

MINIMAL LAGRANGIAN SURFACES IN THE TWO DIMENSIONAL COMPLEX QUADRIC VIA THE LOOP GROUP METHOD

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ABSTRACT. We develop a loop group (DPW-type) representation for minimal Lagrangian surfaces in the complex quadric $Q_2 \cong \mathbb{S}^2 \times \mathbb{S}^2$, formulated via a flat family of connections $\{\nabla^\lambda\}_{\lambda \in \mathbb{S}^1}$ on a trivial bundle. We prove that minimality is equivalent to the flatness of ∇^λ for all λ , describe the associated isometric $\mathbb{S}^1$ -family, and establish a precise correspondence with minimal surfaces in $\mathbb{S}^3$ through their Gauss maps. Our framework unifies and streamlines earlier constructions (e.g., Castro–Urbano) and yields explicit families including $\mathbb{R}$ -equivariant, radially symmetric, and trinoid-type examples.

INTRODUCTION

Minimal surfaces in Kähler manifolds [5] play a fundamental role at the interface of complex and Riemannian geometry, and in the Lagrangian setting, they further connect to symplectic geometry [18]. Thus minimal Lagrangian surfaces in two-dimensional Kähler manifolds are of particular importance. Furthermore, the complex projective plane $\mathbb{C}P^2$ and the product $\mathbb{S}^2 \times \mathbb{S}^2$ are the only two-dimensional Hermitian symmetric spaces of compact type, and both are Kähler–Einstein surfaces. Indeed, I. Castro and F. Urbano studied minimal Lagrangian surfaces in $\mathbb{C}P^2$ [3] and in $\mathbb{S}^2 \times \mathbb{S}^2$ [4], revealing a variety of fundamental properties that highlight their geometric significance. Since $\mathbb{S}^2 \times \mathbb{S}^2$ is isometric to the complex quadric Q_2 in the complex projective space $\mathbb{C}P^3$, minimal (Lagrangian) surfaces in Q_2 are also of interest in algebraic geometry; see [16, 21].

Meanwhile, harmonic maps from a surface into symmetric spaces can be formulated within the framework of integrable systems, via the construction of a family of flat connections on a trivial principal G -bundle; see the loop group method of J. Dorfmeister, F. Pedit, and H. Wu [7] by exploiting loop group decompositions of infinite-dimensional Lie groups (the DPW method).

This method has been successfully applied to Hamiltonian stationary Lagrangian surfaces in $\mathbb{C}^2$ and $\mathbb{C}P^2$ [13, 14], as well as to minimal Lagrangian surfaces in $\mathbb{C}P^2$ [8, 9, 10]. It is therefore natural to extend this approach to minimal Lagrangian surfaces in the complex quadric Q_2 , thereby unveiling their structure via integrable-systems techniques.

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In this paper, we begin with a Lagrangian conformal immersion $f : M \rightarrow Q_2 \subset \mathbb{C}P^3$ from a Riemann surface M , together with a horizontal lift $\mathfrak{f} : \mathbb{D} \subset M \rightarrow \mathbb{S}^7 \subset \mathbb{C}^4$. We then consider a natural $\mathrm{SO}(4)$ -moving frame and its Maurer–Cartan form ω , which takes values in $\mathfrak{so}(4) = \mathrm{Lie}(\mathrm{SO}(4))$. Although the resulting structure equations are involved, we express them in terms of the invariants of $\mathfrak{f}$:

$$2e^u := \langle \mathfrak{f}_z, \bar{\mathfrak{f}}_z \rangle, \quad \alpha := \langle \mathfrak{f}_z, \mathfrak{f}_z \rangle, \quad \beta := \langle \mathfrak{f}_z, \mathfrak{f}_{\bar{z}} \rangle \quad \text{and} \quad \phi := e^{-u} \langle \mathfrak{f}_{z\bar{z}}, \bar{\mathfrak{f}}_{\bar{z}} \rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{C}^4$ and the subscripts z and $\bar{z}$ are the derivatives with respect to the conformal coordinate $z \in \mathbb{D}$ and its conjugate. Note that $ds^2 = 2e^u dzd\bar{z}$ is the induced metric and αdz^2 is the quadratic differential and the vanishing of ϕdz is equivalent to the minimality of f [21, Theorem 3.2].

Since Q_2 is a Hermitian symmetric space $\mathrm{SO}(4)/\mathrm{SO}(2) \times \mathrm{SO}(2)$, we can introduce a special family of connection 1-forms $d + \omega^\lambda$, parametrized by $\lambda \in \mathbb{S}^1$, such that $\omega^\lambda|_{\lambda=1}$ becomes the Maurer–Cartan form of $\mathfrak{f}$. In Theorem 1.4, we show that the flatness of $d + \omega^\lambda$ for all $\lambda \in \mathbb{S}^1$ is equivalent to minimality of a Lagrangian surface. Moreover, minimality is further equivalent to the conditions that the quadratic differential αdz^2 is holomorphic and that the argument of β is constant. Then we can take a new local horizontal lift so that the complex function β becomes a non-negative real function, according to Theorem 1.4.

Next, by choosing a suitable gauge transformation, we show the existence of a family of isometric minimal Lagrangian immersions $\{f^\lambda\}_{\lambda \in \mathbb{S}^1}$ (Theorem 1.8). Moreover, it turns out that the induced metric can be expressed in terms of a function $\hat{u}$ and the holomorphic function α as

$$2e^u dzd\bar{z} = \left(e^{\hat{u}} + |\alpha|^2 e^{-\hat{u}} \right) dzd\bar{z}.$$

where $\hat{u}$ satisfies the sinh-Gordon equation:

$$\hat{u}_{z\bar{z}} + e^{\hat{u}} - |\alpha|^2 e^{-\hat{u}} = 0,$$

It turns out that half the metric of f is induced by the Sasakian metric of the unit tangent bundle US^3 of a minimal surface in the unit three-sphere $\mathbb{S}^3$. Thus we establish a correspondence between the minimal Lagrangian surfaces in Q_2 and the Gauss maps of minimal surfaces in $\mathbb{S}^3$ (Theorem 1.9). This correspondence was previously shown under the additional condition $\alpha \neq 0$ in [4]. Since the structure equation, that is the sinh-Gordon equation, of a minimal Lagrangian surface in Q_2 coincides with those of a harmonic map from a Riemann surface into the unit two-sphere $\mathbb{S}^2$, this suggests that the loop group method can be applied to minimal Lagrangian surfaces in Q_2 through harmonic maps into $\mathbb{S}^2$. Indeed, by the well-known Lie group isomorphism

$$\mathrm{SO}(4) \cong (\mathrm{SU}(2) \times \mathrm{SU}(2)) / \mathbf{Z}_2,$$

this can be achieved (see Section 2).

In Section 3, we construct several examples of minimal Lagrangian surfaces via the loop group method: a family of totally geodesic spheres and totally geodesic tori. As new examples, we also construct families of $\mathbb{R}$ -equivariant surfaces, radially symmetric surfaces, and trinoids which are minimal Lagrangian immersions from a thrice-punctured sphere.

Main results. We demonstrate for the first time that the loop group method can be applied to minimal Lagrangian geometry in the complex quadric $Q_2 \subset \mathbb{C}P^3$, yielding new explicit families such as $\mathbb{R}$ -equivariant and radially symmetric surfaces and trinoids. Finally, we summarize the main results.

- (R1): Characterization (Theorem 1.4). For $f : \mathbb{D} \rightarrow Q_2$, f minimal $\iff d + \omega^\lambda$ are flat for all $\lambda \in \mathbb{S}^1 \iff \alpha$ holomorphic and $\arg \beta$ constant. After a normalized horizontal lift, we can assume $\beta \geq 0$; then $\hat{u}$ solves the sinh-Gordon equation.
- (R2): Associated family (Theorem 1.8). An isometric $\mathbb{S}^1$ -family $\{f^\lambda\}$ exists with the holomorphic quadratic differential $\alpha^\lambda = \lambda^{-2}\alpha$.
- (R3): Correspondence with $\mathbb{S}^3$ (Theorem 1.9). Minimal Lagrangian surfaces in Q_2 correspond to the Gauss maps of minimal surfaces in $\mathbb{S}^3$, without assuming $\alpha \neq 0$.
- (R4): The DPW method and new examples (Sections 2 and 3). The method has been established in Section 2. In Section 3, totally geodesic spheres/tori, and new families of examples: $\mathbb{R}$ -equivariant surfaces, radially symmetric surfaces, and trinoids, obtained by appropriate choices of holomorphic potentials.

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1. MINIMAL LAGRANGIAN SURFACES IN Q_2 , THE FAMILY OF FLAT CONNECTIONS AND MINIMAL SURFACES IN $\mathbb{S}^3$

In this section, we consider a Lagrangian immersion $f : M \rightarrow Q_2$ from a Riemann surface M into the complex quadric Q_2 in $\mathbb{C}P^3$. We explicitly compute the $SO(4)$ -moving frame and its Maurer-Cartan form and equation for f and then we study the special family of connections 1-forms $d + \omega^\lambda$, parametrized by $\lambda \in \mathbb{S}^1$ such that $\omega^\lambda|_{\lambda=1}$ is the Maurer-Cartan form of f . We show that the flatness of the whole family $d + \omega^\lambda$ is equivalent to the minimality of the Lagrangian surfaces f , Theorem 1.4, and thus there exists a one-parameter family of minimal Lagrangian surfaces, Theorem 1.8. Furthermore, we show a correspondence between minimal Lagrangian surfaces and minimal surfaces in $\mathbb{S}^3$, Theorem 1.9.

1.1. Lagrangian surfaces in the two dimensional quadric Q_2 . Let (N, ω) be a Kähler manifold of $\dim_{\mathbb{C}} N = n$ with the Kähler form ω . An immersion $f : M \rightarrow N$ from an m -dimensional manifold M into N is said to be *totally real* if $f^*\omega = 0$. In particular, a totally real immersion f is said to be *Lagrangian* if $m = n$.

Let $\mathbb{C}^n$ be the n -dimensional complex Euclidean space with the complex bilinear form $\langle \cdot, \cdot \rangle$ defined by

$$(1.1) \quad \langle \mathbf{z}, \mathbf{w} \rangle = \sum_{k=1}^n z_k w_k, \quad \text{for } \mathbf{z} = (z_1, \dots, z_n), \mathbf{w} = (w_1, \dots, w_n) \in \mathbb{C}^n.$$

The standard Hermitian inner product $(\cdot, \cdot)$ on $\mathbb{C}^n$ is given by $(\mathbf{z}, \mathbf{w}) = \langle \mathbf{z}, \bar{\mathbf{w}} \rangle$, where $\bar{\mathbf{w}}$ is the conjugate of $\mathbf{w}$. In this paper, three-dimensional complex projective space is denoted by $\mathbb{C}P^3 := \mathbb{C}^4 \setminus \{0\} / \sim$, and the complex quadric Q_2 in $\mathbb{C}P^3$ can be realized by

$$(1.2) \quad Q_2 = \{[\mathbf{z}] \in \mathbb{C}P^3 \mid \mathbf{z} \in \mathbb{C}^4, \langle \mathbf{z}, \mathbf{z} \rangle = 0\}.$$

We introduce the Fubini-Study metric of constant holomorphic sectional curvature 4 on $\mathbb{C}P^3$, which can be explicitly by

$$ds^2 = (dZ - \langle dZ, \bar{Z} \rangle Z) \otimes (d\bar{Z} - \langle d\bar{Z}, Z \rangle \bar{Z}),$$

where Z in $\mathbb{S}^7 \subset \mathbb{C}^4$ is a local holomorphic section of the tautological bundle, and thus $\mathbb{C}P^3$ becomes a Kähler manifold. The metric on Q_2 is naturally induced by the Fubini-Study metric, and thus Q_2 becomes a Kähler surface. It is well known that Q_2 is isometric to $\mathbb{S}^2 \times \mathbb{S}^2$, where the curvatures of the two 2-sphere $\mathbb{S}^2$ are normalized to 4, [4, 22].

Let $f : M \rightarrow Q_2$ be a Lagrangian conformal immersion from a Riemann surface M into Q_2 . Moreover, let $\mathbb{D} \subset M$ be a simply connected domain with conformal coordinate $z = x + iy$. Then the induced metric on $\mathbb{D}$ can be computed as

$$ds_M^2 = 2e^u dz d\bar{z}.$$

Let $\mathbf{f} : \mathbb{D} \rightarrow \mathbb{S}^7 \subset \mathbb{C}^4$ be a local lift of f , i.e. $f = \pi \circ \mathbf{f}$, where $\pi : \mathbb{S}^7 \rightarrow \mathbb{C}P^3$ is the Hopf fibration. In fact, the projection f can be realized as $[\mathbf{f}]$. Since f is conformal and Lagrangian, we obtain

$$\langle \mathbf{f}_z, \bar{\mathbf{f}}_{\bar{z}} \rangle = 0, \quad \langle \mathbf{f}_z, \bar{\mathbf{f}}_z \rangle = \langle \mathbf{f}_{\bar{z}}, \bar{\mathbf{f}}_{\bar{z}} \rangle = e^u.$$

Here $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ and $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$ are the complex differentiations, and the imaginary unit $\sqrt{-1}$ is denoted by i . Since $f(M) \subset Q_2$, we have

$$\langle \mathbf{f}, \mathbf{f} \rangle = 0, \quad \langle \mathbf{f}_z, \mathbf{f} \rangle = \langle \mathbf{f}_{\bar{z}}, \mathbf{f} \rangle = 0.$$

Definition 1.1. If a local lift $\mathbf{f}$ defined above satisfies

$$\langle \mathbf{f}_z, \bar{\mathbf{f}} \rangle = \langle \mathbf{f}_{\bar{z}}, \bar{\mathbf{f}} \rangle = 0,$$

then we call $\mathbf{f}$ a *horizontal lift*.

