A UNIVERSAL P-GROUP OF WEIGHT $\aleph$

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Abstract. We show that under the Continuum Hypothesis, the topological group of all homeomorphisms of the Čech-Stone remainder of ω with the G_δ-topology, is a universal object for all P-groups of weight at most $\mathfrak{c}$.

1. Introduction

All topological spaces under discussion are Tychonoff

Uspenskiy [23] proved in 1986 that there is a universal topological group G with a countable base. Universal in the sense that every topological group with a countable base is (topologically isomorphic to) a subgroup of G. It is the homeomorphism group $\mathcal{H}(Q)$ of the Hilbert cube $Q = \mathbb{I}^\omega$, endowed with the compact-open topology. (Another such group is the group of isometries of the Urysohn universal space, with the topology of pointwise convergence, [24].) The obvious question from [24] – whether there are universal topological groups of uncountable weight – is still open. Natural candidates are the homeomorphism groups of the Tychonoff cubes $\mathbb{I}^\tau$, where τ is uncountable. Uspenskiy's proof in [23] is based on Keller's Theorem [11] from 1931 that all compact convex and infinite-dimensional subsets of Banach spaces, are homeomorphic to Q. There are good topological characterizations of the Tychonoff cubes $\mathbb{I}^\tau$ for uncountable τ by Ščepin [19] (see also [2, 7.2.9]). However, a necessary condition in these characterizations is that the spaces under consideration are Absolute Retracts, which is problematic for arbitrary compact convex subsets of $\mathbb{I}^\tau$. (There are even compact convex subsets of $\mathbb{I}^{\omega_1}$ that do not satisfy the countable chain condition.) This blocks the attempt to prove that the spaces $\mathcal{H}(\mathbb{I}^\tau)$ are universal for all uncountable τ by a generalization of the arguments in Uspenskiy [23].

Shkarin [21] proved that there are universal Abelian topological groups with a countable base. And if τ is an infinite cardinal such that $2^\lambda \leq \tau$ for every cardinal $\lambda < \tau$, then there is a group that is universal in the class of all Abelian topological groups of weight at most τ.

As usual, we denote the Čech-Stone compactification of the discrete space ω by $\beta\omega$, and $\omega^* = \beta\omega \setminus \omega$, its Čech-Stone remainder. A compact, zero-dimensional F-space of weight $\mathfrak{c}$ in which each non-empty G_δ-subset has infinite interior will be called a *Parovičenko space*. Here an F-space is a space in which cozero-subsets are C^*-embedded. For normal spaces,

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this simply boils down to the statement that disjoint open F_σ-subsets have disjoint closures. Under the Continuum Hypothesis (CH), all Parovičenko spaces are homeomorphic to ω^*, as was shown by Parovičenko [16]. Subsequently, van Douwen and van Mill [4] obtained the converse: Parovičenko's characterization of ω^* implies CH. Parovičenko also showed that under CH, every compact space of weight at most $\mathfrak{c}$ is a continuous image of ω^*, hence ω^* is *universal* in the sense of 'mapping onto'. Dow and Hart [5] proved that under CH, the Čech-Stone remainder $[0,\infty)^*$ of the half line $[0,\infty)$ is universal among the continua of weight $\mathfrak{c}$, again in the sense of 'mapping onto'. (It is an open problem whether in some model there is a topological characterization of $[0,\infty)^*$ in the style of Parovičenko.)

We are interested in $\mathcal{H}(\omega^*)$, the homeomorphism group of ω^* with the compact-open topology. Like ω^* itself, it is a mysterious object. Rudin [18] proved that under CH, its cardinality is $2^{\mathfrak{c}}$. And, under the same condition, it is simple by Fuchino [8]. It even remains simple if $\aleph_2$ Cohen reals are added to a model of ZFC + CH, [9]. But Shelah [20] proved that it is consistent that all homeomorphisms of ω^* are 'trivial' in the sense that they are induced by bijections between cofinite subsets of ω; hence, in this model, $\mathcal{H}(\omega^*)$ has cardinality $\mathfrak{c}$ in contrast to Rudin's result, and is by van Douwen [3] not simple, in contrast to Fuchino's result.

The goal of this note is to prove that $\mathcal{H}(\omega^*)$ exhibits similar behavior within the class of suitable topological groups as ω^* does within the class of suitable topological spaces.

Following Arhangel'skii and Tkachenko [1], we will call a topological group whose underlying topological space is a P-space – that is a space in which every G_δ-subset is open – a *P-group.* If G is a topological group, then $G_{(\delta)}$ denotes its *G_δ-modification.* The underlying set of $G_{(\delta)}$ is G, and the G_δ-subsets of G form a basis (for the open subsets) of $G_{(\delta)}$. It is easy to see that $G_{(\delta)}$ is a topological group, [1, 3.6.H], and a P-space, hence is a P-group.

Theorem 1.1. *Let G be a topological group of weight $\mathfrak{c}$. Then there is a Parovičenko space X such that $G_{(\delta)}$ is (topologically isomorphic to) a subgroup of $\mathcal{H}(X)$.*

This is in line with van Douwen's unpublished result that every P-space of weight at most $\mathfrak{c}$ embeds in ω^*; see [6] for details and references.

