On the topology of complex map-germs and a general Lê-Greuel formula

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Abstract

Consider a singular holomorphic map-germ $f : (X, \underline{0}) \to (\mathbb{C}, 0)$ where X is a singular complex analytic variety in $\mathbb{C}^N$, and another holomorphic map-germ $g : (X, \underline{0}) \to (\mathbb{C}, 0)$ which is "sufficiently good" relatively to f. We use stratified Morse theory to determine up to homeomorphism, the topology of the Milnor fiber F_f out from the slice $F_{g,f}$ and the Morse data of a Morsification of the restriction of g to F_f. This generalizes classical results for the case where X is non-singular, and it provides a general formula comparing the Euler characteristics of F_f and $F_{g,f}$. Restricting to the case where the singularity of X at $\underline{0}$ is isolated, the formula for the difference of the Euler characteristics becomes algebraic and easily computable, generalizing in two directions the classical Lê-Greuel formula for the Milnor number of isolated complete intersection germs (ICIS): Firstly, X can have an isolated singularity, and secondly f can have arbitrary critical set. This unifies several known formulae in this vein: i) Lê-Greuel for ICIS of arbitrary codimension; ii) the formula relating the Milnor number of a curve with that of a function on it, and an extension of it for surfaces; iii) the formula for determinantal singularities; iv) and the one for the image Milnor number. All of these are special cases of our general formula.

Introduction

In his seminal work L'Analysis situs et la géométrie algébrique, Solomon Lefschetz viewed algebraic manifolds as being swept out by their intersections with planes varying within a pencil, deriving from this perspective profound topological consequences. René Thom introduced ideas along these lines that Bernard Teissier, Lê Dũng Tráng, and others later developed into major advances in singularity theory. This line of thought led to Lê's article [31], where he described the Milnor fiber of a holomorphic map-germ by considering non-singular slices defined by a linear form. That work also inspired the classical Lê–Greuel formula for the Milnor number of isolated complete intersection singularities (ICIS).

In this work, we study the topology of Milnor fibers of functions defined on arbitrary singular varieties, continuing in that vein.

Consider a complex analytic singular variety $(X, 0) \subset \mathbb{C}^N$ and a non-constant holomorphic germ $f : (X, 0) \to (\mathbb{C}, 0)$ with arbitrary critical locus. By definition, its *Milnor fiber* is the local geneal fiber, *i.e.* the intersection $F_f := f^{-1}(t) \cap \mathbb{B}_\varepsilon$ where $\mathbb{B}_\varepsilon$ is a local Milnor ball for X and for f and t is sufficiently close to $\underline{0}$ with respect to ε. The homeomorphism type of the local Milnor fiber of f can be determined from the topology of the slice $(\ell, f) : (X, 0) \to (\mathbb{C}^2, 0)$ associated with a general linear form ℓ, together with the stratified Morse data of the restriction $\ell|_{F_f}$.

In fact, there exists a Zariski-dense open set Ω of linear forms on $\mathbb{C}^N$ such that, for every $\ell \in \Omega$, the polar variety of (ℓ, f) is a curve Γ. Setting $\Phi = (\ell, f)$, one observes that $\Gamma \cap \Phi^{-1}(0,0) = \{\underline{0}\}$; by Serre's results (cf. [28]), $\Phi|_\Gamma$ is therefore a finite morphism. Moreover, ℓ can be chosen so that $\Phi|_\Gamma$ is injective onto its image Δ, the *Cerf diagram*, which is analytic by Remmert's proper mapping theorem [16, 1.18]. Finally, Φ satisfies the Thom condition at the origin, and consequently admits a local Milnor–Lê fibration.

Recall that a stratified mapping $X \to V$ between Whitney stratified spaces satisfies the *Thom condition* if for any pair of strata S_α and S_β in X such that $S_\alpha \subset \overline{S}_\beta$, and for any sequence $\{x_n\} \subset S_\beta$ converging to $x \in S_\alpha$ such that the sequence of kernels $\ker d_{x_n}(f|_{S_\beta})$ converges to some subspace T (in the corresponding Grassmannian), we have $\ker d_x(f|_{S_\alpha}) \subset T$.

We refer to the introduction in [20] and to the original text of R. Thom [54] for further discussions on the Thom condition. In the literature this is also called the Thom a_f-condition or the Thom property. Once we know that a given map $X \to V$ between Whitney stratified complex analytic spaces has the Thom condition, then we can argue as in [32] and conclude that the map has a local Milnor-Lê fibration as stated above for the map Φ .

We use these facts to study and determine the topology of the Milnor fiber F_f out from a slice by a general linear form. Then we extend this to slices by holomorphic functions on X which have an isolated critical point with respect to f, a concept that we make precise below, in Definition 5.6. As a consequence we get a formula that determines the difference between the Euler characteristics of the Milnor fibers F_f of the germ f and that of the pair (g, f). In the particular case where the ambient variety X has an isolated singularity, this difference $\chi(F_f) - \chi(F_{g,f})$ can be determined algebraically. This leads to a generalization of the classical Lê-Greuel formula for the Milnor number of ICIS that does not require f to have an isolated singularity, and the ambient space can be singular. This also unifies several known Lê-Greuel type formulas.

We first show that for a general linear form ℓ of $\mathbb{C}^N$, the restriction of $|\ell|$ to the fiber F_f is a Morse function with ordinary quadratic singularities in the sense of Section 1. We then use stratified Morse theory as described by Goresky and MacPherson in [22, 21], to build up the Milnor fiber F_f out from the slice $F_{\ell,f}$ and the Morse data of $|\ell|$ on F_f (Theorem 4.5).

Recall that in classical Morse theory (see for instance [39]) one considers a smooth manifold M and a Morse function h on it; one defines the Morse data for h at a critical point $p \in M$ to be a pair of topological spaces A, B with $B \subset A$, determined by the corresponding Morse index. As we cross the level $a = h(p)$, the local change in the topology of $M_{\leq a}$ to the level $M_{\leq a+\varepsilon}$ is given by attaching (A, B) to $M_{\leq a}$ along B. Given a Whitney stratified space Z, say in a manifold M, and $h : Z \to \mathbb{R}$, restriction of a smooth function on M, a stratified Morse point of h means a point p which is critical for $h_{|S\alpha}$, the stratum that contains p, and such that its differential does not annihilate any limit of tangent spaces at points in other strata having p in their closure. Now, as described by M. Goresky and R. MacPherson in [22] the Morse data one needs to build up the topology of Z is the product of the tangent Morse data on S_α, defined as in the smooth case, and the normal Morse data. The latter is a pair (A', B') where A' is homeomorphic to a normal slice, or to the cone over the link L_{S_α} of the stratum, and B' is the *lower half-link* (we refer to [22, 21] for more on the subject; see p. 66 in [22] for the definition of the half links and the normal Morse data).

Coming back to our setting, we notice too that the critical points of $|\ell|$ in each Whitney stratum $S_\alpha \subset F_f$ are the intersection points of S_α with the polar curve Γ. At each such point the Morse data is $(D_\alpha, \partial D_\alpha) \times D_\alpha \times (\mathrm{Cone}\,(\mathrm{L}_\alpha^+ \cup_\partial \mathrm{L}_\alpha^-), \mathrm{L}_\alpha^-)$, where $L_a^\pm$ are the half links and $L_\alpha^+ \cup_\partial L_\alpha^-$ is homeomorphic to the link of the stratum (see [22, p. 66]); so the cone over $L_\alpha^+ \cup_\partial L_\alpha^-$ is homeomorphic to a normal slice.

It follows (Theorem 4.5) that the Milnor fiber F_f is homeomorphic to $F_{\ell,f} \times \mathbb{D}^2$ to which one attaches, at each stratum, δ_α disjoint copies of the local Morse data

$$(D_\alpha, \partial D_\alpha) \times D_\alpha \times (\mathrm{Cone}\,(L_\alpha^+ \cup_\partial L_\alpha^-), L_\alpha^-)\,,$$

where δ_α is the number of critical points of $|\ell|$ on the stratum S_α of complex dimension α, and D_α is a closed disc in S_α of real dimension α, which equals the Morse index of each Whitney stratified Morse critical point in S_α. Moreover, we may replace each Morse data by

$$(D_\alpha, \partial D_\alpha) \times D_\alpha \times (\mathrm{Cone}\,(\mathcal{L}_\alpha), \mathcal{L}_\alpha),$$

where $\mathcal{L}_\alpha$ is the complex link, and we get F_f up to homotopy.

We remark that each stratum S_α where $\ell|_{F_f}$ has a critical point contributes with one or more branches Δ_α to the Cerf diagram Δ, and the number δ_α above can be computed as $(\Delta_\alpha \cdot \{t = 0\})_0$, the intersection number of the curves Δ_α and $\{t = 0\}$ at the origin $0 \in \mathbb{C}^2$.

This theorem is a generalization of Theorem (2.3) in [30] obtained in the smooth case.

More generally, we may replace the linear form ℓ with a holomorphic germ g having an isolated critical point relative to f. Then the polar set of $\Phi = (g, f)$ is a one-dimensional curve intersecting $\Phi^{-1}(0)$ only at $\underline{0}$, and Φ satisfies the Thom condition. Hence Φ admits a local Milnor fibration and, by [6], a Morsification of the restriction $g|F_f$ to the Milnor fiber F_f of f, which may itself be singular if X has a non-isolated singularity at $\underline{0}$. Applying stratified Morse theory, we reconstruct $F_f = f^{-1}(t_0) \cap \mathbb{B}\varepsilon$ from the slice $F_{g,f} = g^{-1}(u_0) \cap f^{-1}(t_0) \cap \mathbb{B}_\varepsilon$, a Milnor fiber for $g|F_f$, together with the normal Morse data of a Morsification of $g|F_f$. We then establish Theorem 5.13 for (g, f) in the same spirit as Theorem 4.5. As before, up to homotopy each local normal Morse datum may be replaced by $(\mathrm{Cone}(\mathcal{L}_\alpha), \mathcal{L}_\alpha)$, where $\mathcal{L}_\alpha$ is the complex link of the stratum.

This allows us to compare the Euler characteristic of the Milnor fibers of f and (g, f). In the case where F_f and $F_{(g,f)}$ are equidimensional, we get:

$$\chi(F_f) - \chi(F_{g,f}) = \sum_\alpha \delta_\alpha m_\alpha, \tag{0.1}$$

with $n = \dim(X, \underline{0}) - 1$, δ_α is the number of critical points of the Morsification of g on each stratum $S_\alpha \cap F_f$ and $m_\alpha = (-1)^{n_\alpha} \chi(\mathrm{Cone}(\mathcal{L}_\alpha), \mathcal{L}_\alpha)$ where $\mathcal{L}_\alpha$ is the complex link.

In the particular case where X is non-singular away from $\underline{0}$, this formula becomes Theorem 6.1:

$$\chi(F_f) - \chi(F_{g,f}) = (-1)^n \dim_{\mathbb{C}} \frac{\mathscr{O}_{X,\underline{0}}}{(f) + J_X(f, g) \colon f^\infty}\,. \tag{0.2}$$

where $J_X(f, g)$ is the ideal in $\mathscr{O}_{X,\underline{0}}$ given by

$$J_X(f) = \frac{I_{X,\underline{0}} + J_{N-d+r}(\phi, \bar{f})}{I_{X,\underline{0}}}\,.$$

Here, $\bar{f}$ is a holomorphic extension of f to a neighborhood of $\underline{0}$ in $\mathbb{C}^N$, $J_{N-d+r}(\phi, \bar{f})$ is the ideal in $\mathscr{O}_N$ generated by the $(N - d + r)$-minors of the Jacobian matrix of $(\phi, \bar{f}) = (\phi_1, \ldots, \phi_k, \bar{f}_1, \ldots, \bar{f}_r)$, and $J_X(f,g)\colon f^\infty$ is the saturation of $J_X(f,g)$ with respect to (f) (see Section 6 for a definition of the saturation). When $(X,\underline{0})$ is smooth, the zero locus $V(J_X(f))$ is precisely the set of critical points of $f\colon (X,\underline{0}) \to (\mathbb{C}^r,0)$.

Lê-Greuel type formulas have been considered previously in various setting, as for instance in [14, 10, 49, 50, 58]. The formula above (0.2) improves in two important ways the classical Lê-Greuel formula [31, 24], which dates from the early 1970s. Firstly, the function f can have arbitrary singular locus. Secondly, the ambient space X can have an isolated singularity. In Section 7.1 we show how the classical case, which is for ICIS of arbitrary codimension, follows from (0.2).

This also unifies other known Lê-Greuel type formulas for the Milnor number:

1. The formula relating the Bassein and Buchweitz-Greuel Milnor number of a curve singularity $(C,\underline{0})$ defined in [1, 8], and the Goryunov and Mond-Van Straten Milnor number of a function on C [23, 46]. We also get an extension of this for normal and smoothable Gorenstein surface singularities; this uses the Milnor number of Wahl [56] and Greuel-Steenbrink [26].
2. The Lê-Greuel formula in [49] for the vanishing Euler characteristic of determinantal singularities.
3. The Lê-Greuel formula in [50] for the image Milnor number in the corank 1 case. This invariant was defined by Mond [45] for the image of a holomorphic map germ $f\colon (\mathbb{C}^n,S) \to (\mathbb{C}^{n+1},\underline{0})$ with isolated instability, where $S \subset \mathbb{C}^n$ is a finite set.

We remark that the formula in [10] expresses the difference $\chi(F_f)-\chi(F_{g,f})$ as the total radial index of the gradient vector field of g restricted to the fiber F_f (see the last section for details). The formula (0.2) tells us what that index is and it rises an open question in that vein.

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1 Preliminaries on singular points of complex analytic functions

Throughout this work we denote by $\underline{0}$ the origin in the source $X \subset \mathbb{C}^N$ of a map and by 0 the origin in the target. Similarly, we use $\mathbb{B}$ for balls in the (ambient space of the) source and $\mathbb{D}$ for those in the target, unless it is stated otherwise.

1.1 The smooth case

Let $(X, \underline{0})$ be the germ of an analytic space contained in an open neighborhood U of the origin $\underline{0}$ in $\mathbb{C}^N$, and let $f : (X, \underline{0}) \to (\mathbb{C}, 0)$ be a germ of complex analytic function. We shall denote also by $f : X \to V$ a representative of the germ $f : (X, \underline{0}) \to (\mathbb{C}, 0)$.

In the case where X is smooth, J. Milnor in his book [40] studied properties of complex analytic functions with isolated critical points. In particular, if $f : X \to \mathbb{C}$ has an isolated critical point at $x \in X$ then, there is an open neighborhood U_x such that the differential of f is $\neq 0$ in $U_x - \{x\}$. Then:

- there is a sufficiently small positive real number ε such that for any ε' such that $0 < \varepsilon' \leq \varepsilon$, the sphere in $\mathbb{C}^N$ centered at x and with radius ε' intersects X transversally; any such sphere is called a Milnor sphere for f at x. And,
- Given such an ε', there exists a positive real number η so that if $\mathbb{B}_\varepsilon$ is the ball bounded by the Milnor sphere $\mathbb{S}_\varepsilon$, then the restriction of f to $\mathbb{B}_\varepsilon \cap f^{-1}(\mathbb{D}_\eta \setminus \{0\})$ is a localy trivial fibration with a fiber which has the bouquet of $\mu(f, x)$ real spheres of dimension $\dim X - 1$.

The number $\mu(f, x)$ of spheres is called the Milnor number of f at x. There is another way to understand the Milnor number. First notice that if we choose local coordinates $x_1, \dots, x_n$ of X in a neighborhood of x, and corresponding coordinates $(\xi_1, \dots, \xi_n)$ in the cotangent bundle of $T^*(X)$ over this neighborhood, then we have a map of $X \to T^*(X)$ given in this neighborhood by:

$$\xi_1 = \frac{\partial f}{x_1}, \dots, \xi_n = \frac{\partial f}{x_n}.$$

Thus one has the well-known:

Lemma 1.1. *The image of df in the cotangent bundle $T^*(X)$ of X is non singular.*

It is clear that a critical point is isolated if and only if locally the image $\operatorname{Im} df$ intersects the zero section of the cotangent bundle of X at an isolated point. Using the Milnor number we have:

Lemma 1.2. *The Milnor number of f at x equals the intersection number of the zero section of the cotangent bundle $T^*(X)$ of X and the image of df in $T^*(X)$.*

We recall that a critical point x of f is called *non-degenerate* or *ordinary quadratic* when the Hessian matrix of f at x in some local coordinates is non-degenerate (i.e., has non-zero determinant). By the Morse lemma, this is equivalent to the fact that f can be written as $\sum_1^n z_i^2$ in local coordinates $z_1, \dots, z_n$. In terms of the Milnor number, it is well known that x is an ordinary quadratic point of f if and only if $\mu(f, x) = 1$. In particular, any ordinary quadratic point is an isolated critical point. Thus, we have from Lemma 1.2:

Corollary 1.3. *The critical point of f at x is ordinary quadratic if and if the image* $\operatorname{Im} df$ *of the differential and the zero section of the cotangent bundle $T^*(X)$ of X intersect transversally at $(x, 0)$.*

1.2 Complex analytic functions on a singular complex analytic set

We now extend this discussion to complex analytic functions on a singular complex analytic set following [34] and [30] (see also [22]). In this situation one has to consider a Whitney complex stratification $\mathcal{S} = (S_\alpha)_{\alpha \in A}$ of X. Then we say that:

Definition 1.4. A point $x \in X$ is a *regular* point of $f : X \to \mathbb{C}$ relatively to the stratification $\mathcal{S}$ if the restriction of f to the stratum S_{α_x} which contains x is regular. Otherwise, we say that x is *critical* or *singular* for the function f.