Since horizontal lifts of f are not unique, we fix one horizontal lift $\mathbf{f}$ for time being. Since the special orthogonal Lie group $\text{SO}(4) = \{A \in M_{4 \times 4}(\mathbb{R}) \mid A^T = A^{-1}, \det A = 1\}$ acts transitively on Q_2 as the orientation-preserving isometry, Q_2 is isomorphic to the symmetric space:

$$(1.3) \quad Q_2 = \text{SO}(4)/\text{SO}(2) \times \text{SO}(2),$$

see [23]. Indeed, by choosing the involution $\sigma = \text{Ad} \text{diag}(1, 1, -1, -1)$ on $\text{SO}(4)$, the fixed point set of σ is exactly $\text{SO}(2) \times \text{SO}(2)$. Let $f : M \rightarrow Q_2 = \text{SO}(4)/\text{SO}(2) \times \text{SO}(2)$ and let $\mathcal{F}$ be a local lift of f as

$$\mathcal{F} := \left(\frac{1}{\sqrt{2}} (\mathbf{f} + \bar{\mathbf{f}}), -\frac{i}{\sqrt{2}} (\mathbf{f} - \bar{\mathbf{f}}), e_3, e_4 \right) : \mathbb{D} \subset M \rightarrow \text{SO}(4),$$

where $\mathbf{f}$ is a horizontal lift defined above, and e_3 and e_4 are defined by

$$e_3 = \frac{\mathbf{f}_z + \bar{\mathbf{f}}_{\bar{z}}}{\sqrt{2e^u + \alpha + \bar{\alpha}}}, \quad e_4 = -\frac{i \left\{ \mathbf{f}_z (e^u + \bar{\alpha}) - \bar{\mathbf{f}}_{\bar{z}} (e^u + \alpha) \right\}}{\sqrt{(2e^u + \alpha + \bar{\alpha})(e^{2u} - \alpha\bar{\alpha})}},$$

with

$$(1.4) \quad \alpha := \langle \mathbf{f}_z, \mathbf{f}_z \rangle.$$

By direct computation $e^{2u} - \alpha\bar{\alpha} \geq 0$ and we assume that $e^{2u} - \alpha\bar{\alpha} > 0$ for the time being so that the frame $\mathcal{F}$ is well-defined. Note that this assumption is not necessary for minimal

Lagrangian surfaces, as we will see later, see Remark 1.5. Its Maurer-Cartan form can be computed as follows:

$$(1.5) \quad \omega = \mathcal{F}^{-1}d\mathcal{F} = \mathcal{F}^{-1}\mathcal{F}_z dz + \mathcal{F}^{-1}\mathcal{F}_{\bar{z}} d\bar{z} = \mathcal{U}dz + \mathcal{V}d\bar{z},$$

where

$$(1.6) \quad \mathcal{U} = \begin{pmatrix} 0 & 0 & p_1 & p_2 \\ 0 & 0 & p_3 & p_4 \\ -p_1 & -p_3 & 0 & q \\ -p_2 & -p_4 & -q & 0 \end{pmatrix}, \quad \mathcal{V} = \begin{pmatrix} 0 & 0 & \bar{p}_1 & \bar{p}_2 \\ 0 & 0 & \bar{p}_3 & \bar{p}_4 \\ -\bar{p}_1 & -\bar{p}_3 & 0 & \bar{q} \\ -\bar{p}_2 & -\bar{p}_4 & -\bar{q} & 0 \end{pmatrix},$$

$$\begin{cases} p_1 = \frac{1}{\sqrt{2}} (\langle e_{3z}, \mathbf{f} \rangle + \langle e_{3z}, \bar{\mathbf{f}} \rangle) = -\frac{1}{\sqrt{2}} \left(\frac{\alpha + e^u + \bar{\beta}}{\sqrt{2e^u + \alpha + \bar{\alpha}}} \right), \\ p_2 = \frac{1}{\sqrt{2}} (\langle e_{4z}, \mathbf{f} \rangle + \langle e_{4z}, \bar{\mathbf{f}} \rangle) = \frac{i}{\sqrt{2}} \left[\frac{(\alpha\bar{\alpha} - e^{2u}) - \bar{\beta}(e^u + \alpha)}{\sqrt{(2e^u + \alpha + \bar{\alpha})(e^{2u} - \alpha\bar{\alpha})}} \right], \\ p_3 = -\frac{i}{\sqrt{2}} (\langle e_{3z}, \mathbf{f} \rangle - \langle e_{3z}, \bar{\mathbf{f}} \rangle) = \frac{i}{\sqrt{2}} \left(\frac{\alpha + e^u - \bar{\beta}}{\sqrt{2e^u + \alpha + \bar{\alpha}}} \right), \\ p_4 = -\frac{i}{\sqrt{2}} (\langle e_{4z}, \mathbf{f} \rangle - \langle e_{4z}, \bar{\mathbf{f}} \rangle) = \frac{1}{\sqrt{2}} \left[\frac{(\alpha\bar{\alpha} - e^{2u}) + \bar{\beta}(e^u + \alpha)}{\sqrt{(2e^u + \alpha + \bar{\alpha})(e^{2u} - \alpha\bar{\alpha})}} \right], \\ q = \langle e_{4z}, e_3 \rangle = i \left[\frac{\frac{1}{2}e^u(\alpha_z - \bar{\alpha}_z) + \frac{1}{2}(\alpha_z\bar{\alpha} - \bar{\alpha}_z\alpha) - e^u\phi(2e^u + \bar{\alpha} + \alpha) - u_z e^u(e^u + \alpha)}{(2e^u + \alpha + \bar{\alpha})\sqrt{e^{2u} - \alpha\bar{\alpha}}} \right], \end{cases}$$

and

$$(1.7) \quad \beta := \langle \mathbf{f}_z, \mathbf{f}_{\bar{z}} \rangle, \quad \phi := e^{-u} \langle \mathbf{f}_{z\bar{z}}, \bar{\mathbf{f}}_{\bar{z}} \rangle.$$

From the above expressions, it is easy to see that αdz^2 , $\beta dzd\bar{z}$ and ϕdz are well-defined differentials for the horizontal lift $\mathbf{f}$. Moreover it is known that

$$\Phi = \phi dz$$

characterizes minimality of the surface, i.e., $\Phi = 0$ if and only if the surface f is minimal [21, Theorem 3.2].

It is clear that the compatibility condition of $\mathcal{F}$, that is, $\mathcal{F}_{z\bar{z}} = \mathcal{F}_{\bar{z}z}$ is equivalent to $\mathcal{U}_{\bar{z}} - \mathcal{V}_z = [\mathcal{U}, \mathcal{V}]$, which is also equivalent to the Maurer-Cartan equation $d\omega + \frac{1}{2}[\omega \wedge \omega] = 0$. Moreover, it can also be interpreted as a flatness of the connection $d + \omega$. Then a straightforward computation shows that the Maurer-Cartan equation $d\omega + \frac{1}{2}[\omega \wedge \omega] = 0$ is equivalent to the following equations:

$$\begin{aligned} (1.8a) & \quad p_1\bar{p}_3 + p_2\bar{p}_4 = \bar{p}_1p_3 + \bar{p}_2p_4, \\ (1.8b) & \quad \bar{p}_2q - p_2\bar{q} = p_1\bar{z} - \bar{p}_1z, \\ (1.8c) & \quad p_1\bar{q} - \bar{p}_1q = p_2\bar{z} - \bar{p}_2z, \\ (1.8d) & \quad \bar{p}_4q - p_4\bar{q} = p_3\bar{z} - \bar{p}_3z, \\ (1.8e) & \quad p_3\bar{q} - \bar{p}_3q = p_4\bar{z} - \bar{p}_4z, \\ (1.8f) & \quad p_2\bar{p}_1 + p_4\bar{p}_3 - \bar{p}_2p_1 - \bar{p}_4p_3 = q_z - \bar{q}_z. \end{aligned}$$

By substituting p_1, p_2, p_3, p_4 and q , (1.8a) can be simplified as follows:

$$(1.9) \quad e^{2u} - \beta\bar{\beta} = \alpha\bar{\alpha}.$$

From the above equation, the assumption $e^{2u} - \alpha\bar{\alpha} > 0$ is equivalent to $|\beta| > 0$. And (1.8b) and (1.8d) are equivalent to the following equation:

$$\frac{\sqrt{2}}{2}(\bar{p}_2 - i\bar{p}_4)q + \frac{\sqrt{2}}{2}(\bar{p}_1 - i\bar{p}_3)_z = \frac{\sqrt{2}}{2}(p_1 - ip_3)_{\bar{z}} + \frac{\sqrt{2}}{2}(p_2 - ip_4)\bar{q}.$$

By substituting p_1, p_2, p_3, p_4 and q , we have

$$(1.10) \quad \frac{1}{2}\bar{\alpha}_{\bar{z}}\alpha - e^{2u}\bar{\phi} - u_{\bar{z}}e^{2u} + \beta\bar{\beta}_{\bar{z}} = \frac{1}{2}\bar{\alpha}_z\beta.$$

Similarly, (1.8c) and (1.8e) are equivalent to the following equation:

$$\frac{\sqrt{2}}{2}(p_1 - ip_3)\bar{q} - \frac{\sqrt{2}}{2}(p_2 - ip_4)_{\bar{z}} = \frac{\sqrt{2}}{2}(\bar{p}_1 - i\bar{p}_3)q - \frac{\sqrt{2}}{2}(\bar{p}_2 - i\bar{p}_4)_z.$$

By substituting p_1, p_2, p_3, p_4 and q , we obtain

$$(1.11) \quad \phi\beta = \bar{\phi}\alpha + \frac{1}{2}\alpha_{\bar{z}}.$$

Note that (1.8f) can also be expressed by the function α , β and ϕ , however, the expression is rather involved and will not be used later, so we omit it here.

1.2. Minimality of a Lagrangian surface and the family of flat connections. Since Q_2 is a symmetric space as in (1.3) and minimal surfaces in Q_2 can be thought as conformal harmonic maps, thus the integrable system method can be applied. We consider the following family of connection 1-forms $d + \omega^\lambda$:

$$(1.12) \quad \omega^\lambda = \lambda^{-1}\omega'_{\mathfrak{p}} + \omega_{\mathfrak{k}} + \lambda\omega''_{\mathfrak{p}}, \quad (\lambda \in \mathbb{S}^1),$$

where $\mathfrak{g} = \text{Lie}(\text{SO}(4)) = \mathfrak{so}(4)$ can be decomposed as

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

with the fixed point subalgebra $\mathfrak{k} = \text{Fix}(d\sigma) = \mathfrak{so}(2) \times \mathfrak{so}(2)$ and its complement $\mathfrak{p}$, and $\omega_{\mathfrak{k}}$ and $\omega_{\mathfrak{p}}$ are the $\mathfrak{k}$ - and the $\mathfrak{p}$ -valued 1-forms. Moreover $'$ and $''$ denote the $(1,0)$ - and the $(0,1)$ -parts, respectively. Since the flatness of the connection $d + \omega$ is only the flatness of the family $d + \omega^\lambda$ at $\lambda = 1$, the flatness of a family $d + \omega^\lambda$ parameterized by $\lambda \in \mathbb{S}^1$ gives an additional constraint, that is, harmonicity, for a Lagrangian surface f .

More explicitly, (1.12) can be written as follows:

$$\omega^\lambda = \mathcal{U}^\lambda dz + \mathcal{V}^\lambda d\bar{z},$$

where

$$\mathcal{U}^\lambda = \begin{pmatrix} 0 & 0 & \lambda^{-1}p_1 & \lambda^{-1}p_2 \\ 0 & 0 & \lambda^{-1}p_3 & \lambda^{-1}p_4 \\ -\lambda^{-1}p_1 & -\lambda^{-1}p_3 & 0 & q \\ -\lambda^{-1}p_2 & -\lambda^{-1}p_4 & -q & 0 \end{pmatrix}, \quad \mathcal{V}^\lambda = \begin{pmatrix} 0 & 0 & \lambda\bar{p}_1 & \lambda\bar{p}_2 \\ 0 & 0 & \lambda\bar{p}_3 & \lambda\bar{p}_4 \\ -\lambda\bar{p}_1 & -\lambda\bar{p}_3 & 0 & \bar{q} \\ -\lambda\bar{p}_2 & -\lambda\bar{p}_4 & -\bar{q} & 0 \end{pmatrix}.$$

The condition $d\omega^\lambda + \frac{1}{2}[\omega^\lambda \wedge \omega^\lambda] = 0$ is now equivalent to $\mathcal{U}_z^\lambda - \mathcal{V}_z^\lambda = [\mathcal{U}^\lambda, \mathcal{V}^\lambda]$, and it is equivalent to the following equations:

$$\begin{aligned} (1.13a) & \quad p_1\bar{p}_3 + p_2\bar{p}_4 = \bar{p}_1p_3 + \bar{p}_2p_4, \\ (1.13b) & \quad \lambda\bar{p}_2q - \lambda^{-1}p_2\bar{q} = \lambda^{-1}p_{1\bar{z}} - \lambda\bar{p}_{1z}, \\ (1.13c) & \quad \lambda^{-1}p_1\bar{q} - \lambda\bar{p}_1q = \lambda^{-1}p_{2\bar{z}} - \lambda\bar{p}_{2z}, \\ (1.13d) & \quad \lambda\bar{p}_4q - \lambda^{-1}p_4\bar{q} = \lambda^{-1}p_{3\bar{z}} - \lambda\bar{p}_{3z}, \\ (1.13e) & \quad \lambda^{-1}p_3\bar{q} - \lambda\bar{p}_3q = \lambda^{-1}p_{4\bar{z}} - \lambda\bar{p}_{4z}, \\ (1.13f) & \quad p_2\bar{p}_1 + p_4\bar{p}_3 - \bar{p}_2p_1 - \bar{p}_4p_3 = q_{\bar{z}} - \bar{q}_z. \end{aligned}$$

First, (1.13a) is just (1.8a) and it is $e^{2u} - \beta\bar{\beta} = \alpha\bar{\alpha}$. Next, (1.13b) and (1.13d) are equivalent to the following equations:

$$\frac{\sqrt{2}}{2}(\bar{p}_2 - i\bar{p}_4)q = -\frac{\sqrt{2}}{2}(\bar{p}_1 - i\bar{p}_3)_z, \quad \frac{\sqrt{2}}{2}(\bar{p}_2 + i\bar{p}_4)q = -\frac{\sqrt{2}}{2}(\bar{p}_1 + i\bar{p}_3)_z.$$

By substituting p_1, p_2, p_3, p_4 and q , the following equations are obtained:

$$(1.14) \quad \frac{1}{2}\bar{\alpha}_z = e^u\phi,$$

$$(1.15) \quad \frac{1}{2}\bar{\alpha}(\bar{\alpha}_z - \alpha_z) + e^u\bar{\alpha}_z + u_z e^{2u} = \bar{\beta}\beta_z.$$

Similarly, (1.13c) and (1.13e) are equivalent to the following equations:

$$\frac{\sqrt{2}}{2}(\bar{p}_1 + i\bar{p}_3)q = \frac{\sqrt{2}}{2}(\bar{p}_2 + i\bar{p}_4)_z, \quad \frac{\sqrt{2}}{2}(\bar{p}_1 - i\bar{p}_3)q = \frac{\sqrt{2}}{2}(\bar{p}_2 - i\bar{p}_4)_z.$$

By substituting p_1, p_2, p_3, p_4 and q , we have $\bar{\alpha}_z = 0$. Thus (1.14) and (1.15) can be simplified as follows:

$$(1.16) \quad \alpha_{\bar{z}} = \bar{\alpha}_z = \phi = 0,$$

$$(1.17) \quad \frac{1}{2}\bar{\alpha}\alpha_z + \bar{\beta}\beta_z - u_z e^{2u} = 0.$$

In addition, the equation (1.13f) can be simplified as follows:

$$(1.18) \quad u_{z\bar{z}}e^u|\beta|^2 - \frac{1}{4}|\alpha_z|^2e^u - |u_z|^2e^u|\alpha|^2 + \frac{1}{2}\alpha_z u_{\bar{z}}e^u\bar{\alpha} + \frac{1}{2}\bar{\alpha}_{\bar{z}}u_z e^u\alpha + 2|\beta|^4 = 0.$$

Thus (1.13a)-(1.13f) can be simplified to the following system of equations:

$$(1.19) \quad \begin{cases} e^{2u} - \beta\bar{\beta} = \alpha\bar{\alpha}, \\ \alpha_{\bar{z}} = \bar{\alpha}_z = \phi = 0, \\ \frac{1}{2}\bar{\alpha}\alpha_z + \bar{\beta}\beta_z - u_z e^{2u} = 0, \\ u_{z\bar{z}}e^u|\beta|^2 - \frac{1}{4}|\alpha_z|^2e^u - |u_z|^2e^u|\alpha|^2 + \frac{1}{2}\alpha_z u_{\bar{z}}e^u\bar{\alpha} + \frac{1}{2}\bar{\alpha}_{\bar{z}}u_z e^u\alpha + 2|\beta|^4 = 0. \end{cases}$$

Without loss of generality, we assume that α and β are not identically zero.