Theorem 1.2 (CH). *$\mathcal{H}(\omega^*)_{(\delta)}$ is universal for the class of all P-groups of weight at most $\mathfrak{c}$.*

The following is in line with the results in [16, 4] that we discussed above.

Theorem 1.3. CH *is equivalent to the statement that for all Parovičenko spaces X and Y, $\mathcal{H}(X)$ and $\mathcal{H}(Y)$ are (algebraically) isomorphic.*

We also discuss some topological properties of the group $\mathcal{H}(\omega^*)$. We show, among other things, that $\mathcal{H}(\omega^*)$ is not a P-group but that all of its countable subsets are closed and discrete.

The class of 'small' P-groups is rather limited, so our results on universality are not a strong addition to what is known. But they may indicate that the homeomorphism group of ω^* is by itself an interesting object that is worth studying.

2. Preliminaries

2.1. **Topology.** For any space X, we let τX denote its topology, and $\mathcal{H}(X)$ its group of homeomorphisms. We let $e_{\mathcal{H}(X)}$ denote the identity element of the group $\mathcal{H}(X)$, that is, the identity function id_X of X. If no confusion is likely, we suppress the index. We use standard terminology for cardinal functions. For example, if X is a space, then $w(X)$ and $\chi(X)$ denote its weight and character, respectively (see Juhász [10]).

If X is a space, then βX and $X^* = \beta X \setminus X$ denote its Čech-Stone compactification and Čech-Stone remainder, respectively. If X is normal, then disjoint closed subsets of X have disjoint closures in βX, [7, 3.6.4]. In fact, this property characterizes βX.

It is well-known, and easy to prove, that the Čech-Stone compactification βX of X is zero-dimensional iff X is strongly zero-dimensional; that is, disjoint zero-sets in X can be separated by disjoint clopen subsets of X, [7, 6.2.12].

If $C \subseteq X$ is clopen, then $C^* = (\mathrm{cl}_{\beta X} C) \cap X^*$. It is easy to see that if C and D are clopen in X and $C \setminus D$ is compact, then $C^* \subseteq D^*$. Hence if $C \triangle D$ is compact, then $C^* = D^*$; here $\triangle$ denotes symmetric difference of course. Moreover, if X is locally compact and strongly zero-dimensional, and $E \subseteq X^*$ is (relatively) clopen, then there is a clopen subset C of X such that $C^* = E$. This easily implies the well-known fact that all nonempty (relatively) clopen subsets of ω^* are homeomorphic to ω^*. This will be used frequently in the remaining part of this note.

2.2. **Groups.** If G and H are groups, then by an *isomorphism* from G to H we will mean a group isomorphism (no topology involved). If G and H are topological groups, then a *topological isomorphism* between G and H is an isomorphism that is also a homeomorphism.

Let G be a topological group with subgroup H. It is easily seen that $H_{(\delta)}$ is (topologically isomorphic to) the subgroup H of $G_{(\delta)}$.

If $\mathcal{K}$ is a class of topological groups, then $G \in \mathcal{K}$ is said to be *universal* (for $\mathcal{K}$) provided that every member of $\mathcal{K}$ is (topologically isomorphic to) a subgroup of G.

2.3. **The topological groups $\mathcal{H}(X)$ for compact zero-dimensional X.** Let X be a compact space. We endow $\mathcal{H}(X)$ with the standard *compact-open* topology. That is the topology for which the collection of all $[K, U]$'s is a subbase for the open sets; here K is compact, $U \in \tau X$, and

$$[K, U] = \{f \in \mathcal{H}(X) : f(K) \subseteq U\}. \tag{†}$$

It is well-known that by compactness of X, this topology makes $\mathcal{H}(X)$ a topological group. In fact, the natural action $\mathcal{H}(X) \times X \to X$ defined by $(g, x) \mapsto g(x)$, $x \in X$, is a (continuous) action of the topological group $\mathcal{H}(X)$ on X.

If X is compact and zero-dimensional, then we can simplify (†), as follows. If $f \in [K, U]$, then we may pick a clopen subset F of X such that $K \subseteq F \subseteq f^{-1}(U)$. Hence $f \in [F, G] \subseteq [K, U]$, where both F and $G = f(F)$ are clopen. This implies that $w(\mathcal{H}(X)) \leq w(X)$. For $C \subseteq X$ clopen, we put

$$\hat{C} = [C, C] \cap [X \setminus C, X \setminus C] = \{f \in \mathcal{H}(X) : f(C) = C\}.$$

Clearly, $\hat{C}$ is an open subgroup of $\mathcal{H}(X)$, hence even a clopen subgroup. A moments reflection shows that if $\mathcal{E}$ is a finite clopen partition of X, then $N(\mathcal{E}) = \{f \in \mathcal{H}(X) : (\forall\, E \in \mathcal{E})(f(E) = E)\}$ is a clopen subgroup of $\mathcal{H}(X)$, and that the collection of all $N(\mathcal{E})$'s is a neighborhood base at e in $\mathcal{H}(X)$. This is well-known, see e.g., [13, 3.2]. (There is a typo in the proof, though: op page 272, line 13. *'base'* should be changed to *'subbase'*.)