So, when we speak of a singular point of a function on X this means a point x of X such that the restriction of f to the stratum S_{α_x} that contains x has a critical point at x. If one stratum has dimension 0 we shall say that the point of this stratum is singular for f relatively to the stratification $\mathcal{S}$.

Naturally we have that:

Definition 1.5. A point $x \in X$ is an *isolated* singular point of the function f if and only if there is an open neighborhood U_x such that every point of $U_x - \{x\}$ is a regular point of f relatively to the stratification $\mathcal{S}$.

We recall the construction of conormal spaces, as in [34]. Let U be an open set in $\mathbb{C}^N$ and denote by $\pi \colon T^*U \to U$ its cotangent bundle. Let $Z \subset U$ be a locally closed analytic subset which is smooth. The *conormal* space of Z in U is

$$T^*_Z U := \{(x, \xi) \in T^*U \mid x \in Z \,\text{and}\; \xi(T_x Z) = 0\}.$$

Now suppose that $Z \subset U$ is a closed analytic subset. Then the *conormal* space of Z in U is

$$T^*_Z U := \overline{T^*_{Z_{reg}} U},$$

where Z_{reg} is the subset of regular points of Z and the bar means the closure in $T^*_Z U$. We also denote by $\pi_Z \colon T^*_Z U \to Z$ the induced projection given by $\pi_Z(x, \pi) = x$.

Finally, let $X \subset U$ be also closed analytic and we fix a Whitney complex stratification $\mathcal{S} = (S_\alpha)_{\alpha \in A}$ of X, as above. The *conormal* space of X in U (as a stratified space) is defined as

$$T^*_X U := \bigcup_{\alpha \in A} T^*_{S_\alpha} U = \bigcup_{\alpha \in A} T^*_{\overline{S_\alpha}} U,$$

where the second equality follows easily from the Whitney condition (a), see [34]. We also denote the induced projection by $\pi_X \colon T^*_X U \to X$. Observe that $T^*_X U$ is closed analytic in T^*U of dimension N and that it has an induced stratification with strata $T^*_{S_\alpha} U$, $\alpha \in A$.

We may think of an isolated singular point as follows:

Proposition 1.6. *Let X be a closed analytic subset in an open subset U of $\mathbb{C}^N$, and let $f : X \to \mathbb{C}$ be a complex analytic function. A point $x \in X$ is an isolated singular point of f if there exists a neighborhood U_x of x in $\mathbb{C}^N$ and a function $\tilde{f}$ which extends f to U_x such that the intersection of the space $\operatorname{Im} d\tilde{f}$ and the conormal space $T^*_X U_x$ is only the point $(x, d\tilde{f}(x))$. That is:*

$$(\operatorname{Im} d\tilde{f}) \cap T^*_X U_x, = \;\{(x, d\tilde{f}(x))\} \;\subset T^*U_x.$$

See [34] for a proof.

1.3 Morsification

As above, let $X \subset U$ be a closed analytic subset with a Whitney complex stratification $\mathcal{S} = (S_\alpha)_{\alpha \in A}$. Let $f: X \to \mathbb{C}$ be a holomorphic function and assume that $\tilde{f}: U \to \mathbb{C}$ is a holomorphic extension of f. Our definition of Morse function is taken from [6, Definition 4.1] (see also [22, page 52] in the real case).

Definition 1.7. We say that $f: X \to \mathbb{C}$ is a (complex) Morse function, in the stratified sense, if the two following conditions hold:

1. For any critical point $x \in S_\alpha$, such that $\dim S_\alpha \geq 1$, the restriction $f|_{S_\alpha}$ has an ordinary quadratic point at x.

2. For any critical point $x \in S_\alpha$ and for any sequence $\{x_n\} \subset S_\beta$ such that $x_n \to x$, $\lim_n T_{x_n} S_\beta = Q$, we have $d\tilde{f}(x)(Q) \neq 0$.

Remark 1.8.
1. The condition (1) in Definition 1.7 is equivalent to saying that $\operatorname{Im} d\tilde{f}$ intersects transversally all the strata of the conormal space $T^*_X U$ in the cotangent bundle T^*U. In particular, all critical points of a Morse fuinction are isolated.

2. If $f: X \to \mathbb{C}$ is a Morse function, then the absolute value $|f|: X \setminus f^{-1}(t) \to \mathbb{R}$ is also a real Morse function in the sense of Goresky-McPherson [22, page 52]. This will be necessary later in order to use stratified Morse theory.

3. The condition (2) in Definition 1.7 implies that if f is a complex Morse function, then all critical points of $|f|$ are nondepraved in the sense of [22, page 55] (see Definition 5.6 below).

Definition 1.9. Given a holomorphic function $f: X \to \mathbb{C}$, a *(complex) Morsification* of f, in the stratified sense, is a holomorphic function $F: X \times \mathbb{D} \to \mathbb{C}$ such that for all $t \in \mathbb{D}$, the function $f_t: X \to \mathbb{C}$ satisfies that $f_0 = f$ and $f_t: X \to \mathbb{C}$ is a Morse function, if $t \neq 0$.

The proof of the following proposition can be found in [6, Proposition 4.3]

Proposition 1.10. *If $f: (X, \underline{0}) \to (\mathbb{C}, 0)$ is the germ of a holomorphic function with an isolated critical point, then there exists a Morsification F of f, for some representative $f: X \to \mathbb{C}$.*

Remark 1.11.
1. A careful revision of the proof in [6, Proposition 4.3] shows that the Morsification $F: X \times \mathbb{D} \to \mathbb{C}$ can be taken as $F(x,t) = f(x) + t\ell(x)$, where $\ell: \mathbb{C}^N \to \mathbb{C}$ is a generic linear form

2. The same arguments in the proof of [6, Proposition 4.3] can be used to see that given a stratified subset X, the restriction of a general linear form $\ell: \mathbb{C}^N \to \mathbb{C}$ defines a Morse function on X.

3. When F is a Morsification of f, then the number of critical points of f_t near a critical point $x \in S_\alpha$ is equal to the Milnor number of the restriction $\mu(f|_{S_\alpha}, x)$.

Example 1.12. The condition 2 in Definition 1.7 is necessary. For instance, take X as the subset given by $xy = 0$ in $\mathbb{C}^3$ with three strata $S_1 = \{x = y = 0\}$, $S_2 = \{x = 0, y \neq 0\}$ and $S_3 = \{y = 0, x \neq 0\}$ and the function $f: X \to \mathbb{C}$, $f(x,y,z) = x^2 + y^2 + z^2$.

We see that f has only a critical point at $\underline{0} \in S_1$ and the restriction of f to S_1 is $f(0,0,z) = z^2$, so $\underline{0}$ is an ordinary quadratic point. However, for any sequence $x_n \in S_2$ such that $x_n \to \underline{0}$, we have $\lim_n T_{x_n} S_2 = \{0\} \times \mathbb{C}^2 \neq T_{\underline{0}} S_1$, but $d\tilde{f}(\underline{0}) = 0$, for any extension $\tilde{f}$ of f, so condition 2 does not hold.

2 Relative Polar Curves

This section concerns well-known results about polar curves relative to stratifications and linear forms. The main properties we need in the sequel are summarized in propositions 2.3 and 2.9.

2.1 Settings

Let X be an equidimensional analytic subset of an open neighbourhood U of the point 0 in $\mathbb{C}^N$ and let:

$$f : (X, \underline{0}) \to (\mathbb{C}, 0)$$

be the germ of a complex analytic function.We still call $f : X \to \mathbb{C}$ a representative of this germ. Let $\mathcal{S} = (S_\alpha)_{\alpha \in A}$ be a Whitney stratification adapted to $f^{-1}(0)$ of a sufficiently small representative X of $(X, \underline{0})$. By a Whitney stratification we mean a regular complex analytic stratification as defined by Whitney in [57, Section 19, p. 540]. In particular the strata S_α and their closures $\overline{S}_\alpha$ are complex analytic spaces. We can assume that $\underline{0}$ is in the closure $\overline{S}_\alpha$ of all the strata. So, the set A of indices is finite. We say that the stratification is adapted to a complex analytic subspace if this is a union of strata.

One can prove (see for instance [35] or [36, Theorem 6.7.1]) that there is a non-empty Zariski subset Ω_α of the space of linear functions such that for every ℓ in Ω_α we have that $\ell(\underline{0}) = 0$ and the critical locus C_α of $(\ell, f)|_{S_\alpha \setminus f^{-1}(0)}$, the restriction of (ℓ, f) to the stratum, is either empty or a non-singular curve. Then, the closure Γ_α of C_α in X is either empty or a reduced curve. Furthermore, one can choose the Ω_α so that the restriction $(\ell, f)|_{C_\alpha \setminus f^{-1}(0)}$ is finite for all strata. Since the number of strata is locally finite, the intersection $\cap_{\alpha \in A} \Omega_\alpha$ is a Zariski dense open set. For any linear form $\ell \in \cap_\alpha \Omega_\alpha$, we have that the union $\Gamma_\ell = \cup_{\alpha \in A} \overline{\Gamma}_\alpha$ of the closures of the Γ_α is either empty or a reduced curve. We define (see for instance [33, p. 310]):

Definition 2.1. For every $\ell \in \Omega_{\mathcal{S}} := \cap_{\alpha \in A} \Omega_\alpha$, the above curve $\Gamma_\ell = \Gamma_\ell(f, \mathcal{S}, \underline{0})$ is *the polar curve of f at $\underline{0}$ relative to ℓ and to the stratification $\mathcal{S}$.*

For $\ell \in \Omega_{\mathcal{S}}$, the contribution of each stratum S_α to Γ_α is either empty or a union of branches of the polar curve Γ.

Now let $\Delta_\ell = \Delta_\ell(f, \mathcal{S}, \underline{0}) := \cup_{\alpha \in A} \overline{\Delta}_\alpha$ be the image of Γ_ℓ in $\mathbb{C}^2$ by $\Phi := (\ell, f)$.

Definition 2.2. When $\Delta_\ell(f, \mathcal{S}, \underline{0}) = \cup_{\alpha \in A} \overline{\Delta}_\alpha$ is empty or a curve, it is called *the Cerf diagram, of f at $\underline{0}$ relative to ℓ and to the stratification $\mathcal{S}$.*

For simplicity, when there is no ambiguity we call Γ_ℓ the polar curve of f at $\underline{0}$, we write $\Delta_\ell = \Delta_\ell(f, \mathcal{S}, \underline{0})$ and we call it the Cerf diagram of f at $\underline{0}$. Similarly, when the stratification is fixed, we shall speak of the relative polar curve $\Gamma_\ell = \Gamma_\ell(f, \underline{0})$ and the Cerf diagram $\Delta_\ell = \Delta_\ell(f, \underline{0})$ without mentioning the stratification.

In [36, Theorem 6.7.1] one proves that for any α there is $\varepsilon, 1 \gg \varepsilon > 0$, and a Zariski dense open set Ω^1_α of linear forms ℓ such that the map $\Phi : X \cap \mathring{\mathbb{B}}_{\varepsilon'} \to \mathbb{C}^2$, with $\varepsilon \geq \varepsilon' > 0$, is one-to-one map from $\Gamma_\alpha \cap \mathring{\mathbb{B}}_{\varepsilon'}$ onto its image by Φ. Also, the contribution of each stratum S_α to Γ_α is either empty or a component of the polar curve Γ (see [35, Lemma 21] or [36, Theorem 6.7.1]). Since the set of indices α is finite, one has:

Proposition 2.3. *There is a Zariski dense open set $\Omega^1 = \Omega^1(\mathcal{S}, f, \ell)$ of linear forms such that for each $\ell \in \Omega^1$ one has:*

1. *The critical locus of Φ is a curve Γ_ℓ and its restriction to each stratum that contains $\underline{0}$ in its closure is either empty or a curve.*
2. *The image $\Delta_\ell := \Phi(\Gamma_\ell)$ is either empty or a curve in $\mathbb{C}^2$, called the Cerf diagram of f.*
3. *The restriction of Φ to Γ_ℓ is one-to-one.*
4. *The contribution of each stratum S_α to Γ_α is either empty or a union of components of the polar curve Γ.*

Remark 2.4. In the above discussion we may ask what happens if we replace the linear function ℓ by a holomorphic germ $g : (X, \underline{0}) \to (\mathbb{C}, 0)$, say with an isolated critical point at 0 with respect to the given stratification. Then, at each stratum S_α, the critical set $C_{g,f,\alpha}$ of the restriction to S_α of the map (g, f) consists of the points in the stratum where either f or g, or both, have a critical point, or else the fibers of f and g in S_α are tangent. In this case the previous discussion goes through if we assume that all these contacts are "good enough". This will be used later in the text.

2.2 Special choice of the linear form

Later in the text we will use stratified Morse theory to study the topology of the Milnor fiber. For this we need linear forms which are transverse to the limits of tangent hyperplanes:

Definition 2.5. Let $(X, \underline{0})$ be a germ in $\mathbb{C}^N$ as before, and let X_{reg} be the regular part. *The conormal space* (see [53, II 4.1]) *of X in $\mathbb{C}^N$ is the closure:*

$$C(X) = \overline{\{(x, H') \in X \times \mathbb{P}^{N-1} \mid x \in X_{\text{reg}} \text{ and } T_x X \subset H'\}}\,.$$

There is the projection $\nu : C(X) \to X$ and the fiber over $\underline{0}$ is the set of pairs of limits of points in X_{reg} and hyperplanes containing the tangent space. These are called *tangent hyperplanes.*

According to [37] the space of all limits of tangent hyperplanes of X at 0 has dimension $\leq N-2$, since this is the space of hyperplanes tangent to the cones of the aureole of X at the point 0 (see [38], Definition 2.1.4 p. 562). Similarly, for any stratum S_α which contains $\underline{0}$ in its closure, the space $\Omega_{\overline{S}_{\alpha,\underline{0}}}$ of hyperplanes of $\mathbb{C}^N$ which do not contain a limit of tangent hyperplanes to $\overline{S}_{\alpha,\underline{0}}$ at $\underline{0}$ is a Zariski dense open set of the space of all hyperplanes. Set $\Omega^2 = \cap_{S_\alpha \neq \{\underline{0}\}} \Omega_{\overline{S}_{\alpha,\underline{0}}}$, which is a Zariski dense open subset of the space of linear forms of $\mathbb{C}^N$. We have:

Lemma 2.6. *For every $\ell \in \Omega^2$ there is an open neighbourhood V of $\underline{0}$ in X such that ℓ is transverse in U to the tangent spaces of all strata $S_\alpha \cap V$, and to all limits of tangent hyperplanes.*

2.3 Stratified critical points

In [34, Definition (2.7)], see also [22], is defined a stratified critical point of a complex analytic function $\varphi : Z \to \mathbb{C}$ relatively to a Whitney stratification (T_β). This means a point z of a stratum T_β such that the restriction of φ to T_β has a critical point at z. We have the following observation:

Lemma 2.7. *Let X and f be as before and let $\ell \in \Omega^2$ as in the preceding subsection. Let ε be small enough such that $\bigcap_\alpha \Gamma_\alpha \cap \mathbb{B}_\varepsilon = \{\underline{0}\}$. Let x be a point in $X \cap \mathring{\mathbb{B}}_\varepsilon \cap \Gamma_\alpha$. Then, the restriction of ℓ to the stratified space $\{f^{-1}(f(x))\} \cap \mathring{\mathbb{B}}_\varepsilon$ has an isolated stratified critical point at x.*

We shall call Milnor fiber of f at $\underline{0}$ the space $\{f^{-1}(f(t))\} \cap \mathbb{B}_\varepsilon$ whenever $0 < |t| \ll \varepsilon \ll 1$ (see 5.3 below. Also, *e.g.*, [36, Theorem 6.11.3]).

Lemma 2.8. *For every let $\ell \in \Omega^2$, the germ at $\underline{0} \in X$ of $\Phi = (\ell, f)$ satisfies the Thom condition.*

We recall that the Thom condition was defined in the introduction. See [20, Theorem 2.6] for a proof of 2.8.

Summarizing one has:

Proposition 2.9. *Let V be an open neighbourhood of $\underline{0}$ in X. There is a Zariski dense open set Ω^2 of linear forms such that for each $\ell \in \Omega^2$ one has:*

1. *ℓ is transverse in $\mathbb{B}_\varepsilon$ to the tangent spaces of all strata $S_\alpha \cap V$ and to the limits of all tangent hyperplanes of V at $\underline{0}$.*
2. *If x is a point in $X \cap \mathbb{B}_\varepsilon \cap \Gamma_\ell$, then the restriction of ℓ to the stratified space $f^{-1}(f(x))$ has an isolated stratified critical point at x.*
3. *The germ at $\underline{0}$ of the map Φ satisfies the Thom condition.*

Convention 2.10. The linear forms that we will consider from now on are those in $\Omega := \Omega^1 \cap \Omega^2$. This is a Zariski dense open subset of the space of all linear forms in $\mathbb{C}^N$, and each $\ell \in \Omega$ satisfies all properties in 2.3 and 2.9.

3 Preparation Lemmas

Let $\mathcal{S} = (S_\alpha)_{\alpha \in A}$ be as above, a Whitney stratification adapted to $f^{-1}(0)$ of a sufficiently small representative X of $(X, \underline{0})$. We assume that $\underline{0}$ is in the closure $\overline{S}_\alpha$ of all the strata. Choose a linear function ℓ in $\Omega := \Omega^1 \cap \Omega^2$ as above, so that we can define a polar curve at $\underline{0}$ of f relatively to ℓ and to the Whitney stratification (S_α). The strata of the space $\ell^{-1}(0) \cap f^{-1}(0)$ are $(S_\alpha \cap \ell^{-1}(0) \cap f^{-1}(0))$.

