/3}|\log\nu|^2|k|^{-2}\|\omega_{\mathrm{in};k}\|_{H_k^2}^2 \\ &\lesssim \sum_{j'=0}^{N} |\log\nu|^2\Big[\nu^{-1/3}|k|^{4/3}e^{2\delta_* j'}\|\mathbb{f}_1+\mathbb{f}_{\mathrm{Disc}[j']}\|_{L^2(\mathcal{I}_{[j']};L^2)}^2 \\ &\quad +\nu^{-1}e^{2\delta_* j'}\|\mathbb{f}_2\|_{L^2(\mathcal{I}_{[j']};L^2)}^2+\nu^{1/3}e^{2\delta_* j'}Y_{[j']}^2\mathbb{1}_{j'\geq 1}\Big]+|\log\nu|^2|k|^{-2}\|\omega_{\mathrm{in};k}\|_{H_k^2}^2. \end{aligned}$$

Using Lemma B.3, we obtain

$$\begin{aligned} \mathcal{E} \lesssim & |\log\nu|^2\Big[\nu^{-1/3}|k|^{4/3}\|e^{\delta_*\nu^{1/3}t}\mathbb{f}_1\|_{L^2([0,T];L^2)}^2+\nu^{-1}\|e^{\delta_*\nu^{1/3}t}\mathbb{f}_2\|_{L^2([0,T];L^2)}^2\Big] \\ & +\nu^{1/3}|\log\nu|^2\mathcal{E}+|\log\nu|^2|k|^{-2}\|\omega_{\mathrm{in};k}\|_{H_k^2}^2. \end{aligned}$$

We take $\nu>0$ sufficiently small to obtain

$$\begin{aligned} \mathcal{E} \lesssim & |\log\nu|^2\Big[\nu^{-1/3}|k|^{4/3}\|e^{\delta_*\nu^{1/3}t}\mathbb{f}_1\|_{L^2([0,T];L^2)}^2+\nu^{-1}\|e^{\delta_*\nu^{1/3}t}\mathbb{f}_2\|_{L^2([0,T];L^2)}^2 \\ & +|k|^{-2}\|\omega_{\mathrm{in};k}\|_{H_k^2}^2\Big]. \end{aligned} \tag{3.80}$$

This concludes the proof of the estimates for the last three terms on the left-hand side of (3.74).

The L_t^∞ estimates. It remains to bound the first term on the left-hand side of (3.74). We note that $\omega_{[j']}$ is a solution of (2.25) if $j'=0$ and is a solution of (2.27) if $j'\geq 1$. The space-time estimates for the time-independent flow problem (Proposition 2.5) and the $\mathbb{f}_{\mathrm{Frozen}[j]}$-estimate in Lemma B.3 imply

$$\begin{aligned} &\|e^{\delta(j-j')}(1-|y|)^{1/2}\omega_{[j']}\|_{L^\infty(\mathcal{I}_{[j]};L^2)} \\ &\lesssim |\log\nu|\Big(\nu^{-1/6}|k|^{2/3}\|\mathbb{f}_1+\mathbb{f}_{\mathrm{Disc}[j']}+\mathbb{1}_{j'\geq 1}\mathbb{f}_{\mathrm{Frozen}[j']}\|_{L^2(\mathcal{I}_{[j']};L^2)}+\nu^{-1/2}\|\mathbb{f}_2\|_{L^2(\mathcal{I}_{[j']};L^2)} \\ &\quad +|k|^{-1}\|\omega_{\mathrm{in};k}\|_{H_k^2}\mathbb{1}_{j'=0}\Big) \\ &\lesssim |\log\nu|\Big(\nu^{1/6}Y_{[j']}\mathbb{1}_{j'\geq 1}+\nu^{-1/6}|k|^{2/3}\|\mathbb{f}_1+\mathbb{f}_{\mathrm{Disc}[j']}\|_{L^2(\mathcal{I}_{[j']};L^2)}+\nu^{-1/2}\|\mathbb{f}_2\|_{L^2(\mathcal{I}_{[j']};L^2)} \\ &\quad +|k|^{-1}\|\omega_{\mathrm{in};k}\|_{H_k^2}\mathbb{1}_{j'=0}\Big), \end{aligned} \tag{3.81}$$

for $0 \le j' \le j$. Hence, using (3.81) and the decomposition of solution ω_k in (2.30), we obtain

$$\begin{aligned}
&\|e^{\delta_*\nu^{1/3}t}(1-|y|)^{1/2}\omega_k\|_{L^\infty(\mathcal{I}_{[j]};L^2)}\\
&\lesssim \sum_{j'=0}^{j} e^{\delta_* j-\delta(j-j')}\|e^{\delta(j-j')}(1-|y|)^{1/2}\omega_{[j']}\|_{L^\infty(\mathcal{I}_{[j]};L^2)}\\
&\lesssim |\log\nu|\sum_{j'=0}^{j} e^{(\delta_*-\delta)(j-j')}e^{\delta_* j'}\big[\nu^{1/6}Y_{[j']}+\nu^{-1/6}|k|^{2/3}\|\mathbb{f}_1+\mathbb{f}_{\mathrm{Disc}[j']}\|_{L^2(\mathcal{I}_{[j']};L^2)}\\
&\quad+\nu^{-1/2}\|\mathbb{f}_2\|_{L^2(\mathcal{I}_{[j']};L^2)}\big]+e^{(\delta_*-\delta)j}|\log\nu||k|^{-1}\|\omega_{\mathrm{in};k}\|_{H^2_k}.
\end{aligned}$$

We observe that $e^{\delta_* j'}\|\mathbb{f}\|_{L^2(\mathcal{I}_{[j']};L^2)}\approx\|e^{\delta_*\nu^{1/3}t}\mathbb{f}\|_{L^2(\mathcal{I}_{[j']};L^2)}$ and the sum $\sum_{j'\le j}e^{2(\delta_*-\delta)(j-j')}\le C_\delta$. Applying the Cauchy-Schwarz inequality and the $\mathbb{f}_{\mathrm{Disc}[j]}$-estimate in Lemma B.3 yields that

$$\begin{aligned}
&\|e^{\delta_*\nu^{1/3}t}(1-|y|)^{1/2}\omega_k\|_{L^\infty((0,T);L^2)}\lesssim\sup_{0\le j\le N}\|e^{\delta_*\nu^{1/3}t}(1-|y|)^{1/2}\omega_k\|_{L^\infty(\mathcal{I}_{[j]};L^2)}\\
&\lesssim|\log\nu|\Big(\nu^{-1/6}|k|^{2/3}\|e^{\delta_*\nu^{1/3}t}\mathbb{f}_1\|_{L^2([0,T];L^2)}+\nu^{-1/2}\|e^{\delta_*\nu^{1/3}t}\mathbb{f}_2\|_{L^2([0,T];L^2)}\\
&\quad+|k|^{-1}\|\omega_{\mathrm{in};k}\|_{H^2_k}\Big)+|\log\nu|\nu^{1/6}\sqrt{\mathcal{E}}.
\end{aligned}$$

Using (3.80) and taking $\nu>0$ sufficiently small, we complete the proof of the proposition. $\square$

3.6 Estimates in the High Frequency Regime

In this section, we prove Proposition 2.3, which captures the high mode dynamics.

Proof of Proposition 2.3. We take the L^2 inner product of $(2.2)_1$ with $-\psi_k$ to obtain

$$\langle\partial_t\omega_k-\nu(\partial_y^2-k^2)\omega_k+iUk\omega_k-U''ik\psi_k,-\psi_k\rangle=\langle ik\mathbb{f}_{1;k}+\partial_y\mathbb{f}_{2;k},\psi_k\rangle,$$

which leads to

$$\langle\partial_t u_k,u_k\rangle+\nu\|\omega_k\|_{L^2}^2+\langle iUk\omega_k,-\psi_k\rangle+ik\int_{-1}^1U''|\psi_k|^2=\langle ik\mathbb{f}_{1;k}+\partial_y\mathbb{f}_{2;k},\psi_k\rangle.$$

We take the real part of the above identity to get

$$\frac12\frac{d}{dt}\|u_k\|_{L^2}^2+\nu\|\omega_k\|_{L^2}^2=\mathrm{Re}\int_{-1}^1Uik\omega_k\overline{\psi_k}\,dy+\mathrm{Re}\int_{-1}^1(ik\mathbb{f}_{1;k}+\partial_y\mathbb{f}_{2;k})\overline{\psi_k}\,dy.\quad(3.82)$$

For the first term on the right-hand side of (3.82), we integrate by parts to get

$$\begin{aligned}\left|\mathrm{Re}\int_{-1}^1Uik\omega_k\overline{\psi_k}\,dy\right|&=\left|\mathrm{Re}\,ik\int_{-1}^1U'\partial_y\psi_k\overline{\psi_k}\,dy\right|\le\|U'\|_{L^\infty_{t,y}}|k|\,\|\partial_y\psi_k\|_{L^2}\|\psi_k\|_{L^2}\\&\le\frac43\nu k^2\|\partial_y\psi_k\|_{L^2}^2+\frac34\nu k^4\|\psi_k\|_{L^2}^2,\end{aligned}\quad(3.83)$$

where we used $\frac{1}{2}\|U'\|_{L^\infty_{t,y}} \le \nu k^2$ in the last step. For the second term on the right-hand side of (3.82), we again integrate by parts to obtain

$$\left|\mathrm{Re}\int_{-1}^{1}(ik\mathbb{f}_{1;k}+\partial_y\mathbb{f}_{2,k})\overline{\psi_k}\,dy\right| \le \frac{2}{\nu k^2}\left(\|\mathbb{f}_{1;k}\|_{L^2}^2+\|\mathbb{f}_{2;k}\|_{L^2}^2\right) + \frac{1}{8}\nu k^2\left(\|\partial_y\psi_k\|_{L^2}^2+k^2\|\psi_k\|_{L^2}^2\right). \tag{3.84}$$