Lemma 1.2. *When $\alpha \equiv 0$ (resp. $\beta \equiv 0$), then the minimal Lagrangian surface is the totally geodesic Lagrangian sphere (resp. totally geodesic flat Lagrangian torus).*

Proof. When $\alpha \equiv 0$, the equation (1.18) can be simplified to

$$(1.20) \quad u_{z\bar{z}} + 2e^u = 0,$$

which is known as Liouville's equation, see [17]. Since the induced metric on $\mathbb{D}$ is given by $ds_M^2 = 2e^u dz d\bar{z}$, the Gaussian curvature K can be computed by $K = -e^{-u} u_{z\bar{z}}$. Then by (1.20), the Gaussian curvature $K = 2$ when $\alpha \equiv 0$.

On the contrary, when $\beta \equiv 0$, we have $\alpha(z) \neq 0$ by $e^{2u} = \alpha\bar{\alpha}$. Notice that $\alpha_z = 2u_z\alpha$, so we have $\alpha_{z\bar{z}} = 2u_{z\bar{z}}\alpha + 2u_z\alpha_{\bar{z}} = 2u_{z\bar{z}}\alpha = 0$. Thus the Gaussian curvature $K = 0$ when $\beta \equiv 0$.

It is well known that Q_2 is isometric to the product space $\mathbb{S}^2 \times \mathbb{S}^2$, and using Theorem 4.3 (1) [4] for minimal Lagrangian surfaces in $\mathbb{S}^2 \times \mathbb{S}^2$, then $f(M)$ is congruent to some open subset of the totally geodesic Lagrangian sphere when $\alpha \equiv 0$ or totally geodesic flat Lagrangian torus when $\beta \equiv 0$, see [4] in detail. $\square$

Remark 1.3. The last PDE of (1.19) is an involved second-order equation with respect to u , however, introducing a function $\hat{u}$ such that $2e^u = e^{\hat{u}} + |\alpha|^2 e^{-\hat{u}}$ holds, it becomes the elliptic sinh-Gordon equation, as we will see in Proposition 1.7.

We now characterize minimal Lagrangian surfaces in terms of the family of flat connections $d + \omega^\lambda$.

Theorem 1.4. *Let $f : M \rightarrow Q_2$ be a Lagrangian immersion and let $d + \omega^\lambda$ be the family of connections in (1.12). Then the following statements are equivalent:*

- (1) *The Lagrangian immersion f is minimal.*
- (2) *The connections $d + \omega^\lambda$ are flat for all $\lambda \in \mathbb{S}^1$, i.e. (1.13a)-(1.13f) holds for all $\lambda \in \mathbb{S}^1$.*
- (3) *The quadratic differential αdz^2 is holomorphic and $\varphi = \arg(\beta)$ is constant, where α, β are defined in (1.4), (1.7) and $\arg(\beta)$ denotes the argument of β .*

Proof. We would like to prove this theorem by $(1) \Rightarrow (2) \Rightarrow (3) \Rightarrow (1)$.

$(1) \Rightarrow (2)$: We assume $\Phi = \phi dz = 0$. Thus the equation (1.11) can be simplified to (1.16) and the equation (1.10) can be simplified to (1.17). Thus (1.13a)-(1.13f) holds for all $\lambda \in \mathbb{S}^1$.

$(2) \Rightarrow (3)$: Note that $\alpha_{\bar{z}} = 0$ in (1.19). By $e^{2u} - \beta\bar{\beta} = \alpha\bar{\alpha}$ and $\alpha_{\bar{z}} = 0$, we have

$$(1.21) \quad 2u_z e^{2u} = \alpha\bar{\alpha}_{\bar{z}} + \beta_{\bar{z}}\bar{\beta} + \beta\bar{\beta}_{\bar{z}}.$$

Thus the third equation of (1.19), $\frac{1}{2}\bar{\alpha}\alpha_z + \bar{\beta}\beta_z - u_z e^{2u} = 0$, can be simplified to

$$(1.22) \quad \beta\bar{\beta}_{\bar{z}} = \beta_{\bar{z}}\bar{\beta}.$$

And if we put $\beta = |\beta|e^{i\varphi}$, then (1.22) can be simplified to $2i\varphi_{\bar{z}}|\beta|^2 = 0$. Since β is not identically zero, we obtain $\varphi_{\bar{z}} = 0$. Since φ takes values in $\mathbb{R}$, it follows that φ is constant.

$(3) \Rightarrow (1)$: By $\alpha_{\bar{z}} = 0$, we still have (1.21). Since φ is constant, it is easy to check $\bar{\beta}_{\bar{z}}\beta = \bar{\beta}\beta_{\bar{z}} = |\beta||\beta|_{\bar{z}}$, and then (1.21) becomes

$$(1.23) \quad u_z e^{2u} = \frac{1}{2}\alpha\bar{\alpha}_{\bar{z}} + |\beta||\beta|_{\bar{z}}.$$

If we substitute this into the equation (1.10), then we obtain $\phi = 0$. $\square$

We now consider a new local horizontal lift of a minimal Lagrangian surface f :

$$(1.24) \quad \hat{\mathbf{f}} = e^{-\frac{i\varphi}{2}} \mathbf{f},$$

where $\varphi(z, \bar{z}) = \arg(\beta)$ is constant with respect to z and $\bar{z}$ by Theorem 1.4. It is easy to check that $\hat{\mathbf{f}}$ satisfies the horizontal condition, and the new complex functions $\hat{\alpha}$ and $\hat{\beta}$ can be given by

$$(1.25) \quad \hat{\alpha} := \langle \hat{\mathbf{f}}_z, \hat{\mathbf{f}}_z \rangle = e^{-i\varphi} \alpha, \quad \hat{\beta} := \langle \hat{\mathbf{f}}_z, \hat{\mathbf{f}}_{\bar{z}} \rangle = |\beta|,$$

i.e. $|\hat{\alpha}| = |\alpha|$, and $\hat{\beta}$ is a non-negative real function. By the relation $e^{2u} - |\alpha|^2 = |\beta|^2$, we obtain that

$$(1.26) \quad \hat{\beta} = \sqrt{e^{2u} - |\hat{\alpha}|^2},$$

i.e., the function $\hat{\beta}$ is completely determined by u and $\hat{\alpha}$. Note that $\hat{\alpha}$ is a holomorphic function for a minimal Lagrangian surface. All the data in Maurer-Cartan form now can be represented by u and $\hat{\alpha}$, and the new complex functions $\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4, \hat{q}$ are introduced as follows:

$$\begin{aligned} \hat{p}_1 &:= -\frac{1}{\sqrt{2}} \left(\frac{\hat{\alpha} + e^u + \hat{\beta}}{\sqrt{2e^u + \hat{\alpha} + \bar{\hat{\alpha}}}} \right), \quad \hat{p}_2 := -\frac{i}{\sqrt{2}} \left(\frac{\hat{\alpha} + e^u + \hat{\beta}}{\sqrt{2e^u + \hat{\alpha} + \bar{\hat{\alpha}}}} \right), \quad \hat{p}_3 := \frac{i}{\sqrt{2}} \left(\frac{\hat{\alpha} + e^u - \hat{\beta}}{\sqrt{2e^u + \hat{\alpha} + \bar{\hat{\alpha}}}} \right), \\ \hat{p}_4 &:= \frac{1}{\sqrt{2}} \left(\frac{\hat{\alpha} + e^u - \hat{\beta}}{\sqrt{2e^u + \hat{\alpha} + \bar{\hat{\alpha}}}} \right), \quad \hat{q} := i \left(\frac{e^{-u} \frac{1}{2} \hat{\alpha}_z \hat{\beta} - e^{-u} \hat{\alpha} \hat{\beta}_z - \hat{\beta}_z}{2e^u + \hat{\alpha} + \bar{\hat{\alpha}}} \right). \end{aligned}$$

Remark 1.5. Since the factor $e^{2u} - \alpha\bar{\alpha}$ in the denominator of p_1, p_2, p_3, p_4 and q can be eliminated by the same factor in the numerator, thus $\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_4$ and $\hat{q}$ are well-defined even when $e^{2u} - \alpha\bar{\alpha} = 0$. Therefore, we do not need to assume the condition $e^{2u} - \alpha\bar{\alpha} > 0$ for any minimal Lagrangian surface.

1.3. Minimal Lagrangian surfaces and the elliptic sinh-Gordon equation. To simplify the Maurer-Cartan form, we now introduce complex functions p and r as follows:

$$(1.27) \quad p := \frac{1}{\sqrt{2}} \left(\frac{\hat{\alpha} + e^u - \hat{\beta}}{\sqrt{2e^u + \hat{\alpha} + \bar{\hat{\alpha}}}} \right), \quad r := -\frac{1}{\sqrt{2}} \left(\frac{\hat{\alpha} + e^u + \hat{\beta}}{\sqrt{2e^u + \hat{\alpha} + \bar{\hat{\alpha}}}} \right).$$

Namely, we have

$$(1.28) \quad \hat{p}_1 = r, \quad \hat{p}_2 = ir, \quad \hat{p}_3 = ip, \quad \hat{p}_4 = p.$$

Definition 1.6. Denote the new frame of the lift $\hat{\mathbf{f}}$ in (1.24) of a minimal Lagrangian immersion by $\hat{\mathcal{F}}$. Then by Theorem 1.4, there exists a family of frames $\hat{\mathcal{F}}_\lambda$ such that $\hat{\mathcal{F}}_\lambda|_{\lambda=1} = \hat{\mathcal{F}}$, and we call $\hat{\mathcal{F}}_\lambda$ the *extended frame*.

The new family of connection 1-forms $d + \hat{\omega}^\lambda$ parameterized by $\lambda \in \mathbb{S}^1$ can be explicitly written as follows:

$$(1.29) \quad \hat{\omega}^\lambda = \hat{\mathcal{F}}_\lambda^{-1} d\hat{\mathcal{F}}_\lambda = \hat{\mathcal{U}}^\lambda dz + \hat{\mathcal{V}}^\lambda d\bar{z},$$

where

$$\hat{\mathcal{U}}^\lambda = \begin{pmatrix} 0 & 0 & \lambda^{-1}r & \lambda^{-1}ir \\ 0 & 0 & \lambda^{-1}ip & \lambda^{-1}p \\ -\lambda^{-1}r & -\lambda^{-1}ip & 0 & \hat{q} \\ -\lambda^{-1}ir & -\lambda^{-1}p & -\hat{q} & 0 \end{pmatrix}, \quad \hat{\mathcal{V}}^\lambda = \begin{pmatrix} 0 & 0 & \lambda\bar{r} & -\lambda i\bar{r} \\ 0 & 0 & -\lambda i\bar{p} & \lambda\bar{p} \\ -\lambda\bar{r} & \lambda i\bar{p} & 0 & \bar{\hat{q}} \\ \lambda i\bar{r} & -\lambda\bar{p} & -\bar{\hat{q}} & 0 \end{pmatrix}.$$

Then the flatness condition leads to

$$(1.30) \quad p_{\bar{z}} = ip\bar{\hat{q}}, \quad r_{\bar{z}} = -ir\bar{\hat{q}} \quad \text{and} \quad -i\frac{\hat{q}_{\bar{z}}}{2} + i\frac{\bar{\hat{q}}_z}{2} = |r|^2 - |p|^2.$$

By the first and second equations of (1.30), we obtain that

$$(1.31) \quad i\bar{\hat{q}} = (\log p)_{\bar{z}} = -(\log r)_{\bar{z}}.$$

On the other hand, note that

$$(1.32) \quad |p|^2 = \frac{1}{2}(e^u - \hat{\beta}), \quad |r|^2 = \frac{1}{2}(e^u + \hat{\beta})$$

hold. These imply that

$$(1.33) \quad |r|^2 - |p|^2 = \hat{\beta}, \quad |p|^2|r|^2 = \frac{1}{4}|\hat{\alpha}|^2.$$

Thus by the third equation of (1.30), (1.31) and (1.33), we have $-(\log |r|^2)_{z\bar{z}} = 2\hat{\beta}$, which can be written as follows:

$$(1.34) \quad -\frac{1}{2}(\log |r|^2)_{z\bar{z}} - \hat{\beta} = -\frac{1}{2}(\log |r|^2)_{z\bar{z}} + \frac{1}{4}|\hat{\alpha}|^2|r|^{-2} - |r|^2 = 0.$$

By choosing a real function $\hat{u}$ by $\hat{u} := \log(2|r|^2)$, (1.34) leads to

$$(1.35) \quad \hat{u}_{z\bar{z}} + e^{\hat{u}} - |\hat{\alpha}|^2 e^{-\hat{u}} = 0,$$

that is the *elliptic sinh-Gordon equation*. Note that by (1.32), we can also represent the metric on $\mathbb{D} \subset M$ by

$$(1.36) \quad 2e^u dz d\bar{z} = (e^{\hat{u}} + |\hat{\alpha}|^2 e^{-\hat{u}}) dz d\bar{z}.$$

By this correspondence, we have established the following proposition.