We let $\varepsilon > 0$ be small enough so that the sphere $\mathbb{S}_\varepsilon := \partial \mathbb{B}_\varepsilon$ of radius ε and center at $\underline{0}$ is a Milnor sphere for X, $f^{-1}(0)$ and $f^{-1}(0) \cap \ell^{-1}(0)$. This means that for any $0 < \varepsilon' \leq \varepsilon \ll 1$, the sphere $\mathbb{S}_{\varepsilon'}$ intersects transversally the strata of X, $f^{-1}(0)$ and $f^{-1}(0) \cap \ell^{-1}(0)$. This is possible because of the local conic structure of analytic sets (see [9, Lemma 3.2], *cf.* Lemma 3.3 below).

Under these assumptions, we have:

Lemma 3.1. *The restriction $\ell_{|f^{-1}(0)}$ has a stratified isolated critical point $\underline{0}$ relatively to the stratification (S_α) and there is η, $1 \gg \varepsilon \gg \eta > 0$, such that, for any $(u, t) \in \mathring{\mathbb{D}}_\eta \subset \mathbb{C}^2$, the fiber $\ell^{-1}(u) \cap f^{-1}(t)$ is transverse to $\mathbb{S}_\varepsilon$ and only contains isolated stratified critical points of $\ell_{|f^{-1}(t)}$ relatively to the stratification $(S_\alpha \cap f^{-1}(t))$.*

The proof is straightforward.

We can assume that η is such that if Δ_α is not empty, then the open ball $\mathring{\mathbb{D}}_\eta$ of $\mathbb{C}^2$ centered at 0 with radius η contains only the point 0 of the intersection $\Delta_\ell \cap \{t = 0\}$. If, for all the $\alpha \in A$, Δ_a is empty, then one can prove by properly using rugose vector fields, that the space $X \cap \mathbb{B}_\varepsilon$ is homeomorphic to $X \cap \ell^{-1}(0) \cap \mathbb{B}_\varepsilon \times \mathbb{D}_\eta$ (see Lemma 3.5 below).

Definition 3.2. *We denote by $\delta_\alpha = (\Delta_\alpha \cdot \{t = 0\})_0$ the intersection number of the curves Δ_α and $\{t = 0\}$ at the origin 0. One can choose $|t_0|$ small enough such that the line $\{t = t_0\}$, for $0 < |t_0'| \leq |t_0|$, intersects the component Δ_α at distinct points contained in the open ball $\mathring{\mathbb{D}}_\eta$. Then δ_α is the number of such points.*

We notice that δ_α is the number of stratified critical points of ℓ on the stratum.

The following lemma is well-known and we omit the proof:

Lemma 3.3. *For $\varepsilon > 0$ small enough, the neighborhood $\mathbb{B}_\varepsilon \cap X$ of $\underline{0}$ in X is good in the sense of Prill [51]: it is stratified homeomorphic to the cone over its link $L_X := \mathbb{S}_\varepsilon \cap X$, and if we remove the vertex $\underline{0}$, it is stratified homeomorphic to the cylinder $L_X \times [0, 1)$. Also, for $0 < |t_0| \ll \varepsilon$, $f^{-1}(t_0) \cap \mathbb{B}_\varepsilon$ is homeomorphic to the Milnor fiber of f at the point $\underline{0}$.*

Notation 3.4. *We let $a, b \in \mathbb{R}$, $0 < a < b$, be such that all the points of $\Delta_\alpha \cap \{t = t_0\} \cap \mathbb{D}_\eta$ lie in the region $a < |\ell| < b$.*

The proof of the next lemma is an exercise using the fibration theorem in [32] (see 5.3 below) and rugose vector fields with to a Whitney-strong, or w-regular, stratification (see [55]). These are used again in the proof of lemma 3.6.

In [53, p. 395] Teissier called w-regularity *"Condition (a) stricte avec exposant 1"*, and proved that in the complex analytic setting, the Whitney conditions are equivalent to w-regularity [53, Theorem 1.2, page 455]. These w-regular stratifications are used by Verdier in [55] to define and construct *rugose vector fields*. These are stratified, continuous, smooth on each stratum and they are integrable.

Lemma 3.5. *Let a be as in 3.4 and let B_a be the ball in $\mathbb{C}$ centered at 0 with radius a. Then the subset $f^{-1}(t_0) \cap \mathbb{B}_\varepsilon \cap \ell^{-1}(B_a)$ of the Milnor fiber $F_f := f^{-1}(t_0) \cap \mathbb{B}_\varepsilon$ is homeomorphic to $F_{\ell,f} \times B_a$ by a stratum preserving homeomorphism that is smooth on each stratum, where a is as in 3.4 and B_a is the ball in $\mathbb{C}$ of radius b.*

Next we have (see Figure 1):

Lemma 3.6. *The subset $f^{-1}(t_0) \cap \mathbb{B}_\varepsilon \cap \ell^{-1}(B_b)$ of the Milnor fiber $F_f := f^{-1}(t_0) \cap \mathbb{B}_\varepsilon$ is homeomorphic to the whole fiber F_f by a stratum preserving homeomorphism that is smooth on each stratum, where b is as in 3.4 and B_b is the ball in $\mathbb{C}$ of radius b.*

Proof. To prove Lemma 3.6 we proceed as Milnor in [40, Chap. 4], but since the space X may have non-isolated singularities, the Milnor fiber itself can be singular, so we build up a rugose vector field whose integration gives the homeomorphism.

Consider the following open space (see Figure 2):

$$A(\varepsilon', b') := f^{-1}(t_0) \cap \mathring{\mathbb{B}}_{\varepsilon'} - \ell^{-1}(B_{b'}) \, ,$$

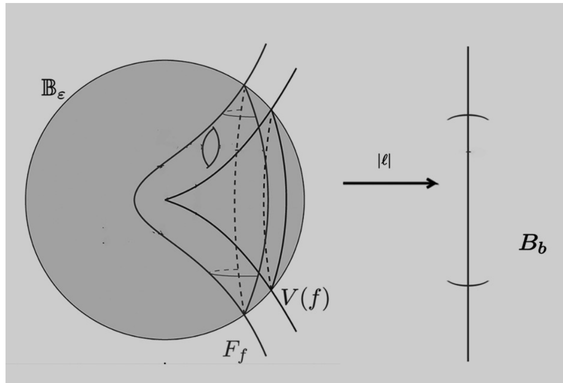

Figure 1:

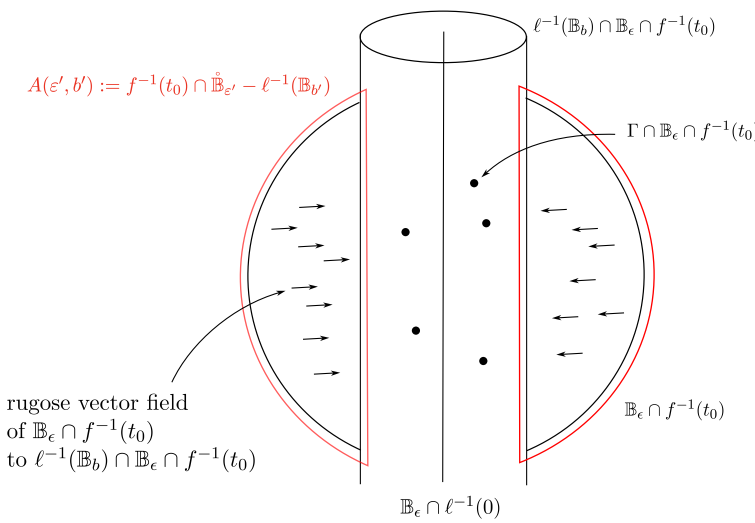

Figure 2:

where $0 < \varepsilon' - \varepsilon \ll 1$ and $0 < b - b' \ll 1$.

The Whitney stratification (S_α) of X adapted to the fiber $f^{-1}(0)$ induces a Whitney stratification of the space $A(\varepsilon', b')$ (see Theorem 2.6 of [20]). Since, by the choice of b, for well chosen ε' and b' the space $A(\varepsilon', b')$ does not intersect the polar curve Γ, the strata are $T_\alpha := S_\alpha \cap A(\varepsilon', b')$, which have dimension ≥ 1. Let us fix such well chosen ε' and b'. There is a finite number of indices α such that $T_\alpha = S_\alpha \cap A(\varepsilon', b') \neq \emptyset$.

Let $grad_\alpha \|x\|$ and $grad_\alpha |\ell|(x)$ be the gradients of the restrictions of $\|x\|$ and $|\ell|$ to T_α.

Let $x \in T_\alpha \cap \{f^{-1}(u_0) \cap \mathbb{B}_\varepsilon\}$. Then, by the Curve Selection Lemma as in [40], either:

1. the vectors $grad_\alpha \|x\|$ and $grad_\alpha |\ell|(x)$ are linearly independant; or,

2. the vectors $grad_\alpha \|x\|$ and $grad_\alpha |\ell|(x)$ are colinear and $grad\, \|x\| = \lambda\, grad\, |\ell|(x)$ for some $\lambda \in \mathbb{R}$.

In the first case, for each point x of $A(\varepsilon', b')$, there is an open neighbourhood U_x of x in $A(\varepsilon', b')$ and a rugose stratified vector field v_x (see [55, S4]) relatively to the stratification (T_α) in $A(\varepsilon', b')$, such that at any point y which belongs to $T_{\alpha_y} \cap U_x$, the vector $v_x(y)$ is tangent to T_{α_y} and satisfies:

$$Re\langle v_x(y), grad_{\alpha_y}\|y\|\rangle < 0 \qquad \text{and} \qquad Re\langle v_x(y), grad_{\alpha_y}|\ell|(y)\rangle < 0$$

where $Re\langle\,,\,\rangle$ is the real part of the hermitian product in $\mathbb{C}^N$.

In the second case we have that if the vectors $grad_\alpha\|x\|$ and $grad_\alpha|\ell|(x)$ are collinear, so that:

$$grad_\alpha\|x\| = \lambda_x grad_\alpha|\ell|(x) \quad , \quad \text{with } \lambda_x \in \mathbb{R} \, ,$$

then, $\lambda_x > 0$ or $\lambda_x = 0$ if x is in a sufficiently small neighbourhood U_0 of $\underline{0}$ in X (see [40] corollary 3.4). We may assume that $\mathbb{B}_\varepsilon \subset U_0$ and we have $\lambda_x \neq 0$ in $U_0 \cap A'$.

We may choose ε' and b' such that $A(\varepsilon', b')$ contains $A := f^{-1}(t_0) - \ell^{-1}(\mathring{B}_b)$, and $A(\varepsilon', b')$ is contained in U_0.

Now, consider a point $x \in A(\varepsilon', b')$. We suppose that $x \in T_\alpha$. Then, there is an open neighbourhood U_x of x contained in $A(\varepsilon', b')$ and a rugose vector field v_x defined in U_x which vanishes nowhere and $Re\langle v_x(y), grad_{\alpha_y}\|y\|\rangle < 0$ and $Re\langle v_x(y), grad_{\alpha_y}|\ell|(y)\rangle < 0$ at every $y \in U_x$ contained in the stratum $T_{\alpha_y} = S_{\alpha_y} \cap A(\varepsilon', b')$.

Now consider a rugose partition of unity (φ_x) associated to the covering $\cup_x U_x$ of A'. The vector field $v := \sum_x \varphi_x v_x$ gives a rugose vector field on $A(\varepsilon', b')$. Now let Ψ be a positive smooth function on U whose value at $U - (\mathring{\mathbb{B}}_\varepsilon \cup \ell^{-1}(\mathring{B}_\eta))$ is 0. Then $\Psi\, v$ also is a rugose vector field and it vanishes on $U - (\mathring{\mathbb{B}}_\varepsilon \cup \ell^{-1}(\mathring{B}_\eta))$.

Since we have for any $x \in A(\varepsilon', b')$ the inequality $Re\langle v_x(y), grad_{\alpha_y}\|y\|\rangle < 0$, the integral path of x remains in the interior of $\mathbb{B}_\varepsilon$ and the inequality $Re\langle v_x(y), grad_{\alpha_y}|\ell|(y)\rangle < 0$ shows that this integral path is hitting the space $f^{-1}(t_o) \cap \{|\ell| \leq b\}$ for some value of the parameter of the path (eventually at ∞).

The integration of the rugose vector field Ψv gives the proof of Lemma 3.6. $\square$

4 Stratified Morse Theory

Let us choose a linear form ℓ as in 2.10. That'll is, $\ell \in \Omega^1 \cap \Omega^2$. We suppose that $\mathbb{B}_\varepsilon$ has small enough radius ε such that the mapping $\Phi := (\ell, f)$ of 2.1 restricted to the polar curve Γ is one to one onto the Cerf diagram Δ.

Consider the real valued function $|\ell|$ and its restriction to the Milnor fiber $F_f := f^{-1}(t_0) \cap \mathbb{B}_\varepsilon$.

In what follows, when we write F, we mean the Milnor fiber of f, if there is no ambiguity. The following lemma is an exercise and it is a special case of a general result for holomorphic functions (see for instance [47, S. 8.3, p. 268]):

Lemma 4.1. *The stratified critical points of the restriction $|\ell|_{|F\cap\ell^{-1}((a,b))}$, a real valued function, are the stratified critical points of the complex valued function $\ell_{|F\cap\ell^{-1}((a,b))}$.*

The next result implies that with our choice of the linear form ℓ, the critical points of $|\ell|_{|F\cap\ell^{-1}((a,b))}$ are Morse singularities:

Lemma 4.2. *A critical point x of the restriction of ℓ to a stratum S_α is an ordinary quadratic point of the restriction $|\ell|_{|S_\alpha}$, if $\ell(x) \neq 0$.*

Proof. We may suppose that the critical point is $z = (a, 0, \dots, 0) \in \mathbb{C}^N$ for $a \neq 0$. and we look at the level $\ell = z_1$, so z is a critical point of the restriction $\ell|_{S_\alpha}$ where S_α is the stratum that contains z.. Then the tangent space $T_z(S_\alpha)$ is contained in $\ell^{-1}(0) = \{z_1 = 0\}$.

In the proof of Theorem 2.6 of [20] it is shown that the Whitney stratification of X adapted to $f^{-1}(0)$ can be refined in such a way that it is adapted to the polar curve Γ of f relatively to ℓ. The intersection of this Whitney stratification of X with $F_f \cap |\ell|^{-1}((a, b))$ gives a Whitney stratification of $F_f \cap |\ell|^{-1}((a, b))$ whose strata are $F_f \cap S_\alpha - \cup_{\alpha,j}\{z_{\alpha,j}\}$ and the points $\{z_{\alpha,j}\}$, for $\alpha, 0 \leq j \leq j_\alpha$.

The stratified critical points of the restriction $|\ell|_{|F_f \cap \ell^{-1}((a,b))}$ are the points of $F_f \cap \ell^{-1}(a, b)$ where $|\ell|$ is not transverse to the strata of $F_f \cap (S_\alpha) \cap \ell^{-1}((a, b))$ of $F_f \cap |\ell|^{-1}((a, b))$ induced by the stratification (S_α) of X.

The only strata of dimension 0 are the points $\{z_{\alpha,j}\}$, $\alpha, 0 \leq j \leq j_\alpha$, and $|\ell|$ is transverse to all the other strata which have dimension ≥ 1 of $F_f \cap |\ell|^{-1}((a, b))$. Therefore the only stratified critical points of $|\ell|_{|F_f \cap |\ell|^{-1}((a,b))}$ are the points $\{z_{\alpha,j}\}$, $\alpha, 0 \leq j \leq j_\alpha$, which are the intersection of the polar curve Γ with F_f.

The critical space of $\Phi_{|S_\alpha \cap \mathring{\mathbb{B}}_\varepsilon} = (\ell_{|S_\alpha \cap \mathring{\mathbb{B}}_\varepsilon}, f_{|S_\alpha \cap \mathring{\mathbb{B}}_\varepsilon})$ on the stratum S_α is the reduced curve:

$$\Gamma_\alpha \cap \mathring{\mathbb{B}}_\varepsilon - \{\underline{0}\}.$$

Therefore, the fibers of $\Phi_{|S_\alpha \cap \mathring{\mathbb{B}}_\varepsilon}$ at the points of $\Gamma_\alpha \cap \mathring{\mathbb{B}}_\varepsilon$ are ordinary critical points. Then the restriction of $\ell_{|F \cap |\ell|^{-1}((a,b))}$ to $S_\alpha \cap F_f$ has ordinary critical points at $\{z_{\alpha,j}\}, \alpha, 0 \leq j \leq j_\alpha$ (see [36, Theorem 6.7.6]). □

Now let a and b are as in 3.4, so the open interval (a, b) contains all $|z_{\alpha,j}|$ where $(u_0, z_{\alpha,j})$ are the intersection points of the line $u = u_0$ and the Cerf diagram Δ in $\mathbb{D}_\eta \subset \mathbb{C}^2$. Then we have:

Lemma 4.3. *With ℓ, η and t_0 defined as above, the function $|\ell|_{|\{f=t_0\} \cap \mathbb{B}_\varepsilon}$ is a stratified real Morse function on $f^{-1}(t_0) \cap \mathbb{B}_\varepsilon \cap |\ell|^{-1}((a', b'))$, for $0 < a' \leq a < b \leq b' \leq b + \delta$, where $\delta > 0$ is sufficiently small.*

Before proving this lemma we recall that a map $X \to \mathbb{R}$ is a Morse map if it is proper, restricted to each stratum it has only isolated non-degenerate critical points and its differential at any critical point x does not annihilate any limit of tangent spaces to any stratum S_β other than the stratum S_α containing x (see for instance [22]).