From (3.82), (3.83), (3.84), and (3.8), it follows that

$$\frac{d}{dt}\|u_k\|_{L^2}^2+\nu\|\omega_k\|_{L^2}^2 \lesssim \frac{1}{\nu k^2}\left(\|\mathbb{f}_{1;k}\|_{L^2}^2+\|\mathbb{f}_{2;k}\|_{L^2}^2\right).$$

We integrate in time to obtain

$$\begin{aligned} &e^{2\zeta\nu^{1/3}t}\|u_k\|_{L^2}^2+\nu\int_0^t e^{2\zeta\nu^{1/3}s}\|\omega_k\|_{L^2}^2\,ds \\ &\quad\le 2\zeta\nu^{1/3}\int_0^t e^{2\zeta\nu^{1/3}s}\|u_k\|_{L^2}^2+\frac{C}{\nu k^2}\int_0^t e^{2\zeta\nu^{1/3}s}\left(\|\mathbb{f}_{1;k}\|_{L^2}^2+\|\mathbb{f}_{2;k}\|_{L^2}^2\right)+C\|u_{\mathrm{in},k}\|_{L^2}^2. \end{aligned}\tag{3.85}$$

Since $k^2\|u_k\|_{L^2}^2 \le \|\omega_k\|_{L^2}^2$, we infer that

$$2\zeta\nu^{1/3}\int_0^t e^{2\zeta\nu^{1/3}s}\|u_k\|_{L^2}^2 ds \le \frac{\nu}{2}\int_0^t e^{2\zeta\nu^{1/3}s}\|\omega_k\|_{L^2}^2\,ds, \tag{3.86}$$

by using $\nu k^2 \ge 1/4$. Combining (3.85) with (3.86), we obtain

$$\begin{aligned} &k^2\|e^{\zeta\nu^{1/3}t}u_k\|_{L^\infty L^2}^2+\nu k^2\|e^{\zeta\nu^{1/3}t}\omega_k\|_{L^2L^2}^2 \\ &\quad\lesssim \nu^{-1}(\|e^{\zeta\nu^{1/3}s}\mathbb{f}_{1;k}\|_{L^2L^2}^2+\|e^{\zeta\nu^{1/3}s}f_k^2\|_{L^2L^2}^2)+\|\omega_{\mathrm{in},k}\|_{L^2}^2. \end{aligned}$$

Using $k^2\|u_k\|_{L^2}^2 \lesssim \|\omega_k\|_{L^2}^2 \lesssim \nu k^2\|\omega_k\|_{L^2}$, we infer that

$$\begin{aligned} &k^2\|e^{\zeta\nu^{1/3}t}u_k\|_{L^\infty L^2}^2+\nu k^2\|e^{\zeta\nu^{1/3}t}\omega_k\|_{L^2L^2}^2+k^2\|e^{\zeta\nu^{1/3}t}u_k\|_{L^2L^2}^2 \\ &\quad\lesssim \nu^{-1}(\|e^{\zeta\nu^{1/3}s}\mathbb{f}_{1;k}\|_{L^2L^2}^2+\|e^{\zeta\nu^{1/3}s}\mathbb{f}_{2;k}\|_{L^2L^2}^2)+\|\omega_{\mathrm{in},k}\|_{L^2}^2. \end{aligned}\tag{3.87}$$

It remains to estimate the term $\|e^{\zeta\nu^{1/3}t}(1-|y|)^{1/2}\omega_k\|_{L^\infty L^2}$. Denote $G=\partial_t\psi_k+ikU\psi_k$. Direct computation shows that

$$\begin{cases} \Delta_k G=\partial_t\omega_k+ikU\omega_k+2ikU'\partial_y\psi_k+ikU''\psi_k, \\ G|_{y=\pm1}=\partial_y G|_{y=\pm1}=0. \end{cases}\tag{3.88}$$

We take the L^2 inner product of $(2.2)_1$ with $-G$ and use (3.88) to obtain

$$\begin{aligned} \langle ik\mathbb{f}_{1;k}+\partial_y\mathbb{f}_{2;k},-G\rangle &= \langle\partial_t\omega_k-\nu\Delta_k\omega_k+ikU\omega_k-U''ik\psi_k,-G\rangle \\ &= \langle\Delta_k G-2ikU'\partial_y\psi_k-2ikU''\psi_k-\nu\Delta_k\omega_k,-G\rangle \\ &= \|\partial_y G\|_{L^2}^2+k^2\|G\|_{L^2}^2+\nu\langle\omega_k,\Delta_k G\rangle+2ik\langle U'\partial_y\psi_k,G\rangle+2ik\langle U''\psi_k,G\rangle \\ &= \|\partial_y G\|_{L^2}^2+k^2\|G\|_{L^2}^2+\nu\langle\omega_k,\partial_t\omega_k+ikU\omega_k+2ikU'\partial_y\psi_k+ikU''\psi_k\rangle \\ &\quad+2ik\langle U'\partial_y\psi_k,G\rangle+2ik\langle U''\psi_k,G\rangle. \end{aligned}$$

Taking the real part, we get

$$\begin{aligned}
&\frac{\nu}{2}\frac{d}{dt}\|\omega_k\|_{L^2}^2+\|\partial_y G\|_{L^2}^2+k^2\|G\|_{L^2}^2\\
&\lesssim \nu|k|\|\omega_k\|_{L^2}\|\partial_y\psi_k\|_{L^2}+\nu|k|\|\partial_y\psi_k\|_{L^2}\|\psi_k\|_{L^2}+|k|\|\partial_y\psi_k\|_{L^2}\|G\|_{L^2}\\
&\qquad+|k|\|\psi_k\|_{L^2}\|G\|_{L^2}+|k|\|\mathbb{f}_{1;k}\|_{L^2}\|G\|_{L^2}+\|\mathbb{f}_{2;k}\|_{L^2}\|\partial_y G\|_{L^2}\\
&\lesssim \nu\|\omega_k\|_{L^2}^2+\nu k^2\|\partial_y\psi_k\|_{L^2}^2+|k|\|\partial_y\psi_k\|_{L^2}\|G\|_{L^2}+|k|\|\psi_k\|_{L^2}\|G\|_{L^2}\\
&\qquad+|k|\|\mathbb{f}_{1;k}\|_{L^2}\|G\|_{L^2}+\|\mathbb{f}_{2;k}\|_{L^2}\|\partial_y G\|_{L^2},
\end{aligned}$$

where we used Lemma B.2 and (3.8). Therefore, using the Young and Poincaré inequalities, we obtain

$$\nu\frac{d}{dt}\|\omega_k\|_{L^2}^2+\|\partial_y G\|_{L^2}^2\lesssim\|\mathbb{f}_{1;k}\|_{L^2}^2+\|\mathbb{f}_{2;k}\|_{L^2}^2+\nu\|\omega_k\|_{L^2}^2.$$

We integrate in time and use (3.87) to obtain

$$\|e^{\zeta\nu^{1/3}t}\omega_k\|_{L^\infty L^2}^2\lesssim\nu^{-1}\|e^{\zeta\nu^{1/3}t}\mathbb{f}_{1;k}\|_{L^2L^2}^2+\nu^{-1}\|e^{\zeta\nu^{1/3}t}\mathbb{f}_{2;k}\|_{L^2L^2}^2+\|\omega_{\mathrm{in},k}\|_{L^2}^2.$$

The proof of the proposition is thus completed. $\square$

4 Nonlinear Estimates

In this section, we prove nonlinear stability. We denote

$$E_k:=\|e^{\zeta\nu^{1/3}t}(1-|y|)^{1/2}\omega_k\|_{L_t^\infty L_y^2}+\nu^{1/4}|k|^{1/2}\|e^{\zeta\nu^{1/3}t}\omega_k\|_{L_t^2L_y^2}+|k|\|e^{\zeta\nu^{1/3}t}u_k\|_{L_t^2L_y^2}$$

for $k\in\mathbb{Z}\setminus\{0\}$, and

$$E_0:=\|\omega_0\|_{L_t^\infty L_y^2}.$$

Proof of Theorem 1.1. We apply the Fourier transform in the x-variable and rewrite the equation (1.6) into the linearized form (2.2), where

$$\mathbb{f}_{1;k}=-f_k^1:=\sum_{\ell\in\mathbb{Z}}u_\ell^{(1)}\omega_{k-\ell},\quad \mathbb{f}_{2;k}=-f_k^2:=\sum_{\ell\in\mathbb{Z}}u_\ell^{(2)}\omega_{k-\ell}. \tag{4.1}$$

By the boundary condition $u^{(2)}|_{y=\pm1}=0$ and incompressibility $\nabla\cdot u=0$, we have $u_0^{(2)}=0$. Therefore, from $(1.5)_1$ it follows that

$$\begin{aligned}
(\partial_t-\nu\partial_y^2)u_0^{(1)}&=-\sum_{\ell\in\mathbb{Z}\setminus\{0\}}u_\ell^{(2)}\partial_y u_{-\ell}^{(1)}=-\sum_{\ell\in\mathbb{Z}\setminus\{0\}}u_\ell^{(2)}(\partial_y u_{-\ell}^{(1)}-i\ell u_{-\ell}^{(2)})\\
&=\sum_{\ell\in\mathbb{Z}\setminus\{0\}}u_\ell^{(2)}\omega_{-\ell}=-f_0^2.
\end{aligned}\tag{4.2}$$

Multiplying (4.2) by $\partial_y^2u_0^{(1)}$ and integrating by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\partial_y u_0^{(1)}\|_{L^2}^2=-\nu\|\partial_y^2u_0^{(1)}\|_{L^2}^2+\langle f_0^2,\partial_y^2u_0^{(1)}\rangle\le-\frac{1}{2}\nu\|\partial_y^2u_0^{(1)}\|_{L^2}^2+(2\nu)^{-1}\|f_0^2\|_{L^2}^2, \tag{4.3}$$

where we used the boundary conditions $u_0^{(1)}|_{y=\pm1}=0$. Since $\omega_0=-\partial_y u_0^{(1)}$, integrating (4.3) in time yields

$$E_0^2=\|\omega_0\|_{L_t^\infty L_y^2}^2\lesssim \nu^{-1}\|f_0^2\|_{L^2L^2}^2+\|\omega_{\mathrm{in};0}\|_{L^2}^2. \tag{4.4}$$

