Fisher discord as a quantifier of quantum complexity

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Two classically equivalent expressions of mutual information of probability distributions (classical bipartite states) diverge when extended to quantum systems, and this difference has been employed to define quantum discord, a quantifier of quantum correlations beyond entanglement. Similarly, equivalent expressions of classical Fisher information of parameterized probability distributions diverge when extended to quantum states, and this difference may be exploited to characterize the complex nature of quantum states. By complexity of quantum states, we mean some hybrid nature which intermingles the classical and quantum features. It is desirable to quantify complexity of quantum states from various perspectives. In this work, we pursue the idea of discord and introduce an information-theoretic quantifier of complexity for quantum states (relative to the Hamiltonian that drives the evolution of quantum systems) via the notion of Fisher discord, which is defined by the difference between two important versions of quantum Fisher information: the quantum Fisher information defined via the symmetric logarithmic derivatives and the Wigner-Yanase skew information defined via the square roots of quantum states. We reveal basic properties of the quantifier of complexity, and compare it with some other quantifiers of complexity. In particular, we show that equilibrium states (or stable states, which commute with the Hamiltonian of the quantum system) and all pure states exhibit zero complexity in this setting. As illustrations, we evaluate the complexity for various prototypical states in both discrete and continuous-variable quantum systems.

Keywords: Classical Fisher information, quantum Fisher information, Wigner-Yanase skew information, Fisher discord, quantum complexity

I. INTRODUCTION

Complexity is a multifaceted and ubiquitous concept that quantifies the intricacy of systems and their divergence from idealization or simplification. It holds prominence in numerous realms of science [1–9], such as engineering [1, 2], financial [3], computer science [4], physics [5–8], and mathematics [9], and so on.

In quantum information theory, complexity is crucial for understanding the interplay between classicality and quantumness [10, 11]. A variety of quantifiers of complexity have been extensively investigated and applied [12–24], such as computational complexity [12, 13], algorithmic complexity [14], circuit complexity [15], statistical complexity [16–21], phase-space complexity [22], quantum phase transitions complexity [23], and so on [24]. Each quantifier of complexity offers unique perspectives. Complexity may carry different meanings in different contexts. In this work, we will regard a quantum state as simple if it is a quantum pure state, or a purely classical state in some sense, and regard it as possessing some complexity if it mixes both classical and quantum features. Given the intricate nature of complexity, it is desirable to characterize and quantify complexity in the quantum context from a wide range of angles.

Originated from statistical inference, the fundamental notion of Fisher information was first proposed by Fisher in 1925 as a quantifier of the information content of an observable random variable about an unknown parameter in a statistical model [25, 26]. Its quantum analogue, the quantum Fisher information, plays a crucial role in quantum metrology due to the celebrated Cramér-Rao inequality in quantum parameter estimation [27–40]. Recent studies have revealed that quantum Fisher information is closely linked to other aspects of quantum mechanics, such as entanglement [41, 42], quantum uncertainty relation [43–45], and quantum non-Markovianity [46], and quantum coherence [47, 48].

Unlike classical Fisher information, quantum Fisher information is not unique due to the intrinsic noncommutative nature of quantum mechanics. Various versions of quantum Fisher information have been introduced ever since 1960s [27–34]. The Wigner-Yanase skew information is a distinguished version of quantum Fisher information with respect to the time parameter encoded in the evolution of quantum state driven by the conserved quantity (Hamiltonian) [33, 34], which was originally introduced by Wigner and Yanase [27]. The skew information exhibits numerous notable properties and has become increasingly significant in quantum information theory [49–51]. It is a pivotal tool for characterizing quantum resources [52–61], such as coherence [52–54], correlations

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[55–57], asymmetry [58], uncertainty relation [59, 60], and nonclassicality [61].

In this work, motivated by the idea of quantum discord [62, 63], which is defined by the difference between two quantum natural extensions of equivalent expressions of the classical mutual information, and inspired by the non-uniqueness of quantum Fisher information as quantum extensions of the classical Fisher information [64, 65], we introduce an information-theoretic quantifier of complexity for quantum states via Fisher discord, defined as the difference between prominent versions of quantum Fisher information: One is defined via symmetric logarithmic derivative, and the other is the Wigner-Yanase skew information defined via square root of quantum states.

The remainder of the work is organized as follows: In Sec. II, we review two versions of quantum Fisher information, i.e., the quantum Fisher information based on the symmetric logarithmic derivative and the Wigner-Yanase skew information. We then propose the difference between them as a quantifier of complexity and investigate its properties in Sec. III. In Sec. IV, we evaluate it for paradigmatic Hamiltonians, and illustrate its utility in quantifying the complexity of qubit states and single-mode Bosonic states. We perform a comprehensive comparative analysis of our complexity quantifier against several existing quantifiers of complexity in Sec. V. Finally, we conclude with a summary in Sec. VI.

II. QUANTUM FISHER INFORMATION

Let us review the mathematical definition of Fisher information and its two natural extensions to the quantum context. As outlined in Refs. [25, 26], the Fisher information for a parametric classical probability distribution p_θ is defined as [25, 26, 34]

$$I_{\rm F}(p_\theta) = \int \Big(\frac{\partial}{\partial\theta}\ln p_\theta(x)\Big)^2 p_\theta(x){\rm d}x, \tag{1}$$

$$= 4\int \Big(\frac{\partial}{\partial\theta}\sqrt{p_\theta(x)}\Big)^2 {\rm d}x. \tag{2}$$

To extend classical Fisher information to quantum systems, parametric probability distributions p_θ are replaced with parametric density operators ρ_θ, and integration is substituted with the trace operation. Inspired by the following trivial identity

$$\frac{\partial}{\partial\theta}p_\theta = \frac{1}{2}(l_\theta p_\theta + p_\theta l_\theta) \tag{3}$$

for the statistical score function $l_\theta = \partial \ln p_\theta/\partial\theta$ in Eq. (1), Helstrom [28] first proposed the following quantum analogue of Fisher information

$$I_{\rm F}(\rho_\theta) = \frac{1}{4}\,{\rm tr}(\rho_\theta L_\theta^2),$$

where L_θ is the symmetric logarithmic derivative (SLD) of ρ_θ determined by

$$\frac{\partial}{\partial\theta}\rho_\theta = \frac{1}{2}(L_\theta\rho_\theta + \rho_\theta L_\theta), \qquad \theta\in\mathbb{R},$$

which is a quantum analogue of Eq. (3).

Another natural approach to quantum Fisher information is to utilize Eq. (2) to make the classical-to-quantum extension, and the following quantum analogue of classical Fisher information

$$I_{\rm H}(\rho_\theta) = 4{\rm tr}\Big(\frac{\partial}{\partial\theta}\sqrt{\rho_\theta}\Big)^2 \tag{4}$$

is obtained.

Let H be the Hamiltonian of a quantum system, which encodes interactions driving quantum state evolution. The dynamics of a quantum state, described by the density matrix ρ_θ, are governed by the Liouville-von Neumann equation

$$i\frac{\partial}{\partial\theta}\rho_\theta = [H,\rho_\theta], \qquad \theta\in\mathbb{R},$$

where $[X,Y]=XY-YX$ denotes commutator between operators X and Y. The solution of this equation is the quantum state obtained after the evolution

$$\rho_\theta = e^{-i\theta H}\rho e^{i\theta H}. \tag{5}$$

For pure state $\rho = |\psi\rangle\langle\psi|$, the Liouville-von Neumann equation reduces to the Schrödinger equation

$$i\frac{\partial}{\partial\theta}|\psi_\theta\rangle = H|\psi_\theta\rangle, \qquad \theta\in\mathbb{R},$$

with solution $|\psi_\theta\rangle = e^{-i\theta H}|\psi\rangle$. This equation is fundamental and important in quantum information, enabling predictions of state evolution in quantum systems.