Proof. First, notice that the restriction $|\ell|_{|f^{-1}(t_0) \cap \{a < |\ell| < b\} \cap \mathbb{B}_\varepsilon}$ is proper.

Let $\partial F_f := F_f \cap \mathbb{S}_\varepsilon$. The strata of $F_f - \partial F_f$ are intersection of strata S_α of X and $F_f - \partial F_f$. The strata of ∂F_f are intersections of the strata S_α with ∂F_f.

By the choice of $\varepsilon > 0$ and the choice of η as in the preceding section, we know (Lemma 3.1) that ℓ is transverse to the strata $S_\alpha \cap \partial F_f$ of $\partial \mathbb{B}_\varepsilon \cap f^{-1}(t)$ over any t such that (u, t) is in $\mathbb{D}_\eta$. Therefore the only stratified critical points of the restriction of $|\ell|$ to $F_f \cap \{a < |\ell| < b\}$ are in $F_f - \partial F_f$.

The stratified critical points of the restriction of $|\ell|$ to $(F_f - \partial F_f) \cap \{a < |\ell| < b\}$ are precisely the points where F_f intersects the polar curve $\Gamma = \cup_\alpha \Gamma_\alpha$ of f at $\underline{0}$. Since each Γ_α is non singular outside $\underline{0}$ (see [35, Lemma 21 (1)] or [36, Theorem 6.7.6]), the restriction of ℓ to $S_\alpha \cap \mathring{\mathbb{B}}_\varepsilon$ is ordinary quadratic at every point of S_α where the restriction of ℓ to S_α is critical. Now, because of Lemma 4.2 above, the restriction of $|\ell|$ to the stratum S_α is real ordinary quadratic at any point of $S_\alpha \cap \Gamma_\alpha$.

The points $x_{\alpha,j}$ $(j = 1, \dots, j_\alpha)$ in $F_f \cap \Gamma_\alpha$ are the stratified critical points of the restriction of $|\ell|$ to F_f and satisfy $a < |\ell|(x_{\alpha,j}) < b$.

It remains to prove the third condition of the definition of a stratified Morse function, see definition 5.5.2 of [21]. Since $\mathbb{B}_\varepsilon \subset V$ (see notations in Lemma 2.6), as we assumed above, the linear form ℓ is transverse in V to the limit of tangent spaces of $\overline{S}_\alpha$ at $z_{\alpha,j}$, it implies that at $z_{\alpha,j}$ the real tangent space of the restriction of $|\ell|$ to the stratum of A which contains the stratified critical point $z_{\alpha,j}$, for all $\alpha, 0 \leq j \leq j_\alpha$, is transverse to all the limit of tangent spaces to all the (complex) strata which contain the stratified critical point of $|\ell|$ in its closure.

From §5.5.2 of [21] we get that the restriction of $|\ell|$ to F_f is a stratified Morse function in:

$$f^{-1}(t_0) \cap \mathbb{B}_\varepsilon \cap |\ell|^{-1}((a, b)).$$

The choice of α' and β' comes from the observation that:

$$0 < \alpha' \leq \alpha < \beta \leq \beta' < \beta' + \delta$$

and all the points $z_{\alpha.j}$ are contained in $\{a' \leq |\ell| \leq b'\}$.

This proves Lemma 4.3. □

We are now going to apply stratified Morse Theory. We assume we are given a Whitney stratification for which ℓ is a general form as in 2.10. We shall use the function $|\ell|_{F_f}$ to pass from the space $\{|\ell| \leq a\} \cap F$ to the space $\{|\ell| \leq b\} \cap F_f$. The starting point (see for instance Theorem 5.5.1 in [21])) is a basic consequence of Thom's First Isotopy Lemma [42, §11] is : if $a_1 > a$ is such that there are no critical points of $|\ell|_{|F_f}$ in the levels $|\ell|_{|F_f} > a$ and $|\ell|_{|F_f} \leq a_1$, then the sets in F_f defined by $|\ell|_{|F_f} \leq a$ and $|\ell|_{|F_f} \leq a_1$ are homeomorphic by a stratum preserving homeomorphism that is smooth on each stratum.

The point now is describing what happens as we pass by a level of $|\ell|$ where its restriction to F_f has a Morse critical point; we follow [22] (see [21, Theorem 5.5.3]). Recall that the critical points of $|\ell|$ in F_f are the points where the polar curve meets the fiber F_f. We get:

Lemma 4.4. *The space $f^{-1}(t_0) \cap \mathbb{B}_\varepsilon \cap \ell^{-1}(B_b)$ is homeomorphic to the space obtained from $f^{-1}(t_0) \cap \mathbb{B}_\varepsilon \cap \ell^{-1}(B_a)$ by attaching $\sum_\alpha j_\alpha$ Morse data, one for each critical point $z_{\alpha,j}$ of $\ell|_{F \cap \{a<|\ell|<b\}}$.*

We now describe the corresponding Morse data. These are, by definition, the product of the tangential and the normal Morse data.

Since the critical points of ℓ are all ordinary quadratic singularities, at a point $z_{\alpha,j}$ in a stratum $S_\alpha \subset F_f$ of complex dimension α, the tangential Morse data is $(D_\alpha, \partial D_\alpha) \times D_\alpha$, where each D_α is a closed disc in S_α of middle dimension, equal to the Morse index of the stratified critical point (see the remark following Proposition 3.6.2 in p. 65 of [22]).

The Normal Morse data are independent of the function, depending on the space X and the corresponding stratum S_α where the critical point $z_{\alpha,j}$ is. The total space of the normal Morse data actually is homeomorphic to a normal slice at a critical point $z_{\alpha,j}$, which is homeomorphic to the cone over the link L_α of the stratum S_α at p. To determine the Morse data we must say also how this total space is attached to the variety, and for this we need more information, we need *the half link* (somehow the real version of the complex link). In fact one has an upper half link $(L_\alpha^+, \partial L_\alpha^+)$ and a lower one $(L_\alpha^-, \partial L_\alpha^-)$, we refer to [22, 3.9.1] for the definition. These are stratified spaces and their union along the boundary is homeomorphic to the link: $L_p = (L_\alpha^+ \cup_\partial L_\alpha^-)$.

Using this, Theorem 3.11.1 in [22] says that the normal Morse data for the Morse function at the critical points is: $(\mathrm{Cone}(L_\alpha^+ \cup_\partial L_\alpha^-), L_\alpha^-)$.

Furthermore, if we are dealing with complex analytic varieties, as we do, this has a simpler description up to homotopy (see [22, Corollary 1, p. 166]: the normal Morse data has the homotopy of the pair $(\mathrm{Cone}(\mathcal{L}_\alpha), \mathcal{L}_\alpha)$, where $\mathcal{L}_\alpha$ is the complex link of the stratum.

Let us summarize what we have done so far. Given a Whitney stratified complex analytic map-germ $f : (X, \underline{0}) \to (\mathbb{C}, 0)$, $X \subset \mathbb{C}^N$, and a Milnor fiber $F_f = f^{-1}(t_0) \cap \mathbb{B}_\varepsilon$, equipped with the induced Whitney stratification, we know already that there is a Zariski dense open set Ω of linear functions on $\mathbb{C}^N$ such that for every $\ell \in \Omega$, if we set $\Phi := (\ell, f)$, then:

- The polar set of (ℓ, f) is a curve Γ that meets $\Phi^{-1}(0,0)$ only at the origin, Φ satisfies the Thom condition, restricted to Γ it is one-to-one over its image.
- The function $|\ell|$ restricted to F_f is a stratified Morse function and its critical points in each stratum $S_\alpha \subset F_f$ are the intersection points of S_α with the polar curve Γ;
- At each critical point ℓ has an ordinary quadratic singularity, so the restriction of $|\ell|$ to a stratum S_α has Morse index α in the stratum and its tangential Morse data is $(D_\alpha, \partial D_\alpha) \times D_\alpha$, while the normal Morse data is $(\mathrm{Cone}(L_\alpha^+ \cup_\partial L_\alpha^-), L_\alpha^-)$, where $L_a^\pm$ are the half links.
- Since the interior of F_f is complex analytic, the normal Morse data is homotopic to $(Cone(\mathcal{L}_\alpha), \mathcal{L}_\alpha)$, where $\mathcal{L}_\alpha$ is the complex link.
- The level $\ell_{F_f} \leq a$ is homeomorphic to $F_{\ell,f} \times \mathbb{D}^2$, the Milnor fiber of (ℓ, f)
- The level $\ell_{F_f} \leq b$ is homeomorphic to the whole Milnor fiber F_f.

So we are now ready to prove:

Theorem 4.5. *Let F_f be the Milnor fiber of f at $\underline{0}$, let ℓ be a general enough linear form (as above), and let $F_{\ell,f}$ be the Milnor fiber of $\Phi := (\ell, f)$. For each stratum $S_\alpha \subset F_f$, let δ_α be the number of critical points of the restriction of $|\ell|$ to the stratum. Then F_f is homeomorphic to $F_{\ell,f} \times \mathbb{D}^2$ to which one attaches at each stratum S_α, δ_α copies of the Morse data*

$$(D_\alpha, \partial D_\alpha) \times D_\alpha \times (\mathrm{Cone}\,(\mathrm{L}_\alpha^+ \cup_\partial \mathrm{L}_\alpha^-), \mathrm{L}_\alpha^-)\,.$$

Moreover, we may replace each Morse data by $(D_\alpha, \partial D_\alpha) \times D_\alpha \times (\mathrm{Cone}\,(\mathcal{L}_\alpha), \mathcal{L}_\alpha)$, where $\mathcal{L}_\alpha$ is the complex link, and we get F_f up to homotopy.

Theorem 4.5 stated in the introduction. In fact, from Lemma 4.3 by using Stratified Morse theory we obtain that the space: $f^{-1}(t_0) \cap \mathbb{B}_\varepsilon \cap \ell^{-1}(B_b)$ is obtained from $f^{-1}(u_0) \cap \mathbb{B}_\varepsilon \cap \ell^{-1}(B_a)$ by adding Morse data for each stratified critical point x of $|\ell|_{|f^{-1}(u_0)}$ such that $a < |\ell|(x) < b$.

By stratified Morse theory, we already know that at each stratified critical point x of the restriction of $|\ell|$ to $F = f^{-1}(t_0)$ the Morse data is

$$(D_\alpha, \partial D_\alpha) \times D_\alpha \times (\mathrm{Cone}(L_\alpha^+ \cup_\partial L_\alpha^-), L_\alpha^-) \, .$$

Therefore the space $f^{-1}(t_0) \cap \mathbb{B}_\varepsilon \cap \ell^{-1}(B_b)$ is homeomorphic to the space obtained from $f^{-1}(t_0) \cap \mathbb{B}_\varepsilon \cap \ell^{-1}(B_a)$ by attaching the Morse data as stated. Finally, we know that this space is homeomorphic to all of F_f, and Theorem 4.5 follows.

5 Milnor-Lê fibrations and relative Morsifications

We consider a germ $(X, \underline{0})$ as before, where X is a closed analytic subset of an open neighbourhood U of the origin $\underline{0}$ in $\mathbb{C}^N$. We fix an analytic Whitney stratification (S_α) of X. We also fix a Milnor ball $\mathbb{B}_\epsilon$ for X at $\underline{0}$, which means that for all ϵ' with $0 < \epsilon' \leq \epsilon$, the sphere $\mathbb{S}_{\epsilon'}$ is transverse to all the strata S_α. In particular, we have a Whitney stratification induced on $\mathbb{B}_\epsilon \cap X$.

5.1 The Thom condition

Suppose for now that we have a holomorphic map germ $f = (f_1, \ldots, f_k)\colon (X, \underline{0}) \to (\mathbb{C}^k, 0)$. We fix a representative $f\colon X \to V$, where $V \subset \mathbb{C}^k$ is an open neighbourhood of the origin. A *stratification* of $f\colon X \to V$ is a pair of analytic Whiney stratifications (S_α) and (T_β) of X and V, respectively, such that for each S_α there exists T_β with $f(S_\alpha) \subset T_\beta$ and the restriction $f\colon S_\alpha \to T_\beta$ is a submersion. It is known that every map between (real or complex) analytic spaces can be made a stratified map taking appropriate stratifications (see [13] in the semi-algebraic case, the analytic one being similar).

The *discriminant* of f with respect to the stratification is defined as

$$\widehat{\Delta} := \bigcup_{\dim T_\beta < k} \overline{T_\beta} \, .$$

By construction, $\widehat{\Delta}$ is a closed analytic subset of V of dimension $< k$. Moreover, the restriction

$$f\colon X \cap f^{-1}(V \setminus \widehat{\Delta}) \longmapsto V \setminus \widehat{\Delta}$$

is a stratified submersion and for any $u \in V \setminus \widehat{\Delta}$, the fibre $f^{-1}(u)$ is a stratified subspace of codimension k in X with the induced stratification. Moreover, if V is connected, then so is $V \setminus \widehat{\Delta}$. A problem is that this mapping is not proper in general, so we cannot use the Thom-Mather First Isotopy Lemma to conclude that it is a locally C^0-trivial fibration.

Definition 5.1. Let $f = (f_1, \ldots, f_k)\colon (X, \underline{0}) \to (\mathbb{C}^k, 0)$ be a holomorphic map germ. We say that f admits a *Milnor-Lê fibration* if there exists a stratification of a representative $f\colon X \to V$ such that for any ϵ, δ, with $0 < \delta \ll \epsilon \ll 1$, the restriction

$$f\colon \mathbb{B}_\epsilon \cap X \cap f^{-1}(\mathring{\mathbb{D}}_\delta \setminus \widehat{\Delta}) \longmapsto \mathring{\mathbb{D}}_\delta \setminus \widehat{\Delta}, \tag{5.1}$$

is a locally trivial stratified fibration with the induced stratification.

Lemma 5.2. *Assume that $f\colon X \to V$ admits a stratification which satisfies the Thom condition. Then f admits a Milnor-Lê fibration.*

Proof. Observe that by the Thom-Mather First Isotopy Lemma [42, §11], the condition that a proper stratified map be a locally trivial stratified fibration is equivalent to saying that it is a stratified submersion.

Notice that for any $\epsilon, \delta > 0$ and for any stratum $S_\alpha \subset X$, the restriction of f to $S_\alpha \cap \mathring{\mathbb{B}}_\epsilon \cap f^{-1}(\mathring{\mathbb{D}}_\delta \setminus \widehat{\Delta})$ is a submersion. Hence, we must prove that we can choose $\epsilon, \delta > 0$ conveniently, so that when we consider the restriction of f to strata of the form $S_\alpha \cap \mathbb{S}_\epsilon \cap f^{-1}(\mathring{\mathbb{D}}_\delta \setminus \widehat{\Delta})$, f is a submersion. This is equivalent to proving that choosing $\epsilon, \delta > 0$ conveniently, the fibers of the map 5.1 in Definition 5.1 are stratified transversal to the sphere $\mathbb{S}_\epsilon$.

We first choose $\epsilon > 0$ such that $\mathbb{S}_\epsilon$ is transverse to all the strata S_α such that $S_\alpha \subset f^{-1}(0)$. We claim that there exists $\delta > 0$ such that for any $x \in \mathbb{S}_\epsilon \cap f^{-1}(\mathring{\mathbb{B}}_\delta \setminus \widehat{\Delta})$, $\mathbb{S}_\epsilon$ is transverse to $f^{-1}(f(x)) \cap S_\beta$, where S_β is the stratum which contains x. In particular, this implies that f is a submersion on $S_\beta \cap \mathbb{S}_\epsilon \cap f^{-1}(\mathring{\mathbb{D}}_\delta \setminus \widehat{\Delta})$.

If the claim were false we could find a sequence $\{x_n\}$ in $\mathbb{S}_\epsilon \cap f^{-1}(V \setminus \widehat{\Delta})$ such that $x_n \to x$, $f(x) = 0$ and

$$\ker d_{x_n}(f|_{S_\beta}) \subset T_{x_n}\mathbb{S}_\epsilon,$$

where S_β is the stratum which contains x_n. By taking subsequences if necessary, we can assume that all terms x_n belong to the same stratum S_β and that $\ker d_{x_n}(f|_{S_\beta})$ converges to some subspace T in the corresponding Grassmannian. This implies that $T \subset T_x\mathbb{S}_\epsilon$. Then by Thom's condition we would have:

$$T_xS_\alpha = \ker d_x(f|_{S_\alpha}) \subset T \subset T_x\mathbb{S}_\epsilon,$$

where $S_\alpha \subset f^{-1}(0)$ is the stratum such that $x \in S_\alpha$, in contradiction with the choice of ϵ. □

When $k = 1$, given a holomorphic function $f\colon (X, \underline{0}) \to (\mathbb{C}, 0)$, we can always choose a small enough representative such that $\widehat{\Delta}$ is either $\{0\}$ or empty for any stratification. Moreover, we know that any Whitney stratification satisfies Thom's condition (see [5]), so we recover Lê's theorem:

Corollary 5.3 ([32]). *Any holomorphic function $f\colon (X, \underline{0}) \to (\mathbb{C}, 0)$ admits a Milnor-Lê fibration.*

Another well known case where f admits a Milnor-Lê fibration is when X is smooth and f defines an isolated complete intersection singularity (ICIS):

Corollary 5.4 (Hamm [27]). *Any holomorphic mapping $f\colon (\mathbb{C}^{n+k}, \underline{0}) \to (\mathbb{C}^k, 0)$ which defines an ICIS admits a Milnor-Lê fibration where $\widehat{\Delta} = f(C)$ and C i s the subset of critical points of f.*

However, when $k > 1$, there are holomorphic mappings $f\colon (X, \underline{0}) \to (\mathbb{C}^k, 0)$ which do not admit a Milnor-Lê fibration, even if X is smooth and f a complete intersection (with non-isolated singularity). The following example is due to Lê and it can be found in [52].