For $k\neq 0$, we have

$$\begin{aligned}\left\|\frac{e^{\zeta\nu^{1/3}t}u_k^{(2)}(t,y)}{(1-|y|)^{1/2}}\right\|_{L_t^2L_y^\infty}^2&=\left\|e^{2\zeta\nu^{1/3}t}\sup_{y\in[-1,1]}\frac{|u_k^{(2)}(t,y)|^2}{1-|y|}\right\|_{L_t^1}\\&=\left\|e^{2\zeta\nu^{1/3}t}\max\left\{\sup_{y\in[0,1]}\frac{|\int_1^y\partial_z u_k^{(2)}(t,z)dz|^2}{1-y},\sup_{y\in[-1,0]}\frac{|\int_{-1}^y\partial_z u_k^{(2)}(t,z)dz|^2}{1+y}\right\}\right\|_{L_t^1}\\&\leq\|e^{\zeta\nu^{1/3}t}\partial_y u_k^{(2)}\|_{L_t^2L_y^2}^2=|k|^2\|e^{\zeta\nu^{1/3}t}u_k^{(1)}\|_{L_t^2L_y^2}^2\leq E_k^2,\end{aligned}\tag{4.5}$$

where we used the Cauchy-Schwarz inequality. For $k\in\mathbb{Z}$, combining (4.1) with (4.5) gives

$$\begin{aligned}\|e^{\zeta\nu^{1/3}t}f_k^2\|_{L_t^2L_y^2}&\leq\sum_{\ell\in\mathbb{Z}\setminus\{0\}}\|e^{\zeta\nu^{1/3}t}u_\ell^{(2)}\omega_{k-\ell}\|_{L_t^2L_y^2}\\&\leq C\sum_{\ell\in\mathbb{Z}\setminus\{0\}}\left\|\frac{e^{\zeta\nu^{1/3}t}u_\ell^{(2)}}{(1-|y|)^{1/2}}\right\|_{L_t^2L_y^\infty}\|(1-|y|)^{1/2}\omega_{k-\ell}\|_{L_t^\infty L_y^2}\\&\leq C\sum_{\ell\in\mathbb{Z}\setminus\{0\}}E_\ell E_{k-\ell},\end{aligned}\tag{4.6}$$

where we used $u_0^{(2)}=0$. In particular,

$$\|f_0^2\|_{L_t^2L_y^2}\leq C\sum_{\ell\in\mathbb{Z}\setminus\{0\}}E_\ell E_{-\ell}. \tag{4.7}$$

Combining (4.4) with (4.7), we obtain

$$E_0\lesssim\nu^{-1/2}\sum_{\ell\in\mathbb{Z}\setminus\{0\}}E_\ell E_{-\ell}+\|\omega_{\mathrm{in};0}\|_{L^2}. \tag{4.8}$$

Next we estimate E_k for $k\neq0$. By Proposition 2.1, for $k\neq 0$,

$$\begin{aligned}E_k\lesssim|\log\nu|\big(&|k|^{-1}\|\omega_{\mathrm{in};k}\|_{H_k^2}+\min\{\nu^{-1/6}|k|^{2/3},\nu^{-1/2}\}\|e^{\zeta\nu^{1/3}t}f_k^1\|_{L^2L^2}\\&+\nu^{-1/2}\|e^{\zeta\nu^{1/3}t}f_k^2\|_{L^2L^2}\big).\end{aligned}\tag{4.9}$$

For $k\neq0$, using (4.1) we have

$$\begin{aligned}\|e^{\zeta\nu^{1/3}t}f_k^1\|_{L_t^2L_y^2}&\leq\|u_0^{(1)}\|_{L_t^\infty L_y^\infty}\|e^{\zeta\nu^{1/3}t}\omega_k\|_{L_t^2L_y^2}+\|e^{\zeta\nu^{1/3}t}u_k^{(1)}\|_{L_t^2L_y^\infty}\|\omega_0\|_{L_t^\infty L_y^2}\\&\quad+\sum_{\ell\in\mathbb{Z}\setminus\{0,k\}}\|e^{\zeta\nu^{1/3}t}u_\ell^{(1)}\omega_{k-\ell}\|_{L_t^2L_y^2}\\&=:T_1+T_2+T_3.\end{aligned}\tag{4.10}$$

By the Poincaré and Agmon inequalities,

$$\|u_0^{(1)}\|_{L_t^\infty L_y^\infty} \lesssim \|u_0^{(1)}\|_{L_t^\infty L_y^2}^{1/2} \|\partial_y u_0^{(1)}\|_{L_t^\infty L_y^2}^{1/2} \lesssim \|\partial_y u_0^{(1)}\|_{L_t^\infty L_y^2} = \|\omega_0\|_{L_t^\infty L_y^2}.$$

Applying Poincaré and Agmon inequalities again gives

$$\begin{aligned} T_1 + T_2 &\lesssim \|\omega_0\|_{L_t^\infty L_y^2} \left(\|e^{\zeta\nu^{1/3}t}\omega_k\|_{L_t^2 L_y^2} + \|\partial_y u_k^{(1)}\|_{L_t^2 L_y^2}^{1/2} \|e^{\zeta\nu^{1/3}t} u_k^{(1)}\|_{L_t^2 L_y^2}^{1/2} \right) \\ &\lesssim \|\omega_0\|_{L_t^\infty L_y^2} \left(\|e^{\zeta\nu^{1/3}t}\omega_k\|_{L_t^2 L_y^2} + (\|\omega_k\|_{L_t^2 L_y^2} + \||k| u_k^{(2)}\|_{L_t^2 L_y^2})^{1/2} \|e^{\zeta\nu^{1/3}t} u_k^{(1)}\|_{L_t^2 L_y^2}^{1/2} \right) \\ &\lesssim \|\omega_0\|_{L_t^\infty L_y^2} \left(\|e^{\zeta\nu^{1/3}t}\omega_k\|_{L_t^2 L_y^2} + |k|^{1/2} \|e^{\zeta\nu^{1/3}t} u_k\|_{L_t^2 L_y^2} \right) \\ &\lesssim \nu^{-1/4} |k|^{-1/2} E_0 E_k. \end{aligned} \tag{4.11}$$

A similar argument to (4.5) yields

$$\left\| \frac{e^{\zeta\nu^{1/3}t} u_m^{(1)}(t,y)}{(1-|y|)^{1/2}} \right\|_{L_t^2 L_y^\infty}^2 \le \|e^{\zeta\nu^{1/3}t} \partial_y u_m^{(1)}\|_{L_t^2 L_y^2}^2, \quad \forall m \in \mathbb{Z}.$$

Therefore,

$$\begin{aligned} T_3 &\lesssim \sum_{\ell \in \mathbb{Z}\setminus\{0,k\}} \left\| \frac{e^{\zeta\nu^{1/3}t} u_\ell^{(1)}}{(1-|y|)^{1/2}} \right\|_{L_t^2 L_y^\infty} \|(1-|y|)^{1/2}\omega_{k-\ell}\|_{L_t^\infty L_y^2} \\ &\lesssim \sum_{\ell \in \mathbb{Z}\setminus\{0,k\}} \|e^{\zeta\nu^{1/3}t} \partial_y u_\ell^{(1)}\|_{L_t^2 L_y^2} \|(1-|y|)^{1/2}\omega_{k-\ell}\|_{L_t^\infty L_y^2} \end{aligned}$$

which leads to

$$\begin{aligned} T_3 &\lesssim \sum_{\ell \in \mathbb{Z}\setminus\{0,k\}} \left(\|e^{\zeta\nu^{1/3}t}\omega_\ell\|_{L_t^2 L_y^2} + \ell^2 \|e^{\zeta\nu^{1/3}t} \Delta_\ell^{-1}\omega_\ell\|_{L_t^2 L_y^2} \right) \|(1-|y|)^{1/2}\omega_{k-\ell}\|_{L_t^\infty L_y^2} \\ &\lesssim \sum_{\ell \in \mathbb{Z}\setminus\{0,k\}} \|e^{\zeta\nu^{1/3}t}\omega_\ell\|_{L_t^2 L_y^2} \|e^{\zeta\nu^{1/3}t}(1-|y|)^{1/2}\omega_{k-\ell}\|_{L_t^\infty L_y^2} \\ &\lesssim \sum_{\ell \in \mathbb{Z}\setminus\{0,k\}} \nu^{-1/4} |k|^{-1/2} E_\ell E_{k-\ell}. \end{aligned} \tag{4.12}$$

Using (4.6), (4.8), (4.9), (4.10), (4.11), and (4.12), we obtain

$$\begin{aligned} \sum_{k\in\mathbb{Z}} E_k &\lesssim |\log\nu| \sum_{k\in\mathbb{Z}} |k|^{-1} \|\omega_{\mathrm{in};k}\|_{H_k^2} + \nu^{-1/2} |\log\nu| \sum_{k\in\mathbb{Z}} \sum_{\ell\in\mathbb{Z}} E_\ell E_{k-\ell} \\ &\lesssim |\log\nu| \sum_{k\in\mathbb{Z}} |k|^{-1} \|\omega_{\mathrm{in};k}\|_{H_k^2} + \nu^{-1/2} |\log\nu| \left(\sum_{k\in\mathbb{Z}} E_k \right)^2, \end{aligned}$$

Noting that the initial data $\|\omega_{\mathrm{in}}\|_{H^2} \le \epsilon\nu^{1/2}|\log\nu|^{-2}$ implies

$$\sum_{k\in\mathbb{Z}} |k|^{-1} \|\omega_{\mathrm{in};k}\|_{H_k^2} \lesssim \epsilon\nu^{1/2} |\log\nu|^{-2}.$$

A continuity argument yields

$$\sum_{k\in\mathbb{Z}} E_k \leq C\epsilon\nu^{1/2}|\log\nu|^{-1},$$

concluding the proof of the theorem. $\square$

A Properties of the Singular Integral Operator $\mathfrak{J}_k$

This section collects the necessary estimates related to the physical side ghost singular integral operator defined as

$$\mathfrak{J}_k[f](y) := |k|\,\text{p.v.}\frac{k}{|k|}\int_{-1}^{1}\frac{1}{2i(y-y')}G_k(y,y')f(y')\mathrm{d}y', \tag{A.1}$$

where G_k is the Green's function for $\Delta_k := -|k|^2+\partial_{yy}$ with homogeneous Dirichlet boundary conditions and has the explicit form (3.13).