Proposition 1.7. *Let u be the conformal factor of the metric of a minimal Lagrangian surface f , and let $\hat{u}$ be a function defined by (1.36). Then u satisfies the last PDE of (1.19) if and only if $\hat{u}$ satisfies (1.35).*

1.4. A family of minimal Lagrangian surfaces. To obtain a family of minimal Lagrangian surfaces of a given minimal Lagrangian surface f , let us consider the *gauge transformation* of the extended frame $\hat{\mathcal{F}}_\lambda$. For a family of smooth maps $\mathcal{G}_\lambda : \mathbb{D} \rightarrow \text{SO}(2) \times \text{SO}(2)$ parametrized by $\lambda \in \mathbb{S}^1$, let $\tilde{\mathcal{F}}_\lambda := \hat{\mathcal{F}}_\lambda \mathcal{G}_\lambda$. Then a straightforward computation shows that

$$\tilde{\omega}^\lambda := \tilde{\mathcal{F}}_\lambda^{-1} d\tilde{\mathcal{F}}_\lambda = \mathcal{G}_\lambda^{-1} \hat{\mathcal{F}}_\lambda^{-1} (d\hat{\mathcal{F}}_\lambda \mathcal{G}_\lambda + \hat{\mathcal{F}}_\lambda d\mathcal{G}_\lambda) = \mathcal{G}_\lambda^{-1} \hat{\omega}^\lambda \mathcal{G}_\lambda + \mathcal{G}_\lambda^{-1} d\mathcal{G}_\lambda$$

holds. We explicitly define $\mathcal{G}_\lambda$ as follows: For p and r in (1.27), it is easy to check $pr = -\frac{1}{2}\hat{\alpha}$, i.e. $p = -\frac{1}{2}\hat{\alpha}r^{-1}$. Thus by setting $r = |r|e^{i\arg r}$, where $\arg r \in \mathbb{R}$ is the argument of r , then

$p = -\frac{1}{2}\hat{\alpha}|r|^{-1}e^{-i\arg r}$ holds. Moreover, let $\lambda = e^{i\theta}$ ($\theta \in \mathbb{R}$) and we define a gauge $\mathcal{G}_\lambda$ by

$$\mathcal{G}_\lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos(\arg r - \theta) & \sin(\arg r - \theta) \\ 0 & 0 & -\sin(\arg r - \theta) & \cos(\arg r - \theta) \end{pmatrix} \in \mathrm{SO}(2) \times \mathrm{SO}(2).$$

Since $|r| = \frac{1}{\sqrt{2}}e^{\frac{\hat{u}}{2}}$ holds, we obtain that

$$(1.37) \quad \tilde{\omega}^\lambda = \tilde{\mathcal{U}}^\lambda dz + \tilde{\mathcal{V}}^\lambda d\bar{z},$$

where

$$(1.38) \quad \tilde{\mathcal{U}}^\lambda = \frac{\sqrt{2}}{2} \begin{pmatrix} 0 & 0 & e^{\hat{u}/2} & ie^{\hat{u}/2} \\ 0 & 0 & -i\lambda^{-2}\hat{\alpha}e^{-\hat{u}/2} & -\lambda^{-2}\hat{\alpha}e^{-\hat{u}/2} \\ -e^{\hat{u}/2} & i\lambda^{-2}\hat{\alpha}e^{-\hat{u}/2} & 0 & -\frac{i}{\sqrt{2}}\hat{u}_z \\ -ie^{\hat{u}/2} & \lambda^{-2}\hat{\alpha}e^{-\hat{u}/2} & \frac{i}{\sqrt{2}}\hat{u}_z & 0 \end{pmatrix},$$

and

$$(1.39) \quad \tilde{\mathcal{V}}^\lambda = \frac{\sqrt{2}}{2} \begin{pmatrix} 0 & 0 & e^{\hat{u}/2} & -ie^{\hat{u}/2} \\ 0 & 0 & i\lambda^2\bar{\hat{\alpha}}e^{-\hat{u}/2} & -\lambda^2\bar{\hat{\alpha}}e^{-\hat{u}/2} \\ -e^{\hat{u}/2} & -i\lambda^2\bar{\hat{\alpha}}e^{-\hat{u}/2} & 0 & \frac{i}{\sqrt{2}}\hat{u}_{\bar{z}} \\ ie^{\hat{u}/2} & \lambda^2\bar{\hat{\alpha}}e^{-\hat{u}/2} & -\frac{i}{\sqrt{2}}\hat{u}_{\bar{z}} & 0 \end{pmatrix}.$$

From the form of $\tilde{\omega}^\lambda$, it is easy to see that there exists a family of minimal Lagrangian surfaces f^λ such that the following fundamental quantities

$$\hat{u}^\lambda = \hat{u}, \quad \hat{\alpha}^\lambda = \lambda^{-2}\hat{\alpha}, \quad \hat{\beta}^\lambda = \hat{\beta}.$$

Note that the metric of f^λ is given by

$$(e^{\hat{u}} + |\hat{\alpha}^\lambda|^2 e^{-\hat{u}})dzd\bar{z} = (e^{\hat{u}} + |\hat{\alpha}|^2 e^{-\hat{u}})dzd\bar{z},$$

and the surface f^λ is obtained by the sum of the first two columns for $\tilde{\mathcal{F}}_\lambda$, i.e.,

$$f^\lambda = [\hat{\mathbf{f}}^\lambda] \in Q_2, \quad \hat{\mathbf{f}}^\lambda = \frac{\sqrt{2}}{2}(\tilde{\mathcal{F}}_\lambda^1 + i\tilde{\mathcal{F}}_\lambda^2) \in \mathbb{C}^4,$$

where $\tilde{\mathcal{F}}_\lambda^j$ ($j = 1, 2$) denotes the j -th column of $\tilde{\mathcal{F}}_\lambda$. We summarize the above discussion as the following theorem:

Theorem 1.8. *If $f : M \rightarrow Q_2$ is a minimal Lagrangian immersion with holomorphic quadratic differential $\hat{\alpha} dz^2$, then there exists a family of minimal Lagrangian immersions $\{f^\lambda\}$ parameterized by $\lambda \in \mathbb{S}^1$, which are isometric to f , and with the holomorphic quadratic differential $\hat{\alpha}^\lambda dz^2 = \lambda^{-2}\hat{\alpha} dz^2$.*

1.5. Minimal Lagrangian surfaces and minimal surfaces in $\mathbb{S}^3$. We now compare the Maurer-Cartan form of a minimal Lagrangian surface with that of a minimal surface in $\mathbb{S}^3$. Then we establish the following theorem.

Theorem 1.9. Any minimal surface $f_{\min}$ in $\mathbb{S}^3$ with unit normal N , the metric $2e^{\hat{u}} dz d\bar{z}$ and the Hopf differential $\mathcal{Q} dz^2$, gives a minimal Lagrangian surface $f = [f_{\min} + iN] \in Q_2$ with the holomorphic differential $-i\mathcal{Q} dz^2$ and the metric

$$(1.40) \quad 2e^u dz d\bar{z} = (e^{\hat{u}} + |\mathcal{Q}|^2 e^{-\hat{u}}) dz d\bar{z}.$$

Conversely, for any minimal Lagrangian surface f in Q_2 with the holomorphic differential $\hat{\alpha} dz^2$ and the metric $2e^u dz d\bar{z}$, there exists a unique map $g = (f_{\min}, N)$ which takes values of the unit tangent bundle of $\mathbb{US}^3 = \mathbb{S}^3 \times \mathbb{S}^2$ with the metric of g is $4e^u dz d\bar{z}$. Furthermore, both projections $f_{\min}$ and N share the Hopf differential $i\hat{\alpha} dz^2$ and the metrics $2e^{\hat{u}} dz d\bar{z}$ of $f_{\min}$ and $2e^{\tilde{u}} dz d\bar{z}$ of N are given by

$$e^{\hat{u}} = e^u + \sqrt{e^{2u} - |\hat{\alpha}|^2}, \quad e^{\tilde{u}} = e^u - \sqrt{e^{2u} - |\hat{\alpha}|^2}.$$

Proof. Let $f_{\min}$ be a minimal surface in $\mathbb{S}^3$ and $\mathbb{D} \subset M$ be a simply connected domain with conformal coordinate z . Note that the induced metric on $\mathbb{D}$ can be written as $ds_M^2 = 2e^{\hat{u}} dz d\bar{z}$. Now let $\mathcal{Q} dz^2 = i\hat{\alpha} dz^2$ be the Hopf differential, that is the $(2, 0)$ -part of the second fundamental form and it is holomorphic for $f_{\min}$, and choose the $\text{SO}(4)$ -frame of $f_{\min}$ as follows:

$$(1.41) \quad \sigma = \left(f_{\min}, N, -\frac{(f_{\min})_z + (f_{\min})_{\bar{z}}}{\sqrt{2}e^{\hat{u}/2}}, \frac{i((f_{\min})_z - (f_{\min})_{\bar{z}})}{\sqrt{2}e^{\hat{u}/2}} \right),$$

where N is the normal vector of $f_{\min}$. By $\mathcal{Q} dz^2 = i\hat{\alpha} dz^2$ and the results in [1, 20], the Maurer-Cartan form obtained by (1.41) is the same as (1.37) for $\lambda = 1$. It follows that the minimal Lagrangian surface f and its local lift $\hat{f}$ defined above can be obtained by

$$(1.42) \quad f = [\hat{f}] \in Q_2, \quad \hat{f} := \frac{\sqrt{2}}{2} (f_{\min} + iN) \in \mathbb{C}^4.$$

By direct computation, we have

$$\langle \hat{f}_z, \hat{f}_z \rangle = \frac{1}{2} \langle (f_{\min})_z + iN_z, (f_{\min})_z + iN_z \rangle = -i\mathcal{Q} = \hat{\alpha}.$$

Thus the first claim follows.

Note that, the normal vector N can be viewed as the dual minimal surface of $f_{\min}$. After choosing the $\text{SO}(4)$ -frame of N properly as follows:

$$(1.43) \quad \tilde{\sigma} = \left(N, f_{\min}, -\frac{N_z + N_{\bar{z}}}{\sqrt{2}e^{\tilde{u}/2}}, \frac{i(N_z - N_{\bar{z}})}{\sqrt{2}e^{\tilde{u}/2}} \right),$$

the minimal Lagrangian surface $\tilde{f}$ can be obtained by

$$(1.44) \quad \tilde{f} = [N + if_{\min}] = [f_{\min} - iN] = [\bar{\hat{f}}] \in Q_2.$$

It shows that $\{f_{\min}, N\}$ corresponds to the same minimal Lagrangian surface f in Q_2 up to a conjugation.

Conversely, let f be a minimal Lagrangian surface in Q_2 with the holomorphic differential $\hat{\alpha} dz^2$ and $\tilde{\omega}^\lambda$ be the Maurer-Cartan form defined in (1.37). Since $\tilde{\omega}^\lambda|_{\lambda=1}$ is the same as the

Maurer-Cartan form of the $\mathrm{SO}(4)$ -frame (1.41) of f_{min} , there exists a pair of minimal surfaces in $\mathbb{S}^3$ as follows:

$$(1.45) \quad f_{min} = \frac{1}{\sqrt{2}}(\hat{\mathbf{f}} + \bar{\hat{\mathbf{f}}}) = \sqrt{2} \operatorname{Re} \hat{\mathbf{f}}, \quad N = -\frac{i}{\sqrt{2}}(\hat{\mathbf{f}} - \bar{\hat{\mathbf{f}}}) = \sqrt{2} \operatorname{Im} \hat{\mathbf{f}},$$

where $\hat{\mathbf{f}}$ is the local lift defined above. On the other hand, since the Maurer-Cartan form $\tilde{\omega}^\lambda|_{\lambda=1}$ could be also identified by another Maurer-Cartan form of the $\mathrm{SO}(4)$ -frame (1.43) of N , there exists another pair of minimal surfaces in $\mathbb{S}^3$ as follows:

$$(1.46) \quad N = \sqrt{2} \operatorname{Re} \hat{\mathbf{f}}, \quad f_{min} = \sqrt{2} \operatorname{Im} \hat{\mathbf{f}}.$$

In conclusion, there exists a unique pair of minimal surfaces $\{(f_{min}, 2e^{\hat{u}}dzd\bar{z}), (N, 2e^{\tilde{u}}dzd\bar{z})\}$ in $\mathbb{S}^3$. By similar computation, we have

$$\mathcal{Q} = \langle (f_{min})_{zz}, N \rangle = -\langle (f_{min})_z, N_z \rangle = \langle N_{zz}, f_{min} \rangle = i\hat{\alpha}.$$

This completes the proof. $\square$

Remark 1.10.

- (1) The metric defined in (1.36) can be thought as half the Sasakian metric on the unit tangent bundle $\mathrm{US}^3 = \mathbb{S}^3 \times \mathbb{S}^2$ for a minimal surface f_{min} in $\mathbb{S}^3$, that is, the following relation holds:

$$2ds_{Lag}^2 = 4e^u dzd\bar{z} = (2e^{\hat{u}} + 2|\mathcal{Q}|^2 e^{-\hat{u}}) dzd\bar{z} = ds_{Sasaki}^2.$$

- (2) The converse statement in Theorem 1.9 has been shown in [4, Theorem 4.4] under the assumption $|\hat{\alpha}| > 0$, through the PDE point view.

In order to highlight how minimal surfaces in $\mathbb{S}^3$ can be constructed from minimal Lagrangian surfaces in Q_2 , we state the following corollary.