Example 5.5. Consider $\Phi\colon (\mathbb{C}^3, 0) \to (\mathbb{C}^2, 0)$ given by $\Phi(x, y, z) = (y, x^2 - y^2 z)$. Then f does not admit a Milnor-Lê fibration and hence, it does not admit a stratification which satisfies Thom's condition.

5.2 Functions with isolated singularities relatively to f

We start with a holomorphic function $f\colon (X,\underline{0}) \to (\mathbb{C},0)$. We already know that this always satisfies the Thom condition, by [5], and therefore it admits a Milnor-Lê fibration [32]. We look at functions $g\colon (X,\underline{0}) \to (\mathbb{C},0)$ such that we have a well-defined polar curve and $\Phi = (g,f)\colon (X,\underline{0}) \to (\mathbb{C}^2,0)$ also admits a Milnor-Lê fibration. We fix the following objects:

1. A representative $\Phi = (g,f)\colon X \to V$, for some open neighbourhood V of 0 in $\mathbb{C}^2$.
2. A Whitney stratification of $f\colon X \to \mathbb{C}$ with strata $\mathcal{S} = (S_\alpha)$ in X and with only two strata $\{\mathbb{C}\setminus\{0\},\{0\}\}$ in $\mathbb{C}$

We denote by $C(\Phi)$ the set of critical points of Φ respect to $\mathcal{S}$.

Definition 5.6. We say that $g\colon (X,\underline{0}) \to (\mathbb{C},0)$ has an *isolated* critical point (or singularity) with respect to $f\colon (X,\underline{0}) \to (\mathbb{C},0)$ if $g\colon X \to \mathbb{C}$ has an isolated critical point at the origin $x = \underline{0}$ with respect to $\mathcal{S}$ (see Definition 1.5).

The main theorem of this section says that if $g\colon (X,\underline{0}) \to (\mathbb{C},0)$ has an isolated critical point respect to $f\colon (X,\underline{0}) \to (\mathbb{C},0)$ then $\Phi = (g,f)\colon (X,\underline{0}) \to (\mathbb{C}^2,0)$ satisfies the Thom condition and hence, admits a Milnor-Lê fibration.

Theorem 5.7. *Suppose that $g\colon (X,\underline{0}) \to (\mathbb{C},0)$ has isolated critical point respect to $f\colon (X,\underline{0}) \to (\mathbb{C},0)$. Then:*

1. $\Gamma := \overline{C(\Phi)\setminus f^{-1}(0)}$ *is empty or a curve in X (i.e., closed analytic of dimension one);*
2. *the restriction $\Phi|_\Gamma\colon \Gamma \to V$ is finite, in particular, $\Delta := \Phi(\Gamma)$ is an analytic curve in V (by Remmert's finite mapping theorem);*
3. *$\Phi\colon X \to V$ admits a stratification $\{\mathcal{S}',\mathcal{T}\}$, such that $\mathcal{S}'$ is a refinement of $\mathcal{S}$, it has discriminant $\widehat{\Delta} = \Delta \cup (\mathbb{C}\times\{0\})$ and it has the Thom condition.*

In particular, $\Phi = (g,f)\colon (X,\underline{0}) \to (\mathbb{C}^2,0)$ admits a Milnor-Lê fibration.

Proof. We assume that $U \subset \mathbb{C}^N$ is an open neighbourhood of $\underline{0}$ such that X is closed analytic in U and $\tilde{g}\colon U \to \mathbb{C}$ is a holomorphic extension of $g\colon X \to \mathbb{C}$. Since $\underline{0}$ is an isolated critical point, we can also assume that $\underline{0}$ is the only critical point of $g\colon X \to \mathbb{C}$.

We consider the relative conormal bundle of $f\colon X \to \mathbb{C}$, defined as

$$T^*_{X,f}U := \bigcup T^*_{S_\alpha,f}U,$$

where

$$T^*_{S_\alpha,f}U := \{(x,\xi) \in T^*U \mid x \in S_\alpha \text{ and } \xi\,(\ker d_x(f|_{S_\alpha})) = 0\}.$$

By [5], the Thom condition holds for $f\colon X \to \mathbb{C}$ and this implies that $T^*_{X,f}U$ is closed analytic in T^*U. Let $d\tilde{g}\colon U \to T^*U$ be the differential mapping given by $x \mapsto (x, d_x\tilde{g})$. It follows that the set

$$C := (d\tilde{g})^{-1}(T^*_{X,f}U)$$

is also closed analytic in X.

Observe that given a point $x \in S_\alpha$, $x \in C$ if and only if x is a critical point of the restriction of g to the fibre $S_\alpha \cap f^{-1}(f(x))$. In particular, when $f(x) = 0$, $S_\alpha \subset f^{-1}(0)$ and hence $x \in C$ if and only if x is a critical point of the restriction of g to S_α. Since $\underline{0}$ is the only critical point of $g \colon X \to \mathbb{C}$, we have $C \cap f^{-1}(0) = \{\underline{0}\}$ and thus $\dim C \leq 1$.

Otherwise, if $f(x) \neq 0$, then $f|_{S_\alpha}$ is a submersion and hence, $x \in C$ if and only if x is a critical point of $\Phi|_{S_\alpha}$. We deduce that $\Gamma = \overline{C \setminus f^{-1}(0)} = \overline{C \setminus \{\underline{0}\}} = C$ and also $\dim \Gamma \leq 1$. Moreover, $T^*_{S_\alpha, f} U$ has codimension $N-1$ in T^*U, so $(d\tilde{g})^{-1}(T^*_{S_\alpha,f}U)$ has also codimension $\leq N-1$ in U. Thus, Γ has dimension 1 (if not empty). This shows item 1. But item 2 also follows, for $\Phi^{-1}(0) \cap \Gamma \subset C \cap f^{-1}(0) = \{\underline{0}\}$, hence $\Phi|_\Gamma \colon \Gamma \to V$ is finite after shrinking the representatives, if necessary.

This proof item 3 is just an adaptation of the proof given in [20, Theorem 2.6] (see also [19, Theorem 5.6.3]) in the particular case that g is the restriction of a submersion. In the target $V \subset \mathbb{C}^2$ we consider the stratification $\mathcal{T}$ with four strata: $V \setminus \widehat{\Delta}$, $\Delta \setminus \{\underline{0}\}$, $\mathbb{C} \times \{0\} \setminus \{\underline{0}\}$ and $\{\underline{0}\}$. In the source X, we consider the stratification $\mathcal{S}'$ with strata of the form $S_\alpha \cap \Phi^{-1}(T) \setminus \Gamma$ and $S_\alpha \cap \Phi^{-1}(T) \cap \Gamma$, for each $S_\alpha \in \mathcal{S}$ and $T \in \mathcal{T}$.

The strata $T \in \mathcal{T}$ are smooth, after shrinking the representatives, if necessary. We prove that all the strata $A \in \mathcal{S}'$ are also smooth. Consider first a stratum of the form $A = S_\alpha \cap \Phi^{-1}(T) \setminus \Gamma$, with $S_\alpha \in \mathcal{S}$ and $T \in \mathcal{T}$. We have three subcases: (a) $A = \{\underline{0}\}$, which is smooth trivially, (b) $A \subset f^{-1}(0) \setminus \{0\}$, with $T = \mathbb{C} \times \{0\} \setminus \{\underline{0}\}$, so $A = S_\alpha$ is also smooth or (c) $A \subset X \setminus f^{-1}(0)$, in this case $T = V \setminus \widehat{\Delta}$, $A = S_\alpha \setminus \Gamma$ is smooth again. Otherwise, we consider a stratum $A = S_\alpha \cap \Phi^{-1}(T) \cap \Gamma$, which is either the point $\{\underline{0}\}$ or has dimension 1 and its closure is analyitic. By the curve selection lemma, we can take a smaller representative such that A is smooth.

By construction, Φ maps strata of $\mathcal{S}'$ onto strata of $\mathcal{T}$. We have to show that Φ maps these strata submersively. If $A = S_\alpha \cap \Phi^{-1}(T) \setminus \Gamma$, the only non-trivial case is when $S_\alpha \subset f^{-1}(0) \setminus \{\underline{0}\}$ and $T = \mathbb{C} \times \{0\} \setminus \{\underline{0}\}$. We see that $A = S_\alpha$ and $\Phi \colon S_\alpha \to T$ is a submersion, because $g|_{S_\alpha}$ has no critical points. Otherwise, $A = S_\alpha \cap \Phi^{-1}(T) \cap \Gamma$ is mapped by Φ onto T. Now, the only non-trivial case is when $T = \Delta \setminus \{\underline{0}\}$. But A and T have dimension 1 and Φ is finite on Γ, so we can assume that $\Phi \colon A \to T$ is a local diffeomorphism.

It is clear that $\mathcal{T}$ is a Whitney stratification of V, but we have to show that $\mathcal{S}'$ satisfies the Whitney conditions. The case of a pair strata contained in $X \setminus \Gamma$ follows from [20, Lemma 2.5]. The case of a pair of strata in Γ is trivial, since one of them must be the stratum $\{\underline{0}\}$. Hence, it only remains to consider the case of a pair of strata $A, B \in \mathcal{S}'$ such that $A \subseteq \overline{B}$, $B \subset X \setminus \Gamma$ and $A \subset \Gamma$ has dimension 1. However, in this case, the set of points x in A such that B is not Whitney regular over A at x is analytic and has dimension 0 (see e.g. [18, Proposition 2.6]). After shrinking the representatives if necessary, we can assume that B is Whitney regular over A.

Finally, it only remains to check the Thom property for $\{\mathcal{S}', \mathcal{T}\}$. Let $A, B \in \mathcal{S}'$ strata such that $A \subseteq \overline{B}$. If $A \subset \Gamma$, the restriction $\Phi : A \to T$ is a local diffeomorphism and the Thom condition is satisfied automatically. Otherwise, suppose that $A \subset X \setminus \Gamma$, we must have also $B \subset X \setminus \Gamma$ and

$$A = S_\alpha \cap \Phi^{-1}(T) \setminus \Gamma, \quad B = S_\beta \cap \Phi^{-1}(T') \setminus \Gamma, \tag{5.2}$$

for some $S_\alpha, S_\beta \in \mathcal{S}$ and $T, T' \in \mathcal{T}$.

We take a sequence $\{x_n\}$ in B which converges to a point $x \in A$ such that $\ker d_{x_n}(\Phi|_B)$ converges to some space P. We must show $\ker d_x(\Phi|_A) \subset P$. It follows from (5.2) that

$$\ker d_x(\Phi|_A) = \ker d_x(\Phi|_{S_\alpha}) = \ker d_x(f|_{S_\alpha}) \cap \ker d_x\tilde{g}$$

and analogously,

$$\ker d_{x_n}(\Phi|_B) = \ker d_{x_n}(\Phi|_{S_\beta}) = \ker d_{x_n}(f|_{S_\beta}) \cap \ker d_{x_n}\tilde{g}.$$

After taking a subsequence if necessary, we can assume that $\ker d_{x_n}(f|_{S_\beta})$ converges to some space Q. Since $f\colon X \to \mathbb{C}$ satisfies the Thom condition for $\mathcal{S}$, $\ker d_x(f|_{S_\alpha}) \subset Q$. Moreover, $x \notin \Gamma = C$, so $d_x\tilde{g}(\ker d_x(f|_{S_\alpha})) \neq 0$ and hence, $d_x\tilde{g}(Q) \neq 0$. But this implies that

$$P = \lim_n \left(\ker d_{x_n}(f|_{S_\beta}) \cap \ker d_{x_n}\tilde{g}\right) = Q \cap \ker d_x(\tilde{g}) \supset \ker d_x(f|_{S_\alpha}) \cap \ker d_x\tilde{g},$$

as required. □

Definition 5.8. Suppose that $g\colon (X,\underline{0}) \to (\mathbb{C},0)$ has a *isolated* at $\underline{0}$ with respect to $f\colon (X,\underline{0}) \to (\mathbb{C},0)$. The curve Γ in Theorem 5.7 is called the *polar curve relatively to* g and the image $\Delta := \Phi(\Gamma)$ is called the *Cerf diagram.*

Example 5.9. Let $f\colon (X,\underline{0}) \to (\mathbb{C},0)$ be as above and let ℓ be a general linear form as in 2.10. Then $\ell\colon (X,\underline{0}) \to (\mathbb{C},0)$ has isolated critical point respect to f.

Example 5.10. Let X be smooth at $\underline{0}$ and assume (g,f) define an ICIS. Then it is easy to show that g has isolated singularity respect to f.

Example 5.11. Consider the mapping of Example 5.5, $(g,f)\colon (\mathbb{C}^3,\underline{0}) \to (\mathbb{C}^2,0)$, with $g(x,y,z) = y$ and $f(x,y,z) = x^2 - y^2z$. In this example, we take the stratification of $f\colon \mathbb{C}^3 \to \mathbb{C}$ with three strata $S_1 = \{x = y = 0\}$, $S_2 = \{x^2 - y^2z = 0\} \setminus S_1$ and $S_3 = \mathbb{C}^3 \setminus (S_1 \cup S_2)$. We have $g|_{S_1} = 0$ and thus, g has not isolated critical point respect to f.

5.3 Morsifications and the theorem

Given a Whitney stratified complex analytic space $Z \subset \mathbb{C}^N$ and a holomorphic function $h\colon Z \to \mathbb{C}$, a Morsification of h is a real analytic 1-parameter deformation $G\colon (Z \times \mathbb{R},\underline{0}) \to (\mathbb{C},0)$ such that if we set $G(x,t) = g_t(x)$, then each g_t is complex analytic, $g_0 = g$ and if $t \neq 0$ is small enough, then g_t is a stratified Morse function on X.

The existence of Morsifications of a function $h\colon Z \to \mathbb{C}$ at an isolated stratified critical point is proved in [6, Proposition 4.2]. Therefore we have:

Lemma 5.12. *Suppose that $g\colon (X,\underline{0}) \to (\mathbb{C},0)$ has an isolated critical point with respect to $f\colon (X,\underline{0}) \to (\mathbb{C},0)$. Then the restriction of g to each Milnor fiber F_f of f admits a Morsification.*

Let $g\colon (X,\underline{0}) \to (\mathbb{C},0)$ have an isolated critical point with respect to $f\colon (X,\underline{0}) \to (\mathbb{C},0)$ and denote by F_f and $F_{g,f}$ the fibers of $f\colon (X,\underline{0}) \to (\mathbb{C},0)$ and $\Phi = (g,f)\colon (X,\underline{0}) \to (\mathbb{C}^2,0)$, respectively. Let $g_t = g + t\ell$ be a Morsification of g with respect to f as in 5.12. For each stratum S_α, we denote by δ_α the number of critical points of the Morsification g_t on the induced stratum $S_\alpha \cap F_f$.

We now determine the topology of the Milnor fibre F_f using the Milnor fiber $F_{g,f}$ and the restriction to F_f of the Morsification of g.

Theorem 5.13. *Let X is a closed analytic subset of an open neighborhood of the origin in $\mathbb{C}^N$, and consider a holomorphic map germ $f\colon (X,\underline{0}) \to (\mathbb{C},0)$. Let $g\colon (X,\underline{0}) \to (\mathbb{C},0)$ have an isolated critical point with respect to f and denote by F_f and $F_{g,f}$ the fibers of f and $(g,f)\colon (X,\underline{0}) \to (\mathbb{C}^2,0)$, respectively. Then F_f is homeomorphic to the space $F_{g,f}\times\mathbb{D}^2$ to which one attaches at each stratum $S_\alpha\cap F_f$, δ_α Morse data*

$$(D_\alpha, \partial D_\alpha) \times D_\alpha \times (\mathrm{Cone}\,(\mathrm{L}^+_\alpha \cup_\partial \mathrm{L}^-_\alpha), \mathrm{L}^-_\alpha)\,,$$

where δ_α is the number of Morse critical points of a Morsification g_t of g with respect to f,

$L^+_\alpha \cup_\partial L^-_\alpha$ is homeomorphic to the link of the stratum $S_\alpha\cap F_f$, and $L^\pm_a$ are the half links; D_α is a real ball of dimension $n_\alpha = \dim(S_\alpha\cap F_f)$.

Proof. We take a representative $\Phi = (g,f)\colon X \to V$. By Theorem 5.7, there exist ϵ,δ, with $0<\delta\ll\epsilon\ll 1$, such that the restriction

$$\Phi\colon \mathbb{B}_\epsilon \cap X \cap \Phi^{-1}(\mathring{\mathbb{D}}_\delta \setminus \widehat{\Delta}) \longmapsto \mathring{\mathbb{D}}_\delta \setminus \widehat{\Delta}, \tag{5.3}$$

is a locally trivial stratified fibration, with $\widehat{\Delta} = \Delta \cup (\mathbb{C}\times\{0\})$, and whose fiber is

$$F_{g,f} = \mathbb{B}_\epsilon \cap X \cap f^{-1}(z) \cap g^{-1}(u),$$

for $(u,z)\in \mathring{\mathbb{B}}_\delta \setminus \widehat{\Delta}$.