Lemma A.1 ([13]). *The singular integral operator $\mathfrak{J}_k$ extends to a bounded linear operator on $L^2\to L^2$ and moreover*

$$\|\mathfrak{J}_k\|_{L^2\to L^2}\lesssim 1.$$

The next lemma captures the commutator between ∂_y and $\mathfrak{J}_k$.

Lemma A.2 ([13]). *If $f\in H^1$ then $\mathfrak{J}_k[f]\in H^1$. We define*

$$\mathfrak{H}_k := [\partial_y,\mathfrak{J}_k].$$

Then

$$\mathfrak{H}_k[f] = |k|\text{p.v.}\int_{-1}^{1}\frac{H_k(y,y')}{2i(y-y')}f(y')\mathrm{d}y',$$

where H_k is given by

$$H_k(y,y') := -\frac{\sinh(k(y+y'))}{\sinh(2k)}$$

We have the following commutator estimate on $\mathfrak{H}_k$.

Lemma A.3 ([13]). *There holds the estimate*

$$\|\mathfrak{H}_k\|_{L^2\to L^2}\lesssim |k|.$$

Lemma A.4 ([13]). *For all $f,g\in L^2$ there holds*

$$\overline{\mathfrak{J}_k[f]} = -\mathfrak{J}_k[\bar{f}]$$

and

$$\int_{-1}^{1}\bar{f}\mathfrak{J}_k[g]dy = -\int_{-1}^{1}\mathfrak{J}_k[\bar{f}]g dy,$$

which in particular, implies $\mathfrak{J}_k = \mathfrak{J}_k^*$.

Lemma A.5 ([13])**.** *Under the regularity hypotheses, for all $\delta > 0$ sufficiently small (depending only on universal constants), the following estimate holds*

$$\begin{aligned}\mathrm{Re}\langle -\mathcal{U}ikw_k, \mathfrak{J}_k w_k\rangle &\le -\frac{|k|^2}{16}\|\nabla_k\phi_k\|_{L^2}^2,\\ |\mathrm{Re}\langle \mathcal{U}''ik\phi_k, \mathfrak{J}_k w_k\rangle| &\le C\delta\|\nabla_k\phi_k\|_{L^2}^2,\\ \mathrm{Re}\langle \nu\Delta_k w_k, \mathfrak{J}_k w_k\rangle &\le C\nu\|\nabla_k w_k\|_{L^2}^2,\end{aligned}$$

where the implicit constant depends on $\|\mathcal{U}\|_{C^3}$.

B Auxiliary Lemmas

Proof of Proposition 2.7. Without loss of generality, we assume that $k > 0$. First, we develop the estimates for the spectral stream function ϕ_{Na} and the spectral vorticity gradient ∇w_{Na} using the singular integral operator $\mathfrak{J}_k$. Next, we use some standard resolvent estimate techniques to derive the enhanced dissipation estimates for the spectral vorticity w_{Na}.

We aim to prove the following key estimates

$$\begin{aligned}&|k|\|\nabla_k\phi_{\mathrm{Na}}\|_{L^2} + \sqrt{\nu}\|\nabla_k w_{\mathrm{Na}}\|_{L^2} + \sqrt{|k|(\mathrm{Im}\lambda)_+}\|w_{\mathrm{Na}}\|_{L^2}\\ &\quad\le C\delta_0^{-\frac12}\nu^{-\frac16}|k|^{-\frac13}\|F_k\|_{L^2} + C\delta_0^{\frac12}\nu^{\frac16}|k|^{\frac13}\|w_{\mathrm{Na}}\|_{L^2}, \quad \text{if } F_k \in L^2, \qquad \text{(B.1)}\\ &|k|\|\nabla_k\phi_{\mathrm{Na}}\|_{L^2} + \sqrt{\nu}\|\nabla_k w_{\mathrm{Na}}\|_{L^2} + \sqrt{|k|(\mathrm{Im}\lambda)_+}\|w_{\mathrm{Na}}\|_{L^2}\\ &\quad\le C|k|^{-1}\|\nabla_k F_k\|_{L^2} + C\delta_0^{\frac12}\nu^{\frac16}|k|^{\frac13}\|w_{\mathrm{Na}}\|_{L^2}, \quad \text{if } F_k \in H_0^1, \qquad \text{(B.2)}\\ &|k|\|\nabla_k\phi_{\mathrm{Na}}\|_{L^2} + \sqrt{\nu}\|\nabla_k w_{\mathrm{Na}}\|_{L^2} + \sqrt{|k|(\mathrm{Im}\lambda)_+}\|w_{\mathrm{Na}}\|_{L^2}\\ &\quad\le C\nu^{-\frac12}\|F_k\|_{H^{-1}} + C\delta_0^{\frac12}\nu^{\frac16}|k|^{\frac13}\|w_{\mathrm{Na}}\|_{L^2}, \quad \text{if } F_k \in H^{-1}. \qquad \text{(B.3)}\end{aligned}$$

First, we multiply the equation $(2.16)_1$ by $\overline{w_{\mathrm{Na}}}$ and take the real part to obtain

$$-\nu\int \Delta_k w_{\mathrm{Na}}\overline{w_{\mathrm{Na}}}dy + \int(\mathcal{U}-\lambda)ikw_{\mathrm{Na}}\overline{w_{\mathrm{Na}}}dy = \int F_k\overline{w_{\mathrm{Na}}}dy + ik\int \mathcal{U}''\phi_{\mathrm{Na}}\overline{\Delta_k\phi_{\mathrm{Na}}}dy.$$

We integrate by parts to get

$$\begin{aligned}&\nu\|\nabla_k w_{\mathrm{Na}}\|_{L^2}^2 + ik\int(\mathcal{U}-\mathrm{Re}\lambda)|w_{\mathrm{Na}}|^2dy + k\mathrm{Im}\lambda\|w_{\mathrm{Na}}\|_{L^2}^2\\ &= \int F_k\overline{w_{\mathrm{Na}}}dy - ik\int\nabla_k(\mathcal{U}''\phi_{\mathrm{Na}})\cdot\overline{\nabla_k\phi_{\mathrm{Na}}}.\end{aligned}$$

Taking the real part, we are led to

$$\begin{aligned}\nu\|\nabla_k w_{\mathrm{Na}}\|_{L^2}^2 + k\max\{0,\mathrm{Im}\lambda\}\|w_{\mathrm{Na}}\|_{L^2}^2 &\le |\mathrm{Re}\langle F_k, w_{\mathrm{Na}}\rangle|\\ &\quad + \|\mathcal{U}'''\|_{L^\infty}\|k\phi_{\mathrm{Na}}\|_{L^2}\|\partial_y\phi_{\mathrm{Na}}\|_{L^2} - k\min\{0,\mathrm{Im}\lambda\}\|w_{\mathrm{Na}}\|_{L^2}^2.\end{aligned} \qquad \text{(B.4)}$$

Thanks to the spectral constraint $k\mathrm{Im}\lambda \ge -\delta_0\nu^{1/3}|k|^{2/3}$ and the fact that $w_{\mathrm{Na}} \in H_0^1$, we apply interpolations based on the regularity of F_k for the first term on the right side of

(B.4), yielding

$$\begin{aligned}\nu\|\nabla_k w_{\mathrm{Na}}\|_{L^2}^2 + |k|\max\{0,\mathrm{Im}\lambda\}\|w_{\mathrm{Na}}\|_{L^2}^2 \le \min\big\{&\|F_k\|_{L^2}\|w_{\mathrm{Na}}\|_{L^2},\\ &\|F_k\|_{H^{-1}}\|w_{\mathrm{Na}}\|_{H^1_k}, \|\nabla_k F_k\|_{L^2}\|\nabla_k\phi_{\mathrm{Na}}\|_{L^2}\big\}\\ &+\|\mathcal{U}'''\|_{L^\infty}\|\nabla_k\phi_{\mathrm{Na}}\|_{L^2}^2 + \delta_0\nu^{1/3}|k|^{2/3}\mathbb{1}_{\mathrm{Im}\lambda\le 0}\|w_{\mathrm{Na}}\|_{L^2}^2.\end{aligned} \tag{B.5}$$

Next, we test the equation $(2.16)_1$ by $\mathfrak{J}_k w_k$ and take the real part to obtain

$$\mathrm{Re}\,\langle -\nu\Delta_k w_{\mathrm{Na}} + ik(\mathcal{U}-\mathrm{Re}\lambda - i\mathrm{Im}\lambda)w_{\mathrm{Na}} - \mathcal{U}''ik\phi_{\mathrm{Na}}, \mathfrak{J}_k w_{\mathrm{Na}}\rangle = \mathrm{Re}\langle F_k, \mathfrak{J}_k w_{\mathrm{Na}}\rangle.$$

Rearranging the terms yields

$$\begin{aligned}\mathrm{Re}\langle \mathcal{U}ikw_{\mathrm{Na}}, \mathfrak{J}_k w_{\mathrm{Na}}\rangle &= \nu\mathrm{Re}\langle\Delta_k w_{\mathrm{Na}}, \mathfrak{J}_k w_{\mathrm{Na}}\rangle + k\mathrm{Re}\lambda\,\mathrm{Re}\left(i\langle w_{\mathrm{Na}}, \mathfrak{J}_k w_{\mathrm{Na}}\rangle\right)\\ -k\mathrm{Im}\lambda\,\mathrm{Re}\langle w_{\mathrm{Na}}, \mathfrak{J}_k w_{\mathrm{Na}}\rangle &+ k\mathrm{Re}\,\langle \mathcal{U}''i\phi_{\mathrm{Na}}, \mathfrak{J}_k w_{\mathrm{Na}}\rangle + \mathrm{Re}\,\langle F_k, \mathfrak{J}_k w_{\mathrm{Na}}\rangle =: \sum_{j=1}^4 T_j.\end{aligned} \tag{B.6}$$