It has been proven that $I_{\rm F}(\rho_\theta)$ is independent of θ when Eq. (5) holds, and thus simplifies to [34]

$$I_{\rm F}(\rho,H) = \frac{1}{4}\,{\rm tr}(\rho L^2), \tag{6}$$

where L satisfies

$$i[\rho,H] = \frac{1}{2}(L\rho+\rho L).$$

The quantum Fisher information $I_{\rm H}(\rho_\theta)$ defined by Eq. (4) is independent of the parameter θ if ρ_θ satisfies Eq. (5), and it holds that [34]

$$I_{\rm H}(\rho_\theta) = 8I_{\rm W}(\rho,H),$$

where

$$I_{\rm W}(\rho,H) = -\frac{1}{2}{\rm tr}[\sqrt{\rho},H]^2 \tag{7}$$

is the celebrated Wigner-Yanase skew information [27]. Thus, the skew information can be regarded as a version of quantum Fisher information. For pure states, it reduces to the variance. It is convex in state ρ, which is proven by Lieb in 1973 and instrumental in establishing the strong subadditivity of von Neumann entropy [49].

The relationship between $I_{\mathrm{F}}(\rho, H)$ and $I_{\mathrm{W}}(\rho, H)$ is given by

$$I_{\mathrm{W}}(\rho, H) \leq I_{\mathrm{F}}(\rho, H) \leq 2 I_{\mathrm{W}}(\rho, H), \tag{8}$$

where the first inequality is saturated, i.e. $I_{\mathrm{W}}(\rho, H) = I_{\mathrm{F}}(\rho, H)$, if ρ is pure or $[\rho, H] = 0$ [34]. Specifically, if $[\rho, H] = 0$, both $I_{\mathrm{W}}(\rho, H) = I_{\mathrm{F}}(\rho, H) = 0$, indicating no information about θ.

For any mixed state ρ with orthogonal spectral representation

$$\rho = \sum_m \lambda_m |\phi_m\rangle\langle\phi_m|, \tag{9}$$

where $\{\lambda_m\}$ are the different positive eigenvalues of ρ and $\{|\phi_m\rangle\}$ are the corresponding eigenstates forming an orthonormal base, the Wigner-Yanase skew information and the quantum Fisher information based on the symmetric logarithmic derivative can be explicitly calculated as [34]

$$I_{\mathrm{W}}(\rho, H) = \frac{1}{2} \sum_{m,n} (\lambda_m^{\frac{1}{2}} - \lambda_n^{\frac{1}{2}})^2 |\langle\phi_m|H|\phi_n\rangle|^2,$$
$$I_{\mathrm{F}}(\rho, H) = \frac{1}{2} \sum_{m,n} \frac{(\lambda_m - \lambda_n)^2}{\lambda_m + \lambda_n} |\langle\phi_m|H|\phi_n\rangle|^2.$$

In 2008, Hansen introduced the metric-adjusted skew information based on operator monotone functions [30, 32]. It is a generalized quantum Fisher information that encompasses the well-known Wigner-Yanase skew information defined by Eq. (7) and the quantum Fisher information based on the symmetric logarithmic derivative defined by Eq. (6) as two important special cases. As a versatile and significant measure of information content, the metric-adjusted skew information has attracted significant interest and found extensive applications in various fields, including uncertainty relations [44, 59, 66, 67], correlations [56, 68], quantum interference [69, 70], coherence [35], and asymmetry [71].

III. QUANTIFIER OF COMPLEXITY

Inspired by the Fisher discord, which is the difference between the quantum Fisher information based on the symmetric logarithmic derivative and the Wigner-Yanase skew information, we define

$$C(\rho, H) = I_{\mathrm{F}}(\rho, H) - I_{\mathrm{W}}(\rho, H) \tag{10}$$

as a quantifier for complexity of a quantum state ρ relative to a Hamiltonian H.

Proposition 1. The quantifier $C(\rho, H)$ for complexity has the following properties:

(1) $0 \leq C(\rho, H) \leq I_{\mathrm{W}}(\rho, H)$.
(2) For mixed states ρ with orthogonal spectral representation defined by Eq. (9), we have

$$C(\rho, H) = \sum_{m,n} \frac{\lambda_m^{\frac{1}{2}} \lambda_n^{\frac{1}{2}} (\lambda_m^{\frac{1}{2}} - \lambda_n^{\frac{1}{2}})^2}{\lambda_m + \lambda_n} |\langle\phi_m|H|\phi_n\rangle|^2. \tag{11}$$

(3) For any unitary operator U, it holds that

$$C(U\rho U^\dagger, H) = C(\rho, U^\dagger H U).$$

In particular,

$$C(e^{i\theta H} \rho e^{-i\theta H}, H) = C(\rho, H), \qquad \theta \in [0, 2\pi).$$

(4) For any $\omega \in \mathbb{R}$, it holds that

$$C(\rho, H + \omega \mathbf{1}) = C(\rho, H),$$
$$C(\rho, \omega H) = \omega^2 C(\rho, H),$$

where $\mathbf{1}$ denotes the identity operator. Moreover,

$$C(\rho, H_1 + H_2) + C(\rho, H_1 - H_2) = 2 \sum_{j=1,2} C(\rho, H_j)$$

for any two Hamiltonians H_1 and H_2 of the quantum system.
(5) $C(\rho, H)$ is additive under tensoring [32], in the sense that for any product states $\otimes_j \rho_j$ and composite Hamiltonian operators $H = \sum_j \omega_j H_j \otimes \mathbf{1}_{j^c}$, where $j^c := \{i : i \neq j\}$ and $\omega_j \in \mathbb{R}$, it holds that

$$C(\otimes_j \rho_j, H) = \sum_j \omega_j^2 C(\rho_j, H_j).$$

Particularly,

$$\begin{aligned} &C(\rho_1 \otimes \rho_2, H_1 \otimes \mathbf{1} + \mathbf{1} \otimes H_2) \\ &= C(\rho_1, H_1) + C(\rho_2, H_2), \end{aligned} \tag{12}$$

for any quantum states ρ_1 and ρ_2, and any Hamiltonian operators H_1 and H_2, on parties 1 and 2, respectively. In particular,

$$C(\rho_1 \otimes \rho_2, H_1 \otimes \mathbf{1}) = C(\rho_1, H_1). \tag{13}$$

(6) For any composite systems with subsystem Hamiltonian operators H_1 and H_2, it holds that

$$\begin{aligned} &C(\rho, H_1 \otimes \mathbf{1} + \mathbf{1} \otimes H_2) + C(\rho, H_1 \otimes \mathbf{1} - \mathbf{1} \otimes H_2) \\ &= 2C(\rho, H_1 \otimes \mathbf{1}) + 2C(\rho, \mathbf{1} \otimes H_2). \end{aligned}$$

(7) $C(\rho, H)$ is additive under direct sum in the sense that

$$C\Big(\bigoplus_i p_i \rho_i, \bigoplus_i H_i\Big) = \sum_i p_i C(\rho_i, H_i)$$

for any quantum states ρ_i, any Hamiltonian operators H_i, and any probability distribution $\{p_i\}$.

The above results follow readily from the properties of the quantum Fisher information $I_{\rm F}(\rho, H)$ and the skew information $I_{\rm W}(\rho, H)$ and direct verification.

Proposition 2. The complexity of a quantum state vanishes if the state is a pure state or a stable state (that is, a state that commutes with the Hamiltonian). In other words, $C(\rho, H) = 0$ if ρ is pure or $[\rho, H] = 0$.

We remark that pure states and stable states are merely sufficient but not necessary conditions for the disappearance of complexity in terms of Fisher discord. To see this, consider the mixture of the Bell state $|\psi_{0,1}\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and pure state $|2\rangle$ in a single-mode Bosonic system, that is

$$\rho = \frac{1}{2}|\psi_{0,1}\rangle\langle\psi_{0,1}| + \frac{1}{2}|2\rangle\langle 2|.$$

We find that

$$C(\rho, a^\dagger a) = 0,$$

while $[\rho, a^\dagger a] \neq 0$. The Hamiltonian $a^\dagger a$ considered here represents the number operator in quantum optics.