We take $z\in\mathbb{C}$, with $0<|z|<\delta$ such that z is a regular value of $f:\mathbb{B}_\epsilon\cap X\to\mathbb{C}$. Then, there exists $\rho>0$ such that for all $u\in\mathbb{C}$, with $0<|u|<\rho$, $(u,z)\notin\widehat{\Delta}$. This implies that (u,z) is a regular value of Φ and hence, u is a regular value of $g\colon f^{-1}(z)\cap\mathbb{B}_\epsilon\cap X\to\mathbb{C}$.

Consider $G\colon (f^{-1}(z)\cap\mathbb{B}_\epsilon\cap X)\times\mathbb{C}\to\mathbb{C}$, given by $G(x,t) = g_t(x)$. We can choose now ρ' with $0<\rho'<\rho$ such that for all u, with $0<|u|<\rho'$, u is also a regular value of G.

Finally, let $\pi\colon G^{-1}(u)\cap\big((f^{-1}(z)\cap\mathbb{B}_\epsilon\cap X)\times\mathbb{C}\big)\to\mathbb{C}$ be the projection $\pi(x,t)=t$. Then t is a regular value of π if and only if u is a regular value of $g_t\colon f^{-1}(z)\cap\mathbb{B}_\epsilon\cap X\to\mathbb{C}$. In particular, 0 is a regular value of π and hence, t here exists $\eta>0$ such that $\pi\colon G^{-1}(u)\cap\big((f^{-1}(z)\cap\mathbb{B}_\epsilon\cap X)\times\mathbb{D}_\eta\big)\to\mathbb{D}_\eta$ is a stratified proper submersion. By Thom-Mather's First Isotopy Lemma [42, §11], π is a locally trivial fibration on $\mathbb{D}_\eta$. We deduce that for $0<|t|<\eta$, the fiber $F_{g,f}\equiv\pi^{-1}(0)$ is homeomorphic to

$$F_{g_t,f} = \mathbb{B}_\epsilon\cap X\cap f^{-1}(z)\cap g_t^{-1}(u) \equiv \pi^{-1}(t).$$

At this point, we can apply now the same arguments of stratified Morse theory as in the proof of Theorem 4.5, but using the function g_t instead of the generic linear function ℓ. □

Corollary 5.14. *We have:*

$$\chi(F_f) - \chi(F_{g,f}) = \sum_\alpha \delta_\alpha m_\alpha\,,$$

where $m_\alpha = (-1)^{n_\alpha}\chi(\mathrm{Cone}(\mathcal{L}_\alpha),\mathcal{L}_\alpha)$, and $\mathcal{L}_\alpha$ is the complex link of the stratum $S_\alpha\cap F_f$.

Proof. Theorem 5.13 implies:

$$\chi(F_f) = \chi(F_{g,f}) + \sum_\alpha \chi\Big(D_\alpha, \partial D_\alpha) \times D_\alpha \times \big(\mathrm{Cone}\,(\mathrm{L}^+_\alpha\cup_\partial \mathrm{L}^-_\alpha), \mathrm{L}^-_\alpha\big)\Big).$$

At each stratum we have:

$$\chi\Big((D_\alpha, \partial D_\alpha) \times D_\alpha \times (\text{Cone}\,(\mathrm{L}_\alpha^+ \cup_\partial \mathrm{L}_\alpha^-), \mathrm{L}_\alpha^-)\Big) = (-1)^{\mathrm{n}_\alpha}\, \chi\big(\mathrm{L}_\alpha^+ \cup_\partial \mathrm{L}_\alpha^-), \mathrm{L}_\alpha^-)\big)$$

and the result follows because for complex varieties the normal Morse data is homotopically equivalent to $(\text{Cone}(\mathcal{L}_\alpha), \mathcal{L}_\alpha)$, see for instance [22, Corollary 1, p. 166]. $\square$

Example 5.15. Consider in $\mathbb{C}^n$ the variety $X = X_1 \cup X_2$, where X_1 is the line $x_1 = \cdots = x_{n-1} = 0$ and X_2 is the hyperplane $x_n = 0$. We take two linear functions $f, g \colon X \to \mathbb{C}$ given by $f(x) = x_1 + x_n$ and $g(x) = a_1 x_1 + \cdots + a_n x_n$, where $a_1, \ldots, a_n \in \mathbb{C}$ are generic.

The fibre of f has two connected components $F_f = (F_f \cap X_1) \sqcup (F_f \cap X_2) \cong \{*\} \sqcup D^{2(n-2)}$, hence $\chi(F_f) = 2$. But the fibre of (g, f) has only one connected component $F_{g,f} = F_{g,f} \cap X_2 \cong D^{2(n-3)}$, which gives $\chi(F_{g,f}) = 1$ and thus, $\chi(F_f) - \chi(F_{g,f}) = 1$.

On the other hand g has only one critical point on F_f corresponding to the one single point stratum. It follows that $\delta_\alpha = 1$ and $m_\alpha = 1$, in concordance with our formula.

The following two examples are taken from [50, Examples 4.1 and 4.2].

Example 5.16. Let X be the surface in $\mathbb{C}^3$ parameterised as the image of the mapping $\mathbb{C}^2 \to \mathbb{C}^3$ given by

$$(t, s) \mapsto (t^3 + st, t^4 + (5/4)st^2, s).$$

The implicit reduced equation of X is:

$$-4s^4 y + 5s^3 x^2 - 32s^2 y^2 - 16sx^2 y + 64x^4 - 64y^3 = 0,$$

where (x, y, s) are the coordinates in $\mathbb{C}^3$. We take the functions $f, g \colon X \to \mathbb{C}$ given by $f(x, y, s) = s$ and $g(x, y, s) = x$, so $\Phi = (g, f) \colon X \to \mathbb{C}^2$ is the projection onto the (x, s)-plane.

The special fibre $f = 0$ in X is the plane curve E_6 and the Milnor fibre $F_f = f^{-1}(s) \cap \mathbb{B}_\varepsilon$, with $0 < |s| < \delta$, is the disentanglement, that is, the image of a stable perturbation of E_6. We see that F_f has three nodes and hence, it has the homotopy type of a bouquet of three 1-spheres with $\chi(F_f) = 1 - 3 = -2$ (see Figure 3). The mapping $\Phi = (g, f) \colon X \to \mathbb{C}^2$ is finite and has local degree three at the origin. Therefore, $F_{g,f}$ is equal to three points and $\chi(F_{g,f}) = 3$. We get $\chi(F_f) - \chi(F_{g,f}) = -5$.

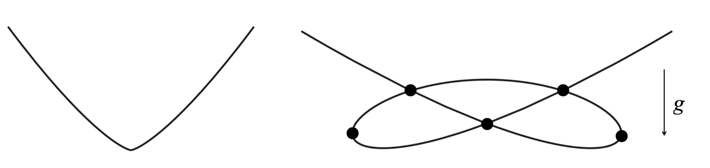

Figure 3:

To compute the sum $\sum_\alpha \delta_\alpha m_\alpha$, we see that F_f has three single point strata which contribute with three critical points and other two critical points on the smooth part of F_f (see again Figure 3). Each single point stratum is a node singularity and has complex link $\mathcal{L}_\alpha \simeq S^0$, which gives $m_\alpha = (-1)^0 \chi(Cone(\mathcal{L}_\alpha), \mathcal{L}_\alpha) = -1$. For each critical point on the smooth part we have $m_\alpha = (-1)^1 \chi(Cone(\mathcal{L}_\alpha), \mathcal{L}_\alpha) = -1$. This gives $\sum_\alpha \delta_\alpha m_\alpha = -5$.

Example 5.17. Consider X as the hypersurface in $\mathbb{C}^4$ parametrised as the image of the mapping $\mathbb{C}^3 \to \mathbb{C}^4$:

$$(u, v, w) \mapsto (u, v, w^2, w^5 + uvw^3 + (u^3 - 5uv - v)w).$$

A simple computation shows that X has the following reduced equation:

$$y^2 - x(x^2 + uvx + u^3 - 5uv - v)^2 = 0,$$

where (u, v, x, y) are the coordinates in $\mathbb{C}^4$. Now we take $f\colon X \to \mathbb{C}$ and $g\colon X \to \mathbb{C}$ as the functions $f(u, v, x, y) = v$ and $g(u, v, x, y) = u$, so $\Phi = (g, f)\colon X \to \mathbb{C}^2$ is the projection onto the (u, v)-plane.

For $v \neq 0$, the fibre $F_f = f^{-1}(v) \cap \mathbb{B}_\varepsilon$ is the disentanglement of the singularity F_4 in Mond's list [44] (see Figure 4). Hence F_f has the homotopy type of a bouquet of four 2-spheres, so that $\chi(F_f) = 1 + 4 = 5$.

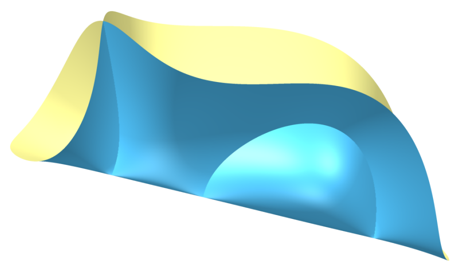

Figure 4:

On the other hand, for generic (u, v), $F_{g,f} = g^{-1}(u) \cap f^{-1}(v) \cap \mathbb{B}_\varepsilon$ is the Morsification ot the plane curve singularity A_4, so it has the homotopy type of a bouquet of two 1-spheres and hence, $\chi(F_{g,f}) = 1 - 2 = -1$. We get $\chi(F_f) - \chi(F_{g,f}) = 6$.

Finally, we see from the computations in [50, Example 4.2] that there are exactly 6 critical points of g restricted to each stratum in the interior of the fibre F_f. Namely, three of them correspond to the 0-dimensional stratum and are the three Whitney umbrellas that appear in the stable perturbation of the F_4 singularity. The complex link $\mathcal{L}_\alpha$ of the Whitney umbrella has the homotopy type of S^1, hence

$$m_\alpha = (-1)^{n_\alpha} \chi(\mathrm{Cone}(\mathcal{L}_\alpha), \mathcal{L}_\alpha) = (-1)^0(1 - 0) = 1.$$

The other three critical points correspond to critical points of g on the 1-dimensional stratum, which is the double point curve. At these points the complex link is $\mathcal{L}_\alpha \simeq S^0$, hence

$$m_\alpha = (-1)^{n_\alpha} \chi(\mathrm{Cone}(\mathcal{L}_\alpha), \mathcal{L}_\alpha) = (-1)^1(1 - 2) = 1.$$

Moreover, there are no critical points of g on the 2-dimensional stratum of regular points. This gives $\sum_\alpha \delta_\alpha m_\alpha = 6$.

6 An algebraic formula

We now give an algebraic formula for the difference $\chi(F_f) - \chi(F_{g,f})$ in the case that $(X, \underline{0})$ has isolated singularity, but we still allow f to have non isolated singularities. In fact, our formula is quite simple and it has several applications that we discuss in the next section.

Suppose that $(X,\underline{0})$ has pure dimension d, with $X\subset\mathbb{C}^N$. Given a map germ $f\colon (X,\underline{0})\to(\mathbb{C}^r,0)$, we denote by $J_X(f)$ the Jacobian ideal of f with respect to X. This is defined as follows: take generators $\phi_1,\dots,\phi_k$ of the ideal $I_{X,0}$ in $\mathscr{O}_N$ and take $\bar f\colon(\mathbb{C}^N,\underline{0})\to(\mathbb{C}^r,0)$ a holomorphic extension of f. Then $J_X(f)$ is the ideal in $\mathscr{O}_{X,\underline{0}}$ given by

$$J_X(f)=\frac{I_{X,\underline{0}}+J_{N-d+r}(\phi,\bar f)}{I_{X,\underline{0}}},$$

where $J_{N-d+r}(\phi,\bar f)$ is the ideal in $\mathscr{O}_N$ generated by the $(N-d+r)$- minors of the Jacobian matrix of $(\phi,\bar f)=(\phi_1,\dots,\phi_k,\bar f_1,\dots,\bar f_r)$. When $(X,\underline{0})$ is smooth, the zero locus $V(J_X(f))$ is precisely the set of critical points of $f\colon(X,\underline{0})\to(\mathbb{C}^r,0)$.

Recall that given ideals I,J in a ring R, the saturation if I with respect to J is the ideal

$$I\colon J^\infty=\left\{a\in R\mid aJ^k\subseteq I,\ \text{for some } k\geq 1\right\}.$$

When $R=\mathscr{O}_{X,0}$ we have a nice geometrical interpretation:

$$V(I\colon J^\infty)=\overline{V(I)\setminus V(J)}\,,$$

where the bar means the Zariski closure (see e.g. [25]).

Theorem 6.1. *Suppose that $(X,\underline{0})$ has pure dimension $n+1$ and it has an isolated singularity. If $g\colon(X,\underline{0})\to(\mathbb{C},0)$ has an isolated critical point with respect to $f\colon(X,\underline{0})\to(\mathbb{C},0)$. Then,*

$$\chi(F_f)-\chi(F_{g,f})=(-1)^n\dim_{\mathbb{C}}\frac{\mathscr{O}_{X,\underline{0}}}{(f)+J_X(f,g)\colon f^\infty}\;.$$

Proof. Since X has an isolated singularity, we have only two strata $X\setminus\{\underline{0}\}$ and $\{\underline{0}\}$. The fibre F_f is in this case a smooth manifold with boundary, hence it follows from Theorem 5.13 (see also [10]) that

$$\chi(F_f)-\chi(F_{g,f})=(-1)^n\delta_\alpha,$$

where δ_α is the number of critical points of a Morsification of g on the interior of F_f. By the conservation of the Milnor number, δ_α is equal to the sum of the Milnor numbers of the critical points of g on the interior of F_f. That is, for $0<\delta\ll\epsilon\ll 1$ and for $u\in\mathbb{C}$, $0<|u|<\delta$, the fibre $F_f=f^{-1}(u)\cap\mathbb{B}_\epsilon$ is smooth at $x\in\mathring{F}_f=f^{-1}(u)\cap\mathring{\mathbb{B}}_\epsilon$, thus:

$$\delta_\alpha=\sum_{x\in\mathring{F}_f}\mu(g|_{F_f};x)=\sum_{x\in\mathring{F}_f}\dim_{\mathbb{C}}\frac{\mathscr{O}_{F_f,x}}{J_{F_f}(g)}.$$

Moreover, X is also smooth at any point $x\in\mathring{F}_f$, which implies that

$$\frac{\mathscr{O}_{F_f,x}}{J_{F_f}(g)}\cong\frac{\mathscr{O}_{X,x}}{(f-u)+J_X(f,g)}$$

Suppose now that at $x=\underline{0}$ the ideal $J_X(f,g)$ has an irredundant primary decomposition in $\mathscr{O}_{X,\underline{0}}$ given by

$$J_X(f,g)=\mathfrak{q}_1\cap\cdots\cap\mathfrak{q}_t.$$

On the one hand, the saturation with respect to (f) can be written as

$$J_X(f,g)\colon f^\infty = \bigcap_i (\mathfrak{q}_i\colon f^\infty).$$

A simple computation shows that $\mathfrak{q}_i\colon f^\infty = \mathscr{O}_{X,\underline{0}}$ when $f \in \sqrt{\mathfrak{q}_i}$ and $\sqrt{\mathfrak{q}_i\colon f^\infty} = \sqrt{\mathfrak{q}_i}$ otherwise. In particular, $J_X(f,g)\colon f^\infty$ does not have any $\mathfrak{m}$-primary component, that is, all its primary components have dimension ≥ 1.

On the other hand, we have

$$V(J_X(f,g)\colon f^\infty) = \overline{V(J_X(f,g)) \setminus V(f))} = \Gamma,$$

the relative polar curve. We consider in Γ the analytic structure given by the ideal $J_X(f,g)\colon f^\infty$, so that Γ has local ring $\mathscr{O}_\Gamma = \mathscr{O}_{X,\underline{0}}/J_X(f,g)\colon f^\infty$.

By condition 1 in Definition 5.8, $\dim \Gamma = 1$, therefore all the primary components of $J_X(f,g)\colon f^\infty$ must have dimension ≤ 1. This shows that in fact, $\mathscr{O}_\Gamma$ is unmixed of dimension 1, that is, all the primary components have dimension 1. Also, by construction, $\Gamma \cap V(f) = \{\underline{0}\}$, which implies that f defines a regular element in $\mathscr{O}_\Gamma$ and thus, $\mathscr{O}_\Gamma$ is Cohen-Macaulay (see [47, Exercise C.3.1]).