Corollary 1.11. *Let $f : M \rightarrow Q_2$ be a minimal Lagrangian surface and $\hat{\mathbf{f}}$ be its local lift defined above. Then the minimal surface f_{min} and its dual minimal surface N in $\mathbb{S}^3$ associated with f are of the form*

$$(1.47) \quad f_{min} = \sqrt{2} \operatorname{Re} \hat{\mathbf{f}}, \quad N = \sqrt{2} \operatorname{Im} \hat{\mathbf{f}}.$$

2. THE LOOP GROUP METHOD FOR MINIMAL LAGRANGIAN SURFACES

The PDE in (1.35) is merely the structure equation for a harmonic map from a Riemann surface into $\mathbb{S}^2$. Thus, it is natural to expect the existence of such a map associated with any minimal Lagrangian surface. Moreover, we will establish the loop group method for minimal Lagrangian surfaces following the approach in [7].

2.1. Minimal Lagrangian surfaces and harmonic maps into $\mathbb{S}^2$. To obtain an associated harmonic map to $\mathbb{S}^2$, we use the following Lie group isomorphism [15]

$$\mathrm{SO}(4) = (\mathrm{SU}(2) \times \mathrm{SU}(2))/\mathbf{Z}_2,$$

see Appendix B for details.

For the Maurer-Cartan form ω , we can use Lie algebra isomorphism $\mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, see also in Appendix B. Accordingly, we can also use $\Lambda\mathfrak{so}(4) \cong \Lambda\mathfrak{su}(2) \oplus \Lambda\mathfrak{su}(2)$, where

$$\Lambda\mathfrak{g} := \{g : \mathbb{S}^1 \rightarrow \mathfrak{g}\}$$

denotes the *loop algebra* of $\mathfrak{g}$, which is an infinite-dimensional Banach Lie algebra with respect to a suitable norm [19]. Hence the associated Maurer-Cartan form $\hat{\omega}^\lambda$ in (1.29) can be represented as follows:

$$(2.1) \quad \hat{U}^\lambda = (\hat{U}_1^\lambda, \hat{U}_2^\lambda) = \left(\begin{pmatrix} \frac{1}{4}\hat{u}_z & -\frac{\sqrt{2}}{2}\lambda^{-1}e^{\hat{u}/2} \\ \frac{\sqrt{2}}{2}\lambda^{-1}\hat{\alpha}e^{-\hat{u}/2} & -\frac{1}{4}\hat{u}_{\bar{z}} \end{pmatrix}, \begin{pmatrix} \frac{1}{4}\hat{u}_z & -i\frac{\sqrt{2}}{2}\lambda^{-1}e^{\hat{u}/2} \\ i\frac{\sqrt{2}}{2}\lambda^{-1}\hat{\alpha}e^{-\hat{u}/2} & -\frac{1}{4}\hat{u}_z \end{pmatrix} \right),$$

and

$$(2.2) \quad \hat{V}^\lambda = (\hat{V}_1^\lambda, \hat{V}_2^\lambda) = \left(\begin{pmatrix} -\frac{1}{4}\hat{u}_{\bar{z}} & -\frac{\sqrt{2}}{2}\lambda\bar{\alpha}e^{-\hat{u}/2} \\ \frac{\sqrt{2}}{2}\lambda e^{\hat{u}/2} & \frac{1}{4}\hat{u}_{\bar{z}} \end{pmatrix}, \begin{pmatrix} -\frac{1}{4}\hat{u}_{\bar{z}} & i\frac{\sqrt{2}}{2}\lambda\bar{\alpha}e^{-\hat{u}/2} \\ -i\frac{\sqrt{2}}{2}\lambda e^{\hat{u}/2} & \frac{1}{4}\hat{u}_{\bar{z}} \end{pmatrix} \right).$$

From the form of $(\hat{U}_1^\lambda, \hat{U}_2^\lambda)$ and $(\hat{V}_1^\lambda, \hat{V}_2^\lambda)$, the corresponding pair of maps is given by $(F_\lambda, F_{-i\lambda})$ such that

$$(2.3) \quad \hat{\omega}_1^\lambda = F_\lambda^{-1}dF_\lambda = \hat{U}_1^\lambda dz + \hat{V}_1^\lambda d\bar{z}, \quad \hat{\omega}_2^\lambda = F_{-i\lambda}^{-1}dF_{-i\lambda} = \hat{U}_2^\lambda dz + \hat{V}_2^\lambda d\bar{z}.$$

Notice that $\hat{U}_2^\lambda$ (resp. $\hat{V}_2^\lambda$) can be obtained from $\hat{U}_1^\lambda$ (resp. $\hat{V}_1^\lambda$) by transforming λ into $-i\lambda$. For further simplification, we can choose $w := \sqrt{2}z$ as our new coordinate and $ds_M^2 = 2e^{\hat{u}}dw d\bar{w}$ as our new metric. And $Qdw^2 := \hat{\alpha}dz^2$ is our holomorphic quadratic differential under the new coordinate w . In addition, the associated holomorphic quadratic differential $\lambda^{-2}\hat{\alpha}dz^2$ mentioned in Theorem 1.8 becomes $\lambda^{-2}Qdw^2$. From the above discussion, we obtain the following proposition.

Proposition 2.1. *Let $\hat{\mathcal{F}}_\lambda$ be the extended frame of $\hat{\mathfrak{f}}$. Then there exists the pair of maps $(F_\lambda, F_{-i\lambda})$ such that the Maurer-Cartan form $\hat{\omega}_1^\lambda$ of F_λ is given by (2.3). Conversely, given a pair of maps $(F_\lambda, F_{-i\lambda})$ with the Maurer-Cartan form $\hat{\omega}_1^\lambda$ of F_λ is given by (2.3), there exists the extended frame $\hat{\mathcal{F}}_\lambda$ of some minimal surface $\hat{\mathfrak{f}}$.*

Now we consider the map $\check{f} : M \rightarrow \mathbb{S}^2$ and focus on the associated Maurer-Cartan form $\check{\omega}^\lambda$ induced by the local lift F_λ , where $\check{\omega}^\lambda$ can be written as follows:

$$(2.4) \quad \check{\omega}^\lambda = F_\lambda^{-1}dF_\lambda = \check{U}^\lambda dw + \check{V}^\lambda d\bar{w},$$

where

$$(2.5) \quad \check{U}^\lambda = \begin{pmatrix} \frac{1}{4}\check{u}_w & -\frac{1}{2}\lambda^{-1}e^{\check{u}/2} \\ \lambda^{-1}Qe^{-\check{u}/2} & -\frac{1}{4}\check{u}_{\bar{w}} \end{pmatrix}, \quad \check{V}^\lambda = \begin{pmatrix} -\frac{1}{4}\check{u}_{\bar{w}} & -\lambda\bar{Q}e^{-\check{u}/2} \\ \frac{1}{2}\lambda e^{\check{u}/2} & \frac{1}{4}\check{u}_{\bar{w}} \end{pmatrix}.$$

The Maurer-Cartan form $\check{\omega}^\lambda$ is the same to the Maurer-Cartan form of the extended frame of a non-conformal harmonic map from a Riemann surface M into $\mathbb{S}^2$. Thus the map $\check{F}_\lambda$ is called the *extended frame* of a harmonic map $\check{f}$. In addition, let $F_\lambda, F_{-i\lambda}$ be two solutions of the system (2.4) and define

$$(2.6) \quad f_{min} := F_\lambda \begin{pmatrix} e^{\pi i/4} & 0 \\ 0 & e^{-\pi i/4} \end{pmatrix} F_{-i\lambda}^{-1}, \quad N := iF_\lambda \begin{pmatrix} e^{\pi i/4} & 0 \\ 0 & -e^{-\pi i/4} \end{pmatrix} F_{-i\lambda}^{-1}.$$

By the results in [1, 12], f_{min} is a minimal surface in $\mathbb{S}^3$ and normal N .

2.2. The DPW method. In Section 2.1, we have obtained the family of Maurer-Cartan forms $\tilde{\omega}^\lambda$ for a harmonic into $\mathbb{S}^2$ from a minimal Lagrangian surface f in Q_2 , thus we can apply the generalized Weierstrass type representation (the DPW method) for harmonic maps in $\mathbb{S}^2$. The basic construction can be found in [6, 12, 20]. First note that the extended frame F_λ takes values in the loop group of $G = \mathrm{SU}(2)$

$$\Lambda G_\sigma = \{g : \mathbb{S}^1 \rightarrow G \mid \sigma g(\lambda) = g(-\lambda)\},$$

where σ is the involution $\sigma(g) = \mathrm{Ad} \, \mathrm{diag}(1, -1)(g)$ is the involution of the symmetric space $\mathbb{S}^2 = \mathrm{SU}(2)/\mathrm{U}_1$. By introducing a suitable topology, ΛG_σ becomes a Banach Lie group and we call it the *loop group* of G , see [19]. The extended frame F_λ clearly takes values in the loop group of $\mathrm{SU}(2)$, and the *Birkhoff decomposition* ([19, 7]) of F_λ

$$F_\lambda = F_- F_+ \quad \text{with } F_- \in \Lambda_*^- G_\sigma^\mathbb{C} \text{ and } F_+ \in \Lambda^+ G_\sigma^\mathbb{C}$$

yields the meromorphic dependence of F_- [7, Lemma 2.6]. Here $G^\mathbb{C} = \mathrm{SL}(2, \mathbb{C})$ and $\Lambda^\pm G_\sigma^\mathbb{C}$ denote the subgroups of $\Lambda G_\sigma^\mathbb{C}$ which can be extended to the inside (resp. outside) of unit disk in $\mathbb{CP}^1$ when the plus sign (resp. the minus sign) is chosen. The subscript $*$ denotes the identity normalization at $\lambda = 0$.

Conversely, minimal Lagrangian surfaces in the complex quadric Q_2 can be constructed in the following four steps:

Step 1: Solve the initial-value problem:

$$(2.7) \quad d\Phi = \Phi\xi, \quad \Phi(z_0) = \Phi_0 \in \Lambda\mathrm{SL}(2, \mathbb{C})_\sigma,$$

to obtain a unique map $\Phi : \mathbb{D} \rightarrow \Lambda\mathrm{SL}(2, \mathbb{C})_\sigma$.

Step 2: Compute the *Iwasawa decomposition* (see [19, 7]) of Φ pointwise on $\mathbb{D}$:

$$(2.8) \quad \Phi = F_\lambda B, \quad F_\lambda \in \Lambda\mathrm{SU}(2)_\sigma, \quad B \in \Lambda^+\mathrm{SL}(2, \mathbb{C})_\sigma,$$

Then by [7, Lemma 4.2], F_λ is the extended frame of a harmonic map into $\mathbb{S}^2$. Set the pair of maps given by another map $F_{-i\lambda}$ as

$$(2.9) \quad (F_\lambda, F_{-i\lambda}) \in \Lambda\mathrm{SU}(2)_\sigma \times \Lambda\mathrm{SU}(2)_\sigma.$$

Step 3: Use the Loop group isomorphism

$$(2.10) \quad \Lambda\mathrm{SO}(4)_\sigma \cong (\Lambda\mathrm{SU}(2)_\sigma \times \Lambda\mathrm{SU}(2)_\sigma) / \mathbf{Z}_2,$$

and Proposition 2.1, one obtains the extended frame $\mathcal{F}_\lambda \in \Lambda\mathrm{SO}(4)_\sigma$ of some minimal Lagrangian immersion into Q_2 .

Step 4: Finally, by using the formula (2.12) in the following proposition, we obtain a family of minimal Lagrangian immersions f^λ into Q_2 .

Remark 2.2. The involution σ of the twisted loop group $\Lambda\mathrm{SO}(4)_\sigma$ is associated with the Hermitian symmetric space $Q_2 = \mathrm{SO}(4)/\mathrm{SO}(2) \times \mathrm{SO}(2)$. It is explicitly given by $\sigma = \mathrm{Ad} \, \mathrm{diag}(1, 1, -1, -1)$ on $\mathrm{SO}(4)$, and it is different from the involution $\mathrm{Ad} \, \mathrm{diag}(1, -1)$ of $\mathrm{SU}(2)$ which is associated with the twisted loop group $\Lambda\mathrm{SU}(2)_\sigma$. However, the homomorphism $\psi : \mathrm{SU}(2) \times \mathrm{SU}(2) \rightarrow \mathrm{SO}(4)$ in (B.2) can be naturally extended to the twisted loop group level as (2.10), see the corresponding Maurer-Cartan form in (2.1) and (2.2).

Proposition 2.3. *Let F_λ be the extended frame defined above. Set*

$$(2.11) \quad X^\lambda = (X_{ij}^\lambda) := F_\lambda F_{-i\lambda}^{-1}, \quad Y^\lambda = (Y_{ij}^\lambda) := iF_\lambda \sigma_3 F_{-i\lambda}^{-1}, \quad \text{with } \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then the associated family $\{f^\lambda\}$ of f can be represented by

$$(2.12) \quad f^\lambda = \left[\left(R(X_{11}^\lambda) + i R(Y_{11}^\lambda), I(X_{11}^\lambda) + i I(Y_{11}^\lambda), R(X_{12}^\lambda) + i R(Y_{12}^\lambda), I(X_{12}^\lambda) + i I(Y_{12}^\lambda) \right) \right],$$

where R and I denotes the real and the imaginary part, respectively.

Proof. By the Lie group isomorphism given in Appendix B and a straightforward computation, we obtain the result. $\square$

Corollary 2.4. *Let F_λ be the extended frame defined above. Define a map*

$$(2.13) \quad \Phi_\lambda = (\phi_\lambda, \psi_\lambda) : \mathbb{D} \rightarrow \mathbb{S}^2 \times \mathbb{S}^2$$

by $(\phi_\lambda, \psi_\lambda) := (iF_\lambda \sigma_3 F_\lambda^{-1}, iF_{-i\lambda} \sigma_3 F_{-i\lambda}^{-1})$. Then $\{\Phi_\lambda\}_{\lambda \in \mathbb{S}^1}$ is a family of minimal Lagrangian surfaces.