IV. ILLUSTRATIONS

In quantum mechanics, the Hamiltonian operator of a system represents the total energy of that system. Hamiltonian is of fundamental importance in quantum theory from its intimate link to the energy spectrum and the time evolution of quantum states via the Schrödinger equation [72–74]. Next, we illustrate the complexity quantifier $C(\rho, H)$ by evaluating it for several paradigmatic Hamiltonians in both discrete-variable systems and continuous-variable quantum systems, revealing quantitative insights of the quantum states from the perspective of their dynamical evolution.

A. Complexity of discrete-variable systems

In this subsection, we illustrate the complexity quantifier through a general qubit state and Hamiltonian, and reveal some quantitative features of complexity from the perspective of quantum dynamic.

Any qubit state can be represented in the basis $\{|0\rangle, |1\rangle\}$ as

$$\rho = \frac{1}{2}\Big(\mathbf{1} + \sum_{j=1}^{3} r_j\sigma_j\Big) = \frac{1}{2}\begin{pmatrix} 1+r_3 & r_1 - ir_2 \\ r_1 + ir_2 & 1 - r_3 \end{pmatrix},$$

where $r = \sqrt{\sum_{j=1}^{3} r_j^2} \leq 1, r_j \in \mathbb{R}$, and σ_j is the Pauli matrices for any $j = 1, 2, 3$. The spectral decomposition of ρ is

$$\rho = \lambda_1|\phi_1\rangle\langle\phi_1| + \lambda_2|\phi_2\rangle\langle\phi_2|,$$

with the eigenvectors

$$|\phi_1\rangle = \frac{1}{\sqrt{2r(r-r_3)}}\Big((r_1 - ir_2)|0\rangle - (r_3 - r)|1\rangle\Big),$$
$$|\phi_2\rangle = \frac{1}{\sqrt{2r(r+r_3)}}\Big((r_1 - ir_2)|0\rangle - (r_3 + r)|1\rangle\Big),$$

and the corresponding eigenvalues

$$\lambda_1 = \frac{1}{2}(1+r), \qquad \lambda_2 = \frac{1}{2}(1-r).$$

Let H be a Hamiltonian for a two-level quantum system. By Eq. (11), we have

$$C(\rho, H) = \frac{1}{2}\big(|r|^2 - 1 + \sqrt{1-|r|^2}\big)|\langle\phi_1|H|\phi_2\rangle|^2.$$

If $\langle\phi_1|H|\phi_2\rangle \neq 0$ which indicates $[\rho, H] \neq 0$ for any two dimensional system, then $C(\rho, H) = 0$ if and only if ρ is a pure state, i.e., $r = 1$.

The above results lead to the following proposition.

Proposition 3. The complexity of a qubit state vanishes if and only if it is a pure state or a stable state (that is, it commutes with the Hamiltonian). In other words, $C(\rho, H) = 0$ if and only if ρ is pure or $[\rho, H] = 0$.

B. Complexity of continuous-variable systems

A single-mode Bosonic field, fundamental in quantum optics, describes a quantized harmonic oscillator, such as a single electromagnetic field mode. It is characterized by the annihilation operator a and creation operator $a^\dagger$ satisfying the commutation relation

$$[a, a^\dagger] = \mathbf{1}.$$

To further illustrate the complexity quantifier based on Fisher discord, we evaluate the complexity of various Bosonic states relative to several prototypical Hamiltonians.

1. *Complexity relative to $a^\dagger a$*

First consider the Hamiltonian $H = a^\dagger a$. In quantum optics, $a^\dagger a$ represents the number operator, whose eigenstates correspond to the Fock states, i.e., $a^\dagger a|n\rangle = n|n\rangle$, $n = 0, 1\cdots$. This operator is fundamental for several reasons: (i) It governs the free evolution of a harmonic oscillator, describing the energy of the field mode (up to a constant). (ii) Its eigenstates, the Fock states $|n\rangle$, $n = 0, 1\cdots$, form a standard orthogonal basis for the Hilbert space and are critical for analyzing photon statistics. (iii) It underpins the definition of coherent and squeezed states, serving as a benchmark for classicality versus nonclassicality. (iv) It plays a central role in characterizing quantum optical phenomena, such as sub-Poissonian statistics and photon bunching. By evaluating $C(\rho, a^\dagger a)$, we gain insights into the complexity of the state ρ in terms of its photon number distribution and nonclassical features under number-conserving dynamics.

Apparently, for the rotation operators $U = e^{i\omega a^\dagger a}, \omega \in [0, 2\pi)$, we have

$$C(U\rho U^\dagger, a^\dagger a) = C(\rho, a^\dagger a).$$

However, $C(\rho, a^\dagger a)$ is not displacement invariant, except for the states defined by Eq. (9) satisfying $\lambda_m \lambda_{m+1} = 0$ for any m.

To further illustrate the complexity quantifier relative to the Hamiltonian $a^\dagger a$, we evaluate the complexity of several prototypical Bosonic states.

Proposition 4. The complexity relative to Hamiltonian $a^\dagger a$ of Fock-diagonal states, displaced Fock-diagonal states and squeezed Fock-diagonal states are as follows.
(1) For any Fock-diagonal state

$$\rho_{\mathrm{D}} = \sum_{n=0}^{\infty} \lambda_n |n\rangle\langle n|, \tag{14}$$

where $\lambda_n \geq 0$ for each n with $\sum_n \lambda_n = 1$, the amount of complexity is

$$C(\rho_{\mathrm{D}}, a^\dagger a) = 0.$$

(2) For any displaced Fock-diagonal state $D_z \rho_{\mathrm{D}} D_z^\dagger$, we have

$$C(D_z \rho_{\mathrm{D}} D_z^\dagger, a^\dagger a) = |z|^2 C(\rho_{\mathrm{D}}, X_\theta), \qquad \theta \in [0, 2\pi).$$

Here, $D_z = e^{z a^\dagger - z^* a}$, $z \in \mathbb{C}$ are displacement operators, and $X_\theta = e^{-i\theta} a + e^{i\theta} a^\dagger$, $\theta \in [0, 2\pi)$.
(3) For any squeezed diagonal state $S_\zeta \rho_{\mathrm{D}} S_\zeta^\dagger$, we have

$$\begin{aligned}
&C(S_\zeta \rho_{\mathrm{D}} S_\zeta^\dagger, a^\dagger a)\\
&= \frac{\sinh^2(2|\zeta|)}{4} C(\rho_{\mathrm{D}}, a^{\dagger 2} + a^2)\\
&= \frac{\sinh^2(2|\zeta|)}{2} \sum_{n=0}^{\infty} \frac{\lambda_n^{\frac{1}{2}} \lambda_{n+2}^{\frac{1}{2}} (\lambda_{n+2}^{\frac{1}{2}} - \lambda_n^{\frac{1}{2}})^2}{\lambda_{n+2} + \lambda_n} (n^2 + 3n + 2)\\
&= \frac{\sinh^2(2|\zeta|)}{4} C(\rho_{\mathrm{D}}, \Lambda_\theta),
\end{aligned}$$

for any $\theta \in [0, 2\pi)$. Here $S_\zeta = e^{(\zeta^* a^2 - \zeta a^{\dagger 2})/2}$, $\zeta = |\zeta| e^{i \arg \zeta} \in \mathbb{C}$ are squeezing operators, and $\Lambda_\theta = e^{-i\theta} a^2 + e^{i\theta} a^{\dagger 2}$, $\theta \in [0, 2\pi)$.

The above results can be directly verified. Next, we specify to the thermal state

$$\tau_\lambda = (1 - \lambda) \sum_{n=0}^{\infty} \lambda^n |n\rangle\langle n|, \qquad \lambda \in [0, 1) \tag{15}$$

as an important example of Fock-diagonal states.