By [17, Proposition 7.1], we compute the local intersection number at $\underline{0}$ of the hypersurface $V(f)$ and the relative polar curve Γ as the length:

$$i(V(f),\Gamma;\underline{0}) = \dim_{\mathbb{C}} \frac{\mathscr{O}_\Gamma}{(f)}.$$

Finally, we use the conservation of the multiplicity (see [47, Corollary E.6]): for $0 < \delta \ll \epsilon \ll 1$ and for $u \in \mathbb{C}$, $0 < |u| < \delta$,

$$\begin{aligned} i(V(f),\Gamma;\underline{0}) &= \sum_{x \in f^{-1}(u)\cap \mathring{\mathbb{B}}_\epsilon} i(V(f-u),\Gamma;x) \\ &= \sum_{x \in f^{-1}(u)\cap \mathring{\mathbb{B}}_\epsilon} \dim_{\mathbb{C}} \frac{\mathscr{O}_{\Gamma,x}}{(f-u)} \\ &= \sum_{x \in f^{-1}(u)\cap \mathring{\mathbb{B}}_\epsilon} \dim_{\mathbb{C}} \frac{\mathscr{O}_{X,x}}{(f-u)+J_X(f,g)\colon f^\infty} \\ &= \sum_{x \in f^{-1}(u)\cap \mathring{\mathbb{B}}_\epsilon} \dim_{\mathbb{C}} \frac{\mathscr{O}_{X,x}}{(f-u)+J_X(f,g)} = \delta_\alpha, \end{aligned}$$

since f is a unit in the ring $\mathscr{O}_{X,x}$ when $x \in f^{-1}(u) \cap \mathring{\mathbb{B}}_\epsilon$. □

Example 6.2. Let $(g,f)\colon (\mathbb{C}^3,\underline{0}) \to (\mathbb{C}^2,0)$ be the mapping $g(x,y,z) = x+y+z$ and $f(x,y,z) = xyz$. In this example, f has a non-isolated critical point, but g has an isolated critical point with respect to f. The fibre F_f has the homotopy type of $\mathbb{S}^1 \times \mathbb{S}^1$, so $\chi(F_f) = 0$. The fibre $F_{g,f}$ is diffeomorphic to the Milnor fibre of the plane curve $xy(x+y) = 0$ at the origin, so it has Milnor number 4 and we have $\chi(F_{g,f}) = 1 - 4 = -3$. Hence $\chi(F_f) - \chi(F_{g,f}) = 3$.

On the other hand, $J_2(f,g)$ is generated by the 2×2-minors of the Jacobian matrix

$$\begin{pmatrix} yz & xz & xy \\ 1 & 1 & 1 \end{pmatrix}$$

which are $yz-xz, yz-xy, xz-xy$. We get that $J_2(f,g)\colon f^\infty$ is generated by $y-z, y-x$ and hence

$$\dim_{\mathbb{C}} \frac{\mathscr{O}_3}{(f)+J_2(f,g)\colon f^\infty} = \dim_{\mathbb{C}} \frac{\mathscr{O}_3}{(xyz, y-z, y-x)} = 3\,.$$

Suppose now that $X \subset \mathbb{C}^N$ has pure dimension $n+1$ and it has isolated singularity at 0. Consider a holomorphic function $f\colon (X,\underline{0}) \to (\mathbb{C},0)$ and choose generic linear functions $\ell_1,\dots,\ell_n : \mathbb{C}^N \to \mathbb{C}$. We define $X_{n+1}=X$, $f_{n+1}=f$ and for $i=1,\dots,n$:

$$X_i = X \cap \ell_n^{-1}(0) \cap \cdots \cap \ell_i^{-1}(0), \quad f_i = f|_{X_i}\colon (X_i,\underline{0}) \to (\mathbb{C},0).$$

Corollary 6.3. *With the hypotheses of Theorem 6.1, one has:*

$$\chi(F_f) = \sum_{i=2}^{n+1} (-1)^{i-1} \dim_{\mathbb{C}} \frac{\mathscr{O}_{X_i,0}}{(f_i)+J_{X_i}(f_i,\ell_{i-1})\colon f_i^\infty} + \dim_{\mathbb{C}} \frac{\mathscr{O}_{X_1,0}}{(f_1)}.$$

Proof. The genericity of the linear forms implies that for each $i=2,\dots,n+1$, $(X_i,\underline{0})$ has pure dimension i and an isolated singularity, and $\ell_{i-1}\colon (X_i,\underline{0}) \to (\mathbb{C},0)$ has an isolated critical with respect to $f_i\colon (X_i,\underline{0}) \to (\mathbb{C},0)$. Moreover, we have $F_{f_i,\ell_{i-1}} = F_{f_{i-1}}$. By Theorem 6.1,

$$\chi(F_{f_i}) - \chi(F_{f_{i-1}}) = (-1)^{i-1} \dim_{\mathbb{C}} \frac{\mathscr{O}_{X_i,\underline{0}}}{(f_i)+J_{X_i}(f_i,\ell_{i-1})\colon f_i^\infty}.$$

Finally, for $i=1$, $(X_1,\underline{0})$ has dimension 1 and $\chi(F_{f_1})$ is precisely the local degree of $f_1\colon (X_1,\underline{0}) \to (\mathbb{C},0)$, that is,

$$\chi(F_{f_1}) = \deg(f_1) = \dim_{\mathbb{C}} \frac{\mathscr{O}_{X_1,\underline{0}}}{(f_1)}.$$

□

Example 6.4. The formula given in Corollary 6.3 can be easily implemented in Singular [12]. For instance, consider X as the union of the 2-planes x,y and z,t in $\mathbb{C}^4$. We see that X has pure dimension 2 and isolated singularity at 0, although it is not Cohen-Macaulay. We also consider the function $f\colon (X,\underline{0}) \to (\mathbb{C},0)$ given by $f(x,y,z,t)=(x+y+z+t)^3$. The following Singular code can by used to compute $\chi(F_f)$ with the formula of Corollary 6.3:

```
 LIB ‘‘primdec.lib’’;
ring r=0,(x,y,z,t),ds;
ideal i1=x,y;
ideal i2=z,t;
ideal X=intersect(i1,i2);
poly f=(x+y+z+t)^3;
poly g=3x-2y+z-t;
ideal F=X,f,g;
module M=jacob(F);
ideal j=minor(M,4)+X;
list l=sat(j,f);
ideal jj=l[1],f;
ideal Xg=X,g;
```

```
ideal X1=radical(Xg);
ideal F1=X1,f;
-vdim(std(jj))+vdim(std(F1));
6
```

The result is that $\chi(F_f) = 6$. In fact, in this example is not difficult to see that F_f is a disjoint union of six complex discs, which coincides with the computation.

7 Applications

In this section we discuss various consequences of Theorem 6.1.

7.1 The classical Lê-Greuel formula

We follow the basic construction, as explained in Looijenga's book [41, Section 5.A]. We start with a holomorphic map germ $f = (f_1, \dots, k_k)\colon (\mathbb{C}^{n+k}, \underline{0}) \to (\mathbb{C}^k, \underline{0})$ which defines an ICIS $(X, \underline{0}) = V(f_1, \dots, k_k)$ of dimension n in $\mathbb{C}^{n+k}$. After a linear change of coordinates in $\mathbb{C}^k$, we can assume that the u_k-axis in $\mathbb{C}^k$ only meets the discriminant $\widehat{\Delta}$ of f at $\underline{0}$. This generic choice of coordinates implies that $f' = (f_1, \dots, f_{k-1})$ also defines an ICIS $(X', \underline{0}) = V(f_1, \dots, k_{k-1})$ of dimension $n+1$. The classical Lê-Greuel formula says:

Theorem 7.1 ([24, 31])**.** *With the above notation,*

$$\mu(X, \underline{0}) + \mu(X', \underline{0}) = \dim_{\mathbb{C}} \frac{\mathscr{O}_{n+k}}{(f_1, \dots, f_{k-1}) + J_k(f_1, \dots, f_k)}.$$

This formula follows easily from Theorem 6.1 as we show below.

Proof. We can perform a generic linear change of coordinates in $\mathbb{C}^{k-1}$ and assume that the u_{k-1}-axis only meets the discriminant Δ_0' of f' at $\underline{0}$. Hence, $(X'', \underline{0}) = V(f_1, \dots, f_{k-2})$ is an ICIS of dimension $n+2$. But this choice of coordinates also forces that $F_f = F_{f''_{k-1}, f''_k}$ and $F_{f'} = F_{f''_{k-1}}$, where $f''_{k-1}, f''_k\colon (X'', \underline{0}) \to (\mathbb{C}, 0)$ are the restrictions of f_{k-1}, f_k to $(X'', \underline{0})$, respectively.

By Theorem 6.1,

$$\chi(F_{f'}) - \chi(F_f) = (-1)^{n+1} \dim_{\mathbb{C}} \frac{\mathscr{O}_{X'', \underline{0}}}{(f''_{k-1}) + J_{X''}(f''_{k-1}, f''_k)\colon (f''_{k-1})^\infty}.$$

Since f defines an ICIS, we have $C(f) \cap f^{-1}(\underline{0}) = \{\underline{0}\}$, where $C(f)$ is the critical locus of f. Hence, $C(f)$ is Cohen-Macaulay of dimension $k-1$ (see e.g. [41, Proposition 4.7]). Thus, the ideal $J_{X''}(f''_{k-1}, f''_k)$ defines a 1-dimensional Cohen-Macaulay variety in $(X'', 0)$. In particular, it is unmixed, that is, we have a primary decomposition

$$J_{X''}(f''_{k-1}, f''_k) = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r,$$

where all its primary components $\mathfrak{q}_i$ have dimension 1. Also, $V(f''_{k-1}) \cap V(J_{X''}(f''_{k-1}, f''_k)) = \{\underline{0}\}$, therefore $f''_{k-1} \notin \sqrt{\mathfrak{q}_i}$. We get:

$$J_{X''}(f''_{k-1}, f''_k)\colon (f''_{k-1})^\infty = J_{X''}(f''_{k-1}, f''_k),$$

and hence

$$\begin{aligned}\chi(F_{f'}) - \chi(F_f) &= (-1)^{n+1} \dim_{\mathbb{C}} \frac{\mathscr{O}_{X'',\underline{0}}}{(f''_{k-1}) + J_{X''}(f''_{k-1}, f''_k)} \\ &= (-1)^{n+1} \dim_{\mathbb{C}} \frac{\mathscr{O}_{n+k}}{(f_1, \dots, f_{k-1}) + J_k(f_1, \dots, f_k)},\end{aligned}$$

where the second equality follows easily from the definition of $J_{X''}(f''_{k-1}, f''_k)$. In this case, the fibres $F_{f'}$ and F_f have the homotopy type of bouquet of spheres of dimensions n and $n-1$, respectively, and the numbers of spheres in each case are the Milnor numbers $\mu(X', \underline{0})$ and $\mu(X, \underline{0})$, respectively. Thus, $F_{f'} = 1 + (-1)^n \mu(X', \underline{0})$, $F_f = 1 + (-1)^{n-1} \mu(X, \underline{0})$ and

$$\chi(F_{f'}) - \chi(F_f) = (-1)^{n+1}(\mu(X, \underline{0}) + \mu(X', \underline{0})).$$

□

7.2 Lê-Greuel formula for smoothable space curves

In this subsection we assume that $(X, \underline{0})$ is a space curve, that is, $\dim(X, \underline{0}) = 1$ and $X \subset \mathbb{C}^N$ for some N. The Milnor number $\mu(X, \underline{0})$ was introduced by Bassein [1] when $(X, \underline{0})$ is smoothable and by Buchweitz-Greuel [8] in general:

$$\mu(X, \underline{0}) = \dim_{\mathbb{C}} \frac{\omega_{X,\underline{0}}}{d\mathscr{O}_{X,\underline{0}}},$$

where $\omega_{X,\underline{0}}$ is the dualizing module of Grothendick and $d\colon \mathscr{O}_{X,\underline{0}} \to \omega_{X,\underline{0}}$ is induced by the exterior derivation. When $(X, \underline{0})$ is smoothable, the smoothing has the homotopy type of a bouquet of 1-spheres and the number of such spheres is $\mu(X, \underline{0})$.

Suppose we have a holomorphic function $f\colon (X, \underline{0}) \to (\mathbb{C}, 0)$ which is finite, that is $f^{-1}(0) = \{\underline{0}\}$. The Milnor number $\mu(f)$ of a function on a space curve was introduced by Goryunov [23] for space curves in $\mathbb{C}^3$ and by Mond-Van Straten [46] in the general case. This is:

$$\mu(f) = \dim_{\mathbb{C}} \frac{\omega_{X,\underline{0}}}{df(\mathscr{O}_{X,\underline{0}})},$$

where now $df\colon \mathscr{O}_{X,\underline{0}} \to \omega_{X,\underline{0}}$ is induced by the exterior product with df. When $(X, \underline{0})$ is smoothable, $\mu(f)$ is equal to the sum of the Milnor numbers of the critical points of f on the smoothing. The main relationship between $\mu(f)$ and $\mu(X, \underline{0})$ is:

$$\mu(f) = \mu(X, \underline{0}) + \deg(f) - 1, \tag{7.1}$$

where $\deg(f)$ is the local degree, that is, the number of points of the fibre F_f of $f\colon (X, \underline{0}) \to (\mathbb{C}, 0)$ (see [29] when $(X, \underline{0})$ is smoothable and [48] for the general case).

We assume that $(X, \underline{0})$ is smoothable and take a smoothing, that is, a flat morphism $\pi\colon (\mathcal{X}, \underline{0}) \to (\mathbb{C}, 0)$ such that $X_t := \pi^{-1}(t)$ is smooth when $t \neq 0$ and $X_0 = X$. We denote by $\bar{f} : (\mathcal{X}, \underline{0}) \to (\mathbb{C}, 0)$ any holomorphic extension of $f : (X, \underline{0}) \to (\mathbb{C}, 0)$. We give another proof of (7.1) based on our Theorems 5.13 and 6.1.

Corollary 7.2. *With the above notation, we have:*

$$\begin{aligned}\mu(f) &= \dim_{\mathbb{C}} \frac{\mathscr{O}_{\mathcal{X},\underline{0}}}{(\pi) + J_{\mathcal{X}}(\pi, \bar{f}) \colon \pi^{\infty}}\,,\\ \mu(f) &= \mu(X,\underline{0}) + \deg(f) - 1\,,\\ \mu(X,\underline{0}) &= \dim_{\mathbb{C}} \frac{\mathscr{O}_{\mathcal{X},\underline{0}}}{(\pi) + J_{\mathcal{X}}(\pi, \ell) \colon \pi^{\infty}} - m_0(X,\underline{0}) + 1\,,\end{aligned}$$

where $\ell\colon \mathbb{C}^N \to \mathbb{C}$ is a generic linear form and $m_0(X,\underline{0})$ is the multiplicity of $(X,\underline{0})$.

Proof. The flatness of $\pi\colon (\mathcal{X},\underline{0}) \to (\mathbb{C},0)$ implies that $\dim(\mathcal{X},\underline{0}) = 2$ and that π is regular in $\mathscr{O}_{\mathcal{X},\underline{0}}$. But f is finite, so it is also regular in $\mathscr{O}_{X,\underline{0}} = \mathscr{O}_{\mathcal{X},\underline{0}}/(\pi)$. Thus $\pi, \bar{f}$ form a regular sequence and hence, $\mathscr{O}_{\mathcal{X},\underline{0}}$ is Cohen-Macaulay. In particular, $(\mathcal{X},\underline{0})$ has pure dimension 2. Moreover, the facts that X_t is smooth for $t \neq 0$ and $(X,\underline{0})$ has isolated singularity give that $(\mathcal{X},\underline{0})$ has also isolated singularity.

We use Theorems 5.13 and 6.1 with the functions $\pi, \bar{f}\colon (\mathcal{X},\underline{0}) \to (\mathbb{C},0)$:

$$\chi(F_\pi) - \chi(F_{\bar{f},\pi}) = -\delta_\alpha = -\dim_{\mathbb{C}} \frac{\mathscr{O}_{\mathcal{X},\underline{0}}}{(\pi) + J_{\mathcal{X}}(\pi, \bar{f}) \colon \pi^{\infty}}$$

where δ_α is the number of critical points of a Morsification of $\bar{f}$ on the interior of F_π. But since $(X,\underline{0})$ is smoothable, the number δ_α is equal to $\mu(f)$ and we get the first formula.

For the second formula, just observe that $\mu(X,\underline{0}) = 1 - \chi(F_\pi)$ and $\deg(f) = \chi(F_{\bar{f},\pi})$.

Finally, the last formula follows from the other two when $f = \ell$, taking into account that the local degree of $\ell : (X,\underline{0}) \to (\mathbb{C},0)$ is the multiplicity $m_0(X,\underline{0})$. □

7.3 Lê-Greuel formula for isolated determinantal singularities

We recall some basic definitions and properties of isolated determinantal singularities (IDS) from [49]. Let $0 < s \leq m \leq n$ be integers. We denote by $M_{m,n} = M_{m,n}(\mathbb{C})$ the space of complex matrices of size $m \times n$ and by $M^s_{m,s}$ the subset of matrices with rank $< s$. We say that $(X,\underline{0})$ is an IDS of type $(m,n;s)$ if $(X,\underline{0}) = F^{-1}(M^s_{m,n})$ for some holomorphic map germ $F\colon (\mathbb{C}^N,\underline{0}) \to M_{m,n}$ such that:

1. $\operatorname{codim}(X,\underline{0}) = (m-s+1)(n-s+1)$,
2. X is smooth at x and $\operatorname{rank} F(x) = s-1$, for all $x \neq \underline{0}$ in a neighbourhood of the origin.

For technical reasons we restrict ourselves to the cases where either $s = 1$ or $N < (m-s+2)(n-s+2)$. Then, we have that $(X,\underline{0})$ is smoothable. In fact, a determinantal smoothing is constructed as follows: choose a generic matrix $A \in M_{m,n}$ and consider the map germ $F_A\colon (\mathbb{C}^N \times \mathbb{C},\underline{0}) \to M_{m,n}$ given by $F_A(x,t) = F(x) + tA$. Then, $(\mathcal{X},\underline{0}) = F_A^{-1}(M^s_{m,n})$ is a determinantal variety in $\mathbb{C}^N \times \mathbb{C}$ and $\pi\colon (\mathcal{X},\underline{0}) \to (\mathbb{C},0)$ is the restriction of the projection $\pi(x,t) = t$. If A is generic, then for all $t \neq 0$, $X_t := \pi^{-1}(t)$ is smooth and F_A has rank $s-1$ at any point in X_t.