A topological group is *non-archimedean* if it has a local base at the identity consisting of open (hence clopen) subgroups. Hence $\mathcal{H}(X)$ for compact zero-dimensional X is non-archimedean. It is easy to see that P-groups are non-archimedean as well, [1, 4.4.1].

2.4. **Teleman's Theorem and generalizations.** In [22], Teleman showed (among other things) that every topological group is topologically isomorphic to a subgroup of $\mathcal{H}(X)$, for some compact space X. We need a version of his result for compact zero-dimensional spaces. Here are the results that we need:

Theorem 2.1 (Megrelishvili and Shlossberg [14, 3.2])**.** *Let G be a topological group. The following statements are equivalent:*

(1) *G is non-archimedean.*
(2) *G is (topologically isomorphic to) a topological subgroup of $\mathcal{H}(X)$, for some compact zero-dimensional space X with $w(X) = w(G)$.*

2.5. **Rubin's Theorem.** A subset of a topological space is *somewhere dense* if its closure has nonempty interior.

Let X be a space, and G a subgroup of $\mathcal{H}(X)$. We say that the pair $langleX, G\rangle$ satisfies *Rubin's condition* $\mathfrak{R}$, abbreviated $\langle X, G\rangle \in \mathfrak{R}$, if the following condition holds:

($\mathfrak{R}$) for every open $U \subseteq X$ and $x \in U$, $\{g(x) : g \in G \text{ and } g{\restriction}(X \setminus U) = \mathrm{id}_{X\setminus U}\}$ is somewhere dense.

We will need the following deep result:

Theorem 2.2 (Rubin [17, Corollary 3.5(c)])**.** *If $\langle X, G\rangle, \langle Y, H\rangle \in \mathfrak{R}$ and both X and Y are locally compact, then for each isomorphism $\varphi\colon G \to H$, there exists a homeomorphism $f\colon X \to Y$ such that for all $g \in G$, $f \circ g = \varphi(g) \circ f$.*

We will us this result in the following case. The space X is locally compact and zero-dimensional, has a clopen base $\mathcal{B}$ of pairwise homeomorphic sets, and $G = \mathcal{H}(X)$. Hence the elements of $\mathcal{B}$ are compact. We claim that $\langle X, \mathcal{H}(X)\rangle \in \mathfrak{R}$. Indeed, if $U \subseteq X$ is open, then for each $x \in U$, $O(x) = \{g(x) : g \in \mathcal{H}(X) \text{ and } g{\restriction}(X \setminus U) = \mathrm{id}_{X\setminus U}\}$, is dense in U. For take an arbitrary nonempty $B_1 \in \mathcal{B}$ such that $B_1 \subseteq U$. We may assume that $x \notin B_1$. Take $B_0 \in \mathcal{B}$ such that $x \in B_0 \subseteq U \setminus B_1$. Let $\xi\colon B_0 \to B_1$ be any homeomorphism, and define $f\colon X \to X$ by

$$f(y) = \begin{cases} \xi(y) & (y \in B_0), \\ \xi^{-1}(y) & (y \in B_1), \\ y & (y \in X \setminus (B_0 \cup B_1). \end{cases}$$

Then $f \in \mathcal{H}(X)$, $f{\restriction}(X \setminus U) = \mathrm{id}_{X\setminus U}$, and $f(x) \in B_1 \cap O(x)$.

We impose similar conditions on Y.

3. Proofs of Theorems ?? and ??

Let X be a fixed nonempty compact and zero-dimensional space. Put $Z = \omega \times X$; here ω has the discrete topology. Then Z is locally compact, and so Z^* is compact. If X has weight at most $\mathfrak{c}$, then Z has at most $\mathfrak{c}^\omega = \mathfrak{c}$ clopen sets, so Z^* is a Parovičenko space by [15, 1.2.5]. Therefore, Z^* and ω^* are homeomorphic under CH. The interplay between $\mathcal{H}(X)$ and $\mathcal{H}(Z^*)$ will be used here for proving 1.1.

Since Z is strongly zero-dimensional, each (relatively) clopen subset of Z^* has the form F^*, where $F \subseteq \omega \times X$ is clopen. So a standard subbasic neighborhood of some $g \in \mathcal{H}(Z^*)$ has, by what we observed in §2.3, the form $[F^*, G^*]$, where F and G are arbitrary clopen subsets of Z.

If $f \in \mathcal{H}(X)$, we define $f^\beta \in \mathcal{H}(Z^*)$, as follows. We first let f^ω denote the homeomorphism of Z defined by $f^\omega(\langle n, x\rangle) = \langle n, f(x)\rangle$ $(n < \omega, x \in X)$. Then we let $\beta f^\omega \colon \beta Z \to \beta Z$ be its Stone extension. Finally, let $f^\beta = \beta f^\omega \restriction Z^*$. Clearly, $f^\beta \in \mathcal{H}(Z^*)$. Define $i \colon \mathcal{H}(X) \to \mathcal{H}(Z^*)$ by $i(f) = f^\beta$.