For any displaced thermal states $D_z \tau_\lambda D_z^\dagger$, we have

$$C(D_z \tau_\lambda D_z^\dagger, a^\dagger a) = \frac{2|z|^2 \sqrt{\lambda}(1 - \sqrt{\lambda})^2}{1 - \lambda^2},$$

which monotonically increases in the displacement parameter $|z|$ for fixed λ, increases first and then decreases in the noise parameter λ for fixed z, and reaches its maximum value at $\lambda = \lambda_0$, where

$$\lambda_0 =: 2 + \sqrt{5} - 2\sqrt{2 + \sqrt{5}} \approx 0.1197. \tag{16}$$

For any squeezed thermal states $S_\zeta \tau_\lambda S_\zeta^\dagger$, we have

$$C(S_\zeta \tau_\lambda S_\zeta^\dagger, a^\dagger a) = \frac{\lambda \sinh^2(2|\zeta|)}{1 + \lambda^2},$$

which is monotonically increasing of the squeezing strength $|\zeta|$ for fixed λ, and of the noise parameter λ for fixed ζ. In particular, for thermal states ($\zeta = 0$, Fock-diagonal as mentioned above), we have $C(\tau_\lambda, a^\dagger a) = 0$.

For the Gaussian states $\rho_g = D_z S_\zeta \tau_\lambda S_\zeta^\dagger D_z^\dagger$, we have

$$C(\rho_g, a^\dagger a) = \frac{\lambda \sinh^2(2|\zeta|)}{1 + \lambda^2} + \frac{2\sqrt{\lambda}(1 - \sqrt{\lambda})^2 |\beta_{z,\zeta}|^2}{1 - \lambda^2},$$

where $\beta_{z,\zeta} = z \cosh|\zeta| - z^* e^{i \arg \zeta} \sinh|\zeta|$. For fixed λ and ζ, $C(\rho_g, a^\dagger a)$ monotonically increases in the displacement parameter $|z|$. For fixed λ, $|\zeta|$ and $|z|$, $C(\rho_g, a^\dagger a)$ monotonically increases in the parameter $(2 \arg z - \arg \zeta)$ in $[0, \pi)$, and monotonically decreases in $[\pi, 2\pi]$. For fixed λ, $(2 \arg z - \arg \zeta)$ and $|z|$, $C(\rho_g, a^\dagger a)$ monotonically increases in the squeezing parameter $|\zeta|$ if $\cos(2 \arg z - \arg \zeta) \leq 0$, and monotonically decreases first and then increases in the squeezing parameter $|\zeta|$ if $\cos(2 \arg z - \arg \zeta) > 0$.

The graphs of $C(\rho_g, a^\dagger a)$ versus $|\zeta|$ with fixed $|z| = 1$, $2 \arg z - \arg \zeta = 0$ and $\lambda = 0.01$, λ_0, 0.3, 0.8 are presented in Fig. 1, with λ_0 defined by Eq. (16), which reveal that $C(\rho_g, a^\dagger a)$ initially decreases monotonically and then increases in $|\zeta|$. However, the minimum point and its corresponding value vary from the parameters λ and $|z|$.

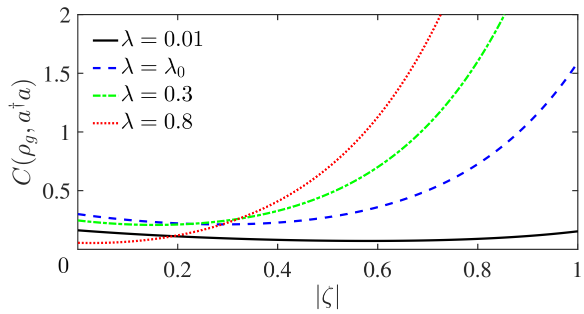

FIG. 1: Complexity $C(\rho_g, a^\dagger a)$ of Gaussian state ρ_g relative to Hamiltonian $a^\dagger a$ versus the squeezing parameter $|\zeta|$ for $\lambda = 0.01$ (black solid line), λ_0 (blue dashed line, defined by Eq. (16)), 0.3 (green dash-dotted line), and 0.8 (red dotted line) with fixed $|z| = 1$ and $2 \arg z - \arg \zeta = 0$.

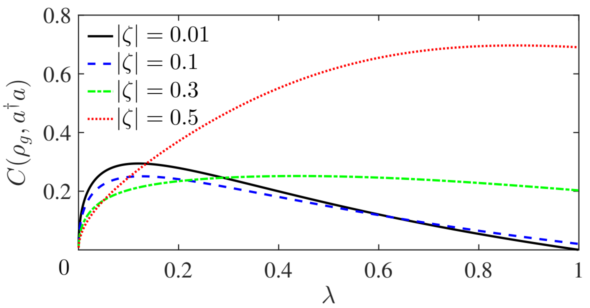

FIG. 2: Complexity $C(\rho_g, a^\dagger a)$ of Gaussian state ρ_g relative to Hamiltonian $a^\dagger a$ versus the noise parameter λ for $|\zeta| = 0.01$ (black solid line), 0.1 (blue dashed line), 0.3 (green dash-dotted line), 0.5 (red dotted line) with fixed $|z| = 1$ and $2\arg z - \arg\zeta = 0$.

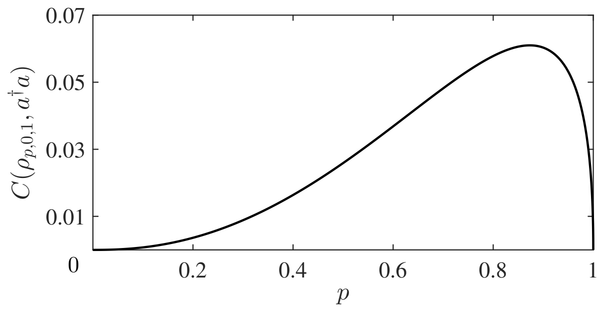

FIG. 3: Complexity $C(\rho_{p,0,1}, a^\dagger a)$ of the mixture state $\rho_{p,0,1}$ relative to Hamiltonian $a^\dagger a$ versus the parameter $p \in [0, 1]$.

For fixed z and ζ, $C(\rho_g, a^\dagger a)$ monotonically increases in the noise parameter λ in $(0, \lambda_0]$. When $\lambda > \lambda_0$, the monotonicity of $C(\rho_g, a^\dagger a)$ in parameter λ depends on the specific parameter values of z and ζ. The graphs of $C(\rho_g, a^\dagger a)$ versus λ with fixed $|z| = 1$, $2\arg z - \arg\zeta = 0$ and $|\zeta| = 0.01, 0.1, 0.3, 0.5$ are shown in Fig. 2, which show that $C(\rho_g, a^\dagger a)$ initially increases monotonically and then decreases in λ. However, the maximum point and its corresponding value vary depending on the parameters ζ and z.

For the mixture of the pure states $|\psi_{m,n}\rangle = (|m\rangle + |n\rangle)/\sqrt{2}$, $m, n \in \mathbb{Z}$, $m \neq n$ and vacuum state $|0\rangle$, that is

$$\rho_{p,m,n} = p|\psi_{m,n}\rangle\langle\psi_{m,n}| + (1-p)|0\rangle\langle 0| \tag{17}$$

with $p \in (0, 1)$, it holds that when $mn \neq 0$,

$$C(\rho_{p,m,n}, a^\dagger a) = 0,$$

and when $mn = 0$ ($n > 0$ without loss of generality),

$$C(\rho_{p,0,n}, a^\dagger a) = \frac{n^2 p^2 f_p}{4\big((1-p)^2 + p^2\big)},$$

where

$$f_p = \sqrt{2p(1-p)}\big(1 - \sqrt{2p(1-p)}\big). \tag{18}$$

$C(\rho_{p,0,n}, a^\dagger a)$ vanishes if $p = 0$ or 1 because the state $\rho_{p,0,n}$ degenerates into a Fock-diagonal state, monotonically increases in n for fixed p, increases first and then decreases related to p for fixed n, and reaches its maximum value at $p \approx 0.8731$. The complexity quantity $C(\rho_{p,0,1}, a^\dagger a)$ of $\rho_{p,0,1}$ in p is shown by Fig. 3. In particular, when $p = 1/2$,

$$C(\rho_{1/2,0,n}, a^\dagger a) = \frac{\sqrt{2}-1}{16} n^2, \qquad n > 0. \tag{19}$$