Proof. By the definition of the maps ϕ_λ and ψ_λ , it is easy to check $\phi_\lambda, \psi_\lambda \in \mathfrak{su}(2) \cong \mathbb{R}^3$ for each $\lambda \in \mathbb{S}^1$ and we have

$$\langle \phi_\lambda, \phi_\lambda \rangle = \frac{1}{2} \text{tr}(\phi_\lambda \sigma_2 \phi_\lambda^T \sigma_2), \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

and similarly for ψ_λ . Then by $F_\lambda, F_{-i\lambda} \in \text{SU}(2)$ for each $\lambda \in \mathbb{S}^1$, it follows that $|\phi_\lambda| = |\psi_\lambda| = 1$, i.e. $\phi_\lambda, \psi_\lambda \in \mathbb{S}^2$. Moreover

$$\begin{aligned} (\phi_\lambda)_z &= iF_\lambda (\hat{U}_1^\lambda \sigma_3 - \sigma_3 \hat{U}_1^\lambda) F_\lambda^{-1} = F_\lambda \begin{pmatrix} 0 & i\sqrt{2}\lambda^{-1}e^{\hat{u}/2} \\ i\sqrt{2}\lambda^{-1}\hat{\alpha}e^{-\hat{u}/2} & 0 \end{pmatrix} F_\lambda^{-1}, \\ (\phi_\lambda)_{\bar{z}} &= iF_\lambda (\hat{V}_1^\lambda \sigma_3 - \sigma_3 \hat{V}_1^\lambda) F_\lambda^{-1} = F_\lambda \begin{pmatrix} 0 & i\sqrt{2}\lambda\hat{\alpha}e^{-\hat{u}/2} \\ i\sqrt{2}\lambda e^{\hat{u}/2} & 0 \end{pmatrix} F_\lambda^{-1}, \end{aligned}$$

and similarly for $(\psi_\lambda)_z$ and $(\psi_\lambda)_{\bar{z}}$. Then a straightforward computation shows that

$$\begin{aligned} |(\phi_\lambda)_z| &= |(\phi_\lambda)_{\bar{z}}| = |(\psi_\lambda)_z| = |(\psi_\lambda)_{\bar{z}}| = \sqrt{2}e^{u/2}, \\ |(\Phi_\lambda)_z|^2 &= \langle (\phi_\lambda)_{\bar{z}}, (\phi_\lambda)_z \rangle + \langle (\psi_\lambda)_{\bar{z}}, (\psi_\lambda)_z \rangle = 2|(\phi_\lambda)_z|^2 = 2|(\psi_\lambda)_z|^2 = 4e^u, \\ \langle (\Phi_\lambda)_z, (\Phi_\lambda)_z \rangle &= \frac{1}{2} \text{tr}((\phi_\lambda)_z \sigma_2 (\phi_\lambda)_z^T \sigma_2) + \frac{1}{2} \text{tr}((\psi_\lambda)_z \sigma_2 (\psi_\lambda)_z^T \sigma_2) = 0 \end{aligned}$$

hold. That is, Φ_λ is a conformal immersion, and moreover, it is Lagrangian. Then by another straightforward computation shows that

$$\begin{aligned} (\phi_\lambda)_{z\bar{z}} &= iF_\lambda \left((\hat{V}_1^\lambda \hat{U}_1^\lambda + (\hat{U}_1^\lambda)_{\bar{z}}) \sigma_3 - \hat{U}_1^\lambda \sigma_3 \hat{V}_1^\lambda - \hat{V}_1^\lambda \sigma_3 \hat{U}_1^\lambda + \sigma_3 (-(\hat{U}_1^\lambda)_{\bar{z}} + \hat{U}_1^\lambda \hat{V}_1^\lambda) \right) F_\lambda^{-1} \\ &= iF_\lambda (-2e^u \sigma_3) F_\lambda^{-1} = -2e^u \phi_\lambda, \end{aligned}$$

holds and similarly $(\psi_\lambda)_{z\bar{z}} = -2e^u \psi_\lambda$ follows. Thus $(\Phi_\lambda)_{z\bar{z}} = -2e^u \Phi_\lambda$ holds, and Φ_λ is harmonic, and it is minimal.¹ $\square$

3. EXAMPLES OF MINIMAL LAGRANGIAN SURFACES THROUGH THE DPW METHOD

By using the DPW method introduced in Section 2, we will construct minimal Lagrangian surfaces in $\mathcal{Q}_2$.

¹It can be also seen by (A.2) in Appendix A.

3.1. Basic examples. The basic examples are the totally geodesic spheres and the totally geodesic tori.

3.1.1. Totally geodesic sphere. Define

$$\xi = \lambda^{-1} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} dz$$

for $z \in \mathbb{C}$. By using this holomorphic potential, we can construct a family of totally geodesic spheres in Q_2 as follows, see also [4]:

It is easy to solve the ODE $d\Phi = \Phi\xi$ by $\Phi = \exp(z\xi/dz)$ with $\Phi(0) = \text{id}$. Moreover, the Iwasawa decomposition of $\Phi = F_\lambda B$ is given by

$$F_\lambda = \frac{1}{\sqrt{1+|z|^2}} \begin{pmatrix} 1 & z\lambda^{-1} \\ -\bar{z}\lambda & 1 \end{pmatrix}.$$

By the loop group isomorphism $\Lambda\text{SO}(4)_\sigma \cong (\Lambda\text{SU}(2)_\sigma \times \Lambda\text{SU}(2)_\sigma)/\mathbf{Z}_2$, we have

$$\mathcal{F}_\lambda = \frac{1}{1+|z|^2} \begin{pmatrix} 1 & -|z|^2 & * & * \\ -|z|^2 & 1 & * & * \\ \frac{1}{2}(z\lambda^{-1} + \bar{z}\lambda - i(z\lambda^{-1} - \bar{z}\lambda)) & \frac{1}{2}(z\lambda^{-1} + \bar{z}\lambda - i(z\lambda^{-1} - \bar{z}\lambda)) & * & * \\ -\frac{1}{2}(z\lambda^{-1} + \bar{z}\lambda + i(z\lambda^{-1} - \bar{z}\lambda)) & -\frac{1}{2}(z\lambda^{-1} + \bar{z}\lambda + i(z\lambda^{-1} - \bar{z}\lambda)) & * & * \end{pmatrix}.$$

Finally, by the first and second columns of $\mathcal{F}_\lambda$, we can obtain a family of totally geodesic spheres $\{f^\lambda\}$ parameterized by $\lambda \in \mathbb{S}^1$

$$f^\lambda = \left[\begin{pmatrix} 1 - i|z|^2 \\ -|z|^2 + i \\ z\lambda^{-1} + i\bar{z}\lambda \\ -\bar{z}\lambda - iz\lambda^{-1} \end{pmatrix} \right].$$

3.1.2. Totally geodesic torus. Define

$$\xi = \lambda^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} dz$$

for $z \in \mathbb{C}$. By using this holomorphic potential, we can construct a family of totally geodesic torus in Q_2 as follows, see also in [4]: It is easy to solve the ODE $d\Phi = \Phi\xi$ by $\Phi = \exp(z\xi/dz)$ with $\Phi(0) = \text{id}$. Moreover, the Iwasawa decomposition of $\Phi = F_\lambda B$ is given by

$$F_\lambda = \begin{pmatrix} \cosh(\lambda^{-1}z - \bar{z}\lambda) & \sinh(\lambda^{-1}z - \bar{z}\lambda) \\ \sinh(\lambda^{-1}z - \bar{z}\lambda) & \cosh(\lambda^{-1}z - \bar{z}\lambda) \end{pmatrix}.$$

By the loop group isomorphism $\Lambda\text{SO}(4)_\sigma \cong (\Lambda\text{SU}(2)_\sigma \times \Lambda\text{SU}(2)_\sigma)/\mathbf{Z}_2$, we have

$$\mathcal{F}_\lambda = \begin{pmatrix} \cos(\lambda^{-1}z + \bar{z}\lambda + i(\lambda^{-1}z - \bar{z}\lambda)) & 0 & * & * \\ 0 & \cos(\lambda^{-1}z + \bar{z}\lambda - i(\lambda^{-1}z - \bar{z}\lambda)) & * & * \\ 0 & \sin(\lambda^{-1}z + \bar{z}\lambda - i(\lambda^{-1}z - \bar{z}\lambda)) & * & * \\ -\sin(\lambda^{-1}z + \bar{z}\lambda + i(\lambda^{-1}z - \bar{z}\lambda)) & 0 & * & * \end{pmatrix}$$

Finally, by the first and second columns of $\mathcal{F}_\lambda$, we obtain a family of totally geodesic tori $\{f^\lambda\}$ parameterized by $\lambda \in \mathbb{S}^1$

$$(3.1) \quad f^\lambda = \left[\begin{pmatrix} \cos(\lambda^{-1}z + \bar{z}\lambda + i(\lambda^{-1}z - \bar{z}\lambda)) \\ i \cos(\lambda^{-1}z + \bar{z}\lambda - i(\lambda^{-1}z - \bar{z}\lambda)) \\ i \sin(\lambda^{-1}z + \bar{z}\lambda - i(\lambda^{-1}z - \bar{z}\lambda)) \\ -\sin(\lambda^{-1}z + \bar{z}\lambda + i(\lambda^{-1}z - \bar{z}\lambda)) \end{pmatrix} \right].$$

3.2. Equivariant and radially symmetric examples. We now show two new examples of minimal Lagrangian surfaces in Q_2 .

3.2.1. $\mathbb{R}$ -equivariant minimal Lagrangian surfaces.

Definition 3.1 ($\mathbb{R}$ -equivariant potentials, [20]). Define

$$(3.2) \quad \xi = D(\lambda) \frac{dz}{z}, \quad \text{where} \quad D(\lambda) = \begin{pmatrix} c & a\lambda^{-1} + b\lambda \\ a\lambda + b\lambda^{-1} & -c \end{pmatrix},$$

with $a, b, c \in \mathbb{R}$ for z in $\mathbb{D} = \mathbb{C} \setminus \{0\}$. We call such potentials the *Equivariant potentials*.

It is easy to see that $\Phi = \exp(\log z \xi)$ is the unique solution of $d\Phi = \Phi \xi$ with the initial condition $\Phi|_{z=0} = \text{id}$. Let

$$\Phi = F_\lambda B$$

be the Iwasawa decomposition of Φ (see below for the explicit form of F_λ for $c = 0$). As discussed in [12, Section 2.5], by the rotation of the domain $z \rightarrow e^{i\theta} \cdot z$, the following transformation rule of F_λ follows:

$$(3.3) \quad F_\lambda(e^{i\theta} \cdot z, e^{-i\theta} \cdot \bar{z}, \lambda) = \exp(i\theta D(\lambda)) \cdot F_\lambda(z, \bar{z}, \lambda).$$

Note that $i\theta D(\lambda)$ takes values in $\Lambda \mathfrak{su}(2)_\sigma$ and thus $\exp(i\theta D(\lambda))$ takes values in $\Lambda \text{SU}(2)_\sigma$.

We now recall the definition of an equivariant surface, see for an example [11]: Let G be a real Lie group and K a closed subgroup. Let M denote a connected Riemann surface and $f : M \rightarrow G/K$ a differentiable map. Let g_t be a one-parameter group of biholomorphic maps of M . Then f is called $\mathbb{R}$ -equivariant with regard to g_t if there exists a one-parameter group $R(t) : \mathbb{R} \rightarrow G$ such that

$$f(g_t \cdot p) = R(t)f(p)$$

for all $p \in M$ and $t \in \mathbb{R}$. Then a straightforward computation shows that the minimal Lagrangian surface constructed by the equivariant potential ξ in (3.2) is an equivariant surface.

Proposition 3.2. *Let ξ be an $\mathbb{R}$ -equivariant potential defined in (3.2) and let $F_\lambda \in \Lambda \text{SU}(2)_\sigma$ for $\lambda \in \mathbb{S}^1$ be the corresponding extended frame. Then the minimal Lagrangian surface $f^\lambda : M \rightarrow Q_2$ constructed by $(F_\lambda, F_{-\lambda})$ is equivariant, that is,*

$$\hat{f}^\lambda(e^{i\theta} \cdot z, e^{-i\theta} \cdot \bar{z}, \lambda) = \psi((\exp(i\theta D(\lambda)), \exp(-i\theta D(-i\lambda))) \hat{f}^\lambda(z, \bar{z}, \lambda))$$

holds, where $\hat{f}^\lambda$ is the horizontal lift of f^λ in $\mathbb{C}^4$ and $\psi : \Lambda \text{SU}(2)_\sigma \times \Lambda \text{SU}(2)_\sigma \rightarrow \Lambda \text{SO}(4)_\sigma$ is the loop group homomorphism.

By Theorem 3 in [20], we can obtain that

$$F_\lambda = \begin{pmatrix} \sqrt{\frac{4ab\lambda^2+v^2}{2v(a\lambda^2+b)}} \cosh(tz - t\mathbf{f}) & \frac{\lambda v' \cosh(tz-t\mathbf{f})}{\sqrt{2v(a\lambda^2+b)(4ab\lambda^2+v^2)}} + \sqrt{\frac{2v(a+b\lambda^2)}{4ab\lambda^2+v^2}} \sinh(tz - t\mathbf{f}) \\ \sqrt{\frac{4ab\lambda^2+v^2}{2v(a+b\lambda^2)}} \sinh(tz - t\mathbf{f}) & \frac{\lambda v' \sinh(tz-t\mathbf{f})}{\sqrt{2v(a+b\lambda^2)(4ab\lambda^2+v^2)}} + \sqrt{\frac{2v(a\lambda^2+b)}{4ab\lambda^2+v^2}} \cosh(tz - t\mathbf{f}) \end{pmatrix},$$

where $a, b \in \mathbb{R} \setminus \{0\}$ and $c = 0$, and the non-constant Jacobian elliptic function $v = v(x)$ and $\mathbf{f} = \mathbf{f}(x)$ are given by

$$v'^2 = -(v^2 - 4a^2)(v^2 - 4b^2), \quad v(0) = 2b, \quad \mathbf{f}(x) = \int_0^x \frac{2ds}{1 + (4ab\lambda^2)^{-1}v^2(s)},$$

and

$$t = \sqrt{(a\lambda + b\lambda^{-1})(a\lambda^{-1} + b\lambda)}.$$

It is also easy to compute the map $F_{-i\lambda}$. Then by Proposition 2.3, we obtain the explicit form of this equivariant surface in Q_2 .

Note that the resulting $\mathbb{R}$ -equivariant minimal Lagrangian surface f^λ is only defined on the universal cover of $\mathbb{C} \setminus \{0\}$ for a general choices of a, b and c . We now give the sufficient conditions such that f^λ is well-defined on $\mathbb{C} \setminus \{0\}$, that is, f^λ is a topologically cylinder:

Proposition 3.3. *The resulting $\mathbb{R}$ -equivariant minimal Lagrangian surface $f^\lambda|_{\lambda=\lambda_0}$ is well-defined on $\mathbb{C} \setminus \{0\}$ if the eigenvalues of $D(\lambda_0)$ and $D(-i\lambda_0)$ are integers or half-integers.*

Remark 3.4.