Lemma 3.1. *i is an injective homomorphism.*

Proof. That i is a homomorphism is clear.

To check that it is injective, take distinct $f, g \in \mathcal{H}(X)$. There exists $x \in X$ such that $f(x) \neq g(x)$. Then $S = \omega \times \{f(x)\}$ and $T = \omega \times \{g(x)\}$ are disjoint closed subsets of Z, hence, by normality of Z, have disjoint closures in βZ. Pick an arbitrary $p \in \mathrm{cl}_{\beta Z}(\omega \times \{x\}) \setminus (\omega \times \{x\})$. Then $\beta f^\omega(p) \in \mathrm{cl}_{\beta X}(S)$ and $\beta g^\omega(p) \in \mathrm{cl}_{\beta X}(T)$, and so $f^\beta(p) \neq g^\beta(p)$. $\square$

Lemma 3.2. *Let F and G be clopen in Z and $f \in \mathcal{H}(X)$ such that $f^\beta \in [F^*, G^*]$. Then $\{n < \omega : f^\omega(F \cap (\{n\} \times X)) \not\subseteq G \cap (\{n\} \times X)\}$ is finite.*

Proof. If not, then we may pick an infinite subset A of ω and for each $n \in A$ an element $x_n \in X$ such that $\langle n, x_n\rangle \in F \cap (\{n\} \times X)$ but $f^\omega(\langle n, x_n\rangle) \notin G \cap (\{n\} \times X)$. Let p be a limit point of $\{\langle n, x_n\rangle : n \in A\}$ in Z^*. Then, clearly, $p \in F^*$. Let $q = f^\beta(p)$. Then $q \in Z^*$ and hence by our assumptions, $q \in G^*$. Moreover, it is a limit point of $L = \{\langle n, f(x_n)\rangle : n \in A\}$. Now L is a closed subset of Z which is disjoint from G. Hence, by normality of Z, $\mathrm{cl}_{\beta X} L \cap \mathrm{cl}_{\beta X} G = \emptyset$. So $q \notin G^*$, which is a contradiction. $\square$

Lemma 3.3. *$i \colon \mathcal{H}(X)_{(\delta)} \to \mathcal{H}(Z^*)$ is continuous.*

Proof. Let F and G be clopen in Z, and assume that $f^\beta \in [F^*, G^*]$. By 3.2, there exists $N < \omega$ such that for each $n \geq N$, $f^\omega(F \cap (\{n\} \times X)) \subseteq G \cap (\{n\} \times X)\}$. For each $n \geq N$, take clopen subsets P_n and Q_n of X such that $\{n\} \times P_n = F \cap (\{n\} \times X)$ and $\{n\} \times Q_n = G \cap (\{n\} \times X)$, respectively. Hence for each $n \geq N$, $f \in [P_n, Q_n]$. Put $S = \bigcap_{n \geq N} [P_n, Q_n]$. Then S is a neighborhood of f in $\mathcal{H}(X)_{(\delta)}$. Now assume that $g \in S$. Then

$$g^\omega(\textstyle\bigcup_{n \geq N} F \cap (\{n\} \times X)) = g^\omega(\bigcup_{n \geq N} \{n\} \times P_n) \subseteq \bigcup_{n \geq N} \{n\} \times Q_n = \bigcup_{n \geq N} G \cap (\{n\} \times X),$$

and so $g^\beta(F^*) \subseteq G^*$. To see that this is true, simply observe that βg^ω is a homeomorphism of βX and that

$$[\bigcup_{n\geq N} F \cap (\{n\} \times X)] \triangle F$$

is compact, and so $(\bigcup_{n\geq N} F \cap (\{n\} \times X))^* = F^*$, and, similarly, for G. ☐

Lemma 3.4. *$i\colon \mathcal{H}(X) \to i(\mathcal{H}(X))$ is open.*

Proof. It suffices to show that i is open at $e_{\mathcal{H}(X)}$.

Let $V \subseteq \mathcal{H}(X)$ be an open neighborhood of $e_{\mathcal{H}(X)} = \mathrm{id}_X$. There is a finite clopen partition $\mathcal{U}$ of X such that $N(\mathcal{U}) \subseteq V$. We have to prove that $i(V)$ is a neighborhood of $i(e_{\mathcal{H}(X)}) = \mathrm{id}_{Z^*}$ in $i(\mathcal{H}(X))$. Put $\mathcal{K} = \{(\omega \times U)^* : U \in \mathcal{U}\}$. Then $\mathcal{K}$ is a finite clopen partition of Z^*. Hence $N(\mathcal{K})$ is a neigborhood of id_{Z^*} in $\mathcal{H}(Z^*)$.

Claim 1. $i(N(\mathcal{U})) = N(\mathcal{K}) \cap i(\mathcal{H}(X))$.

Take an arbitrary $f \in N(\mathcal{U})$, and fix $U \in \mathcal{U}$. Then $f^\omega(\{n\} \times U) = \{n\} \times U$ for each $n < \omega$. Hence $f^\beta((\omega \times U)^*) = (\omega \times U)^*$, and so $i(f)((\omega \times U)^*) = (\omega \times U)^*$. We conclude that $i(f) \in N(\mathcal{K})$.