2. *Complexity relative to* $e^{-i\theta}a + e^{i\theta}a^\dagger$

Consider the complexity quantifier relative to the fixed Hamiltonian $X_\theta = e^{-i\theta}a + e^{i\theta}a^\dagger$, $\theta \in [0, 2\pi)$, which represents a rotated quadrature operator, generalizing the position and momentum observables defined as $X = (a+a^\dagger)/\sqrt{2} = X_0/\sqrt{2}$ and $P = (a-a^\dagger)/i\sqrt{2} = X_{\pi/2}/\sqrt{2}$. Hamiltonian X_θ is fundamental in quantum optics, corresponding to the homodyne rotated quadrature, and enabling measurements of field quadratures at any phase θ. This is crucial in homodyne detection, where the phase θ is tuned to probe position-like or momentum-like properties of the field. By evaluating $C(\rho, X_\theta)$, we can gain quantitative insights into the complexity of Bosonic state under quadrature-dependent dynamics.

The quantifier of complexity $C(\rho, X_\theta)$ relative to X_θ is invariant with respect to any displacement operator D_z in the sense that

$$C(D_z \rho D_z^\dagger, X_\theta) = C(\rho, X_\theta), \qquad z \in \mathbb{C},$$

which follows readily from

$$\begin{aligned} C(D_z \rho D_z^\dagger, X_\theta) &= C(\rho, D_z X_\theta D_z^\dagger) \\ &= C(\rho, X_\theta + 2\mathrm{Re}(e^{-i\theta} z)\mathbf{1}) \\ &= C(\rho, X_\theta). \end{aligned}$$

In contrast, $C(\rho, X_\theta)$ is not rotation invariant, as

$$C(e^{i\omega a^\dagger a} \rho e^{-i\omega a^\dagger a}, X_\theta) = C(\rho, X_{\theta+\omega}),$$

except for the displaced Fock-diagonal states.

In order to gain an intuitive understanding of complexity, we evaluate $C(\rho, X_\theta)$ for various Bosonic states.

Proposition 5.
(1) For the Fock-diagonal state ρ_{D} defined by Eq. (14), we have

$$C(\rho_{\mathrm{D}}, X_\theta) = 2\sum_{n=0}^{\infty} \frac{\lambda_n^{1/2}\lambda_{n+1}^{1/2}(\lambda_n^{1/2} - \lambda_{n+1}^{1/2})^2}{\lambda_n + \lambda_{n+1}}(n+1),$$

which is independent of parameter θ.
(2) For the squeezed diagonal state $S_\zeta \rho_{\mathrm{D}} S_\zeta^\dagger$, we have

$$C(S_\zeta \rho_{\mathrm{D}} S_\zeta^\dagger, X_\theta) = |\beta_{\theta,\zeta}|^2 C(\rho_{\mathrm{D}}, X_\theta),$$

where

$$\beta_{\theta,\zeta} = e^{i\theta}\cosh|\zeta| - e^{-i(\theta-\arg\zeta)}\sinh|\zeta|. \qquad (20)$$

Take some typical diagonal states and squeezed diagonal states for example.

For $\rho_p = p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|$, we have

$$C(\rho_p, X_\theta) = 2\sqrt{p(1-p)}\left(\sqrt{p}-\sqrt{1-p}\right)^2 =: g_p, \qquad (21)$$

which is shown in Fig. 4. Apparently, it is neither convex nor concave in p. Moreover, it is not monotonic in p, reaches its minimum value 0 at $p = 0, 1/2, 1$, and reaches its maximum value 1/4 at $p = (2\pm\sqrt{3})/4$.

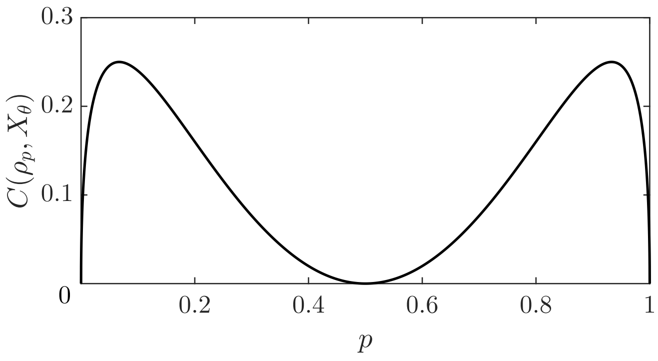

FIG. 4: Complexity $C(\rho_p, X_\theta) = g_p$ of the state ρ_p relative to Hamiltonian X_θ versus the parameter $p \in [0,1]$.

In contrast, for $\rho_{p,k} = p|0\rangle\langle 0| + (1-p)|k\rangle\langle k|,\ k \geq 2$, we have

$$C(\rho_{p,k}, X_\theta) = 0.$$

In general,

$$C(\rho_{p,k}, e^{-i\theta}a^l + e^{i\theta}a^{\dagger l}) = g_p k!\delta_{k,l},$$

where g_p is defined by Eq. (21), and $\delta_{k,l} = 1$ if $k = l$, otherwise, $\delta_{k,l} = 0$. This indicates that the complexity of the quantum state $\rho_{p,k}$ can be detected by the Hamiltonian $e^{-i\theta}a^l + e^{i\theta}a^{\dagger l}$ with $l = k$.

For any thermal states τ_λ defined by Eq. (15), we have

$$C(\tau_\lambda, X_\theta) = \frac{2\sqrt{\lambda}(1-\sqrt{\lambda})^2}{1-\lambda^2},$$

which is monotonically increasing first and then decreasing in the noise parameter λ, and reaches its maximum value 0.3003 when $\lambda = \lambda_0$ defined by Eq. (16).

For any squeezed thermal states $S_\zeta\tau_\lambda S_\zeta^\dagger$, we have

$$C(S_\zeta\tau_\lambda S_\zeta^\dagger, X_\theta) = \frac{2\sqrt{\lambda}(1-\sqrt{\lambda})^2|\beta_{\theta,\zeta}|^2}{1-\lambda^2},$$

where $\beta_{\theta,\zeta}$ is defined by Eq. (20).

For the truncated thermal state (no vacuum component $|0\rangle\langle 0|$)

$$\tau_\lambda^{\rm t} = \frac{1-\lambda}{\lambda}\sum_{n=1}^{\infty}\lambda^n|n\rangle\langle n|, \qquad \lambda \in (0,1),$$

we have

$$C(\tau_\lambda^{\rm t}, X_\theta) = \frac{2\sqrt{\lambda}(1-\sqrt{\lambda})^2(2-\lambda)}{1-\lambda^2},$$

which reaches its maximum value 0.5675 when $\lambda \approx 0.1014$.

For the photon-added thermal state (no vacuum component)

$$\tau_\lambda^{\rm pa} = \frac{a^\dagger\tau_\lambda a}{{\rm tr}a^\dagger\tau_\lambda a} = \frac{(1-\lambda)^2}{\lambda}\sum_{n=1}^{\infty}\lambda^n n|n\rangle\langle n|, \qquad \lambda \in (0,1),$$

we have

$$C(\tau_\lambda^{\rm pa}, X_\theta) = \frac{2(1-\lambda)^2}{\sqrt{\lambda}}\sum_{n=1}^{\infty}p_{\lambda,n}\lambda^n,$$

where

$$p_{\lambda,n} = \sqrt{n(n+1)^3}(\sqrt{(n+1)\lambda}-\sqrt{n})^2/\big((n+1)\lambda+n\big).$$

The comparison between the complexity $C(\tau_\lambda, X_\theta)$, $C(\tau_\lambda^{\rm t}, X_\theta)$ and $C(\tau_\lambda^{\rm pa}, X_\theta)$ as functions of λ is shown in Fig. 5. All three of them are monotonically increasing first and then monotonically decreasing in the noise parameter λ.