It follows that the topology of the Milnor fibre $F_\pi = X_t \cap \mathbb{B}_\epsilon$ of the determinantal smoothing is independent of the choice of A. In general, F_π does not have the homotopy type of a bouquet of spheres (see [11]), so we cannot speak of a Milnor number. We consider instead the vanishing Euler characteristic defined in [49] as $\nu(X,\underline{0}) = (-1)^d(\chi(F_\pi) - 1)$, where $d = \dim(X,\underline{0})$.

Assume now that $f\colon (X,\underline{0}) \to (\mathbb{C},0)$ is a holomorphic function with isolated critical point defined on the IDS $(X,\underline{0})$. We associate a pair of invariants with this function. As above, we denote by $\bar{f}\colon (\mathcal{X},\underline{0}) \to (\mathbb{C},0)$ any holomorphic extension of $f\colon (X,\underline{0}) \to (\mathbb{C},0)$. On the one hand, the Milnor number $\mu(f)$ is the number of critical points of $f_a|_{X_t}$, where f_a is a Morsification of $\bar{f}$ on the determinantal smoothing X_t. On the other hand, the vanishing Euler characteristic of the fiber is also defined as

$$\nu(X \cap f^{-1}(0),\underline{0}) = (-1)^{d-1}(\chi(F_\pi \cap f_a^{-1}(c)) - 1),$$

where c is a regular value of f_a. It is shown in [49] that both numbers are well defined and depend only on the germs of X and f at $\underline{0}$.

In this setting we have again a Lê-Greuel type formula for IDS. The following corollary can be proved by using the same arguments as in Corollary 7.2:

Corollary 7.3. *Let $f\colon (X,\underline{0}) \to (\mathbb{C},0)$ be a holomorphic function with isolated critical point on an IDS $(X,\underline{0})$ of dimension d. Then,*

$$\mu(f) = \dim_{\mathbb{C}} \frac{\mathscr{O}_{\mathcal{X},\underline{0}}}{(\pi) + J_{\mathcal{X}}(\pi,\bar{f})\colon \pi^\infty},$$

$$\mu(f) = \nu(X,\underline{0}) + \nu(X \cap f^{-1}(0),\underline{0}),$$

$$\nu(X,\underline{0}) = (-1)^d \left(\sum_{i=2}^{d+1} (-1)^{i+1} \dim_{\mathbb{C}} \frac{\mathscr{O}_{\mathcal{X}_i,\underline{0}}}{(\pi_i) + J_{\mathcal{X}_i}(\pi_i,\ell_{i-1})\colon \pi_i^\infty} + m_0(X,\underline{0}) - 1 \right),$$

where $\ell_1,\dots,\ell_d\colon \mathbb{C}^N \to \mathbb{C}$ are generic linear forms, $\mathcal{X}_i = \mathcal{X} \cap \ell_d^{-1}(0) \cap \cdots \cap \ell_i^{-1}(0)$ and $\pi_i = \pi|_{\mathcal{X}_i}$.

The second equation in Corollary 7.3 was proved originally in [49].

7.4 Lê-Greuel formula for image Milnor number

The image Milnor number was introduced by Mond [45] for a hypersurface $(X,\underline{0})$ in $\mathbb{C}^{n+1}$ which is defined as the image of a holomorphic map germ $f\colon (\mathbb{C}^n,S) \to (\mathbb{C}^{n+1},\underline{0})$ with isolated instability. Here $S \subset \mathbb{C}^n$ is a finite subset whose cardinality is the number of irreducible components of $(X,\underline{0})$ and the restriction $(\mathbb{C}^n,S) \to (X,\underline{0})$ is the normalisation, since f must be finite and generically one-to-one. The condition that f has isolated instability ensures that f admits a stabilisation, provided that either f has corank one or $(n,n+1)$ are nice dimensions in the sense of Mather [43]. By definition, a stabilisation is a 1-parameter unfolding $F\colon (\mathbb{C}^n \times \mathbb{C},S) \to (\mathbb{C}^{n+1} \times \mathbb{C},\underline{0})$ given by $F(x,s) = (f_s(x),s)$ with the property that $f_0 = f$ and f_s has only stable singularities when $s \neq 0$.

Mond proved in [45] that $X_s \cap \mathbb{B}_\epsilon$ has the homotopy type of a bouquet of n-spheres, where X_s is the image of f_s and $0 < |s| < \delta \ll \epsilon$. The number of such spheres is called the image Milnor number. This is denoted by $\mu_I(f)$ and it is independent of the choice of δ,ϵ and the stabilisation. Since $\mu_I(f)$ is invariant under coordinate changes in the source and target of f, and the normalisation is unique up to coordinate changes in the source, it is also natural to consider it as an invariant of $(X,\underline{0})$: the image Milnor number $\mu_I(X,\underline{0})$ of $(X,\underline{0})$.

There is a long standing conjecture by Mond [45] which claims that

$$\operatorname{codim}_{\mathscr{A}_e}(f) \leq \mu_I(f),$$

with equality when f is weighted homogeneous. Here, $\operatorname{codim}_{\mathscr{A}_e}(f)$ is the extended $\mathscr{A}$-codimension and can be interpreted geometrically as the number of parameters in a miniversal unfolding of f. Hence, the conjecture is an inequality of type $\tau \leq \mu$, where τ is the usual Tjurina number of $(X, \underline{0})$, but for the image Milnor number instead of the classical Milnor number. The main difference is that, instead of deforming the implicit equation of X to get a smooth object, we deform the parametrisation of X to get a stable object. We refer to the survey [50] for a recent account on the Mond conjecture.

The image $(\mathcal{X}, \underline{0})$ of the stabilisation F, together with the projection $\pi\colon (\mathcal{X}, \underline{0}) \to (\mathbb{C}, 0)$, $\pi(y, s) = s$, defines a flat deformation of $(X, \underline{0})$ whose Milnor fibre is $F_\pi = X_s \cap \mathbb{B}_\epsilon$. We also have a natural stratification of $(\mathcal{X}, \underline{0})$ given by the stable types on $\mathcal{X} \setminus \{\underline{0}\}$. We denote such stratification by (S_α).

Assume now that f has corank one and let $\ell\colon \mathbb{C}^{n+1} \to \mathbb{C}$ be a generic linear function, which defines a hyperplane $H = \ell^{-1}(0)$. Then $f^{-1}(H)$ is smooth at S and the restriction $f\colon (f^{-1}(H), S) \to (H, \underline{0})$ has isolated instability. As a consequence, its image $(X \cap H, \underline{0})$ has also a well defined image Milnor number $\mu_I(X \cap H, \underline{0})$. The following Lê-Greuel type formula in this context was proved in [50]:

Theorem 7.4. *With the above notation we have:*

$$\mu_I(X, \underline{0}) + \mu_I(X \cap H, \underline{0}) = \sum_\alpha \delta_\alpha,$$

where δ_α is the number of critical points of ℓ on the stratum $S_\alpha \cap X_s$.

The corank one assumption on f also implies that the multiplicity is $m_\alpha = 1$ for all stratum $S_\alpha \cap X_s$ (see [50]). Hence, this formula can be seen as a particular case of our Theorem 5.13. Unfortunately, we do not have yet an algebraic version of this formula, as in the previous subsections. We presume that such an algebraic version of the formula could help for proving the Mond conjecture, at least in the corank one case.

7.5 Lê-Greuel formula for smoothable Gorenstein surfaces

Let $(X, \underline{0})$ be a normal surface singularity which is also smoothable and Gorenstein. Take a smoothing $\pi\colon (\mathcal{X}, \underline{0}) \to (\mathbb{C}, 0)$ with Milnor fibre F_π. It follows from the works of Wahl [56] and Greuel and Steenbrink [26] that $\beta_1(F_\pi) = 0$ and that $\beta_2(F_\pi)$ is independent of the choice of the smoothing, where β_i is the ith-Betti number. Thus, it makes sense to call Milnor number to $\mu(X, \underline{0}) = \beta_2(F_\pi)$.

Suppose now that $f\colon (X, \underline{0}) \to (\mathbb{C}, 0)$ is a holomorphic function with isolated critical point. Then the especial fibre $(X \cap f^{-1}(0), \underline{0})$ is a space curve and has a well defined Milnor number $\mu(X \cap f^{-1}(0), \underline{0})$, as in Subsection 7.2. We choose any holomorphic extension $\bar{f}\colon (\mathcal{X}, \underline{0}) \to (\mathbb{C}, 0)$ of f. We define the Milnor number $\mu(f)$ as the sum of the Milnor numbers of the critical points of $\bar{f}$ restricted to the interior of F_π.

The next corollary follows again easily from Theorems 5.13 and 6.1 and contains the Lê-Greuel type formulas for smoothable Gorenstein surfaces:

Corollary 7.5. *With the above notation we have*

$$
\begin{aligned}
\mu(f) &= \dim_{\mathbb{C}} \frac{\mathscr{O}_{\mathcal{X},0}}{(\pi) + J_{\mathcal{X}}(\pi, \bar{f})\colon \pi^{\infty}},\\
\mu(f) &= \mu(X, \underline{0}) + \mu(X \cap f^{-1}(0), \underline{0}),\\
\mu(X, \underline{0}) &= \dim_{\mathbb{C}} \frac{\mathscr{O}_{\mathcal{X},0}}{(\pi) + J_{\mathcal{X}}(\pi, \ell_2)\colon \pi^{\infty}} - \dim_{\mathbb{C}} \frac{\mathscr{O}_{\mathcal{X}_2,0}}{(\pi_2) + J_{\mathcal{X}_2}(\pi_2, \ell_1)\colon \pi_2^{\infty}} + m_0(X, \underline{0}) - 1,
\end{aligned}
$$

where $\ell_1, \ell_2 : \mathbb{C}^N \to \mathbb{C}$ are generic linear forms, $\mathcal{X}_2 = \mathcal{X} \cap \ell_2^{-1}(0)$ and $\pi_2 = \pi|_{\mathcal{X}_2}$.

An immediate consequence of the corollary is that $\mu(f)$ is well defined and depends only on $f\colon (X, \underline{0}) \to (\mathbb{C}, 0)$.

Example 7.6. We consider a surface $(X, \underline{0})$ in $\mathbb{C}^5$ which is defined by the Pfaffians $P_1, \ldots, P_{2r+1}$ of a skew-symmetric matrix $F = (f_{ij})$ of size $(2r+1) \times (2r+1)$ and with entries $f_{ij} \in \mathscr{O}_5$ such that $f_{ji} = -f_{ij}$. Recall that the Pfaffian P_i is the square root of the determinant of the submatrix of F obtained after removing the ith-row and ith-column (see e.g. [7]).

If $(X, \underline{0})$ has dimension 2, then it is Gorenstein, by a theorem due to Buchsbaum and Eisenbud [7]. In fact, the same theorem says that any Gorenstein surface $(X, \underline{0})$ in $\mathbb{C}^5$ can be obtained in this way. When $r = 1$, $(X, \underline{0})$ is a complete intersection, but if $r > 1$ and all the entries f_{ij} are in the maximal ideal $\mathfrak{m}_5$, then $(X, \underline{0})$ is not a complete intersection. As far as we know, this is the simplest example of a Gorenstein surface which is not a complete intersection.

If we choose the functions f_{ij} in such a way that $(X, \underline{0})$ has isolated singularity, then $(X, \underline{0})$ is normal. Moreover, $(X, \underline{0})$ is also smoothable since a smoothing is constructed as follows: take a skew-symmetric matrix $A = (a_{ij})$ of the same size whose entries are generic constants $a_{ij} \in \mathbb{C}$ such that $a_{ji} = -a_{ij}$. We consider the 3-fold $(\mathcal{X}, \underline{0})$ in $\mathbb{C}^5 \times \mathbb{C}$ defined by the Pfaffians of $F + tA$ and the projection $\pi\colon (\mathcal{X}, \underline{0}) \to (\mathbb{C}, 0)$, $\pi(x, t) = t$. It follows that $\pi^{-1}(0) = X$ and $\pi^{-1}(t)$ is smooth if $t \neq 0$ and the matrix A is generic enough.

We compute the Milnor number $\mu(X, \underline{0})$ in the case that we take a skew-symmetric matrix $F = (f_{ij})$ of size 5×5 and whose entries f_{ij} are linear forms, i.e., homogenous polynomials of degree one. If the coefficients of f_{ij} are generic enough then $(X, \underline{0})$ has isolated singularity and we can proceed with this example. We use Corollary 7.5 to compute the Milnor number of $(X, \underline{0})$ with the aid of Singular, which returns $\mu(X, \underline{0}) = 4$.

```
 LIB ``matrix.lib'';
LIB ``sing.lib'';
ring r=0,(x,y,z,w,v,t),ds;
ideal i = ideal(randommat(1,10,maxideal(1),9));
matrix A=skewmat(5,i); // Skew-symmetric matrix of generic linear forms
poly Pf1=A[1,2]*A[3,4]-A[1,3]*A[2,4]+A[1,4]*A[2,3]; // The Pfaffians
poly Pf2=A[1,3]*A[4,5]-A[1,4]*A[3,5]+A[1,5]*A[3,4];
poly Pf3=A[1,2]*A[4,5]-A[1,4]*A[2,5]+A[1,5]*A[2,4];
poly Pf4=A[1,2]*A[3,5]-A[1,3]*A[2,5]+A[1,5]*A[2,3];
poly Pf5=A[2,3]*A[4,5]-A[2,4]*A[3,5]+A[2,5]*A[3,4];
ideal X=Pf1,Pf2,Pf3,Pf4,Pf5; // The ideal generated by the Pfaffians
dim(std(X));
3
```

```
mres(X,0); // We check it is Gorenstein of codimension 3
  1     5     5     1
r <-- r <-- r <-- r

0     1     2     3
poly pi=t;
ideal X0=X,pi;
module M0=jacob(X0);
ideal J0=minor(M0,3)+X;
dim(std(J0)); // We check X0 has isolated singularity and that X is a smoothing
0
poly ell2=3x-2y+z-w+v; // A generic linear form
ideal F=X,pi,ell2;
module M=jacob(F);
ideal J=minor(M,4)+X;
list L=sat(J,pi);
ideal K=L[1],pi;
ideal X2=X,ell2;
poly ell1=x+y-2z-3w+v; // Another generic linear form
ideal F2=X2,pi,ell1;
module M2=jacob(F2);
ideal J2=minor(M2,5)+X2;
list L2=sat(J2,pi);
ideal K2=L2[1],pi;
vdim(std(K))-vdim(std(K2))+mult(std(X0))-1; // The Milnor number
4
```

7.6 A remark on indices of vector fields

Let $(X,\underline{0})$ be a complex analytic germ of dimension $n+k$ in $\mathbb{C}^N$ and $f:(X,\underline{0})\to(\mathbb{C}^k,0)$ a holomorphic map-germ, stratified with respect to some complex analytic Whitney stratifications adapted to $V=V(f)$. Assume further that $\underline{0}\in\mathbb{C}^m$ is a stratum and f satisfies the Thom condition with respect to the above stratification. One has a local Milnor-Lê fibration. We let $\mathbb{B}_\varepsilon$ be a Milnor ball for X, small enough so that each stratum in $X\cap\mathbb{B}_\varepsilon$ has $\underline{0}$ in its closure. We let $F_f=f^{-1}(t)\cap B_\varepsilon$ be a Milnor fiber of f.

Let $g:(X,\underline{0})\to(\mathbb{C},0)$ be a germ with an isolated critical point relatively to $V(f)$ in the stratified sense, and that (f,g) satisfies the Thom condition with respect to the given stratification. We denote by $F_{f,g}$ the corresponding Milnor fiber. We equip the fibers F_f and $F_{f,g}$ with the stratification obtained by intersecting these fibers with the Whitney strata of X.

Just as in [3, Section 2], at each point x near the boundary ∂F_f we can project the gradient $\overline{\nabla}g(x)$ to the stratum that contains x, and glue all these vector fields together by a partition of unity in order to get a stratified vector field on U that we denote by $\overline{\nabla}g|_{F_f}$. Now we extend $\overline{\nabla}g|_{F_f}$ over the interior of F_f using radial extension as defined by M. H. Schwartz (see [2, Section 7] or [4, Section 2.3]), and let $\mathrm{Ind}(\overline{\nabla}g|_{F_f})$ be the sum of all the local Schwartz (or radial) indices of this extension.

Then the Lê-Greuel formula in [10] says:

$$\chi(F_f) - \chi(F_{f,g}) = \operatorname{Ind}(\overline{\nabla} g|_{F_f}) \,. \tag{7.2}$$

The problem now is computing the index $\operatorname{Ind}(\overline{\nabla} g|_{F_f})$ when F_f is singular.

If $k = 1$, V is equidimensional and g has an isolated critical point relative to f, then Theorem 5.13 implies:

$$\operatorname{Ind}(\overline{\nabla} g|_{F_f}) = \sum_{\alpha} \delta_\alpha m_\alpha \,,$$

where δ_α is the number of critical points in each stratum in F_f of a Morsification of g, and

$$m_\alpha = (-1)^{n_\alpha} \chi(\operatorname{Cone}(L_\alpha), L_\alpha) \,,$$

where L_α is the complex link of the stratum. Is this a particular case of a general theorem?

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