Conversely, assume that for some $f \in \mathcal{H}(X)$, $i(f) \in N(\mathcal{K})$. Again, fix $U \in \mathcal{U}$. Then $i(f) \in [(\omega \times U)^*, (\omega \times U)^*]$, and so by 3.2 (applied to both f and f^{-1}), $\{n < \omega : f^\omega(\{n\} \times U) \neq \{n\} \times U\}$ is finite. Hence, $\mathcal{U}$ being finite, there exists N such that for all $U \in \mathcal{U}$, $f^\omega(\{N\} \times U) = \{N\} \times U$, and so $f(U) = U$. We conclude that $f \in N(\mathcal{U})$, as desired. ☐

To conclude the proof of 1.1, consider an arbitrary topological group G of weight at most $\mathfrak{c}$. Then $G_{(\delta)}$ has weight at most $\mathfrak{c}$, and is non-archimedean, being a P-group. By 2.1, there is a zero-dimensional compact space X such that $G_{(\delta)}$ is topologically isomorphic to a subgroup H of $\mathcal{H}(X)$. Let Z be as above. We know by the Lemmas that i is an injective homomorphism, $i\colon \mathcal{H}(X)_{(\delta)} \to \mathcal{H}(Z^*)$ is continuous, and that $i\colon \mathcal{H}(X) \to i(\mathcal{H}(X))$ is open. Since H is a P-space, $i{\restriction}H\colon H \to \mathcal{H}(Z^*)$ is continuous. Since i is injective and open, it follows that $i{\restriction}H\colon H \to i(H)$ is open as well. We conclude that $i{\restriction}H\colon H \to i(H)$ is a topological isomorphism.

Observe that we actually proved a stronger result than stated: every P-group G is topologically isomorphic to a subgroup of some $\mathcal{H}(X)$, where X is a compact zero-dimensional F-space in which nonempty G_δ's have infinite interior. The assumption on the weight of G in 1.1 was only used to conclude that the space X can be chosen to be of weight $\mathfrak{c}$, making it a Parovičenko space and hence homeomorphic to ω^* under CH.

4. Proof of 1.3

Since all Parovičenko spaces are homeomorphic under CH, one implication is trivial.

For the proof of the reverse implication, we use the Parovičenko spaces in [4]. Assume that CH fails. Our job is to find a contradiction.

Claim 1. There is a Parovičenko space S having the following properties:

(1) for some $p \in S$, $\chi(p, S) = \omega_1$,
(2) $S \setminus \{p\}$ is homeomorphic to an open subset of ω^*.

This is the space considered in [4, p. 539].

Claim 2. There is a Parovičenko space T having the following properties:

(1) all nonempty clopen subsets of T are homeomorphic to T,
(2) for each $x \in T$, $\chi(x, T) = \mathfrak{c}$.

Indeed, $T = (\omega \times 2^{\mathfrak{c}})^*$, the space considered in [4, p. 540]. We only need to check (1) since (2) can be found in [4, p. 540]. If $C \subseteq T$ is nonempty and clopen, then there is a noncompact clopen $D \subseteq \omega \times 2^{\mathfrak{c}}$ such that $\mathrm{cl}_{\beta T} D = D \cup C$, and so $D^* = C$. There are infinitely many n such that $(\{n\} \times 2^{\mathfrak{c}}) \cap D \neq \emptyset$. Since all nonempty clopen subsets of $2^{\mathfrak{c}}$ are homeomorphic to $2^{\mathfrak{c}}$, $D \approx \omega \times 2^{\mathfrak{c}}$, hence $D^* \approx T$.

Assume that $\mathcal{H}(T)$, $\mathcal{H}(S)$ and $\mathcal{H}(\omega^*)$ are pairwise isomorphic.

Since all nonempty clopen subsets of ω^* are homeomorphic to ω^*, $T \approx \omega^*$ by 2.2. Hence for each $q \in \omega^*$, $\chi(q, \omega^*) = \mathfrak{c}$.

Now consider S, and its point p. Since $S \setminus \{p\}$ is homeomorphic to an open subset of ω^*, it follows from what we know that for each $r \in S \setminus \{p\}$, $\chi(r, S) = \mathfrak{c}$. Since $\chi(p, S) = \omega_1 < \mathfrak{c}$, this implies that for each $f \in \mathcal{H}(S)$, $f(p) = p$. But then since S is the Alexandroff one-point compactification of $S \setminus \{p\}$, $\mathcal{H}(S \setminus \{p\})$ and $\mathcal{H}(S)$ are isomorphic. It is also clear that $S \setminus \{p\}$, being homeomorphic to an open subset of ω^*, has a base of homeomorphic copies of ω^*. Hence both $\langle S \setminus \{p\}, \mathcal{H}(S \setminus \{p\}\rangle$ and $\langle \omega^*, \mathcal{H}(\omega^*)\rangle$ are in $\mathfrak{R}$, while moreover $\mathcal{H}(S \setminus \{p\})$ and $\mathcal{H}(\omega^*)$ are isomorphic. So $S \setminus \{p\}$ and ω^* are homeomorphic by 2.2; this is a contradiction since one of them is compact, while the other one is not.