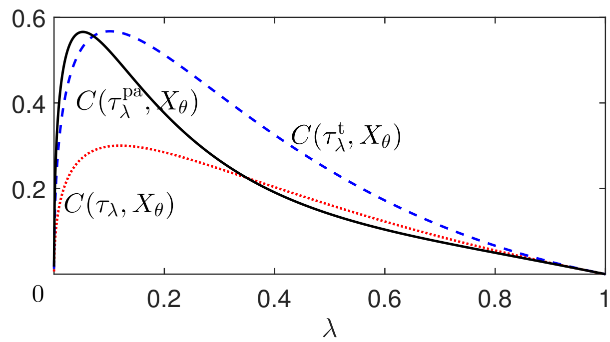

FIG. 5: Complexity $C(\tau_\lambda, X_\theta)$ for thermal state τ_λ (red dotted line), $C(\tau_\lambda^{\rm t}, X_\theta)$ for truncated thermal state $\tau_\lambda^{\rm t}$ (blue dashed line) and $C(\tau_\lambda^{\rm pa}, X_\theta)$ for photon-added thermal state $\tau_\lambda^{\rm pa}$ (black solid line) versus noise parameter $\lambda \in (0,1)$.

For the Gaussian states $\rho_g = D_z S_\zeta\tau_\lambda S_\zeta^\dagger D_z^\dagger$, we have

$$C(\rho_g, X_\theta) = C(S_\zeta\tau_\lambda S_\zeta^\dagger, X_\theta) = \frac{2\sqrt{\lambda}(1-\sqrt{\lambda})^2|\beta_{\theta,\zeta}|^2}{1-\lambda^2},$$

where $\beta_{\theta,\zeta}$ is defined by Eq. (20), which is independent of the displacement parameter z. For fixed θ and ζ, $C(\rho_g, X_\theta)$ initially increases monotonically and then decreases in the noise parameter λ, reaching a maximum value about $0.3|\beta_{\theta,\zeta}|^2$ at $\lambda = \lambda_0$. For fixed λ and $|\zeta|$, $C(\rho_g, X_\theta)$ monotonically increases in the parameter $\theta'_\zeta := 2\theta - \arg\zeta$ in $[0, \pi)$, and decreases in $[\pi, 2\pi]$. For fixed λ, $\arg\zeta$ and θ, $C(\rho_g, X_\theta)$ monotonically increases in the squeezing parameter $|\zeta|$ when $\cos\theta'_\zeta \le 0$ as shown by the green dash-dotted line and red dotted line in Fig. 6, and monotonically decreases when $\cos\theta'_\zeta = 1$ as shown by the black solid line in Fig. 6. However for fixed λ and $2\theta - \arg\zeta$, $C(\rho_g, X_\theta)$ initially decreases monotonically and then increases in $|\zeta|$ when $\cos\theta'_\zeta \in (0, 1)$, reaching its minimum value at $|\zeta| = \frac{1}{4}\ln\frac{1+\cos\theta'_\zeta}{1-\cos\theta'_\zeta}$, as exemplified by the blue dashed line in Fig. 6 with $\theta'_\zeta = \pi/4$, attaining its minimum value at $|\zeta| = \frac{1}{2}\ln(1+\sqrt{2}) \approx 0.4407$.

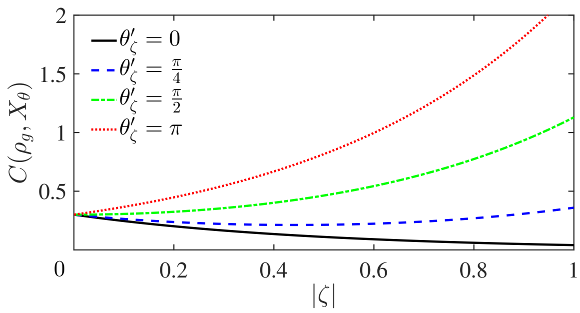

FIG. 6: Complexity $C(\rho_g, X_\theta)$ of Gaussian state ρ_g relative to Hamiltonian X_θ versus the squeezing parameter $|\zeta|$ for $\theta'_\zeta = 0$ (black solid line), $\pi/4$ (blue dashed line), $\pi/2$ (green dash-dotted line), π (red dotted line) with fixed $\lambda = \lambda_0$. Here, $\theta'_\zeta = 2\theta - \arg\zeta$.

For the mixed states $\rho_{p,m,n},\ m \neq n$ defined by Eq. (17), we have

$$\begin{aligned} C(\rho_{p,0,n}, X_\theta) &= \Big(1 - \frac{p^2\cos^2\theta}{2p^2 - 2p + 1}\Big) f_p \delta_{n,1}, && n > 0, \\ C(\rho_{p,1,n}, X_\theta) &= C(\rho_{p,n,1}, X_\theta) = g_p/2, && n > 1, \\ C(\rho_{p,m,n}, X_\theta) &= 0, && n > 1 \,\&\, m > 1, \end{aligned}$$

where f_p and g_p are defined by Eqs. (18) and (21) respectively. The graph of $C(\rho_{p,0,1}, X_\theta)$ versus the parameter p is shown in Fig. 7 when $\theta = 0, \pi/4, \pi/2$. $C(\rho_{p,0,1}, X_0)$ initially increases monotonically and then decreases in p, reaching its maximum about 0.2440 at $p \approx 0.1269$. $C(\rho_{p,0,1}, X_{\pi/4})$ initially increases monotonically and then decreases in p, reaching its maximum about 0.2468 at $p \approx 0.1351$. When $\theta = \pi/2$, $C(\rho_{p,0,1}, X_{\pi/2}) = f_p$ first increases monotonically in $[0, (2-\sqrt{2})/4)$, then decreases in $[(2-\sqrt{2})/4, 1/2)$, then increases again in $[1/2, (2+\sqrt{2})/4)$, and finally decreases once more in $[(2+\sqrt{2})/4, 1]$, and reaches its maximum $1/4$ at $p = (2 \pm \sqrt{2})/4$. In particular,

$$C(\rho_{1/2,0,n}, X_\theta) = \frac{1}{4}(\sqrt{2} - 1)(1 + \sin^2\theta)\delta_{n,1}, \tag{22}$$

in sharp contrast to the result shown in Eq. (19).

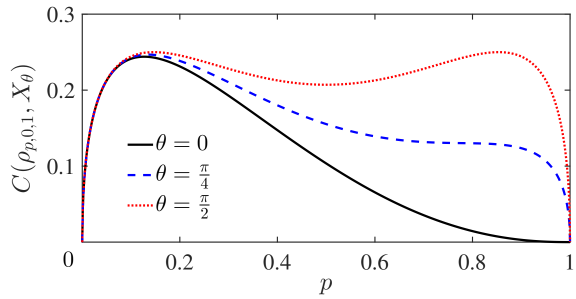

FIG. 7: Complexity of state $\rho_{p,0,1}$ relative to Hamiltonian X_θ versus $p \in (0, 1)$ when $\theta = 0$ (black solid line), $\pi/4$ (blue dashed line) and $\pi/2$ (red dotted line). Concretely, $X_0 = \sqrt{2}X$, $X_{\pi/4} = X + P$, $X_{\pi/2} = \sqrt{2}P$.

3. *Complexity relative to $e^{-i\theta}a^2 + e^{i\theta}a^{\dagger 2}$*

The Hamiltonian $\Lambda_\theta = e^{-i\theta}a^2 + e^{i\theta}a^{\dagger 2},\ \theta \in [0, 2\pi)$, is a quadratic operator. The significance of Λ_θ lies in its ability to generate squeezing transformations, critical for nonclassical state preparation and quantum technologies. Here we illustrate the complexity quantifier $C(\rho, \Lambda_\theta)$ relative to the Hamiltonian Λ_θ by several typical Bosonic states.