The left-hand side of (B.6) and the term T_1 can be estimates using Lemma A.5 as

$$\mathrm{Re}\langle \mathcal{U}ikw_{\mathrm{Na}}, \mathfrak{J}_k w_{\mathrm{Na}}\rangle \ge \frac{k^2}{16}\|\nabla_k\phi_{\mathrm{Na}}\|_{L^2}^2, \tag{B.7}$$

$$T_1 \le C\nu\|\nabla_k w_{\mathrm{Na}}\|_{L^2}^2. \tag{B.8}$$

Thanks to Lemma A.4, we have the identity

$$\overline{\int_{-1}^1 \bar f\mathfrak{J}_k[f]dy} = -\int_{-1}^1 f\mathfrak{J}_k[\bar f]dy = \int_{-1}^1 \bar f\mathfrak{J}_k[f]dy, \quad \forall f\in L^2.$$

Hence, we deduce that $\int_{-1}^1 \bar f\mathfrak{J}_k[f]dy \in \mathbb{R}$ and in particular, $\int_{-1}^1 \overline{w_{\mathrm{Na}}}\mathfrak{J}_k w_{\mathrm{Na}}dy \in \mathbb{R}$. Using Lemma A.1, we obtain

$$\begin{aligned}&T_2 = k\mathrm{Re}\lambda\,\mathrm{Re}\left(i\langle w_{\mathrm{Na}}, \mathfrak{J}_k w_{\mathrm{Na}}\rangle\right) = 0,\\ &|T_3| \le |k|\left(\delta_0\nu^{1/3}|k|^{-1/3}\mathbb{1}_{\mathrm{Im}\lambda\le 0} + \max\{0,\mathrm{Im}\lambda\}\right)\|w_{\mathrm{Na}}\|_{L^2}^2.\end{aligned} \tag{B.9}$$

Now we estimate the term T_4. Based on the type of regularity, we have following cases:

- If $F_k\in L^2$, we invoke Lemma A.1 to obtain

$$|T_4| = |\langle F_k, \mathfrak{J}_k w_{\mathrm{Na}}\rangle| \lesssim \|F_k\|_{L^2}\|w_{\mathrm{Na}}\|_{L^2}.$$

- If $F_k\in H^1_0$, using Lemmas A.1 and A.3 we obtain

$$\begin{aligned}|T_4| = |\langle F_k, \mathfrak{J}_k w_{\mathrm{Na}}\rangle| &\lesssim |\langle |k|F_k, \mathfrak{J}_k|k|\phi_{\mathrm{Na}}\rangle| + |\langle F_k, \mathfrak{J}_k\partial_{yy}\phi_{\mathrm{Na}}\rangle|\\ &\lesssim \|\nabla_k F_k\|_{L^2}\|\nabla_k\phi_{\mathrm{Na}}\|_{L^2} + |\langle F_k, \partial_y\mathfrak{J}_k\partial_y\phi_{\mathrm{Na}}\rangle| + |\langle F_k, [\partial_y,\mathfrak{J}_k]\partial_y\phi_{\mathrm{Na}}\rangle|\\ &\lesssim \|\nabla_k F_k\|_{L^2}\|\nabla_k\phi_{\mathrm{Na}}\|_{L^2} + |\langle \partial_y F_k, \mathfrak{J}_k\partial_y\phi_{\mathrm{Na}}\rangle|\\ &\quad + \|[\mathfrak{J}_k,\partial_y]\|_{L^2\to L^2}\|F_k\|_{L^2}\|\nabla_k\phi_{\mathrm{Na}}\|_{L^2}\\ &\lesssim \|\nabla_k F_k\|_{L^2}\|\nabla_k\phi_{\mathrm{Na}}\|_{L^2}.\end{aligned}$$

- If $F_k \in H^{-1}$, using the definition of $\mathfrak{J}_k$ in (A.1) and $G_k(\pm 1, y') = 0$, we get

$$\mathfrak{J}_k w_k(y = \pm 1) = k \text{ p.v.} \int_{-1}^{1} \frac{1}{2i} \cdot \frac{G_k(\pm 1, y')}{(\pm 1 - y')} w_k(y') dy' = 0.$$

  By Lemmas A.1, A.2, and A.3, we deduce

$$\|\partial_y \mathfrak{J}_k w_{\mathrm{Na}}\|_{L^2} \le \|\mathfrak{J}_k \partial_y w_{\mathrm{Na}}\|_{L^2} + \|[\partial_y, \mathfrak{J}_k] w_{\mathrm{Na}}\|_{L^2} \lesssim \|\nabla_k w_{\mathrm{Na}}\|_{L^2}.$$

  Hence, $\mathfrak{J}_k w_k \in H_0^1$ and

$$|T_4| \lesssim \|F_k\|_{H^{-1}} \|\nabla_k w_{\mathrm{Na}}\|_{L^2}.$$

To summarize, we have the following bound for the term T_4:

$$|T_4| \lesssim \min\Big\{ \|F_k\|_{L^2}\|w_{\mathrm{Na}}\|_{L^2},\ \|\nabla_k F_k\|_{L^2}\|\nabla_k \phi_{\mathrm{Na}}\|_{L^2},\ \|F_k\|_{H^{-1}}\|\nabla_k w_{\mathrm{Na}}\|_{L^2} \Big\}. \tag{B.10}$$

From (B.6), (B.7), (B.8), (B.9), and (B.10), we infer that there exists a constant $C_0 \ge 1$ such that

$$\begin{aligned} |k|^2 \|\nabla_k \phi_{\mathrm{Na}}\|_{L^2}^2 \le C_0 \Big(&\nu \|\nabla_k w_{\mathrm{Na}}\|_{L^2}^2 + \min\left\{ \|F_k\|_{L^2}\|w_{\mathrm{Na}}\|_{L^2},\ \|F_k\|_{H^{-1}}\|\nabla_k w_{\mathrm{Na}}\|_{L^2}, \right. \\ &\left. \|\nabla_k F_k\|_{L^2}\|\nabla_k \phi_{\mathrm{Na}}\|_{L^2} \right\} + |k| \left(\delta_0 \nu^{1/3} |k|^{-1/3} \mathbb{1}_{\mathrm{Im}\lambda \le 0} + \max\{0, \mathrm{Im}\lambda\} \right) \|w_{\mathrm{Na}}\|_{L^2}^2 \Big). \end{aligned} \tag{B.11}$$

We multiply (B.11) by $1/2C_0$ and add the resulting inequality to (B.5) to obtain that

$$\begin{aligned} &\nu \|\nabla_k w_{\mathrm{Na}}\|_{L^2}^2 + |k| \max\{0, \mathrm{Im}\lambda\} \|w_{\mathrm{Na}}\|_{L^2}^2 + \frac{|k|^2}{2C_0} \|\nabla_k \phi_{\mathrm{Na}}\|_{L^2}^2 \\ \le& 2 \min\left\{ \|F_k\|_{L^2}\|w_{\mathrm{Na}}\|_{L^2},\ \|F_k\|_{H^{-1}}\|w_{\mathrm{Na}}\|_{H_k^1},\ \|\nabla_k F_k\|_{L^2}\|\nabla_k \phi_{\mathrm{Na}}\|_{L^2} \right\} \\ &+ \|\mathcal{U}'''\|_{L^\infty} \|\nabla_k \phi_k\|_{L^2}^2 + 2\delta_0 \nu^{1/3} |k|^{2/3} \mathbb{1}_{\mathrm{Im}\lambda \le 0} \|w_{\mathrm{Na}}\|_{L^2}^2 \\ &+ \frac{\nu}{2} \|\nabla_k w_{\mathrm{Na}}\|_{L^2}^2 + \frac{|k|}{2} \max\{0, \mathrm{Im}\lambda\} \|w_{\mathrm{Na}}\|_{L^2}^2. \end{aligned}$$

Consequently,

$$\begin{aligned} &\nu \|\nabla_k w_{\mathrm{Na}}\|_{L^2}^2 + |k| \max\{0, \mathrm{Im}\lambda\} \|w_{\mathrm{Na}}\|_{L^2}^2 + \frac{|k|^2}{4C_0} \|\nabla_k \phi_{\mathrm{Na}}\|_{L^2}^2 \\ \le& 4 \min\left\{ \|F_k\|_{L^2}\|w_{\mathrm{Na}}\|_{L^2},\ \|F_k\|_{H^{-1}}\|w_{\mathrm{Na}}\|_{H_k^1},\ \|\nabla_k F_k\|_{L^2}\|\nabla_k \phi_{\mathrm{Na}}\|_{L^2} \right\} \\ &+ 4\delta_0 \nu^{1/3} |k|^{2/3} \mathbb{1}_{\mathrm{Im}\lambda \le 0} \|w_{\mathrm{Na}}\|_{L^2}^2. \end{aligned} \tag{B.12}$$

Based on (B.12), we conclude that:

- If $F_k \in L^2$, (B.12) implies

$$\nu \|\nabla_k w_{\mathrm{Na}}\|_{L^2}^2 + |k| \max\{0, \mathrm{Im}\lambda\} \|w_{\mathrm{Na}}\|_{L^2}^2 + \frac{|k|^2}{4C_0} \|\nabla_k \phi_{\mathrm{Na}}\|_{L^2}^2$$

$$\leq \frac{8}{\delta_0 \nu^{1/3}|k|^{2/3}} \|F_k\|_{L^2}^2 + 8\delta_0 \nu^{1/3}|k|^{2/3} \mathbb{1}_{\mathrm{Im}\lambda \leq \delta_0 L^{-1}} \|w_{\mathrm{Na}}\|_{L^2}^2$$
$$+ \frac{1}{2} \mathbb{1}_{\mathrm{Im}\lambda \geq \delta_0 L^{-1}} |k| \max\{0, \mathrm{Im}\lambda\} \|w_{\mathrm{Na}}\|_{L^2}^2,$$

which leads to

$$\sqrt{\nu}\|\nabla_k w_{\mathrm{Na}}\|_{L^2} + \sqrt{|k| \max\{0, \mathrm{Im}\lambda\}} \|w_{\mathrm{Na}}\|_{L^2} + \frac{|k|}{2\sqrt{C_0}} \|\nabla_k \phi_{\mathrm{Na}}\|_{L^2}$$
$$\leq C\delta_0^{-1/2} \nu^{-1/6} |k|^{-1/3} \|F_k\|_{L^2} + C \mathbb{1}_{\mathrm{Im}\lambda \leq \delta_0 L^{-1}} \sqrt{\delta_0} \nu^{1/6} |k|^{1/3} \|w_{\mathrm{Na}}\|_{L^2},$$

and (B.1) follows.