5. Some topological properties of $\mathcal{H}(\omega^*)$

We study the topological group $\mathcal{H}(\omega^*)$ here.

5.1. **Cardinal functions.** We discuss the values of a few familiar cardinal functions on $\mathcal{H}(\omega^*)$; as an application, it will follow that $\mathcal{H}(\omega^*)$ is not a P-group.

Lemma 5.1. *Let $B \subseteq \omega^*$ be clopen. and let $\mathcal{Z}$ be a finite clopen partition of ω^* such that $\bigcap_{Z \in \mathcal{Z}} [Z, Z] \subseteq [B, B]$. Then for each $Z \in \mathcal{Z}$, either $B \cap Z = \emptyset$ or $Z \subseteq B$.*

Proof. Let $Z \in \mathcal{Z}$ be arbitrary, and assume that $B \cap Z \neq \emptyset$ and $Z \setminus B \neq \emptyset$. Let $g\colon B \cap Z \to Z \setminus B$ be any homeomorphism, and define $f \in \mathcal{H}(\omega^*)$ by: f restricts to the identity on $\omega^* \setminus Z$, restricts to g on $B \cap Z$, and g^{-1} on $Z \setminus B$.

Then $f \in \bigcap_{Z \in \mathcal{Z}} [Z, Z] \setminus [B, B]$, which is a contradiction. □

An indexed family $\{(A_i^0, A_i^1) : i \in I\}$ of pairs of disjoint subsets of a set X is called *independent* provided that for all $\sigma \in [I]^{<\omega}$ and $\xi \in 2^{\sigma}$, the set $\bigcap_{i \in \sigma} A_i^{\xi(i)}$ is infinite.

Lemma 5.2. *There is an independent family $\{(A_\alpha^0, A_\alpha^1) : \alpha < \mathfrak{c}\}$ of pairs of disjoint subsets of $\mathcal{H}(\omega^*)$ such that*

(1) *for each $\alpha < \mathfrak{c}$, A_α^0 is a clopen subgroup of $\mathcal{H}(\omega^*)$, and A_α^1 is its complement,*
(2) *for each infinite $\sigma \subseteq \mathfrak{c}$, $\bigcap_{\alpha \in \sigma} A_\alpha^0$ is nowhere dense in $\mathcal{H}(\omega^*)$.*

Proof. Let $\mathcal{B}$ be a pairwise disjoint family nonempty clopen subsets of ω^*, [15, 3.1.2(a)] of size $\mathfrak{c}$, and enumerate it 'faithfully' as $\{B_\alpha : \alpha < \mathfrak{c}\}$. We can split this family in two subfamilies, each of cardinality $\mathfrak{c}$. Hence we may assume that there is also a similarly indexed pairwise disjoint family nonempty clopen subsets $\{C_\alpha : \alpha < \mathfrak{c}\}$ of ω^*, such that $\bigcup_{\alpha<\mathfrak{c}} B_\alpha \cap \bigcup_{\alpha<\mathfrak{c}} C_\alpha = \emptyset$.

For each $\alpha < \mathfrak{c}$, put $A^0_\alpha = \hat{B}^0_\alpha$ (see §2.3).

For (1), pick arbitrary $\sigma \in [\mathfrak{c}]^{<\omega}$ and $\xi \in 2^\sigma$. Let $\tau = \{\alpha \in \sigma : \xi(\alpha) = 1\}$. Define an element $f \in \mathcal{H}(\omega^*)$, as follows. If $\alpha \in \tau$, then $f(B_\alpha) = C_\alpha$ and $f(C_\alpha) = B_\alpha$. Moreover, f restricts to the identity on $\omega^* \setminus (\bigcup_{\alpha\in\tau} B_\alpha \cup C_\alpha)$. Then $f \in \bigcap_{\alpha\in\sigma} A^{\xi(\alpha)}_\alpha$. It is easily seen that the intersection is in fact infinite.

For (2), let $\sigma \subseteq \mathfrak{c}$ be infinite. If $\bigcap_{\alpha\in\sigma} A^0_\alpha$ has nonempty interior, then it contains a neighborhood of the identity, being a subgroup. Hence it suffices to prove that this cannot happen. To this end, let $\mathcal{Z}$ be a finite clopen partition of ω^* by nonempty sets such that $\bigcap_{Z\in\mathcal{Z}}[Z, Z]$ is contained in $\bigcap_{\alpha\in\sigma} A^0_\alpha$. By 5.1, for each $\alpha \in \sigma$ and $Z \in \mathcal{Z}$, either $B_\alpha \cap Z = \emptyset$, or $Z \subseteq B_\alpha$. There exists $Z \in \mathcal{Z}$ such that $\{\alpha \in \sigma : B_\alpha \cap Z \neq \emptyset\}$ is infinite. Hence for distinct such δ and γ, $Z \subseteq B_\delta \cap B_\gamma$, which is a contradiction. $\square$

Corollary 5.3. *$\mathcal{H}(\omega^*)$ is not a P-group.*

We mentioned in §1 that the cardinality of $\mathcal{H}(\omega^*)$ cannot be determined in ZFC. We see the same phenomenon in ω^*. For example, both the minimum character and the minimum π-character of points of ω^* cannot be determined in ZFC, see [15]; but of course, its cardinality can, it is $2^{\mathfrak{c}}$. Interestingly, $\mathcal{H}(\omega^*)$ behaves similarly but differently.