Proposition 6.

(1) For any Fock-diagonal state ρ_{D}, we have

$$C(\rho_{\mathrm{D}}, \Lambda_\theta) = 2\sum_{n=0}^{\infty} \frac{\lambda_n^{1/2}\lambda_{n+2}^{1/2}(\lambda_n^{1/2} - \lambda_{n+2}^{1/2})^2}{\lambda_n + \lambda_{n+2}}(n+1)(n+2),$$

which is independent of the parameter θ.

(2) It holds that

$$C(D_z\rho_{\mathrm{D}}D_z^\dagger, \Lambda_\theta) = C(\rho_{\mathrm{D}}, \Lambda_\theta) + 4|z|^2 C(\rho_{\mathrm{D}}, X_\theta),$$

for any $\theta \in [0, 2\pi)$.

(3) It holds that

$$C(S_\zeta\rho_{\mathrm{D}}S_\zeta^\dagger, \Lambda_\theta) = |\beta'_{\theta,\zeta}|^2 C(\rho_{\mathrm{D}}, \Lambda_\theta),$$

where

$$\beta'_{\theta,\zeta} = e^{i\theta}\cosh^2|\zeta| + e^{-i(\theta - 2\arg\zeta)}\sinh^2|\zeta|. \tag{23}$$

Next, we consider some more specific examples.

For $\rho_{p,k} = p|0\rangle\langle 0| + (1-p)|k\rangle\langle k|,\ k \ge 1$, we have

$$C(\rho_{p,k}, \Lambda_\theta) = 4\sqrt{p(1-p)}\big(1 - 2\sqrt{p(1-p)}\big)\delta_{k,2} = 2g_p\delta_{k,2}.$$

In particular, $C(\rho_{p,2}, \Lambda_\theta)$ is twice the value of $C(\rho_p, X_\theta)$ as shown in Fig. 4.

For any thermal states τ_λ defined by Eq. (15), we have

$$C(\tau_\lambda, \Lambda_\theta) = \frac{4\lambda}{1 + \lambda^2},$$

which monotonically increases in the noise parameter λ.

For the displaced thermal states $D_z\tau_\lambda D_z^\dagger$, we have

$$C(D_z\tau_\lambda D_z^\dagger, \Lambda_\theta) = \frac{8\sqrt{\lambda}(1-\sqrt{\lambda})^2}{1-\lambda^2}|z|^2 + \frac{4\lambda}{1+\lambda^2},$$

which monotonically increases in the displacement parameter $|z|$ for fixed λ.

For any squeezed thermal states $S_\zeta\tau_\lambda S_\zeta^\dagger$, we have

$$C(S_\zeta\tau_\lambda S_\zeta^\dagger, \Lambda_\theta) = \frac{4\lambda}{1+\lambda^2}|\beta'_{\theta,\zeta}|^2,$$

where $\beta'_{\theta,\zeta}$ is defined by Eq. (23). The complexity $C(S_\zeta\tau_\lambda S_\zeta^\dagger, \Lambda_\theta)$ is monotonically increasing in the squeezing strength $|\zeta|$ for fixed λ, θ, and in the parameter λ for fixed ζ, θ. For fixed λ and $|\zeta|$, $C(S_\zeta\tau_\lambda S_\zeta^\dagger, \Lambda_\theta)$ is monotonically decreasing in $\theta - \arg\zeta$ in $[k\pi, k\pi+\pi/2)$, and is monotonically increasing in $[k\pi+\pi/2, k\pi+\pi]$.

For the Gaussian states $\rho_g = D_z S_\zeta\tau_\lambda S_\zeta^\dagger D_z^\dagger$, we have

$$C(\rho_g, \Lambda_\theta) = \frac{4\lambda}{1+\lambda^2}|\beta'_{\theta,\zeta}|^2 + \frac{8\sqrt{\lambda}(1-\sqrt{\lambda})^2}{1-\lambda^2}|\beta_{\theta,z,\zeta}|^2,$$

where $\beta'_{\theta,\zeta}$ is defined by Eq. (23) and

$$\begin{aligned}\beta_{\theta,z,\zeta} &= z^* e^{i\theta}\cosh|\zeta| - z e^{-i\theta_\zeta}\sinh|\zeta|,\\ \theta_z &= \theta - 2\arg z,\\ \theta_\zeta &= \theta - \arg\zeta.\end{aligned}$$

For fixed λ, θ and ζ, $C(\rho_g, \Lambda_\theta)$ monotonically increases in the displacement parameter $|z|$. For fixed λ, θ, z and $\arg\zeta$, $C(\rho_g, \Lambda_\theta)$ monotonically increases in the squeezing parameter $|\zeta|$ when $\cos(\theta_z+\theta_\zeta) \le 0$, however monotonically decreases first and then increases in the squeezing parameter when $\cos(\theta_z+\theta_\zeta) > 0$. For fixed θ, z and ζ, $C(\rho_g, \Lambda_\theta)$ monotonically increases in the noise parameter λ when $\lambda \le \lambda_0$ defined by Eq. (16). However, for $\lambda > \lambda_0$, the monotonicity of $C(\rho_g, \Lambda_\theta)$ depends on the specific values of θ, z and ζ. Moreover, as indicated in Fig. 8, $C(\rho_g, \Lambda_\theta)$ does not have a consistent monotonicity in the parameter θ_ζ with fixed $|z| = |\zeta| = 1$, $\lambda = \lambda_0$ and $\theta_z = 0,\ \pi/4,\ \pi/2,\ \pi$.

For the mixed states $\rho_{p,m,n}$ defined by Eq. (17), when $m = 0$ or $n = 0$, we have

$$C(\rho_{p,0,n}, \Lambda_\theta) = C(\rho_{p,n,0}, \Lambda_\theta) = \left(2 - \frac{2p^2\cos^2\theta}{2p^2-2p+1}\right) f_p \delta_{n,2},$$

and

$$C(\rho_{p,n,m}, \Lambda_\theta) = C(\rho_{p,m,n}, \Lambda_\theta) = g_p\delta_{n,2},$$

where f_p and g_p are defined by Eqs. (18) and (21), respectively. In particular,

$$C(\rho_{1/2,0,n}, \Lambda_\theta) = \frac{1}{4}(\sqrt{2}-1)(3-\cos 2\theta)\delta_{n,2}, \qquad (24)$$

in sharp contrast to the result shown in Eqs. (19) and (22). It is easy to observe the following relationship

$$C(\rho_{p,0,2}, \Lambda_\theta) = 2C(\rho_{p,0,1}, X_\theta).$$

The graph of $C(\rho_{p,0,1}, X_\theta)$ is shown in Fig. 7.

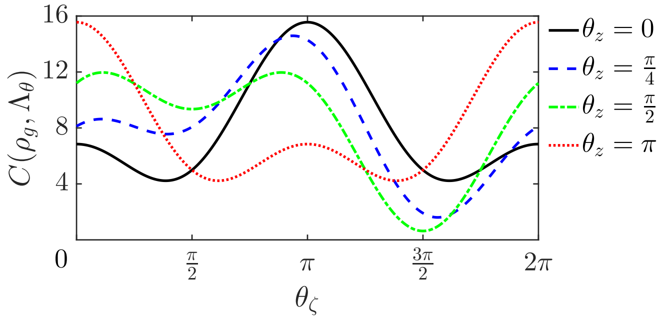

FIG. 8: Complexity $C(\rho_g, \Lambda_\theta)$ of Gaussian state ρ_g relative to Hamiltonian Λ_θ versus parameter $\theta_\zeta = \theta - \arg\zeta$ for $\theta_z = 0$ (black solid line), $\pi/4$ (blue dashed line), $\pi/2$ (green dash-dotted line), and π (red dotted line) with fixed $|z| = 1$, $|\zeta| = 1$ and $\lambda = \lambda_0$. Here $\theta_z = \theta - 2\arg z$.