- (1) The conditions in Proposition 3.3 are explicitly given by

$$2\sqrt{c^2 + |\lambda_0 a + \lambda_0^{-1} b|^2}, \quad 2\sqrt{c^2 + |\lambda_0 a - \lambda_0^{-1} b|^2} \in \mathbb{Z}_{\geq 0}.$$

- (2) In [20, Section 8], the conditions for the well-definedness on $\mathbb{C} \setminus \{0\}$ of $\mathbb{R}$ -equivariant constant mean curvature surfaces in $\mathbb{S}^3$ have been given by that the eigenvalues of $D(\lambda_0)$ and $D(-i\lambda_0)$ are half-integers. Thus even the corresponding $\mathbb{R}$ -equivariant minimal surface is not well-defined, the minimal Lagrangian surface could be well-defined on $\mathbb{C} \setminus \{0\}$. Such cases exactly happen if the one of eigenvalues $D(\lambda_0)$ and $D(-i\lambda_0)$ is an integer and the other is a half-integer, and thus

$$\hat{\mathbf{f}}^\lambda(e^{2\pi i} \cdot z, e^{-2\pi i} \cdot \bar{z}, \lambda_0) = -\hat{\mathbf{f}}^\lambda(z, \bar{z}, \lambda_0)$$

holds for the horizontal lift $\hat{\mathbf{f}}^\lambda$, but it is the same point in Q_2 .

3.2.2. Radially symmetric minimal Lagrangian surfaces.

Definition 3.5 (Radially symmetric potentials, [12]). Define

$$(3.4) \quad \xi = \lambda^{-1} \begin{pmatrix} 0 & 1 \\ cz^k & 0 \end{pmatrix} dz,$$

for $z \in \mathbb{D} = \mathbb{C}$ and $k \in \mathbb{N}$ and some $c \in \mathbb{C} \setminus (\mathbb{S}^1 \cup \{0\})$. Here we call such potentials the *radially symmetric potentials*.

First we recall the result of [12, Lemma 2.6.1]: The constant mean curvature surface in $\mathbb{R}^3$ (the so-called Smyth surfaces) constructed by ξ has reflection symmetries with respect to $k+2$ geodesic planes that meet equiangularly along a geodesic line.

Thus it is natural to think the reflections $R_\ell(z) = e^{\frac{2\pi i \ell}{k+2}} \bar{z}$ of the domain $\mathbb{C}$, for $\ell \in \{0, 1, \dots, k+1\}$. Note that

$$(3.5) \quad \xi(R_\ell(z), \lambda) = A_\ell \xi(\bar{z}, \lambda) A_\ell^{-1}$$

holds, where

$$A_\ell = \begin{pmatrix} e^{\frac{\pi i \ell}{k+2}} & 0 \\ 0 & e^{-\frac{\pi i \ell}{k+2}} \end{pmatrix} \in \text{SU}(2).$$

Let Φ be the solution of $d\Phi = \Phi\xi$ with $\Phi(z_0) = \text{id}$ and consider the Iwasawa decomposition $\Phi = F_\lambda B$. Then by (3.5),

$$F_\lambda(R_\ell(z), \overline{R_\ell(z)}, \lambda) = A_\ell F_\lambda(\bar{z}, z, \lambda) A_\ell^{-1}$$

holds. This leads to the following proposition.

Proposition 3.6. *Let ξ be the radially symmetric potentials defined in (3.4) and let $F_\lambda \in \Lambda\text{SU}(2)_\sigma$ for $\lambda \in \mathbb{S}^1$ be an extended frame obtained by ξ . Then the minimal Lagrangian surface $f^\lambda : M \rightarrow Q_2$ constructed by $(F_\lambda, F_{-i\lambda})$ has discrete rotational symmetries:*

$$\hat{f}^\lambda(R_\ell(z), \overline{R_\ell(z)}, \lambda) = \mathcal{A}_\ell \hat{f}^\lambda(\bar{z}, z, \lambda), \quad \text{with} \quad \mathcal{A}_\ell := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\left(\frac{2\pi\ell}{k+2}\right) & -\sin\left(\frac{2\pi\ell}{k+2}\right) \\ 0 & 0 & \sin\left(\frac{2\pi\ell}{k+2}\right) & \cos\left(\frac{2\pi\ell}{k+2}\right) \end{pmatrix},$$

where $\hat{f}^\lambda$ is the horizontal lift of f^λ . Moreover, the metric of the minimal Lagrangian surface f^λ depends only on the radial direction, that is $|z|$. Such a minimal Lagrangian surface f is said to be radially symmetric.

Proof. Let ψ be a loop group homomorphism $\psi : \Lambda\text{SU}(2)_\sigma \times \Lambda\text{SU}(2)_\sigma \rightarrow \Lambda\text{SO}(4)_\sigma$ as before, and set $\hat{\mathcal{F}}(z, \bar{z}, \lambda) = \psi(F(z, \bar{z}, \lambda), F(z, \bar{z}, -i\lambda))$. By direct computation, we have

$$\begin{aligned} \hat{\mathcal{F}}(R_\ell(z), \overline{R_\ell(z)}, \lambda) &= \psi\left(F(R_\ell(z), \overline{R_\ell(z)}, \lambda), F(R_\ell(z), \overline{R_\ell(z)}, -i\lambda)\right) \\ &= \psi\left((A_\ell, A_\ell) \cdot (F(\bar{z}, z, \lambda), F(\bar{z}, z, -i\lambda)) \cdot (A_\ell, A_\ell)^{-1}\right) \\ &= \psi(A_\ell, A_\ell) \cdot \hat{\mathcal{F}}(\bar{z}, z, \lambda) \cdot (\psi(A_\ell, A_\ell))^{-1}, \end{aligned}$$

where

$$\psi(A_\ell, A_\ell) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\left(\frac{2\pi\ell}{k+2}\right) & -\sin\left(\frac{2\pi\ell}{k+2}\right) \\ 0 & 0 & \sin\left(\frac{2\pi\ell}{k+2}\right) & \cos\left(\frac{2\pi\ell}{k+2}\right) \end{pmatrix}.$$

Then a straightforward computation shows the first claim.

It is shown in [2] that the real function $\hat{u}(z, \bar{z})$ in the metric $2e^{\hat{u}}dzd\bar{z}$ of the surface from the frame F_λ is independent of $\arg(z)$. By (1.36), the real function $u(z, \bar{z})$ in the metric $2e^u dzd\bar{z}$ of the minimal Lagrangian surface f only depends on $|z|$. This completes the proof. $\square$

Note that when $c \in \mathbb{C} \setminus (\mathbb{S}^1 \cup \{0\})$, there is no simple way to perform the Iwasawa decomposition of this solution ϕ , so we cannot write the explicit form of all radially symmetric surfaces.

3.3. Minimal Lagrangian trinoids. Finally, we construct minimal Lagrangian surfaces from thrice-punctured sphere into Q_2 according to [20]. We first discuss the *monodromy problem* of the DPW method: For a given holomorphic potential ξ , there exists a unique solution Φ in **Step 1** in Section 2.2. Note that even ξ is well-defined on a Riemann surface M , the solution Φ may not be well-defined on M , that is, Φ is well-defined on the universal cover of M (see Section 3.2.1 for an example). To simplify the exposition, let us take $M = \mathbb{CP}^1 \setminus \{0, 1, \infty\}$, and set simple loops starting from the base point $z_0 \neq 0, 1, \infty$ going around 0, 1 and ∞ ending to z_0 by γ_j ($j = 0, 1, \infty$). Note that it has a simple relation $\gamma_\infty \gamma_1 \gamma_0 = \text{id}$, and γ_i and γ_j ($i \neq j$) are the generators of $\pi_1(M)$. The fundamental relations are given as follows:

$$\gamma_j^* \Phi = \mathcal{H}(\gamma_j) \Phi, \quad \mathcal{H}(\gamma_j) \in \text{ASL}(2, \mathbb{C})_\sigma,$$

and from the relation of $\gamma_\infty \gamma_1 \gamma_0 = \text{id}$, $\mathcal{H}(\gamma_\infty) \mathcal{H}(\gamma_1) \mathcal{H}(\gamma_0) = \text{id}$ follows. The matrices $\mathcal{H}(\gamma_j)$ are called the *monodromy matrices*. In **Step 2**, we have considered the Iwasawa decomposition to obtain the extended frame F_λ . In general, the behavior of F_λ under γ_j is not obvious, since the monodromy matrix $\mathcal{H}(\gamma_j)$ does not take values in $\text{ASU}(2)_\sigma$. Moreover, even though $\mathcal{H}(\gamma_j)$ takes values $\text{ASU}(2)_\sigma$, it is not clear that the resulting minimal Lagrangian immersion f^λ in **Step 4** is well-defined on M . We now state the following sufficient condition.

Proposition 3.7. *The resulting minimal Lagrangian immersion $f^\lambda|_{\lambda=\lambda_0}$ is well-defined on $M = \mathbb{CP}^1 \setminus \{0, 1, \infty\}$ if the monodromy matrices $\mathcal{H}(\gamma_j)$ ($j = 0, 1, \infty$) satisfy the following two conditions:*

- (1) $\mathcal{H}(\gamma_j)$ takes values in $\text{ASU}(2)_\sigma$ for $j = 0, 1, \infty$.
- (2) $\mathcal{H}(\gamma_j)|_{\lambda=\lambda_0} = \mathcal{H}(\gamma_j)|_{\lambda=-i\lambda_0} = \pm \text{id}$.

Proof. From the assumption (1), it is easy to see

$$\gamma_j^* F_\lambda = \mathcal{H}(\gamma_j) F_\lambda$$

holds. Then the resulting minimal Lagrangian immersion $f^\lambda|_{\lambda=\lambda_0}$ is given by (2.11) and (2.12), and the assumption (2) clearly shows that

$$\gamma_j^* f^\lambda|_{\lambda=\lambda_0} = f^\lambda|_{\lambda=\lambda_0}$$

holds. This completes the proof. $\square$

Remark 3.8. From the relation $\mathcal{H}(\gamma_\infty) \mathcal{H}(\gamma_1) \mathcal{H}(\gamma_0) = \text{id}$, the assumptions of Proposition 3.7 only need to satisfy two matrices of $\{\mathcal{H}(\gamma_0), \mathcal{H}(\gamma_1), \mathcal{H}(\gamma_\infty)\}$.

Now we give the definition of trinoid potential, which can be used to construct constant mean curvature trinoid in $\mathbb{S}^3$. In the following, we use the *untwisted* loop group according to [20], however, it makes no essential difference. Indeed, there is an isomorphism between them: For an untwisted element $g(\lambda) \in \text{ASU}(2)$, define a map

$$g(\lambda) \mapsto \text{Ad} \text{diag}(\sqrt{\lambda}, \sqrt{\lambda}^{-1}) g(\lambda^2),$$

which gives a twisted element $\text{ASU}(2)_\sigma$ and it is an isomorphism between the loop groups.

Definition 3.9 (Minimal trinoid potentials, [20]). Let $M = \mathbb{CP}^1 \setminus \{0, 1, \infty\}$ be the thrice-punctured Riemann sphere. Let $\lambda_0 \in \{\pm\sqrt{-1}\}$, and let

$$h(\lambda) = \lambda^{-1}(\lambda - \lambda_0)(\lambda - \lambda_0^{-1}).$$

Let $v_0, v_1, v_\infty \in \mathbb{R} \setminus \{0\}$. The family of trinoid potentials ξ , parametrized by λ_0 and v_0, v_1, v_∞ , is defined by

$$\xi = \begin{pmatrix} 0 & \lambda^{-1}dz \\ \lambda h(\lambda)Q/dz & 0 \end{pmatrix}, \quad Q = \frac{v_\infty z^2 + (v_1 - v_0 - v_\infty)z + v_0}{16z^2(z-1)^2} dz^2.$$

Here Q will be the Hopf differential of the resulting minimal trinoid, up to a z -independent multiplicative constant. Its three poles are the ends of the minimal trinoid, and its two zeros are the umbilic points.

Remark 3.10. The choice of λ_0 determines the value of constant mean curvature of the resulting trinoids in $\mathbb{S}^3$, and we are only interested in minimal trinoids; thus, we choose $\lambda_0 \in \{\pm\sqrt{-1}\}$.

Theorem 3.11 (Theorem 5 in [20]). *Let $\mathcal{H}(\gamma_j)$ ($j = 0, 1, \infty$) be monodromy matrices of a trinoid potential, and assume that $\mathcal{H}(\gamma_j)$ is holomorphic on $\mathbb{C}^*$. Let the parameters in the potential be $\lambda_0, v_0, v_1, v_\infty$, and for $k \in \{0, 1, \infty\}$ suppose that $1 + v_k h(-1)/4 \geq 0$ and $1 + v_k h(1)/4 \geq 0$. Define*

$$n_k = \frac{1}{2} - \frac{1}{2}\sqrt{1 + v_k h(-1)/4}, \quad m_k = \frac{1}{2} - \frac{1}{2}\sqrt{1 + v_k h(1)/4}.$$

Suppose the following inequalities hold for every permutation (i, j, k) of $(0, 1, \infty)$:

$$\begin{aligned} |n_0| + |n_1| + |n_\infty| &\leq 1 \quad \text{and} \quad |n_i| \leq |n_j| + |n_k|, \\ |m_0| + |m_1| + |m_\infty| &\leq 1 \quad \text{and} \quad |m_i| \leq |m_j| + |m_k|. \end{aligned}$$

Then such a trinoid potential satisfying the above conditions generates via the DPW method a conformal minimal immersion of the thrice-punctured Riemann sphere into $\mathbb{S}^3$.

Applying Proposition 3.7, we obtain the following corollary.

Corollary 3.12. *The trinoid potentials ξ with the assumption in Theorem 3.11 yield minimal Lagrangian trinoids in Q_2 .*

APPENDIX A. COMPARISON TO THE RESULT OF CASTRO-URBANO

In this section, we shall discuss a relation to the result of Castro-Urbano [4] and establish a correspondence with their quantities. The following notations and definitions are all used in [4].