- If $F \in H_0^1$, (B.12) implies

$$\nu\|\nabla_k w_{\mathrm{Na}}\|_{L^2}^2 + \frac{|k|^2}{4C_0} \|\nabla_k \phi_{\mathrm{Na}}\|_{L^2}^2$$
$$\leq 32C_0 |k|^{-2} \|\nabla_k F_k\|_{L^2}^2 + \frac{|k|^2}{8C_0} \|\nabla_k \phi_{\mathrm{Na}}\|_{L^2}^2 + 4\delta_0 \nu^{1/3} |k|^{2/3} \mathbb{1}_{\mathrm{Im}\lambda \leq 0} \|w_{\mathrm{Na}}\|_{L^2}^2,$$

and (B.2) follows.

- If $F_k \in H^{-1}$, from (B.12) it follows that

$$\nu\|\nabla_k w_{\mathrm{Na}}\|_{L^2}^2 + \frac{|k|^2}{4C_0} \|\nabla_k \phi_{\mathrm{Na}}\|_{L^2}^2$$
$$\leq 8\nu^{-1} \|F_k\|_{H^{-1}}^2 + \frac{\nu}{2} \|\nabla_k w_{\mathrm{Na}}\|_{L^2}^2 + 4\delta_0 \nu^{1/3} |k|^{2/3} \mathbb{1}_{\mathrm{Im}\lambda \leq 0} \|w_{\mathrm{Na}}\|_{L^2}^2,$$

which yields (B.3).

Next we estimate the spectral vorticity w_{Na} following the strategy in [19,21]. We introduce the following test function $\chi_1 \in C^\infty([-1,1])$:

$$\chi_1(y) = \frac{1}{(\mathcal{U}(y) - \mathrm{Re}\lambda) + iL^{-1}}.$$

It is readily checked that

$$\begin{aligned} &L^{-1}\|\chi_1\|_{L^\infty} + L^{-\frac{1}{2}}\|\chi_1\|_{L^2} + L^{-2}\|\partial_y \chi_1\|_{L^\infty} \leq C, \\ &\|\chi_1 f\|_{L^2} \leq \|\chi_1\|_{L^2}\|f\|_{L^\infty} \leq CL^{\frac{1}{2}}\|f\|_{L^2}^{\frac{1}{2}}\|\nabla_k f\|_{L^2}^{\frac{1}{2}}, \quad \forall f \in H_0^1([-1,1]). \end{aligned} \tag{B.13}$$

We test the equation $(2.16)_1$ by $\chi_1 w_{\mathrm{Na}}$ and taking the imaginary part, obtaining

$$\begin{aligned} &\mathrm{Im}\Big(ik \int (\mathcal{U}(y) - \lambda)\overline{\chi_1}|w_{\mathrm{Na}}|^2 dy\Big) \\ &= -\nu \mathrm{Im} \int \nabla_k w_{\mathrm{Na}} \cdot \overline{\nabla_k(\chi_1 w_{\mathrm{Na}})} dy + \mathrm{Im} \int F_k \overline{\chi_1 w_{\mathrm{Na}}} dy \\ &\qquad + \mathrm{Im}\Big(ik \int \mathcal{U}'' \phi_{\mathrm{Na}} \overline{\chi_1 w_{\mathrm{Na}}} dy\Big) \\ &=: T_5 + T_6 + T_7. \end{aligned} \tag{B.14}$$

For the left-hand side of the relation (B.14), direct computation and $(\text{B.13})_2$ give

$$\begin{aligned}(\text{B.14})_{\text{L.H.S}} =&k\text{Re}\int |w_{\text{Na}}|^2\frac{(\mathcal{U}-\text{Re}\lambda)^2+L^{-1}\text{Im}\lambda}{(\mathcal{U}-\text{Re}\lambda)^2+L^{-2}}dy\\ =&k\|w_{\text{Na}}\|_{L^2}^2+kL^{-1}(\text{Im}\lambda-L^{-1})\|\chi_1 w_{\text{Na}}\|_{L^2}^2\\ \geq&k\|w_{\text{Na}}\|_{L^2}^2-C\mathbb{1}_{\text{Im}\lambda\leq L^{-1}}kL^{-1}\|w_{\text{Na}}\|_{L^2}\|\nabla_k w_{\text{Na}}\|_{L^2}\\ \geq&\frac{k}{2}\|w_{\text{Na}}\|_{L^2}^2-C\mathbb{1}_{\text{Im}\lambda\leq L^{-1}}|k|^{\frac13}\nu^{\frac23}\|\nabla_k w_{\text{Na}}\|_{L^2}^2.\end{aligned}\tag{B.15}$$

For the term T_5, we use $(\text{B.13})_1$ to get

$$\begin{aligned}|T_5|\leq&\nu\|\nabla_k w_{\text{Na}}\|_{L^2}(\|\chi_1'\|_{L^\infty}\|w_{\text{Na}}\|_{L^2}+\|\chi_1\|_{L^\infty}\|\nabla_k w_{\text{Na}}\|_{L^2})\\ \leq&C\nu^{\frac13}|k|^{\frac23}\|\nabla_k w_{\text{Na}}\|_{L^2}\|w_{\text{Na}}\|_{L^2}+C\nu^{\frac23}|k|^{\frac13}\|\nabla_k w_{\text{Na}}\|_{L^2}^2\\ \leq&C\delta_0^{-\frac14}\nu^{\frac23}|k|^{\frac13}\|\nabla_k w_{\text{Na}}\|_{L^2}^2+\delta_0^{\frac14}|k|\|w_{\text{Na}}\|_{L^2}^2.\end{aligned}\tag{B.16}$$

Similarly, the term T_6 is estimated using (B.13) as follows:

$$|T_6|\leq\begin{cases}\|w_{\text{Na}}\|_{L^2}\|\chi_1 F_k\|_{L^2}\leq C\nu^{-\frac13}|k|^{\frac13}\|w_{\text{Na}}\|_{L^2}\|F_k\|_{L^2}, & \text{if } F_k\in L^2;\\ \|w_{\text{Na}}\|_{L^2}\|\chi_1 F_k\|_{L^2}\leq C\nu^{-\frac16}|k|^{-\frac13}\|w_{\text{Na}}\|_{L^2}\|\nabla_k F_k\|_{L^2}, & \text{if } F_k\in H_0^1;\\ \|\nabla_k(\chi_1 w_{\text{Na}})\|_{L^2}\|F_k\|_{H^{-1}}\leq C\left(\nu^{-\frac23}|k|^{\frac23}\|w_{\text{Na}}\|_{L^2}+\nu^{-\frac13}|k|^{\frac13}\|\nabla_k w_{\text{Na}}\|_{L^2}\right)\|F_k\|_{H^{-1}},\\ \quad\text{if } F_k\in H^{-1}.\end{cases}$$

Finally, we estimate the term T_7 using (B.1), (B.2), (B.3), and the bound $\|\mathcal{U}''\|_{L^\infty}\leq\kappa$ as

$$\begin{aligned}|T_7|\leq&C\|\mathcal{U}''\|_{L^\infty}|k|\|w_{\text{Na}}\|_{L^2}\|\phi_{\text{Na}}\chi_1\|_{L^2}\leq C\|\mathcal{U}''\|_{L^\infty}|k|\|w_{\text{Na}}\|_{L^2}\nu^{-\frac16}|k|^{-\frac13}\|\nabla_k\phi_{\text{Na}}\|_{L^2}\\ \leq&C\kappa\nu^{-\frac16}|k|^{-\frac13}\|w_{\text{Na}}\|_{L^2}\min\{\delta_0^{-\frac12}\nu^{-\frac16}|k|^{-\frac13}\|F_k\|_{L^2},|k|^{-1}\|\nabla_k F_k\|_{L^2},\\ &\nu^{-\frac12}\|F_k\|_{H^{-1}}\}+C\kappa\delta_0^{\frac12}\|w_{\text{Na}}\|_{L^2}^2.\end{aligned}\tag{B.17}$$

Combining (B.1), (B.2), (B.3), (B.14), (B.15), (B.16), and (B.17), we obtain

- If $F_k\in L^2$,

$$\begin{aligned}\|w_{\text{Na}}\|_{L^2}^2\lesssim&\,\delta_0^{-\frac14}\nu^{\frac23}|k|^{-\frac23}\|\nabla_k w_{\text{Na}}\|_{L^2}^2+\delta_0^{\frac14}\|w_{\text{Na}}\|_{L^2}^2+\nu^{-\frac13}|k|^{-\frac23}\|w_{\text{Na}}\|_{L^2}\|F_k\|_{L^2}\\ &+\kappa\nu^{-\frac13}|k|^{-\frac53}\|w_{\text{Na}}\|_{L^2}\|F_k\|_{L^2}+\kappa\delta_0^{\frac12}|k|^{-1}\|w_{\text{Na}}\|_{L^2}^2\\ \lesssim&\,\delta_0^{-\frac54}\nu^{-\frac23}|k|^{-\frac43}\|F_k\|_{L^2}^2+\delta_0^{\frac14}\|w_{\text{Na}}\|_{L^2}^2.\end{aligned}$$

  We take $\delta_0>0$ sufficiently small to obtain

$$\nu^{\frac13}|k|^{\frac23}\|w_{\text{Na}}\|_{L^2}\lesssim\|F_k\|_{L^2}.$$

  The estimates of $\mathbb{E}[w_{\text{Na}},\phi_{\text{Na}}]$ and $\sqrt{|k|(\text{Im}\lambda)_+}\|w_{\text{Na}}\|_{L^2}$ in (2.19) follows by appealing to (B.1).