V. COMPARISON

In this section, we compare our quantifier of complexity with those proposed by Manzano [75] and Tang *et al.* [22].

Recall that the quantifier of complexity proposed by Manzano is defined as the product of Fisher information and the entropic power of the one-dimensional probability distribution for the measurement outcomes of an observable [75]

$$\tilde{X}_t = X\cos t - P\sin t = X_{-t}/\sqrt{2}, \qquad t \in [0,\pi].$$

Denoting the eigenvectors and eigenvalues of operator $\tilde{X}_t$ by $\{|x_t\rangle\}$ and $\{x_t\}$ respectively, then the probability distribution of the measurement of $\tilde{X}_t$ in the pure state $|\psi\rangle$ is $q_t(x_t) = |\langle x_t|\psi\rangle|^2$, and the statistical complexity proposed by Manzano reads [75]

$$M_{\rm FS}(\psi|t) = I(q_t)J(q_t),$$

where $I(q_t)$ is the Fisher information of q_t with respect to the location, i.e., of $p_\theta(x) = q_t(x-\theta)$ as defined by Eq. (1), and

$$J(q_t) = \frac{1}{2\pi e}e^{2H(q_t)}$$

is the entropic power of Shannon entropy $H(q_t) = -\int_{-\infty}^{\infty} q_t(x_t)\ln q_t(x_t){\rm d}x_t$. $M_{\rm FS}(\psi|t)$ depends on the basis of the measurement, in other words, its dependence on the parameter t is non-trivial. Manzano obtained two quantifiers independent of the measurement basis by averaging and minimizing over the parameter t,

$$\begin{aligned}M_{\rm GFS}(\psi) &= \frac{1}{\pi}\int_0^\pi M_{\rm FS}(\psi|t){\rm d}t,\\ M_{\rm MFS}(\psi) &= \min_{t\in[0,\pi]} M_{\rm FS}(\psi|t),\end{aligned}$$

Quantifier of complexity	Minimum achieved by	System
$M_{\mathrm{GFS}}(\psi)$, $M_{\mathrm{MFS}}(\psi)$	Gaussian states	Pure, Continuous-variable
$C_{\mathrm{T}}(\rho)$	(displaced) thermal states	Continuous-variable
$C(\rho, H)$	pure and stable states	Discrete or continuous-variable

TABLE I: Comparison among three different quantifiers of complexity

which are called global Fisher-Shannon quantifier and minimum Fisher-Shannon quantifier of complexity, respectively. Both quantifiers are limited to pure states and reach their minimum values for Gaussian states.

The quantifier of complexity proposed by Tang *et al* is the product of Fisher information and Wehrl entropy power based on the Husimi quasiprobability distribution $Q(\alpha|\rho) = \langle\alpha|\rho|\alpha\rangle$ of any single-mode bosonic state ρ [22], that is

$$C_{\mathrm{T}}(\rho) = e^{S_{\mathrm{W}}(\rho)-1} I_{\mathrm{F}}(\rho),$$

where

$$I_{\mathrm{F}}(\rho) = \frac{1}{4}\int_{\mathbb{C}} \frac{||\nabla Q(\alpha|\rho)||^2}{Q(\alpha|\rho)} \frac{\mathrm{d}^2\alpha}{\pi},$$
$$S_{\mathrm{W}}(\rho) = -\int_{\mathbb{C}} Q(\alpha|\rho) \ln Q(\alpha|\rho) \frac{\mathrm{d}^2\alpha}{\pi},$$

which are the (classical) Fisher information of the Husimi function and the Wehrl entropy, respectively. Here $\mathrm{d}^2\alpha = \mathrm{d}x\mathrm{d}y$ (with $\alpha = x + iy$) is the Lebesgue measure on $\mathbb{C}$. The gradient operator ∇ is taken with respect to the real and imaginary parts of the complex number α. The Wehrl entropy $S_{\mathrm{W}}(\rho)$ represents the spread of ρ, while the Fisher information $I_{\mathrm{F}}(\rho)$ represents the localization of ρ in phase space. The complexity quantifier $C_{\mathrm{T}}(\rho)$ possesses some favorable properties, such as invariance under displacements and phase-space rotations, as well as under uniform phase-space scaling. The quantifier $C_{\mathrm{T}}(\rho)$ reaches its minimum value only for the (displaced) thermal states.

The quantifier of complexity we proposed is defined as the difference between two important versions of quantum Fisher information: the quantum Fisher information defined via the symmetric logarithmic derivatives and the Wigner-Yanase skew information, and incorporates the Hamiltonian to reflect the dynamic variation of quantum states. Unlike the quantifiers proposed by Manzano and Tang *et al*, our quantifier is not limited to continuous-variable quantum systems. It inherits several favorable properties of quantum Fisher information, such as additivity in the sense of Eq. (12).

Another important difference among the three complexities lies in the states with the lowest complexity. The quantifiers $M_{\mathrm{GFS}}(\psi)$ and $M_{\mathrm{MFS}}(\psi)$ of Manzano are based on the one-dimensional probability distribution obtained through measurement. In the one-dimensional variable case, Gaussian states have normal distributions, which saturate the isoperimetric inequality. Consequently, Gaussian states minimize Manzano's complexity. The quantifier $C_{\mathrm{T}}(\rho)$ is based on a two-dimensional probability distribution (i.e., Husimi quasiprobability distribution [76]). In the two-dimensional variable case, the (displaced) thermal states (without squeezing) follow normal distributions which also saturate the relevant isoperimetric inequality, leading them to minimize $C_{\mathrm{T}}(\rho)$ [22]. Our quantifier is based on the fact that quantum dynamics vanishes when the states are pure states or stable states, and can distinguish different Gaussian states.

We summarize the comparisons of the three quantifiers of complexity in Table I.

VI. SUMMARY

In this work, we have proposed a quantifier of complexity for quantum states based on Fisher discord. We have revealed some properties of this complexity quantifier. To illustrate the complexity quantifier, we have evaluated it in both discrete and continuous-variable quantum systems for some paradigmatic states.

We have compared our complexity quantifier with the Fisher-Shannon quantifier and the Fisher-Wehrl quantifier in the literature. Our findings highlight notable distinctions between these approaches. These quantifiers assess complexity from distinct perspectives, offering distinct strengths in various contexts. Further comprehensive studies comparing other complexity quantifiers would be highly beneficial.

Gibilisco *et al.* proved that the metric-adjusted skew information is dominated by the quantum Fisher information based on the symmetric logarithmic derivative [77]. It is also interesting to study the difference between these two (generalized) quantum Fisher information, which is a natural generalization of Fisher discord.

Both the quantum Fisher information based on the symmetric logarithmic derivative and the skew information does not increase under the partial traces, that is,

$$I_{\mathrm{F}}(\rho_{12}, H\otimes\mathbf{1}) \geq I_{\mathrm{F}}(\rho_1, H),$$
$$I_{\mathrm{W}}(\rho_{12}, H\otimes\mathbf{1}) \geq I_{\mathrm{W}}(\rho_1, H),$$

where ρ_{12} is a bipartite state of the composite system, $\rho_1 = \mathrm{tr}_2\rho_{12}$ is the reduced state of system 1 and H is an observable on system 1. It is natural to consider such an open question: Whether quantum Fisher discord

$$C(\rho_{12}, H\otimes\mathbf{1}) = I_{\mathrm{F}}(\rho_{12}, H\otimes\mathbf{1}) - I_{\mathrm{W}}(\rho_{12}, H\otimes\mathbf{1})$$

for bipartite states does not increase under the partial trace. If the answer is affirmative, it is worth investigating correlations of the bipartite states via the difference as $C(\rho_{12}, H\otimes\mathbf{1}) - C(\rho_1, H)$.

Given the significance of quantum Fisher information in metrology and information theory, there is compelling motivation to investigate the theoretical implications and practical applications of Fisher discord further. Applying these quantifiers to study correlations, interference, quantum metrology, and coherence holds significant promise, warranting deeper exploration.

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