Proposition 5.4. *If $\phi \in \{w, \chi, \pi\chi, c, d, t\}$, then for each nonempty open U in $\mathcal{H}(\omega^*)$, $\phi(U) = \mathfrak{c}$.*

Proof. Let $U \in \tau(\mathcal{H}(\omega^*)) \setminus \{\emptyset\}$. We may assume without loss of generality that $e \in U$.

That $w(\mathcal{H}(\omega^*)) \leq \mathfrak{c}$ is clear, see §2.3, hence $w(U) \leq \mathfrak{c}$.

Assume that $\{U_\gamma : \gamma < \kappa\}$, where $\kappa < \mathfrak{c}$, is a local base at e in $\mathcal{H}(\omega^*)$. Let $\{(A^0_\alpha, A^1_\alpha) : \alpha < \mathfrak{c}\}$ be as in 5.2. For each $\alpha < \mathfrak{c}$, there exists $\gamma(\alpha) < \kappa$ such that $U_{\gamma(\alpha)} \subseteq A^0_\alpha$. There is $\delta < \kappa$ such that the set $E = \{\alpha < \mathfrak{c} : \gamma(\alpha) = \delta\}$ is infinite. Hence $U_\delta \subseteq \bigcap_{\alpha\in E} A^0_\alpha$, which contradicts 5.2(2). Hence $\chi(e, \mathcal{H}(\omega^*)) = \mathfrak{c}$, and so $\chi(U) = \mathfrak{c}$.

By the same argument, $\pi\chi(U) = \mathfrak{c}$ as well.

Let $\mathcal{Z}$ be a finite clopen partition of ω^* such that $N(\mathcal{Z}) \subseteq U$. Pick a nonempty $Z \in \mathcal{Z}$. Let $\mathcal{B}$ be a pairwise disjoint family nonempty clopen subsets of Z of size $\mathfrak{c}$, [15, 3.1.2(a)]. We may assume that there exists a nonempty clopen subset C of Z such that $C \cap \bigcup \mathcal{B} = \emptyset$. For each $B \in \mathcal{B}$, let $U_B = \{f \in \mathcal{H}(\omega^*) : (f(B) = C) \,\&\, (f(Z \setminus (B \cup C)) = Z \setminus (B \cup C) \,\&\, (f{\restriction}(\omega^* \setminus Z)$ is the identity on $\omega^* \setminus Z)\}$. It is easy to see that $\{U_B : B \in \mathcal{B}\}$ is pairwise disjoint clopen family of subsets of $N(\mathcal{Z})$ of cardinality $\mathfrak{c}$.

Hence $\mathfrak{c} \leq c(U) \leq d(U) \leq w(U) \leq \mathfrak{c}$, and so we get $c = d = w = \mathfrak{c}$.

Now we deal with the tightness of $\mathcal{H}(\omega^*)$. This is more tricky. By [15, 3.3.4], there are an open F_σ-subset V of ω^* and a point $p \in \overline{V} \setminus V$ such that for every $F \in [V]^{<\mathfrak{c}}$, $p \notin \overline{F}$ (the proof of this result is based on a highly nontrivial matrix of clopen subsets of ω^* due to Kunen [12]). Let $\{\mathcal{Z}_\alpha : \alpha < \mathfrak{c}\}$ enumerate all finite clopen partitions of ω^*. For every $\alpha < \mathfrak{c}$, there is a unique $Z_\alpha \in \mathcal{Z}_\alpha$ that contains p and hence intersects V. Let $f_\alpha \in \mathcal{H}(\omega^*)$ be such that $f_\alpha(Z_\alpha) = Z_\alpha$, $f_\alpha(p) \in V$ (use that nonempty clopen subsets of ω^* are homeomorphic to ω^*), and f_α restricts to the identity on $\omega^* \setminus Z_\alpha$. Then $f_\alpha \in N(\mathcal{Z}_\alpha) \setminus \{e\}$. Hence $e \in \overline{\{f_\alpha : \alpha < \mathfrak{c}\}}$. Let $E \in [\mathfrak{c}]^{<\mathfrak{c}}$, and assume that $e \in \overline{\{f_\alpha : \alpha \in E\}}$.

Claim 3. $p \in \overline{\{f_\alpha(p) : \alpha \in E\}}$.

Indeed, let W be an arbitrary clopen neighborhood of p, and put $\tilde{W} = \{g \in \mathcal{H}(\omega^*) : g(W) = W\}$. Then $\tilde{W}$ is a clopen neighborhood of e in $\mathcal{H}(\omega^*)$, hence there exists $\alpha \in E$ such that $f_\alpha \in \tilde{W}$, and so $f_\alpha(p) \in W$.