- If $F_k \in H_0^1$,

$$
\begin{aligned}
\|w_{\mathrm{Na}}\|_{L^2}^2 &\lesssim \delta_0^{-\frac14}\nu^{\frac23}|k|^{-\frac23}\|\nabla_k w_{\mathrm{Na}}\|_{L^2}^2 + \delta_0^{\frac14}\|w_{\mathrm{Na}}\|_{L^2}^2 + \nu^{-\frac16}|k|^{-\frac43}\|w_{\mathrm{Na}}\|_{L^2}\|\nabla_k F_k\|_{L^2} \\
&\quad + \kappa\nu^{-\frac16}|k|^{-\frac73}\|w_{\mathrm{Na}}\|_{L^2}\|\nabla_k F_k\|_{L^2} + \kappa\delta_0^{\frac12}|k|^{-1}\|w_{\mathrm{Na}}\|_{L^2}^2 \\
&\lesssim \delta_0^{-\frac14}\nu^{-\frac13}|k|^{-\frac83}\|\nabla_k F_k\|_{L^2}^2 + \delta_0^{\frac14}\|w_{\mathrm{Na}}\|_{L^2}^2.
\end{aligned}
$$

We take $\delta_0 > 0$ sufficiently small to get

$$
\nu^{\frac16}|k|^{\frac43}\|w_{\mathrm{Na}}\|_{L^2} \lesssim \|\nabla_k F_k\|_{L^2}. \tag{B.18}
$$

From (B.18) and (B.2) it follows that

$$
|k|\|\nabla_k \phi_{\mathrm{Na}}\|_{L^2} + \sqrt{\nu}\|\nabla_k w_{\mathrm{Na}}\|_{L^2} + \nu^{\frac16}|k|^{\frac13}\|w_{\mathrm{Na}}\|_{L^2} \lesssim |k|^{-1}\|\nabla_k F_k\|_{L^2},
$$

and the estimates of $\mathbb{E}[w_{\mathrm{Na}}, \phi_{\mathrm{Na}}]$ and $\sqrt{|k|(\mathrm{Im}\lambda)_+}\|w_{\mathrm{Na}}\|_{L^2}$ in (2.20) follows.

- If $F_k \in H^{-1}$,

$$
\begin{aligned}
\|w_{\mathrm{Na}}\|_{L^2}^2 &\lesssim \delta_0^{-\frac14}\nu^{\frac23}|k|^{-\frac23}\|\nabla_k w_{\mathrm{Na}}\|_{L^2}^2 + \delta_0^{\frac14}\|w_{\mathrm{Na}}\|_{L^2}^2 + \nu^{-\frac23}|k|^{-\frac13}\|w_{\mathrm{Na}}\|_{L^2}\|F_k\|_{H^{-1}} \\
&\quad + \nu^{-\frac13}|k|^{-\frac23}\|F_k\|_{H^{-1}}\|\nabla_k w_{\mathrm{Na}}\|_{L^2} + \kappa\delta_0^{\frac12}|k|^{-1}\|w_{\mathrm{Na}}\|_{L^2}^2 \\
&\lesssim \delta_0^{-\frac14}\nu^{-\frac43}|k|^{-\frac23}\|F_k\|_{H^{-1}}^2 + \delta_0^{\frac14}\|w_{\mathrm{Na}}\|_{L^2}^2.
\end{aligned}
$$

Similar arguments as in other two cases yield the estimates of $\mathbb{E}[w_{\mathrm{Na}}, \phi_{\mathrm{Na}}]$ and the term $\sqrt{|k|(\mathrm{Im}\lambda)_+}\|w_{\mathrm{Na}}\|_{L^2}$ in (2.21).

To conclude the proof of the proposition, it remains to prove

$$
\nu\|\Delta_k w_{\mathrm{Na}}\|_{L^2} \lesssim \min\{\|F_k\|_{L^2}, \nu^{1/6}|k|^{-2/3}\|\nabla_k F_k\|_{L^2}\}. \tag{B.19}
$$

We test the equation $(2.16)_1$ by $\nu\Delta_k w_{\mathrm{Na}}$ and take the real part to obtain

$$
\begin{aligned}
&\nu^2\|\Delta_k w_{\mathrm{Na}}\|_{L^2}^2 + \nu|k|(\mathrm{Im}\lambda)_+\|\nabla_k w_{\mathrm{Na}}\|_{L^2}^2 \\
&\le |(\mathrm{Im}\lambda)_-|\nu|k|\|\nabla_k w_{\mathrm{Na}}\|_{L^2}^2 \\
&+ C\nu|k|\|\mathcal{U}'\|_{L^\infty}\|w_{\mathrm{Na}}\|_{L^2}\|\nabla_k w_{\mathrm{Na}}\|_{L^2} + C\nu|k|\|\mathcal{U}''\|_{W^{1,\infty}}\|\nabla_k\phi_{\mathrm{Na}}\|_{L^2}\|\nabla_k w_{\mathrm{Na}}\|_{L^2} \\
&\quad + C\min\Big\{\nu\|F_k\|_{L^2}\|\Delta_k w_{\mathrm{Na}}\|_{L^2}, \nu\|\nabla_k F_k\|_{L^2}\|\nabla_k w_{\mathrm{Na}}\|_{L^2}\Big\} \\
&\le C\mathbb{E}[w_{\mathrm{Na}}, \phi_{\mathrm{Na}}]^2 + C\min\{\|F_k\|_{L^2}^2, \nu^{1/3}|k|^{-4/3}\|\nabla_k F_k\|_{L^2}^2\} + \frac12\nu^2\|\Delta_k w_{\mathrm{Na}}\|_{L^2}^2,
\end{aligned} \tag{B.20}
$$

where we used the relation $\nu^{1/3}|k|^{-1/3} \le \nu^{1/6}|k|^{-2/3}\|\mathcal{U}'\|_{L^\infty}^{1/6}$ in the last step, which follows from $\nu|k|^2 \le \|\mathcal{U}'\|_{L^\infty}$. From the already-established bound

$$
\mathbb{E}[w_{\mathrm{Na}}, \phi_{\mathrm{Na}}]^2 \lesssim \min\{\|F_k\|_{L^2}^2, \nu^{1/3}|k|^{-4/3}\|\nabla_k F_k\|_{L^2}^2\},
$$

we deduce (B.19) from (B.20). □

Lemma B.1. *For $\mathcal{E}$ defined in (2.32), we have*

$$\|e^{\delta_* \nu^{1/3} t}\omega\|^2_{L^2([0,T];\mathcal{Z})} \lesssim \mathcal{E},$$

where $\delta_* > 0$ is the constant from Proposition 3.10.

Proof. Using (2.30), we have

$$\|e^{\delta_* \nu^{1/3} t}\omega\|^2_{L^2([0,T];\mathcal{Z})} = \sum_{j=0}^{N} \int_{\mathcal{I}_{[j]}} e^{2\delta_* \nu^{1/3} t} \Big\| \sum_{j'=0}^{j} \omega_{[j']} \Big\|^2_{\mathcal{Z}} \le \sum_{j=0}^{N} e^{2\delta_*(j+1)} \int_{\mathcal{I}_{[j]}} \left(\sum_{j'=0}^{j} \|\omega_{[j']}\|_{\mathcal{Z}} \right)^2 .$$

An application of the Cauchy-Schwarz inequality leads to

$$\begin{aligned}\int_{\mathcal{I}_{[j]}} \left(\sum_{j'=0}^{j} \|\omega_{[j']}\|_{\mathcal{Z}} \right)^2 &\le \int_{\mathcal{I}_{[j]}} \left(\sum_{j'=0}^{j} (j-j'+1)^2 \|\omega_{[j']}\|^2_{\mathcal{Z}} \right) \left(\sum_{j'=0}^{j} \frac{1}{(j-j'+1)^2} \right) \\ &\le C \left(\sum_{j'=0}^{j} (j-j'+1) \|\omega_{[j']}\|_{L^2(\mathcal{I}_{[j]};\mathcal{Z})} \right)^2 = C Y^2_{[j]}.\end{aligned}$$

Therefore, we conclude the proof of the lemma. □

Lemma B.2. *Assume that $U_{\rm in} \in H^4(-1,1)$ satisfies the compatibility condition (1.10). Then for $U(t,y)$ defined in (1.3), we have*

$$\|U(t,\cdot) - y\|_{H^4} \lesssim \|U_{\rm in} - y\|_{H^4}, \quad \forall t \ge 0, \tag{B.21}$$

$$\sup_{y\in[-1,1]} \left| \frac{U(t,y) - U(s,y)}{1-|y|} \right| \lesssim \nu \|U_{\rm in}\|_{H^4} |t-s|, \tag{B.22}$$

$$\|U''(t,\cdot) - U''(s,\cdot)\|_{L^2_y} \lesssim \nu \|U_{\rm in}\|_{H^4} |t-s|, \tag{B.23}$$

for any $t \ge s \ge 0$.