Let $\mathbb{S}^2$ be the unit sphere in the Euclidean space $\mathbb{R}^3$ endowed with its standard Euclidean metric $\langle, \rangle$ and its structure of Riemann surface given by $J_x v = x \times v$. for any $v \in T_x \mathbb{S}^2$, $x \in \mathbb{S}^2$, where $\times$ stands for the vectorial product in $\mathbb{R}^3$. Its Kähler 2-form is the area 2-form ω_0 defined by

$$\omega_0(v, w) = \langle J_x v, w \rangle = \det\{x, v, w\}$$

for any $v, w \in T_x \mathbb{S}^2$. We endow $\mathbb{S}^2 \times \mathbb{S}^2$ with the product metric (also denoted by $\langle, \rangle$) and the product complex structure J given by

$$J_{(x,y)}(v) = (J_x v_1, J_y v_2) = (x \times v_1, y \times v_2),$$

for any $v = (v_1, v_2) \in T_{(x,y)}(\mathbb{S}^2 \times \mathbb{S}^2)$, $(x, y) \in \mathbb{S}^2 \times \mathbb{S}^2$, which becomes $\mathbb{S}^2 \times \mathbb{S}^2$ in a Kähler surface. Its Kähler 2-form is $\omega = \pi_1^* \omega_0 + \pi_2^* \omega_0$, where π_i , $i = 1, 2$, are the projections of $\mathbb{S}^2 \times \mathbb{S}^2$ onto $\mathbb{S}^2$.

Let $\Phi = (\phi, \psi) : \Sigma \rightarrow \mathbb{S}^2 \times \mathbb{S}^2$ be a Lagrangian immersion of an oriented surface with area 2-form ω_Σ . Define the Jacobians of ϕ and ψ by

$$\phi^* \omega_0 = \text{Jac}(\phi) \omega_\Sigma, \quad \psi^* \omega_0 = \text{Jac}(\psi) \omega_\Sigma, \quad \text{Jac}(\phi) = -\text{Jac}(\psi).$$

We will call the function

$$C := \text{Jac}(\phi) = -\text{Jac}(\psi)$$

the *associated Jacobian* of the oriented Lagrangian surface Σ .

We consider a local isothermal parameter $z = x + iy$ on Σ . To begin with, we assume the metric

$$(A.1) \quad ds_\Sigma^2 = 8e^u dz d\bar{z},$$

and the formulas for Lagrangian immersion Φ in $\mathbb{S}^2 \times \mathbb{S}^2$ transform into

$$\begin{aligned} |\Phi_z|^2 &= 2|\phi_z|^2 = 2|\psi_z|^2 = 4e^u, \quad \Phi_{z\bar{z}} = 4e^u \left(H - \frac{1}{2} \Phi \right), \\ \Phi_{zz} &= u_z \Phi_z + \langle H, J\Phi_z \rangle J\Phi_z + \frac{1}{4} e^{-u} \langle \Phi_{zz}, J\Phi_z \rangle J\Phi_{\bar{z}} - \frac{1}{2} \langle \Phi_z, \hat{\Phi}_z \rangle \hat{\Phi}, \\ \hat{\Phi}_z &= \frac{1}{4} e^{-u} \langle \Phi_z, \hat{\Phi}_z \rangle \Phi_{\bar{z}} - 2iC J\Phi_z, \end{aligned}$$

where the derivatives respect to z and $\bar{z}$ are given by $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$, $\partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y)$, and H is the mean curvature of Φ , and $\hat{\Phi} = (\phi, -\psi)$. These quantities correspond to (4.3), (4.4) and (4.6) in [4]. When Φ is minimal, it follows that

$$(A.2) \quad \Phi_{z\bar{z}} = -2e^u \Phi,$$

$$(A.3) \quad \Phi_{zz} = u_z \Phi_z + \frac{1}{4} e^{-u} \langle \Phi_{zz}, J\Phi_z \rangle J\Phi_{\bar{z}} - \frac{1}{2} \langle \Phi_z, \hat{\Phi}_z \rangle \hat{\Phi},$$

$$(A.4) \quad \hat{\Phi}_z = \frac{1}{4} e^{-u} \langle \Phi_z, \hat{\Phi}_z \rangle \Phi_{\bar{z}} - 2iC J\Phi_z,$$

$$(A.5) \quad 16e^{2u}(1 - 4C^2) = \left| \langle \Phi_z, \hat{\Phi}_z \rangle \right|^2 = 4|\langle \phi_z, \phi_z \rangle|^2 = 4|\langle \psi_z, \psi_z \rangle|^2.$$

The equation (A.2) implies that $\phi_{z\bar{z}} = -2e^u \phi$ and $\psi_{z\bar{z}} = -2e^u \psi$. This means that $\phi, \psi : \Sigma \rightarrow \mathbb{S}^2$ are harmonic maps. Thus, the associated Hopf differential

$$\Theta(z) = \langle \phi_z, \phi_z \rangle \otimes (dz)^2 = -\langle \psi_z, \psi_z \rangle \otimes (dz)^2 = \frac{1}{2} \langle \Phi_z, \hat{\Phi}_z \rangle \otimes (dz)^2$$

is holomorphic. From (A.5), we have

$$(A.6) \quad |\Theta|^2 = 4e^{2u} (1 - 4C^2).$$

The squared norm of the gradient of C is given in (4.12) in [4] by

$$|\nabla C|^2 = \frac{(1 - 4C^2)(2C^2 - K)}{2}.$$

Since the Gauss curvature K can be given by $K = -\frac{1}{4}e^{-u}u_{z\bar{z}}$, it can be rephrased as

$$(A.7) \quad 4|C_z|^2 = (1 - 4C^2)(u_{z\bar{z}} + 8e^u C^2).$$

To establish the correspondence between minimal Lagrangian surfaces in Q_2 and $\mathbb{S}^2 \times \mathbb{S}^2$, we first note that Q_2 is isometric to $\mathbb{S}^2 \times \mathbb{S}^2$, where two 2-spheres $\mathbb{S}^2$ have the constant curvature 4 [22]. Thus to compare our surface f and [4], we need to use the metric

$$4ds_M^2 = ds_\Sigma^2 = 8e^u dz d\bar{z}.$$

We now show (A.7) is the Gauss equation for a minimal Lagrangian surface f in Q_2 under suitable identification.

Proposition A.1. *Let $f : M \rightarrow Q_2$ be a minimal Lagrangian immersion, and define $\hat{C} = \frac{1}{2}e^{-u}|\beta|$. Then the last PDE of (1.19) is equivalent to*

$$(A.8) \quad u_{z\bar{z}} + 8e^u \hat{C}^2 - \frac{4|\hat{C}_z|^2}{1 - 4\hat{C}^2} = 0.$$

Proof. Note $|\alpha|^2 = e^{2u}(1 - 4\hat{C}^2)$ holds. When $\beta \equiv 0$, that is, $\hat{C} = 0$, by Lemma 1.2, f becomes the totally geodesic torus and (A.8) clearly holds. From now on we assume that $\beta \not\equiv 0$. By $\hat{C} = \frac{1}{2}e^{-u}|\beta|$, it follows that

$$\frac{1}{4}|\alpha_z|^2 e^u + |u_z|^2 e^u |\alpha|^2 - \frac{1}{2}\alpha_z u_{\bar{z}} e^u \bar{\alpha} - \frac{1}{2}\bar{\alpha}_{\bar{z}} u_z e^u \alpha = e^u |\beta|^2 \left(\frac{4|\hat{C}_z|^2}{1 - 4\hat{C}^2} \right),$$

and

$$2|\beta|^4 = e^u |\beta|^2 (8e^u \hat{C}^2).$$

Note that $e^u |\beta|^2 \left(\frac{4|\hat{C}_z|^2}{1 - 4\hat{C}^2} \right)$ is well-defined even if a point p where $\hat{C}(p) = \frac{1}{2}$, since $\hat{C}_z(p) = 0$ there. Thus the last PDE of (1.19) can be written as follows:

$$(A.9) \quad e^u |\beta|^2 \left(u_{z\bar{z}} + 8e^u \hat{C}^2 - \frac{4|\hat{C}_z|^2}{1 - 4\hat{C}^2} \right) = 0.$$

Since $\beta \not\equiv 0$, we have (A.8). This completes the proof. $\square$

By Proposition A.1, it is natural to have the correspondence of the associated Hopf differential Θ and the associated Jacobian C with the quantities α, β and u by

$$(A.10) \quad \Theta = 2\alpha, \quad C = \hat{C} = \frac{1}{2}e^{-u}|\beta|.$$

It is easy to check the condition $C^2 < 1/4$ in [4, Theorem 4.4] is actually the condition $|\alpha| > 0$. Indeed, a straightforward computation shows that (A.10) holds for surfaces Φ in $\mathbb{S}^2 \times \mathbb{S}^2$ and f in Q_2 .

Remark A.2. The normalization of Θ given in [4] corresponds to the normalization of Θ here, i.e. $|\Theta| = 1$. We then obtain the Gauss and Codazzi equations (4.19) in [4] of the minimal Lagrangian immersion Φ under the new metric $ds_\Sigma^2 = 8e^u dz d\bar{z}$ as follows:

$$(A.11) \quad u_{z\bar{z}} + 8e^u C^2 - 16e^{2u} |C_z|^2 = 0, \quad 1 - 4C^2 = \frac{1}{4}e^{-2u}.$$

APPENDIX B. THE LIE GROUP AND THE LIE ALGEBRA ISOMORPHISM FOR $SO(4)$

We first check that the Lie group isomorphism

$$(B.1) \quad SO(4) \cong (SU(2) \times SU(2))/\mathbf{Z}_2,$$

which is explicitly given in [24].

The set of all quaternions can be defined by

$$\mathbf{H} = \mathbf{R} \oplus \mathbf{R}\mathbf{i} \oplus \mathbf{R}\mathbf{j} \oplus \mathbf{R}\mathbf{k} = \{x + y\mathbf{i} + s\mathbf{j} + t\mathbf{k} \mid x, y, s, t \in \mathbf{R}\},$$

with $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$, $\mathbf{i}\mathbf{j} = -\mathbf{j}\mathbf{i} = \mathbf{k}$, $\mathbf{j}\mathbf{k} = -\mathbf{k}\mathbf{j} = \mathbf{i}$, $\mathbf{k}\mathbf{i} = -\mathbf{i}\mathbf{k} = \mathbf{j}$, where $\mathbf{R}$ is the set of all real numbers. For a quaternion α , it can be expressed as follows:

$$\alpha = x + y\mathbf{i} + s\mathbf{j} + t\mathbf{k} = (x + y\mathbf{i}) + (s + t\mathbf{i})\mathbf{j} \quad x, y, s, t \in \mathbf{R}.$$

Thus the set $\mathbf{H}$ can also be written as $\mathbf{H} = \mathbf{C} \oplus \mathbf{C}\mathbf{j} = \{a + b\mathbf{j} \mid a, b \in \mathbf{C}\}$ with $\mathbf{j}^2 = -1$, where $\mathbf{C}$ is the set of all complex numbers. Now we define the symplectic group by

$$\mathrm{Sp}(n) := \{A \in M(n, \mathbf{H}) \mid A^* A = E\}.$$

In particular, if $\alpha = a + b\mathbf{j} \in \mathrm{Sp}(1)$, it satisfies $|a|^2 + |b|^2 = 1$.

The Lie group isomorphism (B.1) can be established in the following two steps.

- (1) $SU(2) \cong \mathrm{Sp}(1)$. When $p = p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k} \in \mathrm{Sp}(1)$, we can establish the map $k : \mathrm{Sp}(1) \rightarrow SU(2)$ by

$$k(p) = \begin{pmatrix} p_0 + p_1\mathbf{i} & p_2 + p_3\mathbf{i} \\ -p_2 + p_3\mathbf{i} & p_0 - p_1\mathbf{i} \end{pmatrix}.$$

See [24, p. 93, Theorem 10.4] for details.

- (2) $(\mathrm{Sp}(1) \times \mathrm{Sp}(1))/\mathbf{Z}_2 \cong SO(4)$. When $p = p_0 + p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}$, $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \in \mathrm{Sp}(1)$, we can establish the map $\psi : \mathrm{Sp}(1) \times \mathrm{Sp}(1) \rightarrow SO(4)$ by

$$(B.2) \quad \psi(p, q) = \begin{pmatrix} p_0 & -p_1 & -p_2 & -p_3 \\ p_1 & p_0 & -p_3 & p_2 \\ p_2 & p_3 & p_0 & -p_1 \\ p_3 & -p_2 & p_1 & p_0 \end{pmatrix} \begin{pmatrix} q_0 & q_1 & q_2 & q_3 \\ -q_1 & q_0 & -q_3 & q_2 \\ -q_2 & q_3 & q_0 & -q_1 \\ -q_3 & -q_2 & q_1 & q_0 \end{pmatrix}.$$

Clearly, the kernel of ψ is $\pm \mathrm{id}$, and the isomorphism can be established. See [24, p. 99, Theorem 10.7] for details.

Accordingly, we can introduce the Lie algebra isomorphism

$$(B.3) \quad \mathfrak{so}(4) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2).$$

Define the Lie algebra of the symplectic group by

$$\mathfrak{sp}(n) := \{X \in M(n, \mathbf{H}) \mid X^* + X = 0\}.$$

In particular, if $\alpha = a + b\mathbf{j} \in \mathfrak{sp}(1)$, it is easy to check $\bar{\alpha} = \bar{a} - b\mathbf{j}$, and it follows that $\bar{a} = -a$.

The Lie algebra isomorphism (B.3) can be established in the following two steps.

- (1) $\mathfrak{sp}(1) \cong \mathfrak{su}(2)$. When $p = p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k} \in \mathfrak{sp}(1)$, we can establish the map $k : \mathfrak{sp}(1) \rightarrow \mathfrak{su}(2)$ by

$$k(p) = \begin{pmatrix} p_1\mathbf{i} & p_2 + p_3\mathbf{i} \\ -p_2 + p_3\mathbf{i} & -p_1\mathbf{i} \end{pmatrix}.$$

See [24, p. 119, Theorem 11.9] for details.

- (2) $\mathfrak{o}(4) \cong \mathfrak{sp}(1) \oplus \mathfrak{sp}(1)$. When $p = p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}$, $q = q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \in \mathfrak{sp}(1)$, we can establish the map $\psi : \mathfrak{sp}(1) \oplus \mathfrak{sp}(1) \rightarrow \mathfrak{o}(4)$ by

$$\psi(p_1\mathbf{i} + p_2\mathbf{j} + p_3\mathbf{k}, q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}) = \begin{pmatrix} 0 & -p_1 + q_1 & -p_2 + q_2 & -p_3 + q_3 \\ p_1 - q_1 & 0 & -p_3 - q_3 & p_2 + q_2 \\ p_2 - q_2 & p_3 + q_3 & 0 & -p_1 - q_1 \\ p_3 - q_3 & -p_2 - q_2 & p_1 + q_1 & 0 \end{pmatrix}.$$

See [24, p. 120, Theorem 11.10] for details.

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