Hence we arrived at a contradiction since $\{f_\alpha(p) : \alpha \in E\} \in [V]^{<\mathfrak{c}}$ and so, by the properties of p, $p \notin \overline{\{f_\alpha(p) : \alpha \in E\}}$. □

Corollary 5.5. *$\mathcal{H}(\omega^*)$ is crowded.*

5.2. **Universality properties of $\mathcal{H}(\omega^*)$.** In light of 1.2, it is a natural question whether $\mathcal{H}(\omega^*)$ is universal for all non-archimedean groups of small weight. It is not, as we will now show.

Proposition 5.6. *If X is a crowded compact zero-dimensional F-space in which nonempty G_δ's have nonempty interior, then all countable subsets of $\mathcal{H}(X)$ are discrete (and hence closed and discrete).*

Proof. Assume that $\{f_n : n < \omega\} \subseteq \mathcal{H}(X) \setminus \{e\}$. Pick a nonempty clopen subset C^0_0 in X such that $C^0_0 \cap f_0(C^0_0) = \emptyset$, and a point $x_0 \in C^0_0$. By recursion on $1 \le n < \omega$, we will construct $x_n \in X$, and for each $m \le n$ a clopen neighborhood C^n_m of x_m, such that

(1) $\{C^n_m : m \le n\} \cup \{f_m(C^n_m) : m \le n\}$ has cardinality $2(n{+}1)$ and is pairwise disjoint,
(2) for $m \le n{-}1$, $C^n_m \subseteq C^{n-1}_m$.

Assume that for certain $n < \omega$, we constructed $\{x_m : m \le n\}$ and the family $\{C^n_m : m \le n\}$. By (1), $F = \{x_m : m \le n\} \cup \{f_m(x_m) : m \le n\}$ has cardinality $2(n{+}1)$. Put $G = F \cup f^{-1}_{n+1}(F)$, and let U be a nonempty clopen subset of X such that $U \cap f_{n+1}(U) = \emptyset$. Since X is crowded, U is infinite and G is finite, we may pick $x_{n+1} \in U \setminus G$. Hence $H = \{x_m : m \le n{+}1\} \cup \{f_m(x_m) : m \le n{+}1\}$ has cardinality $2(n{+}2)$. For each $m \le n{+}1$, let S_m and T_m be clopen neighborhoods of x_m respectively $f_m(x_m)$ such that $T_m = f_m(S_m)$, the family $\{S_m : m \le n{+}1\} \cup \{T_m : m \le n{+}1\}$ has cardinality $2(n{+}2)$ and is pairwise disjoint. Now for each $m \le n$, put $C^{n+1}_m = C^n_m \cap S_m$, and let $C^{n+1}_{n+1} = S_{n+1} \cap U$. Then our choices are clearly as desired, which completes the recursion.

Now for each n, the intersection of the collection $\{C^k_n : k \ge n\}$ contains x_n. Hence, by assumption, we there is a nonempty clopen $E_n \subseteq \bigcap_{k \ge n} C^k_n$. Then $\bigcup_n E_n \cap \bigcup_n f_n(E_n) = \emptyset$ by (1), and hence, since X is an F-space, there is a clopen subset E of X such that $\bigcup_n E_n \subseteq E \subseteq X \setminus \bigcup_n f_n(E_n)$. Now $U = \{g \in \mathcal{H}(X) : g(E) = E\}$ is an open neighborhood of e in $\mathcal{H}(X)$ not containing any f_n. □

This means that the familiar topological group 2^ω, which is non-archimedean, is not topologically isomorphic to a subgroup of $\mathcal{H}(\omega^*)$.

The assumption on zero-dimensionality in 5.6, is in fact superfluous: a slight adaptation of the proof gives the result for all compact crowded F-spaces in which nonempty G_δ's have nonempty interior.

We do not know whether 5.6 can be generalized for higher cardinals, even for ω^*. By 5.4, $\mathcal{H}(\omega^*)$ is crowded and has density $\mathfrak{c}$, hence there are subsets of $\mathcal{H}(\omega^*)$ of size $\mathfrak{c}$ that are not discrete. Hence our question is for cardinals between ω and $\mathfrak{c}$. An inspection of the proof of 5.6 will reveal that for subsets say of cardinality ω_1, one can run for example into a Hausdorff gap (a nightmare for ω^* aficionados), or perhaps ultrafilters with character ω_1 (another nightmare).

Question 5.7. Is there in ZFC a subset of $\mathcal{H}(\omega^*)$ of size ω_1 that is not closed?

It is also a natural question whether $\mathcal{H}(\omega^*)$ contains all 'small' P-groups in ZFC. It is not universal for 'small' P-groups since it is not a P-group itself, see 5.3. The question cannot be answered in ZFC. As we showed, it does under CH. But it does not in Shelah's model where $\mathcal{H}(\omega^*)$ has cardinality $\mathfrak{c}$ (see §1), simply because the group $2^{\mathfrak{c}}_{(\delta)}$, which has weight $\mathfrak{c}$, is too large.

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