Proof. From (1.3) it follows that $(U-y)$ solves heat equation

$$\begin{cases} \partial_t (U-y) = \nu \partial_y^2 (U-y), \\ (U-y)\big|_{y=\pm1} = 0, \quad (U-y)(t=0) = U_{\rm in} - y. \end{cases} \tag{B.24}$$

A standard basic energy estimate yields (B.21). Hence, $\partial_y^2(U-y)|_{y=\pm1} = 0$ and

$$\int_{-1}^{1} \partial_y (U-y) dy = (U-y)\Big|_{y=-1}^{y=1} = 0.$$

By the Agmon inequality and Poincaré inequality, we have

$$\|\partial_y (U-y)\|_{L^\infty} \lesssim \|\partial_y(U-y)\|^{1/2}_{L^2} \|\partial_y(U-y)\|^{1/2}_{H^1} \lesssim \|\partial_y^2 (U-y)\|_{L^2}, \tag{B.25}$$

for any $t \geq 0$. Differentiating (B.24) gives

$$\begin{cases} \partial_t \partial_y^2 (U - y) = \nu \partial_y^2 (\partial_y^2 (U - y)), \\ \partial_y^2 (U - y)\Big|_{y=\pm 1} = 0, \quad \partial_y^2 (U - y)(t = 0) = \partial_y^2 U_{\text{in}}. \end{cases} \tag{B.26}$$

Using (B.25) and the basic energy inequality of the heat equation (B.26), we infer

$$\|\partial_y (U - y)\|_{L_t^\infty L_y^\infty} \lesssim \|\partial_y^2 (U - y)\|_{L_t^\infty L_y^2} \lesssim \|\partial_y^2 U_{\text{in}}\|_{L_y^2} \lesssim \|U_{\text{in}}\|_{H^2}.$$

Similar argument yields

$$\|\partial_y^3 (U - y)\|_{L_t^\infty L_y^\infty} \lesssim \|\partial_y^4 (U - y)\|_{L_t^\infty L_y^2} \lesssim \|\partial_y^4 U_{\text{in}}\|_{L_y^2} \lesssim \|U_{\text{in}}\|_{H^4}. \tag{B.27}$$

Let $t \geq s \geq 0$. From the fundamental theorem of calculus it follows that

$$U(t, y) - U(s, y) = \int_s^t \partial_\tau U(\tau, y) d\tau = \nu \int_s^t \partial_y^2 U(\tau, y) d\tau, \quad y \in [-1, 1]. \tag{B.28}$$

Using (B.27), (B.28), and the mean value theorem, we arrive at

$$\begin{aligned} \left| \frac{U(t, y) - U(s, y)}{1 - y} \right| &\leq \nu \int_s^t \left| \frac{\partial_y^2 U(\tau, y) - \partial_y^2 U(\tau, 1)}{y - 1} \right| d\tau \\ &\leq \nu \int_s^t \sup_{y \in [-1,1]} |\partial_y^3 U(\tau, y)| d\tau \leq C\nu \|U_{\text{in}}\|_{H^4} |t - s|, \end{aligned} \tag{B.29}$$

for any $y \in [-1, 1]$. A similar argument shows that

$$\left| \frac{U(t, -y) - U(s, -y)}{1 - y} \right| \leq C\nu \|U_{\text{in}}\|_{H^4} |t - s|,$$

for any $y \in [-1, 1]$. Hence, the proof of (B.22) is completed.

To prove (B.23), we proceed similarly as in (B.29) to obtain

$$\begin{aligned} \|U''(t, \cdot) - U''(s, \cdot)\|_{L_y^2} &\leq \int_s^t \|\partial_t U''(\tau, \cdot)\|_{L_y^2} d\tau \leq \nu \int_s^t \|\partial_y^4 U(\tau, \cdot)\|_{L_y^2} d\tau \\ &\leq \nu \int_s^t \|\partial_y^4 U_{\text{in}}\|_{L^2} d\tau \leq \nu \|U_{\text{in}}\|_{H^4} |t - s|, \end{aligned}$$

where we also used the basic energy estimate for the one-dimensional heat equation. $\square$

Lemma B.3. *Suppose that $\nu k^2 \lesssim 1$. Then we have*

$$\|\mathfrak{f}_{\text{Frozen}[j]}\|_{L^2(\mathcal{I}_{[j]}; L^2)} \lesssim \nu^{1/2} |k|^{-1/3} \mathrm{Y}_{[j]}, \quad j \geq 1, \tag{B.30}$$

$$\sum_{j=0}^N |\log \nu|^2 \nu^{-1/3} |k|^{4/3} \|e^{\delta_* \nu^{1/3} t} \mathfrak{f}_{\text{Disc}[j]}\|_{L^2(\mathcal{I}_{[j]}; L^2)}^2 \lesssim |\log \nu|^2 \nu^{1/3} \mathcal{E}. \tag{B.31}$$

Proof. Using (2.29), we write

$$\begin{aligned}\mathfrak{f}_{\mathrm{Frozen}[j]} &= -\sum_{j'=0}^{j-1}(U_{[j]}-U_{[j']})\omega_{[j']} + \sum_{j'=0}^{j-1}(U''_{[j]}-U''_{[j']})\psi_{[j']}\\ &=: \sum_{j'=0}^{j-1}\mathfrak{f}^{(1)}_{\mathrm{Frozen}[j']} + \sum_{j'=0}^{j-1}\mathfrak{f}^{(2)}_{\mathrm{Frozen}[j']}.\end{aligned}\tag{B.32}$$

For $0\le j'\le j-1$, using Lemma B.2 we obtain

$$\begin{aligned}&\|\mathfrak{f}^{(1)}_{\mathrm{Frozen}[j']}\|_{L^2(\mathcal{I}_{[j]};L^2)} = \left\|\frac{U_{[j]}-U_{[j']}}{(1-|y|)^{1/2}}(1-|y|)^{1/2}\omega_{[j']}\right\|_{L^2_t(\mathcal{I}_{[j]};L^2)}\\ &\lesssim \left\|\frac{U_{[j]}-U_{[j']}}{(1-|y|)^{1/2}}\right\|_{L^\infty_y}\|\rho_k^{1/2}\omega_{[j']}\|_{L^2(\mathcal{I}_{[j]};L^2)} \lesssim \nu^{1/2}|k|^{-1/3}|j-j'|\|\omega_{[j']}\|_{L^2(\mathcal{I}_{[j]};\mathcal{Z})}.\end{aligned}\tag{B.33}$$

From Lemma B.2 it follows that

$$\begin{aligned}\|\mathfrak{f}^{(2)}_{\mathrm{Frozen}[j']}\|_{L^2(\mathcal{I}_{[j]};L^2)} &= \left\|(U''_{[j]}-U''_{[j']})\psi_{[j']}\right\|_{L^2(\mathcal{I}_{[j]};L^2)}\\ &\lesssim \left\|U''_{[j]}-U''_{[j']}\right\|_{L^\infty_y}|k|^{-1}\|\nabla_k\Delta_k^{-1}\omega_{[j']}\|_{L^2(\mathcal{I}_{[j]};L^2)}\\ &\lesssim \nu^{2/3}|k|^{-1}|j-j'|\|\omega_{[j']}\|_{L^2(\mathcal{I}_{[j]};\mathcal{Z})}.\end{aligned}\tag{B.34}$$

Combining (B.32), (B.33), and (B.34), we arrive at

$$\|\mathfrak{f}_{\mathrm{Frozen}[j]}\|_{L^2(\mathcal{I}_{[j]};L^2)} \lesssim \sum_{j'=0}^{j-1}(\nu^{1/2}|k|^{-1/3}+\nu^{2/3}|k|^{-1})|j-j'|\|\omega_{[j']}\|_{L^2(\mathcal{I}_{[j]};\mathcal{Z})} \lesssim \nu^{1/2}|k|^{-1/3}Y_{[j]}.$$

completing the proof of (B.30).

For $0\le j\le N$, we use (2.26) and (2.28) to write

$$\mathfrak{f}_{\mathrm{Disc}[j]} = \sum_{j'=0}^{j}(U_{[j]}(y)-U(t,y))\omega_{[j']} - \sum_{j'=0}^{j}(U''_{[j]}(y)-U''(t,y))\psi_{[j']}.\tag{B.35}$$

An application of the Cauchy-Schwarz inequality leads to

$$\begin{aligned}&\left\|e^{\delta_*\nu^{1/3}t}\sum_{j'=0}^{j}(U_{[j]}(y)-U(t,y))\omega_{[j']}\right\|^2_{L^2(\mathcal{I}_{[j]};L^2)}\\ &\lesssim \int_{\mathcal{I}_{[j]}}\left(\sum_{j'=0}^{j}\|e^{\delta_*\nu^{1/3}t}(U_{[j]}(y)-U(t,y))\omega_{[j']}\|_{L^2}\right)^2\\ &\lesssim \int_{\mathcal{I}_{[j]}}\left(\sum_{j'=0}^{j}(j-j'+1)^2\|e^{\delta_*\nu^{1/3}t}(U_{[j]}(y)-U(t,y))\omega_{[j']}\|^2_{L^2}\right)\left(\sum_{j'=0}^{j}\frac{1}{(j-j'+1)^2}\right)\\ &\lesssim \sum_{j'=0}^{j}e^{2\delta_* j}(j-j'+1)^2\int_{\mathcal{I}_{[j]}}\|(U_{[j]}(y)-U(t,y))\omega_{[j']}\|^2_{L^2}.\end{aligned}\tag{B.36}$$

Similar arguments as in (B.33) give

$$\|(U_{[j]}(y)-U(t,y))\omega_{[j']}\|^2_{L^2(\mathcal{I}_{[j]};L^2)} \lesssim \nu|k|^{-2/3}\|\omega_{[j']}\|^2_{L^2(\mathcal{I}_{[j]};\mathcal{Z})}. \tag{B.37}$$

From (B.36) and (B.37) it follows that

$$\begin{aligned}&\sum_{j=0}^{N}|\log\nu|^2\nu^{-1/3}|k|^{4/3}\|e^{\delta_*\nu^{1/3}t}\sum_{j'=0}^{j}(U_{[j]}(y)-U(t,y))\omega_{[j']}\|^2_{L^2(\mathcal{I}_{[j]};L^2)}\\ &\lesssim |\log\nu|^2\nu^{2/3}|k|^{2/3}\sum_{j=0}^{N}\sum_{j'=0}^{j}e^{2\delta_*j}(j-j'+1)^2\|\omega_{[j']}\|^2_{L^2(\mathcal{I}_{[j]};\mathcal{Z})}\lesssim|\log\nu|^2\nu^{1/3}\mathcal{E}.\end{aligned} \tag{B.38}$$

A similar argument as in (B.34) shows that

$$\sum_{j=0}^{N}|\log\nu|^2\nu^{-1/3}|k|^{4/3}\left\|e^{\delta_*\nu^{1/3}t}\sum_{j'=0}^{j}(U''_{[j]}(y)-U''(t,y))\psi_{[j']}\right\|^2_{L^2(\mathcal{I}_{[j]};L^2)}\lesssim|\log\nu|^2\nu^{1/3}\mathcal{E}. \tag{B.39}$$

Combining (B.35), (B.38), and (B.39), we conclude the proof of (B.31). □

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